Mathematical Analysis of a Nonlocal System of Equations Arising in Peridynamics

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To my brother Kevin, an artist whose work and perspectives always inspire me.

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Abstract

We analyze a strongly-coupled system of nonlocal equations. The system comes from a linearization of peridynamics, a nonlocal model in continuum mechanics. It is an analogue of the Navier-Lamé system of classical elasticity. The leading operator is an integro-differential operator characterized by a distinctive matrix kernel which is used to couple differences of components of a vector field.

We first demonstrate the convergence of a class of nonlocal systems to the Navier-Lamé system in the event of vanishing nonlocality via the Fourier transform. We then study the Dirichlet problem associated with a wide class of systems. We prove well-posedness and demonstrate optimal interior Sobolev regularity of solutions.

Next, we analyze the energy space associated to the system. We first show that a certain class of spaces of vector fields whose semi-norms involve projected difference quotients is in fact equivalent to the class of fractional Sobolev spaces. The equivalence can be considered a Korn-type characterization of fractional Sobolev spaces. Second, we show that functions in L^p whose Marcinkiewicz-type integrals are in L^p in fact belong to the Bessel potential space $\mathcal{L}^p_s$. The distinction here is that the Marcinkiewicz-type integral exhibits the coupling from the nonlocal model and does not resemble other classes of potential-type integrals found in the literature.

We use these equivalence results in two ways. First, we show using the Korn-type equivalence that weak solutions to the system enjoy both improved differentiability and improved integrability. Second, we show that weak solutions belong to a potential space with higher integrability via the potential space characterization.

Finally we revisit the question of solvability and prove existence and uniqueness of strong (pointwise) solutions to the time dependent system posed on all of Euclidean space. For a wide class of kernels, we prove the L^2-solvability of the steady-state system in a Bessel potential space using the Fourier transform and *a priori* estimates. This L^2-solvability and the Hille-Yosida theorem is used to prove the well posedness of the time dependent problem. For the fractional Laplacian kernel we extend the solvability to L^p spaces using classical multiplier theorems.

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Chapter 1

Introduction

The objective of this dissertation is to build on the mathematical foundations of a nonlocal model in continuum mechanics known as peridynamics. The first peridynamic model was introduced by Stewart Silling of Sandia National Laboratory [81], with generalizations later in [83], as a reformulation of the classical equations of elasticity that takes discontinuities and long-range forces into account. To describe the mathematical formulation of the model proposed [81], known as the *bond-based* peridynamic model, assume that a solid body occupies a *reference configuration*, or a bounded domain $\Omega \subset \mathbb{R}^3$. Each pair of particles in the solid body interacts through a pairwise force density $\mathbf{F}$ which is a function of relative position and displacement. Summing the forces acting at a point results in an integral operator which represents the force per unit volume. Denoting the *displacement* vector field by $\mathbf{u} : \Omega \to \mathbb{R}^3$, this integral operator is given by

$$\mathcal{L}[\mathbf{u}](\mathbf{x},t) := \int_\Omega \mathbf{F}(\mathbf{u}(\mathbf{y},t) - \mathbf{u}(\mathbf{x},t), \mathbf{y} - \mathbf{x}) \, \mathrm{d}\mathbf{y} \, .$$

The constitutive equations of motion in the bond-based peridynamic model can now be stated via Newton's laws of motion as

$$\partial_{tt}\mathbf{u}(\mathbf{x},t) = \mathcal{L}[\mathbf{u}](\mathbf{x},t) + \mathbf{b}(\mathbf{x},t) \, , \qquad \mathbf{x} \in \Omega \, , \ t \geq 0 \, , \tag{1.1}$$

where $\mathbf{b}$ is some prescribed *external loading force density.* The equilibrium (or steady-state) equation is

$$\mathcal{L}[\mathbf{u}](\mathbf{x}) + \mathbf{b}(\mathbf{x}) = \mathbf{0}\,, \qquad \mathbf{x} \in \Omega\,.$$

A typical example of the function $\mathbf{F}$ is inspired by the Hookean energy for isotropic materials; specifically

$$\mathbf{F}(\boldsymbol{\eta}, \boldsymbol{\xi}) = \left(\frac{\rho(|\boldsymbol{\xi}|)}{|\boldsymbol{\xi}|^2}\right) \left(|\boldsymbol{\xi} + \boldsymbol{\eta}| - |\boldsymbol{\xi}|\right) \left(\frac{\boldsymbol{\xi} + \boldsymbol{\eta}}{|\boldsymbol{\xi} + \boldsymbol{\eta}|}\right)\,. \tag{1.2}$$

The scalar-valued function ρ is called the *kernel* or *influence function*, which represents the strength and extent of interactions between particles $\mathbf{x}$ and $\mathbf{y}$ in Ω, and contains properties of the modeled material. It is typically a function with compact support, usually within a ball of radius δ. This parameter δ is the interaction extent, and is known as the *horizon* of the material.

The defining feature of the peridynamic system is that it use of integrals in place of partial derivatives, avoiding a direct use of objects in classical elasticity like stress tensor and deformation gradient. This feature makes peridynamics extremely useful for modeling systems involving cracks and damage, as has been demonstrated in a variety of settings; see [11] for an overview.

Just as in classical elasticity, the simple peridynamic model described above is nonlinear, making mathematical and numerical analysis as well as practical implementation of the model challenging. For deformations with uniformly small strain, i.e. for $\mathbf{u}$ satisfying

$$\sup_{|\mathbf{x}-\mathbf{y}|<\delta} |\mathbf{u}(\mathbf{x}) - \mathbf{u}(\mathbf{y})| \ll 1\,,$$

the common practice is to linearize the equations. With the choice of (1.2) for $\mathbf{F}$, linearizing (1.1) leads to the *linearized bond-based peridynamic* model

$$\partial_{tt}\mathbf{u}(\mathbf{x}, t) = \mathbb{L}\mathbf{u}(\mathbf{x}, t) + \mathbf{b}(\mathbf{x}, t)\,, \tag{1.3}$$

where $\mathbb{L}\mathbf{u}(\mathbf{x},t)$ is the linearized internal force density given by

$$\mathbb{L}\mathbf{u}(\mathbf{x},t) := \int_\Omega \rho(|\mathbf{x}-\mathbf{y}|)\left(\big(\mathbf{u}(\mathbf{x},t)-\mathbf{u}(\mathbf{y},t)\big)\cdot\frac{\mathbf{x}-\mathbf{y}}{|\mathbf{x}-\mathbf{y}|}\right)\frac{\mathbf{x}-\mathbf{y}}{|\mathbf{x}-\mathbf{y}|}\,\mathrm{d}\mathbf{y}\,. \tag{1.4}$$

The linearized equilibrium equation is then

$$\mathbb{L}\mathbf{u}(\mathbf{x})+\mathbf{b}(\mathbf{x}) := \int_\Omega \rho(|\mathbf{x}-\mathbf{y}|)\left(\big(\mathbf{u}(\mathbf{x})-\mathbf{u}(\mathbf{y})\big)\cdot\frac{\mathbf{x}-\mathbf{y}}{|\mathbf{x}-\mathbf{y}|}\right)\frac{\mathbf{x}-\mathbf{y}}{|\mathbf{x}-\mathbf{y}|}\,\mathrm{d}\mathbf{y}+\mathbf{b}(\mathbf{x})=\mathbf{0}\,. \tag{1.5}$$

Thus strain of the bond $\mathbf{y}-\mathbf{x}$ in (1.4) is given by the nonlocal linearized strain

$$\mathcal{D}(\mathbf{u})(\mathbf{x},\mathbf{y}) = \big(\mathbf{u}(\mathbf{x})-\mathbf{u}(\mathbf{y})\big)\cdot\frac{\mathbf{x}-\mathbf{y}}{|\mathbf{x}-\mathbf{y}|}\,. \tag{1.6}$$

A portion of this strain contributes to the volume changing component of the deformation and the remaining is the shape changing component. The operator $\mathbb{L}\mathbf{u}$ is then the linearized internal force density function due to the deformation $\mathbf{x}\mapsto\mathbf{x}+\mathbf{u}(\mathbf{x})$ and is a weighted average of the linearized strain function associated with the displacement [59, 82]. Indeed, rewriting (1.4) in terms of the nonlocal strain (1.6) we get $\mathbb{L}\mathbf{u}(\mathbf{x})=\int_\Omega\rho(|\mathbf{x}-\mathbf{y}|)\mathcal{D}(\mathbf{u})(\mathbf{x},\mathbf{y})\frac{\mathbf{x}-\mathbf{y}}{|\mathbf{x}-\mathbf{y}|}\,\mathrm{d}\mathbf{y}$.

Nonlocal models like (1.5) are becoming commonplace across application areas. Typically these models involve averaged difference quotients instead of derivatives of quantities. As a result model equations are formulated using integral operators and integral equations, in contrast to classical ones that rely on differential operators and differential equations. This characteristic makes nonlocal models amenable to describe singular and discontinuous physical, social and biological phenomena, see [2, 3, 41, 67] for applications and analysis of nonlocal equations. See the papers [59, 82] for derivation. See also the papers [26, 29, 34, 91] for some mathematical analysis of linearized models. Nonlocal functionals related to the peridynamic model are also studied in [37]. The nonlocal linearized strain has been used in nonlinear models of damage and fracture [51, 52] as well.

1.1 Properties of Peridynamic Models

The usage of the "projected" difference of $\mathbf{u}$, $\mathcal{D}(\mathbf{u})(\mathbf{x},\mathbf{y})$, in (1.5) makes the operator $\mathbb{L}$ distinct from other nonlocal operators that use the full difference $\mathbf{u}(\mathbf{y})-\mathbf{u}(\mathbf{x})$. To see this distinction, it suffices to note that for smooth vector fields

$$\frac{\mathbf{u}(\mathbf{y})-\mathbf{u}(\mathbf{x})}{|\mathbf{y}-\mathbf{x}|} = \nabla\mathbf{u}(\mathbf{x})\frac{(\mathbf{y}-\mathbf{x})}{|\mathbf{y}-\mathbf{x}|} + o(|\mathbf{y}-\mathbf{x}|)\,,$$

whereas

$$\frac{\mathcal{D}(\mathbf{u})(\mathbf{x},\mathbf{y})}{|\mathbf{y}-\mathbf{x}|} = \frac{(\mathbf{y}-\mathbf{x})^\intercal\big(\nabla_{sym}(\mathbf{u})(\mathbf{x})\big)(\mathbf{y}-\mathbf{x})}{|\mathbf{y}-\mathbf{x}|^2} + o(|\mathbf{y}-\mathbf{x}|)\,.$$

Here $\nabla_{sym}(\mathbf{u}) = \frac{1}{2}(\nabla\mathbf{u}(\mathbf{x})+\nabla\mathbf{u}(\mathbf{x})^\intercal)$, the symmetric part of the gradient matrix, commonly called the strain tensor. The action $\{\}^\intercal$ denotes the transpose. As a consequence of this relation, the nonlocal system (1.5) can be seen as a nonlocal analogue of the strongly coupled system of partial differential equations

$$\mathbf{div}\,\mathfrak{C}(\mathbf{x})\nabla_{sym}(\mathbf{u})(\mathbf{x}) = \mathbf{f}(\mathbf{x})\,, \tag{1.7}$$

where $\mathfrak{C}(\mathbf{x})$ is a fourth–order tensor of bounded coefficients, which is not necessarily uniformly elliptic but rather satisfies the weaker Legendre-Hadamard ellipticity condition. Systems of partial differential equations of the above type are are commonly used in the theory of linearized elasticity, and are well-studied, see [40].

For a class of kernels the connection between the nonlocal system (1.5) and the local system (1.7) is often justified using the *vanishing horizon limit*, i.e. taking $\delta\to 0$. This convergence of these linearized nonlocal peridynamic models to classical linearized elasticity models as $\delta\to 0$ has been explored in [1, 29, 33, 60, 84]. The first efforts in [33, 34] established the connection between the classical Navier-Lamé equations of elasticity and peridynamic models for smooth functions via Taylor expansion, see also [59]. To be precise, for $\mathbf{u}\in C^2_c(\mathbb{R}^d;\mathbb{R}^d)$ and for a certain class of kernels ρ supported on a Euclidean ball of radius δ,

$$\mathbb{L}\mathbf{u} \underset{\delta\to 0}{\longrightarrow} \mathfrak{L}_{1,1}\mathbf{u} := \Delta\mathbf{u} + 2\nabla(\operatorname{div}\mathbf{u})\,,$$

where the operator $\mathfrak{L}_{\mu,\lambda}$ is the classical Lamé operator with parameters μ and λ, that is

$$\mathfrak{L}_{\mu,\lambda}\mathbf{u} := \mu\Delta\mathbf{u} + (\mu+\lambda)\nabla(\operatorname{div}\mathbf{u})\,.$$

Later, using the variational technique of Γ-convergence, the authors of [8, 60] proved the convergence of minimizers of the nonlocal energies to minimizers of classical local elastic energies as $\delta \to 0$.

In Chapter 2 we show that peridynamic-type models are closely connected to local elasticity using a different type of limit. We assume that the kernel ρ is nonintegrable with a singularity at $\mathbf{0}$, and is supported on the entire space. To be precise,

$$\rho(|\boldsymbol{\xi}|) = \frac{1}{|\boldsymbol{\xi}|^{d+2s}}\,, \qquad s \in (0,1)\,,\ \boldsymbol{\xi} \in \mathbb{R}^d\,.$$

These kernels are associated to the scalar nonlocal model known as the fractional Laplacian $(-\Delta)^s$, defined for vector fields $\mathbf{u}$ as

$$(-\Delta)^s\mathbf{u}(\mathbf{x}) := c_{d,s}\int_{\mathbb{R}^d} \frac{\mathbf{u}(\mathbf{x})-\mathbf{u}(\mathbf{y})}{|\mathbf{x}-\mathbf{y}|^{d+2s}}\,\mathrm{d}\mathbf{y}\,.$$

It is well-known [68] that $\lim\limits_{s\to 1^-}(-\Delta)^s\mathbf{u} = -\Delta\mathbf{u}$. We will show in fact that for $\mathbf{u}$ smooth and for some normalizing constant C depending only on d and s

$$\lim_{s\to 1^-} C\int_{\mathbb{R}^d}\left(\frac{\mathbf{x}-\mathbf{y}}{|\mathbf{x}-\mathbf{y}|}\otimes\frac{\mathbf{x}-\mathbf{y}}{|\mathbf{x}-\mathbf{y}|}\right)\frac{\mathbf{u}(\mathbf{x})-\mathbf{u}(\mathbf{y})}{|\mathbf{x}-\mathbf{y}|^{d+2s}}\,\mathrm{d}\mathbf{y} = -\mathfrak{L}_{1,1}\mathbf{u}(\mathbf{x})\,.$$

Thus the system (1.5) with kernel $\rho(|\boldsymbol{\xi}|) = |\boldsymbol{\xi}|^{-d-2s}$ may be thought of as a fractional analogue of the system (1.7). Included in our proof is the explicit calculation of the normalizing constant.

1.2 Scope and Objectives of the Dissertation

Following successful demonstrations in computational mechanics of the potential for peridynamics in modeling singularities, efforts have been made to give the model a firm mathematical background. This dissertation builds upon those efforts; its chief focus is the analysis of the equations (1.3) and (1.5). In the following sections we describe the dissertation's contributions in detail. We first show well-posedness of the Dirichlet problem associated to the nonlocal system and prove optimal Sobolev regularity of solutions for a special class of kernels. We then analyze a class of function spaces motivated by an energy space naturally appearing in the Dirichlet problem. We use the function space analysis to prove regularity results for weak solutions to nonlocal systems with more general kernels. A portion of the work in this dissertation appears in several journal articles [47, 78, 79].

1.2.1 Solvability of Boundary-Value Problems

In Chapter 3 we consider the Dirichlet problem

$$\mathbb{L}\mathbf{u} = \mathbf{f} \quad \text{in } \Omega; \quad \mathbf{u} = \mathbf{0} \quad \text{in } \complement\Omega\,, \tag{1.8}$$

where $\Omega \subset \mathbb{R}^d$ is an open bounded set and $d \geq 1$. The complement of Ω is denoted $\complement\Omega$. We write the nonlocal operator $\mathbb{L}$ as

$$\mathbb{L}\mathbf{u}(\mathbf{x}) = \lim_{\varepsilon \to 0^+} \int_{|\mathbf{x}-\mathbf{y}|>\varepsilon} k(\mathbf{x},\mathbf{y}) \left(\frac{\mathbf{x}-\mathbf{y}}{|\mathbf{x}-\mathbf{y}|} \otimes \frac{\mathbf{x}-\mathbf{y}}{|\mathbf{x}-\mathbf{y}|} \right) (\mathbf{u}(\mathbf{x}) - \mathbf{u}(\mathbf{y}))\, \mathrm{d}\mathbf{y}\,, \tag{1.9}$$

where the kernel $k : \mathbb{R}^d \times \mathbb{R}^d \to [0,\infty]$ is assumed to be measurable.

The goal of Chapter 3 is twofold. First, we formulate a variational problem for (1.8), the resolution of which provides solutions to (1.8). For given data $\mathbf{f}$ in an appropriate class, we describe a notion of solution and demonstrate existence of vector-valued solutions $\mathbf{u} : \mathbb{R}^d \to \mathbb{R}^d$ to the system (1.8). The second goal is to prove some results related to the optimal regularity of solutions.

The authors of [33, 34] provided the earliest analytical result on the well-posedness of the linear dynamic model (1.3) as an initial-value problem. The works [27, 29, 58, 59, 92] built on these efforts to deliver results of well-posedness to the Dirichlet problem associated to the equilibrium equations (1.5) both in the linearized bond-based and state-based models using Fourier analysis and Hilbert space techniques. Regularity is also discussed in [29]. The work [59], for instance, studies the Dirichlet problem associated to linearized bond-based peridynamic operators of the form (1.4), or equivalently

$$\int_{\Omega} \rho(|\mathbf{x}-\mathbf{y}|) \left(\frac{\mathbf{x}-\mathbf{y}}{|\mathbf{x}-\mathbf{y}|} \otimes \frac{\mathbf{x}-\mathbf{y}}{|\mathbf{x}-\mathbf{y}|} \right) \big(\mathbf{u}(\mathbf{x}) - \mathbf{u}(\mathbf{y})\big) \, \mathrm{d}\mathbf{y} \,. \tag{1.10}$$

The operator (1.9) is therefore a generalization of the operators described by (1.10). The kernel ρ belongs to the class of models for homogeneous materials, and thus depends only on the relative distance between two material points $|\mathbf{y}-\mathbf{x}|$. The kernel k may depend on their relative position $\mathbf{y}-\mathbf{x}$, or it may depend on $\mathbf{x}$ and $\mathbf{y}$ in a more general way. Thus for general k the equation (1.8) may model heterogeneous and anisotropic materials.

Our study of the nonlocal system (1.8) begins with a mathematically rigorous understanding of the operator $\mathbb{L}$. The focus is to find a large class of kernels k that may not be symmetric ($k(\mathbf{x},\mathbf{y}) \neq k(\mathbf{y},\mathbf{x})$), may have singularity along the diagonal $\{(\mathbf{x},\mathbf{y}) \in \mathbb{R}^d \times \mathbb{R}^d : \mathbf{x} = \mathbf{y}\}$ or degeneracy on some directions such that both the operator $\mathbb{L}$ and associated system of equations (1.8) make sense. Notice that even for smooth functions the limit in (1.9) does not exist in general unless we put a condition on k. As with partial differential equations in divergence form with measurable coefficients, we study variational solutions based on quadratic forms. We use Hilbert space techniques to study the Dirichlet problem (1.8).

To describe some of our results, following [36, 77] we introduce a decomposition of $k(\mathbf{x},\mathbf{y})$ in terms of its symmetric part k_s and its anti-symmetric part k_a. They are given by

$$k_s(\mathbf{x},\mathbf{y}) = \frac{1}{2}(k(\mathbf{x},\mathbf{y}) + k(\mathbf{y},\mathbf{x}))\,, \quad k_a(\mathbf{x},\mathbf{y}) = \frac{1}{2}(k(\mathbf{x},\mathbf{y}) - k(\mathbf{y},\mathbf{x}))\,.$$

Throughout Chapter 3 we consider kernels whose symmetric part has locally integrable second moment; see (3.3). With this, we define the natural energy space of vector fields associated to the problem (1.8) by

$$S(\mathbb{R}^d;k) := \left\{ \mathbf{v} \in L^2(\mathbb{R}^d;\mathbb{R}^d) \,:\, [\mathbf{u},\mathbf{u}]_{S(\mathbb{R}^d;K)} := \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} k_s(\mathbf{x},\mathbf{y}) \mathcal{D}(\mathbf{u})(\mathbf{x},\mathbf{y})^2 \, \mathrm{d}\mathbf{y} \, \mathrm{d}\mathbf{x} < \infty \right\}.$$

It turns out that $S(\mathbb{R}^d;k)$ is a separable Hilbert space with inner product $(\cdot,\cdot)_{L^2} + [\cdot,\cdot]_{S(\mathbb{R}^d;k)}$.

Roughly speaking, we show the following results: for those kernels k whose antisymmetric part is small relative to the symmetric part, the limit in (1.9) exists in the weak-* topology of the dual space $S^*(\mathbb{R}^d;k)$, and therefore $\mathbb{L}\mathbf{u} \in S^*(\mathbb{R}^d;k)$. This interpretation of the operator allows us to define a generalized or weak notion of solution to the system of equations in (1.8). The well-posedness of the problem is demonstrated via the application of the Lax-Milgram theorem. To this end, we introduce a bilinear form on the space $S(\mathbb{R}^d;k) \times S(\mathbb{R}^d;k)$ that is compatible with the system (1.8), and by imposing additional conditions on k we show that this form is continuous and coercive on appropriate subspaces.

Systems of the type (1.8) have been studied extensively in the literature; see [26, 27, 34, 44, 58]. The results in Chapter 3 complement the well-posedness results described in these works. Indeed, this work deals with kernels that give rise to non-symmetric bilinear forms while earlier works are based on kernels associated to symmetric bilinear forms.

We also obtain local regularity results for variational solutions of the system (1.8) corresponding to a special class of kernels. For this aspect of our study, we concentrate on translation invariant operators with kernels of the form $k(\mathbf{x},\mathbf{y}) = k(\mathbf{x}-\mathbf{y})$. Additionally k is assumed to be even and comparable with the standard kernel of fractional order. We allow this comparability to hold true in any double cone Λ with apex at the origin, i.e.

$$k(\mathbf{x}-\mathbf{y}) \approx \frac{1}{|\mathbf{y}-\mathbf{x}|^{d+2s}}, \quad s \in (0,1), \quad \mathbf{x}-\mathbf{y} \in \Lambda\,.$$

For these types of kernels we show that the Hilbert space $S(\mathbb{R}^d;k)$ is equivalent to the standard fractional Sobolev space

$$W^{s,2}(\mathbb{R}^d;\mathbb{R}^d) := \left\{ \mathbf{u} \in L^2(\mathbb{R}^d;\mathbb{R}^d) \,:\, [\mathbf{u}]^2_{W^{s,2}} := \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} \frac{|\mathbf{u}(\mathbf{x})-\mathbf{u}(\mathbf{y})|^2}{|\mathbf{x}-\mathbf{y}|^{d+2s}}\,\mathrm{d}\mathbf{y}\,\mathrm{d}\mathbf{x} < \infty \right\}.$$

Such an equivalence will be proved using the Fourier transform; see [29, 57] for related results. Our main regularity result is the following: Define the non-homogeneous potential space

$$\mathcal{L}^{2s,p}(\mathbb{R}^d;\mathbb{R}^d) := \{\mathbf{u} \in L^p(\mathbb{R}^d;\mathbb{R}^d) : (-\Delta)^s\mathbf{u} \in L^p(\mathbb{R}^d;\mathbb{R}^d)\}$$

for $p \in (1,\infty)$, where the fractional Laplacian $(-\Delta)^s$ is acting on each component. We show that for any $p \in \left[2, \frac{2d}{d-2s}\right]$ and $\mathbf{f} \in L^p(\Omega;\mathbb{R}^d)$, the unique variational solution $\mathbf{u} \in W^{s,2}(\mathbb{R}^d;\mathbb{R}^d)$ to (1.8) in fact belongs to $\mathcal{L}^{2s,p}_{loc}(\mathbb{R}^d;\mathbb{R}^d)$. For nonlocal equations, results of the above type have been proved in [10, 43, 50]. We follow an approach that is used in [9, 10], where a similar but more general result is proved for the Dirichlet problem for the fractional Laplacian equation; see Chapter 3 for a detailed discussion.

1.2.2 Analysis of Function Spaces

The main focus of Chapter 4 is to study two distinct function spaces of vector fields. The first space is characterized by the projected difference quotient, namely

$$\mathcal{X}_{K,p}(\Omega) := \left\{ \mathbf{v} \in L^p(\Omega;\mathbb{R}^d) \;:\; [\mathbf{v}]^p_{\mathcal{X}_{K,p}(\Omega)} < \infty \right\},$$
$$[\mathbf{v}]^p_{\mathcal{X}_{K,p}(\Omega)} := \int_\Omega \int_\Omega K(\mathbf{y}-\mathbf{x}) \left| \big(\mathbf{v}(\mathbf{y}) - \mathbf{v}(\mathbf{x})\big) \cdot \frac{\mathbf{y}-\mathbf{x}}{|\mathbf{y}-\mathbf{x}|} \right|^p \mathrm{d}\mathbf{y}\, \mathrm{d}\mathbf{x}\,.$$

The second space is given by the closure of Schwarz functions in a certain norm, precisely

$$\mathcal{D}^p_s(\mathbb{R}^d) := \overline{\mathcal{S}(\mathbb{R}^d;\mathbb{R}^d)}^{\|\cdot\|_{\mathcal{D}^p_s(\mathbb{R}^d)}},$$

where

$$\|\mathbf{v}\|_{\mathcal{D}^p_s(\mathbb{R}^d)} := \|\mathbf{v}\|_{L^p(\mathbb{R}^d)} + \|D^s(\mathbf{v})\|_{L^p(\mathbb{R}^d)}\,,$$
$$D^s(\mathbf{v})(\mathbf{x}) = \left(\int_{\mathbb{R}^d} \frac{\left| \big(\mathbf{v}(\mathbf{x}) - \mathbf{v}(\mathbf{y})\big) \cdot \frac{(\mathbf{x}-\mathbf{y})}{|\mathbf{x}-\mathbf{y}|} \right|^2}{|\mathbf{x}-\mathbf{y}|^{d+2s}} d\mathbf{y} \right)^{\frac{1}{2}}.$$

Both of these function spaces arise naturally when studying problems like (1.8). Indeed, in the case of the first space, note that for $p = 2$ and $\Omega = \mathbb{R}^d$, the space $\mathcal{X}_{K,2}(\Omega)$ is precisely the space $S(\mathbb{R}^d;k)$. For the second space, the function $D^s(\mathbf{v})$ is the natural measure of the differentiability of $\mathbf{u}$ that is associated to the peridynamic energy. More precisely, a vector field $\mathbf{u} \in L^2(\mathbb{R}^d;\mathbb{R}^d)$ will belong to $\mathcal{X}_{K,2}(\mathbb{R}^d)$ with $K(\boldsymbol{\xi}) = |\boldsymbol{\xi}|^{-d-2s}$ if and only if $D^s(\mathbf{u}) \in L^2(\mathbb{R}^d)$. These and other spaces associated to nonlocal models such as (1.8) have been investigated analytically in [56, 61]. The results in this section of the dissertation extend these works.

Chapter 4 is divided into two sections. For a particular class of kernels, we show in the first section that $\mathcal{X}_{K,p}(\Omega)$ is equivalent to a Sobolev space. In the second section we show that the space $\mathcal{D}^p_s(\mathbb{R}^d)$ is equivalent to a Bessel potential space.

The Sobolev Space

Some basic structural properties of $\mathcal{X}_{K,p}(\Omega)$ have already been investigated in [29, 59, 60]. There it is shown that for any $1 \leq p < \infty$, the space $\mathcal{X}_{K,p}(\Omega)$ is a separable Banach space with norm $\big(\|\mathbf{v}\|^p_{L^p} + [\mathbf{v}]^p_{\mathcal{X}_{K,p}} \big)^{1/p}$, reflexive if $1 < p < \infty$, and is a Hilbert space when $p = 2$. Conditions on the kernel K can be imposed so that a Poincaré-Korn type inequality holds over subsets that contain no nontrivial zeros of the semi-norm $[\cdot]_{\mathcal{X}_{K,p}}$. It is not difficult to see that $[\mathbf{v}]_{\mathcal{X}_{K,p}} = 0$ if and only if $\mathbf{v}$ is an affine map with skew-symmetric gradient.

As a difference-based function space, it may seem that $\mathcal{X}_{K,p}(\Omega)$ contains functions with some "differentiability." This is not in general true, however. Taking a radial K that is compactly supported and with the property that $|\mathbf{x}|^{-p}K(\mathbf{x})$ is integrable, it is shown in [60] that $\mathcal{X}_{K,p}(\Omega) = L^p(\Omega;\mathbb{R}^d)$. In the event that the space $\mathcal{X}_{K,p}(\Omega)$ is a proper subset of $L^p(\Omega;\mathbb{R}^d)$, the fact that the semi-norm utilizes the "directional" or "projected" difference quotient $\big(\mathbf{v}(\mathbf{y}) - \mathbf{v}(\mathbf{x})\big) \cdot \frac{(\mathbf{y}-\mathbf{x})}{|\mathbf{y}-\mathbf{x}|}$ appears to make the space relatively big compared to those that use the full difference quotient. Nevertheless, by averaging the projected difference quotient over enough directions, it is reasonable to think that the semi-norm generated will be comparable with the one that is associated with the full difference quotient. *However, this remains unclear in general.* Finding general conditions on K and Ω so that equivalence holds is an open problem, and so we restrict our discussion to the special class of kernels

$$K(\boldsymbol{\xi}) = \frac{1}{|\boldsymbol{\xi}|^{d+sp}}\,, \quad 0 < s < 1\,, \quad 1 < p < \infty\,.$$

We denote the corresponding space by $\mathcal{X}_p^s(\Omega)$. These kernels are associated with the fractional Sobolev spaces $W^{s,p}(\Omega;\mathbb{R}^d)$ given by

$$W^{s,p}(\Omega;\mathbb{R}^d) := \left\{ \mathbf{v} \in L^p(\Omega;\mathbb{R}^d) : [\mathbf{v}]_{W^{s,p}}^p := \int_\Omega \int_\Omega \frac{|\mathbf{v}(\mathbf{y}) - \mathbf{v}(\mathbf{x})|^p}{|\mathbf{y}-\mathbf{x}|^{d+sp}} \, \mathrm{d}\mathbf{y} \, \mathrm{d}\mathbf{x} < \infty \right\}.$$

The question is now if $\mathcal{X}_p^s(\Omega)$ is the same as $W^{s,p}(\Omega;\mathbb{R}^d)$ for these fractional kernels.

A recent work [57] answers the above question in the affirmative for the special case $p = 2$, and $\Omega = \mathbb{R}^d$ or $\mathbb{R}^d_+$. When $p = 2$, and $\Omega = \mathbb{R}^d$, the question is tractable because both spaces $\mathcal{X}_2^s(\mathbb{R}^d)$, and $W^{s,2}(\mathbb{R}^d;\mathbb{R}^d)$ can be characterized by Fourier symbols which makes the comparison of norms more straightforward; see [29]. For functions defined over the half-space $\mathbb{R}^d_+$ and vanishing near the hyperplane $x_d = 0$, one can use an appropriate extension operator to control the semi-norm $[\cdot]_{W^{s,2}(\mathbb{R}^d_+)}$ by the semi-norm $[\cdot]_{\mathcal{X}_2^s(\mathbb{R}^d_+)}$ of vector fields in the dense class $C_c^1(\mathbb{R}^d_+;\mathbb{R}^d)$. This section of the dissertation is devoted to extending these results to any $p \in (1,\infty)$ again providing an affirmative answer to the question of the equivalence of these spaces. Let us introduce the function space

$$\mathring{\mathcal{X}}_p^s(\Omega) = \text{Closure of } C_c^\infty(\Omega;\mathbb{R}^d) \text{ in } \mathcal{X}_p^s(\Omega)\,.$$

Theorem 1.1 (Fractional Korn's inequality). *For any $s \in (0,1)$ and $1 < p < \infty$,*

$$\mathcal{X}_p^s(\mathbb{R}^d) = W^{s,p}(\mathbb{R}^d;\mathbb{R}^d)\,, \quad \textit{and} \quad \mathring{\mathcal{X}}_p^s(\mathbb{R}^d_+) = W_0^{s,p}(\mathbb{R}^d_+;\mathbb{R}^d)\,.$$

Moreover, there exists a universal constant $C = C(d,p,s)$ such that for all $\mathbf{f} \in W^{s,p}(\mathbb{R}^d;\mathbb{R}^d)$

$$[\mathbf{f}]_{\mathcal{X}_p^s} \leq [\mathbf{f}]_{W^{s,p}} \leq C[\mathbf{f}]_{\mathcal{X}_p^s}\,. \tag{1.11}$$

While the first inequality in (1.11) is trivial, the second inequality is the interesting one, as it gives a control of the integral norm of a pointwise larger function by the integral norm of a pointwise smaller function. We call the second inequality a *fractional Korn's inequality* for the following reason. For a smooth vector field $\mathbf{f}$, the semi-norm $[\mathbf{f}]_{W^{s,p}}$ uses the full difference quotient which locally behaves as

$$\left|\frac{\mathbf{f}(\mathbf{y})-\mathbf{f}(\mathbf{x})}{|\mathbf{y}-\mathbf{x}|}\right|^p \approx \left|\nabla \mathbf{f}(\mathbf{x})\frac{\mathbf{y}-\mathbf{x}}{|\mathbf{y}-\mathbf{x}|}\right|^p + O(|\mathbf{y}-\mathbf{x}|)$$

while the semi-norm $[\mathbf{f}]_{\mathcal{X}_p^s}$ uses the projected difference quotient and locally behaves as

$$\left|\frac{\mathbf{f}(\mathbf{y})-\mathbf{f}(\mathbf{x})}{|\mathbf{y}-\mathbf{x}|}\cdot\frac{(\mathbf{y}-\mathbf{x})}{|\mathbf{y}-\mathbf{x}|}\right|^p \approx \left|\left\langle (\nabla_{sym}(\mathbf{f})(\mathbf{x}))\frac{\mathbf{y}-\mathbf{x}}{|\mathbf{y}-\mathbf{x}|}, \frac{\mathbf{y}-\mathbf{x}}{|\mathbf{y}-\mathbf{x}|}\right\rangle\right|^p + O(|\mathbf{y}-\mathbf{x}|)$$

The connection between the projected difference quotient and ∇_{sym} runs deeper; multiplying the semi-norm by the proper correcting constant $(1-s)$ it has been shown in [56], following the argument in [12], that the space $\mathcal{X}_p^s(\Omega)$ "converges" to

$$W^{1,p}_{Sym}(\Omega;\mathbb{R}^d) := \{\mathbf{u}\in L^p(\Omega;\mathbb{R}^d) : \nabla_{sym}(\mathbf{u}) \in L^p(\Omega;\mathbb{R}^{d\times d})\}$$

as $s \to 1^-$. This association suggests that $\mathcal{X}_p^s(\Omega)$ is the fractional analogue of $W^{1,p}_{Sym}(\Omega;\mathbb{R}^d)$. In turn for $1 < p < \infty$, the space $W^{1,p}_{sym}(\Omega;\mathbb{R}^d)$ is known to coincide with $W^{1,p}(\Omega;\mathbb{R}^d)$ via the classical Korn's inequality, a fundamental tool in the theory of linearized elasticity; see [24] for a complete proof. This equivalence is false when $p = 1$, see [70], and when $p = \infty$, see [23]. As such, establishing $\mathcal{X}_p^s(\Omega) = W^{s,p}(\Omega;\mathbb{R}^d)$ when $1 < p < \infty$ in the affirmative amounts to proving a version of Korn's inequality for fractional Sobolev spaces.

Our proof of Theorem 1.1 makes use of the classical characterization of functions in the fractional spaces in terms of their Poisson integrals. See Chapter 4, Section 1 for a detailed discussion.

The Bessel Space

We introduce the Bessel potential space, $\mathcal{L}^q_s(\mathbb{R}^d)$. For $0 < s < 1$, $1 < q < \infty$

$$\mathcal{L}^q_s(\mathbb{R}^d) := \left\{\mathbf{f}\in\mathcal{S}'(\mathbb{R}^d;\mathbb{R}^d) \;:\; \left(\left(1+4\pi^2|\boldsymbol{\xi}|^2\right)^{s/2}\widehat{\mathbf{f}}\right)^\vee \in L^q(\mathbb{R}^d;\mathbb{R}^d)\right\}.$$

The norm in $\mathcal{L}^q_s(\mathbb{R}^d)$ is given by

$$\|\mathbf{f}\|_{\mathcal{L}^q_s(\mathbb{R}^d)} := \left\| \left(\left(1+4\pi^2|\boldsymbol{\xi}|^2\right)^{s/2} \widehat{\mathbf{f}} \right)^{\vee} \right\|_{L^q(\mathbb{R}^d)}.$$

We will establish that for $p \in \left(\frac{2d}{d+2s}, \infty\right)$, the space of functions in which both $\mathbf{u}$ and $D^s(\mathbf{u})$ are in L^p is precisely the space $\mathcal{L}^p_s$. That is, $\mathcal{D}^p_s(\mathbb{R}^d) = \mathcal{L}^p_s(\mathbb{R}^d)$.

Theorem 1.2. *Let $\frac{2d}{d+2s} \leq p < \infty$ and $0 < s < 1$. Then $\mathbf{f} \in \mathcal{L}^p_s(\mathbb{R}^d)$ if and only if $\mathbf{f} \in \mathcal{D}^p_s(\mathbb{R}^d)$. Moreover, there exist positive constants C_1 and C_2 such that for any $\mathbf{f} \in L^p(\mathbb{R}^d;\mathbb{R}^d)$,*

$$C_1\|\mathbf{f}\|_{\mathcal{L}^p_s(\mathbb{R}^d)} \leq \|\mathbf{f}\|_{L^p(\mathbb{R}^d)} + \|D^s(\mathbf{u})\|_{L^p(\mathbb{R}^d)} \leq C_2\|\mathbf{f}\|_{\mathcal{L}^p_s(\mathbb{R}^d)}.$$

We note that this characterization is in the same spirit as classical characterizations of vector fields in the Bessel potential space. One such characterization is given by Stein in [85, Theorem 1] or [86, Chapter V] in terms of smallness of the difference $\mathbf{f}(\mathbf{x}+\mathbf{y})-\mathbf{f}(\mathbf{x})$ via the Marcinkiewicz integral $\Upsilon^s(\mathbf{f})$. The Marcinkiewicz integral $\Upsilon^s(\mathbf{u})$ (also called the *carré du champ* operator associated to $(-\Delta)^s$) is a measure of the local smoothness of $\mathbf{u}$ given by

$$\Upsilon^s(\mathbf{u})(\mathbf{x}) := \left(\int_{\mathbb{R}^d} \frac{|\mathbf{u}(\mathbf{x})-\mathbf{u}(\mathbf{y})|^2}{|\mathbf{x}-\mathbf{y}|^{d+2s}}\, d\mathbf{y} \right)^{\frac{1}{2}}.$$

The result in [85, 86] states that for $0 < s < 1$ and $\frac{2d}{d+2s} \leq p < \infty$, $\mathbf{f} \in \mathcal{L}^p_s(\mathbb{R}^d;\mathbb{R}^d)$ if and only if $\mathbf{f} \in L^p(\mathbb{R}^d;\mathbb{R}^d)$ and $\Upsilon^s(\mathbf{f}) \in L^p(\mathbb{R}^d)$. A finer characterization that uses means over balls is also given in [87] that is valid for the full range of $p \in (1,\infty)$. Various other characterizations have also been explored throughout the literature. However, the essence of Theorem 1.2 lies in the fact that to determine if $\mathbf{f}$ is a Bessel potential of an L^p vector field we do not need the smallness of a measure of $\mathbf{f}(\mathbf{x}+\mathbf{y})-\mathbf{f}(\mathbf{x})$ but rather a measure of the quantity $[\mathbf{f}(\mathbf{x}+\mathbf{y})-\mathbf{f}(\mathbf{x})]\cdot\frac{\mathbf{y}}{|\mathbf{y}|}$ via another Marcinkiewicz-type integral of $\mathbf{f}$, $D^s(\mathbf{f})$. The function $D^s(\mathbf{u})$ is the natural measure of the s-differentiability of $\mathbf{u}$ directly related to the peridynamic energy and satisfies the pointwise estimate $D^s(\mathbf{u})(\mathbf{x}) \leq \Upsilon^s(\mathbf{u})(\mathbf{x})$ for all $\mathbf{x} \in \mathbb{R}^d$. From this pointwise estimate, Theorem 1.2 follows from the characterization in [85, Theorem 1]. However, the left-hand side inequality is a refined version of [85, Theorem 1] since we are estimating $\|\mathbf{f}\|_{\mathcal{L}^p_s(\mathbb{R}^d)}$ in terms of the smaller function $D^s(\mathbf{f})$ in place of $\Upsilon^s(\mathbf{f})$. We will in fact show that the left-hand side inequality in the theorem is valid for $p \in (1, \infty)$.

