To the Graduate Council:

I am submitting herewith a thesis written by Lawrence William Burns III entitled "On Moore Families in Semigroups." I recommend that it be accepted in partial fulfillment of the requirements for the degree of Master of Arts, with a major in Mathematics.

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ON MOORE FAMILIES IN SEMIGROUPS

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ABSTRACT

The properties of Moore families and dual Moore families are studied in general and several specific occurrences of each in semigroups are shown. In particular, the family of all regular subsets of a semigroup is shown to be a dual Moore family and, based on this fact, the regular core of a subset of a semigroup is defined. The regular core of a Green's class of a semigroup is described and an example is given to show that a $\mathcal{J}$-class J may have a non-empty regular core that is a proper subset of J. The thesis is concluded with a study of congruence relations on a semigroup by use of the properties of Moore families.

PREFACE

In this thesis a definition for a dual Moore family of subsets of an arbitrary set is given, based on the definition for a Moore family of subsets as given in [2]. The purpose of this thesis is to study the properties of Moore families and dual Moore families in general, and specific occurrences of each in the study of semigroups.

Generally we adhere to the notation of Clifford and Preston's treatise [1]. In particular, inclusion of a set A in a set B is denoted by $A \subseteq B$, proper inclusion by $A \subset B$, and the complement of A in B by $B \setminus A$; we write mapping symbols to the right of their arguments. In addition to the above conventions, we denote by $P(X)$ the power set of X (that is, the set of all subsets of X); and by $\phi|A$, where ϕ is a function and A is a subset of the domain of ϕ , we shall mean the restriction of ϕ to the set A .

In [2] a Moore family of subsets of a set X is defined to be a family of subsets of X that contains X and is closed under arbitrary intersections. Associated with a Moore family is the Moore closure mapping, from $P(X)$ onto the Moore family which maps a subset A of X to the intersection of all members of the Moore family that contain A . In Chapter I we define a dual Moore family of subsets of a set X to be a family of subsets that contains the empty set $\square$ and is closed under arbitrary unions. Associated with any dual Moore family is a mapping from $P(X)$ onto the dual Moore family and this mapping, called a Moore core mapping, maps a subset A of X to the union of all members of the dual Moore family that are contained in A . The correspondence between each type of family and its associated mapping is discussed and properties of Moore families and dual Moore families are studied.

Several examples of Moore families and dual Moore families are presented in Chapter II. In particular, the family of all regular subsets of a semigroup S (that is, subsets A of S such that if $a \in A$ then there exists $b \in A$ such that $aba = a$) is shown to be a dual Moore family.

Recall that the five Green's relations on a semigroup S as defined in [1, section 2.1] are denoted by $\mathcal{L}$, $\mathcal{R}$, $\mathcal{D}$, $\mathcal{H}$, and $\mathcal{J}$. Since these are equivalence relations, they partition S into Green's classes called $\mathcal{L}$-classes, $\mathcal{R}$-classes, $\mathcal{D}$-classes, $\mathcal{H}$-classes, and $\mathcal{J}$-classes, respectively. In Chapter III we define the regular core of a subset X of a semigroup S to be the image of X under the Moore core mapping associated with the dual Moore family of regular subsets of S . The regular cores of the Green's classes of a semigroup are studied and we show that the regular core of an $\mathcal{L}$-class L [$\mathcal{R}$-class R] is the union of all $\mathcal{H}$-classes in L[R] that contain idempotents, and therefore is either empty or a regular subsemigroup of S . We also include an example to show that a $\mathcal{J}$-class may have a non-empty proper regular core (unlike $\mathcal{D}$-classes or $\mathcal{H}$-classes).

We conclude this thesis with Chapter IV in which congruences on a semigroup are studied using the properties of Moore families.

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CHAPTER 1

PROPERTIES OF MOORE FAMILIES AND DUAL MOORE FAMILIES

Let X be a non-empty set and F a non-empty family of subsets of X such that (i) $X \in F$ and (ii) if F^* is a non-empty subfamily of F then $\cap F^* \in F$. Such a family F we shall call a Moore family of sets.

Example. Let X be a non-empty set and let ϕ be any mapping of X into X. Let S_ϕ be the family of all subsets T of X that are stable under ϕ (i.e. $T\phi \subseteq T$). Since $X\phi \subseteq X$ we have $X \in S_\phi$. Now, let $\{T_i : i \in I\}$ be a non-empty subfamily of S_ϕ. Then

$$(\cap \{T_i : i \in I\})\phi \subseteq \cap \{T_i\phi : i \in I\} \subseteq \cap \{T_i : i \in I\}$$

whence $\cap \{T_i : i \in I\}$ is an element of S_ϕ, and we have shown that S_ϕ is a Moore family.

Let X be a set and let F be a Moore family of subsets of X. If $Y \subseteq X$ let $\bar{Y} = \cap\{F \in F : Y \subseteq F\}$ and call $\bar{Y}$ the closure (more precisely, the F-closure) of Y. Clearly (i) implies that every subset of X has an F-closure, and it is immediate that

(1) $\qquad Y \subseteq \bar{Y}$ for all $Y \subseteq X$.

Let $Z \subseteq Y \subseteq X$. Then $Z \subseteq Y \subseteq \bar{Y} \subseteq X$ by (1). But, by (ii), $\bar{Y} \in F$, whence $Z \subseteq \bar{Y}$ implies that $\bar{Z} \subseteq \bar{Y}$. We have thus established

(2) if $Z \subseteq Y$ then $\overline{Z} \subseteq \overline{Y}$.

Clearly $Z = \overline{Z}$ if and only if $Z \in F$; in particular

(3) $\overline{\overline{Y}} = \overline{Y}$ for all $Y \subseteq X$.

Any mapping $Y \mapsto Y'$ of $P(X)$ that is <u>extensive</u> (i.e., $Y \subseteq Y'$), <u>monotone</u> (i.e., if $Z \subseteq Y$ then $Z' \subseteq Y'$) , and <u>idempotent</u> (i.e., $Y'' = Y'$) will be called a <u>closure mapping</u> (called a <u>Moore closure mapping</u> in [2]), and we have just shown that to a given Moore family we may associate a closure mapping $Y \mapsto \overline{Y}$. Conversely, let X be a non-empty set and $\mu: Y \mapsto Y'$ a closure mapping of $P(X)$ into $P(X)$, and let

$$F_\mu = \{F \subseteq X: \ F = Y' \text{ for some } Y \subseteq X\} .$$

Since $X \in P(X)$ and μ is extensive, $X' = X$; hence $X \in F_\mu$. Let F^*_μ be a non-empty subfamily of F_μ , let $A = \cap F^*_\mu$, and let $Z \in F^*_\mu$. Then $Z \in F_\mu$, whence $Z = Y'$ for some $Y \subseteq X$, and since μ is idempotent we have $Z' = Y'' = Y' = Z$. Since μ is monotone and $A \subseteq Z$, $A' \subseteq Z'$. Thus $A' \subseteq Z$, and since Z is an arbitrary member of F^*_μ it follows that $A' \subseteq \cap F^*_\mu = A$. But, μ being extensive, $A \subseteq A'$. Hence $A = A' \in F_\mu$ and we conclude that F_μ is a Moore family.

We have now shown, reproducing almost verbatim the discussion in [2], that there is a one-to-one correspondence between the class of all Moore families of subsets of a set X and the class of all closure mappings of $\mathcal{P}(X)$ into itself. A mapping in the latter class may be called a closure operation in X. It is immediate that the $\mathcal{F}$-closure of a subset Y of X is the least member of $\mathcal{F}$ containing Y.

Let X be a non-empty set and $\mathcal{G}$ a non-empty family of subsets of X such that (i') $\square \in \mathcal{G}$ and (ii') if $\mathcal{G}^*$ is a subfamily of $\mathcal{G}$ then $\cup\, \mathcal{G}^* \in \mathcal{G}$. Such a family $\mathcal{G}$ we shall call a dual Moore family of sets.

Example. Let X be a non-empty set, let ϕ be any mapping of X into X, and let $\mathcal{S}_\phi$ be the family of all subsets of X that are stable under ϕ. Since $\square\phi = \square$ we have $\square \in \mathcal{S}_\phi$. Now, let $\{T_i\colon i \in I\}$ be a subfamily of $\mathcal{S}_\phi$. Then

$$(\cup\, \{T_i\colon i \in I\})\phi = \cup\, \{T_i\phi\colon i \in I\} \subseteq \cup\, \{T_i\colon i \in I\},$$

whence $\cup\, \{T_i\colon i \in I\}$ is an element of $\mathcal{S}_\phi$, and we have shown that $\mathcal{S}_\phi$ is a dual Moore family. A family $\mathcal{F}$ of subsets of a set X that is both a Moore family and a dual Moore family will be called a double Moore family. Thus, by this example and the previous one, we have established the following

Lemma 1. *Let X be a non-empty set, let ϕ be any mapping of X into X, and let $\mathcal{S}_\phi$ be the family of all subsets of X that are stable under ϕ. Then $\mathcal{S}_\phi$ is a double Moore family.*

Let X be a set and let $\mathcal{G}$ be a dual Moore family of subsets of X. If $Y \subseteq X$ let $\breve{Y} = \cup\, \{G \in \mathcal{G}\colon G \subseteq Y\}$ and call $\breve{Y}$ the core (more precisely, the $\mathcal{G}$-core) of Y. Clearly (i') implies that every subset of X has a $\mathcal{G}$-core, and it is immediate that

(1') $\breve{Y} \subseteq Y$ for all $Y \subseteq X$.

Let $Z \subseteq Y \subseteq X$. Then $\breve{Z} \subseteq Z \subseteq Y$ by (1') . But, by (ii'), $\breve{Z} \in G$, whence $\breve{Z} \subseteq Y$ implies that $\breve{Z} \subseteq \breve{Y}$. We have thus established

(2') if $Z \subseteq Y$ then $\breve{Z} \subseteq \breve{Y}$.

Clearly $Z = \breve{Z}$ if and only if $Z \in G$; in particular,

(3') $\breve{\breve{Y}} = \breve{Y}$ for all $Y \subseteq X$.

Any mapping $Y \mapsto Y'$ of $P(X)$ that is <u>intensive</u> (i.e., $Y' \subseteq Y$) , monotone, and idempotent will be called a <u>core mapping</u>, and we have just shown that to any dual Moore family we may associate a core mapping $Y \mapsto Y'$. Conversely, let X be a non-empty set and μ: $Y \mapsto Y'$ a core mapping of $P(X)$ into $P(X)$, and let $G_\mu = \{G \subseteq X\colon G = Y'$ for some $Y \subseteq X\}$. Since $\square \in P(X)$ and μ is intensive, $\square' = \square$; hence $\square \in G_\mu$. Let G^*_μ be a subfamily of G_μ , let $U = \cup G^*_\mu$, and let $Z \in G^*_\mu$. Then $Z \in G_\mu$, whence $Z = Y'$ for some $Y \subseteq X$, and since μ is idempotent we have $Z' = Y'' = Y' = Z$. Since μ is monotone and $Z \subseteq U$, $Z' \subseteq U'$. Thus $Z \subseteq U'$, and since Z is an arbitrary member of G^*_μ it follows that $U = \cup G^*_\mu \subseteq U'$. But, μ being intensive, $U' \subseteq U$. Hence $U = U' \in G_\mu$ and we conclude that G_μ is a dual Moore family.

We have now shown that there is a one-to-one correspondence between the class of all dual Moore families of subsets of a set X and the class of all core mappings of $\mathcal{P}(X)$ into itself; by a slight abuse of language, mappings in the latter class might be called <u>core operations in</u> X (by analogy with the dual concept of closure operation in X). It is immediate that the $\mathcal{G}$-core of a subset Y of X is the greatest member of $\mathcal{G}$ contained in Y.

We note that a core mapping need be neither union- nor intersection-preserving. Let $X = \{a, b, c\}$, let $U = \{a, b\}$ and $V = \{b, c\}$; then the family $\mathcal{G} = \{X, U, V, \square\}$ is a dual Moore family and therefore is associated with a core mapping $Y \mapsto \overset{\smile}{Y}$ where $\overset{\smile}{Y}$ is the greatest member of $\mathcal{G}$ contained in Y. Now $\overset{\smile}{\{a\}} = \square$ and $\overset{\smile}{\{b\}} = \square$, whence $\overset{\smile}{\{a\}} \cup \overset{\smile}{\{b\}} = \square$. However $\overset{\smile}{\{a\} \cup \{b\}} = \overset{\smile}{\{a, b\}} = \overset{\smile}{U} = U$ and thus the core mapping is not union-preserving. Since U and V are in $\mathcal{G}$, we have $\overset{\smile}{U} \cap \overset{\smile}{V} = U \cap V = \{b\}$; but $\overset{\smile}{U \cap V} = \overset{\smile}{\{b\}} = \square$, and therefore this core mapping is not intersection-preserving. Similarly, it can be shown that a closure mapping may be neither union- nor intersection-preserving.

Again let $X = \{a, b, c\}$. Let $F_1 = \{X, \{a, b\}\}$ and $F_2 = \{X, \{b, c\}\}$; then it is clear that F_1 and F_2 are Moore families of subsets of X. But $F_1 \cup F_2$ is not a Moore family since $\{b\}$ is not a member of $F_1 \cup F_2$. However, we do have the following

<u>Theorem</u> 2. <u>Let</u> X <u>be a set and</u> $\{F_\alpha : \alpha \in \Lambda\}$ <u>be a collection of Moore families of subsets of</u> X. <u>Then</u> $\cap \{F_\alpha : \alpha \in \Lambda\}$ <u>is a Moore family of subsets of</u> X.

Proof. Let $F = \cap\{F_\alpha : \alpha \in \Lambda\}$. Then X is an element of F since $X \in F_\alpha$ for each $\alpha \in \Lambda$. Now, let F' be a subfamily of F and let $F' = \cap\{F : F \in F'\}$. Then for each $\alpha \in \Lambda$ we have F' a subfamily of F_α . Thus $F' \in F_\alpha$ for each α in Λ , since each F_α is a Moore family and therefore closed under intersection. Hence F' is an element of F and, since we know that $X \in F$, we have shown F to be a Moore family.

Let $X = \{a, b\}$, and let $G_1 = \{\{a\}, \square\}$ and $G_2 = \{\{b\}, \square\}$. Then G_1 and G_2 are dual Moore families of subsets of X but $G_1 \cup G_2$ is not a dual Moore family since X is not a member of $G_1 \cup G_2$. However, we do have the following

Theorem 3. *Let* X *be a set and* $\{G_\alpha : \alpha \in \Lambda\}$ *a collection of dual Moore families of subsets of* X . *Then* $\cap\{G_\alpha : \alpha \in \Lambda\}$ *is a dual Moore family of subsets of* X .

Proof. Let $G = \cap\{G_\alpha : \alpha \in \Lambda\}$ and let G' be a subfamily of G . Then, for each α in Λ , we have G' a subfamily of G_α since G is contained in G_α and therefore $\cup\{G : G \in G'\}$ is an element of G_α because G_α is a dual Moore family. Since $\cup\{G : G \in G'\}$ is a member of each G_α , it is a member of G . Thus we have shown that G is closed under arbitrary unions. Since we also have $\square$ an element of G , we have established that G is a dual Moore family.

CHAPTER II

MOORE FAMILIES AND DUAL MOORE FAMILIES IN SEMIGROUPS

In this chapter we study several examples of Moore families and dual Moore families that occur in a semigroup.

Let S be a semigroup. In [1], a subsemigroup T of S is a non-empty subset of S in which a and b are elements of T only if their product ab is also in T. Note that S is a subsemigroup of S and it is well known that the intersection of subsemigroups of S is either empty or a subsemigroup of S. Thus we have established

Proposition 4. *If S is a semigroup and F is the family consisting of the null set and all subsemigroups of S then F is a Moore family.*

In the next examples we consider the right [left, two-sided] ideals of a semigroup. Recall that ideals are defined in [1] to be non-empty. It is well known that the intersection of a finite number of two-sided ideals is non-empty since it contains their product. However, this may not be the case for an arbitrary intersection of two-sided ideals, nor for the intersection of even two right [left] ideals. For this reason, and because we want the family of right [left, two-sided] ideals to be a dual Moore family (whence it must contain the null set), we adjoin the null set to the family of all right [left, two-sided] ideals.