Our proof of Theorem 1.2 follows the steps presented in the proof of [85, Theorem 1]. We first develop the necessary technical tools that allow us to relate the Marcinkiewicz-type integral $D^s(\mathbf{f})$ with the potential function of $\mathbf{f}$. It turns out this is possible by using the Poisson-type integral of $\mathbf{f}$ defined in the previous section and a corresponding Littlewood-Paley g-function. We will show that, in parallel with classical results, this new g-function can be used to characterize L^p norms of vector fields.

1.2.3 Regularity of Solutions

In Chapter 5 we use the function space analysis from Chapter 4 to obtain two regularity results for weak solutions to peridynamic-type systems. The first regularity result states that weak solutions to systems of the type (1.5) belong to certain Bessel potential spaces, and the second regularity result states that weak solutions belong to certain fractional Sobolev-Slobodeckij spaces. The latter result also holds for weak solutions to nonlinear nonlocal equations. We describe both results below.

Bessel Space Regularity

For the first result we return our attention to nonlocal operators that use kernels $\rho(\mathbf{x}, \mathbf{y})$ comparable to $|\mathbf{x} - \mathbf{y}|^{-(d+2s)}$. We write the strongly coupled system of equations as

$$\mathbb{L}_{\mathfrak{h}} \mathbf{u}(\mathbf{x}) := c_{\mathfrak{h}} \int_{B_{\mathfrak{h}}(\mathbf{x})} \frac{A(\mathbf{x}, \mathbf{y})}{|\mathbf{x} - \mathbf{y}|^{d+2s}} \frac{(\mathbf{x} - \mathbf{y})}{|\mathbf{x} - \mathbf{y}|} \otimes \frac{(\mathbf{x} - \mathbf{y})}{|\mathbf{x} - \mathbf{y}|} \left(\mathbf{u}(\mathbf{x}) - \mathbf{u}(\mathbf{y}) \right) \, \mathrm{d}\mathbf{y} = \mathbf{F}(\mathbf{x}) \,, \tag{1.12}$$

where $d \geq 2$, $0 < s < 1$ are fixed, and $A(\mathbf{x}, \mathbf{y})$ is a measurable function, which we refer to as the *coefficient*, that is elliptic and symmetric in the sense that

$$0 < \alpha_1 \leq A(\mathbf{x}, \mathbf{y}) \leq \alpha_2 < \infty \,, \qquad A(\mathbf{x}, \mathbf{y}) = A(\mathbf{y}, \mathbf{x}) \qquad \text{for all } \mathbf{x}, \mathbf{y} \in \mathbb{R}^d.$$

The number $\mathfrak{h}$ is a positive constant. Weak solutions to (1.12) are defined in Chapter 5.

Our goal in the first part of this chapter is to establish the higher integrability of the measure of smoothness $D^s(\mathbf{u})$ of weak solutions $\mathbf{u}$ to the peridynamic system of nonlocal equations given in (1.12). Roughly speaking, we show that there exists an exponent $p > 2$ such that for any weak solution $\mathbf{u}$ to (1.12) corresponding to rough data $\mathbf{F} \in L^r(\mathbb{R}^d;\mathbb{R}^d)$ with $r < 2$, both $\mathbf{u}$ and $D^s(\mathbf{u})$ are in L^p. The higher integrability result holds under no additional assumption on the coefficient $A(\mathbf{x},\mathbf{y})$ other than ellipticity and measurability. Then by Theorem 1.2 it immediately follows that $\mathbf{u} \in \mathcal{L}^p_s(\mathbb{R}^d;\mathbb{R}^d)$ for some $p > 2$.

In light of the connection to local elasticity (see the discussion in Section 1.1) the higher integrability result for weak solutions to (1.12) is a fractional analogue of Meyers inequality for systems of differential equations. Meyers inequality states that weak solutions of the strongly coupled systems (1.7) with measurable coefficients satisfying the Legendre-Hadamard ellipticity condition and corresponding to highly integrable data are in $W^{1,p}(\mathbb{R}^d;\mathbb{R}^d)$ for some $p > 2$, see [62, 63]. The result we obtain is the fractional analogue of this inequality in the Bessel potential spaces. This inequality will be proved using invertibility of the operator $\mathbb{I} + \mathbb{L}_{\mathfrak{h}}$ over a range of Sobolev spaces; see Chapter 5 for details.

Sobolev Estimates for Nonlinear Equations

We use Theorem 1.1 and classical embedding estimates to obtain Sobolev regularity of weak solutions to nonlinear nonlocal peridynamic-type systems of equations with elliptic measurable coefficients. The system is given as

$$\mathbb{L}^s_{p,\Omega}(\mathbf{u})(\mathbf{x}) = \mathbf{F}(\mathbf{x})\,, \tag{1.13}$$

where the operator $\mathbb{L}^s_{p,\Omega}$ is defined as

$$\mathbb{L}^s_{p,\Omega}(\mathbf{u})(\mathbf{x}) := \int_\Omega \frac{A(\mathbf{x},\mathbf{y})}{|\mathbf{x}-\mathbf{y}|^{d+2s}} |\mathcal{D}(\mathbf{u})(\mathbf{x},\mathbf{y})|^{p-2} \mathcal{D}(\mathbf{u})(\mathbf{x},\mathbf{y}) \frac{\mathbf{x}-\mathbf{y}}{|\mathbf{x}-\mathbf{y}|}\, \mathrm{d}\mathbf{y}\,.$$

We assume that $s \in (0,1)$, $p \geq 2$, and that $A(\mathbf{x},\mathbf{y})$ is a measurable function such that $\alpha_1 \leq A(\mathbf{x},\mathbf{y}) \leq \alpha_2$ and symmetric in the sense that $A(\mathbf{x},\mathbf{y}) = A(\mathbf{y},\mathbf{x})$ for any $\mathbf{x}$, $\mathbf{y} \in \mathbb{R}^d$. Weak solutions to (1.13) are defined in Chapter 5.

The operator $\mathbb{L}_{p,\Omega}$ is a nonlinear p-Laplacian-type version of the linear peridynamic operator defined in (1.4). Indeed, for $p = 2$ the operator $\mathbb{L}_{p,\Omega}$ takes the form of the operator (1.4). For any $1 < p < \infty$ and $A(\mathbf{y}, \mathbf{x}) = a(|\mathbf{y} - \mathbf{x}|)$, well posedness of the coupled system (1.13) has been established in [60] by using variational methods and imposing appropriate volumetric conditions. Moreover, by exploiting the connection between the spaces $\mathcal{X}_p^s(\Omega)$ and $W^{1,p}_{Sym}(\Omega; \mathbb{R}^d)$ it has been shown that (1.13) is a fractional analogue of a strongly coupled nonlinear system of partial differential equations of the type

$$\mathbf{div}\left(|(\nabla \mathbf{u})_{sym}|^{p-2}\, (\nabla \mathbf{u})_{sym}(\mathbf{x})\right) = \mathbf{F}(\mathbf{x})\,.$$

In fact, for specific variational problems, this relationship has been established via Γ-convergence in [60] in the event of vanishing nonlocality (that is, $s \to 1^-$). Regularity of solutions of the nonlinear system has been the subject of recent works, see [90].

The second main result of this chapter is an *a priori* estimate for the self-improving properties of solutions to the nonlocal nonlinear coupled system (1.13). This result is stated in the following theorem.

Theorem 1.3. *Let $s \in (0, 1)$, $p \geq 2$, and $\Omega \subset \mathbb{R}^d$ be a bounded domain. Let $\mathbf{F} \in [\mathcal{X}_p^{s-\varepsilon(p-1)}(\Omega)]^*$, and let $\mathbf{u} \in C^1_{loc}(\Omega; \mathbb{R}^d) \cap \mathcal{X}_p^s(\Omega)$ be a weak solution to the coupled system of nonlocal equations (1.13) corresponding to $\mathbf{F}$. Then there exists a positive constant ε_0 such that for all $\varepsilon \in (0, \varepsilon_0)$ the weak solution $\mathbf{u}$ belongs to $W^{s+\varepsilon,p}_{loc}(\Omega)$. Moreover, for any $\eta \in C_c^\infty(\Omega)$, there exists a positive constant C such that*

$$[\eta \mathbf{u}]_{W^{s+\varepsilon,p}(\mathbb{R}^d)} \leq C \|\mathbf{F}\|^{\frac{1}{p-1}}_{\left[\mathcal{X}_p^{s-\varepsilon(p-1)}(\Omega)\right]^*} + C \|\mathbf{u}\|_{\mathcal{X}_p^s(\Omega)}\,.$$

The potential implication of the regularity result in the theorem is that, with no additional smoothness conditions on the coefficient $A(\mathbf{x}, \mathbf{y})$, a weak solution to the coupled system (1.13) has improved fractional differentiability in response to improved regularity in the data. For scalar equations, this type of self-improving property of solutions has been verified in [49] using reverse Hölder inequalities and nonlocal Gehring-type lemmas, obtained in [76] via a commutator estimate and later obtained in [4] via a functional analytic approach. The main contribution of this section is the extension of the *a priori* estimate for the self-improving properties of solutions obtained in the above cited works to the nonlocal nonlinear system (1.13). We should note that the application of appropriate embedding estimates imply both improved differentiability and higher integrability.

To prove Theorem 1.3, we follow the approach in [76]; our strategy is close in spirit with the technique of "differentiating the equation," and finding relations between higher derivatives of solutions and test functions in order to estimate derivatives of the solution. These estimates are based on the norm $[\cdot]_{\mathcal{X}_p^s}$. We then use Theorem 1.1 to conclude that the estimates are also valid using the larger semi-norm $[\cdot]_{W^{s,p}}$. See Chapter 5 for further discussion.

1.2.4 Solvability Revisited: The Wave Equation

The last chapter of this dissertation is devoted to showing the solvability of the time dependent system of wave-type equations

$$\begin{cases} \partial_{tt}\mathbf{u}(\mathbf{x},t) + \mathbb{L}\mathbf{u}(\mathbf{x},t) = \mathbf{f}(\mathbf{x},t)\,, & (\mathbf{x},t) \text{ in } \mathbb{R}^d \times [0,T]\,, \\ \mathbf{u}(\mathbf{x},0) = \mathbf{u}_0(\mathbf{x})\,, & \mathbf{x} \text{ in } \mathbb{R}^d\,, \\ \partial_t\mathbf{u}(\mathbf{x},0) = \mathbf{v}_0(\mathbf{x})\,, & \mathbf{x} \text{ in } \mathbb{R}^d\,, \end{cases} \tag{1.14}$$

where the operator $\mathbb{L}$ is given by

$$-\mathbb{L}\mathbf{u}(\mathbf{x},t) = \text{P.V.}\int_{\mathbb{R}^d} \left(\frac{\mathbf{y}\otimes\mathbf{y}}{|\mathbf{y}|^2}\right)\big(\mathbf{u}(\mathbf{x}+\mathbf{y},t) - \mathbf{u}(\mathbf{x},t) - \varepsilon(\mathbf{u})(\mathbf{x},t)\mathbf{y}\,\chi^{(s)}(\mathbf{y})\big)\;\rho(\mathbf{y})\,\mathrm{d}\mathbf{y}\,. \tag{1.15}$$

The definition of the operator $\mathbb{L}$ will be explained in a moment. We analyze (1.14) by proving results for the equilibrium equations associated to the corresponding steady-state system, which are given in (1.5) but we rewrite here for convenience:

$$\int_{\mathbb{R}^d} \rho(\mathbf{x},\mathbf{y})\left(\big(\mathbf{u}(\mathbf{x}) - \mathbf{u}(\mathbf{y})\big)\cdot\frac{\mathbf{x}-\mathbf{y}}{|\mathbf{x}-\mathbf{y}|}\right)\frac{\mathbf{x}-\mathbf{y}}{|\mathbf{x}-\mathbf{y}|}\,\mathrm{d}\mathbf{y} = \mathbf{f}(\mathbf{x})\,. \tag{1.16}$$

The majority of Chapter 6 is devoted to obtaining results for (1.16), which are then leveraged via semigroups to obtain results for (1.14). This semigroup approach will be explained in detail in Section 6.5, and at present we focus on the system (1.16). Mathematical analysis of (1.16) often considers the case when the kernel is compactly supported, radially symmetric, $\rho(\mathbf{x},\mathbf{y}) = \rho(|\mathbf{x}-\mathbf{y}|)$, and bounded away from zero in a neighborhood of the origin [29, 56, 59, 91]. The time dependent linearized equation of motion (1.14) is also studied in [33, 34] as an evolution equation in various spaces when the kernel is radial. For the well posedness of the nonlinear peridynamic equations of motion we refer to [16, 30, 31]. In this case, the integral operator is well-defined either as a convolution-type operator, when the kernel is integrable, or in the *principal value* sense in when the kernel has strong singularity.

Similarly to Chapter 3, here we study models associated with kernels that are translation-invariant but may be rotationally variant. To be precise, we assume $\rho(\mathbf{x},\mathbf{y}) = \rho(\mathbf{x}-\mathbf{y})$, where ρ is not necessarily symmetric. This assumption combined with a formal change of variables leads to the nonlocal system

$$\int_{\mathbb{R}^d} \rho(\mathbf{y}) \left(\frac{\mathbf{y} \otimes \mathbf{y}}{|\mathbf{y}|^2} \right) \big(\mathbf{u}(\mathbf{x}+\mathbf{y}) - \mathbf{u}(\mathbf{x}) \big) \, \mathrm{d}\mathbf{y} = -\mathbf{f}(\mathbf{x}) \, .$$

The integral operator on the left hand side converges when $\rho \in L^1(\mathbb{R}^d)$. However, when ρ is not integrable, it is not clear that the integral converges, even when $\mathbf{u}$ is smooth. We therefore modify the integral operator to ensure that it is well-defined. To illustrate the modification, first consider the specific choice of kernel $\rho(\mathbf{y}) := |\mathbf{y}|^{-d-2s}$ for $s \in (0,1)$, and define the integral operator

$$-(-\mathring{\Delta})^s \mathbf{u}(\mathbf{x}) := \mathrm{P.V.} \int_{\mathbb{R}^d} \frac{C}{|\mathbf{y}|^{d+2s}} \left(\frac{\mathbf{y} \otimes \mathbf{y}}{|\mathbf{y}|^2} \right) \big(\mathbf{u}(\mathbf{x}+\mathbf{y}) - \mathbf{u}(\mathbf{x}) \big) \, \mathrm{d}\mathbf{y} \, ,$$

where C is a normalizing constant. As indicated, the integral converges in the principal value sense for $\mathbf{u}$ smooth enough, say C^2. It is in fact a fractional analogue of the Láme operator in linearized elasticity [59]. Since the kernel is radial, we can subtract any odd function in the integrand and get, for example,

$$-(-\mathring{\Delta})^s \mathbf{u}(\mathbf{x}) = \text{P.V.} \int_{\mathbb{R}^d} \frac{C}{|\mathbf{y}|^{d+2s}} \left(\frac{\mathbf{y}\otimes\mathbf{y}}{|\mathbf{y}|^2}\right) \Big(\mathbf{u}(\mathbf{x}+\mathbf{y}) - \mathbf{u}(\mathbf{x}) - \nabla\mathbf{u}(\mathbf{x})\mathbf{y}\chi^{(s)}(\mathbf{y})\Big)\, \mathrm{d}\mathbf{y}\,,$$

where the function $\chi^{(s)}$ is defined as

$$\chi^{(s)}(\mathbf{y}) = \begin{cases} 0 & \text{if } s \in (0, 1/2) \\ \mathbb{1}_{B_1}(\mathbf{y}) & \text{if } s = 1/2 \\ 1 & \text{if } s \in (1/2, 1)\,. \end{cases}$$

Because of the presence of the rank-one matrix $\left(\frac{\mathbf{y}\otimes\mathbf{y}}{|\mathbf{y}|^2}\right)$, the last term can be rewritten, giving

$$-(-\mathring{\Delta})^s \mathbf{u}(\mathbf{x}) = \text{P.V.} \int_{\mathbb{R}^d} \frac{C}{|\mathbf{y}|^{d+2s}} \left(\frac{\mathbf{y}\otimes\mathbf{y}}{|\mathbf{y}|^2}\right) \Big(\mathbf{u}(\mathbf{x}+\mathbf{y}) - \mathbf{u}(\mathbf{x}) - \varepsilon(\mathbf{u})(\mathbf{x})\mathbf{y}\chi^{(s)}(\mathbf{y})\Big)\, \mathrm{d}\mathbf{y}\,,$$

where $\varepsilon(\mathbf{u})(\mathbf{x})$ is the symmetric part of the gradient matrix given by

$$\varepsilon(\mathbf{u})(\mathbf{x}) = \frac{1}{2}(\nabla\mathbf{u}(\mathbf{x}) + \nabla\mathbf{u}(\mathbf{x})^{\intercal}).$$

With this motivation at hand, we may now replace the kernel $|\mathbf{y}|^{-d-2s}$ with any translation-invariant kernel $\rho(\mathbf{y})$ and introduce the integral operator $\mathbb{L}$ by

$$-\mathbb{L}\mathbf{u}(\mathbf{x}) = \text{P.V.} \int_{\mathbb{R}^d} \left(\frac{\mathbf{y}\otimes\mathbf{y}}{|\mathbf{y}|^2}\right) \big(\mathbf{u}(\mathbf{x}+\mathbf{y}) - \mathbf{u}(\mathbf{x}) - \varepsilon(\mathbf{u})(\mathbf{x})\mathbf{y}\,\chi^{(s)}(\mathbf{y})\big)\ \rho(\mathbf{y})\, \mathrm{d}\mathbf{y}$$

which is precisely the operator defined in (1.15). With this new modification, it is clear now that even for radially nonsymmetric kernels comparable to $|\mathbf{y}|^{-d-2s}$ the integral converges absolutely for smooth $\mathbf{u}$. More generally, the kernel ρ that we will consider in Chapter 6 will come from two classes.

Class A: Integrable kernels: ρ is a nonnegative integrable function in $\mathbb{R}^d$. That is, $\rho(\mathbf{y}) \in L^1(\mathbb{R}^d)$. In this case, in the definition of $\mathbb{L}$ in (1.15), we take $\chi^{(s)}(\mathbf{y}) \equiv 0$. Integrable kernels commonly used in applications have compact support.

Class B: Nonintegrable kernels: ρ is a nonnegative singular kernel that is comparable to $|\mathbf{y}|^{-d-2s}$. More precisely, we assume that the kernel ρ is of the form

$$\rho(\mathbf{y}) := \frac{m(\mathbf{y})}{|\mathbf{y}|^{d+2s}} \mathbb{1}_{\Lambda_r}(\mathbf{y}),$$

where for some $0 < r \leq \infty$ the set $\Lambda_r := \Lambda \cap B_r$ is a truncated double cone $\Lambda := \{\mathbf{x} \in \mathbb{R}^d \,\big|\, \frac{\mathbf{x}}{|\mathbf{x}|} \in \Gamma \cup -\Gamma\}$ corresponding to a given measurable subset $\Gamma \subseteq \mathbb{S}^{d-1}$ that has positive Hausdorff measure, and the positive measurable function $m : \mathbb{R}^d \to [0, \infty)$ satisfies

$$0 < \alpha_1 \leq m(\mathbf{y}) \leq \alpha_2 < \infty,$$

for positive constants α_1 and α_2. We also assume that when $s = 1/2$, ρ satisfies the cancellation condition

$$\int_{\partial B_\mu} y_i y_j y_k \rho(\mathbf{y}) \, \mathrm{d}\sigma(\mathbf{y}) = 0, \quad \forall\, i, j, k \in \{1, 2, \dots, d\}, \quad \forall \mu > 0. \tag{1.17}$$

Note that (1.17) is always satisfied when $m(\mathbf{y})$ is an even function.

To show well posedness of the time dependent problem, we establish the solvability of the linear system

$$\mathbb{L}\mathbf{u}(\mathbf{x}) + \lambda \mathbf{u}(\mathbf{x}) = \mathbf{f}(\mathbf{x}) \quad \text{ in } \mathbb{R}^d, \tag{1.18}$$

which is the first main goal of Chapter 6. The constant λ is nonnegative and the operator $\mathbb{L}$ is assumed to possess a kernel from one of the above classes of kernels. For operators that use kernels in the nonintegrable class, we prove the existence and uniqueness of a strong solution in the Bessel potential space $\mathbf{u} \in H^{2s,2}$ corresponding to data $\mathbf{f} \in L^2$ satisfying (1.18) almost everywhere. Further, when $\mathbb{L} = (-\mathring{\Delta})$, for each $\mathbf{f} \in L^p$ there exists a unique $\mathbf{u} \in H^{2s,p}$ solving (1.18). Finally, we will show the well posedness of a wave equation closely resembling (1.14) as a consequence of the solvability obtained for the steady-state problem. These results described here will be precisely stated in Chapter 6.

1.3 Notation

Generally, notation will be set within each chapter, but we state several conventions used throughout the dissertation here. Given vectors $\mathbf{a} = (a_1, a_2, \cdots, a_d)$, and $\mathbf{b} = (b_1, b_2, \cdots, b_d)$ in $\mathbb{R}^d$, the tensor $\mathbf{a} \otimes \mathbf{b}$ is the rank-one matrix with $a_i b_j$ as its ij^{th} entry. We define the Fourier Transform of a function f as

$$\mathcal{F} f(\boldsymbol{\xi}) = \widehat{f}(\boldsymbol{\xi}) = \int\limits_{\mathbb{R}^d} \mathrm{e}^{-2\pi\imath \mathbf{x}\cdot\boldsymbol{\xi}} f(\mathbf{x}) \, \mathrm{d}\mathbf{x} \, .$$

The notation $\mathcal{F}^{-1}$ or $\cdot^{\vee}$ denotes the inverse Fourier transform

$$\mathcal{F}^{-1} f(\mathbf{x}) = f^{\vee}(\mathbf{x}) = \int\limits_{\mathbb{R}^d} \mathrm{e}^{2\pi\imath \mathbf{x}\cdot\boldsymbol{\xi}} f(\boldsymbol{\xi}) \, \mathrm{d}\boldsymbol{\xi} \, .$$

We will retain the same Fourier transform notations even for distributions.

We denote the surface measure of the unit sphere $\mathbb{S}^{d-1} \subset \mathbb{R}^d$ as

$$\omega_{d-1} = \frac{2\pi^{\frac{d}{2}}}{\Gamma(\frac{d}{2})} \, .$$

Chapter 2

A Peridynamic Operator With Fractional Kernel

The convergence of peridynamics to local elasticity has been thoroughly explored in the case of integrable kernels. Here we show a similar connection in the case of non-integrable kernels, specifically the fractional kernel $|\mathbf{x}-\mathbf{y}|^{-d-2s}$.

For this chapter, we define the operator $\mathbb{L}$ as

$$\mathbb{L}\mathbf{u}(\mathbf{x}) := \kappa_{d,s} \int_{\mathbb{R}^d} \left(\frac{(\mathbf{x}-\mathbf{y}) \otimes (\mathbf{x}-\mathbf{y})}{|\mathbf{x}-\mathbf{y}|^2} \right) \frac{\mathbf{u}(\mathbf{x}) - \mathbf{u}(\mathbf{y})}{|\mathbf{x}-\mathbf{y}|^{d+2s}} \, \mathrm{d}\mathbf{y} \,, \tag{2.1}$$

for $d \geq 2$ and $s \in (0,1)$. The positive constant $\kappa_{d,s}$ depends only on d and s, and is defined as

$$\begin{aligned} \kappa_{d,s} &:= (d+2s) \frac{2^{2s} s \Gamma(\frac{d}{2}+s)}{\pi^{d/2}\Gamma(1-s)} = (d+2s) c_{d,s} \,, \text{ where} \\ c_{d,s} &:= \frac{2^{2s} s \Gamma(\frac{d}{2}+s)}{\pi^{d/2}\Gamma(1-s)} = \left(\int_{\mathbb{R}^d} \frac{1-\cos(h_1)}{|\mathbf{h}|^{d+2s}} \, \mathrm{d}\mathbf{h} \right)^{-1} \,. \end{aligned} \tag{2.2}$$

Here, Γ denotes the gamma function

$$\Gamma(t) = \int_0^\infty y^{t-1} \mathrm{e}^{-y} \, \mathrm{d}y \,.$$

To see that the two definitions of $c_{d,s}$ are the same we refer the interested reader to [14].

2.1 A Formula via the Fourier Transform

We will show that $\mathbb{L}\mathbf{u} \longrightarrow_{s\to 1^-} \mathfrak{L}_{1,1}\mathbf{u}$ using the Fourier transform. We begin by finding closed expressions for two constants that appear in the Fourier transform of the operator $\mathbb{L}$.

2.1.1 Calculation of Constants

We recall the following well-known identity concerning the Gamma function:

$$\Gamma(t+1) = t\Gamma(t) \text{ for every } t > 0\,. \tag{2.3}$$

We also recall the definition of the Beta function with parameters a and b as $\mathrm{B}(a,b)$ via the formula

$$\mathrm{B}(a,b) := \int_0^\infty \frac{t^{a-1}}{(1+t)^{a+b}}\,\mathrm{d}t\,, \text{ where } a,b > 0\,. \tag{2.4}$$

The Beta function and Gamma function are related via the formula

$$\mathrm{B}(a,b) = \frac{\Gamma(a)\Gamma(b)}{\Gamma(a+b)}\,, \text{ where } a,b > 0\,. \tag{2.5}$$

Finally, from [14, Appendix A.2],

$$\frac{1}{(2\pi)^{2s}}\int_{-\infty}^\infty \frac{1-\cos(2\pi t)}{|t|^{1+2s}}\,\mathrm{d}t = \frac{2}{(2\pi)^{2s}}\int_0^\infty \frac{1-\cos(2\pi t)}{t^{1+2s}}\,\mathrm{d}t = \frac{\pi^{1/2}\Gamma(1-s)}{2^{2s}s\Gamma(s+\frac{1}{2})}\,. \tag{2.6}$$

Definition 2.1. Define two constants ℓ_1 and ℓ_2 as

$$\ell_1 := \int_{\mathbb{R}^d} \frac{1-\cos(h_1)}{|\mathbf{h}|^{d+2s}}\frac{h_1^2}{|\mathbf{h}|^2}\,\mathrm{d}\mathbf{h}\,, \qquad \ell_2 := \int_{\mathbb{R}^d} \frac{1-\cos(h_1)}{|\mathbf{h}|^{d+2s}}\frac{h_2^2}{|\mathbf{h}|^2}\,\mathrm{d}\mathbf{h}\,.$$

Lemma 2.2. *For $d \geq 2$ and $0 < s < 1$*

$$\ell_2 = \frac{1}{(d+2s)c_{d,s}} = \frac{1}{\kappa_{d,s}} = \frac{1}{d+2s}\cdot\frac{\pi^{d/2}\Gamma(1-s)}{2^{2s}s\Gamma(\frac{d}{2}+s)}\,, \tag{2.7}$$

and

$$\ell_1 = (2s+1)\ell_2 = \frac{2s+1}{\kappa_{d,s}} = \frac{2s+1}{d+2s}\cdot\frac{\pi^{d/2}\Gamma(1-s)}{2^{2s}s\Gamma(\frac{d}{2}+s)}\,. \tag{2.8}$$

Proof. We first prove (2.7). By a change of variables $\mathbf{h} = 2\pi\mathbf{x}$, we have

$$\ell_2 = \frac{1}{(2\pi)^{2s}} \int_{\mathbb{R}^d} \frac{1-\cos(2\pi x_1)}{|\mathbf{x}|^{d+2s}} \frac{x_2^2}{|\mathbf{x}|^2}\,\mathrm{d}\mathbf{x} = \frac{1}{(2\pi)^{2s}} \int_{\mathbb{R}} \int_{\mathbb{R}^{d-1}} \frac{(1-\cos(2\pi x_1))\,x_2^2}{(x_1^2+|\mathbf{x}'|^2)^{\frac{d+2s+2}{2}}}\,\mathrm{d}\mathbf{x}'\,\mathrm{d}x_1\,,$$

where $\mathbf{x} = (x_1, \mathbf{x}')$. We change variables in the inner integral; let $\boldsymbol{\eta} = (\eta_2, \ldots, \eta_d) = \frac{(x_2,\ldots,x_d)}{|x_1|}$. Then using (2.6),

$$\begin{aligned}
\ell_2 &= \frac{1}{(2\pi)^{2s}} \int_{\mathbb{R}} \int_{\mathbb{R}^{d-1}} \frac{1-\cos(2\pi x_1)}{|x_1|^{d+2+2s}(1+|\boldsymbol{\eta}|^2)^{\frac{d+2+2s}{2}}} |x_1|^2\eta_2^2 \cdot |x_1|^{d-1}\,\mathrm{d}\boldsymbol{\eta}\,\mathrm{d}x_1 \\
&= \frac{1}{(2\pi)^{2s}} \int_{\mathbb{R}} \int_{\mathbb{R}^{d-1}} \frac{(1-\cos(2\pi x_1))\,\boldsymbol{\eta}_2^2}{|x_1|^{1+2s}(1+|\eta|^2)^{\frac{d+2+2s}{2}}}\,\mathrm{d}\boldsymbol{\eta}\,\mathrm{d}x_1 \\
&= \left(\frac{1}{(2\pi)^{2s}} \int_{\mathbb{R}} \frac{1-\cos(2\pi x_1)}{|x_1|^{1+2s}}\,\mathrm{d}x_1\right)\left(\int_{\mathbb{R}^{d-1}} \frac{\eta_2^2}{(1+|\boldsymbol{\eta}|^2)^{\frac{d+2+2s}{2}}}\,\mathrm{d}\boldsymbol{\eta}\right) \\
&= \left(\frac{\pi^{1/2}\Gamma(1-s)}{2^{2s}s\Gamma(s+\frac{1}{2})}\right)\left(\int_{\mathbb{R}^{d-1}} \frac{\eta_2^2}{(1+|\boldsymbol{\eta}|^2)^{\frac{d+2+2s}{2}}}\,\mathrm{d}\boldsymbol{\eta}\right).
\end{aligned} \tag{2.9}$$

To handle to remaining integral in (2.9), changing to polar coordinates gives

$$\begin{aligned}
\int_{\mathbb{R}^{d-1}} \frac{\eta_2^2}{(1+|\boldsymbol{\eta}|^2)^{\frac{d+2+2s}{2}}}\,\mathrm{d}\eta &= \int_0^\infty \int_{\mathbb{S}^{d-2}} \frac{(r\zeta_2)^2 r^{d-2}}{(1+r^2)^{\frac{d+2+2s}{2}}}\,\mathrm{d}\sigma(\zeta)\,\mathrm{d}r \\
&= \int_{\mathbb{S}^{d-2}} \zeta_2^2\,\mathrm{d}\sigma(\zeta) \int_0^\infty \frac{r^d}{(1+r^2)^{\frac{d+2+2s}{2}}}\,\mathrm{d}r \\
&= \frac{\omega_{d-2}}{d-1} \int_0^\infty \frac{r^d}{(1+r^2)^{\frac{d+2+2s}{2}}}\,\mathrm{d}r\,,
\end{aligned}$$

Where ω_{d-2} denotes the surface measure of the unit sphere $\mathbb{S}^{d-2} \subset \mathbb{R}^{d-1}$. Letting $t = r^2$ and using the integral definition (2.4) of the Beta function,

$$\begin{aligned}
\frac{\omega_{d-2}}{d-1} \int_0^\infty \frac{r^d}{(1+r^2)^{\frac{d+2+2s}{2}}}\,\mathrm{d}r &= \frac{\omega_{d-2}}{2(d-1)} \int_0^\infty \frac{t^{\frac{d-1}{2}}}{(1+t)^{\frac{d+2+2s}{2}}}\,\mathrm{d}t \\
&= \frac{\omega_{d-2}}{2(d-1)} \mathrm{B}\left(\frac{d+1}{2}, \frac{1}{2}+s\right).
\end{aligned}$$

Using (2.5) and using the definition of surface measure,

$$\begin{aligned}
\omega_{d-2}\frac{1}{2(d-1)}\mathrm{B}\left(\frac{d+1}{2},\frac{1}{2}+s\right) &= \frac{2\pi^{\frac{d-1}{2}}}{\Gamma(\frac{d-1}{2})}\cdot\frac{1}{2(d-1)}\cdot\frac{\Gamma(\frac{d+1}{2})\Gamma(\frac{1}{2}+s)}{\Gamma(\frac{d}{2}+1+s)} \\
&= \frac{\pi^{\frac{d-1}{2}}}{\Gamma(\frac{d-1}{2})}\cdot\frac{1}{(d-1)}\cdot\frac{\frac{d-1}{2}\Gamma(\frac{d-1}{2})\Gamma(\frac{1}{2}+s)}{\Gamma(\frac{d}{2}+1+s)} \\
&= \frac{\pi^{\frac{d-1}{2}}}{2}\cdot\frac{\Gamma(\frac{1}{2}+s)}{\Gamma(\frac{d}{2}+1+s)}\,.
\end{aligned}$$

In the second equality we additionally used (2.4). Thus

$$\int_{\mathbb{R}^{d-1}}\frac{\eta_2^2}{(1+|\boldsymbol{\eta}|^2)^{\frac{d+2+2s}{2}}}\,\mathrm{d}\eta = \frac{\pi^{\frac{d-1}{2}}}{2}\cdot\frac{\Gamma(\frac{1}{2}+s)}{\Gamma(\frac{d}{2}+1+s)}\,. \tag{2.10}$$

Combining (2.10) with (2.9), we get using (2.3) that

$$\begin{aligned}
\ell_2 &= \left(\frac{\pi^{1/2}\Gamma(1-s)}{2^{2s}s\Gamma(s+\frac{1}{2})}\right)\left(\frac{\pi^{\frac{d-1}{2}}}{2}\cdot\frac{\Gamma(\frac{1}{2}+s)}{\Gamma(\frac{d}{2}+1+s)}\right) \\
&= \frac{\pi^{d/2}\Gamma(1-s)}{2^{2s}s}\cdot\frac{1}{2\Gamma(\frac{d}{2}+1+s)} \\
&= \frac{\pi^{d/2}\Gamma(1-s)}{2^{2s}s}\cdot\frac{1}{2(\frac{d+2s}{2})\Gamma(\frac{d}{2}+s)} = \frac{1}{d+2s}\cdot\frac{\pi^{d/2}\Gamma(1-s)}{2^{2s}s\Gamma(\frac{d}{2}+s)}\,,
\end{aligned}$$

which is (2.7).

To obtain (2.8), note that by the integral definitions of the constants ℓ_1, ℓ_2 and $c_{d,s}$, and by using rotations, that

$$\begin{aligned}
(c_{d,s})^{-1} &= \int_{\mathbb{R}^d}\frac{1-\cos(h_1)}{|\mathbf{h}|^{d+2s}}\,\mathrm{d}\mathbf{h} \\
&= \sum_{j=1}^{d}\int_{\mathbb{R}^d}\frac{1-\cos(h_1)}{|\mathbf{h}|^{d+2s}}\frac{h_j^2}{|\mathbf{h}|^2}\,\mathrm{d}\mathbf{h} \\
&= \int_{\mathbb{R}^d}\frac{1-\cos(h_1)}{|\mathbf{h}|^{d+2s}}\frac{h_2^2}{|\mathbf{h}|^2}\,\mathrm{d}\mathbf{h} + \sum_{j=2}^{d}\int_{\mathbb{R}^d}\frac{1-\cos(h_1)}{|\mathbf{h}|^{d+2s}}\frac{h_2^2}{|\mathbf{h}|^2}\,\mathrm{d}\mathbf{h} = \ell_1 + (d-1)\ell_2\,.
\end{aligned}$$

Thus, by (2.7)

$$\ell_1 = (c_{d,s})^{-1} - (d-1)\ell_2 = \big((d+2s) - (d-1)\big)\ell_2 = (2s+1)\ell_2\,.$$

$\square$

2.1.2 Fourier Transform of the Peridynamic Operator

Theorem 2.3. *Let* $\mathbf{u} \in \left[\mathcal{S}(\mathbb{R}^d)\right]^d$. *Then*

$$\widehat{\mathbb{L}\mathbf{u}}(\boldsymbol{\xi}) = \big(2\pi|\boldsymbol{\xi}|\big)^{2s}\left(2s\frac{\boldsymbol{\xi}\otimes\boldsymbol{\xi}}{|\boldsymbol{\xi}|^2} + \mathbb{I}\right)\widehat{\mathbf{u}}(\boldsymbol{\xi})\,. \tag{2.11}$$

Proof. By a change of variables $\mathbf{h} = \mathbf{y} - \mathbf{x}$, write

$$\mathbb{L}\mathbf{u}(\mathbf{x}) = \kappa_{d,s}\int_{\mathbb{R}^d}\frac{1}{|\mathbf{h}|^{d+2s}}\left(\frac{\mathbf{h}\otimes\mathbf{h}}{|\mathbf{h}|^2}\right)\big(\mathbf{u}(\mathbf{x}) - \mathbf{u}(\mathbf{x}+\mathbf{h})\big)\,\mathrm{d}\mathbf{h}\,. \tag{2.12}$$

Thus,

$$\widehat{\mathbb{L}\mathbf{u}}(\boldsymbol{\xi}) = \kappa_{d,s}\left(\int_{\mathbb{R}^d}\frac{1-e^{2\pi\imath\mathbf{h}\cdot\boldsymbol{\xi}}}{|\mathbf{h}|^{d+2s}}\left(\frac{\mathbf{h}\otimes\mathbf{h}}{|\mathbf{h}|^2}\right)\,\mathrm{d}\mathbf{h}\right)\hat{\mathbf{u}}(\boldsymbol{\xi}) := \kappa_{d,s}\mathbb{M}(\boldsymbol{\xi})\hat{\mathbf{u}}(\boldsymbol{\xi})\,. \tag{2.13}$$

We proceed to find a closed-form expression for $\mathbb{M}(\boldsymbol{\xi})$. For each $\boldsymbol{\xi} \neq \mathbf{0}$, choose a rotation $R(\boldsymbol{\xi}) \in \mathcal{O}(d)$ such that $R(\boldsymbol{\xi})\mathbf{e}_1 = \frac{\boldsymbol{\xi}}{|\boldsymbol{\xi}|}$, where $\mathbf{e}_1 = (1,0,\ldots,0)$. By a change of variables $\mathbf{y} = R^{\intercal}\mathbf{h}$, we have

$$\begin{aligned}\mathbb{M}(\boldsymbol{\xi}) &= \int_{\mathbb{R}^d}\frac{1-\mathrm{e}^{2\pi\imath R\mathbf{y}\cdot R|\boldsymbol{\xi}|\mathbf{e}_1}}{|\mathbf{y}|^{d+2s}}R(\boldsymbol{\xi})\left(\frac{\mathbf{y}\otimes\mathbf{y}}{|\mathbf{y}|^2}\right)R^{\intercal}(\boldsymbol{\xi})\,\mathrm{d}\mathbf{y}\\ &= R(\boldsymbol{\xi})\left[\int_{\mathbb{R}^d}\frac{1-\mathrm{e}^{2\pi\imath|\boldsymbol{\xi}|y_1}}{|\mathbf{y}|^{d+2s}}\left(\frac{\mathbf{y}\otimes\mathbf{y}}{|\mathbf{y}|^2}\right)\,\mathrm{d}\mathbf{y}\right]R^{\intercal}(\boldsymbol{\xi})\,.\end{aligned}$$

Again changing variables $\mathbf{h} = \mathbf{y}|\boldsymbol{\xi}|$,

$$\mathbb{M}(\boldsymbol{\xi}) = |\boldsymbol{\xi}|^{2s}R(\boldsymbol{\xi})\left[\int_{\mathbb{R}^d}\frac{1-\mathrm{e}^{2\pi\imath h_1}}{|\mathbf{h}|^{d+2s}}\left(\frac{\mathbf{h}\otimes\mathbf{h}}{|\mathbf{h}|^2}\right)\,\mathrm{d}\mathbf{h}\right]R^{\intercal}(\boldsymbol{\xi})\,. \tag{2.14}$$

Now,

$$\Im\mathbb{M}(\boldsymbol{\xi}) = |\boldsymbol{\xi}|^{2s}R(\boldsymbol{\xi})\left[\int_{\mathbb{R}^d}\frac{-\sin(2\pi h_1)}{|\mathbf{h}|^{d+2s}}\left(\frac{\mathbf{h}\otimes\mathbf{h}}{|\mathbf{h}|^2}\right)\,\mathrm{d}\mathbf{h}\right]R^{\intercal}(\boldsymbol{\xi})\,,$$

and since sine is an odd function the integral $\int_{\mathbb{R}^d} \frac{\sin(2\pi h_1)}{|\mathbf{h}|^{d+2s}} \frac{h_i h_j}{|\mathbf{h}|^2} \,\mathrm{d}\mathbf{h} = 0$ for every $i, j \in \{1, 2, \ldots, d\}$. Thus $\mathbb{M}(\boldsymbol{\xi}) = \Re\mathbb{M}(\boldsymbol{\xi})$.