Proposition 5. *The family consisting of the null set and the collection of all right [left, two-sided] ideals of a semigroup* S *is a double Moore family.*

Proof. Let S be a semigroup and let $\mathcal{P}$ be the family consisting of all subsets of S that are either right ideals or empty. Then S and $\square$ are both members of $\mathcal{P}$. Now let $\mathcal{P}'$ be an arbitrary subfamily of $\mathcal{P}$. Then

$$(\cap \{I: I \in \mathcal{P}'\})\ S \subseteq \cap \{IS: I \in \mathcal{P}'\} \subseteq \cap \{I: I \in \mathcal{P}'\},$$

whence $\cap \{I: I \in \mathcal{P}'\}$ is either empty or a right ideal of S and therefore is an element of $\mathcal{P}$. We also have

$$(\cup \{I: I \in \mathcal{P}'\})S = \cup \{IS: I \in \mathcal{P}'\} \subseteq \cup \{I: I \in \mathcal{P}'\} ,$$

whence $\cup \{I: I \in \mathcal{P}'\}$ is either a right ideal or empty, and therefore is an element of $\mathcal{P}$. Thus $\mathcal{P}$ is a double Moore family.

The proofs for left ideals and two-sided ideals are similar to the argument above.

An alternate proof may be obtained by an application of Lemma 1. Consider ρ_a , the inner right translation of S by an element a of S . Recall that if $x \in S$ then $x\rho_a = xa$. Let $\mathcal{S}(\rho_a)$ be the family of subsets of S that are stable under the mapping ρ_a . Note that $\mathcal{S}(\rho_a)$ is a double Moore family by Lemma 1. Let $\mathcal{P} = \cap \{\mathcal{S}(\rho_a): a \in S\}$; then, by Theorems 2 and 3, $\mathcal{P}$ is a double Moore family. Now, note that a subset X of S is an element of $\mathcal{P}$ if and only if $X\rho_a \subseteq X$ for all a in S ; that is, if and only if either X is a right ideal of S or X is empty. Therefore, the family consisting of the null set and the collection of all right ideals of S is P, which we have shown to be a double Moore family. Let $Q = \cap \{S(\lambda_a): a \in S\}$ where λ_a is the inner left translation of S by a. By a proof that is similar to the one for P, we can show that Q is the family consisting of the null set and all left ideals of S, and that Q is a double Moore family. Thus $P \cap Q$ is a double Moore family; but $P \cap Q$ is the family consisting of the null set and all two-sided ideals of S.

We say that a two-sided [left, right] ideal of a semigroup S is <u>completely prime</u> if $S \setminus A$ is a subsemigroup of S.

<u>Proposition</u> 6. <u>Let S be a semigroup and let G be the family consisting of the null set, S, and all completely prime two-sided [left, right] ideals of S. Then G is a dual Moore family but is not, in general, a Moore family.</u>

<u>Proof</u>. We prove only the case for two-sided ideals since right or left ideals may be handled by the same technique.

Let I be the family consisting of the null set and all two-sided ideals (henceforth referred to as "ideals") of S. Then I is a dual Moore family by Proposition 5.

Let K be the family of all subsets X of S such that $S \setminus X$ is empty or a subsemigroup of S. Then the null set is an element of K since $S \setminus \square$ is S. Now, let K' be a subfamily of K. Then

$$S \setminus [\cup \{K: K \in K'\}] = \cap \{S \setminus K: K \in K'\} ,$$

which is either empty or a subsemigroup of S . Therefore $\cup\{K: K \in \mathcal{K}'\}$ is an element of $\mathcal{K}$ and we have shown that $\mathcal{K}$ is a dual Moore family.

Now, a subset of S is an element of G if and only if it is empty or it is an ideal whose complement in S is either empty or a subsemigroup of S . Therefore $G = I \cap K$ which is a dual Moore family by Theorem 3.

To see that G need not be a Moore family, consider the following example from [1, p. 204, Exercise 1].

Let S be the multiplicative semigroup of positive integers. Let Ω be a subset of Π , the set of prime numbers, and consider the set P_Ω of all positive integers divisible by at least one member of Ω . It is not difficult to show that P_Ω is a completely prime ideal of S , and that every completely prime ideal of S is a P_Ω for some subset Ω of Π . Thus the family $G = \{S, \square\} \cup \{P_\Omega: \Omega \subseteq \Pi\}$ is a dual Moore family by the first part of this proposition. However, if we take $\Lambda = \{2\}$ and $\Psi = \{3\}$ then $P_\Lambda \cap P_\Psi$ is not completely prime, since both 2 and 3 are elements of $S \setminus \{P_\Lambda \cap P_\Psi\}$ but $2 \cdot 3 = 6$ is an element of $P_\Lambda \cap P_\Psi$, whence $S \setminus \{P_\Lambda \cap P_\Psi\}$ is not a subsemigroup of S . Since $P_\Lambda \cap P_\Psi$ is neither empty nor all of S , it cannot be in G and therefore G is not a Moore family.

Definition. Let S be a semigroup and let T be a subset of S such that if a is an element of T then there exists an element b in T such that $aba = a$. We shall call T a regular subset of S .

Proposition 7. *Let* $\mathcal{G}$ *be the family of all regular subsets of a semigroup* S . *Then* $\mathcal{G}$ *is a dual Moore family*.

Proof. First, note that $\square$ is vacuously a regular subset of S and thus an element of $\mathcal{G}$. Now, let $\mathcal{G}'$ be a non-empty subfamily of $\mathcal{G}$ and suppose $a \in \cup \{G: G \in \mathcal{G}'\}$. Then $a \in G'$ for some G' of $\mathcal{G}'$ and, since G' is a regular subset of S , there exists $b \in G'$ such that $aba = a$. Since $G' \in \mathcal{G}'$, we have $b \in \cup \{G: G \in \mathcal{G}'\}$ and, since a was arbitrarily chosen, $\cup \{G: G \in \mathcal{G}'\}$ is a regular subset of S . Hence $\mathcal{G}$ is a dual Moore family.

Regular subsets will be discussed again in Chapter III. The concept of regular subset is motivated by the definition of regular semigroup. Along the same lines, the definition of inverse semigroup motivates the following

Definition. Let S be a semigroup and let T be a subset of S such that if a is an element of T then there exists a unique element b in T that is a semigroup-inverse of a ; that is, $aba = a$ and $bab = b$. Such a subset T will be called an *inversive subset* of S .

If we omit the word "unique" from the above definition then we shall call T an *almost inversive subset of* S .

Clearly, if T is an inversive subset then T is an almost inversive subset, and if T is an almost inversive subset then it is a regular subset. We recall that if T is a subsemigroup of S and a is a regular element of T then there exists a semigroup-inverse of a in T [1, Lemma 1.14]. Therefore a subsemigroup of S is a regular subset if and only if it is an almost inversive subset.

However, the family of almost inversive subsets of a semigroup S and the dual Moore family of regular subsets of S do not coincide, in general, as the following example will illustrate.

Example. Let S be the semigroup which is represented by the Cayley table below.

	a	b	c	d	e	f
a	a	a	a	a	a	a
b	a	a	a	b	c	c
c	a	b	c	b	c	c
d	a	a	a	d	e	e
e	a	d	e	d	e	e
f	a	d	e	d	e	f

Note that f is an idempotent and bfb equals b ; thus {b, f} is a regular subset of S . However, fbf is e , not f , and b^3 equals a , not b ; hence b has no inverse in {b, f} , whence {b, f} is not an almost inversive subset of S . This semigroup also provides an example of an almost inversive subset which is not an inversive subset. The set {b, e} is almost inversive because b and e are inverses of each other. However, e is idempotent and therefore {b, e} contains two different inverses of e , whence {b, e} is not an inversive subset.

Proposition 8. Let G be the family consisting of the almost inversive subsets of a semigroup S . Then G is a dual Moore family.

Proof. Let G' be a non-empty subfamily of G and suppose $a \in \cup \{G: G \in G'\}$. Then $a \in G'$ for some member G' of G' . Since G' is an almost inversive subset of S , there exists an inverse b of a in G' , whence a has an inverse in $\cup \{G: G \in G'\}$. Since a was arbitrary, we have shown that $\cup \{G: G \in G'\}$ is an almost inversive subset. Because the null set is vacuously an almost inversive subset, we have shown that G is a dual Moore family.

We note that the family of inversive subsets of a semigroup is not a dual Moore family, in general. For an example, consider T_4 , the full transformation semigroup on the set $\{1, 2, 3, 4\}$. We shall use the notation of [1] in which $(i\ j\ k\ \ell)$ represents the mapping which sends 1 to i , 2 to j , 3 to k and 4 to ℓ . Recall that the binary operation in T_4 is composition of functions. Let $a = (3\ 1\ 3\ 3)$, $b = (2\ 2\ 4\ 2)$, and $c = (2\ 2\ 1\ 2)$. Then both b and c are inverses of a and, since $a^3 = (3\ 3\ 3\ 3)$ and $b^3 = (2\ 2\ 2\ 2) = c^3$, the sets $\{a, b\}$ and $\{a, c\}$ are inversive subsets of T_4 . However their union, the set $\{a, b, c\}$, is not an inversive subset because it contains two different inverses of the element a . Hence the inversive subsets of T_4 do not form a dual Moore family. We should also note that the intersection of $\{a, b\}$ and $\{a, c\}$ is $\{a\}$, which is not an inversive subset because it contains no inverse of a . Thus, the family consisting of the inversive subsets of T_4 cannot be a Moore family, even if T_4 is adjoined to the family. Since inversive subsets are also almost inversive and regular subsets, we have just shown that neither of these collections of subsets of T_4 is a Moore family, even if we adjoin T_4 to the collections. However, we do have the following concerning the inversive subsets of an inverse semigroup.

Proposition 9. *Let* F *be the family of all inversive subsets of an inverse semigroup* S . *Then* F *is a double Moore family.*

Proof. Let $\{A_\alpha : \alpha \in \Lambda\}$ be an arbitrary non-empty subfamily of the family of all inversive subsets of S and let $A = \cup \{A_\alpha : \alpha \in \Lambda\}$ and $B = \cap \{A_\alpha : \alpha \in \Lambda\}$.

If $a \in A$ then, since a is a member of some inversive subset A_α with $\alpha \in \Lambda$, there exists an inverse of a , say a' in A . However, a has only one inverse in S , whence a' is the unique inverse of a in A . Since a was arbitrary, we have shown that A is an inversive subset of S . Now, S is itself an inversive set because it is an inverse semigroup and, since we have shown that an arbitrary union of inversive sets is inversive, F is a dual Moore family.

If $b \in B$ then $b \in S$, whence there exists a unique inverse b' of b in S . For each $\alpha \in \Lambda$ we have $b \in A_\alpha$ and A_α inversive, whence $b' \in A_\alpha$ for each $\alpha \in \Lambda$ and therefore $b' \in B$. Since b was an arbitrary member of B , we have shown that B is an inversive set. Since the null set is vacuously an inversive subset and we have shown that the intersection of inversive sets is again inversive, F is a Moore family. Since F has been shown to be both a Moore family and a dual Moore family, it is a double Moore family.

CHAPTER III

REGULAR CORES OF GREEN'S CLASSES

By Proposition 7, the family of all regular subsets of a semigroup is a dual Moore family. Thus we can associate a core mapping from the power set of the semigroup into the family of regular subsets. For a subset X of a semigroup S, we shall denote the image of X under this core mapping by $\breve{X}$, called the *regular core of* X. In this chapter we study the regular cores of the Green's classes of a semigroup.

Let S be a semigroup and let D be a $\mathcal{D}$-class of S. It is well known that if D contains a single regular element of S then every element of D is regular in S [1, Theorem 2.11]. Furthermore, if a is a regular element of S in D then there exists a semigroup-inverse of a in D [1, Theorem 2.18 and Lemma 1.14]. Therefore, if D contains a regular element of S then every element of D is regular and has an inverse in D, whence D is a regular subset of S; that is, $D = \breve{D}$. If D contains no regular elements then D clearly contains no non-empty regular subsets of S and thus $\breve{D}$ is empty. We have therefore established

Theorem 10. *The regular core of a* $\mathcal{D}$-*class* D *is either empty or* D *itself*.

We note that, in the terminology of [1], a $\mathcal{D}$-class is called regular if and only if it contains a regular element. Thus, by the proof of the theorem above, a $\mathcal{D}$-class D is regular if and only if $D = \breve{D}$.

We next consider the regular core of an $\mathcal{R}$-class of a semigroup. The following technical result will be used in the discussion which follows.

Lemma 11. *Let* S *be a semigroup and let* a *and* b *be elements of* S *such that* b *is in* $R_a[L_a]$ *and* $aba = a$. *Then*

(i) H_a *contains an idempotent if and only if* H_b *contains an idempotent, and*

(ii) $ba \in L_a$ *and* $ab \in R_a$.

Proof. To prove (i), assume first that H_a contains an idempotent and that $b \in R_a$. Then $R_a = R_b$ and thus $H_a = R_a \cap L_a = R_b \cap L_a$. Therefore $R_b \cap L_a$ contains an idempotent and, by [1, Theorem 2.17], $R_a \cap L_b$ contains ab , which is an idempotent. But $R_a \cap L_b = R_b \cap L_b = H_b$ whence H_b contains an idempotent.

Conversely, assume H_b contains an idempotent and $b \in R_a$. Then $R_a = R_b$, whence $H_b = R_b \cap L_b = R_a \cap L_b$. Thus $R_a \cap L_b$ contains an idempotent and, by [1, Theorem 2.17], $R_b \cap L_a$ contains ba , which is idempotent. However $R_b \cap L_a = R_a \cap L_a = H_a$, whence H_a contains an idempotent element.

The proof for $b \in L_a$ is dual. One can show that if H_a contains an idempotent then $ba \in H_b$ and if H_b contains an idempotent then $ab \in H_a$.

For part (ii), note that $a(ba) = a$ and $b \cdot a = ba$, whence $ba \in L_a$. Similarly, $(ab)a = a$ and $a \cdot b = ab$, whence $ab \in R_a$.

Before approaching the problem of finding the regular core of an $\mathcal{R}$-class of an arbitrary semigroup, let us consider an example of some significance, the full transformation semigroup $\mathcal{T}_X$ on a set X . We shall adhere to the notation of [1] . In particular, for all $\alpha \in \mathcal{T}_X$, we shall denote by $X\alpha$ the range of α and by π_α the equivalence relation defined on X by $x \, \pi_\alpha \, y$ if and only if $x\alpha = y\alpha$. The next lemma is a combination of three facts proved in [1, lemmas 2.5 and 2.6 and Theorem 2.10].

Lemma 12. *Let* X *be a set and* α *and* β *elements of* T_X .

(i) $\alpha L \beta$ *if and only if* $X\alpha = X\beta$.

(ii) $\alpha R \beta$ *if and only if* $\pi_\alpha = \pi_\beta$.

(iii) *We have* $|X\alpha \cap C| = 1$ *for all members* $C \in X/\pi_\alpha$ *if and only if* H_α *contains an idempotent*.

The following results are technical and will be used to find the regular core of an R-class of T_X .

Lemma 13. *Let* α *and* β *be elements of* T_X *such that* $\alpha R \beta$ *and* $\alpha\beta\alpha = \alpha$. *If* $C \cap X\alpha$ *is non-empty for each* $C \in X/\pi_\alpha$ *then* H_α *contains an idempotent*.

Proof. Let C be an arbitrary member of X/π_α . Then, by Lemma 12(iii), it suffices to show that $|X\alpha \cap C| = 1$. By the hypothesis that $X\alpha \cap C$ is non-empty we have $|X\alpha \cap C| \geq 1$. Now, suppose s and t are elements of $C \cap X\alpha$. Because $s \in X\alpha$, there exists $y \in X$ such that $y\alpha = s$, whence $s\beta\alpha = y\alpha\beta\alpha = y\alpha = s$. Thus $s\beta \in C_s$ where $C_s = \{x \in X\colon x\alpha = s\} \in X/\pi_\alpha$. Similarly, $t\beta \in C_t = \{x \in X\colon x\alpha = t\} \in X/\pi_\alpha$. Since s and t are both members of the same π_α-class C , we have $s \; \pi_\alpha \; t$, and, since $\alpha R \beta$, we have $s \; \pi_\beta \; t$ by Lemma 12(ii). But $s \; \pi_\beta \; t$ implies $s\beta = t\beta \in C_t$; that is, $(s\beta)\alpha = t$. We have already shown that $(s\beta)\alpha = s$, and therefore $s = t$. Since s and t were arbitrary, we have shown that $|X\alpha \cap C| = 1$ as we wished.