Now, we claim that

$$\int_{\mathbb{R}^d} \frac{1 - \cos(2\pi h_1)}{|\mathbf{h}|^{d+2s}} \frac{h_i h_j}{|\mathbf{h}|^2} \,\mathrm{d}\mathbf{h} = 0\,, \qquad i \neq j\,. \tag{2.15}$$

Indeed, writing the left-hand side of (2.15) in polar coordinates,

$$\int_{\mathbb{R}^d} \frac{1 - \cos 2\pi h_1}{|\mathbf{h}|^{d+2s}} \frac{h_i h_j}{|\mathbf{h}|^2} \,\mathrm{d}\mathbf{h} = \int_0^\infty \frac{1}{r^{1+2s}} \int_{\mathbb{S}^{d-1}} (1 - \cos(2\pi r w_1)) w_i w_j \,\mathrm{d}\sigma(\mathbf{w}) \,\mathrm{d}r$$

Choose a rotation R_{ij} taking w_j to $-w_j$ and fixing w_1 and w_i. Changing coordinates and letting $\mathbf{z} = R_{ij}\mathbf{w}$,

$$\int_{\mathbb{S}^{d-1}} (1 - \cos(2\pi r w_1)) w_i w_j \,\mathrm{d}\sigma(\mathbf{w}) = -\int_{\mathbb{S}^{d-1}} (1 - \cos(2\pi r w_1)) w_i w_j \,\mathrm{d}\sigma(\mathbf{w})\,,$$

which proves (2.15). Therefore,

$$\mathbb{M}(\boldsymbol{\xi}) = (2\pi|\boldsymbol{\xi}|)^{2s} R(\boldsymbol{\xi}) \,\mathrm{diag}(\ell_1, \ldots, \ell_d) R^\intercal(\xi) \tag{2.16}$$

where

$$\ell_i = \frac{1}{(2\pi)^{2s}} \int_{\mathbb{R}^d} \frac{1 - \cos(2\pi h_1)}{|\mathbf{h}|^{d+2s}} \frac{h_i^2}{|\mathbf{h}|^2} \,\mathrm{d}\mathbf{h} = \int_{\mathbb{R}^d} \frac{1 - \cos(h_1)}{|\mathbf{h}|^{d+2s}} \frac{h_i^2}{|\mathbf{h}|^2} \,\mathrm{d}\mathbf{h}, \quad i = 1, \ldots, d\,. \tag{2.17}$$

Observe by using rotations and change of coordinates that $\ell_2 = \ell_3 = \ldots = \ell_d$. By Lemma 2.2,

$$\begin{aligned} \mathbb{M}(\boldsymbol{\xi}) &= \big(2\pi|\boldsymbol{\xi}|\big)^{2s} R(\boldsymbol{\xi}) \Big((\ell_1 - \ell_2)(\mathbf{e}_1 \otimes \mathbf{e}_1) + \ell_2 \mathbb{I} \Big) R^\intercal(\boldsymbol{\xi}) \\ &= \frac{1}{\kappa_{d,s}} \big(2\pi|\boldsymbol{\xi}|\big)^{2s} R(\boldsymbol{\xi}) \Big(2s(\mathbf{e}_1 \otimes \mathbf{e}_1) + \mathbb{I} \Big) R^\intercal(\boldsymbol{\xi})\,. \end{aligned}$$

Therefore, since

$$R(\boldsymbol{\xi})(\mathbf{e}_1 \otimes \mathbf{e}_1) R^\intercal(\boldsymbol{\xi}) = \left(\frac{\boldsymbol{\xi} \otimes \boldsymbol{\xi}}{|\boldsymbol{\xi}|^2} \right)\,,$$

we have

$$\widehat{\mathbb{L}\mathbf{u}}(\boldsymbol{\xi}) = \kappa_{d,s}\mathbb{M}(\boldsymbol{\xi})\widehat{\mathbf{u}}(\boldsymbol{\xi}) = \big(2\pi|\boldsymbol{\xi}|\big)^{2s}\left(2s\left(\frac{\boldsymbol{\xi}\otimes\boldsymbol{\xi}}{|\boldsymbol{\xi}|^2}\right)+\mathbb{I}\right)\widehat{\mathbf{u}}(\boldsymbol{\xi})\,.$$

□

With Theorem 2.3, it is apparent that

$$\begin{aligned}
&\lim_{s\to 0^+}\mathbb{L}\mathbf{u} = \mathbf{u}\,,\ \text{and}\\
&\lim_{s\to 1^-}\mathbb{L}\mathbf{u} = -\Delta\mathbf{u} - 2\nabla(\operatorname{div}\mathbf{u})\,.
\end{aligned}$$

Chapter 3

On A Variational Formulation

3.1 Introduction

We study the Dirichlet problem associated with the nonlocal system of equations

$$\mathbb{L}\mathbf{u} = \mathbf{f} \quad \text{in } \Omega; \quad \mathbf{u} = \mathbf{0} \quad \text{in } \complement\Omega\,, \tag{3.1}$$

where the vector-valued nonlocal operator $\mathbb{L}$ is of the form

$$\mathbb{L}\mathbf{u}(\mathbf{x}) = \lim_{\varepsilon\to 0^+}\int_{|\mathbf{x}-\mathbf{y}|>\varepsilon} k(\mathbf{x},\mathbf{y})\left(\frac{\mathbf{x}-\mathbf{y}}{|\mathbf{x}-\mathbf{y}|}\otimes\frac{\mathbf{x}-\mathbf{y}}{|\mathbf{x}-\mathbf{y}|}\right)(\mathbf{u}(\mathbf{x})-\mathbf{u}(\mathbf{y}))\,\mathrm{d}\mathbf{y}, \tag{3.2}$$

when the limit exists. The functions $\mathbf{u}$ and $\mathbf{f}$ are vector fields defined in their respective domain. The kernel $k:\mathbb{R}^d\times\mathbb{R}^d\to[0,\infty]$ is measurable. From the very definition of the nonlocal operator $\mathbb{L}$, it is clear that (3.1) is a strongly coupled system of equations.

The goal of this chapter is twofold. First, we formulate a variational problem for (3.1), the resolution of which provides solutions to (3.1). We treat more general kernels than those covered in the literature. For given data $\mathbf{f}$ in an appropriate class, we describe a notion of solution and demonstrate existence of vector-valued solutions $\mathbf{u}:\mathbb{R}^d\to\mathbb{R}^d$ to the nonlocal coupled system (3.1). The second goal is to prove some results related to the optimal regularity of solutions. This will be carried out for a specific class of kernels.

Recalling the discussion in Chapter 1, define the symmetric and anti-symmetric parts of $k(\mathbf{x},\mathbf{y})$

$$k_s(\mathbf{x},\mathbf{y}) = \frac{1}{2}(k(\mathbf{x},\mathbf{y}) + k(\mathbf{y},\mathbf{x}))\,, \quad k_a(\mathbf{x},\mathbf{y}) = \frac{1}{2}(k(\mathbf{x},\mathbf{y}) - k(\mathbf{y},\mathbf{x}))\,.$$

Throughout the chapter we consider kernels whose symmetric part has locally integrable second moment, i.e., we assume

$$\mathbf{x} \mapsto \int_{\mathbb{R}^d} \min\{1, |\mathbf{x}-\mathbf{y}|^2\} k_s(\mathbf{x},\mathbf{y})\, \mathrm{d}\mathbf{y} \in L^1_{loc}(\mathbb{R}^d). \tag{3.3}$$

We also define the function space of vector fields

$$S(\mathbb{R}^d;k) = \left\{ \mathbf{v} \in L^2(\mathbb{R}^d;\mathbb{R}^d)) : \mathcal{D}(\mathbf{v})(\mathbf{x},\mathbf{y}) k_s^{1/2}(\mathbf{x},\mathbf{y}) \in L^2(\mathbb{R}^d \times \mathbb{R}^d) \right\}.$$

The mapping $[\mathbf{u},\mathbf{v}]_{H(\mathbb{R}^d;k)} := \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} k_s(\mathbf{x},\mathbf{y})\mathcal{D}(\mathbf{u})(\mathbf{x},\mathbf{y})\mathcal{D}(\mathbf{v})(\mathbf{x},\mathbf{y})\,\mathrm{d}\mathbf{y}\,\mathrm{d}\mathbf{x}$ defines a bilinear form on $S(\mathbb{R}^d;k)$. One can easily show that the function $\|\cdot\|_{S(\mathbb{R}^d;k)}$, defined via the relation

$$\|\mathbf{v}\|^2_{S(\mathbb{R}^d;k)} = \|\mathbf{v}\|^2_{L^2(\mathbb{R}^d)} + \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} k_s(\mathbf{x},\mathbf{y})(\mathcal{D}(\mathbf{v})(\mathbf{x},\mathbf{y}))^2\,\mathrm{d}\mathbf{y}\,\mathrm{d}\mathbf{x},$$

serves as a norm for $S(\mathbb{R}^d;k)$. Moreover, adapting the argument used in the proof of [36, Lemma 2.3], we can actually show that $S(\mathbb{R}^d;k)$ is a separable Hilbert space with inner product $(\cdot,\cdot)_{L^2(\mathbb{R}^d)} + [\cdot,\cdot]_{S(\mathbb{R}^d;k)}$. See also similar results in [29, 58, 59]. We denote the dual space of $S(\mathbb{R}^d;k)$ by $S^*(\mathbb{R}^d;k)$.

Roughly speaking, we show the following results: for those kernels k whose antisymmetric part is small relative to the symmetric part (e.g. the function $\mathbf{x} \mapsto \int_{\mathbb{R}^d} \frac{(k_a(\mathbf{x},\mathbf{y}))^2}{k_s(\mathbf{x},\mathbf{y})}\,\mathrm{d}\mathbf{y}$ is uniformly bounded for any $\mathbf{u} \in S(\mathbb{R}^d;k)$), the limit in (3.2) exists in the weak-* topology of the dual space $S^*(\mathbb{R}^d;k)$, and therefore $\mathbb{L}\mathbf{u} \in S^*(\mathbb{R}^d;k)$. This interpretation of the operator allows us to define a generalized or weak notion of solution to the system of equations in (3.1). The well-posedness of the problem is demonstrated via the application of the Lax-Milgram theorem. To this end, we introduce a bilinear form on the space $S(\mathbb{R}^d;k) \times S(\mathbb{R}^d;k)$ that is compatible with the system (3.1), and by imposing additional conditions on k we show that this form is continuous and coercive on appropriate subspaces. Systems of the type (3.1) have been studied extensively in the literature, cf. in [26, 27, 34, 44, 58]. Our results complement the well-posedness result in the above cited papers. Indeed, our work deals with kernels that give rise to non-symmetric bilinear forms while earlier works are based on kernels associated to symmetric bilinear forms. As we will see in the next section clearly, the non-symmetric bilinear forms we study account for the the presence of lower order terms that may involve "lower order fractional" derivatives, while the results in [58] deal with linear problems with lower order terms that involve the unknown function without any derivatives.

Let us comment on the case where the vector fields are scalar. In this case, the quadratic form under consideration becomes a regular Dirichlet form in the sense of [38]. For this reason there is an associated strong Markov jump process, which can be used to study the Dirichlet problem. In the particular case of translation invariant operators, i.e., when $k(\mathbf{x}, \mathbf{y})$ depends only on $(\mathbf{x} - \mathbf{y})$, the process has stationary independent increments and is called a Lévy process. The potential theory of Markov jump processes including fine properties of heat kernels has been developed in great detail in recent years. It can be shown that our notion of a variational solution coincides with the probabilistic notion of harmonicity [15, 53] if the source term $\mathbf{f}$ vanishes. For the theory of nonlocal non-symmetric Dirichlet forms we refer to [39, 45, 77]. In the case of scalar fields, the variational approach to the Dirichlet problem has been used by several authors [36, 72, 80]. Note that we only comment on nonlocal operators in bounded domains which are related to quadratic forms. For a survey of results on nonlocal Dirichlet problem in the non-variational context, see [71].

Our study of the nonlocal system (3.1) for general kernels follows the variational approach taken in [36] adapting it to the system of equations. This adaptation is not trivial because of the structure of the operator. For instance, one can easily check that the seminorm $[\mathbf{u}, \mathbf{u}]_{S(\mathbb{R}^d;k)}$ vanishes over a class of affine maps of the type $\mathbf{u}(\mathbf{x}) = \mathbb{B}\mathbf{x} + \mathbf{c}$, where $\mathbb{B}$ is skew-symmetric matrix. When proving coercivity of the form over a subspace, one has to find a mechanism to remove this large class of maps, as opposed to constants in the case of equations. We will see that we need to use fractional Poincaré-Korn-type inequalities for the system in contrast to the standard fractional Poincaré inequality for problems involving scalar fields.

Let us mention that the system arising in (3.1) is related to the Euler-Lagrange system generated by fractional harmonic maps. Those systems were studied first in [20] for the half-Laplacian and then extended to more general situations [19, 21, 65, 73, 75]. In these works, the systems arise as Euler-Lagrange equations for critical points of functionals like $\|(-\Delta)^{\frac{s}{2}}\mathbf{u}\|_{L^p}$ for $\mathbf{u} \in \dot{H}^{s,p}(\mathbb{R}^d;\mathcal{M})$ where $M \subset \mathbb{R}^N$ is a smooth closed manifold. Obviously, these systems are nonlinear in general, which makes the regularity theory very challenging. However, the systems generated by harmonic maps do not possess a strong coupling in the main part of the operator as in (3.1).

In this chapter, we also obtain local regularity results for variational solutions of the system (3.1) corresponding to a special class of kernels. For this aspect of our study, we concentrate on translation invariant operators with kernels of the form $k(\mathbf{x},\mathbf{y}) = k(\mathbf{x}-\mathbf{y})$, that is even and comparable with the standard kernel of fractional order. We allow this comparability to hold true in any double cone Λ with apex at the origin, i.e.

$$k(\mathbf{x}-\mathbf{y}) \asymp \frac{1}{|\mathbf{y}-\mathbf{x}|^{d+2s}}, \quad s \in (0,1), \quad \mathbf{x}-\mathbf{y} \in \Lambda.$$

For these types of kernels we show that the Hilbert space $S(\mathbb{R}^d;k)$ is equivalent to the standard fractional Sobolev space

$$H^s(\mathbb{R}^d;\mathbb{R}^d) := \left\{\mathbf{u} \in L^2(\mathbb{R}^d;\mathbb{R}^d) \,:\, [\mathbf{u}]^2_{W^{s,2}} := \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} \frac{|\mathbf{u}(\mathbf{x})-\mathbf{u}(\mathbf{y})|^2}{|\mathbf{x}-\mathbf{y}|^{d+2s}}\,\mathrm{d}\mathbf{y}\,\mathrm{d}\mathbf{x} < \infty\right\}.$$

Such an equivalence will be proved using the Fourier transform. See [29, 57] for related results. For such kernels we show that actually the operator $\mathbb{L} : H^{2s}(\mathbb{R}^d;\mathbb{R}^d) \to L^2(\mathbb{R}^d;\mathbb{R}^d)$ is continuous. More generally, for any $p \in (1,\infty)$, if we define the non-homogeneous potential space

$$\mathcal{L}^{2s,p}(\mathbb{R}^d;\mathbb{R}^d) = \{\mathbf{u} \in L^p(\mathbb{R}^d;\mathbb{R}^d) : (-\Delta)^s\mathbf{u} \in L^p(\mathbb{R}^d;\mathbb{R}^d)\}$$

where the fractional Laplacian $(-\Delta)^s$ is acting on each component, then it can be shown that the nonlocal matrix-valued operator $\mathbb{L}$ is continuous from $\mathcal{L}^{2s,p}(\mathbb{R}^d;\mathbb{R}^d) \to L^p(\mathbb{R}^d;\mathbb{R}^d)$. Most importantly, we show in this chapter that for any $2 \leq p \leq \frac{2d}{d-2s}$ and $\mathbf{f} \in L^p(\Omega;\mathbb{R}^d)$, the unique variational solution $\mathbf{u} \in H^s(\mathbb{R}^d;\mathbb{R}^d)$ to the zero Dirichlet problem

$$\mathbb{L}\mathbf{u} = \mathbf{f} \quad \text{in } \Omega, \quad \mathbf{u} = 0 \quad \text{in } \complement\Omega. \tag{3.4}$$

belongs to $\mathcal{L}^{2s,p}_{loc}(\mathbb{R}^d;\mathbb{R}^d)$. We say $\mathbf{u} \in \mathcal{L}^{2s,p}_{loc}(\mathbb{R}^d;\mathbb{R}^d)$ if $\mathbf{u}\eta \in \mathcal{L}^{2s,p}(\mathbb{R}^d;\mathbb{R}^d)$ for any $\eta \in C^\infty_c(\mathbb{R}^d)$. For nonlocal equations, results of the above type have been proved in [10, 43, 50]. We follow an approach that is used in [9, 10], where a similar but more general result is proved for the Dirichlet problem for fractional Laplacian equation when the right hand side comes from L^p for any $1 < p < \infty$. In the case of vector fields, we could not cover all ranges of p but only with the additional assumption that the weak solution $\mathbf{u} \in L^p$. In the scalar case such an assumption is not necessary since it can be proven that a solution to the Dirichlet problem of the fractional Laplacian with right hand side in L^p must also be in L^p, see [10, Lemma 2.5]. A similar Calderón-Zygmund type estimate for solutions is also proved in [50, Theorem 16]. Unfortunately we are unable to extend their proof to the vector-valued case because the argument in [10] relies on a monotonicity property of an associated semigroup and in the case of [50] it uses a Moser-type argument where a nonlinear function of the solution is used as a test function. Neither of these arguments can be applied for systems.

The organization of the chapter is as follows: In Section 3.2 we introduce additional notation, provide some auxiliary results, and show well-posedness of the Dirichlet problem (3.1) using Hilbert space methods. We present sufficient conditions that imply the validity of fractional Poincaré-Korn type estimates for a larger class of kernels. We also provide examples of kernels for which the theorem is applicable. For a smaller class of kernels we also link the energy space $S(\mathbb{R}^d;k)$ with classical Sobolev spaces. In Section 3.3 we prove higher-order interior regularity of solutions to the Dirichlet problem corresponding to a particular class of kernels.

3.2 The Variational Formulation

In this section we set up the variational approach to solve the system (3.1).

3.2.1 Notation and Definitions

Throughout this chapter we will be using the following function spaces and their associated norms. We assume that $D \subset \mathbb{R}^d$ is an open subset, and $\complement D$ denotes its complement. We begin with the function spaces

$$L^p_D(\mathbb{R}^d) = \{\mathbf{u} \in L^p(\mathbb{R}^d;\mathbb{R}^d) \,|\, \mathbf{u} = \mathbf{0} \text{ a.e. on } \complement D\}$$

which collects L^p functions defined over $\mathbb{R}^d$ that vanish outside of D. We also use the notation $S_D(\mathbb{R}^d;k)$ to denote the space of functions in $S(\mathbb{R}^d;k)$ that vanish outside of D:

$$S_D(\mathbb{R}^d;k) = \{\mathbf{u} \in S(\mathbb{R}^d;k) : \mathbf{u} = \mathbf{0} \text{ a.e. on } \complement D\}.$$

It is not difficult to show that $S_D(\mathbb{R}^d;k)$ is a closed subset of $S(\mathbb{R}^d;k)$ and that from the definition, $\left(S_D(\mathbb{R}^d;k), \|\cdot\|_{S(\mathbb{R}^d;k)}\right) \hookrightarrow \left(S(\mathbb{R}^d;k), \|\cdot\|_{S(\mathbb{R}^d;k)}\right)$. We denote the dual space of $S_D(\mathbb{R}^d;k)$ by $S^*_D(\mathbb{R}^d;k)$.

To set up a variational problem, we will make necessary preparations. To begin with, we introduce a bilinear form that will be used to define a generalized notion of a solution to the nonlocal systems of equations.

Definition 3.1. Given two measurable functions $\mathbf{u}$ and $\mathbf{v}$, we define

$$\begin{aligned}\mathcal{F}^k(\mathbf{u},\mathbf{v}) = \frac{1}{2}\int_{\mathbb{R}^d}\int_{\mathbb{R}^d} k_s(\mathbf{x},\mathbf{y})\mathcal{D}(\mathbf{u})(\mathbf{x},\mathbf{y})\mathcal{D}(\mathbf{v})(\mathbf{x},\mathbf{y})\,\mathrm{d}\mathbf{y}\,\mathrm{d}\mathbf{x}\\ + \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} k_a(\mathbf{x},\mathbf{y})\mathcal{D}(\mathbf{u})(\mathbf{x},\mathbf{y})\left(\mathbf{v}(\mathbf{x})\cdot\frac{\mathbf{x}-\mathbf{y}}{|\mathbf{x}-\mathbf{y}|}\right)\mathrm{d}\mathbf{y}\,\mathrm{d}\mathbf{x},\end{aligned}$$

whenever the integrals exist.

We notice that the form is not necessarily symmetric. We aim to find conditions on k that allow us to have good control on the quadratic functional $\mathcal{F}^k(\mathbf{u},\mathbf{u})$ for $\mathbf{u}$ in the function space $S(\mathbb{R}^d;k)$. To that end, following [36, 77] let us assume that there exists a symmetric kernel $\tilde{k}$ and constants $A_1 \geq 1$, $A_2 \geq 1$ such that for all $\mathbf{x} \in \mathbb{R}^d$, the measure

$|\{\mathbf{y} \in \mathbb{R}^d : k_a^2(\mathbf{x},\mathbf{y}) \neq 0 \,\text{and}\, \tilde{k}(\mathbf{x},\mathbf{y}) = 0\}| = 0$, and

$$\int_{\mathbb{R}^d}\int_{\mathbb{R}^d} \tilde{k}(\mathbf{x},\mathbf{y})(\mathcal{D}(\mathbf{u})(\mathbf{x},\mathbf{y}))^2 \,\mathrm{d}\mathbf{y}\,\mathrm{d}\mathbf{x} \leq A_1 \|\mathbf{u}\|^2_{S(\mathbb{R}^d;k)} \tag{3.5}$$

for all $\mathbf{u} \in S(\mathbb{R}^d;k)$, and that

$$\sup_{\mathbf{x}\in\mathbb{R}^d} \int_{\mathbb{R}^d} \frac{k_a^2(\mathbf{x},\mathbf{y})}{\tilde{k}(\mathbf{x},\mathbf{y})}\,\mathrm{d}\mathbf{y} \leq A_2. \tag{3.6}$$

Note that we can choose $\tilde{k} = k_s$, see [77] where it is used for scalar equations. The next lemma describes the proper definition of $\mathcal{F}^k(\mathbf{u},\mathbf{v})$ and its continuity on $S(\mathbb{R}^d;k)$. It also clarifies in what sense the operator (3.2) is defined.

Proposition 3.2. *Let $\Omega \subset \mathbb{R}^d$ be open and assume that k satisfies (3.3) and (3.5)-(3.6). For $n \in \mathbb{N}$, define the subset $D_n = \{(\mathbf{x},\mathbf{y}) \in \mathbb{R}^d \times \mathbb{R}^d \,:\, |\mathbf{x}-\mathbf{y}| > 1/n\}$ and let*

$$\mathbb{L}_n\mathbf{u}(\mathbf{x}) = \int_{|\mathbf{x}-\mathbf{y}|>1/n} k(\mathbf{x},\mathbf{y})\mathcal{D}(\mathbf{u})(\mathbf{x},\mathbf{y})\frac{\mathbf{y}-\mathbf{x}}{|\mathbf{y}-\mathbf{x}|}\,\mathrm{d}\mathbf{y},$$

$$\mathcal{F}_n^k(\mathbf{u},\mathbf{v}) = \iint_{D_n} k(\mathbf{x},\mathbf{y})\mathcal{D}(\mathbf{u})(\mathbf{x},\mathbf{y})\left(\mathbf{v}(\mathbf{x})\cdot\frac{\mathbf{x}-\mathbf{y}}{|\mathbf{x}-\mathbf{y}|}\right)\mathrm{d}\mathbf{y}\,\mathrm{d}\mathbf{x}\,.$$

Then we have that

i) $(\mathbb{L}_n\mathbf{u},\mathbf{v})_{L^2(\mathbb{R}^d)} = \mathcal{F}_n^k(\mathbf{u},\mathbf{v})$ *and* $\lim_{n\to\infty}\mathcal{F}_n^k(\mathbf{u},\mathbf{v}) = \mathcal{F}^k(\mathbf{u},\mathbf{v})$ *for all* $\mathbf{u},\mathbf{v} \in C_c^\infty(\Omega)$.

ii) Moreover, $\mathcal{F}^k : S(\mathbb{R}^d;k)\times S(\mathbb{R}^d;k) \to \mathbb{R}$ *is continuous, and thus on* $S_\Omega(\mathbb{R}^d;k)$.

Proof. We begin by noticing that if $\mathbf{u} \in C_c^\infty(\mathbb{R}^d)$, the expression $\mathbb{L}_n\mathbf{u}(\mathbf{x})$ is finite for almost all $\mathbf{x} \in \mathbb{R}^d$. This follows from the fact that for almost all $(\mathbf{x},\mathbf{y}) \in \mathbb{R}^{d\times d}$, $k(\mathbf{x},\mathbf{y}) \leq k_s(\mathbf{x},\mathbf{y})$, assumption (3.3), and that the integration is over D_n. Similarly, for $\mathbf{u},\mathbf{v} \in C_c^\infty(\mathbb{R}^d)$, $\mathcal{F}_n^k(\mathbf{u},\mathbf{v})$ is finite as well.

Now, for $\mathbf{u},\mathbf{v} \in C_c^\infty(\mathbb{R}^d)$ we have by Fubini's theorem that

$$\begin{aligned}(\mathbb{L}_n\mathbf{u},\mathbf{v})_{L^2(\mathbb{R}^d)} &= \int_{\mathbb{R}^d}\left[\int_{\complement B(\mathbf{x},1/n)} k(\mathbf{x},\mathbf{y})\left(\frac{\mathbf{x}-\mathbf{y}}{|\mathbf{x}-\mathbf{y}|}\otimes\frac{\mathbf{x}-\mathbf{y}}{|\mathbf{x}-\mathbf{y}|}\right)(\mathbf{u}(\mathbf{x})-\mathbf{u}(\mathbf{y}))\,\mathrm{d}\mathbf{y}\right]\mathbf{v}(\mathbf{x})\,\mathrm{d}\mathbf{x} \\ &= \iint_{D_n} k(\mathbf{x},\mathbf{y})\left((\mathbf{u}(\mathbf{x})-\mathbf{u}(\mathbf{y}))\cdot\frac{\mathbf{x}-\mathbf{y}}{|\mathbf{x}-\mathbf{y}|}\right)\left(\mathbf{v}(\mathbf{x})\cdot\frac{\mathbf{x}-\mathbf{y}}{|\mathbf{x}-\mathbf{y}|}\right)\mathrm{d}\mathbf{y}\,\mathrm{d}\mathbf{x}\,.\end{aligned}$$

Split the last integral using the decomposition of k into k_s and k_a, and interchange $\mathbf{x}$ and $\mathbf{y}$ to obtain that

$$
\begin{aligned}
(\mathbb{L}_n \mathbf{u}, \mathbf{v})_{L^2(\mathbb{R}^d)} = \frac{1}{2} \iint_{D_n} & k_s(\mathbf{x},\mathbf{y}) \mathcal{D}(\mathbf{u})(\mathbf{x},\mathbf{y}) \mathcal{D}(\mathbf{v})(\mathbf{x},\mathbf{y}) \, \mathrm{d}\mathbf{y} \, \mathrm{d}\mathbf{x} \\
+ \iint_{D_n} & k_a(\mathbf{x},\mathbf{y}) \mathcal{D}(\mathbf{u})(\mathbf{x},\mathbf{y}) \left(\mathbf{v}(\mathbf{x}) \cdot \frac{\mathbf{x}-\mathbf{y}}{|\mathbf{x}-\mathbf{y}|} \right) \mathrm{d}\mathbf{y} \, \mathrm{d}\mathbf{x} \, .
\end{aligned}
\tag{3.7}
$$

We will be using the Lebesgue dominated convergence theorem to pass to the limit in both terms in (3.7). To pass to the limit in the first term we use the function $(\mathbf{x},\mathbf{y}) \mapsto k_s(\mathbf{x},\mathbf{y}) \mathcal{D}(\mathbf{u})(\mathbf{x},\mathbf{y}) \mathcal{D}(\mathbf{v})(\mathbf{x},\mathbf{y})$ as a majorant. It is integrable and by the Cauchy-Schwarz inequality,

$$
\int_{\mathbb{R}^d} \int_{\mathbb{R}^d} k_s(\mathbf{x},\mathbf{y}) \mathcal{D}(\mathbf{u})(\mathbf{x},\mathbf{y}) \, \mathcal{D}(\mathbf{v})(\mathbf{x},\mathbf{y}) \, \mathrm{d}\mathbf{y} \, \mathrm{d}\mathbf{x} \leq [\mathbf{u},\mathbf{u}]_{H(\mathbb{R}^d;k)} [\mathbf{v},\mathbf{v}]_{H(\mathbb{R}^d;k)} < \infty \tag{3.8}
$$

due to (3.3), since $\mathbf{u},\mathbf{v} \in S(\mathbb{R}^d, k)$. We next bound the integrand in the second term in (3.7) as follows. For $\mathbf{x},\mathbf{y} \in \mathbb{R}^d$, using Young's inequality we have that

$$
\begin{aligned}
k_a(\mathbf{x},\mathbf{y}) \mathcal{D}(\mathbf{u})(\mathbf{x},\mathbf{y}) \left(\mathbf{v}(\mathbf{x}) \cdot \frac{\mathbf{x}-\mathbf{y}}{|\mathbf{x}-\mathbf{y}|} \right) & \leq |k_a(\mathbf{x},\mathbf{y})| \tilde{k}^{-1/2}(\mathbf{x},\mathbf{y}) |\mathcal{D}(\mathbf{u})(\mathbf{x},\mathbf{y})| \, |\mathbf{v}(\mathbf{x})| \tilde{k}^{1/2}(\mathbf{x},\mathbf{y}) \\
& \leq \frac{1}{2} \left(\mathbf{v}(\mathbf{x})^2 \frac{k_a^2(\mathbf{x},\mathbf{y})}{\tilde{k}(\mathbf{x},\mathbf{y})} + \tilde{k}(\mathbf{x},\mathbf{y}) |\mathcal{D}(\mathbf{u})(\mathbf{x},\mathbf{y})|^2 \right),
\end{aligned}
$$

where assumption (3.5)-(3.6) guarantees that both functions on the right hand side are integrable in the product space $\mathbb{R}^d \times \mathbb{R}^d$. It is now clear that

$$
\lim_{n \to \infty} (\mathbb{L}_n \mathbf{u}, \mathbf{v})_{L^2(\mathbb{R}^d)} = \mathcal{F}^k(\mathbf{u},\mathbf{v}) \, .
$$

To prove the continuity of the bilinear form $\mathcal{F}^k : S(\mathbb{R}^d;k) \times S(\mathbb{R}^d;k) \to \mathbb{R}^d$ we estimate the two terms of $\mathcal{F}^k$ separately. As has been shown in (3.8), the first term of $\mathcal{F}^k(\mathbf{u},\mathbf{v})$ cannot exceed $\frac{1}{2}[\mathbf{u},\mathbf{u}]_{S(\mathbb{R}^d;k)}[\mathbf{v},\mathbf{v}]_{S(\mathbb{R}^d;k)}$. To estimate the second term, we use (3.5)-(3.6) with

$A = \max\{A_1, A_2\}$ and the Cauchy-Schwarz inequality to obtain that

$$\begin{aligned}
&\int_{\mathbb{R}^d}\int_{\mathbb{R}^d} k_a(\mathbf{x},\mathbf{y})\mathcal{D}(\mathbf{u})(\mathbf{x},\mathbf{y})\left(\mathbf{v}(\mathbf{x})\cdot\frac{\mathbf{x}-\mathbf{y}}{|\mathbf{x}-\mathbf{y}|}\right)\mathrm{d}\mathbf{y}\,\mathrm{d}\mathbf{x}\\
&\qquad\leq \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} |k_a(\mathbf{x},\mathbf{y})|\tilde{k}^{-1/2}(\mathbf{x},\mathbf{y})|\mathcal{D}(\mathbf{u})(\mathbf{x},\mathbf{y})||\mathbf{v}(\mathbf{x})|\tilde{k}^{1/2}(\mathbf{x},\mathbf{y})\,\mathrm{d}\mathbf{y}\,\mathrm{d}\mathbf{x}\\
&\qquad\leq \left(\int_{\mathbb{R}^d}\mathbf{v}(\mathbf{x})^2\int_{\mathbb{R}^d}\frac{k_a^2(\mathbf{x},\mathbf{y})}{\tilde{k}(\mathbf{x},\mathbf{y})}\,\mathrm{d}\mathbf{y}\,\mathrm{d}\mathbf{x}\right)^{1/2}\\
&\qquad\quad\times\left(\int_{\mathbb{R}^d}\int_{\mathbb{R}^d}\tilde{k}(\mathbf{x},\mathbf{y})(\mathcal{D}(\mathbf{u})(\mathbf{x},\mathbf{y}))^2\,\mathrm{d}\mathbf{y}\,\mathrm{d}\mathbf{x}\right)^{1/2}\\
&\qquad\leq A\,\|\mathbf{v}\|_{L^2(\mathbb{R}^d)}\,\|\mathbf{u}\|_{S(\mathbb{R}^d;k)}.
\end{aligned}$$

Combining the above estimates we have that

$$|\mathcal{F}^k(\mathbf{u},\mathbf{v})| \leq \frac{1}{2}[\mathbf{u},\mathbf{u}]_{S(\mathbb{R}^d;k)}[\mathbf{v},\mathbf{v}]_{S(\mathbb{R}^d;k)} + A\,\|\mathbf{u}\|_{S(\mathbb{R}^d;k)}\,\|\mathbf{v}\|_{L^2(\mathbb{R}^d)} \leq C\,\|\mathbf{u}\|_{S(\mathbb{R}^d;k)}\,\|\mathbf{v}\|_{S(\mathbb{R}^d;k)}\,, \tag{3.9}$$

proving that $\mathcal{F}^k$ is indeed a continuous bilinear form on the space $S(\mathbb{R}^d;k)$. □

Remark 3.3. A discussion on the nature of the "limiting operator" $\mathbb{L} = \lim_{n\to\infty}\mathbb{L}_n$ is in order. First, in the event that the *kernel* $k(\mathbf{x},\mathbf{y})$ *is integrable in the sense that if for every* $\mathbf{x}\in\mathbb{R}^d$, $\int_{\mathbb{R}^d} k_s(\mathbf{x},\mathbf{y})\,\mathrm{d}\mathbf{y} < \infty$ *and the function* $\mathbf{x}\mapsto\int_{\mathbb{R}^d} k_s(\mathbf{x},\mathbf{y})\,\mathrm{d}\mathbf{y}\in L^1_{loc}(\mathbb{R}^d)$, then for any $\mathbf{u}\in S(\mathbb{R}^d;k)$ and for each $n\in\mathbb{N}$, the value $\mathbb{L}_n\mathbf{u}(\mathbf{x})$ is finite for almost all $\mathbf{x}\in\mathbb{R}^d$ and for almost all $\mathbf{x}\in\mathbb{R}^d$ we have

$$\begin{aligned}
\lim_{n\to\infty}\mathbb{L}_n\mathbf{u}(\mathbf{x}) &= \mathbb{L}\mathbf{u}(\mathbf{x})\\
&= \int_{\mathbb{R}^d} k(\mathbf{x},\mathbf{y})\mathcal{D}(\mathbf{u})(\mathbf{x},\mathbf{y})\frac{\mathbf{x}-\mathbf{y}}{|\mathbf{x}-\mathbf{y}|}\,\mathrm{d}\mathbf{y}\\
&= \int_{\mathbb{R}^d} k_s(\mathbf{x},\mathbf{y})\mathcal{D}(\mathbf{u})(\mathbf{x},\mathbf{y})\frac{\mathbf{x}-\mathbf{y}}{|\mathbf{x}-\mathbf{y}|}\,\mathrm{d}\mathbf{y} + \int_{\mathbb{R}^d} k_a(\mathbf{x},\mathbf{y})\mathcal{D}(\mathbf{u})(\mathbf{x},\mathbf{y})\frac{\mathbf{x}-\mathbf{y}}{|\mathbf{x}-\mathbf{y}|}\,\mathrm{d}\mathbf{y}.
\end{aligned}$$

Moreover, the above proposition implies that the sequence $\{\mathbb{L}_n\mathbf{u}\}$ is bounded in the dual space of $S(\mathbb{R}^d;k)$, and converges in the weak-* topology to $\mathcal{F}^k(\mathbf{u},\cdot)$. In this case, since for any $\mathbf{v}\in C_c^\infty(\mathbb{R}^d)$ one can verify using Fubini's theorem that

$$\langle\mathbb{L}\mathbf{u},\mathbf{v}\rangle_{L^2(\mathbb{R}^d)} = \mathcal{F}^k(\mathbf{u},\mathbf{v})\,,$$

so we can identify $\mathcal{F}^k(\mathbf{u},\cdot)$ with the measurable vector field $\mathbb{L}\mathbf{u}$.

More generally, for any kernel satisfying (3.3) and (3.5)-(3.6), and for any $\mathbf{u} \in S(\mathbb{R}^d;k)$, one may replace the L^2 inner product by the duality pairing to define the sequence of functionals $\mathbb{L}_n u$ defined by

$$\begin{aligned}\langle \mathbb{L}_n \mathbf{u}, \mathbf{v}\rangle := \frac{1}{2}\iint_{D_n} k_s(\mathbf{x},\mathbf{y})\mathcal{D}(\mathbf{u})(\mathbf{x},\mathbf{y})\mathcal{D}(\mathbf{v})(\mathbf{x},\mathbf{y})\,\mathrm{d}\mathbf{y}\,\mathrm{d}\mathbf{x}\\ + \iint_{D_n} k_a(\mathbf{x},\mathbf{y})\mathcal{D}(\mathbf{u})(\mathbf{x},\mathbf{y})\left(\mathbf{v}(\mathbf{x})\cdot\frac{\mathbf{x}-\mathbf{y}}{|\mathbf{x}-\mathbf{y}|}\right)\mathrm{d}\mathbf{y}\,\mathrm{d}\mathbf{x}\,,\end{aligned}$$

for $\mathbf{v} \in S(\mathbb{R}^d, k)$. The proposition proved above shows that $\{\mathbb{L}_n\mathbf{u}\}$ is bounded in the dual space of $S(\mathbb{R}^d, k)$ and converges in the weak-* topology to $\mathcal{F}^k(\mathbf{u},\cdot)$. For $\mathbf{u} \in C_c^\infty(\mathbb{R}^n)$, the limiting functional $\mathcal{F}^k(\mathbf{u},\cdot)$ can be identified with the function

$$\mathbb{L}\mathbf{u}(\mathbf{x}) = P.V.\int_{\mathbb{R}^d} k_s(\mathbf{x},\mathbf{y})\mathcal{D}(\mathbf{u})(\mathbf{x},\mathbf{y})\frac{\mathbf{x}-\mathbf{y}}{|\mathbf{x}-\mathbf{y}|}\,\mathrm{d}\mathbf{y} + \int_{\mathbb{R}^d} k_a(\mathbf{x},\mathbf{y})\mathcal{D}(\mathbf{u})(\mathbf{x},\mathbf{y})\frac{\mathbf{x}-\mathbf{y}}{|\mathbf{x}-\mathbf{y}|}\,\mathrm{d}\mathbf{y}.$$

In the event that k is not integrable and not necessarily symmetric, the second term in the above expression corresponds to a term with “lower order derivatives;” see [36] for a detailed discussion.

3.2.2 The Dirichlet Problem for the System of Nonlocal Equations

In this subsection we use the bilinear form we introduced earlier to define what we mean by a variational solution to the Dirichlet problem of the nonlocal system of equations.

Zero Dirichlet Data

Definition 3.4. Assume that k satisfies both (3.3) and (3.5)-(3.6). Let $\Omega \subset \mathbb{R}^d$ be open, bounded. Let $\mathbf{f} \in S^*_\Omega(\mathbb{R}^d;k)$. We say that $\mathbf{u} \in S_\Omega(\mathbb{R}^d;k)$ is a solution of

$$\mathbb{L}\mathbf{u} = \mathbf{f} \quad \text{in } \Omega, \qquad \mathbf{u} = \mathbf{0} \quad \text{on } \complement\Omega, \tag{D_0}$$

if

$$\mathcal{F}^k(\mathbf{u},\varphi) = (\mathbf{f},\varphi)_{L^2(\mathbb{R}^d)} \quad \text{for all } \varphi \in S_\Omega(\mathbb{R}^d;k). \tag{3.10}$$

The main result of this section is the following well-posedness of the Dirichlet problem (D_0).

Theorem 3.5. *Let $\Omega \subset \mathbb{R}^d$ be open, bounded. Let k satisfy (3.3) and (3.5)-(3.6). Assume further that*

i) there exists $C_P \geq 1$ such that for all $\mathbf{u} \in L^2_\Omega(\mathbb{R}^d)$,

$$\|\mathbf{u}\|^2_{L^2(\Omega)} \leq C_P \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} k_s(\mathbf{x},\mathbf{y})(\mathcal{D}(\mathbf{u})(\mathbf{x},\mathbf{y}))^2 \, \mathrm{d}\mathbf{y} \, \mathrm{d}\mathbf{x}, \quad \textit{and} \tag{PK}$$

ii) for every $\varepsilon > 0$, there exists $C_\varepsilon \geq 0$ such that

$$C_\varepsilon = \sup_{\mathbf{x}\in\Omega} \int_{\complement B(\mathbf{x},\varepsilon)} |k_a(\mathbf{x},\mathbf{y})| \, \mathrm{d}\mathbf{y} < \infty, \quad \textit{and} \tag{3.11}$$

iii)

$$\inf_{\mathbf{x}\in\mathbb{R}^d} \liminf_{\varepsilon\to 0^+} \int_{\complement B(\mathbf{x},\varepsilon)} k_a(\mathbf{x},\mathbf{y}) \left(\frac{\mathbf{x}-\mathbf{y}}{|\mathbf{x}-\mathbf{y}|} \otimes \frac{\mathbf{x}-\mathbf{y}}{|\mathbf{x}-\mathbf{y}|} \right) \mathrm{d}\mathbf{y} \geq 0 \tag{3.12}$$

in the sense of quadratic forms.