Lemma 14. *Let* α *and* β *be elements of* T_X *such that* $\beta \in \breve{R}_\alpha$ *and* $\alpha\beta\alpha = \alpha$. *Then* H_α *contains an idempotent*.

Proof. First, let C be an arbitrary member of X/π_α . Since $\alpha R \beta$ we have $\pi_\alpha = \pi_\beta$ by Lemma 12(ii), whence $C \in X/\pi_\beta$. Let $x \in C$ and let $t = x\alpha$. Then $(t\beta)\alpha = x\alpha\beta\alpha = x\alpha$, whence $t\beta \in C$.

Thus, $C \cap X\beta \neq \square$ and, since C was arbitrary in $X/\pi_\alpha = X/\pi_\beta$, we have shown that each member of X/π_β has non-empty intersection with $X\beta$. Now, $\beta \in \breve{R}_\alpha$, whence there exists $\gamma \in \breve{R}_\alpha \subseteq R_\alpha$ such that $\beta\gamma\beta = \beta$. Therefore H_β contains an idempotent by Lemma 13. Finally, by Lemma 11(i), with $a = \alpha$ and $b = \beta$, we have H_α containing an idempotent as we wished to show.

Now let R be an $\mathcal{R}$-class and let $\alpha \in \breve{R}$. Then there exists $\beta \in \breve{R}$ such that $\alpha\beta\alpha = \alpha$. By Lemma 14, H_α contains an idempotent; hence α is a member of an $\mathcal{H}$-class that contains an idempotent. Note that if an $\mathcal{H}$-class contains an idempotent then it is a group and therefore a regular subset. Since the union of the $\mathcal{H}$-classes of R that contain idempotents is always within $\breve{R}$ by Proposition 7, we have proved the following proposition.

Proposition 15. *Let* $\mathcal{T}_X$ *be the full transformation semigroup on a set* X. *If* R *is an* $\mathcal{R}$*-class of* $\mathcal{T}_X$ *then* $\breve{R}$ *is the union of all* $\mathcal{H}$*-classes in* R *that contain idempotents*.

This result leads one to believe that the following theorem may be true.

Theorem 16. *Let* S *be a semigroup. Then the regular core of an* $\mathcal{R}$*-class* R *is the union of all* $\mathcal{H}$*-classes in* R *that contain idempotents*.

Two proofs of this theorem will be given. The first is an application of Proposition 15 above and the properties of the extended regular representation of a semigroup. The second is a direct proof. Before proceeding with the first proof, let us note some important properties of the extended regular representation of a semigroup.

Let S be a semigroup and let ϕ be the extended regular representation of S; that is, for c in S^1, we have $c\phi = \rho_c$ where ρ_c

is the inner right translation of S^1 corresponding to c (i.e., $x\rho_c = xc$ for all $x \in S^1$). It is well known that the extended regular representation is faithful. If c and d are elements of S such that $cdc = c$ then

$$\rho_c\rho_d\rho_c = (c\phi)(d\phi)(c\phi) = (cdc)\phi = c\phi = \rho_c .$$

Now suppose T is a regular subset of S and let ρ_c be an element of $T\phi = \{x\phi : x \in T\}$. Then there exists $d \in T$ such that $cdc = c$, whence ρ_d is an element of $T\phi$ such that $\rho_c\rho_d\rho_c = \rho_c$. Since ρ_c was arbitrary in $T\phi$ we have just shown $T\phi$ to be a regular subset of $\mathcal{T}_{S'}$.

Let s and t be elements of S such that $s\mathcal{R}t$. Then there exist x and y in S such that $sx = t$ and $ty = s$. Thus $\rho_t = t\phi = (sx)\phi = (s\phi)(x\phi) = \rho_s\rho_t$ and similarly $\rho_s = s\phi = (ty)\phi = (t\phi)(y\phi) = \rho_t\rho_y$, whence $\rho_s \mathcal{R} \rho_t$ in $\mathcal{T}_{S'}$. We combine the facts proved above in the following lemma.

Lemma 17. *Let S be a semigroup and let ϕ be the extended regular representation of S.*

(i) *If $cdc = c$ in S then $\rho_c\rho_d\rho_c = \rho_c$ in $\mathcal{T}_{S'}$.*

(ii) *If T is a regular subset of S then $T\phi$ is a regular subset of $\mathcal{T}_{S'}$.*

(iii) *If $c\mathcal{R}d$ in S then $\rho_c \mathcal{R} \rho_d$ in $\mathcal{T}_{S'}$.*

We may now proceed with the proof of Theorem 16.

Proof 1. Let R be an $\mathcal{R}$-class of S. Note that if $\breve{R} = \square$ then R contains no subgroups and therefore the conclusion of the theorem holds. Now assume $\breve{R} \neq \square$ and let $a \in \breve{R}$. Let ϕ be the extended regular representation of S by elements of $T_{S'}$ as described above. Since $a \in \breve{R}$, there exists $b \in \breve{R} \subseteq R$ such that $aba = a$. Let $\alpha = a\phi$ and $\beta = b\phi$; then, by Lemma 17(i), with $a = c$ and $b = d$, we have $\alpha\beta\alpha = \alpha$. Now, $\breve{R}$ is a regular subset of S, and therefore $\breve{R}\phi$ is a regular subset of $T_{S'}$ by Lemma 17(ii). Since $\breve{R} \subseteq R$, we have $\breve{R}\phi \subseteq R\phi = R_a\phi \subseteq R_\alpha$ and, since $R_a = R_b$, we have $R_\alpha = R_\beta$. Since $\breve{R}\phi \subseteq R_\alpha$ and $\breve{R}\phi$ is a regular subset of $T_{S'}$, we have $\breve{R}\phi \subseteq \breve{R}_\alpha$. Now, $H_\alpha = R_\alpha \cap L_\alpha = R_\beta \cap L_\alpha$, and H_α contains an idempotent by Proposition 15 because $\alpha \in \breve{R}_\alpha$. Therefore $\alpha\beta \in R_\alpha \cap L_\beta$ by [1, Theorem 2.17], whence $\alpha\beta \mathcal{L} \beta$, and thus $T_{S'}\alpha\beta = T_{S'}\beta$ by Lemma 12(i). Therefore $S'(ab) = S'b$ and thus $ab \in L_b$. We also have $ab \in R_b$ because, by Lemma 11(ii), $ab \in R_a$ and, by hypothesis, $R_a = R_b$. Therefore $(ab)^2 = ab \in R_b \cap L_b = H_b$. By Lemma 11(i), there exists an idempotent in H_a, and therefore a is a member of the union of all $\mathcal{H}$-classes in R that contain idempotents. Since the union of these $\mathcal{H}$-classes is a union of regular subsets of R, it must be contained in $\breve{R}$. Hence $\breve{R}$ equals the union of all $\mathcal{H}$-classes of R that contain idempotents.

<u>Proof</u> 2. Let R be an $\mathcal{R}$-class of S. As before, the theorem is true if $\breve{R} = \square$. Therefore, assume $a \in \breve{R}$; then there exist b and c, members of $\breve{R}$, such that $aba = a$ and $bcb = b$. Note that $ab \in R_a = R$ by Lemma 11(ii). Since ab is idempotent, it is a left identity of R by [1, Lemma 2.14]. Therefore, since $c \in \breve{R} \subseteq R$, we have $(ab)c = c$, whence $(ba)(bc) = b(ab)c = bc$. But we also have $(bc)(ba) = (bcb)a = ba$, and therefore $ba \mathcal{R} bc$. Since $bc \mathcal{R} b$, by Lemma 11(ii), and $b\mathcal{R}a$, we have $ba \mathcal{R} a$. But $ba \mathcal{L} a$ by

Lemma 11(ii), and therefore $ba \in H_a$. Since ba is idempotent, we have shown that a is an element of an $\mathcal{H}$-class that contains an idempotent and, since a was an arbitrary member of $\breve{R}$, we conclude that $\breve{R}$ is a subset of the union of all $\mathcal{H}$-classes in R that contain idempotents. As noted in the first proof, this union is always a subset of $\breve{R}$ and therefore $\breve{R}$ equals the union of all $\mathcal{H}$-classes in R that contain idempotents.

The next theorem is the dual to Theorem 16, and only the direct proof is given.