Then corresponding to any $\mathbf{f} \in S^*_\Omega(\mathbb{R}^d;k)$ there exists a unique solution $\mathbf{u} \in S_\Omega(\mathbb{R}^d;k)$ to (D_0). Moreover, there exists a constant $c > 0$ such that

$$[\mathbf{u},\mathbf{u}]_{S_\Omega(\mathbb{R}^d;k)} \leq c\|\mathbf{f}\|_{S^*_\Omega(\mathbb{R}^d;k)}.$$

Remark 3.6. Condition (PK) in the theorem is called a Poincaré-Korn type inequality. In the theorem it appears as an assumption that restricts the choice of the kernel k. Later, we provide sufficient conditions that guarantee the validity of (PK) for a class of kernels. Conditions (3.11)-(3.12) should be treated as cancellation conditions on the antisymmetric part of the kernel. Indeed, condition (3.11) is an integrability requirement on k_a away from the diagonal which allows us to apply Fubini's theorem and use other properties of the integral. Condition (3.12) on the other hand says that the term in the energy $\mathcal{F}^k[\mathbf{u},\mathbf{u}]$ involving the antisymmetric part $k_a(\mathbf{x},\mathbf{y})$ should not be "too negative." This condition can be relaxed a little bit, but verifying it may be a challenge. A relaxed condition is given in

[36, Remark 3.3]. See also nonlocal variational problems that involve sign changing kernels in a different sense in [58].

Proof of Theorem 3.5. We use the Lax-Milgram theorem to prove the result. Conditions (3.11)-(3.12) will be used to show that $\mathcal{F}^k[\mathbf{u},\mathbf{u}]$ is positive semidefinite, while (PK) implies positive definiteness of the energy. We show step by step that all the assumptions in the Lax-Milgram theorem are satisfied. We begin by noting that as in Proposition 3.2 the conditions (3.3), (3.5)-(3.6) imply that the bilinear form $\mathcal{F}^k$ is a continuous form on $S(\mathbb{R}^d;k)$. Next, we will show that $\mathcal{F}^k$ is coercive on the closed subspace $S_\Omega(\mathbb{R}^d;k)$ of $S(\mathbb{R}^d;k)$. We begin by showing that

$$\mathcal{F}^k(\mathbf{u},\mathbf{u}) \geq \frac{1}{2}[\mathbf{u},\mathbf{u}]_{S(\mathbb{R}^d;k)} \quad \text{for all } \mathbf{u}\in S_\Omega(\mathbb{R}^d;k). \tag{3.13}$$

For any $\varepsilon>0$, and for any $\mathbf{u}\in S_\Omega(\mathbb{R}^d;k)$ we have that

$$\begin{aligned}
&\iint_{|\mathbf{x}-\mathbf{y}|>\varepsilon} \mathcal{D}(\mathbf{u})(\mathbf{x},\mathbf{y})\left(\mathbf{u}(\mathbf{y})\cdot\frac{\mathbf{x}-\mathbf{y}}{|\mathbf{x}-\mathbf{y}|}\right)k_a(\mathbf{x},\mathbf{y})\,\mathrm{d}\mathbf{y}\,\mathrm{d}\mathbf{x}\\
&=\frac{1}{2}\iint_{|\mathbf{x}-\mathbf{y}|>\varepsilon} \mathcal{D}(\mathbf{u})(\mathbf{x},\mathbf{y})\left((\mathbf{u}(\mathbf{x})+\mathbf{u}(\mathbf{y}))\cdot\frac{\mathbf{x}-\mathbf{y}}{|\mathbf{x}-\mathbf{y}|}\right)k_a(\mathbf{x},\mathbf{y})\,\mathrm{d}\mathbf{y}\,\mathrm{d}\mathbf{x}\\
&=\frac{1}{2}\iint_{|\mathbf{x}-\mathbf{y}|>\varepsilon}\left(\left(\mathbf{u}(\mathbf{x})\cdot\frac{\mathbf{x}-\mathbf{y}}{|\mathbf{x}-\mathbf{y}|}\right)^2-\left(\mathbf{u}(\mathbf{y})\cdot\frac{\mathbf{x}-\mathbf{y}}{|\mathbf{x}-\mathbf{y}|}\right)^2\right)k_a(\mathbf{x},\mathbf{y})\,\mathrm{d}\mathbf{y}\,\mathrm{d}\mathbf{x}\,.
\end{aligned}$$

where we have used the anti-symmetry of k_a in the first equality. We use the integrability assumption (3.11) in the last integral to apply Fubini's theorem to obtain that

$$\begin{aligned}
&\iint_{|\mathbf{x}-\mathbf{y}|>\varepsilon} \mathcal{D}(\mathbf{u})(\mathbf{x},\mathbf{y})\left(\mathbf{u}(\mathbf{y})\cdot\frac{\mathbf{x}-\mathbf{y}}{|\mathbf{x}-\mathbf{y}|}\right)k_a(\mathbf{x},\mathbf{y})\,\mathrm{d}\mathbf{y}\,\mathrm{d}\mathbf{x}\\
&\qquad=\iint_{|\mathbf{x}-\mathbf{y}|>\varepsilon}\left(\mathbf{u}(\mathbf{x})\cdot\frac{\mathbf{x}-\mathbf{y}}{|\mathbf{x}-\mathbf{y}|}\right)^2 k_a(\mathbf{x},\mathbf{y})\,\mathrm{d}\mathbf{y}\,\mathrm{d}\mathbf{x}\\
&\qquad=\int_{\mathbb{R}^d}\mathbf{u}(\mathbf{x})^\intercal\left[\int_{\complement B(\mathbf{x},\varepsilon)} k_a(\mathbf{x},\mathbf{y})\left(\frac{\mathbf{x}-\mathbf{y}}{|\mathbf{x}-\mathbf{y}|}\otimes\frac{\mathbf{x}-\mathbf{y}}{|\mathbf{x}-\mathbf{y}|}\right)\mathrm{d}\mathbf{y}\right]\mathbf{u}(\mathbf{x})\,\mathrm{d}\mathbf{x}\,.
\end{aligned}$$

We then conclude from (3.12) that

$$\begin{aligned}\int_{\mathbb{R}^d}\int_{\mathbb{R}^d} &\mathcal{D}(\mathbf{u})(\mathbf{x},\mathbf{y})\left(\mathbf{u}(\mathbf{y})\cdot\frac{\mathbf{x}-\mathbf{y}}{|\mathbf{x}-\mathbf{y}|}\right)k_a(\mathbf{x},\mathbf{y})\,\mathrm{d}\mathbf{y}\,\mathrm{d}\mathbf{x}\\
&= \lim_{\varepsilon\to 0}\iint_{|\mathbf{x}-\mathbf{y}|>\varepsilon}\mathcal{D}(\mathbf{u})(\mathbf{x},\mathbf{y})\left(\mathbf{u}(\mathbf{y})\cdot\frac{\mathbf{x}-\mathbf{y}}{|\mathbf{x}-\mathbf{y}|}\right)k_a(\mathbf{x},\mathbf{y})\,\mathrm{d}\mathbf{y}\,\mathrm{d}\mathbf{x}\\
&= \int_{\mathbb{R}^d}\mathbf{u}(\mathbf{x})^{\intercal}\left[\int_{\complement B(\mathbf{x},\varepsilon)}k_a(\mathbf{x},\mathbf{y})\left(\frac{\mathbf{x}-\mathbf{y}}{|\mathbf{x}-\mathbf{y}|}\otimes\frac{\mathbf{x}-\mathbf{y}}{|\mathbf{x}-\mathbf{y}|}\right)\mathrm{d}\mathbf{y}\right]\mathbf{u}(\mathbf{x})\,\mathrm{d}\mathbf{x}\geq 0.\end{aligned}$$

Hence, from the definition of the bilinear form we have that

$$\mathcal{F}^k(\mathbf{u},\mathbf{u})\geq\frac{1}{2}\int_{\mathbb{R}^d}\int_{\mathbb{R}^d}(\mathcal{D}(\mathbf{u})(\mathbf{x},\mathbf{y}))^2k_s(\mathbf{x},\mathbf{y})\,\mathrm{d}\mathbf{y}\,\mathrm{d}\mathbf{x} \tag{3.14}$$

and (3.13) is proved. Therefore, by the Poincaré-Korn inequality (PK) and (3.13),

$$\mathcal{F}^k(\mathbf{u},\mathbf{u})\geq\frac{1}{4C_P}\|\mathbf{u}\|^2_{L^2(\Omega)}+\frac{1}{4}[\mathbf{u},\mathbf{u}]_{S(\mathbb{R}^d;k)}\geq\frac{1}{4C_P}\|\mathbf{u}\|^2_{S(\mathbb{R}^d;k)}.$$

Finally, the Lax-Milgram Lemma implies that there exists a unique $\mathbf{u}\in S_\Omega(\mathbb{R}^d;k)$ such that

$$\mathcal{F}^k(\mathbf{u},\varphi)=\langle\mathbf{f},\varphi\rangle\qquad\text{for all }\varphi\in S_\Omega(\mathbb{R}^d;k).$$

$\square$

A Sufficient Condition for the Poincaré-Korn Inequality

We emphasize that the Poincaré-Korn inequality (PK) is an assumption in Theorem 3.5. Here we present a theorem that gives sufficient conditions on the kernel k for the validity of thee Poincaré-Korn inequality. Given $\mathcal{I}$ an open subset of the unit sphere $\mathbb{S}^{d-1}$ such that the Hausdorff measure $\mathcal{H}^{d-1}(\mathcal{I})>0$, we call the set $\Lambda=\left\{\mathbf{h}\in\mathbb{R}^d\setminus\{\mathbf{0}\}\,:\,\frac{\mathbf{h}}{|\mathbf{h}|}\in\mathcal{I}\cup(-\mathcal{I})\right\}$ a double cone with apex at the origin. Note that for any such cone $\Lambda=-\Lambda$. Denote $\Lambda_{B_r}:=\Lambda\cap B_r(\mathbf{0})$, a part of a double cone with apex at the origin in $B_r(\mathbf{0})$.

Proposition 3.7. *Let $\Omega\subset\mathbb{R}^d$ be open, bounded. Assume that there is an even, nonnegative function $\rho\in L^1(\mathbb{R}^d)$ satisfying the following conditions:*

i) There exists $\delta_0>0$ and a cone Λ with apex at the origin such that $\Lambda_{B_{\delta_0}}\subset\{\rho>0\}$.

ii) There exists $c_0 > 0$ such that for all $\mathbf{u} \in S_\Omega(\mathbb{R}^d; k)$

$$\begin{aligned}\int_{\mathbb{R}^d}\int_{\mathbb{R}^d} k_s(\mathbf{x},\mathbf{y}) \left((\mathbf{u}(\mathbf{x}) - \mathbf{u}(\mathbf{y})) \cdot \frac{\mathbf{x}-\mathbf{y}}{|\mathbf{x}-\mathbf{y}|} \right)^2 \mathrm{d}\mathbf{y}\, \mathrm{d}\mathbf{x} \\ \geq c_0 \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} \rho(\mathbf{x}-\mathbf{y}) \left((\mathbf{u}(\mathbf{x}) - \mathbf{u}(\mathbf{y})) \cdot \frac{\mathbf{x}-\mathbf{y}}{|\mathbf{x}-\mathbf{y}|} \right)^2 \mathrm{d}\mathbf{y}\, \mathrm{d}\mathbf{x}\,.\end{aligned} \tag{3.15}$$

Then, there exists $C_P = C_P(\Omega, c_0, \rho, \delta_0, \Lambda) > 0$ such that for all $\mathbf{u} \in S_\Omega(\mathbb{R}^d; k)$

$$\|\mathbf{u}\|_{L^2(\mathbb{R}^d)} \leq C_P \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} k_s(\mathbf{x},\mathbf{y}) \left((\mathbf{u}(\mathbf{x}) - \mathbf{u}(\mathbf{y})) \cdot \frac{\mathbf{x}-\mathbf{y}}{|\mathbf{x}-\mathbf{y}|} \right)^2 \mathrm{d}\mathbf{y}\, \mathrm{d}\mathbf{x}. \tag{3.16}$$

In the next subsection we give a number of examples of kernels that satisfy the hypothesis of the proposition. We need the following lemma which generalizes [89, Proposition 1.2] and [29, Lemma 2.2] that give a characterization of infinitesimal rigid motions.

Lemma 3.8. *Suppose that $\mathbf{u} : \mathbb{R}^d \to \mathbb{R}^d$ is a measurable vector-valued function such that for some fixed $\delta_0 > 0$, and $\mathscr{J} \subset \mathbb{S}^{d-1}$, an open subset of the unit sphere $\mathbb{S}^{d-1}$ with $\mathcal{H}^{d-1}(\mathscr{J}) > 0$, it holds that for almost every $\mathbf{x} \in \mathbb{R}^d$,*

$$(\mathbf{u}(\mathbf{x}) - \mathbf{u}(\mathbf{y})) \cdot (\mathbf{x}-\mathbf{y}) = 0 \quad \textit{for almost every } \mathbf{y} \in \left\{ \mathbf{y} \in B_{\delta_0}(\mathbf{x}) : \frac{\mathbf{x}-\mathbf{y}}{|\mathbf{x}-\mathbf{y}|} \in \mathscr{J} \right\}.$$

Then $\mathbf{u}$ is an affine map of the form $\mathbf{u}(\mathbf{x}) = \mathbb{A}\mathbf{x} + \mathbf{b}$, almost everywhere, where $\mathbb{A}$ is a constant skew symmetric matrix ($\mathbb{A}^\intercal = -\mathbb{A}$), and $\mathbf{b} \in \mathbb{R}^d$.

Proof. For $\mathbf{x} \in \mathbb{R}^d$, define $\Gamma(\mathbf{x}) = \left\{ \mathbf{y} \in B_{\delta_0}(\mathbf{x}) : \frac{\mathbf{x}-\mathbf{y}}{|\mathbf{x}-\mathbf{y}|} \in \mathscr{J} \right\}$. For each $\mathbf{x}$, the set $\Gamma(\mathbf{x})$ is an open set and is in fact the intersection of the ball $B_{\delta_0}(\mathbf{x})$ with the cone whose directions lie in $\mathscr{J}$ with apex at $\mathbf{x}$. Let $\{\mathbf{e}_i\}_{i=1}^d$ denote a basis for $\mathbb{R}^d$ contained in $\mathscr{J}$; such a basis exists because $\mathscr{J}$ is open and nontrivial. Then since the Lebesgue integral is continuous with respect to translations, there exists a $\delta_1 > 0$ such that the function

$$\delta \mapsto \int_{\mathbb{R}^d} \left(\prod_{i=1}^d \chi_{\Gamma(\delta \mathbf{e}_i)} \right) \chi_{\Gamma(0)}\, \mathrm{d}\mathbf{x}$$

is positive. For $\mathbf{x} \in \mathbb{R}^d$, set $\tilde{\Gamma}(\mathbf{x}) := \left(\bigcap_{i=1}^d \Gamma(x + \delta_1 \mathbf{e}_i)\right) \cap \Gamma(\mathbf{x})$. By the discussion above, $\tilde{\Gamma}(\mathbf{x})$ is an open set of positive measure.

Now fix $\mathbf{x}_0 \in \mathbb{R}^d$ (up to a set of measure zero). Then by the main assumption in the lemma, for almost every $\mathbf{x} \in \tilde{\Gamma}(\mathbf{x}_0)$ we have

$$((\mathbf{u}(\mathbf{x}) - \mathbf{u}(\mathbf{x}_0)) \cdot (\mathbf{x} - \mathbf{x}_0)) = 0 \tag{3.17}$$

and

$$((\mathbf{u}(\mathbf{x}) - \mathbf{u}(\mathbf{x}_0 + \delta_1 \mathbf{e}_i)) \cdot (\mathbf{x} - \mathbf{x}_0 - \delta_1 \mathbf{e}_i)) = 0\,. \tag{3.18}$$

Therefore, adding and subtracting $\mathbf{u}(\mathbf{x}_0)$ in the first argument of (3.18) and $\mathbf{x}_0$ in the second and using (3.17) we see that

$$(\mathbf{u}(\mathbf{x}) - \mathbf{u}(\mathbf{x}_0)) \cdot \delta_1 \mathbf{e}_i = -(\mathbf{u}(\mathbf{x}_0 + \delta_1 \mathbf{e}_i) - \mathbf{u}(\mathbf{x}_0)) \cdot (\mathbf{x} - \mathbf{x}_0).$$

So,

$$\mathbf{u}(\mathbf{x}) \cdot \mathbf{e}_i = \frac{1}{\delta_1} \left((\mathbf{u}(\mathbf{x}_0 + \delta_1 \mathbf{e}_i) - \mathbf{u}(\mathbf{x}_0)) \cdot (\mathbf{x} - \mathbf{x}_0)\right) + \mathbf{u}(\mathbf{x}_0) \cdot \mathbf{e}_i$$

for every $\mathbf{x} \in \tilde{\Gamma}(\mathbf{x}_0)$ and every i, which is clearly a linear map. Then, letting $\mathbb{E} = [\mathbf{e}_i]$ be the matrix of basis vectors, and $\mathbf{u} = (u_1, u_2, \cdots, u_d)$, we have that

$$u_i(\mathbf{x}) = \left(\mathbb{E}^{-1}(\mathbb{E}\mathbf{u})\right)_i = \sum_j e_{ij}^{-1}(\mathbf{e}_j \cdot \mathbf{u}(\mathbf{x}))$$

which, being a sum of of linear maps, is still linear. We conclude that for almost all $\mathbf{x} \in \tilde{\Gamma}(\mathbf{x}_0)$ the vector field $\mathbf{u}$ is of the form $\mathbb{A}(\mathbf{x}_0)\mathbf{x} + \mathbf{b}(\mathbf{x}_0)$, where $\mathbb{A}$ is matrix with constant entries (depending possibly on $\mathbf{x}_0$) and $\mathbf{b}$ is a constant vector (also depending on $\mathbf{x}_0$) in $\mathbb{R}^d$.

Next given any two points in $\mathbb{R}^d$, outside of a set of measure zero, we connect them by finitely many sets of the form $\tilde{\Gamma}(\mathbf{x})$, i.e. for any two points $\mathbf{x}_0$ and $\mathbf{x}_1$ in $\mathbb{R}^d$ there exists a finite subcover of $(\tilde{\Gamma}(\mathbf{x}))_{\mathbf{x} \in \mathbb{R}^d}$, denoted $(\tilde{\Gamma}(\mathbf{x}_k))_{k=1}^N$, such that $\tilde{\Gamma}(\mathbf{x}_k) \cap \tilde{\Gamma}(\mathbf{x}_{k+1}) \neq \emptyset$ and $\mathbf{x}_0 \in \tilde{\Gamma}(\mathbf{x}_0)$, $x_1 \in \tilde{\Gamma}(\mathbf{x}_N)$. This is possible, since the line segment connecting $\mathbf{x}_0$ and $\mathbf{x}_1$ is compact. Therefore the $\mathbf{u}$ given above is the same in neighboring intersecting open sets and so $\mathbf{u} = \mathbb{A}\mathbf{x} + \mathbf{b}$ on $\mathbb{R}^d$ where $\mathbb{A}$, $\mathbf{b}$ are now constants. Again from the main assumption, the matrix $\mathbb{A}$ must be skew symmetric. $\square$

Corollary 3.9. *Let $\Omega \subset \mathbb{R}^d$ be open, bounded. Assume that there is a nonnegative even function $\rho \in L^1(\mathbb{R}^d)$ satisfying the following:*

There exist $\delta_0 > 0$ and a symmetric cone Λ with vertex at the origin such that
$$\Lambda \cap B_{\delta_0}(0) \subset \operatorname{supp} \rho.$$

Suppose that $\mathbf{u} \in L^2(\mathbb{R}^d; \mathbb{R}^d)$ satisfies

$$\int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \rho(\mathbf{x}-\mathbf{y}) \left(\left(\mathbf{u}(\mathbf{x}) - \mathbf{u}(\mathbf{y})\right) \cdot \frac{\mathbf{x}-\mathbf{y}}{|\mathbf{x}-\mathbf{y}|} \right)^2 \mathrm{d}\mathbf{y}\, \mathrm{d}\mathbf{x} = 0.$$

Then $\mathbf{u} = \mathbf{0}$ almost everywhere.

Proof. Since the integrand is nonnegative, we see that for almost every $\mathbf{x} \in \mathbb{R}^d$,

$$(\mathbf{u}(\mathbf{x}) - \mathbf{u}(\mathbf{y})) \cdot (\mathbf{x} - \mathbf{y}) = 0$$

for almost every $\mathbf{y} \in \operatorname{supp} \rho + \mathbf{x} := \{\mathbf{z} : \mathbf{z} - \mathbf{x} \in \operatorname{supp} \rho\}$. By assumption and Lemma 3.8, $\mathbf{u}$ is an affine map. But since $\mathbf{u} \in L^2(\mathbb{R}^d)$, it follows that $\mathbf{u}$ must be the zero vector field. $\square$

Now, we are ready to prove the sufficiency for Poincaré-Korn inequality. The proof follows the argument presented in the proof of [59, Proposition 2], that applies the case when ρ is radial.

Proof of Lemma 3.7. Without loss of generality we assume that ρ has compact support of positive measure. (else replace ρ by $\rho\chi_{B(0,r)}$). Then ρ satisfies (3.3). To prove the lemma, it suffices to show that there exists a constant $C > 0$ such that for all $\mathbf{u} \in L^2_\Omega(\mathbb{R}^d)$,

$$\|\mathbf{u}\|_{L^2} \le C[\mathbf{u}, \mathbf{u}]_{H(\mathbb{R}^d;\rho)}.$$

Suppose to the contrary; that there exists $\{\mathbf{u}_n\} \subset S_\Omega(\mathbb{R}^d; \rho)$ such that $\forall\, n \in \mathbb{N}$ $\|\mathbf{u}_n\|_{L^2(\mathbb{R}^d)} = 1$ and $[\mathbf{u}_n, \mathbf{u}_n]_{S(\mathbb{R}^d;\rho)} \to 0$ as $n \to \infty$. Let $\mathbf{u}$ be the weak $L^2(\mathbb{R}^d)$ limit of $\{\mathbf{u}_n\}$. We first show that $\mathbf{u} = \mathbf{0}$. Note that because of the properties of ρ the operator

$$\mathbb{L}_\rho \mathbf{u}(\mathbf{x}) := \int_{\mathbb{R}^d} \rho(\mathbf{x}-\mathbf{y}) \left(\frac{\mathbf{x}-\mathbf{y}}{|\mathbf{x}-\mathbf{y}|} \otimes \frac{\mathbf{x}-\mathbf{y}}{|\mathbf{x}-\mathbf{y}|} \right) (\mathbf{u}(\mathbf{x}) - \mathbf{u}(\mathbf{y}))\, \mathrm{d}\mathbf{y}$$

is a bounded linear map from $L^2(\mathbb{R}^d;\mathbb{R}^d)$ to $L^2(\mathbb{R}^d;\mathbb{R}^d)$. Let $\varphi \in C_c^\infty(\mathbb{R}^d)$. Then, by the Cauchy-Schwartz inequality,

$$\begin{aligned}
\mathcal{F}^\rho(\mathbf{u}_n, \varphi) &= \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \rho(\mathbf{x}-\mathbf{y}) \left((\mathbf{u}_n(\mathbf{x}) - \mathbf{u}_n(\mathbf{y})) \cdot \frac{\mathbf{x}-\mathbf{y}}{|\mathbf{y}-\mathbf{x}|} \right) \left(\varphi(\mathbf{x}) \cdot \frac{\mathbf{x}-\mathbf{y}}{|\mathbf{y}-\mathbf{x}|} \right) \mathrm{d}\mathbf{y}\, \mathrm{d}\mathbf{x} \\
&\leq \left(\int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \rho(\mathbf{x}-\mathbf{y}) \left((\mathbf{u}_n(\mathbf{x}) - \mathbf{u}_n(\mathbf{y})) \cdot \frac{\mathbf{x}-\mathbf{y}}{|\mathbf{y}-\mathbf{x}|} \right)^2 \mathrm{d}\mathbf{y}\, \mathrm{d}\mathbf{x} \right)^{1/2} \\
&\quad \times \left(\int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \rho(\mathbf{x}-\mathbf{y}) |\varphi(\mathbf{x})|^2 \, \mathrm{d}\mathbf{y}\, \mathrm{d}\mathbf{x} \right)^{1/2} \\
&= \|\rho\|_{L^1(\mathbb{R}^d)} \, [\mathbf{u}_n, \mathbf{u}_n]^{1/2}_{S(\mathbb{R}^d;\rho)} \, \|\varphi\|_{L^2(\mathbb{R}^d)} \\
&\to 0 \text{ as } n \to \infty.
\end{aligned}$$

Now, since ρ is symmetric, $\mathcal{F}^\rho$ is symmetric. Thus,

$$(\mathbb{L}_\rho \mathbf{u}_n, \varphi)_{L^2} = \mathcal{F}^\rho(\mathbf{u}_n, \varphi) = \mathcal{F}^\rho(\varphi, \mathbf{u}_n) = (\mathbf{u}_n, \mathbb{L}_\rho \varphi)_{L^2} \quad \forall\, n \in \mathbb{N}.$$

Since $\mathbb{L}_\rho \varphi \in L^2(\mathbb{R}^d;\mathbb{R}^d)$, it follows that $\mathcal{F}^\rho(\varphi, \mathbf{u}_n) \to \mathcal{F}^\rho(\varphi, \mathbf{u})$ as $n \to \infty$. Therefore, for all $\varphi \in C_c^\infty(\mathbb{R}^d;\mathbb{R}^d)$, $(\mathbb{L}_\rho \varphi, \mathbf{u})_{L^2} = (\mathbb{L}_\rho \mathbf{u}, \varphi)_{L^2} = 0$. Thus $\mathbb{L}_\rho \mathbf{u} = \mathbf{0}$ a.e. As a consequence, since ρ is even and assumption $ii)$

$$\begin{aligned}
(\mathbb{L}\mathbf{u}, \mathbf{u})_{L^2} = \mathcal{F}^\rho(\mathbf{u}, \mathbf{u}) &= \frac{1}{2} [\mathbf{u}, \mathbf{u}]_{S(\mathbb{R}^d;\rho)} \\
&= \frac{1}{2} \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \rho(\mathbf{x}-\mathbf{y}) \left(\big(\mathbf{u}(\mathbf{x}) - \mathbf{u}(\mathbf{y})\big) \cdot \frac{\mathbf{x}-\mathbf{y}}{|\mathbf{x}-\mathbf{y}|} \right)^2 \mathrm{d}\mathbf{y}\, \mathrm{d}\mathbf{x} = 0\,.
\end{aligned}$$

We can now apply Corollary 3.9 to conclude that $\mathbf{u} \equiv \mathbf{0}$ on $\mathbb{R}^d$.

Next we show that in fact, up to a subsequence, $\mathbf{u}_n \to 0$ strongly in L^2, and arrive at our contradiction. To show this it suffices to demonstrate that $\|\mathbf{u}_n\|_{L^2(\Omega)} \to 0$ as $n \to \infty$. Define $\mathbb{K}(\boldsymbol{\xi}) = \rho(\boldsymbol{\xi}) \frac{\boldsymbol{\xi} \otimes \boldsymbol{\xi}}{|\boldsymbol{\xi}|^2}$. Note that $\mathbb{K} \in L^1(\mathbb{R}^d; \mathbb{R}^d \times \mathbb{R}^d)$ since $\rho \in L^1 \cap L^\infty(\mathbb{R}^d)$. Then, define

$\mathbb{K} * \mathbf{u}(\mathbf{x})$ and $\mathbb{B}$ as

$$(\mathbb{K} * \mathbf{u}(\mathbf{x}))_i = \sum_{j=1}^{d} \int_{\mathbb{R}^d} (\mathbb{K}(\mathbf{x}-\mathbf{y}))_{ij} \mathbf{u}(\mathbf{y})_j \, \mathrm{d}\mathbf{y}, \qquad \mathbb{B} = \int_{\mathbb{R}^d} \mathbb{K}(\boldsymbol{\xi}) \, \mathrm{d}\boldsymbol{\xi}.$$

Both quantities converge absolutely and are well-defined. Further, $\mathbb{L}_\rho \mathbf{u}(\mathbf{x}) = \mathbb{K} * \mathbf{u}(\mathbf{x}) - \mathbb{B}\mathbf{u}(\mathbf{x})$. Note that $\mathbb{B}$ is a positive definite constant matrix, which follows from the fact that $\Phi(\mathbf{v}) = \mathbf{v}^\intercal \mathbb{B} \mathbf{v} = \int_{\mathbb{R}^d} \rho(\boldsymbol{\xi}) \left| \frac{\boldsymbol{\xi}}{|\boldsymbol{\xi}|} \cdot \mathbf{v} \right|^2 \mathrm{d}\boldsymbol{\xi}$ is a continuous and positive function on the unit sphere $\mathbb{S}^{d-1}$. From the above estimate, we have that

$$(\mathbb{L}_\rho \mathbf{u}_n, \mathbf{u}_n)_{L^2} \leq \|\rho\|_{L^1(\mathbb{R}^d)} [\mathbf{u}_n, \mathbf{u}_n]_{S(\mathbb{R}^d;\rho)} \|\mathbf{u}_n\|_{L^2(\mathbb{R}^d)} \to 0 \quad \text{ as } n \to \infty.$$

Since $\mathbf{u}_n \rightharpoonup \mathbf{0}$ weakly in $L^2(\mathbb{R}^d)$, by compactness of the convolution operator [13, Corollary 4.28] we have that

$$\mathbb{K} * \mathbf{u}_n(\mathbf{x}) \to \mathbf{0} \text{ strongly in } L^2(\Omega; \mathbb{R}^d).$$

Therefore, if $\mathbb{B} \geq \gamma \mathbb{I}$ in the sense of quadratic forms, we have that

$$\begin{aligned}
\gamma \lim_{n\to\infty} \int_{\mathbb{R}^d} |\mathbf{u}_n|^2 \, \mathrm{d}\mathbf{x} \leq \lim_{n\to\infty} (\mathbb{B}\mathbf{u}_n, \mathbf{u}_n)_{L^2(\mathbb{R}^d)} &= \lim_{n\to\infty} (\mathbb{B}\mathbf{u}_n, \mathbf{u}_n)_{L^2(\mathbb{R}^d)} + \lim_{n\to\infty} (\mathbb{K} * \mathbf{u}_n, \mathbf{u}_n)_{L^2(\Omega)} \\
&= \lim_{n\to\infty} (\mathbb{L}_\rho \mathbf{u}_n, \mathbf{u}_n)_{L^2(\Omega)} = 0.
\end{aligned}$$

That completes the proof. □

Examples of Kernels

There are several examples of kernels that satisfy all the conditions of the theorem; a number of them are discussed in detail in [36] in connection with the solvability of the Dirichlet problem of nonlocal equations. For some of these examples, the verification of (PK) is nontrivial. We list several examples of nontrivial kernels, for which we can verify all the conditions. This shows that the nonlocal Dirichlet problem for the corresponding equations is well-posed. Given $\mathcal{I}$ an open subset of the unit sphere $\mathbb{S}^{d-1}$ such that Hausdorff measure $\mathcal{H}^{d-1}(\Lambda) > 0$, we call the set $\Lambda = \left\{ \mathbf{h} \in \mathbb{R}^d \setminus \{\mathbf{0}\} : \frac{\mathbf{h}}{|\mathbf{h}|} \in \mathcal{I} \cup (-\mathcal{I}) \right\}$ a double cone with apex at the origin. Note that for such cone $\Lambda = -\Lambda$. Denote $\Lambda_{B_r} := \Lambda \cap B_r(\mathbf{0})$, a part of a double cone with apex at the origin in $B_r(\mathbf{0})$.

Example 1: Suppose that $\rho \in L^1(\mathbb{R}^d)$ is nonnegative and even. Define now

$$k(\mathbf{x}, \mathbf{y}) = \rho(\mathbf{x} - \mathbf{y})\chi_{\Lambda_{B_1}}(\mathbf{x} - \mathbf{y})\,.$$

Since K is symmetric, we only need to verify the Poincaré-Korn type inequality (PK). But this is a consequence of Lemma 3.7. See also [59, Proposition 2] for a similar result that is valid for radial kernels. Note that for these types of kernels the space $S(\mathbb{R}^d; k)$ is just $L^2(\mathbb{R}^d; \mathbb{R}^d)$.

Example 2: More generally, if $\mathcal{C} = \left\{\mathbf{h} \in B_1(\mathbf{0}) : \frac{\mathbf{h}}{|\mathbf{h}|} \in \mathcal{J}\right\}$, and $\mathcal{J}$ is an open subset of the unit sphere $\mathbb{S}^{d-1}$ with Hausdorff measure $\mathcal{H}^{d-1}(\mathcal{J}) > 0$, then $k(\mathbf{x}, \mathbf{y}) = \rho(\mathbf{x}-\mathbf{y})\chi_{\mathcal{C}}(\mathbf{x}-\mathbf{y})$ satisfies all the conditions of the theorem. In this case the kernel is not symmetric. However, its symmetric part is $k_s(\mathbf{x}, \mathbf{y}) = \rho(\mathbf{x} - \mathbf{y})\chi_{\mathcal{C}\cup(-\mathcal{C})}(\mathbf{x} - \mathbf{y})$, and the union $\mathcal{C} \cup (-\mathcal{C})$ is now a double cone with apex at the origin. The antisymmetric part of k is given by $k_a(\mathbf{x}, \mathbf{y}) = \frac{1}{2}\left(\rho(\mathbf{x} - \mathbf{y})\chi_{\mathcal{C}}(\mathbf{x} - \mathbf{y}) - \rho(\mathbf{x} - \mathbf{y})\chi_{(-\mathcal{C})}(\mathbf{x} - \mathbf{y})\right)$ and satisfies both conditions (3.11) and (3.12) as can easily be seen.

Before we give other examples let us first prove a lemma that helps us compare function spaces. The result is an improvement of [29, Lemma 2.12], where the same result is shown for radial kernels that are supported on $\Lambda_{B_r} = B_r$.

Lemma 3.10 **(Fractional Korn inequality).** *Let $s \in (0, 1)$ and let $m(\boldsymbol{\xi})$ be an even function defined on $B_r(\mathbf{0})$ with the property that $0 < \alpha_1 \le m(\boldsymbol{\xi}) \le \alpha_2 < \infty$ for some positive constants α_1 and α_2. For a given Λ a double cone with apex at the origin and a given $r > 0$ define the kernel*

$$k_r(\mathbf{x}, \mathbf{y}) = \frac{m(\mathbf{x} - \mathbf{y})}{|\mathbf{x} - \mathbf{y}|^{d+2s}}\chi_{\Lambda_{B_r}}(\mathbf{x} - \mathbf{y}).$$

Then the function space $S(\mathbb{R}^d; k_r)$ is precisely $H^s(\mathbb{R}^d; \mathbb{R}^d)$. Moreover, there exists a function $\beta(r)$ with the property that $\beta(r) \to 0$ as $r \to \infty$, and positive constants C_1, C_2 such that

$$C_1[\mathbf{u}, \mathbf{u}]_{S(\mathbb{R}^d;k_r)} \le |\mathbf{u}|^2_{H^s} \le C_2[\mathbf{u}, \mathbf{u}]_{S(\mathbb{R}^d;k_r)} + C_2\beta(r)\|\mathbf{u}\|^2_{L^2}. \tag{3.19}$$

If Λ_{B_r} is replaced by Λ, then β can be taken to be the zero function. The constants C_1, C_2 and the function β depend on α_i, Λ, d and s.