Theorem 18. *Let* S *be a semigroup. Then the regular core of an* $\mathcal{L}$*-class* L *is the union of all* $\mathcal{H}$*-classes in* L *that contain idempotents*.

Proof. Let L be an $\mathcal{L}$-class of S . First, note that if $\breve{L} = \square$ then L contains no subgroups and therefore the conclusion of the theorem holds. Now, assume $\breve{L} \neq \square$ and let a be an arbitrary member of $\breve{L}$. Then there exist elements b and c of $\breve{L}$ such that $aba = a$ and $bcb = b$. By Lemma 11(ii), we have $ba \in L_a = L$ and $cb \in L_b$. Now, since ba is an idempotent of L , it is a right identity of L by [1, Lemma 2.14]. Therefore $c(ba) = c$, and $c \in \breve{L} \subseteq L$, whence $(cb)(ab) = [c(ba)]b = cb$. But we also have $(ab)(cb) = a(bcb) = ab$ and therefore $ab \,\mathcal{L}\, cb$. We have now shown that $ab \,\mathcal{L}\, cb$ and $cb \,\mathcal{L}\, b$, and we have $b \,\mathcal{L}\, a$ by hypothesis; thus we may conclude that $ab \,\mathcal{L}\, a$. However, $ab \,\mathcal{R}\, a$ by Lemma 11(ii) , and therefore $ab \in H_a$. Since ab is idempotent, we have shown that a is an element of an $\mathcal{H}$-class that contains idempotents and since a was arbitrary in $\breve{L}$, we have proved that $\breve{L}$ is contained in the union of all H-classes in L that contain idempotents. However, each H-class in L that contains an idempotent is a group and therefore a regular subset of L, whence the union of all such H-classes is contained in $\breve{L}$, and therefore equals $\breve{L}$, as we wished to show.

Corollary 19. *The regular core of an R-class [L-class] of a semigroup* S *is a subsemigroup of* S *if it is non-empty.*

Proof. Let R be an R-class of a semigroup S and let a and b be elements of $\breve{R}$, the regular core of R. Then $R_a = R_b = R$ and, by Theorem 16, both H_a and H_b contain idempotents. Now, $R_b \cap L_a = R_a \cap L_a = H_a$ whence $R_b \cap L_a$ contains an idempotent and therefore, by [1, Theorem 2.17], the product $ab \in R_a \cap L_b$. However $R_a \cap L_b = R_b \cap L_b = H_b$ and $H_b \subseteq \breve{R}$ by Theorem 16. Hence $\breve{R}$ is a subsemigroup of S.

The proof for an L-class L of S is similar. Let a and b be elements of $\breve{L}$; then $L_a = L_b = L$ and by Theorem 18, both H_a and H_b contain idempotents. Now, $R_b \cap L_a = R_b \cap L_b = H_b$ whence $R_b \cap L_a$ contains an idempotent. Thus $ab \in R_a \cap L_b$ by [1, Theorem 2.17]. But $R_a \cap L_b = R_a \cap L_a = H_a$ and $H_a \subseteq \breve{L}$ by Theorem 18, whence $ab \in \breve{L}$. Since a and b were arbitrary members of $\breve{L}$, we have shown that $\breve{L}$ is a subsemigroup of S.

Corollary 20. *Let* $R[L]$ *be an R-class [L-class] of a semigroup* S *and let* a *and* b *be elements of* $\breve{R}[\breve{L}]$ *such that* $aba = a$. *Then* $bab = b$ *and therefore* a *and* b *are semigroup inverses of each other.*

Proof. In Corollary 19 it was shown that if R is an R-class and a and b are elements of $\breve{R}$ then $ab \in H_b$. Now, if $aba = a$ then ab is an idempotent in the group H_b and thus is the identity of H_b. In particular, $bab = b$ as we wished to show.

For the dual, note that H_a contains an idempotent and $H_a = R_a \cap L_a = R_a \cap L_b$ whence $ba \in R_b \cap L_a$ by [1, Theorem 2.17]. Since ba is idempotent and $R_b \cap L_a = R_b \cap L_b = H_b$, we may conclude that ba is the identity of the group H_b, whence $bab = b$.

We next consider the regular core of an $\mathcal{H}$-class H. If H does not contain an idempotent then H contains no regular subsets except $\square$, for if it contained a non-empty regular subset we would have a contradiction to Theorem 16. If H contains an idempotent then H is a group and therefore a regular subset; that is, $H = \breve{H}$. We have just proved the following.

Theorem 21: *If H is an $\mathcal{H}$-class of a semigroup S then $\breve{H} = \square$ if H does not contain an idempotent and $\breve{H} = H$ if H contains an idempotent.*

We now consider the regular core of a $\mathcal{J}$-class. Recall that a $\mathcal{J}$-class is a disjoint union of $\mathcal{D}$-classes.

Theorem 22. *Let J be a $\mathcal{J}$-class of a semigroup S. Then $\breve{J}$ is the union of all regular $\mathcal{D}$-classes of S that are contained in J.*

Proof. Let $x \in \breve{J}$; then x is a regular element of S, whence D_x is a regular $\mathcal{D}$-class of S by [1, Theorem 2.11(i)]. Since $\mathcal{D} \subseteq \mathcal{J}$, if $s \in S$ and $s\mathcal{D}x$ then $s\mathcal{J}x$. Thus $D_x \subseteq J$ and x is an element of the union of the regular $\mathcal{D}$-classes of S in J. Since x was an arbitrary member of $\breve{J}$, we have shown that $\breve{J}$ is contained in the union of these regular $\mathcal{D}$-classes.

Conversely, assume D is a regular $\mathcal{D}$-class of S contained in J. Then $D = \breve{D}$; that is, D is a regular subset of S contained in J. Therefore $D \subseteq \breve{J}$ since $\breve{J}$ is the union of all regular subsets of S that are contained in J. Since D was an arbitrary regular $\mathcal{D}$-class in J, we have shown that $\breve{J}$ contains the union of all regular $\mathcal{D}$-classes of S contained in J, and therefore $\breve{J}$ equals the union of these regular $\mathcal{D}$-classes.

We know that the regular core of an $\mathcal{H}$-class or a $\mathcal{D}$-class is either empty or the whole class. The following example shows that a $\mathcal{J}$-class may have a regular core that is a non-empty proper subset of that $\mathcal{J}$-class.

Let $S = S^1$ be a semigroup that contains both regular and irregular elements. Let $\mathbb{N}$ be the set of non-negative integers and let θ be a homomorphism of S into the group of units of S. The <u>Bruck-Reilly extension of S determined by θ</u>, denoted by $BR(S, \theta)$, is the semigroup obtained when we define an operation on $\mathbb{N} \times S \times \mathbb{N}$ as follows:

$$(m, a, n)(p, b, q) = (m - n + t\ ,\ a\theta^{t-n}\, b\theta^{t-p},\ q - p + t)$$

where $t = \max\{n, p\}$ and θ° is the identity mapping of S.

To verify that $BR(S)$ $[= BR(S, \theta)]$ is a semigroup we must show that the operation defined above is associative.

Let (m, a, n), (p, b, q) and (i, c, j) be elements of $\mathbb{N} \times S \times \mathbb{N}$, and let $x = [(m, a, n)(p, b, q)](i, c, j)$. If we let $t = \max\{n, p\}$ then

$$x = (m - n + t,\ a\theta^{t-n}\, b\theta^{t-p},\ q - p + t)(i, c, j)\ .$$

Now, let $u = \max\{q - p + t, i\} = \max\{q - p + n, q, i\}$. Then

$$x = (m - n + t - (q - p + t) + u ,$$

$$(a\,\theta^{t-n}\, b\,\theta^{t-p})\theta^{u-(q-p+t)}\, c\,\theta^{u-i},\ j - i + u) .$$

Since θ is a homomorphism,

$$(a\,\theta^{t-n}\, b\,\theta^{t-p})\theta^{u-(q-p+t)} = a\,\theta^{t-n+u-q+p-t}\, b\,\theta^{t-p+u-q+p-t}$$

$$= a\,\theta^{u-n-q+p}\, b\,\theta^{u-q} .$$

Thus we have

(1) $$x = (m - n - q + p + u ,\ a\,\theta^{u-n-q+p}\, b\,\theta^{u-q}\, c\,\theta^{u-i},\ j - i + u) .$$

Now, let $y = (m, a, n)[(p, b, q)(i, c, j)]$ and let $v = \max\{q, i\}$. Then

$$y = (m, a, n)(p - q + v,\ b\,\theta^{v-q}\, c\,\theta^{v-i},\ j - i + v) .$$

If we let $w = \max\{n, p - q + v\} = \max\{n, p, p - q + i\}$ then

$$y = (m - n + w,\ a\,\theta^{w-n}(b\,\theta^{v-q}\, c\,\theta^{v-i})\,\theta^{w-(p-q+v)},$$
$$j - i - p + q + w) .$$

Since θ is a homomorphism,

$$(b\,\theta^{v-q}\,c\,\theta^{v-i})\,\theta^{w-(p-q+v)} = b\,\theta^{v-q+w-p+q-v}\,c\,\theta^{v-i+w-p+q-v}$$
$$= b\,\theta^{w-p}\,c\,\theta^{w-i-p+q}\,.$$

Thus,

$$y = (m - n + w,\ a\,\theta^{w-n}\,b\,\theta^{w-p}\,c\,\theta^{w-i-p+q}\,,\ j - i - p + q + w).$$

Now $u = \max\{q - p + n,\ q,\ i\}$ and therefore

$$u - q + p = \max\{q - p + n - q + p\,,\ q - q + p\,,\ i - q + p\}$$
$$= \max\{n,\ p,\ i - q + p\} = w\ .$$

Using the identity $w = u - q + p$ we have

$$m - n + w = m - n - q + p + u\ ,$$
$$j - i - p + q + w = j - i + u\ ,$$
$$w - n = u - n - q + p\ ,$$
$$w - p = u - q\ , \text{ and}$$
$$w - i - p + q = u - i\ ,$$

whence $x = y$. Therefore the operation is associative and $BR(S)$ is a semigroup.

<u>Lemma</u> 23. <u>Let</u> s <u>be an element of a semigroup</u> $S = S'$ <u>and let</u> i <u>and</u> j <u>be non-negative integers</u>. <u>Then</u> (i, s, j) <u>is regular in</u> $BR(S)$ <u>if and only if</u> s <u>is regular in</u> S .

Proof. First, assume that (i, s, j) is a regular element of $BR(S)$. Then there exists (p, t, q) in $\mathbb{N} \times S \times \mathbb{N}$ such that $(i, s, j)(p, t, q)(i, s, j) = (i, s, j)$. For notational purposes, let $\alpha = (i, s, j)$ and $\beta = (p, t, q)$. From equation (1) above, if we take $m = i$, $n = j$, $a = c = s$, and $b = t$, we obtain

$$(2) \qquad \alpha\beta\alpha = (i - j - q + p + u\ ,\ s\,\theta^{u-j-q+p}\ t\,\theta^{u-q}\ s\,\theta^{u-i},\ j - i + u)$$

where $u = \max\{q - p + j, q, i\}$. Now, $\alpha\beta\alpha = \alpha$ by hypothesis whence, by equating the first components of $\alpha\beta\alpha$ and α, we obtain $i = i - j - q + p + u$ and thus $u - j - q + p = 0$. Therefore

$$s\,\theta^{u-j-q+p} = s\,\theta^{\circ} = s\ .$$

By equating the last components of $\alpha\beta\alpha$ and α, we have $j = j - i + u$ whence $u - i = 0$, and thus

$$s\,\theta^{u-i} = s\,\theta^{\circ} = s\ .$$

Note that since $u = \max\{q - p + j\ ,\ q,\ i\}$, we have $u \geq q$. The middle component of $\alpha\beta\alpha$ is now seen to be $s(t\theta^{u-q})s$, and since by hypothesis it must equal the middle component s of α we now have $s = s(t\theta^{u-q})s$, whence s is a regular element.

Conversely, assume that s is a regular element of S. Then there exists $t \in S$ such that $sts = s$. Now, let $\alpha = (i, s, j)$ and $\beta = (j, t, i)$. Then, from equation (2) above, with $p = j$ and

$q = i$, we have $\alpha\beta\alpha = (i - j - i + j + u,\ s\theta^{u-j-i+j}\ t\theta^{u-i}\ s\theta^{u-i}$, $j - i + u)$, where $u = \max\{i - j + j\ ,\ i,\ i\} = i$. Therefore $\alpha\beta\alpha = (i,\ s\theta^{\circ}\ t\theta^{\circ}\ s\theta^{\circ},\ j) = (i,\ sts,\ j)$. But $sts = s$ by hypothesis, and thus $\alpha\beta\alpha = (i,\ sts,\ j) = (i,\ s,\ j) = \alpha$, whence α is a regular element of BR(S) as we wished to show.

By Lemma 23, if S contains both regular and irregular elements then so must BR(S) . But BR(S) is simple; for if $\alpha = (i,\ s,\ j)$ and $\beta = (m,\ t,\ n)$ then $\alpha = (i,\ (t\theta)^{-1},\ m + 1)\ \beta(n + 1,\ s,\ j)$, where $(t\theta)^{-1}$ is the inverse of $t\theta$ in the group of units of S . Hence BR(S) consists of a single $\mathcal{J}$-class, and BR(S) contains both regular and irregular $\mathcal{D}$-classes when S contains both regular and irregular elements. Thus if S contains both regular and irregular elements then the regular core of BR(S) as a $\mathcal{J}$-class is neither empty nor all of BR(S) itself.

CHAPTER IV

MOORE FAMILIES AND CONGRUENCES

Throughout this chapter we shall study binary relations on a semigroup S, with particular attention given to congruence relations on S. We shall denote by ω the universal relation $S \times S$ and by ι the diagonal of $S \times S$ defined by $(a, b) \in \iota$ if and only if $a = b$. The symbol "$\square$" will be used to represent the empty relation.

It is well known that the properties of reflexivity, symmetry, transitivity and compatibility are preserved under intersections of binary relations. From this fact, it is clear that the intersection of covering relations (reflexive and symmetric binary relations) is a covering relation, the intersection of quasi-orders (reflexive and transitive binary relations) is a quasi-order, the intersection of equivalence relations (reflexive, symmetric, and transitive binary relations) is an equivalence relation, and the intersection of congruence relations (compatible equivalence relations) is a congruence relation. We note that the intersection of reflexive relations is non-empty (since it contains ι) .

Since ω is a congruence relation and the intersection of congruences is a congruence, we have the following

Proposition 24. *If S is a semigroup and F is the family of all congruence relations on S then F is a Moore family.*

If S is a semigroup and ρ is a congruence on S then it is of interest to study S/ρ, the semigroup of ρ-classes of S. We shall denote by $a\rho$ the congruence class of S which contains the element a in S. By [1, Proposition 1.7], if the intersection of all congruences of a given type is also of that type, then S modulo the intersection is a maximal homomorphic image of S of the given type. Now, if the family of all congruences of a given type is closed under arbitrary intersections, as when the family is a Moore family, the above result holds. Thus it is of interest to study families of congruences which are Moore families.

If ρ is a congruence such that S/ρ is an inverse semigroup then we say that ρ is an <u>inverse semigroup congruence</u>. If S is a regular semigroup then the inverse semigroup congruences on S are characterized by the following lemma.

<u>Lemma</u> 25. <u>Let S be a regular semigroup and let ρ be a congruence on S. Then ρ is an inverse semigroup congruence if and only if for each pair of idempotents e and f of S we have (ef, fe) an element of ρ.</u>

<u>Proof</u>. First assume ρ is an inverse semigroup congruence and let e and f be arbitrary idempotents of S. Then $e\rho$ and $f\rho$ are idempotents of S/ρ and, by hypothesis, S/ρ is an inverse semigroup, whence $(e\rho)(f\rho) = (f\rho)(e\rho)$. Thus $(ef)\rho = (fe)\rho$ and therefore $(ef, fe) \in \rho$.

Conversely, assume that if e and f are idempotents of S then $(ef, fe) \in \rho$. Since ρ is a congruence and S is regular, S/ρ is regular. Thus, to show S/ρ is an inverse semigroup it suffices to show that idempotents of S/ρ commute. Now, let E and F be idempotents of S/ρ. By Lallement's lemma [3, p. 52], there exist e and f, idempotents of S, such that $e \in E$ and $f \in F$. By hypothesis $ef \; \rho \; fe$, whence $EF = FE$. Therefore, since E and F

were arbitrary idempotents of S/ρ, we have shown that S/ρ is an inverse semigroup.