Proof. We prove the lemma using the Fourier transform. First let us introduce the following modification

$$\tilde{m}(\boldsymbol{\xi}) = \begin{cases} m(\boldsymbol{\xi}) & \boldsymbol{\xi} \in \Lambda_{B_r}\,, \\ \alpha_1 & \boldsymbol{\xi} \in \Lambda \cap \complement B_r(\mathbf{0})\,. \end{cases}$$

Then $\tilde{m}$ is even, and $\tilde{m}(\boldsymbol{\xi}) \geq \alpha_1$ for all $\boldsymbol{\xi} \in \Lambda$. Now, for $\mathbf{u} \in S(\mathbb{R}^d; k_r)$

$$\begin{aligned}
&[\mathbf{u}, \mathbf{u}]_{S(\mathbb{R}^d;k_r)} + \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} \alpha_1 \frac{\left|(\mathbf{u}(\mathbf{y}) - \mathbf{u}(\mathbf{x})) \cdot \frac{(\mathbf{y}-\mathbf{x})}{|\mathbf{y}-\mathbf{x}|}\right|^2}{|\mathbf{y}-\mathbf{x}|^{d+2s}} \chi_{\Lambda\cap\complement B_r(\mathbf{0})}(\mathbf{x}-\mathbf{y})\,\mathrm{d}\mathbf{y}\,\mathrm{d}\mathbf{x} \\
&= \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} \tilde{m}(\mathbf{x}-\mathbf{y}) \frac{\left|(\mathbf{u}(\mathbf{y}) - \mathbf{u}(\mathbf{x})) \cdot \frac{(\mathbf{y}-\mathbf{x})}{|\mathbf{y}-\mathbf{x}|}\right|^2}{|\mathbf{y}-\mathbf{x}|^{d+2s}} \chi_{\Lambda}(\mathbf{x}-\mathbf{y})\,\mathrm{d}\mathbf{y}\,\mathrm{d}\mathbf{x} \\
&= \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} \tilde{m}(\mathbf{x}-\mathbf{y}) \frac{\left|(\mathbf{u}(\mathbf{y}) - \mathbf{u}(\mathbf{x})) \cdot \frac{(\mathbf{y}-\mathbf{x})}{|\mathbf{y}-\mathbf{x}|}\right|^2}{|\mathbf{y}-\mathbf{x}|^{d+2s}} \chi_{\Lambda}(\mathbf{x}-\mathbf{y})\,\mathrm{d}\mathbf{y}\,\mathrm{d}\mathbf{x} \\
&= \int_{\Lambda} \frac{\tilde{m}(\mathbf{h})}{|\mathbf{h}|^{\mathbf{d+2s}}} \|\tau_{\mathbf{h}}\mathbf{u}\|^2_{L^2(\mathbb{R}^d)}\,\mathrm{d}\mathbf{h}
\end{aligned}$$

where $\tau_{\mathbf{h}}\mathbf{u}(\mathbf{x}) = (\mathbf{u}(\mathbf{x}+\mathbf{h}) - \mathbf{u}(\mathbf{x})) \cdot \frac{\mathbf{h}}{|\mathbf{h}|}$. Note that the Fourier transform of $\tau_{\mathbf{h}}\mathbf{u}(\mathbf{x})$ is given by

$$\mathcal{F}(\tau_{\mathbf{h}}\mathbf{u})(\boldsymbol{\xi}) = (e^{\imath 2\pi\boldsymbol{\xi}\cdot\mathbf{h}} - 1)\mathcal{F}(\mathbf{u})(\boldsymbol{\xi}) \cdot \frac{\mathbf{h}}{|\mathbf{h}|}.$$

Using Parseval's identity and after a simple calculation we see that

$$\|\tau_{\mathbf{h}}\mathbf{u}\|^2_{L^2(\mathbb{R}^d)} = 2\int_{\mathbb{R}^d} \left|\mathcal{F}(\mathbf{u})(\boldsymbol{\xi}) \cdot \frac{\mathbf{h}}{|\mathbf{h}|}\right|^2 (1 - \cos(2\pi\boldsymbol{\xi}\cdot\mathbf{h}))\,\mathrm{d}\boldsymbol{\xi}.$$

Plugging the last expression in the above semi-norm and interchanging the integral we get that

$$\begin{aligned}\int_{\Lambda} \frac{\tilde{m}(\mathbf{h})}{|\mathbf{h}|^{d+2s}} \|\tau_{\mathbf{h}}\mathbf{u}\|^2_{L^2(\mathbb{R}^d)} \, \mathrm{d}\mathbf{h} &= 2\int_{\mathbb{R}^d} \left[\int_{\Lambda} \frac{\tilde{m}(\mathbf{h})}{|\mathbf{h}|^{d+2s}} \left| \mathcal{F}(\mathbf{u})(\boldsymbol{\xi}) \cdot \frac{\mathbf{h}}{|\mathbf{h}|} \right|^2 (1-\cos(2\pi\boldsymbol{\xi}\cdot\mathbf{h}))\, \mathrm{d}\mathbf{h} \right] \mathrm{d}\boldsymbol{\xi} \\ &\geq 2\alpha_1 \int_{\mathbb{R}^d} \left[\int_{\Lambda} \frac{(1-\cos(2\pi\boldsymbol{\xi}\cdot\mathbf{h}))}{|\mathbf{h}|^{d+2s}} \left| \mathcal{F}(\mathbf{u})(\boldsymbol{\xi}) \cdot \frac{\mathbf{h}}{|\mathbf{h}|} \right|^2 \mathrm{d}\mathbf{h} \right] \mathrm{d}\boldsymbol{\xi} \\ &= 2\alpha_1 \int_{\mathbb{R}^d} |\boldsymbol{\xi}|^{2s} \left[\int_{\Lambda} \frac{(1-\cos(2\pi\frac{\boldsymbol{\xi}}{|\boldsymbol{\xi}|}\cdot\mathbf{h}))}{|\mathbf{h}|^{d+2s}} \left| \mathcal{F}(\mathbf{u})(\boldsymbol{\xi}) \cdot \frac{\mathbf{h}}{|\mathbf{h}|} \right|^2 \mathrm{d}\mathbf{h} \right] \mathrm{d}\boldsymbol{\xi},\end{aligned}$$

where in the last step we have made a change of variables $\mathbf{h} \mapsto |\boldsymbol{\xi}|\mathbf{h}$ and used the fact that Λ remains invariant under scaling. Notice that the last inequality can be written as

$$\begin{aligned}&[\mathbf{u},\mathbf{u}]_{S(\mathbb{R}^d;k)} + \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} \alpha_1 \frac{\left|(\mathbf{u}(\mathbf{y})-\mathbf{u}(\mathbf{x}))\cdot\frac{(\mathbf{y}-\mathbf{x})}{|\mathbf{y}-\mathbf{x}|}\right|^2}{|\mathbf{y}-\mathbf{x}|^{d+2s}} \chi_{\Lambda\cap\complement B_r(\mathbf{0})}(\mathbf{x}-\mathbf{y})\, \mathrm{d}\mathbf{y}\, \mathrm{d}\mathbf{x} \\ &\qquad \geq 2\alpha_1 \int_{\mathbb{R}^d} |\boldsymbol{\xi}|^{2s} |\mathcal{F}(\mathbf{u})(\boldsymbol{\xi})|^2 \Psi\left(\frac{\mathcal{F}(\mathbf{u})(\boldsymbol{\xi})}{|\mathcal{F}(\mathbf{u})(\boldsymbol{\xi})|}, \frac{\boldsymbol{\xi}}{|\boldsymbol{\xi}|} \right) \mathrm{d}\boldsymbol{\xi},\end{aligned}$$

where the map $\Psi : \mathbb{S}^{d-1}\times\mathbb{S}^{d-1} \to [0,\infty)$ is given by $\Psi(\nu,\eta) = \int_{\Lambda} \frac{(1-\cos(2\pi\nu\cdot\mathbf{h}))}{|\mathbf{h}|^{\mathbf{d+2s}}} \left|\eta\cdot\frac{\mathbf{h}}{|\mathbf{h}|}\right|^2 \mathrm{d}\mathbf{h}$. It is not difficult to see that Ψ is a continuous positive function on the compact set $\mathbb{S}^{d-1}\times\mathbb{S}^{d-1}$, and therefore has a positive minimum, Ψ_{min}. As a consequence we have

$$\begin{aligned}2\alpha_1\Psi_{min}|\mathbf{u}|^2_{H^s} &= 2\alpha_1\Psi_{min} \int_{\mathbb{R}^d} |\boldsymbol{\xi}|^{2s} |\mathcal{F}(\mathbf{u})(\boldsymbol{\xi})|^2 \, \mathrm{d}\boldsymbol{\xi} \\ &\leq [\mathbf{u},\mathbf{u}]_{S(\mathbb{R}^d;k)} + \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} \alpha_1 \frac{\left|(\mathbf{u}(\mathbf{y})-\mathbf{u}(\mathbf{x}))\cdot\frac{(\mathbf{y}-\mathbf{x})}{|\mathbf{y}-\mathbf{x}|}\right|^2}{|\mathbf{y}-\mathbf{x}|^{d+2s}} \chi_{\Lambda\cap\complement B_r(\mathbf{0})}(\mathbf{x}-\mathbf{y})\, \mathrm{d}\mathbf{y}\, \mathrm{d}\mathbf{x}.\end{aligned}$$

Next, we estimate the second term on the right hand side of the above inequality. Again using the Fourier transform we have that

$$\begin{aligned}
&\int_{\mathbb{R}^d}\int_{\mathbb{R}^d}\alpha_1\frac{\left|(\mathbf{u}(\mathbf{y})-\mathbf{u}(\mathbf{x}))\cdot\frac{(\mathbf{y}-\mathbf{x})}{|\mathbf{y}-\mathbf{x}|}\right|^2}{|\mathbf{y}-\mathbf{x}|^{d+2s}}\chi_{\Lambda\cap\complement B_r(\mathbf{0})}(\mathbf{x}-\mathbf{y})\,\mathrm{d}\mathbf{y}\,\mathrm{d}\mathbf{x}\\
&\qquad=2\alpha_1\int_{\mathbb{R}^d}\int_{\Lambda\cap\complement B_r(\mathbf{0})}\frac{(1-\cos(2\pi\boldsymbol{\xi}\cdot\mathbf{h}))}{|\mathbf{h}|^{\mathbf{d+2s}}}\left|\mathcal{F}(\mathbf{u})(\boldsymbol{\xi})\cdot\frac{\mathbf{h}}{|\mathbf{h}|}\right|^2\,\mathrm{d}\mathbf{h}\,\mathrm{d}\boldsymbol{\xi}\\
&\qquad\le 2\alpha_1\beta(r)\int_{\mathbb{R}^d}|\mathcal{F}(\mathbf{u})(\boldsymbol{\xi})|^2\,\mathrm{d}\boldsymbol{\xi},
\end{aligned}$$

where $\beta(r)=\int_{\Lambda\cap\complement B_r(\mathbf{0})}\frac{\mathrm{d}\mathbf{h}}{|\mathbf{h}|^{d+2s}}\to 0$, as $r\to\infty$ and depends only on d, s, and Λ. We conclude that there exists a constant $C>0$ such that for every $\mathbf{u}\in S(\mathbb{R}^d;k)$ we have $|\mathbf{u}|^2_{H^s}\le C[\mathbf{u},\mathbf{u}]_{S(\mathbb{R}^d;k)}+C\beta(r)\|\mathbf{u}\|^2_{L^2}$. The bound

$$[\mathbf{u},\mathbf{u}]_{S(\mathbb{R}^d;k)}\le 2\alpha_2\Psi_{max}|\mathbf{u}|^2_{H^s}$$

can be proved in a similar fashion. □

Let us now continue discussing examples of kernels that may satisfy our well-posedness result.

Example 3: Let k_r be as in Lemma 3.10. Since the kernel is symmetric, to check the applicability of Theorem 3.5 for this kernel, we need to verify only the Poincaré-Korn type inequality. But this follows from Lemma 3.7 by taking $\rho(\boldsymbol{\xi})=|\boldsymbol{\xi}|^2k_r(\boldsymbol{\xi})$, for any $r>0$. By above lemma, the space $S(\mathbb{R}^d;k_r)$ is in fact $H^s(\mathbb{R}^d;\mathbb{R}^d)$.

Example 4: Another nontrivial non-symmetric kernel given in [36] is the following. For $s\in(0,1)$, fix $\alpha\in\left(0,\frac{s}{2}\right)$. Let $\mathcal{J}$ be a double cone with apex at the origin. Given the cone $\mathcal{C}=\left\{\mathbf{h}\in B_1(\mathbf{0}):\frac{\mathbf{h}}{|\mathbf{h}|}\in\mathcal{J}\right\}$, and $\mathcal{J}$ is a nontrivial open subset of the unit sphere $\mathbb{S}^{d-1}$, such that $-\mathcal{J}\neq\mathcal{J}$, let us consider the kernel

$$k(\mathbf{x},\mathbf{y})=\frac{1}{|\mathbf{x}-\mathbf{y}|^{d+2s}}\chi_\Lambda(\mathbf{y}-\mathbf{x})+\frac{1}{|\mathbf{x}-\mathbf{y}|^{d+2\alpha}}\chi_{\mathcal{C}}(\mathbf{y}-\mathbf{x}).$$

Then the symmetric and antisymmetric part of k are given by

$$\begin{aligned}
k_s(\mathbf{x},\mathbf{y}) &= \frac{1}{|\mathbf{x}-\mathbf{y}|^{d+2s}}\chi_\Lambda(\mathbf{y}-\mathbf{x}) + \frac{1}{2}\frac{1}{|\mathbf{x}-\mathbf{y}|^{d+2\alpha}}\chi_{\mathcal{C}\cup(-\mathcal{C})}(\mathbf{y}-\mathbf{x})\\
k_a(\mathbf{x},\mathbf{y}) &= \frac{1}{2}\frac{1}{|\mathbf{x}-\mathbf{y}|^{d+2\alpha}}\chi_{\mathcal{C}}(\mathbf{y}-\mathbf{x}) - \frac{1}{2}\frac{1}{|\mathbf{x}-\mathbf{y}|^{d+2\alpha}}\chi_{(-\mathcal{C})}(\mathbf{y}-\mathbf{x}).
\end{aligned}$$

Conditions (3.3) and (3.11)-(3.12) can be shown as in [36]. Let us show (3.5)-(3.6) with $\tilde{k}(\mathbf{x},\mathbf{y}) = \frac{1}{|\mathbf{x}-\mathbf{y}|^{d+2s}}$. Again, (3.6) is shown in [36] where the constant A_2 depends on $\mathcal{C}$ and $s-2\alpha$, but to show (3.5) we use the fact that $k_s(\mathbf{x},\mathbf{y}) \geq \frac{1}{|\mathbf{x}-\mathbf{y}|^{d+2s}}\chi_\Lambda(\mathbf{y}-\mathbf{x})$. Indeed, using Lemma 3.10 and the remark following it, there exist constants $c_1, c_2 > 0$ such that

$$\begin{aligned}
\int_{\mathbb{R}^d}\int_{\mathbb{R}^d} \tilde{k}(\mathbf{x},\mathbf{y}) \left|(\mathbf{u}(\mathbf{y})-\mathbf{u}(\mathbf{x}))\cdot\frac{\mathbf{y}-\mathbf{x}}{|\mathbf{y}-\mathbf{x}|}\right|^2 \mathrm{d}\mathbf{x}\,\mathrm{d}\mathbf{y} &\leq c_1|\mathbf{u}|^2_{H^s}\\
&\leq c_2\int_{\mathbb{R}^d}\int_{\mathbb{R}^d}\frac{\chi_\Lambda(\mathbf{y}-\mathbf{x})}{|\mathbf{x}-\mathbf{y}|^{d+2s}}|\mathcal{D}(\mathbf{u})(\mathbf{x},\mathbf{y})|^2\,\mathrm{d}\mathbf{x}\,\mathrm{d}\mathbf{y}\\
&\leq 2c_2[\mathbf{u},\mathbf{u}]_{S(\mathbb{R}^d;k)}\,.
\end{aligned}$$

The Poincaré-Korn inequality (PK) now follows from the standard Fractional Poincaré inequality, because the function space $S(\mathbb{R}^d;k)$ coincides with $H^s(\mathbb{R}^d;\mathbb{R}^d)$, and because by Lemma 3.10

$$|\mathbf{u}|^2_{H^s} \leq C[\mathbf{u},\mathbf{u}]_{S(\mathbb{R}^d;k)}.$$

Variants of the Dirichlet Problem

As indicated earlier in the proof of Theorem 3.5, conditions (3.11)-(3.12) on the kernel k are used to show the positive semi-definiteness of the bilinear form on $S(\mathbb{R}^d;k)$. There are however kernels for which either these conditions are not true or difficult to verify. For this class of kernels, well-posedness of the Dirichlet problem corresponding to the addition of a positive multiple of the identity operator can be obtained.

Proposition 3.11. *Let $\Omega \subset \mathbb{R}^d$ be open, bounded. Let k satisfy (3.3) and (3.5)-(3.6). Assume also that (PK) holds. Then there exists $\beta_0 > 0$ such that for any $\beta > \beta_0$ and any*

$\mathbf{f} \in S^*_\Omega(\mathbb{R}^d; k)$, *there exists a unique solution* $\mathbf{u} \in S_\Omega(\mathbb{R}^d; k)$ *to*

$$\begin{cases} \mathbb{L}\mathbf{u} + \beta\mathbf{u} = \mathbf{f} & \text{in } \Omega\,, \\ \mathbf{u} = \mathbf{0} & \text{on } \complement\Omega\,. \end{cases} \tag{3.20}$$

Moreover, there exists a constant $c > 0$ *independent of* $\mathbf{f}$ *such that*

$$[\mathbf{u}, \mathbf{u}]_{S_\Omega(\mathbb{R}^d;k)} \le c\|\mathbf{f}\|_{S^*_\Omega(\mathbb{R}^d;k)}.$$

Proof. The proof follows from standard arguments once Gårding-type estimates are established. To that end, we show that there is a constant $\gamma = \gamma(A_1, A_2) > 0$ such that

$$\mathcal{F}^k(\mathbf{u}, \mathbf{u}) \ge \frac{1}{4}\|\mathbf{u}\|^2_{S(\mathbb{R}^d;k)} - \gamma\|\mathbf{u}\|^2_{L^2(\mathbb{R}^d)} \quad \text{for all } \mathbf{u} \in S(\mathbb{R}^d; k). \tag{3.21}$$

To prove this, let $\mathbf{u} \in S(\mathbb{R}^d; k)$. From (3.5)-(3.6) and by Young's inequality,

$$\begin{aligned} \mathcal{F}^k(\mathbf{u}, \mathbf{u}) &\ge \frac{1}{2}\int_{\mathbb{R}^d}\int_{\mathbb{R}^d} k_s(\mathbf{x}, \mathbf{y})(\mathcal{D}(\mathbf{u})(\mathbf{x}, \mathbf{y}))^2 \,\mathrm{d}\mathbf{y}\,\mathrm{d}\mathbf{x} \\ &\qquad - \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} k_a(\mathbf{x}, \mathbf{y})|\mathcal{D}(\mathbf{u})(\mathbf{x}, \mathbf{y})| \left|\mathbf{u}(\mathbf{x}) \cdot \frac{\mathbf{x} - \mathbf{y}}{|\mathbf{x} - \mathbf{y}|}\right| \,\mathrm{d}\mathbf{y}\,\mathrm{d}\mathbf{x} \\ &\ge \frac{1}{2}[\mathbf{u}, \mathbf{u}]_{S(\mathbb{R}^d;k)} - \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} |\mathcal{D}(\mathbf{u})(\mathbf{x}, \mathbf{y})|\tilde{k}^{1/2}(\mathbf{x}, \mathbf{y})|\mathbf{u}(\mathbf{x})|k_a(\mathbf{x}, \mathbf{y})\tilde{k}^{-1/2}(\mathbf{x}, \mathbf{y}) \,\mathrm{d}\mathbf{y}\,\mathrm{d}\mathbf{x} \\ &\ge \frac{1}{2}[\mathbf{u}, \mathbf{u}]_{S(\mathbb{R}^d;k)} \\ &\qquad - \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} \left(\varepsilon|\mathcal{D}(\mathbf{u})(\mathbf{x}, \mathbf{y})|^2\tilde{k}(\mathbf{x}, \mathbf{y}) + \frac{1}{4\varepsilon}|\mathbf{u}(\mathbf{x})|^2 k_a^2(\mathbf{x}, \mathbf{y})\tilde{k}^{-1}(\mathbf{x}, \mathbf{y})\right) \,\mathrm{d}\mathbf{y}\,\mathrm{d}\mathbf{x} \\ &\ge \frac{1}{4}[\mathbf{u}, \mathbf{u}]_{S(\mathbb{R}^d;k)} - C(\varepsilon)\|\mathbf{u}\|^2_{L^2(\mathbb{R}^d)} \\ &\ge \frac{1}{4}\|\mathbf{u}\|^2_{S(\mathbb{R}^d;k)} - \gamma\|\mathbf{u}\|^2_{L^2(\mathbb{R}^d)}, \end{aligned}$$

if ε is chosen sufficiently small such that $A_1\varepsilon < 1/4$ and then $\gamma = \gamma(A_1, A_2)$ chosen sufficiently large. □

We next discuss an example of a nontrivial kernel that satisfies all the conditions of the proposition. The example is taken from [36, 39, 77] and discussed in detail there. For two given positive numbers $0 < \alpha_1 \leq \alpha_2 < 2$, let $\alpha : \mathbb{R}^d \to [\alpha_1, \alpha_2]$ be a continuous function, with its modulus of continuity $\omega[\alpha]$ satisfying $\int_0^1 \frac{(\omega[\alpha](r)\ln(r))^2}{r^{1+\alpha_2}} \mathrm{d}r < \infty$. We introduce the non-symmetric kernel

$$k(\mathbf{x}, \mathbf{y}) = \frac{b(\mathbf{x})}{|\mathbf{x} - \mathbf{y}|^{d+\alpha(\mathbf{x})}},$$

where $b(\mathbf{x})$ is a continuous function bounded from below and above by positive numbers and satisfying the inequality $|b(\mathbf{x}) - b(\mathbf{y})| \leq c|\alpha(\mathbf{x}) - \alpha(\mathbf{y})|$ for some $c > 0$ provided $|\mathbf{x} - \mathbf{y}| < 1$. To see if Proposition 3.11 applies to this kernel, we need to verify (3.3), (3.5)-(3.6) and (PK). It has been shown in [77] that this kernel satisfies (3.3) and (3.5)-(3.6), with $\tilde{k}$ taken to be the symmetric part k_s of k. What remains is the show the Poincaré-Korn inequality (PK) holds for k. But this follows from Lemma 3.7 and the fact that $k_s(\mathbf{x}, \mathbf{y}) \geq \frac{b_{min}}{|\mathbf{x}-\mathbf{y}|^{d+\alpha_1}}$ when $|\mathbf{x} - \mathbf{y}| < 1$.

We would like to mention that in [36] for the modified kernel $k'(\mathbf{x}, \mathbf{y}) = \chi_{B_R(\mathbf{0})}(\mathbf{y} - \mathbf{x})k(\mathbf{x}, \mathbf{y})$, for $1 \ll R$, the Dirichlet problem for scalar equations is shown to be well-posed even for $\beta = 0$, see [36, Theorem 4.4]. This was possible using the Fredholm Alternative theorem via the application of the weak maximum principle that is used to prove uniqueness of the solution to the Dirichlet problem with zero right hand side. Following the argument in [36], one can write a Fredholm Alternative theorem for the Dirichlet problem (D_0) of the system of nonlocal equations. However, since we are dealing with system of equations a maximum principle is not applicable and we are unable to show uniqueness of the solution of the linear system of equations (D_0). The uniqueness of the zero solution (D_0) corresponding to $\mathbf{f} = \mathbf{0}$ under the assumption of Proposition 3.11 or even the stronger assumption on k given in [36, Theorem 4.4] remains an open problem.

We end this section by noting that well-posedness of the Dirichlet problem with nonzero complementary data can also be proved. To that end, again following the set up in [36], let us introduce the function space

$$V(D; k) = \left\{ \mathbf{v} : \mathbb{R}^d \to \mathbb{R}^d \ : \ \begin{array}{c} \mathbf{v}|_D \in L^2(D; \mathbb{R}^d)\,, \\ (\mathbf{v}(\mathbf{x}) - \mathbf{v}(\mathbf{y})) \cdot \frac{(\mathbf{x}-\mathbf{y})}{|\mathbf{x}-\mathbf{y}|} k_s^{1/2}(\mathbf{x}, \mathbf{y}) \in L^2(D \times \mathbb{R}^d) \end{array} \right\}.$$

The mapping $[\mathbf{u},\mathbf{v}]_{V(D;k)}$ given by

$$[\mathbf{u},\mathbf{v}]_{V(D;k)} := \int_D\int_{\mathbb{R}^d} k_s(\mathbf{x},\mathbf{y})\left(\big(\mathbf{u}(\mathbf{x})-\mathbf{u}(\mathbf{y})\big)\cdot\frac{\mathbf{x}-\mathbf{y}}{|\mathbf{x}-\mathbf{y}|}\right)\left(\big(\mathbf{v}(\mathbf{x})-\mathbf{v}(\mathbf{y})\big)\cdot\frac{\mathbf{x}-\mathbf{y}}{|\mathbf{x}-\mathbf{y}|}\right)\mathrm{d}\mathbf{y}\,\mathrm{d}\mathbf{x}$$

defines a bilinear form on $V(D;k)$. In the event that $D=\mathbb{R}^d$, it is clear that $V(\mathbb{R}^d;k) = S(\mathbb{R}^d;k)$, where $S(\mathbb{R}^d;k)$ is defined in the introduction. For a given $\mathbf{g}\in V(\Omega;k)$, we say $\mathbf{u}\in V(\Omega;k)$ is called a solution of

$$\mathbb{L}\mathbf{u}=\mathbf{f}\quad\text{in }\Omega,\qquad \mathbf{u}=\mathbf{g}\quad\text{on }\complement\Omega, \tag{D}$$

if $\mathbf{u}-\mathbf{g}\in S_\Omega(\mathbb{R}^d;k)$ and (3.10) holds.

We now state the well-posedness of the Dirichlet problem. We omit the proof here as it can be done following the argument in [36].

Theorem 3.12. *Let $\Omega\subset\mathbb{R}^d$ be open, bounded. Let k be a kernel that satisfies (3.3), (3.11)-(3.12), and (PK). Assume further that there exists a $\tilde{k}$ such that for all $\mathbf{u}\in V(\Omega;k)$*

$$\begin{aligned}&\int_\Omega\int_{\mathbb{R}^d}\left(\big(\mathbf{u}(\mathbf{x})-\mathbf{u}(\mathbf{y})\big)\cdot\frac{\mathbf{x}-\mathbf{y}}{|\mathbf{x}-\mathbf{y}|}\right)^2\tilde{k}(\mathbf{x},\mathbf{y})\,\mathrm{d}\mathbf{y}\,\mathrm{d}\mathbf{x}\\&\quad\le A_1\int_\Omega\int_{\mathbb{R}^d}\left(\big(\mathbf{u}(\mathbf{x})-\mathbf{u}(\mathbf{y})\big)\cdot\frac{\mathbf{x}-\mathbf{y}}{|\mathbf{x}-\mathbf{y}|}\right)^2 k_s(\mathbf{x},\mathbf{y})\,\mathrm{d}\mathbf{y}\,\mathrm{d}\mathbf{x}\end{aligned}\tag{3.22}$$

and such that (3.6) holds for this $\tilde{k}$. Then (D) has a unique solution $\mathbf{u}\in V(\Omega;k)$, with

$$[\mathbf{u},\mathbf{u}]_{V(\Omega;k)}\le C\left(\|f\|^2_{S^*_\Omega(\mathbb{R}^d;k)}+[g,g]_{V(\Omega;k)}\right), \tag{3.23}$$

where $C=C(C_P,A_1,A_2)>0$.

Remark 3.13. Condition (3.22) obviously holds if one chooses $\tilde{k}(\mathbf{x},\mathbf{y}) = k_s(\mathbf{x},\mathbf{y})$. The integration allows for more flexibility here, see [36] for examples. Note that Theorem 3.12 opens up an interesting question concerning data on $\complement\Omega$. The result requires $\mathbf{g}\in V(\Omega;k)$, i.e., the data is given in all of $\mathbb{R}^d$. This condition is similar to the condition $g\in H^1(\Omega)$ when searching for a solution v solving some partial differential equation of second order in Ω with $v-g\in H^1_0(\Omega)$. From point of view of applications it is desirable to prescribe g only on the complement $\complement\Omega$ and to have some extension theorem. Such results are nowadays standard for classical Sobolev function spaces. They put into relation the trace space $H^{\frac{1}{2}}(\Omega)$ with $H^1(\Omega)$. For spaces with fractional order of derivative, a similar relation has been addressed in [46].

3.3 Interior Regularity of Solutions

3.3.1 Setup and Main Results

We now turn to the question of regularity of solutions. We want to answer the following question: if the data $\mathbf{f}$ are in $L^p(\Omega;\mathbb{R}^d)$, what is the optimal space for the weak solution $\mathbf{u}$ of the Dirichlet problem of the system of nonlocal equations (D_0)? From the existence result proved in the previous section, if $\mathbf{f} \in S^*_\Omega(\mathbb{R}^d;k)$ then $\mathbf{u} \in S_\Omega(\mathbb{R}^d;k)$, which is the largest space to which the solution can belong. This space is, in general, not the optimal space. Moreover, for general kernels k there is no good characterization of the space or other finer subspaces in which the solution may live. With this in mind, in this section we give a partial result concerning regularity of solutions. The result applies to systems of equations with leading operator $\mathbb{L}$ defined using an even function comparable with the fractional kernel. To be precise, let $s \in (0,1)$ and $m(\boldsymbol{\xi})$ be an even function with the property that $0 < \alpha_1 \leq m(\boldsymbol{\xi}) \leq \alpha_2 < \infty$ for some positive constants α_1 and α_2. For a given Λ, a double cone with apex at the origin, and $0 < r \leq \infty$ we consider translation-invariant kernels that may be supported on Λ:

$$k_r(\mathbf{x}-\mathbf{y}) = \frac{m(\mathbf{x}-\mathbf{y})}{|\mathbf{x}-\mathbf{y}|^{d+2s}}\chi_{\Lambda_{B_r}}(\mathbf{x}-\mathbf{y}). \tag{3.24}$$

For kernels of this form we have shown in Lemma 3.10 that $S(\mathbb{R}^d;k) = H^s(\mathbb{R}^d;\mathbb{R}^d)$. Noting that $H_\Omega(\mathbb{R}^d;k) = L^2_\Omega(\mathbb{R}^d;\mathbb{R}^d) \cap S(\mathbb{R}^d;k)$, we have that $S_\Omega(\mathbb{R}^d;k) = L^2_\Omega(\mathbb{R}^d;\mathbb{R}^d) \cap H^s(\mathbb{R}^d;\mathbb{R}^d)$. We denote the latter set by $H^s_\Omega(\mathbb{R}^d;\mathbb{R}^d)$. We also denote

$$H^{2s}_{loc}(\Omega;\mathbb{R}^d) = \{\mathbf{u} \in L^2(\Omega;\mathbb{R}^d) : \eta\mathbf{u} \in H^{2s}(\mathbb{R}^d;\mathbb{R}^d), \quad \forall \eta \in C^\infty_c(\Omega)\}.$$

We also need the following potential spaces. Denote the space of tempered distributions by $\mathcal{S}(\mathbb{R}^d;\mathbb{R}^d)$. If $\mathbf{u} \in \mathcal{S}'(\mathbb{R}^d;\mathbb{R}^d)$, then

$$\mathcal{L}^{s,p}(\mathbb{R}^d) := \{\mathbf{u} \in \mathcal{S}'(\mathbb{R}^d;\mathbb{R}^d) : \mathcal{F}^{-1}[(1+|\boldsymbol{\xi}|^2)^{s/2}\mathcal{F}(\mathbf{u})] \in L^p(\mathbb{R}^d;\mathbb{R}^d)\},$$

where $1 < p < \infty$ and $s \geq 0$. The space is equivalent to the space

$$\{\mathbf{u} \in L^p(\mathbb{R}^d;\mathbb{R}^d) : (-\Delta)^{s/2}\mathbf{u} \in L^p(\mathbb{R}^d;\mathbb{R}^d)\},$$

where $(-\Delta)^{s/2}$ is the standard fractional Laplacian operator applied componentwise. We also denote

$$\mathcal{L}^{s,p}_{loc}(\Omega;\mathbb{R}^d) := \{\mathbf{u} \in L^p(\Omega;\mathbb{R}^d) : \eta\mathbf{u} \in \mathcal{L}^{s,p}(\mathbb{R}^d;\mathbb{R}^d), \quad \forall \eta \in C_c^\infty(\Omega)\}.$$

When $p = 2$, $\mathcal{L}^{s,2}(\mathbb{R}^d) = H^s(\mathbb{R}^d;\mathbb{R}^d)$.

The following theorem contains one of the results of this section.

Theorem 3.14. *Assume that k_r has the form (3.24). Let $\Omega \subset \mathbb{R}^d$ be a bounded open set, $\mathbf{f} \in L^2_\Omega(\mathbb{R}^d;\mathbb{R}^d)$ and $\mathbf{u} \in S_\Omega(\mathbb{R}^d;k)$ be the unique weak solution to the system*

$$\begin{cases} \mathbb{L}\mathbf{u} = \mathbf{f} & \text{on } \Omega \\ \mathbf{u} = 0 & \text{on } \complement\Omega. \end{cases} \tag{3.25}$$

Then $\mathbf{u} \in H^{2s}_{loc}(\Omega;\mathbb{R}^d)$. Moreover, for any $\eta \in C_c^\infty(\Omega)$, there exists a constant $C > 0$ depending on η such that

$$|\eta\mathbf{u}|_{H^{2s}} \leq C\|\mathbf{f}\|_{L^2}.$$

Our second regularity result corresponds to the case when $\mathbf{f} \in L^p_\Omega(\mathbb{R}^d;\mathbb{R}^d)$ for $p \geq 2$. For this result, we only study $\mathbb{L}$ corresponding to $m = 1$ and $\Lambda = \mathbb{R}^d$. That is, k is the standard fractional kernel given by $k(\mathbf{x},\mathbf{y}) = \kappa_{d,s}|\mathbf{x}-\mathbf{y}|^{-d-2s}$, where $\kappa_{d,s}$ is the normalizing constant defined in (2.2). To separate this special operator from generic ones, we introduce the notation $(-\mathring{\boldsymbol{\Delta}})^s$ to denote the matrix operator. That is,

$$(-\mathring{\boldsymbol{\Delta}})^s \mathbf{u}(\mathbf{x}) = \kappa_{d,s} \int_{\mathbb{R}^d} \left(\frac{\mathbf{x}-\mathbf{y}}{|\mathbf{x}-\mathbf{y}|} \otimes \frac{\mathbf{x}-\mathbf{y}}{|\mathbf{x}-\mathbf{y}|} \right) \frac{\mathbf{u}(\mathbf{x}) - \mathbf{u}(\mathbf{y})}{|\mathbf{x}-\mathbf{y}|^{d+2s}} \, \mathrm{d}\mathbf{y} \, .$$

See Chapter 2 for the limit cases $s \nearrow 1$ or $s \searrow 0$.

Theorem 3.15. *Let $p \in [2, \infty)$, $\mathbf{f} \in L^p_\Omega(\mathbb{R}^d)$ and $\mathbf{u} \in S_\Omega(\mathbb{R}^d; k)$ be the unique weak solution to the Dirichlet problem (3.25) with $\mathbb{L}$ replaced by $(-\mathring{\boldsymbol{\Delta}})^s$. Then $\mathbf{u} \in \mathcal{L}^{2s,p}_{loc}(\Omega)$ provided*

a) $2 \leq p \leq 2^{*_s}$, where $2^{*_s} := \frac{2d}{d-2s}$;

or

b) $p > 2^{*_s}$ and $\mathbf{u} \in L^p(\Omega; \mathbb{R}^d)$.

As we describe in the introduction, to prove Theorem 3.14 and Theorem 3.15 we follow an argument used in [10], where a similar but more general result is proved for the Dirichlet problem for fractional Laplacian equation when the right hand side comes from L^p for any $1 < p < \infty$. The argument relies on an optimal regularity result for weak solutions of the same system posed on the entire space. Multiplying the weak solution of the Dirichlet problem by a cutoff function, the product becomes a weak solution of a system of equations posed on $\mathbb{R}^d$ with a perturbed right hand side. The task is then to show that the perturbed force term lives in the same space as the original right hand side function. In implementing the strategy of [10] to our case, although the cutoff function argument remains the same, we have to demonstrate the optimal regularity result for weak solutions of the strongly coupled system in $\mathbb{R}^d$. For strong solutions of nonlocal equations defined on $\mathbb{R}^d$, optimal regularity is obtained in [25].

We should mention that, for the scalar case, the result [10, Theorem 1.4] does not require $\mathbf{u}$ be in $L^p(\Omega)$ for large p as we do in Theorem 3.15. Roughly speaking, they prove that a solution to the Dirichlet problem of the fractional Laplacian with right hand side in L^p must also be in L^p, see [10, Lemma 2.5]. A similar Calderon-Zygmund type estimate is also proved in [50, Theorem 16]. Unfortunately we are unable to extend their proof to the vector-valued case because the argument in [10] relies on a monotonicity property of an associated semigroup and in the case of [50] it uses a Moser-type argument where a function that is a power of the solution is used as a test function. which we cannot do for systems.

3.3.2 Interior H^{2s} Regularity for the Dirichlet Problem of the System of Equations

Now we turn to the main point. We recall that for a given $\mathbf{f} \in L^2(\mathbb{R}^d;\mathbb{R}^d)$, we say that $\mathbf{u} \in S(\mathbb{R}^d;k)$ is a weak solution to $\mathbb{L}\mathbf{u} = \mathbf{f}$ in $\mathbb{R}^d$ if

$$\begin{aligned}\langle \mathbb{L}\mathbf{u}, \psi \rangle &:= \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} \mathcal{D}(\mathbf{u})(\mathbf{x},\mathbf{y})\mathcal{D}(\psi)(\mathbf{x},\mathbf{y})k(\mathbf{x}-\mathbf{y})\,\mathrm{d}\mathbf{x}\,\mathrm{d}\mathbf{y} \\ &= \int_{\mathbb{R}^d} \mathbf{f}(\mathbf{x})\cdot\psi(\mathbf{x})\,\mathrm{d}\mathbf{x}, \quad \forall \psi \in C_c^\infty(\mathbb{R}^d;\mathbb{R}^d).\end{aligned} \tag{3.26}$$

The following lemma gives an optimal regularity result for weak solutions of the system.

Lemma 3.16. *Assume that k_r has the form (3.24). Suppose that $\mathbf{f} \in L^2(\mathbb{R}^d;\mathbb{R}^d)$. Let $\mathbf{u} \in H^s(\mathbb{R}^d;\mathbb{R}^d)$ be a weak solution to the system of nonlocal equations*

$$\mathbb{L}\mathbf{u} = \mathbf{f}, \quad \textit{in } \mathbb{R}^d.$$

Then $\mathbf{u} \in H^{2s}(\mathbb{R}^d;\mathbb{R}^d)$, and $|\mathbf{u}|_{H^{2s}} \le c\,(\|\mathbf{f}\|_{L^2} + \|\mathbf{u}\|_{L^2})$ for some constant c depending only on r, s, d, and Λ.

Remark 3.17. Note that [18] establishes very similar regularity results.

Proof. Let $\psi \in C_c^\infty(\mathbb{R}^d;\mathbb{R}^d)$. Then iterating the integral in (3.26) and changing variables we have that

$$\int_{\mathbb{R}^d} k_r(\mathbf{h}) \int_{\mathbb{R}^d} \left((\mathbf{u}(\mathbf{x}+\mathbf{h}) - \mathbf{u}(\mathbf{x}))\cdot\frac{\mathbf{h}}{|\mathbf{h}|}\right)\left((\psi(\mathbf{x}+\mathbf{h}) - \psi(\mathbf{x}))\cdot\frac{\mathbf{h}}{|\mathbf{h}|}\right)\mathrm{d}\mathbf{x}\,\mathrm{d}\mathbf{h} = \langle \mathbf{f}, \psi \rangle_{L^2(\mathbb{R}^d)}\,.$$

We apply the Fourier transform and Plancherel theorem to rewrite the above integral in the frequency space as

$$\int_{\mathbb{R}^d} (\mathbb{M}_r(\boldsymbol{\xi})\hat{\mathbf{u}}(\boldsymbol{\xi}))\cdot\hat{\psi}(\boldsymbol{\xi})\,\mathrm{d}\boldsymbol{\xi} = \int_{\mathbb{R}^d} \hat{\mathbf{f}}(\boldsymbol{\xi})\cdot\hat{\psi}(\boldsymbol{\xi})\,\mathrm{d}\boldsymbol{\xi},$$

where $\mathbb{M}_r(\boldsymbol{\xi})$ is the matrix of Fourier symbols given by

$$\mathbb{M}_r(\boldsymbol{\xi}) = \int_{\mathbb{R}^d} k_r(\mathbf{h})(e^{\imath 2\pi \boldsymbol{\xi}\cdot\mathbf{h}} - 1)^2 \frac{\mathbf{h}}{|\mathbf{h}|} \otimes \frac{\mathbf{h}}{|\mathbf{h}|}\, \mathrm{d}\mathbf{h}.$$

We now use the density of the Fourier transform of $C_c^\infty(\mathbb{R}^d;\mathbb{R}^d)$ in $L^2(\mathbb{R}^d;\mathbb{R}^d)$ to conclude that

$$\mathbb{M}_r(\boldsymbol{\xi})\hat{\mathbf{u}}(\boldsymbol{\xi}) = \hat{\mathbf{f}}(\boldsymbol{\xi}), \quad \text{almost everywhere in } \mathbb{R}^d. \tag{3.27}$$

Let us write $k_r = k - \tilde{k}_r$ where $k(\mathbf{z}) = \frac{m(\mathbf{z})}{|\mathbf{z}|^{d+2s}}\chi_\Lambda(\mathbf{z})$. Notice that $\tilde{k}_r$ is supported outside of the ball B_r. If we denote the matrix of symbols by $\mathbb{M}$ and $\tilde{\mathbb{M}}_r$, we have that

$$\mathbb{M}(\boldsymbol{\xi})\hat{\mathbf{u}}(\boldsymbol{\xi}) = \tilde{\mathbb{M}}_r(\boldsymbol{\xi})\hat{\mathbf{u}}(\boldsymbol{\xi}) + \hat{\mathbf{f}}(\boldsymbol{\xi}), \quad \text{almost everywhere in } \mathbb{R}^d. \tag{3.28}$$

To estimate the relevant norms of $\mathbf{u}$, let us first estimate the eigenvalues of the matrix $\mathbb{M}(\boldsymbol{\xi})$. To that end, for any $\eta \in \mathbb{S}^{d-1}$, noting the form of k we have that

$$\begin{aligned}(\mathbb{M}(\boldsymbol{\xi})\eta)\cdot\eta = \int_{\mathbb{R}^d} k(\mathbf{h})(e^{\imath 2\pi \boldsymbol{\xi}\cdot\mathbf{h}} - 1)^2 \left|\eta\cdot\frac{\mathbf{h}}{|\mathbf{h}|}\right|^2 \mathrm{d}\mathbf{h} &\geq 2\alpha_1 \int_\Lambda \frac{(1-\cos(2\pi\boldsymbol{\xi}\cdot\mathbf{h}))}{|\mathbf{h}|^{d+2s}} \left|\eta\cdot\frac{\mathbf{h}}{|\mathbf{h}|}\right|^2 \mathrm{d}\mathbf{h} \\ &\geq 2\alpha_1\Psi_{min}|\boldsymbol{\xi}|^{2s},\end{aligned}$$

where the last inequality is from the proof of Lemma 3.10. As a consequence the eigenvalues of the matrix function $|\boldsymbol{\xi}|^{-2s}\mathbb{M}(\boldsymbol{\xi})$ are uniformly bounded from below by a positive number. We also note that since $\mathbb{M}(\boldsymbol{\xi})$ is symmetric and positive definite for each $\boldsymbol{\xi}$, the eigenvalues of the square of $\mathbb{M}(\boldsymbol{\xi})$ are precisely the squares of the eigenvalues of $\mathbb{M}(\boldsymbol{\xi})$. It then follows that for any vector $\mathbf{w}$

$$|\mathbb{M}(\boldsymbol{\xi})\mathbf{w}|^2 = \mathbb{M}(\boldsymbol{\xi})\mathbf{w}\cdot(\mathbb{M}(\boldsymbol{\xi})\mathbf{w}) = \mathbb{M}(\boldsymbol{\xi})\mathbb{M}(\boldsymbol{\xi})\mathbf{w}\cdot\mathbf{w} \geq |\mathbf{w}|^2 \min_\beta\{\beta(\boldsymbol{\xi})^2\},$$

where the minimum is taken over the eigenvalues $\beta(\boldsymbol{\xi})$ of $\mathbb{M}(\boldsymbol{\xi})$. We conclude that there exists a positive number α_0 that depends only on α_1, s, Λ, such that for all vectors $\boldsymbol{\xi}$ and $\mathbf{w}$ in $\mathbb{R}^d$ we have that

$$|\mathbb{M}(\boldsymbol{\xi})\mathbf{w}|^2 \geq \alpha_0(|\boldsymbol{\xi}|^{2s}|\mathbf{w}|)^2.$$

We now easily see from (3.27) that

$$\|(-\Delta)^s \mathbf{u}\|_{L^2(\mathbb{R}^d)} = \||\boldsymbol{\xi}|^{2s}\hat{\mathbf{u}}\|_{L^2} \le \alpha_0^{-1/2}\|\mathbb{M}(\boldsymbol{\xi})\hat{\mathbf{u}}\|_{L^2} = \alpha_0^{-1/2}\left(\|\hat{\mathbf{f}}\|_{L^2(\mathbb{R}^d)} + +\|\tilde{\mathbb{M}}_r(\boldsymbol{\xi})\hat{\mathbf{u}}(\boldsymbol{\xi})\|_{L^2}^2\right)$$
$$\le \alpha_0^{-1/2}\left(\|\hat{\mathbf{f}}\|_{L^2(\mathbb{R}^d)} + \beta(r)\|\hat{\mathbf{u}}\|_{L^2}^2\right),$$

where in the last inequality $\beta(r)$ is from Lemma 3.10. Thus, since we already know that $\mathbf{u} \in L^2(\mathbb{R}^d;\mathbb{R}^d)$, we get that $\mathbf{u} \in \mathcal{L}^{2s,2}$. □

Proof of Theorem 3.14. Let $\Omega_1 \Subset \Omega_2 \Subset \Omega$. Let $\eta \in C_c^\infty(\Omega_2)$ be a real-valued function such that

$$\eta(\mathbf{x}) \equiv 1,\ \mathbf{x} \in \Omega_1, \quad \eta(\mathbf{x}) \in [0,1],\, \mathbf{x} \in \Omega_2 \setminus \Omega_1, \quad \text{and}\, \eta(\mathbf{x}) = 0,\ \mathbf{x} \in \mathbb{R}^d \setminus \Omega_2.$$

Let $\mathbf{f} \in L^2_\Omega(\mathbb{R}^d;\mathbb{R}^d)$ and let $\mathbf{u} \in H_\Omega(\mathbb{R}^d;k)$ be the unique weak solution to the Dirichlet problem (3.25). Notice that because of the form of k, $\mathbf{u}$ is in fact in $H^s_\Omega(\mathbb{R}^d;\mathbb{R}^d)$. Now, it is clear that the function $\eta\mathbf{u} \in H^s_\Omega(\mathbb{R}^d;\mathbb{R}^d)$. Using the identity

$$\mathcal{D}(\mathbf{u}\eta)(\mathbf{y},\mathbf{x}) = (\eta(\mathbf{x}) - \eta(\mathbf{y}))\mathbf{u}(\mathbf{x}) \cdot \frac{\mathbf{y}-\mathbf{x}}{|\mathbf{y}-\mathbf{x}|} + \eta(\mathbf{x})\mathcal{D}(\mathbf{u})(\mathbf{x},\mathbf{y}) + (\eta(\mathbf{y}) - \eta(\mathbf{x}))\mathcal{D}(\mathbf{u})(\mathbf{x},\mathbf{y})$$

we see that for every $\mathbf{v} \in C_c^\infty(\mathbb{R}^d;\mathbb{R}^d)$,

$$\mathcal{F}^k(\eta\mathbf{u},\mathbf{v}) - \mathcal{F}^k(\mathbf{u},\eta\mathbf{v}) = ([\mathbb{L}\eta]\mathbf{u},\mathbf{v})_{L^2(\mathbb{R}^d)} - (I_s(\mathbf{u},\eta),\mathbf{v})_{L^2(\mathbb{R}^d)}. \tag{3.29}$$

where for almost all $\mathbf{x} \in \mathbb{R}^d$ the vector valued function is

$$I_s(\mathbf{u},\eta)(\mathbf{x}) = \int_{\mathbb{R}^d} k(\mathbf{y}-\mathbf{x})(\eta(\mathbf{x}) - \eta(\mathbf{y}))\mathcal{D}(\mathbf{u})(\mathbf{x},\mathbf{y})\frac{\mathbf{x}-\mathbf{y}}{|\mathbf{x}-\mathbf{y}|}\,\mathrm{d}\mathbf{y},$$

which is finite via Hölder's inequality. In the above and hereafter we suppress the dependence of k on r. The matrix valued function $\mathbb{L}\eta(\mathbf{x})$ is given by

$$\mathbb{L}\eta(\mathbf{x}) = \int_{\mathbb{R}^d} k(\mathbf{x}-\mathbf{y})(\eta(\mathbf{x}) - \eta(\mathbf{y}))\left(\frac{\mathbf{x}-\mathbf{y}}{|\mathbf{x}-\mathbf{y}|} \otimes \frac{\mathbf{x}-\mathbf{y}}{|\mathbf{x}-\mathbf{y}|}\right)\mathrm{d}\mathbf{y}.$$

Let us justify the L^2 inner products on the right hand side of (3.29). To that end, we introduce the vector field

$$\mathbf{g} := (\mathbb{L}\eta)\mathbf{u} - I_s(\mathbf{u}, \eta),$$

and show that $\mathbf{g} \in L^2(\mathbb{R}^d; \mathbb{R}^d)$. In fact, we also show that there exists a constant $C > 0$ independent of $\mathbf{u}$ (but depends on η) such that

$$\|\mathbf{g}\|_{L^2(\mathbb{R}^d)} \leq C \|\mathbf{u}\|_{H^s}. \tag{3.30}$$

The rest of the argument is similar to that given in [10] adjusted to the system case. We include it here fore clarity and completeness. We begin by noting that $\mathbb{L}\eta$ is uniformly bounded in $\mathbb{R}^d$. Indeed, using the fact that $\eta \in C_c(\mathbb{R}^d)$ and k is even, we can easily show that $\|\mathbb{L}\eta\|_\infty \leq C(\|D^2\eta\|_{L^\infty} + \|\eta\|_{L^\infty}.)$ As a consequence of this and the Poincaré-Korn inequality, since $\mathbf{u} \in H^s_\Omega(\mathbb{R}^d, \mathbb{R}^d)$ we have

$$\|(\mathbb{L}\eta)\mathbf{u}\|^2_{L^2(\mathbb{R}^d)} \leq \|\mathbb{L}\eta\|^2_{L^\infty(\mathbb{R}^d)} \|\mathbf{u}\|^2_{L^2(\mathbb{R}^d)} \leq C|\mathbf{u}|^2_{H^s}. \tag{3.31}$$

To bound the L^2 norm of $I_s(\mathbf{u}, \eta)(\mathbf{x})$, we begin by breaking the region of integration as

$$\begin{aligned}
I_s(\mathbf{u}, \eta)(\mathbf{x}) &= \int_\Omega k(\mathbf{y} - \mathbf{x})(\eta(\mathbf{x}) - \eta(\mathbf{y}))\mathcal{D}(\mathbf{u})(\mathbf{x}, \mathbf{y})\frac{\mathbf{x} - \mathbf{y}}{|\mathbf{x} - \mathbf{y}|}\, \mathrm{d}\mathbf{y} \\
&\quad + \int_{\mathbb{R}^d \setminus \Omega} k(\mathbf{y} - \mathbf{x})(\eta(\mathbf{x}) - \eta(\mathbf{y}))\mathcal{D}(\mathbf{u})(\mathbf{x}, \mathbf{y})\frac{\mathbf{x} - \mathbf{y}}{|\mathbf{x} - \mathbf{y}|}\, \mathrm{d}\mathbf{y} \\
&= \int_\Omega k(\mathbf{y} - \mathbf{x})(\eta(\mathbf{x}) - \eta(\mathbf{y}))\mathcal{D}(\mathbf{u})(\mathbf{x}, \mathbf{y})\frac{\mathbf{x} - \mathbf{y}}{|\mathbf{x} - \mathbf{y}|}\, \mathrm{d}\mathbf{y} \\
&\quad + \eta(\mathbf{x}) \int_{\mathbb{R}^d \setminus \Omega} k(\mathbf{y} - \mathbf{x})\mathcal{D}(\mathbf{u})(\mathbf{x}, \mathbf{y})\frac{\mathbf{x} - \mathbf{y}}{|\mathbf{x} - \mathbf{y}|}\, \mathrm{d}\mathbf{y} \\
&:= I_1(\mathbf{x}) + I_2(\mathbf{x}).
\end{aligned}$$

Let us estimate the first integral $I_1(\mathbf{x})$. Using Cauchy-Schwarz,

$$|I_1(\mathbf{x})| \leq \left(\int_\Omega k(\mathbf{y} - \mathbf{x})|\eta(\mathbf{x}) - \eta(\mathbf{y})|^2\, \mathrm{d}\mathbf{y}\right)^{1/2} \left(\int_\Omega k(\mathbf{y} - \mathbf{x})|\mathcal{D}(\mathbf{u})(\mathbf{x}, \mathbf{y})|^2\, \mathrm{d}\mathbf{y}\right)^{1/2}.$$

Since Ω is bounded, taking $R = \operatorname{diam}(\Omega)$, we see that for any $\mathbf{x} \in \Omega$, $\Omega \subset B_{2R}(\mathbf{x})$. We now use the fact that η is smooth and the kernel is comparable with the fractional kernel to obtain that a constant $C > 0$ depending on η such that for any $\mathbf{x} \in \Omega$

$$\begin{aligned}\int_{\Omega} k(\mathbf{y}-\mathbf{x})|\eta(\mathbf{x}) - \eta(\mathbf{y})|^2 \,\mathrm{d}\mathbf{y} &\le \|\nabla \eta\|_{L^\infty} \int_{B_{2R}(\mathbf{x})} k(\mathbf{y}-\mathbf{x})|\mathbf{y}-\mathbf{x}|^2 \,\mathrm{d}\mathbf{y} \\ &= \|\nabla \eta\|_{L^\infty} \int_{B_{2R}(\mathbf{0})} k(\boldsymbol{\xi})|\boldsymbol{\xi}|^2 \,\mathrm{d}\boldsymbol{\xi} \le C\,.\end{aligned}$$

Since $\eta(\mathbf{x}) = 0$ for $\mathbf{x} \in \complement\Omega$, $\operatorname{supp}(\eta) \subset \Omega_2$, and $\delta = \operatorname{dist}(\Omega_2, \partial\Omega) > 0$,

$$\int_{\Omega} k(\mathbf{y}-\mathbf{x})|\eta(\mathbf{x}) - \eta(\mathbf{y})|^2 \,\mathrm{d}\mathbf{y} = \int_{\Omega_2} k(\mathbf{y}-\mathbf{x})|\eta(\mathbf{y})|^2 \,\mathrm{d}\mathbf{y} \le \|\eta\|_{L^\infty}^2 \int_{\{|\boldsymbol{\xi}|>\delta\}} k(\boldsymbol{\xi}) \,\mathrm{d}\boldsymbol{\xi} \le C.$$

Using the two preceding estimates, we see that there exists a positive constant $C > 0$, that depends on η such that

$$\int_{\mathbb{R}^d} |I_1(\mathbf{x})|^2 \,\mathrm{d}\mathbf{x} \le C \int_{\mathbb{R}^d} \int_{\Omega} k(\mathbf{y}-\mathbf{x})|\mathcal{D}(\mathbf{u})(\mathbf{x},\mathbf{y})|^2 \,\mathrm{d}\mathbf{y}\,\mathrm{d}\mathbf{x} \le C|\mathbf{u}|_{H^s}^2. \tag{3.32}$$

To estimate the L^2 norm of I_2, again using Cauchy-Schwarz we get that

$$|I_2(\mathbf{x})|^2 \le |\eta(\mathbf{x})|^2 \left(\int_{\mathbb{R}^d \setminus \Omega} k(\mathbf{y}-\mathbf{x}) \,\mathrm{d}\mathbf{y} \right) \left(\int_{\mathbb{R}^d \setminus \Omega} k(\mathbf{y}-\mathbf{x})|\mathcal{D}(\mathbf{u})(\mathbf{x},\mathbf{y})|^2 \,\mathrm{d}\mathbf{y} \right).$$

As before, since $\eta(\mathbf{x}) = 0$, and $\operatorname{supp}(\eta) \subset \Omega_2$, and that $\delta = \operatorname{dist}(\Omega_2, \partial\Omega) > 0$, we have that the function

$$\mathbf{x} \mapsto |\eta(\mathbf{x})|^2 \int_{\mathbb{R}^d \setminus \Omega} k(\mathbf{y}-\mathbf{x}) \,\mathrm{d}\mathbf{y}$$

is bounded. Thus,

$$\int_{\mathbb{R}^d} |I_2(\mathbf{x})|^2 \,\mathrm{d}\mathbf{x} = \int_{\Omega_2} |I_2(\mathbf{x})|^2 \,\mathrm{d}\mathbf{x} \le C \int_{\Omega_2} \int_{\complement\Omega} k(\mathbf{y}-\mathbf{x})|\mathcal{D}(\mathbf{u})(\mathbf{x},\mathbf{y})|^2 \,\mathrm{d}\mathbf{y}\,\mathrm{d}\mathbf{x} \le C|\mathbf{u}|_{H^s}^2. \tag{3.33}$$

Therefore, the estimate (3.30) of $\mathbf{g}$ follows from (3.32) and (3.33). We have shown that $\eta\mathbf{u}$ is a weak solution to the equation

$$\mathbb{L}(\eta\mathbf{u}) = \mathbf{F} \quad \text{in } \mathbb{R}^d,$$

where $\mathbf{F} = \eta\mathbf{f} + (\mathbb{L}\eta)\mathbf{u} - I_s(\mathbf{u},\eta) \in L^2(\mathbb{R}^d;\mathbb{R}^d)$. By Lemma 3.16, $\eta\mathbf{u} \in H^{2s,2}(\mathbb{R}^d;\mathbb{R}^d)$. Thus $\mathbf{u} \in H^{2s,2}_{loc}(\Omega)$ and the proof is complete. $\square$

3.3.3 Interior $\mathcal{L}^{2s,p}$ Regularity for $p > 2$

In this section we prove Theorem 3.15. As before we start with an optimal regularity estimate for the system of equations posed on $\mathbb{R}^d$.

Lemma 3.18. *Let $p \in (1,\infty)$. For $\mathbf{f} \in L^p(\mathbb{R}^d;\mathbb{R}^d) \cap L^2(\mathbb{R}^d;\mathbb{R}^d)$, if $\mathbf{u} \in H^s_\Omega(\mathbb{R}^d;\mathbb{R}^d) \cap L^p(\mathbb{R}^d;\mathbb{R}^d)$ is a weak solution of*

$$(-\mathring{\mathbf{\Delta}})^s\mathbf{u} = \mathbf{f} \quad \text{in } \mathbb{R}^d.$$

Then $\mathbf{u} \in \mathcal{L}^{2s,p}(\mathbb{R}^d;\mathbb{R}^d)$, Moreover, there exists a constant $C = C(d,s) > 0$ such that

$$\|(-\mathring{\mathbf{\Delta}})^s\mathbf{u}\|_{L^p} \le C\|\mathbf{f}\|_{L^p}.$$

Proof. We proceed as in the proof of Lemma 3.16 to obtain that in the Fourier space the equation is $\mathbb{M}(\boldsymbol{\xi})\hat{\mathbf{u}}(\boldsymbol{\xi}) = \hat{\mathbf{f}}(\boldsymbol{\xi})$ almost everywhere, where $\mathbb{M}(\boldsymbol{\xi})$ is as given in (3.28) with k replaced by the kernel $\frac{\kappa_{d,s}}{|\mathbf{x}-\mathbf{y}|^{d+2s}}$. For the particular form of the kernel, we can explicitly compute the matrix of symbols; see Chapter 2. $\mathbb{M}(\boldsymbol{\xi})$ is given by

$$\mathbb{M}(\boldsymbol{\xi}) = \big(2\pi|\boldsymbol{\xi}|\big)^{2s}\left(\mathbb{I} + 2s\frac{\boldsymbol{\xi}\otimes\boldsymbol{\xi}}{|\boldsymbol{\xi}|^2}\right),$$

where $\mathbb{I}$ is the $d\times d$ identity matrix. The matrix $\mathbb{I} + 2s\frac{\boldsymbol{\xi}\otimes\boldsymbol{\xi}}{|\boldsymbol{\xi}|^2}$ is invertible for any $\boldsymbol{\xi} \ne \mathbf{0}$, with

$$\left(\mathbb{I} + 2s\frac{\boldsymbol{\xi}\otimes\boldsymbol{\xi}}{|\boldsymbol{\xi}|^2}\right)^{-1} = \left(\mathbb{I} - \frac{2s}{1+2s}\frac{\boldsymbol{\xi}\otimes\boldsymbol{\xi}}{|\boldsymbol{\xi}|^2}\right).$$

Using this formula, we can rewrite $\mathbb{M}(\boldsymbol{\xi})\hat{\mathbf{u}}(\boldsymbol{\xi}) = \hat{\mathbf{f}}(\boldsymbol{\xi})$ as

$$\big(2\pi|\boldsymbol{\xi}|\big)^{2s}\hat{\mathbf{u}}(\boldsymbol{\xi}) = \left(\mathbb{I} - \frac{2s}{1+2s}\frac{\boldsymbol{\xi}\otimes\boldsymbol{\xi}}{|\boldsymbol{\xi}|^2}\right)\hat{\mathbf{f}}(\boldsymbol{\xi})$$

The conclusion of the lemma now follows from the assumption that $\mathbf{u} \in L^p(\mathbb{R}^d;\mathbb{R}^d)$ and for any real numbers a and b, the matrix multiplier $a\mathbb{I} + b\frac{\boldsymbol{\xi}\otimes\boldsymbol{\xi}}{|\boldsymbol{\xi}|^2}$ is an L^p-multiplier for any $1 < p < \infty$. The latter follows immediately from the L^p-boundedness of the Riesz Transforms.

□

We use this regularity theorem to prove the second main result of this section. Let begin by review the standard fractional Sobolev spaces. For $p \in [1,\infty)$, Ω an open subset of $\mathbb{R}^d$ and $s \in (0,1)$, we define

$$W^{s,p}(\Omega;\mathbb{R}^d) := \left\{\mathbf{u} \in L^p(\Omega) \,:\, \int_\Omega\int_\Omega \frac{|\mathbf{u}(\mathbf{x}) - \mathbf{u}(\mathbf{y})|^p}{|\mathbf{x}-\mathbf{y}|^{d+sp}}\,\mathrm{d}\mathbf{x}\,\mathrm{d}\mathbf{y} < \infty\right\}.$$

With norm $\|\mathbf{u}\|^p = \|\mathbf{u}\|_{L^p} + \int_\Omega\int_\Omega \frac{|\mathbf{u}(\mathbf{x})-\mathbf{u}(\mathbf{y})|^p}{|\mathbf{x}-\mathbf{y}|^{d+sp}}\,\mathrm{d}\mathbf{x}\,\mathrm{d}\mathbf{y}$, it is well known that $W^{s,p}(\Omega;\mathbb{R}^d)$ is a Banach space.

If $s > 1$, then let we write $s = m + \sigma$, where m is the largest integer less than s, and define

$$W^{s,p}(\Omega;\mathbb{R}^d) = \{u \in W^{m,p}(\Omega;\mathbb{R}^d) : D^\alpha\mathbf{u} \in W^{\sigma,p}(\Omega;\mathbb{R}^d), \quad |\alpha| = m\}.$$

With the norm $\|\mathbf{u}\|^p = \|\mathbf{u}\|_{W^{m,p}} + \sum_{|\alpha|=m}\|D^\alpha\|_{W^{\sigma,p}}$, the space $W^{s,p}(\Omega;\mathbb{R}^d)$ is known to be a Banach space.

We also need the Sobolev embedding result

$$W^{r,p}(\mathbb{R}^d;\mathbb{R}^d) \hookrightarrow W^{s,q}(\mathbb{R}^d;\mathbb{R}^d), \text{ provided } 0 < s \leq r,\ 1 < p \leq q < \infty, \text{ and } r - \frac{d}{p} = s - \frac{d}{q}$$

If Ω is open and bounded with smooth enough boundary then

$$W^{r,p}(\Omega;\mathbb{R}^d) \hookrightarrow W^{s,q}(\Omega;\mathbb{R}^d), \text{ provided } 0 < s \leq r,\ 1 < p \leq q < \infty, \text{ and } r - \frac{d}{p} \geq s - \frac{d}{q}.$$

Let us recall the relation between the potential spaces and the fractional Sobolev spaces, [86, Chapter 5, Theorem 5]. For our purpose it suffices to recall

$$\mathcal{L}^{s,p}(\mathbb{R}^d;\mathbb{R}^d) \subset W^{s,p}(\mathbb{R}^d;\mathbb{R}^d), \text{ if } p \geq 2.$$

For $p = 2$, the spaces are the same; $H^s(\mathbb{R}^d;\mathbb{R}^d) = \mathcal{L}^{s,2}(\mathbb{R}^d;\mathbb{R}^d) = W^{s,2}(\mathbb{R}^d;\mathbb{R}^d)$.

Proof of Theorem 3.15. Let $\mathbf{f} \in L^p_\Omega(\mathbb{R}^d)$. Since $p \geq 2$, and Ω is bounded, $\mathbf{f} \in L^2_\Omega(\mathbb{R}^d)$. Therefore a unique weak solution $\mathbf{u}$ in $H^s_\Omega(\mathbb{R}^d)$ exists. Let η be the cutoff function constructed in the proof of Theorem 3.14. Then we have that $\eta\mathbf{u} \in H^{2s}(\mathbb{R}^d;\mathbb{R}^d) = W^{2s,2}(\mathbb{R}^d)$. By Sobolev embedding, $\eta\mathbf{u} \in W^{s,2^{*s}}(\mathbb{R}^d;\mathbb{R}^d)$. Now, let ω_1, ω_2 be open sets such that $\omega \Subset \omega_1 \Subset \omega_2 \Subset \Omega$. The cutoff function η was arbitrary, so therefore $\mathbf{u} \in W^{s,2^{*s}}(\omega_2;\mathbb{R}^d)$.

Part a. ($p \leq 2^{*_s}$) Since ω_2 is bounded and $p \leq 2^{*_s}$, again by Sobolev Embedding $\mathbf{u} \in W^{s,p}(\omega_2;\mathbb{R}^d)$. Moreover, since $\mathbf{u} \in H^s_\Omega(\mathbb{R}^d;\mathbb{R}^d)$, we have that $\mathbf{u} \in W^{s;2}(\Omega) \hookrightarrow L^p_\Omega(\mathbb{R}^d)$. In summary, $\mathbf{u} \in H^s(\mathbb{R}^d;\mathbb{R}^d) \cap W^{s,p}(\omega_2;\mathbb{R}^d) \cap L^p_\Omega(\mathbb{R}^d;\mathbb{R}^d)$. Now we proceed as in the proof of Theorem 3.14. With the same reasoning, $\eta\mathbf{u}$ is a weak solution of

$$(-\mathring{\boldsymbol{\Delta}})^s(\eta\mathbf{u}) = \mathbf{F} \quad \text{in } \mathbb{R}^d,$$

where $\mathbf{F} = \eta\mathbf{f} + ((-\mathring{\boldsymbol{\Delta}})^s\eta)\mathbf{u} - I_s(\mathbf{u},\eta)$. Let $\mathbf{g} := ((-\mathring{\boldsymbol{\Delta}})^s\eta)\mathbf{u} - I_s(\mathbf{u},\eta)$. We have shown already that $\mathbf{g} \in L^2(\mathbb{R}^d;\mathbb{R}^d)$. Noting that $\eta\mathbf{u} \in H^s(\mathbb{R}^d;\mathbb{R}^d) \cap L^p_\Omega(\mathbb{R}^d;\mathbb{R}^d)$, we can now apply Lemma 3.18 to conclude that $\eta\mathbf{u} \in \mathcal{L}^{2s,p}(\mathbb{R}^d;\mathbb{R}^d)$ provided we successfully show $\mathbf{g} \in L^p(\mathbb{R}^d;\mathbb{R}^d)$. In fact, we demonstrate that for some constant $C > 0$ independent of $\mathbf{u}$

$$\|\mathbf{g}\|_{L^p(\mathbb{R}^d)} \leq C(\|\mathbf{u}\|_{W^{s,p}(\omega_2;\mathbb{R}^d)} + \|\mathbf{u}\|_{L^p(\Omega;\mathbb{R}^d)}). \tag{3.34}$$

The last estimate follows from a similar argument as in Theorem 3.14. We sketch its proof. More detail can be found in [10]. As before, the matrix function $(-\mathring{\boldsymbol{\Delta}})^s\eta \in L^\infty(\mathbb{R}^d)$. Thus,

$$\begin{aligned}\left\|((-\mathring{\boldsymbol{\Delta}})^s\eta)\mathbf{u}\right\|^p_{L^p(\mathbb{R}^d)} &= \int_{\mathbb{R}^d}\left|\int_{\mathbb{R}^d}\frac{\eta(\mathbf{x})-\eta(y)}{|\mathbf{x}-\mathbf{y}|^{d+2s}}\left(\frac{\mathbf{x}-\mathbf{y}}{|\mathbf{x}-\mathbf{y}|}\otimes\frac{\mathbf{x}-\mathbf{y}}{|\mathbf{x}-\mathbf{y}|}\right)\mathrm{d}\mathbf{y}\,\mathbf{u}(\mathbf{x})\right|^p\mathrm{d}\mathbf{x}\\ &\leq \left\|(-\mathring{\boldsymbol{\Delta}})^s\eta\right\|^p_{L^\infty(\mathbb{R}^d)}\|\mathbf{u}\|^p_{L^p(\Omega)}.\end{aligned}$$

The second term $I_s(\mathbf{u},\eta)$ can also be dealt with in the same way as in the proof of Theorem 3.14. We begin by breaking the integral as

$$I_s(\mathbf{u},\eta)(\mathbf{x}) = \int_{\omega_1}\frac{\eta(\mathbf{x})-\eta(\mathbf{y})}{|\mathbf{x}-\mathbf{y}|^{d+2s}}\mathcal{D}\mathbf{u}(\mathbf{x},\mathbf{y})\,\mathrm{d}\mathbf{y} + \eta(\mathbf{x})\int_{\mathbb{R}^d\setminus\omega_1}\frac{\mathcal{D}(\mathbf{u})(\mathbf{x},\mathbf{y})}{|\mathbf{x}-\mathbf{y}|^{d+2s}}\,\mathrm{d}\mathbf{y} := I_1(\mathbf{x}) + I_2(\mathbf{x}).$$

We estimate $I_1(\mathbf{x})$: Using Holder's inequality with conjugate $p' = p/(p-1)$,

$$
\begin{aligned}
|I_1(\mathbf{x})| &\leq \int_{\omega_1} \frac{|\eta(\mathbf{x}) - \eta(\mathbf{y})|}{|\mathbf{x}-\mathbf{y}|^{d+2s}} \left| \big(\mathbf{u}(\mathbf{x}) - \mathbf{u}(\mathbf{y})\big) \cdot \frac{\mathbf{x}-\mathbf{y}}{|\mathbf{x}-\mathbf{y}|} \right| \mathrm{d}\mathbf{y} \\
&\leq \left(\int_{\omega_1} \frac{|\eta(\mathbf{x}) - \eta(\mathbf{y})|^{p'}}{|\mathbf{x}-\mathbf{y}|^{d+sp'}} \mathrm{d}\mathbf{y} \right)^{1/p'} \left(\int_{\omega_1} \left| \big(\mathbf{u}(\mathbf{x}) - \mathbf{u}(\mathbf{y})\big) \cdot \frac{\mathbf{x}-\mathbf{y}}{|\mathbf{x}-\mathbf{y}|} \right|^p \frac{1}{|\mathbf{x}-\mathbf{y}|^{d+sp}} \mathrm{d}\mathbf{y} \right)^{1/p},
\end{aligned}
$$

from which we get that

$$
\int_{\mathbb{R}^d} |I_1(\mathbf{x})|^p \, \mathrm{d}\mathbf{x} \leq C \left(\|\mathbf{u}\|^p_{W^{s,p}(\omega_2)} + \|\mathbf{u}\|^p_{L^p(\Omega)} \right). \tag{3.35}
$$

Similarly, we also get that

$$
\int_{\mathbb{R}^d} |I_2(\mathbf{x})|^p \, \mathrm{d}\mathbf{x} \leq C \, \|\mathbf{u}\|^p_{L^p(\Omega)}. \tag{3.36}
$$

Therefore, the estimate (3.34) of $\mathbf{g}$ follows from (3.35) and (3.36).

Part b. ($p > 2^{*_s}$) From Part a) we have that $\mathbf{u} \in W^{2s,2^{*_s}}_{loc}(\Omega;\mathbb{R}^d)$, and so $\mathbf{u} \in W^{2s,2^{*_s}}(\omega_2;\mathbb{R}^d)$. By Sobolev Embedding we have that $\mathbf{u} \in W^{s,q_1}(\omega_2;\mathbb{R}^d)$ with $q_1 = \min\left\{p, \frac{N2^{*_s}}{N-s2^{*_s}}\right\} = \min\left\{p, \frac{2N}{N-4s}\right\}$. By assumption $\mathbf{u} \in L^{q_1}(\Omega;\mathbb{R}^d)$. With this information, we can now repeat the argument in Part a) to conclude that $\mathbf{u} \in W^{2s,q_1}_{loc}(\Omega;\mathbb{R}^d)$. Now, if $2 \leq p \leq \frac{2N}{N-4s}$, the proof is completed. Otherwise we iterate the above procedure to obtain $\mathbf{u} \in W^{2s,q_j}_{loc}(\Omega;\mathbb{R}^d)$ with $q_j = \min\left\{p, \frac{2N}{N-js}\right\}$, for all $j \geq 2$. For $p \geq 2^{*_s}$, we can now choose $j \in \mathbb{N}$ such that $2 \leq p \leq \frac{2N}{N-js}$. That completes the proof of the theorem. $\square$

Chapter 4

Analysis of Function Spaces

4.1 A Korn-Type Inequality

4.1.1 Introduction

The main focus of this section is to study the function space of vector fields given by

$$\mathcal{X}_{K,p}(\Omega) := \left\{ \mathbf{v} \in L^p(\Omega;\mathbb{R}^d) \, : \, [\mathbf{v}]^p_{\mathcal{X}_{K,p}(\Omega)} < \infty \right\} ,$$
$$[\mathbf{v}]^p_{\mathcal{X}_{K,p}(\Omega)} := \int_\Omega \int_\Omega K(\mathbf{y}-\mathbf{x}) \left| \frac{(\mathbf{v}(\mathbf{y})-\mathbf{v}(\mathbf{x}))}{|\mathbf{y}-\mathbf{x}|} \cdot \frac{(\mathbf{y}-\mathbf{x})}{|\mathbf{y}-\mathbf{x}|} \right|^p \mathrm{d}\mathbf{y} \, \mathrm{d}\mathbf{x} ,$$

where $\Omega \subset \mathbb{R}^d$ is an open subset and the kernel K is a nonnegative function with appropriate integrability. For a particular class of kernels, our main result states that $\mathcal{X}_{K,p}(\Omega)$ is equivalent to a Sobolev space.

As discussed in Chapter 1, we consider the special class of kernels

$$K(|\boldsymbol{\xi}|) = \frac{1}{|\boldsymbol{\xi}|^{d+ps-p}} , \quad 0 < s < 1 , \quad 1 < p < \infty .$$

We denote the corresponding space by $\mathcal{X}^s_p(\Omega)$. These kernels are associated with the fractional Sobolev spaces $W^{s,p}(\Omega;\mathbb{R}^d)$ via the Gagliardo semi-norm, where $W^{s,p}(\Omega;\mathbb{R}^d)$ is given by

$$W^{s,p}(\Omega;\mathbb{R}^d) := \left\{ \mathbf{v} \in L^p(\Omega;\mathbb{R}^d) : [\mathbf{v}]^p_{W^{s,p}} := \int_\Omega \int_\Omega \frac{|\mathbf{v}(\mathbf{y}) - \mathbf{v}(\mathbf{x})|^p}{|\mathbf{y}-\mathbf{x}|^{d+sp}} \, \mathrm{d}\mathbf{y} \, \mathrm{d}\mathbf{x} < \infty \right\} .$$

The question is now if $\mathcal{X}^s_p(\Omega)$ is the same as $W^{s,p}(\Omega;\mathbb{R}^d)$ for these fractional kernels.

The recent work [57] answers the above question in the affirmative for the special case $p = 2$, and $\Omega = \mathbb{R}^d$ or $\mathbb{R}^d_+$. When $p = 2$, and $\Omega = \mathbb{R}^d$, the question is tractable because both spaces $\mathcal{X}^s_2(\mathbb{R}^d)$, and $W^{s,2}(\mathbb{R}^d;\mathbb{R}^d)$ can be characterized by Fourier symbols which made the comparison of norms more straightforward; see [29]. For functions defined over the half-space $\mathbb{R}^d_+$ and vanishing near the hyperplane $x_d = 0$, one can use an appropriate extension operator to control the semi-norm $[\cdot]_{W^{s,2}(\mathbb{R}^d_+)}$ by the semi-norm $[\cdot]_{\mathcal{X}^s_2(\mathbb{R}^d_+)}$ of vector fields in the dense class $C^1_c(\mathbb{R}^d_+;\mathbb{R}^d)$. Here we extend these results to any $p \in (1,\infty)$, again providing an affirmative answer to the question of the equivalence of these spaces. Let us introduce the function space

$$\mathring{\mathcal{X}}^s_p(\Omega) = \text{Closure of } C^\infty_c(\Omega;\mathbb{R}^d) \text{ in } \mathcal{X}^s_p(\Omega) .$$

Theorem 4.1 (Fractional Korn's inequality). *For any $s \in (0,1)$ and $1 < p < \infty$,*

$$\mathcal{X}^s_p(\mathbb{R}^d) = W^{s,p}(\mathbb{R}^d;\mathbb{R}^d) , \quad \textit{and} \quad \mathring{\mathcal{X}}^s_p(\mathbb{R}^d_+) = W^{s,p}_0(\mathbb{R}^d_+;\mathbb{R}^d) .$$

Moreover, there exists a universal constant $C = C(d,p,s)$ such that for all $\mathbf{f} \in W^{s,p}(\mathbb{R}^d;\mathbb{R}^d)$

$$[\mathbf{f}]_{\mathcal{X}^s_p} \leq [\mathbf{f}]_{W^{s,p}} \leq C[\mathbf{f}]_{\mathcal{X}^s_p} . \tag{4.1}$$

Our proof of Theorem 4.1 makes use of the classical characterization of functions in the fractional spaces in terms of their Poisson integrals. Given a vector function $\mathbf{f}$, its Poisson integral is defined as $\mathbf{u}(\mathbf{x},t) = p_t * \mathbf{f}(\mathbf{x})$, where for each $t > 0$, the function $p_t(\mathbf{x})$ is the standard Poisson kernel. For $s \in (0,1)$, $1 < p < \infty$ it is well-known [86, Proposition 7', Chapter V] that $\mathbf{f} \in W^{s,p}(\mathbb{R}^d;\mathbb{R}^d)$ if and only if

$$\int_0^\infty \left(t^{1-s} \|\partial_t \mathbf{u}(\cdot,t)\|_{L^p} \right)^p \frac{\mathrm{d}t}{t} < \infty .$$

Moreover, the semi-norm $[\mathbf{f}]_{W^{s,p}}$ is equivalent with $\left(\int_0^\infty \left(t^{1-s}\|\partial_t \mathbf{u}(\cdot,t)\|_{L^p}\right)^p \frac{\mathrm{d}t}{t}\right)^{1/p}$. To prove Theorem 4.1, we compare the latter semi-norm with that of $[\mathbf{f}]_{\mathcal{X}_p^s}$. The key idea is the introduction of a "Poisson-type" integral $\mathbf{U}(\mathbf{x},t)$ of a vector field $\mathbf{f}$. We construct $\mathbf{U}$ using a convolution with a modified "Poisson-type" *matrix* kernel whose components are some linear combination of convolutions of components of the vector field $\mathbf{f}$. The structure of the Poisson-type kernel reveals that each component of $\mathbf{U}$ is related with components of the Poisson integral $\mathbf{u}$ via Riesz transforms leading to the norm relations

$$\|\partial_t \mathbf{u}(\cdot,t)\|_{L^p} \leq C(d,p,s)\|\partial_t \mathbf{U}(\cdot,t)\|_{L^p} \text{ for all } t>0\,,$$
$$\int_0^\infty \left(t^{1-s}\|\partial_t \mathbf{U}(\cdot,t)\|_{L^p}\right)^p \frac{dt}{t} \leq C(d,p,s)[\mathbf{f}]^p_{\mathcal{X}_p^s}\,.$$

Combining these inequalities with the characterization of $W^{s,p}(\mathbb{R}^d;\mathbb{R}^d)$ in terms of Poisson integrals we obtain the equivalence of spaces. Interestingly, this approach also leads to a characterization of the whole Besov scale of spaces $\Lambda^s_{p,q}$ in terms of the newly defined Poisson-type integrals.

This first part of the chapter is devoted to proving Theorem 4.1. In the next section we recall the classical Poisson kernel and present some preliminaries. We will also review how the Poisson kernel is used in the characterization of functions belonging to the fractional Sobolev spaces. In Section 4.1.3 we introduce a Poisson-type kernel that is central to our result. Its properties as well the relationship between associated Poisson-type integrals and classical Poisson integral will be established. This relationship will be used to in Section 4.1.4 to prove the main result.

4.1.2 Preliminaries: Poisson Integrals and the Riesz Transforms

We recall the classical Poisson kernel and some of its properties that we will use in this chapter. We begin with the formula

$$p_t(\mathbf{x}) := \frac{2}{\omega_d}\frac{t}{\left(|\mathbf{x}|^2+t^2\right)^{\frac{d+1}{2}}}\,,\quad t>0\,,\quad \mathbf{x}\in\mathbb{R}^d\,,$$

where ω_d is the surface measure of the unit sphere $\mathbb{S}^d \subset \mathbb{R}^{d+1}$. It is easy to check that p_t is an approximation to the identity. Its Fourier transform is given by $\mathcal{F}(p_t)(\boldsymbol{\xi}) = \mathrm{e}^{-2\pi|\boldsymbol{\xi}|t}$ for every $t > 0$, where the Fourier transform operator $\mathcal{F}$ is given by the formula

$$\mathcal{F}(g)(\boldsymbol{\xi}) = \int_{\mathbb{R}^d} e^{-\imath 2\pi \boldsymbol{\xi}\cdot\mathbf{x}} f(\mathbf{x})\,\mathrm{d}\mathbf{x}\,.$$

It then follows from the Fourier transform expression that the Poisson kernel has the semigroup property $p_{t_1} * p_{t_2} = p_{t_1+t_2}$ for every t_1, $t_2 > 0$. Using the notation ∇ for the vector of partial derivative operators $(\nabla_{\mathbf{x}}, \partial_t) = (\partial_1, \partial_2, \dots, \partial_d, \partial_t)$, we have that $\nabla p_t \in L^1(\mathbb{R}^d)$ with the estimates

$$\int_{\mathbb{R}^d} |\partial_t p_t(\mathbf{x})|\,\mathrm{d}\mathbf{x} \leq \frac{c}{t}\,, \qquad \int_{\mathbb{R}^d} |\partial_{x_j} p_t(\mathbf{x})|\,\mathrm{d}\mathbf{x} \leq \frac{c}{t}\,, \qquad t > 0\,, \quad j = 1, \cdots, d\,,$$

for some constant $c > 0$. For any $f \in L^p$, $1 \leq p \leq \infty$, its *Poisson integral* is given by

$$u(\mathbf{x}, t) := p_t * f(\mathbf{x}) = \int_{\mathbb{R}^d} p_t(\mathbf{y}) f(\mathbf{x}-\mathbf{y})\,\mathrm{d}\mathbf{y}\,.$$

The Poisson integral is a C^∞ harmonic function in $\mathbb{R}^{d+1}_+ := \mathbb{R}^d \times (0, \infty)$, with the property that $u(\cdot, t) \to f$ in $L^p(\mathbb{R}^d)$ as $t \to 0$. For a vector field $\mathbf{f}$ its vector-valued Poisson integral $\mathbf{u}(\mathbf{x}, t) = p_t * \mathbf{f}(\mathbf{x})$ will be defined where the convolution is taken component-wise.