Theorem 26. Let S be a regular semigroup and let F be the family consisting of all inverse semigroup congruences on S. Then F is a Moore family.

Proof. First, note that S/ω is a one-element semigroup and therefore ω is an inverse semigroup congruence, whence $\omega \in F$. Now assume that $\{\rho_\alpha : \alpha \in A\}$ is a subfamily of F and let $\rho = \bigcap\{\rho_\alpha : \alpha \in A\}$, a congruence by Proposition 24. Let e and f be arbitrary idempotents of S. For each $\alpha \in A$, we have ρ_α an inverse semigroup congruence by hypothesis, whence $(ef, fe) \in \rho_\alpha$ for each $\alpha \in A$. Hence $(ef, fe) \in \rho$ and thus ρ is an inverse semigroup congruence by Lemma 25. Therefore F is a Moore family.

(Note: An example to show that if a semigroup S is not regular then the family of inverse semigroup congruences may not be a Moore family follows Corollary 28.)

If ρ is a congruence on a semigroup S such that if e and f are idempotents of S then (e, f) is an element of ρ, we shall call ρ an idempotent-identifying congruence. For a regular semigroup S, if ρ is an idempotent-identifying congruence then S/ρ contains at most one idempotent, for if S/ρ contained two distinct idempotent classes then, by Lallement's lemma [3, p. 52], there would exist idempotents e and f in S such that $e\rho \neq f\rho$, which is impossible since $(e, f) \in \rho$ by hypothesis. The converse is true for any semigroup S. If S/ρ contains no idempotents then S contains no idempotents and therefore every congruence on S is vacuously idempotent-identifying. If S/ρ contains exactly one idempotent then all idempotents are contained in a single ρ-class and again ρ is idempotent-identifying. Note that by making suitable alterations to the proof of Theorem 25 (substitute (e, f) for (ef, fe) and "idempotent-identifying" for "inverse semigroup", and omit "by Theorem 25"), we have the following

Theorem 27. *Let S be a semigroup. The family F of all idempotent-identifying congruences on S is a Moore family.*

We next consider group congruences; that is, congruences ρ such that S/ρ is a group. It is well known that a regular semigroup is a group if and only if it has only one idempotent. Thus, if S is a regular semigroup and ρ is a congruence on S then S/ρ is a group if and only if ρ is an idempotent-identifying congruence. Therefore, we have the following corollary to Theorem 27.

Corollary 28. *Let S be a regular semigroup. Then the family F of all group congruences on S is a Moore family.*

The following example [3, Chapter V, Exercise 26] shows that for a semigroup that is not regular the family of all group congruences need not be a Moore family.

Let $S = \langle a \rangle$, the infinite cyclic semigroup generated by a. Let $\mathbb{P}$ denote the positive integers. For each $n \in \mathbb{P}$, define $\rho_n = \{(a^p, a^q): p \equiv q(\text{mod } n)\}$. That ρ_n is a congruence is easily verified and since S/ρ_n is isomorphic to $\mathbb{Z}_n$, the additive group of integers modulo n, we have S/ρ_n a group, whence ρ_n is a group congruence for each $n \in \mathbb{P}$. Let $\rho = \cap \{\rho_n: n \in \mathbb{P}\}$. Clearly the identity relation ι is a subset of ρ since ρ is a congruence. But also $\rho \subseteq \iota$; for if

$(a^p, a^q) \in \rho$ then, for all $n \in \mathbb{P}$, we have $p \equiv q (\mathrm{mod}\ n)$ and thus $p = q$, whence (a^p, a^q) is an element of ι. Therefore $\rho = \iota$ and, since S/ρ is isomorphic to S which is not a group, ρ is not a group congruence. Note that this example illustrates that, in general, the family of all inverse semigroup congruences need not be a Moore family, for if S/ρ is a group then S/ρ is an inverse semigroup. However S was not an inverse semigroup, and S/ρ was isomorphic to S.

Let S be a semigroup and let ρ be a congruence on S. We say that ρ is an <u>idempotent-separating congruence</u> if and only if each ρ-class contains at most one idempotent. That is, ρ is idempotent-separating if and only if, for idempotents e and f of S, if (e, f) is an element of ρ then $e = f$. Note that idempotent-separating congruences are in some sense opposite to the idempotent-identifying congruences mentioned above. The idempotent-separating congruences on a semigroup containing more than one idempotent do not themselves form a Moore family. However, we do have the following result.

<u>Theorem</u> 29. <u>Let S be a semigroup and let F be the family consisting of the universal relation ω and all idempotent-separating congruences. Then F is a Moore family.</u>

<u>Proof</u>. Let $\{\rho_\alpha : \alpha \in A\}$ be a subfamily of F and let $\rho = \cap\{\rho_\alpha : \alpha \in A\}$. Note that if $\rho_\alpha = \omega$ for each $\alpha \in A$ then $\rho = \omega$ and thus $\rho \in F$. Now, assume that there exists $\beta \in A$ such that ρ_β is an idempotent-separating congruence. Let e and f be idempotents of S and suppose that $(e, f) \in \rho$. Then $(e, f) \in \rho_\beta$

and, since ρ_β is idempotent-separating, $\dot{e} = f$. Thus ρ separates idempotents and, since the intersection of congruences is a congruence by Proposition 24, we have shown that ρ is an idempotent-separating congruence. Hence $\rho \in F$ and, because $\omega \in F$, we have shown F to be a Moore family.

If S is a semigroup and ρ is a congruence on S such that S/ρ is a band (a semigroup of idempotents) then we call ρ a <u>band congruence</u>. Clearly ρ is a band congruence if and only if $(x^2, x) \in \rho$ for all $x \in S$. Note that the universal congruence ω is a band congruence since S/ω is a one-element semigroup. Now, let $\{\rho_\alpha : \alpha \in A\}$ be a family of band congruences on a semigroup S. If $\rho = \cap \{\rho_\alpha : \alpha \in A\}$ and x is an element of S then, for each α in A, we have (x^2, x) in ρ_α and therefore (x^2, x) is an element of ρ. Since the intersection of congruences is a congruence, we have shown that ρ is a band congruence. Thus we have proved the following

<u>Theorem</u> 30: <u>Let S be a semigroup and let F be the family of all band congruences on S. Then F is a Moore family</u>.

If S is a semigroup and ρ is a congruence on S such that S/ρ is commutative, that is, if $(xy, yx) \in \rho$ for all x and y in S, we call ρ a <u>commutative semigroup congruence</u>.

<u>Theorem</u> 31. <u>Let S be a semigroup. The family F of all commutative semigroup congruences on S is a Moore family</u>.

<u>Proof</u>. First, note that the universal congruence ω is an element of F since S/ω is a one-element semigroup. Now, let $\{\rho_\alpha : \alpha \in A\}$ be a non-empty subfamily of F and let $\rho = \cap \{\rho_\alpha : \alpha \in A\}$. Then ρ is a congruence by Proposition 24. Let x and y be arbitrary

elements of S ; then, since each ρ_α is a commutative semigroup congruence, $(xy, yx) \in \rho_\alpha$ for each $\alpha \in A$, whence $(xy, yx) \in \rho$ and thus $\rho \in F$. Therefore F has been shown to be a Moore family.

Let S be a semigroup and let ρ be a congruence on S . We say that ρ is a semilattice congruence if S/ρ is a semilattice; that is, if S/ρ is a commutative band. Thus ρ is a semilattice congruence if and only if S/ρ is commutative and a band; that is, if and only if ρ is both a commutative semigroup congruence and a band congruence. Thus the family of all semilattice congruences on S is just the intersection of the family of all commutative semigroup congruences with the family of all band congruences. Since the intersection of Moore families is a Moore family by Theorem 2, we have the following

Theorem 32. *Let S be a semigroup. The family F of all semilattice congruences on S is a Moore family.*

BIBLIOGRAPHY

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1. A.H. Clifford and G.B. Preston, The Algebraic Theory of Semigroups, Vol. I, Mathematical Surveys 7, Second Edition, American Mathematical Society, Providence, R.I., 1964.

2. P. Dubreil and M.L. Dubreil-Jacotin, Lectures on Modern Algebra, Trans. by A. Geddes, Hafner Publishing Company, New York, 1967.

3. J.M. Howie, An Introduction to Semigroup Theory, Academic Press Inc., New York, 1976.

VITA

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