The Riesz transforms will be used frequently throughout this work. We recall that for $1 \leq j \leq d$ and $f \in \mathcal{S}(\mathbb{R}^d)$ the class of Schwartz functions, the j^{th} *Riesz transform* is an operator defined as

$$R_j(f)(\mathbf{x}) := \frac{2}{\omega_d}\mathrm{P.V.}\int_{\mathbb{R}^d} \frac{y_j}{|\mathbf{y}|^{d+1}} f(\mathbf{x}-\mathbf{y})\,\mathrm{d}\mathbf{y}\,.$$

For any $f \in \mathcal{S}(\mathbb{R}^d)$ we have $\mathcal{F}(R_j(f))(\boldsymbol{\xi}) = -\imath\frac{\xi_j}{|\boldsymbol{\xi}|}\mathcal{F}(f)(\boldsymbol{\xi})$. From this formula it is immediately clear that the Riesz transforms commute with partial differential operators ∂_{x_j}. We recall also the celebrated result of L^p boundedness (c.f. [86, Chapter III]), namely

$$\|R_j f\|_{L^p(\mathbb{R}^d)} \leq C(p)\,\|f\|_{L^p(\mathbb{R}^d)}\,, \qquad j = 1, \dots, d\,, \quad 1 < p < \infty\,.$$

The Riesz transforms can be used to establish relations between the partial derivatives of functions. Let us display such relations for the Poisson integral of a function now. First note that for $\mathbf{f} \in \mathcal{S}(\mathbb{R}^d;\mathbb{R}^d)$, its Poisson integral $u = p_t * \mathbf{f}$ belongs to $\mathcal{S}(\mathbb{R}^d;\mathbb{R}^d)$. Further, for any $\mathbf{x} \in \mathbb{R}^d$ and $t > 0$ we have

$$\partial_t \mathbf{u}(\mathbf{x},t) = -\sum_{j=1}^{d} R_j(\partial_{x_j}\mathbf{u})(\mathbf{x},t)\,, \qquad \partial_{x_j}\mathbf{u}(\mathbf{x},t) = R_j(\partial_t \mathbf{u})(\mathbf{x},t) \quad \text{for } j = 1,\cdots,d\,. \tag{4.2}$$

We can verify the above identities by taking the Fourier transform in the $\mathbf{x}$ variable as follows.

$$\begin{aligned}
\mathcal{F}_{\mathbf{x}}\big(\partial_t \mathbf{u}(\cdot,t)\big)(\boldsymbol{\xi}) = \partial_t\, \mathrm{e}^{-2\pi|\boldsymbol{\xi}|t}\mathcal{F}(\mathbf{f})(\boldsymbol{\xi}) &= -2\pi|\boldsymbol{\xi}|\mathrm{e}^{-2\pi|\boldsymbol{\xi}|t}\mathcal{F}(\mathbf{f})(\boldsymbol{\xi}) \\
&= -\sum_{j=1}^{d}\left(-\frac{\imath\xi_j}{|\boldsymbol{\xi}|}\right)(2\pi\imath\xi_j)\mathrm{e}^{-2\pi|\boldsymbol{\xi}|t}\mathcal{F}(\mathbf{f})(\boldsymbol{\xi}) \\
&= \mathcal{F}\left(-\sum_{j=1}^{d} R_j\big[\partial_{x_j}\mathbf{u}(\cdot,t)\big]\right)(\boldsymbol{\xi})\,,
\end{aligned}$$

demonstrating the first relation in (4.2). Conversely,

$$\begin{aligned}
\mathcal{F}_{\mathbf{x}}\big(\partial_{x_j}\mathbf{u}(\cdot,t)\big)(\boldsymbol{\xi}) &= (2\pi\imath\xi_j)\mathrm{e}^{-2\pi|\boldsymbol{\xi}|t}\mathcal{F}(\mathbf{f})(\boldsymbol{\xi}) \\
&= \left(-\imath\frac{\xi_j}{|\boldsymbol{\xi}|}\right)(-2\pi|\boldsymbol{\xi}|)\mathrm{e}^{-2\pi|\boldsymbol{\xi}|t}\mathcal{F}(\mathbf{f})(\boldsymbol{\xi}) \\
&= \left(-\imath\frac{\xi_j}{|\boldsymbol{\xi}|}\right)\partial_t\Big(\mathrm{e}^{-2\pi|\boldsymbol{\xi}|t}\mathcal{F}(\mathbf{f})(\boldsymbol{\xi})\Big) = \mathcal{F}_{\mathbf{x}}\Big(R_j\big[\partial_t\mathbf{u}(\cdot,t)\big]\Big)(\boldsymbol{\xi})\,,
\end{aligned}$$

establishing the second identity in (4.2). The pointwise relation in (4.2) and the L^p boundedness of the Riesz transforms implies that for every Schwartz vector field $\mathbf{f} \in \mathcal{S}(\mathbb{R}^d;\mathbb{R}^d)$ and $1 < p < \infty$,

$$\|\partial_t\mathbf{u}(\cdot,t)\|_{L^p(\mathbb{R}^d)} \approx \sum_{j=1}^{d}\big\|\partial_{x_j}\mathbf{u}(\cdot,t)\big\|_{L^p(\mathbb{R}^d)} \tag{4.3}$$

where $\approx$ represents equivalence of norms up to a constant independent of $\mathbf{f}$. Using density of $\mathcal{S}(\mathbb{R}^d;\mathbb{R}^d)$ in $L^p(\mathbb{R}^d;\mathbb{R}^d)$ and the fact that $|\nabla p_t| \in L^1(\mathbb{R}^d)$, (4.3) remains true for all $\mathbf{f} \in L^p(\mathbb{R}^d;\mathbb{R}^d)$.

Poisson integrals can be used to give a characterization of the L^p norm of a function; see [86, Chapter IV] for details. Given a function f we introduce the Littlewood-Paley g-function of f in terms of its Poisson integral u as

$$g(f)(\mathbf{x}) = \left(\int_0^\infty t\, |\nabla u(\mathbf{x},t)|^2 \, \mathrm{d}t \right)^{1/2}, \qquad \nabla u(\mathbf{x},t) = (\nabla_{\mathbf{x}} u, \partial_t u)\,.$$

Theorem 1 of [86, Chapter IV] states that for $1 < p < \infty$, if $f \in L^p(\mathbb{R}^d)$ so is $g(f)$, and its L^p norm is comparable with that of f. Most important to our work is the usefulness of Poisson integrals in characterizing fractional Sobolev spaces $W^{s,p}(\mathbb{R}^d;\mathbb{R}^d)$.

Proposition 4.2 (Proposition 7′ in [86])**.** *Let $s \in (0,1)$ and $1 < p < \infty$. Let $\mathbf{f} = (f_1, f_2, \dots, f_d) \in L^p(\mathbb{R}^d;\mathbb{R}^d)$. Then $\mathbf{f} \in W^{s,p}(\mathbb{R}^d;\mathbb{R}^d)$ if and only if*

$$\int_0^\infty t^{p(1-s)} \, \|\partial_t \mathbf{u}(\cdot,t)\|^p_{L^p(\mathbb{R}^d)} \, \frac{\mathrm{d}t}{t} < \infty\,.$$

Moreover, there exists constants C_1 and C_2 depending only on s, p, and d such that

$$C_1 \int_0^\infty t^{p(1-s)} \, \|\partial_t \mathbf{u}(\cdot,t)\|^p_{L^p(\mathbb{R}^d)} \, \frac{\mathrm{d}t}{t} \le [\mathbf{f}]^p_{W^{s,p}} \le C_2 \int_0^\infty t^{p(1-s)} \, \|\partial_t \mathbf{u}(\cdot,t)\|^p_{L^p(\mathbb{R}^d)} \, \frac{\mathrm{d}t}{t}\,. \tag{4.4}$$

Proof. The inequality on the left-hand side in (4.4) is proved in [86, 88], and the right-hand proved in [88]. However the inequality on the right-hand side is the one that we need later and so for completeness we present its proof here. We will prove it for scalar functions, and for the vector case it follows easily by making the comparison component wise. We let $u(\mathbf{x},t) = p_t * f(\mathbf{x})$. Let $\mathbf{x}$, $\mathbf{y}$ be such that $\mathbf{x}$ and $\mathbf{x}+\mathbf{y}$ are Lebesgue points of f. We choose $t = |\mathbf{y}|$ and write

$$\begin{aligned} f(\mathbf{x}+\mathbf{y}) - f(\mathbf{x}) = &\Big(u(\mathbf{x}+\mathbf{y}, |\mathbf{y}|) - u(\mathbf{x}, |\mathbf{y}|) \Big) \\ &+ \Big(f(\mathbf{x}+\mathbf{y}) - u(\mathbf{x}+\mathbf{y}, |\mathbf{y}|) \Big) - \Big(f(\mathbf{x}) - u(\mathbf{x}, |\mathbf{y}|) \Big)\,. \end{aligned}$$

We estimate each of the integrals associated with the three differences separately. We denote these integrals by I, II and III. Using the mean value theorem,

$$u(\mathbf{x}+\mathbf{y},|\mathbf{y}|)-u(\mathbf{x},|\mathbf{y}|)=\int_0^1 \nabla_{\mathbf{x}} u(\mathbf{x}+\tau\mathbf{y},|\mathbf{y}|)\cdot\mathbf{y}\,\mathrm{d}\tau\,. \tag{4.5}$$

It then follows from Minkowski's inequality that

$$\begin{aligned}&\left(\int_{\mathbb{R}^d}\left|u(\mathbf{x}+\mathbf{y},|\mathbf{y}|)-u(\mathbf{x},|\mathbf{y}|)\right|^p\mathrm{d}\mathbf{x}\right)^{1/p}\\ &=\left(\int_{\mathbb{R}^d}\left|\int_0^1\nabla_{\mathbf{x}}u(\mathbf{x}+\tau\mathbf{y},|\mathbf{y}|)\cdot\mathbf{y}\,\mathrm{d}\tau\right|^p\mathrm{d}\mathbf{x}\right)^{1/p}\le|\mathbf{y}|\ \|\nabla_{\mathbf{x}}u(\cdot,|\mathbf{y}|)\|_{L^p(\mathbb{R}^d)}\ .\end{aligned}$$

Then using polar coordinates ($t=|\mathbf{y}|$) we get that

$$\begin{aligned}(\mathrm{I})^p:=\int_{\mathbb{R}^d}\int_{\mathbb{R}^d}\frac{|u(\mathbf{x}+\mathbf{y},|\mathbf{y}|)-u(\mathbf{x},|\mathbf{y}|)|^p}{|\mathbf{y}|^{d+sp}}\,\mathrm{d}\mathbf{x}\,\mathrm{d}\mathbf{y}&\le\int_{\mathbb{R}^d}\frac{\|\nabla_{\mathbf{x}}u(\cdot,|\mathbf{y}|)\|^p_{L^p(\mathbb{R}^d)}}{|\mathbf{y}|^{d+sp-p}}\,\mathrm{d}\mathbf{y}\\ &\le C\int_0^\infty\frac{\|\nabla_{\mathbf{x}}u(\cdot,t)\|^p_{L^p(\mathbb{R}^d)}}{t^{sp-p+1}}\mathrm{d}t\,.\end{aligned}$$

Now we repeat the same argument for the second difference; using (4.5),

$$f(\mathbf{x}+\mathbf{y})-u(\mathbf{x}+\mathbf{y},|\mathbf{y}|)=-\int_0^1\partial_t u(\mathbf{x}+\mathbf{y},\tau|\mathbf{y}|)\,|\mathbf{y}|\,\mathrm{d}\tau\,.$$

Then Minkowski's inequality gives us

$$\left(\int_{\mathbb{R}^d}\left|f(\mathbf{x}+\mathbf{y})-u(\mathbf{x}+\mathbf{y},|\mathbf{y}|)\right|^p\mathrm{d}\mathbf{x}\right)^{1/p}\le\int_0^1|\mathbf{y}|\ \|\partial_t u(\cdot,\tau|\mathbf{y}|)\|_{L^p(\mathbb{R}^d)}\mathrm{d}\tau\,.$$

Calculations similar to the one above along with a second application of Minkowski's inequality show that

$$\begin{aligned}(\mathrm{II})^p&:=\int_{\mathbb{R}^d}\frac{\|f(\cdot+\mathbf{y})-u(\cdot+\mathbf{y},|\mathbf{y}|)\|^p_{L^p(\mathbb{R}^d)}}{|\mathbf{y}|^{d+sp}}\,\mathrm{d}\mathbf{y}\\ &\le C\left(\int_0^1\left(\int_0^\infty\frac{\|\partial_t u(\cdot,\tau r)\|^p_{L^p(\mathbb{R}^d)}}{r^{sp-p+1}}\,\mathrm{d}r\right)^{1/p}\mathrm{d}\tau\right)^p\,.\end{aligned}$$

Changing variables $t = \tau r$ in the inner integral we obtain that

$$\begin{aligned}
\mathrm{II}^p &\leq C \left(\int_0^1 \left(\int_0^\infty \frac{\|\partial_t u(\cdot,t)\|^p_{L^p(\mathbb{R}^d)}}{t^{sp-p+1}} \tau^{sp-p} \, \mathrm{d}t \right)^{1/p} \mathrm{d}\tau \right)^p \\
&\leq C \left(\int_0^1 \tau^{s-1} \, \mathrm{d}\tau \left(\int_0^\infty \frac{\|\partial_t u(\cdot,t)\|^p_{L^p(\mathbb{R}^d)}}{t^{sp-p+1}} \, \mathrm{d}t \right)^{1/p} \right)^p \\
&= \frac{C}{s^p} \int_0^\infty \frac{\|\partial_t u(\cdot,t)\|^p_{L^p(\mathbb{R}^d)}}{t^{sp-p+1}} \, \mathrm{d}t \,.
\end{aligned}$$

The quantity $f_j(\mathbf{x}) - u(\mathbf{x}, |\mathbf{y}|)$ can be estimated exactly the same way, and so we obtain

$$(\mathrm{III})^p := \int_{\mathbb{R}^d} \frac{\|f - u(\cdot, |\mathbf{y}|)\|^p_{L^p(\mathbb{R}^d)}}{|\mathbf{y}|^{d+sp}} \mathrm{d}\mathbf{y} \leq \frac{C}{s^p} \int_0^\infty \frac{\|\partial_t u(\cdot,t)\|^p_{L^p(\mathbb{R}^d)}}{t^{sp-p+1}} \, \mathrm{d}t \,.$$

We now invoke the comparison estimate (4.3) to conclude the proof. $\square$

4.1.3 Poisson-Type Integrals

In this section we introduce a Poisson-type matrix kernel $\mathbb{P}_t(\mathbf{x})$ that we convolve with vector fields so that the resulting Poisson-type integral can be used to characterize $\mathcal{X}^s_p(\mathbb{R}^d)$ in the same spirit as Proposition 4.2.

Definition of the Poisson-Type Kernel and Integral

Poisson-type kernel Denoting $\mathbb{M}_J(\mathbb{R})$ to be the space of $J \times J$ matrices with real entries, we introduce the $\mathbb{M}_{d+1}(\mathbb{R})$-valued function $\mathbb{P}(\mathbf{x}) = \left(\mathfrak{p}^{jk}(\mathbf{x})\right)_{j,k=1}^{d+1}$ as

$$\mathfrak{p}^{jk}(\mathbf{x}) := \frac{2(d+1)}{\omega_d} \frac{(\mathbf{x},1)_j (\mathbf{x},1)_k}{(|\mathbf{x}|^2+1)^{\frac{d+3}{2}}} \,, \quad \mathbf{x} \in \mathbb{R}^d, \quad j,k = 1, \cdots, d+1$$

where $(\mathbf{x}, 1)$ is a $d+1$ vector whose $d+1$ component is 1. For $t > 0$, we denote the the Poisson-type kernel $\mathbb{P}_t : \mathbb{R}^d \to \mathbb{M}_{d+1}(\mathbb{R})$ by

$$\mathbb{P}_t(\mathbf{x}) := \frac{1}{t^d} \mathbb{P}\left(\frac{\mathbf{x}}{t}\right) \,, \ \mathbf{x} \in \mathbb{R}^d.$$

We notice that the $(d+1)\times(d+1)$ matrix $\mathbb{P}_t(\mathbf{x})$ has the form

$$\mathbb{P}_t(\mathbf{x}) = \frac{2(d+1)}{\omega_d}\frac{t}{(|\mathbf{x}|^2+t^2)^{\frac{d+3}{2}}}\begin{bmatrix}\mathbf{x}\otimes\mathbf{x} & t\mathbf{x}\\ t\mathbf{x} & t^2\end{bmatrix},$$

where $\mathbf{x}$ is considered both a row and column vector in $\mathbb{R}^d$.

Poisson-type integrals Given $\mathbf{F}:\mathbb{R}^d\to\mathbb{R}^{d+1}$ with $\mathbf{F}=(F_1,\dots,F_{d+1})\in L^p(\mathbb{R}^d)$, we define $\mathbf{U}=(U_1,U_2,\dots,U_{d+1}):\mathbb{R}^{d+1}_+\to\mathbb{R}^{d+1}$ as the convolution

$$\mathbf{U}(\mathbf{x},t) := \mathbb{P}_t * \mathbf{F}(\mathbf{x}).$$

The convolution in the above equation is taken in the sense of matrix multiplication. That is the i^{th} entry component of $\mathbf{U}$ is given by $U_i=\sum_{j=1}^{d+1}\mathfrak{p}_t^{ij}*F_j$.

Properties of the Poisson-type Kernel

We next establish basic but fundamental properties of $\mathbb{P}_t$ which are analogues of the properties of the classical Poisson kernel. We begin by noting that $\mathbb{P}\in C^\infty(\mathbb{R}^d;\mathbb{M}_{d+1}(\mathbb{R}))$, and

$$(\mathbf{x},t)\mapsto\mathbb{P}_t(\mathbf{x})\in C^\infty\left(\overline{\mathbb{R}^{d+1}_+}\setminus B(0,\varepsilon)\,;\,\mathbb{M}_{d+1}(\mathbb{R})\right),\quad \text{for every } \varepsilon>0.$$

Moreover, it is immediate from the definition to see that for every $1\le p\le\infty$, $\mathbb{P}\in L^p(\mathbb{R}^d;\mathbb{M}_{d+1}(\mathbb{R}))$ with the pointwise estimate

$$|\mathbb{P}(\mathbf{x})|\le\frac{C}{(1+|\mathbf{x}|^2)^{\frac{d+1}{2}}},\quad \mathbf{x}\in\mathbb{R}^d.$$

By the norm of $|\mathbb{A}|$ for a matrix $\mathbb{A}=(a^{ij})$ we mean the Frobenius norm $|\mathbb{A}|=\sqrt{\sum_{i,j}|a^{ij}|^2}$. In the following lemma we prove that the matrix kernel $\mathbb{P}_t$ is in fact an approximation to the identity.

Lemma 4.3. *For any $t > 0$, if $\mathbb{I}_{d+1}$ denotes the $(d+1) \times (d+1)$ identity matrix, then*

$$\int_{\mathbb{R}^d} \mathbb{P}_t(\mathbf{x})\,\mathrm{d}\mathbf{x} = \int_{\mathbb{R}^d} \mathbb{P}(\mathbf{x})\,\mathrm{d}\mathbf{x} = \mathbb{I}_{d+1}\,.$$

For any $\mathbf{F} \in L^1_{loc}(\mathbb{R}^d; \mathbb{R}^{d+1})$, the function $(\mathbf{x}, t) \mapsto (\mathbb{P}_t * \mathbf{F})(\mathbf{x}) \in C^\infty(\mathbb{R}^{d+1}_+)$ with

$$\lim_{t \to 0^+} (\mathbb{P}_t * \mathbf{F})(\mathbf{x}) = \mathbf{F}(\mathbf{x}) \quad \text{for all Lebesgue points } \mathbf{x} \in \mathbb{R}^d \text{ of } \mathbf{F}\,.$$

Moreover, for $1 \le p < \infty$ if $\mathbf{F} \in L^p(\mathbb{R}^d; \mathbb{R}^{d+1})$, then

$$\lim_{t \to 0^+} \|(\mathbb{P}_t * \mathbf{F}) - \mathbf{F}\|_{L^p(\mathbb{R}^d)} \to 0\,.$$

Proof. The conclusion of the lemma can be deduced from [55, Lemma 3.3] where it is shown that similar properties are enjoyed by the Poisson kernel for the Lamé system. To be specific, given constants μ, λ satisfying $3\mu + \lambda > 0$, $\mu + \lambda > 0$, the matrix kernel $\mathbb{K} : \mathbb{R}^d \to \mathbb{M}_{d+1}(\mathbb{R})$ defined by

$$\mathbb{K}(\mathbf{x}) := \frac{4\mu}{3\mu + \lambda} \frac{1}{\omega_d} \frac{1}{(|\mathbf{x}|^2 + 1)^{\frac{d+1}{2}}} \mathbb{I}_{d+1} + \frac{\mu + \lambda}{3\mu + \lambda} \frac{2(d+1)}{\omega_d} \frac{(\mathbf{x}, 1) \otimes (\mathbf{x}, 1)}{(|\mathbf{x}|^2 + 1)^{\frac{d+3}{2}}}$$

is shown to be the Poisson kernel for the Lamé system in the upper half space, see [54, Lemma 5.1]. In addition the scaled kernel $\mathbb{K}_t(\mathbf{x}) := t^{-d}\mathbb{K}(\mathbf{x}/t)$ is shown to be an approximation to the identity. As a consequence, to prove the above lemma it suffices to note that for every $\mathbf{x} \in \mathbb{R}^d$, $t > 0$,

$$\frac{\mu + \lambda}{3\mu + \lambda} \mathbb{P}_t(\mathbf{x}) = \mathbb{K}_t(\mathbf{x}) - \frac{2\mu}{3\mu + \lambda} p_t(\mathbf{x}) \mathbb{I}_{d+1}, \tag{4.6}$$

where we recall that $p_t(\mathbf{x})$ the Poisson kernel of the Laplacian in the upper half space. For each $t > 0$, we can now integrate both sides of (4.6) to get

$$\frac{\mu + \lambda}{3\mu + \lambda} \int_{\mathbb{R}^d} \mathbb{P}_t(\mathbf{x})\,\mathrm{d}\mathbf{x} = \int_{\mathbb{R}^d} \mathbb{K}_t(\mathbf{x})\,\mathrm{d}\mathbf{x} - \frac{2\mu}{3\mu + \lambda} \left(\int_{\mathbb{R}^d} p_t(\mathbf{x})\,\mathrm{d}\mathbf{x} \right) \mathbb{I}_{d+1}$$

which yields $\int_{\mathbb{R}^d} \mathbb{P}_t(\mathbf{x})\mathrm{d}\mathbf{x} = \frac{3\mu+\lambda}{\mu+\lambda} \left(1 - \frac{2\mu}{3\mu+\lambda}\right) \mathbb{I}_{d+1} = \mathbb{I}_{d+1}\,.$

The smoothness and the convergences of the matrix convolutions also follow from the same results for $\mathbb{K}$ and p_t. It can also be easily verified using Minkowski's inequality and the Lebesgue Dominated Convergence Theorem as follows:

$$\begin{aligned}
\|\mathbb{P}_t * \mathbf{F} - \mathbf{F}\|^p_{L^p(\mathbb{R}^d)} &= \int_{\mathbb{R}^d} \left| \int_{\mathbb{R}^d} \mathbb{P}_t(\mathbf{y})(\mathbf{F}(\mathbf{x}-\mathbf{y}) - \mathbf{F}(\mathbf{x}))\,\mathrm{d}\mathbf{y} \right|^p \mathrm{d}\mathbf{x} \\
&\le C(d) \int_{\mathbb{R}^d} |\mathbb{P}_t(\mathbf{y})|\, \|\tau_{\mathbf{y}}\mathbf{F} - \mathbf{F}\|^p_{L^p(\mathbb{R}^d)}\,\mathrm{d}\mathbf{y} \\
&= C(d) \int_{\mathbb{R}^d} |\mathbb{P}(\mathbf{y})|\, \|\tau_{t\mathbf{y}}\mathbf{F} - \mathbf{F}\|^p_{L^p(\mathbb{R}^d)}\,\mathrm{d}\mathbf{y} \to 0\,,
\end{aligned}$$

where $\tau_{\mathbf{y}}\mathbf{F} = \mathbf{F}(\mathbf{x}-\mathbf{y})$. The integrand in the last term is bounded by the L^1 function $C\,|\mathbb{P}(\mathbf{y})|\,\|\mathbf{F}\|^p_{L^p}$, and for each $\mathbf{y}$ the integrand converges to zero by continuity of translations in L^p. $\square$

Remark 4.4. A connection between $\mathbb{P}_t$ and the semi-norm $|\cdot|_{\mathcal{X}^s_p}$ is obtained through the following important relation that we use below. For any $\mathbf{z}$, $\mathbf{x} \in \mathbb{R}^d$, we have

$$\mathbb{P}_t(\mathbf{x}) \begin{bmatrix} \mathbf{z} \\ 0 \end{bmatrix} = \overline{\mathbf{P}}(\mathbf{x},t) \left(\mathbf{z} \cdot \frac{\mathbf{x}}{|\mathbf{x}|} \right)\,, \qquad \mathbf{z} \in \mathbb{R}^d\,,$$

where the vector function $\overline{\mathbf{P}}(\mathbf{x},t)$ is given by $\overline{\mathbf{P}}(\mathbf{x},t) := \dfrac{2(d+1)}{\omega_d} \dfrac{t|\mathbf{x}|}{(|\mathbf{x}|^2+t^2)^{\frac{d+3}{2}}} \begin{bmatrix} \mathbf{x} \\ t \end{bmatrix}$. As a consequence of this and the approximation to the identity result, we see that if $\mathbf{F} = (\mathbf{f}, 0)$, then

$$\begin{aligned}
\mathbf{U}(\mathbf{x},t) &= \mathbf{F}(\mathbf{x}) + \int_{\mathbb{R}^d} \mathbb{P}_t(\mathbf{y})(\mathbf{F}(\mathbf{x}+\mathbf{y}) - \mathbf{F}(\mathbf{x}))\,\mathrm{d}\mathbf{y} \\
&= \mathbf{F}(\mathbf{x}) + \int_{\mathbb{R}^d} \overline{\mathbf{P}}(\mathbf{y},t)(\mathbf{f}(\mathbf{x}+\mathbf{y}) - \mathbf{f}(\mathbf{x})) \cdot \frac{\mathbf{y}}{|\mathbf{y}|}\,\mathrm{d}\mathbf{y}\,.
\end{aligned}$$

The matrix Poisson kernel $\mathbb{P}_t$ also satisfies a semigroup property as documented in the next lemma. The following arguments rely centrally on an explicit formula for the Fourier transform of $\mathbb{P}_t$.

Lemma 4.5. *For each $t > 0$, the Fourier transform of $\mathbb{P}_t(\mathbf{x})$ is given by*

$$\mathcal{F}_{\mathbf{x}}(\mathbb{P}_t)(\boldsymbol{\xi}) := \mathrm{e}^{-2\pi|\boldsymbol{\xi}|t}\left(\mathbb{I}_{d+1} + (2\pi|\boldsymbol{\xi}|t)\begin{bmatrix} -\frac{\boldsymbol{\xi}\otimes\boldsymbol{\xi}}{|\boldsymbol{\xi}|^2} & -\imath\frac{\boldsymbol{\xi}}{|\boldsymbol{\xi}|} \\ -\imath\frac{\boldsymbol{\xi}}{|\boldsymbol{\xi}|} & 1 \end{bmatrix}\right). \tag{4.7}$$

As a consequence $\mathbb{P}_t$ satisfies the semigroup property: for every $t_1, t_2 > 0$

$$\mathbb{P}_{t_1} * \mathbb{P}_{t_2} = \mathbb{P}_{t_1+t_2}\,,$$

where the convolution is understood in the sense of matrix multiplication.

Proof. To prove the semigroup property of $\mathbb{P}_t$ we use the property of convolution and the explicit Fourier transform formula given in (4.7). To preserve the flow of the presentation in this section, the Fourier transform of $\mathbb{P}_t$ is computed in the appendix. We carry out this calculation via matrix multiplication. For any positive t_1, t_2 we have that

$$\begin{aligned}\mathcal{F}_{\mathbf{x}}\left(\mathbb{P}_{t_1} * \mathbb{P}_{t_2}\right)(\boldsymbol{\xi}) &= \mathcal{F}_{\mathbf{x}}(\mathbb{P}_{t_1})(\boldsymbol{\xi})\,.\,\mathcal{F}_{\mathbf{x}}(\mathbb{P}_{t_2})(\boldsymbol{\xi}) \\ &= \mathrm{e}^{-2\pi|\boldsymbol{\xi}|(t_1+t_2)}\Bigg(\mathbb{I}_{d+1} + (2\pi|\boldsymbol{\xi}|t_1)\begin{bmatrix} -\frac{\boldsymbol{\xi}\otimes\boldsymbol{\xi}}{|\boldsymbol{\xi}|^2} & -\imath\frac{\boldsymbol{\xi}}{|\boldsymbol{\xi}|} \\ -\imath\frac{\boldsymbol{\xi}}{|\boldsymbol{\xi}|} & 1 \end{bmatrix} \\ &\qquad + (2\pi|\boldsymbol{\xi}|t_2)\begin{bmatrix} -\frac{\boldsymbol{\xi}\otimes\boldsymbol{\xi}}{|\boldsymbol{\xi}|^2} & -\imath\frac{\boldsymbol{\xi}}{|\boldsymbol{\xi}|} \\ -\imath\frac{\boldsymbol{\xi}}{|\boldsymbol{\xi}|} & 1 \end{bmatrix}\Bigg) \\ &= \mathcal{F}_{\mathbf{x}}(\mathbb{P}_{t_1+t_2})(\boldsymbol{\xi})\,,\end{aligned}$$

where in the second equality we have multiplied the matrix of Fourier symbols and have also used the fact that $\begin{bmatrix} -\frac{\boldsymbol{\xi}\otimes\boldsymbol{\xi}}{|\boldsymbol{\xi}|^2} & -\imath\frac{\boldsymbol{\xi}}{|\boldsymbol{\xi}|} \\ -\imath\frac{\boldsymbol{\xi}}{|\boldsymbol{\xi}|} & 1 \end{bmatrix}^2 = \mathbf{0}$, the zero matrix, which can be verified easily by computation. □

The next lemma summarizes integrability properties of the first derivatives of $\mathbb{P}_t$ that we will be using later. The proof is purely computational and can be done following similar calculations for the Poisson kernel. We omit it here.

Lemma 4.6. *For each j, k, and $\ell \in \{1, \dots, d, d+1\}$ and for every $t > 0$ we have that $\partial_t \mathfrak{p}_t^{jk}(\mathbf{x}) \in L^1(\mathbb{R}^d)$ and $\partial_{x_\ell} \mathfrak{p}_t^{jk}(\mathbf{x}) \in L^1(\mathbb{R}^d)$. In addition we have the following pointwise estimates: There exists a constant $c = c(d) > 0$ such that*

$$|\partial_t \mathfrak{p}_t^{jk}(\mathbf{x})| \le c\,|\mathbf{x}|^{-d-1}\,, \qquad |\partial_t \mathfrak{p}_t^{jk}(\mathbf{x})| \le c\,t^{-d-1}\,, \qquad \mathbf{x} \in \mathbb{R}^d\,, \quad t > 0\,,$$

for any $j, k = 1, 2, \dots d+1$.

Norm Equivalence of Poisson Integrals

We begin first by establishing relations between the Poisson integrals obtained from p_t and $\mathbb{P}_t$.

Lemma 4.7. *Suppose that $\mathbf{F} = (\mathbf{f}, 0) \in \mathcal{S}(\mathbb{R}^d; \mathbb{R}^{d+1})$. Then for every $t > 0$ both Poisson integrals $\mathbf{u}(\mathbf{x}, t) = p_t * \mathbf{F}(\mathbf{x})$ and $\mathbf{U}(\mathbf{x}, t) = \mathbb{P}_t * \mathbf{F}(\mathbf{x})$ are in $\mathcal{S}(\mathbb{R}^d; \mathbb{R}^{d+1})$. Moreover we have the following relations between $\mathbf{u}$ and $\mathbf{U}$.*

- *For any $j = 1, \dots, d$, we have*

$$U_j(\mathbf{x}, t) = u_j(\mathbf{x}, t) + R_j(U_{d+1})(\mathbf{x}, t)\,, \qquad \mathbf{x} \in \mathbb{R}^d\,, \quad t > 0\,. \tag{4.8}$$

-

$$\partial_t U_{d+1}(\mathbf{x}, t) = -\mathrm{div}_{\mathbf{x}} \mathbf{u}(\mathbf{x}, t) - \sum_{j=1}^{d} R_j(\partial_{x_j} U_{d+1})(\mathbf{x}, t)\,, \qquad \mathbf{x} \in \mathbb{R}^d\,, \quad t > 0\,. \tag{4.9}$$

- *For any $j = 1, \dots, d$, we have*

$$\partial_{x_j} U_{d+1}(\mathbf{x}, t) = R_j \left(\partial_t U_{d+1} + \sum_{\ell=1}^{d} R_\ell \left(\partial_t u_\ell \right) \right) (\mathbf{x}, t)\,, \qquad \mathbf{x} \in \mathbb{R}^d\,, \quad t > 0\,, \tag{4.10}$$

where R_j is the j^{th} Riesz transform.

Proof. We prove first the identity (4.8). For a fixed $t > 0$, since all the functions involved are in $\mathcal{S}(\mathbb{R}^d)$, it suffices to check that the Fourier transform of the right-hand side agrees with that of the left-hand side in (4.8). From the definition of $\mathbf{U}$, we see that for any $t > 0$, $\mathcal{F}_{\mathbf{x}}(\mathbf{U}(\cdot,t))(\boldsymbol{\xi}) = \mathcal{F}(\mathbb{P}_t)(\boldsymbol{\xi})\mathcal{F}(\mathbf{F})(\boldsymbol{\xi})$. Now from the particular form of $\mathbf{F}$ and using the explicit formula (4.7) for the Fourier transform of $\mathbb{P}_t$ we see that for any $j = 1, \ldots, d$ we have

$$\mathcal{F}_{\mathbf{x}}(U_j)(\boldsymbol{\xi},t) = \mathrm{e}^{-2\pi|\boldsymbol{\xi}|t}\mathcal{F}(f_j)(\boldsymbol{\xi}) + \mathrm{e}^{-2\pi|\boldsymbol{\xi}|t}(2\pi|\boldsymbol{\xi}|t)\left(-\frac{\xi_j}{|\boldsymbol{\xi}|}\right)\left(\frac{\boldsymbol{\xi}}{|\boldsymbol{\xi}|}\cdot\mathcal{F}(\mathbf{f})(\boldsymbol{\xi})\right) \tag{4.11}$$

and that after simplification $\mathcal{F}_{\mathbf{x}}(U_{d+1})(\boldsymbol{\xi},t) = -\imath 2\pi t\,\mathrm{e}^{-2\pi|\boldsymbol{\xi}|t}\boldsymbol{\xi}\cdot\mathcal{F}(\mathbf{f})(\boldsymbol{\xi})$. To complete the proof of the identity (4.8) we notice that the first term in (4.11) is precisely $\mathcal{F}_{\mathbf{x}}(u_j)(\boldsymbol{\xi},t)$, whereas the second term can be rewritten to obtain

$$\begin{aligned}
\mathrm{e}^{-2\pi|\boldsymbol{\xi}|t}(2\pi|\boldsymbol{\xi}|t)\left(-\frac{\xi_j}{|\boldsymbol{\xi}|}\right)\left(\frac{\boldsymbol{\xi}}{|\boldsymbol{\xi}|}\cdot\mathcal{F}(\mathbf{f})(\boldsymbol{\xi})\right) &= \left(-\imath\frac{\xi_j}{|\boldsymbol{\xi}|}\right)\left(-\imath(2\pi t)\mathrm{e}^{-2\pi|\boldsymbol{\xi}|t}\left(\boldsymbol{\xi}\cdot\mathcal{F}(\mathbf{f})(\boldsymbol{\xi})\right)\right)\\
&= \mathcal{F}_{\mathbf{x}}\left(R_j(U_{d+1})\right)(\boldsymbol{\xi},t)\,.
\end{aligned}$$

Let us proceed to show (4.9). Using a direct calculation and some rearrangement we get

$$\partial_t\mathcal{F}_{\mathbf{x}}(U_{d+1})(\boldsymbol{\xi},t) = (-\imath 2\pi)\mathrm{e}^{-2\pi|\boldsymbol{\xi}|t}\boldsymbol{\xi}\cdot\mathcal{F}(\mathbf{f})(\boldsymbol{\xi}) + \imath(4\pi^2|\boldsymbol{\xi}|t)\mathrm{e}^{-2\pi|\boldsymbol{\xi}|t}\boldsymbol{\xi}\cdot\mathcal{F}(\mathbf{f})(\boldsymbol{\xi})\,. \tag{4.12}$$

The identity follows easily once we realize that the first term of (4.12) can be rewritten as

$$(-\imath 2\pi)\mathrm{e}^{-2\pi|\boldsymbol{\xi}|t}\boldsymbol{\xi}\cdot\mathcal{F}(\mathbf{f})(\boldsymbol{\xi}) = -2\pi\imath\boldsymbol{\xi}\cdot\mathrm{e}^{-2\pi|\boldsymbol{\xi}'|t}\mathcal{F}(\mathbf{f})(\boldsymbol{\xi}) = -\mathcal{F}_{\mathbf{x}}\left(\mathrm{div}_{\mathbf{x}}\mathbf{u}(\cdot,t)\right)(\boldsymbol{\xi})\,,$$

while the second term in (4.12) can also be rewritten as

$$\begin{aligned}
\imath(4\pi^2|\boldsymbol{\xi}|t)\mathrm{e}^{-2\pi|\boldsymbol{\xi}|t}\boldsymbol{\xi}\cdot\mathcal{F}(\mathbf{f})(\boldsymbol{\xi}) &= (2\pi|\boldsymbol{\xi}|)(\imath 2\pi t)\mathrm{e}^{-2\pi\boldsymbol{\xi}t}\left(\boldsymbol{\xi}\cdot\mathcal{F}(\mathbf{f})(\boldsymbol{\xi})\right)\\
&= -\sum_{j=1}^{d}\left(-\imath\frac{\xi_j}{|\boldsymbol{\xi}|}\right)2\pi\imath\xi_j(-\imath 2\pi t)\mathrm{e}^{-2\pi|\boldsymbol{\xi}|t}\left(\boldsymbol{\xi}\cdot\mathcal{F}(\mathbf{f})(\boldsymbol{\xi})\right)\\
&= -\mathcal{F}\left(\sum_{j=1}^{d}R_j\left(\partial_j U_{d+1}\right)\right)(\boldsymbol{\xi},t)\,.
\end{aligned}$$

Next we prove the identity (4.10). Again, by a direct calculation

$$
\begin{aligned}
\mathcal{F}(\partial_{x_j} U_{d+1})(\boldsymbol{\xi}, t) &= (\imath 2\pi \xi_j)(-\imath 2\pi t)\mathrm{e}^{-2\pi|\boldsymbol{\xi}|t} \boldsymbol{\xi} \cdot \mathcal{F}(\mathbf{f})(\boldsymbol{\xi}) \\
&= \left(-\imath \frac{\xi_j}{|\boldsymbol{\xi}|}\right) (4\pi^2 |\boldsymbol{\xi}| t)\mathrm{e}^{-2\pi|\boldsymbol{\xi}|t} \big(\imath \boldsymbol{\xi} \cdot \mathcal{F}(\mathbf{f})(\boldsymbol{\xi})\big) .
\end{aligned} \tag{4.13}
$$

We need to connect the last expression in (4.13) with $\partial_t U_{d+1}$. To do so, we observe from (4.12) that

$$
(4\pi^2 |\boldsymbol{\xi}|^2 t)\mathrm{e}^{-2\pi|\boldsymbol{\xi}|t} \left(\imath \frac{\boldsymbol{\xi}}{|\boldsymbol{\xi}|} \cdot \mathcal{F}(\mathbf{f})(\boldsymbol{\xi})\right) = \mathcal{F}_{\mathbf{x}}(\partial_t U_{d+1})(\boldsymbol{\xi}, t) + \mathcal{F}\left(\sum_{k=1}^{d} R_k \left[\partial_t u_k(\cdot, t)\right]\right)(\boldsymbol{\xi}) ,
$$

where the last term is a rewriting of the expression $(-\imath 2\pi)\mathrm{e}^{-2\pi|\boldsymbol{\xi}|t}\boldsymbol{\xi} \cdot \mathcal{F}(\mathbf{f})(\boldsymbol{\xi})$ in (4.12). Substituting this into (4.13) we get the desired result. $\square$

Proposition 4.8. *Let $1 < p < \infty$, $\mathbf{f} \in L^p(\mathbb{R}^d; \mathbb{R}^d)$. Set $\mathbf{F} = (\mathbf{f}, 0) \in L^p(\mathbb{R}^d; \mathbb{R}^{d+1})$, $\mathbf{u}(\mathbf{x}, t) = p_t * \mathbf{F}(\mathbf{x})$ and set $\mathbf{U}(\mathbf{x}, t) = \mathbb{P}_t * \mathbf{F}(\mathbf{x})$. Then there exists a positive constant $C = C(d, p)$ such that for any $t > 0$ we have*

$$
\|\partial_t \mathbf{u}(\cdot, t)\|_{L^p(\mathbb{R}^d)} \leq C \|\partial_t \mathbf{U}(\cdot, t)\|_{L^p(\mathbb{R}^d)} \tag{4.14}
$$

and for each $k = 1, \ldots, d$ we have

$$
\|\partial_{x_k} \mathbf{u}(\cdot, t)\|_{L^p(\mathbb{R}^d)} \leq C \|\partial_{x_k} \mathbf{U}(\cdot, t)\|_{L^p(\mathbb{R}^d)} . \tag{4.15}
$$

Proof. We prove both inequalities for $\mathbf{f} \in \mathcal{S}(\mathbb{R}^d; \mathbb{R}^d)$ and then the general case follows by density. Both inequalities (4.14) and (4.15) follow from identity (4.8) in Proposition 4.7. Indeed, for $j = 1, \ldots, d$, we can differentiate the equation (4.8) in t to obtain that for any $\mathbf{x} \in \mathbb{R}^d$ and $t > 0$

$$
\partial_t u_j(\mathbf{x}, t) = \partial_t U_j(\mathbf{x}, t) - R_j\big(\partial_t U_{d+1}\big)(\mathbf{x}, t) .
$$

In a similar fashion, if we differentiate the equation (4.8) in x_k we obtain that

$$
\partial_{x_k} u_j(\mathbf{x}, t) = \partial_{x_k} U_j(\mathbf{x}, t) - R_j\big(\partial_{x_k} U_{d+1}\big)(\mathbf{x}, t) .
$$

Both inequalities (4.14) and (4.15) now follow by taking the L^p norm on both sides of the above two equations and summing over $j = 1, \dots, d$. Note that we have used both the fact that the Riesz transforms commute with differential operators and that the Riesz transforms are L^p bounded.

For general $\mathbf{f} \in L^p(\mathbb{R}^d;\mathbb{R}^d)$ take a sequence $(\mathbf{f}_n) \subset \mathcal{S}(\mathbb{R}^d;\mathbb{R}^d)$ converging to $\mathbf{f}$ in $L^p(\mathbb{R}^d;\mathbb{R}^d)$. Set $\mathbf{F}_n = (\mathbf{f}_n, 0)$. Then apply Lemma 4.3, Lemma 4.6 and Young's inequality to see that $\partial_t\big[\mathbb{P}_t * \mathbf{F}_n\big]$ converges to $\partial_t\big[\mathbb{P}_t * \mathbf{F}\big]$ in $L^p(\mathbb{R}^d;\mathbb{R}^d)$ and that $\nabla_{\mathbf{x}}\big[\mathbb{P}_t * \mathbf{F}_n\big]$ converges to $\nabla_{\mathbf{x}}\big[\mathbb{P}_t * \mathbf{F}\big]$ in $L^p(\mathbb{R}^d;\mathbb{R}^d)$. $\square$

4.1.4 A Characterization of Fractional Sobolev Spaces

Equivalence of Spaces

In this subsection we prove the main result of Section 4.1, which is the equivalence of the spaces $\mathcal{X}^s_p(\mathbb{R}^d)$ and $W^{s,p}(\mathbb{R}^d;\mathbb{R}^d)$. We paraphrase it in the following theorem.

Theorem 4.9. *Let $s \in (0,1)$ and $1 < p < \infty$. Then $\mathcal{X}^s_p(\mathbb{R}^d) = W^{s,p}(\mathbb{R}^d;\mathbb{R}^d)$. Moreover, there are constants C_i, $i = 1,2,3$ all depending only on s, p, and d such that for any $\mathbf{f} = (f_1, f_2, \dots, f_d) \in L^p(\mathbb{R}^d)$*

$$\begin{aligned}
[\mathbf{f}]_{W^{s,p}} &\overset{(\mathbf{EQ_1})}{\leq} C_1 \int_0^\infty t^{p(1-s)} \, \|\partial_t \mathbf{u}(\cdot,t)\|^p_{L^p(\mathbb{R}^d)} \, \frac{1}{t} dt \\
&\overset{(\mathbf{EQ_2})}{\leq} C_2 \int_0^\infty t^{p(1-s)} \, \|\partial_t \mathbf{U}(\cdot,t)\|^p_{L^p(\mathbb{R}^d)} \, \frac{1}{t} dt \\
&\overset{(\mathbf{EQ_3})}{\leq} C_3 [\mathbf{f}]_{\mathcal{X}^s_p} \,,
\end{aligned}$$

where $\mathbf{F} = (\mathbf{f}, 0)$, $\mathbf{u}(\mathbf{x},t) = p_t * \mathbf{f}(\mathbf{x})$, and $\mathbf{U}(\mathbf{x},t) = \mathbb{P}_t * \mathbf{F}(\mathbf{x})$.

Proof. In the above $(\mathbf{EQ_1})$ is proved in (4.4) of Proposition 4.2 and $(\mathbf{EQ_2})$ follows from the pointwise-in-t estimate proved in Proposition 4.8. What remains is the proof of the inequality $(\mathbf{EQ_3})$. We prove it as follows. Recalling that $\int_{\mathbb{R}^d} \mathbb{P}_t(\mathbf{x}) \, \mathrm{d}\mathbf{x} = \mathbb{I}_{d+1}$, we have

$$\partial_t \mathbf{U}(\mathbf{x},t) = \int_{\mathbb{R}^d} \partial_t \mathbb{P}_t(\mathbf{y}) \mathbf{F}(\mathbf{x}-\mathbf{y}) \, \mathrm{d}\mathbf{y} = \int_{\mathbb{R}^d} \partial_t \mathbb{P}_t(\mathbf{y}) \left(\mathbf{F}(\mathbf{x}-\mathbf{y}) - \mathbf{F}(\mathbf{x})\right) \mathrm{d}\mathbf{y} \,.$$

To reveal the connection with the integrand in the semi-norm $[\mathbf{f}]_{\mathcal{X}_p^s}$ we compute the derivative $\partial_t \mathbb{P}_t(\mathbf{y})$ in the above convolution directly. For $j = 1, \ldots, d$ the j^{th} term is given by

$$
\begin{aligned}
&\partial_t U_j(\mathbf{x}, t) \\
=&\frac{2(d+1)}{\omega_d} \sum_{k=1}^{d} \int_{\mathbb{R}^d} \frac{y_j}{(|\mathbf{y}|^2 + t^2)^{\frac{d+3}{2}}} \left(1 - \frac{(d+3)t^2}{|\mathbf{y}|^2 + t^2}\right) \Big(y_k \big(f_k(\mathbf{x} - \mathbf{y}) - f_k(\mathbf{x})\big)\Big) \,\mathrm{d}\mathbf{y} \\
=&\frac{2(d+1)}{\omega_d} \int_{\mathbb{R}^d} \frac{y_j |\mathbf{y}|}{(|\mathbf{y}|^2 + t^2)^{\frac{d+3}{2}}} \left(1 - \frac{(d+3)t^2}{|\mathbf{y}|^2 + t^2}\right) \left(\big(\mathbf{f}(\mathbf{x} + \mathbf{y}) - \mathbf{f}(\mathbf{x})\big) \cdot \frac{\mathbf{y}}{|\mathbf{y}|}\right) \mathrm{d}\mathbf{y}\,.
\end{aligned}
\tag{4.16}
$$

A similar computation also shows that

$$
\begin{aligned}
&\partial_t U_{d+1}(\mathbf{x}, t) \\
=&\frac{2(d+1)}{\omega_d} \sum_{k=1}^{d} \int_{\mathbb{R}^d} \frac{t}{(|\mathbf{y}|^2 + t^2)^{\frac{d+3}{2}}} \left(2 - \frac{(d+3)t^2}{|\mathbf{y}|^2 + t^2}\right) \Big(y_k \big(f_k(\mathbf{x} - \mathbf{y}) - f_k(\mathbf{x})\big)\Big) \,\mathrm{d}\mathbf{y} \\
=&-\frac{2(d+1)}{\omega_d} \int_{\mathbb{R}^d} \frac{t |\mathbf{y}|}{(|\mathbf{y}|^2 + t^2)^{\frac{d+3}{2}}} \left(2 - \frac{(d+3)t^2}{|\mathbf{y}|^2 + t^2}\right) \left(\big(\mathbf{f}(\mathbf{x} + \mathbf{y}) - \mathbf{f}(\mathbf{x})\big) \cdot \frac{\mathbf{y}}{|\mathbf{y}|}\right) \mathrm{d}\mathbf{y}\,.
\end{aligned}
\tag{4.17}
$$

Notice that the differentiated kernel expressions inside the integrals in (4.16) and (4.17) are linear combinations of the $\partial_t \mathfrak{p}_t^{jk}$ after factoring the unit vector $\frac{\mathbf{y}}{|\mathbf{y}|}$. As a result, these expressions enjoy the same pointwise estimates as $\partial_t \mathfrak{p}_t^{jk}$ stated in the Lemma 4.6. That is, the expressions are majorized by t^{-d-1} as well as by $|\mathbf{y}|^{-d-1}$. We will make use of these pointwise estimates below.

By splitting the convolution integrals in (4.16) and (4.17) into an integral over $B_t(\mathbf{0})$ and $\mathbb{R}^d \setminus B_t(\mathbf{0})$, we obtain that for any $t > 0$ and $\mathbf{x} \in \mathbb{R}^d$

$$
\begin{aligned}
|\partial_t \mathbf{U}(\mathbf{x}, t)| \leq \frac{C}{t^{d+1}} \int_{|\mathbf{y}| \leq t} \left| \big(\mathbf{f}(\mathbf{x} + \mathbf{y}) - \mathbf{f}(\mathbf{x})\big) \cdot \frac{\mathbf{y}}{|\mathbf{y}|} \right| \mathrm{d}\mathbf{y} \\
+ C \int_{|\mathbf{y}| > t} \left| \big(\mathbf{f}(\mathbf{x} + \mathbf{y}) - \mathbf{f}(\mathbf{x})\big) \cdot \frac{\mathbf{y}}{|\mathbf{y}|} \right| \frac{1}{|\mathbf{y}|^{d+1}} \,\mathrm{d}\mathbf{y}\,.
\end{aligned}
$$

Now, using Minkowski's integral inequality we obtain that

$$\|\partial_t \mathbf{U}(\cdot,t)\|_{L^p(\mathbb{R}^d)} \le \frac{C}{t^{d+1}} \int_{|\mathbf{y}|\le t} \left\| (\mathbf{f}(\cdot+\mathbf{y}) - \mathbf{f}(\cdot)) \cdot \frac{\mathbf{y}}{|\mathbf{y}|} \right\|_{L^p(\mathbb{R}^d)} \mathrm{d}\mathbf{y}$$
$$+ C \int_{|\mathbf{y}|>t} \frac{\left\| (\mathbf{f}(\cdot+\mathbf{y}) - \mathbf{f}(\cdot)) \cdot \frac{\mathbf{y}}{|\mathbf{y}|} \right\|_{L^p(\mathbb{R}^d)}}{|\mathbf{y}|^{d+1}} \mathrm{d}\mathbf{y}\,. \quad (4.18)$$

The remaining part of the argument that estimates the right-hand side of the above inequality by the semi-norm $[\mathbf{f}]_{\mathcal{X}_p^s}$ follows that of [86, Page 152] where it was done for classical Besov spaces. We repeat it here for clarity. Changing to polar coordinates, write $\mathbf{y} = r\mathbf{w} \in \mathbb{R}^d$, with $r = |\mathbf{y}|$ and $\mathbf{w} \in \mathbb{S}^{d-1}$. Define

$$\Psi(r) := \int_{\mathbb{S}^{d-1}} \|(\mathbf{f}(\cdot + r\mathbf{w}) - \mathbf{f}(\cdot)) \cdot \mathbf{w}\|_{L^p(\mathbb{R}^d)} \,\mathrm{d}\sigma(\mathbf{w})\,.$$

Then the last inequality in (4.18) can be rewritten in terms of $\Psi(r)$ to obtain

$$\|\partial_t \mathbf{U}(\cdot,t)\|_{L^p(\mathbb{R}^d)} \le \frac{C}{t^{d+1}} \int_0^t \Psi(r) r^{d-1}\,\mathrm{d}r + C\int_t^\infty \Psi(r) r^{-2}\,\mathrm{d}r\,.$$

We multiply both sides by t^{1-s} and estimate the norm in $L^p((0,\infty); t^{-1}\,\mathrm{d}t)$ on both sides. The right hand side can be estimates using Hardy's inequalities [86, Appendix A] on the functions $\Psi(r)r^{d-1}$, and $\Psi(r)r^{-2}$ to obtain

$$\left(\int_0^\infty \left(t^{1-s}\,\|\partial_t \mathbf{U}(\cdot,t)\|_{L^p}\right)^p \frac{1}{t}\,\mathrm{d}t\right)^{1/p} \le \frac{C}{(1-s)(s+d)} \left(\int_0^\infty \left(\Psi(r) r^{-s}\right)^p \frac{1}{r}\mathrm{d}r\right)^{1/p}\,.$$

By Hölder's inequality we have $\Psi(r)^p \leq C \int_{\mathbb{S}^{d-1}} \|(\mathbf{f}(\cdot + r\mathbf{w}) - \mathbf{f}(\cdot)) \cdot \mathbf{w}\|^p_{L^p(\mathbb{R}^d)} \, \mathrm{d}\sigma(\mathbf{w})$ and so we have

$$
\begin{aligned}
&\left(\int_0^\infty \left(t^{1-s} \|\partial_t \mathbf{U}(\cdot, t)\|_{L^p} \right)^p \frac{1}{t} \, \mathrm{d}t \right)^{1/p} \\
&\quad \leq \frac{C}{(1-s)(s+d)} \left(\int_0^\infty \int_{\mathbb{S}^{d-1}} \|(\mathbf{f}(\cdot + r\mathbf{w}) - \mathbf{f}(\cdot)) \cdot \mathbf{w}\|^p_{L^p} r^{-sp-1} \, \mathrm{d}\sigma(w) \, \mathrm{d}r \right)^{1/p} \\
&\quad \leq \frac{C}{(1-s)(s+d)} \left(\int_0^\infty \int_{\mathbb{S}^{d-1}} \frac{\|(\mathbf{f}(\cdot + r\mathbf{w}) - \mathbf{f}(\cdot)) \cdot \mathbf{w}\|^p_{L^p}}{r^{d+sp}} r^{d-1} \, \mathrm{d}\sigma(w) \, \mathrm{d}r \right)^{1/p} \\
&\quad \leq \frac{C}{(1-s)(s+d)} \left(\int_{\mathbb{R}^d} \frac{\left\| (\mathbf{f}(\cdot + \mathbf{y}) - \mathbf{f}(\cdot)) \cdot \frac{\mathbf{y}}{|\mathbf{y}|} \right\|^p_{L^p}}{|\mathbf{y}|^{d+sp}} \, \mathrm{d}\mathbf{y} \right)^{1/p} \\
&\quad = C[\mathbf{f}]_{\mathcal{X}^s_p(\mathbb{R}^d)} \, .
\end{aligned}
$$

where the last C depends only only on s, p, and d. This completes the proof of the theorem.

$\square$

Applications

One may now use the equivalence of spaces we have established to obtain inequalities that are important in applications. The simplest of all is the fractional Poincaré-Korn inequality.

Corollary 4.10. *Let $s \in (0,1)$ and $p \in (1, \infty)$. Let $B \subset \mathbb{R}^d$ be a ball. Then there exists a constant $C = C(p, s, d, B) > 0$ such that*

$$
\|\mathbf{f}\|^p_{L^p(\mathbb{R}^d)} \leq C \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \frac{\left| (\mathbf{f}(\mathbf{x}) - \mathbf{f}(\mathbf{y})) \cdot \frac{\mathbf{x}-\mathbf{y}}{|\mathbf{x}-\mathbf{y}|} \right|^p}{|\mathbf{x} - \mathbf{y}|^{d+sp}} \, \mathrm{d}\mathbf{x} \, \mathrm{d}\mathbf{y} \, .
$$

for any $\mathbf{f} \in W^{s,p}(\mathbb{R}^d; \mathbb{R}^d)$ such that $\mathrm{supp}(\mathbf{f}) \subset B$.

The corollary follows from the fact that when $\mathbf{f}$ is compactly supported, its L^p-norm is controlled by its $W^{s,p}$-semi-norm which in turn is controlled by its $\mathcal{X}^s_p$-semi-norm via the now-proved fractional Korn's inequality.

Another corollary of Theorem 4.9 is a fractional Sobolev embedding [68] that uses the $\mathcal{X}^s_p$-semi-norm.

Corollary 4.11. *Let $s \in (0,1)$ and $p \in (1,\infty)$ such that $sp < d$. Then there exists a constant $C = C(d,p,s)$ such that for $\mathbf{f} \in \mathcal{X}_p^s(\mathbb{R}^d)$ we have*

$$\|\mathbf{f}\|^p_{L^{p^{*s}}(\mathbb{R}^d)} \leq C \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} \frac{\left|(\mathbf{f}(\mathbf{x})-\mathbf{f}(\mathbf{y}))\cdot \frac{\mathbf{x}-\mathbf{y}}{|\mathbf{x}-\mathbf{y}|}\right|^p}{|\mathbf{x}-\mathbf{y}|^{d+sp}}\, d\mathbf{x}\, d\mathbf{y}\,,$$

where $p^{*s} = \frac{dp}{d-sp}$. As a consequence the space $\mathcal{X}_p^s(\mathbb{R}^d)$ is continuously embedded in $L^q(\mathbb{R}^d;\mathbb{R}^d)$, for any $q \in [p, p^{*s}]$, with the estimate that for $q \in [p, p^{*s})$ there exists a constant $C = C(q,p,d) > 0$ such that $\|\mathbf{u}\|_{L^q(\mathbb{R}^d)} \leq C\|\mathbf{u}\|_{\mathcal{X}_p^s(\mathbb{R}^d)}$.

Proof. Suppose that $\mathbf{f} \in \mathcal{X}_p^s(\mathbb{R}^d)$. Then by Theorem 4.9, $\mathbf{f} \in W^{s,p}(\mathbb{R}^d;\mathbb{R}^d)$. By fractional Sobolev embedding, which can be found in [68, Theorem 6.5 followed by Theorem 2.4], we have that $\mathbf{f} \in L^{p^{*s}}(\mathbb{R}^d;\mathbb{R}^d)$ and that

$$\|\mathbf{f}\|^p_{L^{p^{*s}}(\mathbb{R}^d)} \leq C\,\|\mathbf{f}\|^p_{W^{s,p}(\mathbb{R}^d)} \leq C\,\|\mathbf{f}\|_{\mathcal{X}_p^s(\mathbb{R}^d)}\,.$$

The embedding of $\mathcal{X}_p^s(\mathbb{R}^d)$ in $L^q(\mathbb{R}^d;\mathbb{R}^d)$ for any $q \in [p, p^{*s})$ follows via interpolation. □

We will also use Theorem 4.9 to prove fractional Korn-type inequalities for functions defined on the half-space $\mathbb{R}^d_+$. The argument to prove such results is standard. We first extend vector fields to be defined over $\mathbb{R}^d$ such that the norm of the extended vector field is controlled by the original one. Such an extension theorem is recently proved in [57], which we state below.

Theorem 4.12 (Extension operator). *Let $d \geq 1$, $p \in [1,\infty)$ and $0 < s < 1$ and $ps \neq 1$. There exists an extension operator*

$$E : \mathring{\mathcal{X}}_p^s(\mathbb{R}^d_+) \to \mathcal{X}_p^s(\mathbb{R}^d)$$

and a positive constant $C = C(p,d,s)$ such that for any $\mathbf{f} \in \mathring{\mathcal{X}}_p^s(\mathbb{R}^d_+)$, and $\tilde{\mathbf{f}} = E\mathbf{f}$ we have that $\tilde{\mathbf{f}} = \mathbf{f}$ a.e. in $\mathbb{R}^d_+$, $\tilde{\mathbf{f}} \in \mathcal{X}_p^s(\mathbb{R}^d)$ and

$$|\tilde{\mathbf{f}}|_{\mathcal{X}_p^s(\mathbb{R}^d)} \leq C|\mathbf{u}|_{\mathcal{X}_p^s(\mathbb{R}^d_+)}.$$

The theorem is proved in [57]. We emphasize that the proof of the above extension theorem is nontrivial as the commonly-used reflection across the hyperplane $x_d = 0$ would not preserve the semi-norm $|\cdot|_{\mathcal{X}_p^s}$. Extending by zero is also not appropriate, since it is not clear how to control the norm of the extended function. Rather we use an extension operator that has been used by J. A. Nitsche in [69] in his simple proof of Korn's second inequality. In showing the boundedness of the extension operator with respect to the semi-norm $|\cdot|_{\mathcal{X}_p^s}$ we need to first establish the fractional Hardy-type inequality. See [57] for more details. With this extension at hand the proof of the result below is standard.

Proposition 4.13. *For $s \in (0,1)$, $1 < p < \infty$, $sp \neq 1$, there exists a constant $C = C(d,p,s) > 0$ such that for any $\mathbf{f} \in \mathring{\mathcal{X}}_p^s(\mathbb{R}^d_+)$ it holds that*

$$\int_{\mathbb{R}^d_+}\int_{\mathbb{R}^d_+} \frac{|\mathbf{f}(\mathbf{x}) - \mathbf{f}(\mathbf{y})|^p}{|\mathbf{x}-\mathbf{y}|^{d+sp}} \mathrm{d}\mathbf{x}\, \mathrm{d}\mathbf{y} \leq C \int_{\mathbb{R}^d_+}\int_{\mathbb{R}^d_+} \frac{\left|(\mathbf{f}(\mathbf{x}) - \mathbf{f}(\mathbf{y})) \cdot \frac{\mathbf{x}-\mathbf{y}}{|\mathbf{x}-\mathbf{y}|}\right|^p}{|\mathbf{x}-\mathbf{y}|^{d+sp}} \mathrm{d}\mathbf{x}\, \mathrm{d}\mathbf{y}\,.$$

In future work we hope to report on the natural next step of establishing the equivalence of the spaces $\mathcal{X}_p^s(\Omega)$ and $W^{s,p}(\Omega;\mathbb{R}^d)$ defined over domains with sufficiently regular boundary.

4.1.5 The Fourier Transform of the Poisson-Type Kernel $\mathbb{P}_t$

Here we obtain the Fourier transform of the Poisson-type kernel $\mathbb{P}_t$ that has been used to establish relations between various Poisson integrals. Recall that the $(d+1)\times(d+1)$ matrix $\mathbb{P}_t(\mathbf{x}) = (\mathfrak{p}_t^{jk})$ has the form

$$\begin{bmatrix} \widetilde{\mathbb{P}}_t(\mathbf{x}) & \mathbf{P}_t^{[d+1]}(\mathbf{x}) \\ \mathbf{P}_t^{[d+1]}(\mathbf{x}) & \mathfrak{p}_t^{d+1,d+1}(\mathbf{x}) \end{bmatrix}.$$

Here, $\widetilde{\mathbb{P}}_t(\mathbf{x}) : \mathbb{R}^d \to \mathbb{M}_d(\mathbb{R})$ is a $d\times d$ matrix function given by

$$\widetilde{\mathbb{P}}_t(\mathbf{x}) = \frac{2(d+1)}{\omega_d}\frac{t}{(|\mathbf{x}|^2+t^2)^{\frac{d+3}{2}}}\mathbf{x}\otimes\mathbf{x}\,.$$

The function $\mathbf{P}_t^{d+1} : \mathbb{R}^d \to \mathbb{R}^d$ is a vector valued function, which we consider both a row and column vector, given by

$$\mathbf{P}_t^{[d+1]}(\mathbf{x}) = \frac{2(d+1)}{\omega_d} \frac{t^2 \mathbf{x}}{(|\mathbf{x}|^2 + t^2)^{\frac{d+3}{2}}} .$$

Finally the $(d+1) \times (d+1)$ entry is given by the function $\mathfrak{p}_t^{d+1,d+1} : \mathbb{R}^d \to \mathbb{R}$ defined as

$$\mathfrak{p}_t^{d+1,d+1}(\mathbf{x}) = \frac{2(d+1)}{\omega_d} \frac{t^3}{(|\mathbf{x}|^2 + t^2)^{\frac{d+3}{2}}} .$$

We compute the Fourier transform of each of these functions and put those transforms together to obtain the Fourier transform of $\mathbb{P}_t$. We begin by writing some useful Fourier transform formulas. Rather than write the calculations explicitly in-line each time during a proof, we instead reference the formulas in their full generality. For completeness they are listed here and their proofs can be found in many textbooks, for example [66].

- Let $n \in \mathbb{N}$, $\lambda \in (0, n)$, $f_\lambda(\mathbf{x}) = |\mathbf{x}|^{-\lambda}$, $\mathbf{x} \in \mathbb{R}^n$. Then,

$$\mathcal{F}\left(f_\lambda\right)(\boldsymbol{\xi}) = \frac{\Gamma(\frac{n-\lambda}{2})}{\Gamma(\frac{\lambda}{2})} \pi^{\lambda - n/2} |\boldsymbol{\xi}|^{\lambda - n} .$$

- Let $n \in \mathbb{N}$, $n \geq 3$. Then for each j, $k \in \{1, 2, \ldots, n\}$ we have

$$\mathcal{F}^{-1}\left(\frac{\xi_j \xi_k}{|\boldsymbol{\xi}|^4}\right) = 4\pi^2 \left[\frac{1}{2(n-2)\omega_{n-1}} \cdot \frac{\delta_{jk}}{|\mathbf{x}|^{n-2}} - \frac{1}{2\omega_{n-1}} \cdot \frac{x_j x_k}{|\mathbf{x}|^n}\right] \qquad \text{in } \mathcal{S}'(\mathbb{R}^n)\,, \tag{4.19}$$

  and as a consequence we have that

$$\mathcal{F}\left(\frac{x_j x_k}{|\mathbf{x}|^n}\right) = \frac{1}{4\pi^2}\left[\omega_{n-1} \frac{\delta_{jk}}{|\boldsymbol{\xi}|^2} - 2\omega_{n-1} \frac{\xi_j \xi_k}{|\boldsymbol{\xi}|^4}\right] \qquad \text{in } \mathcal{S}'(\mathbb{R}^n)\,. \tag{4.20}$$

- For $a \in (0, \infty)$ and $x \in \mathbb{R}$, define $f(x) = \frac{1}{x^2 + a^2}$ and $g(x) = \frac{x}{x^2 + a^2}$. Then

$$\mathcal{F}(f)(\xi) = \frac{\pi}{a} \mathrm{e}^{-2\pi a |\xi|} \qquad \text{for every } \xi \in \mathbb{R} \text{ and in } \mathcal{S}'(\mathbb{R})\,, \tag{4.21}$$

and

$$\mathcal{F}(g)(\xi) = -\pi\imath\,\mathrm{sgn}(\xi)\,\mathrm{e}^{-2\pi a|\xi|} \quad \text{in } \mathcal{S}'(\mathbb{R})\,. \tag{4.22}$$

We assume throughout that $d \geq 2$.

Proposition 4.14. *For every $t > 0$, we have the following:*

1) $\mathcal{F}_{\mathbf{x}}(\widetilde{\mathbb{P}}_t)(\boldsymbol{\xi}) = \mathrm{e}^{-2\pi|\boldsymbol{\xi}|t}\,\mathbb{I}_d - (2\pi|\boldsymbol{\xi}|t)\mathrm{e}^{-2\pi|\boldsymbol{\xi}|t}\left(\frac{\boldsymbol{\xi}\otimes\boldsymbol{\xi}}{|\boldsymbol{\xi}|^2}\right)$ *in* $\mathcal{S}'(\mathbb{R}^d)\,.$

2) $\mathcal{F}_{\mathbf{x}}(\mathbf{P}_t^{[d+1]})(\boldsymbol{\xi}) = (2\pi|\boldsymbol{\xi}|t)\mathrm{e}^{-2\pi|\boldsymbol{\xi}|t}\left(-\imath\frac{\boldsymbol{\xi}}{|\boldsymbol{\xi}|}\right)$ *in* $\mathcal{S}'(\mathbb{R}^d)\,.$

3) $\mathcal{F}_{\mathbf{x}}(\mathfrak{p}_t^{d+1,d+1})(\boldsymbol{\xi}) = (1 + 2\pi|\boldsymbol{\xi}|t)\,\mathrm{e}^{-2\pi|\boldsymbol{\xi}|t}$ *in* $\mathcal{S}'(\mathbb{R}^d)\,.$

Proof of Item 1). Let $j,k \in \{1,2,\ldots,d\}$. Since $\widetilde{\mathbb{P}}_t = (\mathfrak{p}_t^{jk}) \in L^p(\mathbb{R}^d;\mathbb{M}_d(\mathbb{R}))$ for every $1 \leq p \leq \infty$ we have that $\mathfrak{p}_t^{jk} \in \mathcal{S}'(\mathbb{R}^d)$ for every $t > 0$. Thus, its Fourier transform is a well-defined object in $\mathcal{S}'(\mathbb{R}^d)$ and agrees with its Fourier transform as a function in $L^1(\mathbb{R}^d)$. The plan is to make use of partial Fourier transforms. Specifically, we will compute $\mathcal{F}_{\mathbf{x}}\left(\mathfrak{p}_t^{jk}\right)(\boldsymbol{\xi})$ in $\mathcal{S}'(\mathbb{R}^d)$ by using the Fourier transform of $\mathfrak{p}^{jk}$ in $\mathcal{S}'(\mathbb{R}^{d+1})$. To that end, we first compute the Fourier transform of $\mathfrak{p}_t^{jk}(\mathbf{x}) = \frac{2(d+1)}{\omega_d}\frac{x_jx_kt}{(|\mathbf{x}|^2+t^2)^{\frac{d+3}{2}}}$ in $\mathcal{S}'(\mathbb{R}^{d+1})$. We use several properties of the Fourier transform. We denote the Fourier variables in $\mathbb{R}^{d+1}$ by $(\boldsymbol{\xi},\eta)$.

Using the observation that $\mathfrak{p}_t^{jk}(\mathbf{x}) = -\frac{2}{\omega_d}\frac{\partial}{\partial t}\left(\frac{x_jx_k}{(|\mathbf{x}|^2+t^2)^{\frac{d+1}{2}}}\right)$, we have that

$$\mathcal{F}_{\mathbf{x},t}(\overline{\mathfrak{p}_t}^{jk})(\boldsymbol{\xi},\eta) = \frac{4\pi\imath\eta}{\omega_d}\mathcal{F}_{\mathbf{x},t}\left(\frac{x_jx_k}{(|\mathbf{x}|^2+t^2)^{\frac{d+1}{2}}}\right)$$

$$\begin{aligned} &= -\frac{4\pi\imath\eta}{\omega_d}\cdot\frac{1}{4\pi^2}\left[\omega_d\frac{\delta_{jk}}{|\boldsymbol{\xi}|^2+\eta^2} - 2\omega_d\frac{(\boldsymbol{\xi},\eta)_j(\boldsymbol{\xi},\eta)_k}{(|\boldsymbol{\xi}|^2+\eta^2)^2}\right] \\ &= -\frac{\imath}{\pi}\eta\left[\frac{\delta_{jk}}{|\boldsymbol{\xi}|^2+\eta^2} - 2\frac{(\boldsymbol{\xi},\eta)_j(\boldsymbol{\xi},\eta)_k}{(|\boldsymbol{\xi}|^2+\eta^2)^2}\right]. \end{aligned} \tag{4.23}$$

where in the second equality we have applied the Fourier transform formula (4.20) with $n = d+1$. By taking partial inverse Fourier transform in η and using the definition of $\mathbb{P}_t$, we see that for $j,k \in \{1,2,\ldots,d\}$

$$\mathcal{F}_{\mathbf{x}}(\mathfrak{p}_t{}^{jk})(\boldsymbol{\xi}) = \mathcal{F}_\eta^{-1}\big(\mathcal{F}_{\mathbf{x},t}(\mathfrak{p}_t{}^{jk})(\boldsymbol{\xi},\eta)\big)(t)$$

for every $\boldsymbol{\xi} \in \mathbb{R}^d \setminus \{0\}$ and for every $t > 0$. Our task is to compute the right hand side. Applying $\mathcal{F}_\eta^{-1}$ to the right hand side of (4.23) we obtain for $j, k \in \{1, 2, \ldots, d\}$ that

$$\begin{aligned}
&\mathcal{F}_\eta^{-1}\left(-\frac{\imath}{\pi}\left[\frac{\eta\delta_{jk}}{(|\boldsymbol{\xi}|^2+\eta^2)} - 2\frac{\eta\xi_j\xi_k}{(|\boldsymbol{\xi}|^2+\eta^2)^2}\right]\right)(t) \\
=&\frac{-\imath}{\pi}\mathcal{F}_\eta^{-1}\left(\frac{\eta\delta_{jk}}{(|\boldsymbol{\xi}|^2+\eta^2)}\right)(t) + \frac{2\imath}{\pi}\mathcal{F}_\eta^{-1}\left(\frac{\eta\xi_j\xi_k}{(|\boldsymbol{\xi}|^2+\eta^2)^2}\right)(t) \\
=&\frac{-\imath\delta_{jk}}{\pi}\mathcal{F}_\eta\left(\frac{-\eta}{|\boldsymbol{\xi}|^2+\eta^2}\right)(t) + \frac{2\imath\xi_j\xi_k}{\pi}\mathcal{F}_\eta\left(\frac{-\eta}{(|\boldsymbol{\xi}|^2+\eta^2)^2}\right)(t)
\end{aligned} \tag{4.24}$$

where in the in the second inequality we used that $\mathcal{F}(\mathcal{F}(f))(\mathbf{x}) = f(-\mathbf{x})$. We will use the formula (4.22) to compute and simplify the first term of (4.24) as

$$\frac{-i\delta_{jk}}{\pi}\mathcal{F}_\eta\left(\frac{-\eta}{|\boldsymbol{\xi}|^2+\eta^2}\right)(t) = \frac{i\delta_{jk}}{\pi}\left(-\pi i\,\mathrm{sgn}(t)\mathrm{e}^{-2\pi|\boldsymbol{\xi}||t|}\right) = \delta_{jk}\,\mathrm{sgn}(t)\mathrm{e}^{-2\pi|\boldsymbol{\xi}||t|}$$

The second term in the expression (4.24) can be computed using (4.21) and can be simplified as

$$\begin{aligned}
\frac{2i\xi_j\xi_k}{\pi}\mathcal{F}_\eta\left(\frac{-\eta}{(|\boldsymbol{\xi}|^2+\eta^2)^2}\right)(t) &= \frac{2i\xi_j\xi_k}{\pi}\mathcal{F}_\eta\left(\frac{\mathrm{d}}{\mathrm{d}\eta}\left(\frac{1}{2(|\boldsymbol{\xi}|^2+\eta^2)}\right)\right)(t) \\
&= \frac{2i\xi_j\xi_k}{\pi}\cdot\frac{(2\pi i t)}{2}\mathcal{F}_\eta\left(\frac{1}{|\boldsymbol{\xi}|^2+\eta^2}\right)(t) \\
&= -2\xi_j\xi_k t\left(\frac{\pi}{|\boldsymbol{\xi}|}\mathrm{e}^{-2\pi|\boldsymbol{\xi}||t|}\right) \\
&= -\frac{\xi_j\xi_k}{|\boldsymbol{\xi}|^2}(2\pi|\boldsymbol{\xi}|t)\mathrm{e}^{-2\pi|\boldsymbol{\xi}||t|}.
\end{aligned}$$

In the above calculation we have use the fact that $|\boldsymbol{\xi}|$, $\frac{1}{|\boldsymbol{\xi}|}$ are in $\mathcal{S}'(\mathbb{R}^d)$ and that for any multi-index α and any number $k < |\alpha|$ the quantity $\frac{(\boldsymbol{\xi})^\alpha}{|\boldsymbol{\xi}|^k}\mathrm{e}^{-|\boldsymbol{\xi}|} \in L^1(\mathbb{R}^d)$, hence in $\mathcal{S}'(\mathbb{R}^d)$. Finally, plugging the last expressions into (4.24) we obtain that for each $j, k \in \{1, 2, \ldots, d\}$

$$\mathcal{F}(\mathfrak{p}_t)^{jk}(\boldsymbol{\xi}) = \mathrm{e}^{-2\pi|\boldsymbol{\xi}|t}\delta_{jk} - (2\pi|\boldsymbol{\xi}'|t)\mathrm{e}^{-2\pi|\boldsymbol{\xi}|t}\,\frac{\xi_j\xi_k}{|\boldsymbol{\xi}|^2}$$

for every $\boldsymbol{\xi} \in \mathbb{R}^d \setminus \{0\}$ and $t > 0$. $\square$

Proof of item 2). This proof is much the same as the last one. Let $j \in \{1, 2, \dots, d\}$. As before, we compute the Fourier transform of $\mathfrak{p}_t^{j,d+1}(\mathbf{x}) = \frac{2(d+1)}{\omega_d} \frac{x_j t^2}{(|\mathbf{x}|^2+t^2)^{\frac{d+3}{2}}}$ in $\mathcal{S}'(\mathbb{R}^d)$. First notice again that

$$\mathfrak{p}_t^{j,d+1}(\mathbf{x}) = \frac{\partial}{\partial_{x_j}} \left[\frac{-2t^2}{\omega_d (|\mathbf{x}|^2 + t^2)^{\frac{d+1}{2}}} \right]$$

Now we can use (4.20) with $n = d + 1$ to obtain

$$\begin{aligned} \mathcal{F}_{\mathbf{x},t}(\mathfrak{p}_t^{j,d+1})(\boldsymbol{\xi}) &= \frac{-2}{\omega_d} \cdot (2\pi\imath\xi_j) \mathcal{F}_{\mathbf{x},t} \left(\frac{t^2}{(|\mathbf{x}|^2 + t^2)^{\frac{d+1}{2}}} \right) \\ &= -\frac{4\pi\imath\xi_j}{\omega_d} \cdot \frac{1}{4\pi^2} \left[\omega_d \frac{1}{|\boldsymbol{\xi}|^2 + \eta^2} - 2\omega_d \frac{\eta^2}{(|\boldsymbol{\xi}|^2 + \eta^2)^2} \right] \\ &= -\frac{\imath}{\pi}\xi_j \left[\frac{1}{|\boldsymbol{\xi}|^2 + \eta^2} - 2\frac{\eta^2}{(|\boldsymbol{\xi}|^2 + \eta^2)^2} \right] . \end{aligned} \tag{4.25}$$

Again taking partial Fourier transforms we see that

$$\mathcal{F}(\mathfrak{p}_t^{j,d+1})(\boldsymbol{\xi}) = \mathcal{F}_\eta^{-1}\big(\mathcal{F}_{\mathbf{x},t}(\mathfrak{p}^{j,d+1})(\boldsymbol{\xi}, \eta)\big)(t)$$

for every $\boldsymbol{\xi} \in \mathbb{R}^d \setminus \{0\}$ and for every $t > 0$. Our task is to compute the right hand side. Applying $\mathcal{F}_\eta^{-1}$ to the right hand side of (4.25) we obtain

$$\begin{aligned} &\mathcal{F}_\eta^{-1} \left(-\frac{\imath}{\pi} \left[\frac{\xi_j}{(|\boldsymbol{\xi}|^2 + \eta^2)} - 2\frac{\xi_j \eta^2}{(|\boldsymbol{\xi}|^2 + \eta^2)^2} \right] \right) \\ =&\frac{-\imath}{\pi} \mathcal{F}_\eta^{-1} \left(\frac{\xi_j}{(|\boldsymbol{\xi}|^2 + \eta^2)} \right) + \frac{2\imath}{\pi} \mathcal{F}_\eta^{-1} \left(\frac{\eta^2 \xi_j}{(|\boldsymbol{\xi}|^2 + \eta^2)^2} \right) \\ =&\frac{-\imath\xi_j}{\pi} \mathcal{F}_\eta \left(\frac{1}{|\boldsymbol{\xi}|^2 + \eta^2} \right) + \frac{2i\xi_j}{\pi} \mathcal{F}_\eta \left(\frac{\eta^2}{(|\boldsymbol{\xi}|^2 + \eta^2)^2} \right) \\ =&\frac{-\imath\xi_j}{\pi} \left(\frac{\pi}{|\boldsymbol{\xi}|} \mathrm{e}^{-2\pi|\boldsymbol{\xi}||t|} \right) + \frac{2\imath\xi_j}{\pi} \frac{1}{-2\pi\imath} \frac{\mathrm{d}}{\mathrm{d}t} \mathcal{F}_\eta \left(\frac{\eta}{(|\boldsymbol{\xi}|^2 + \eta^2)^2} \right) . \end{aligned} \tag{4.26}$$

In the third equality we used the identities (4.21) and (4.22). We also used the identity $\mathcal{F}(xf)(\xi) = -\frac{1}{2\pi i}\frac{\mathrm{d}}{\mathrm{d}\xi}\mathcal{F}(f)(\xi)$. Now, just as in the proof of item 1) we can show that

$$\mathcal{F}_\eta \left(\frac{\eta}{(|\boldsymbol{\xi}'|^2 + \eta^2)^2} \right)(t) = -\pi\imath t \left(\frac{\pi}{|\boldsymbol{\xi}'|} \mathrm{e}^{-2\pi|\boldsymbol{\xi}'||t|} \right) . \tag{4.27}$$

Now putting together (4.26) and (4.27) we obtain that

$$\begin{aligned}
\mathcal{F}(\mathfrak{p}_t^{j,d+1})(\boldsymbol{\xi}) &= -\imath\frac{\xi_j}{|\boldsymbol{\xi}'|}\mathrm{e}^{-2\pi|\boldsymbol{\xi}||t|} - \frac{\xi_j}{\pi^2}\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{-\pi^2\imath t}{|\boldsymbol{\xi}'|}\mathrm{e}^{-2\pi|\boldsymbol{\xi}||t|}\right) \\
&= -\imath\frac{\xi_j}{|\boldsymbol{\xi}|}\mathrm{e}^{-2\pi|\boldsymbol{\xi}||t|} + \imath\frac{\xi_j}{|\boldsymbol{\xi}|}\left(\mathrm{e}^{-2\pi|\boldsymbol{\xi}||t|} - (2\pi|\boldsymbol{\xi}|t)\mathrm{e}^{-2\pi|\boldsymbol{\xi}||t|}\right) \\
&= (2\pi|\boldsymbol{\xi}|t)\mathrm{e}^{-2\pi|\boldsymbol{\xi}|t}\left(-\imath\frac{\xi_j}{|\boldsymbol{\xi}|}\right).
\end{aligned}$$

□

Proof of item 3). Writing $\frac{t^3}{(|\mathbf{x}|^2+t^2)^{\frac{d+3}{2}}} = \frac{t}{(|\mathbf{x}|^2+t^2)^{\frac{d+1}{2}}} - \sum_{j=1}^d \frac{tx_j^2}{(|\mathbf{x}|^2+t^2)^{\frac{d+3}{2}}}$. we notice that

$$\mathfrak{p}_t^{d+1,d+1}(\mathbf{x}) = \frac{2(d+1)}{\omega_d}\frac{t}{(|\mathbf{x}|^2+t^2)^{\frac{d+1}{2}}} - \sum_{j=1}^d \mathfrak{p}_t^{jj}(\mathbf{x}).$$

That is, $\mathfrak{p}_t^{d+1,d+1}(\mathbf{x}) = (d+1)p_t(\mathbf{x}) - \sum_{j=1}^d \mathfrak{p}_t^{jj}(\mathbf{x})$. Taking the Fourier Transform on both sides we get

$$\begin{aligned}
\mathcal{F}_{\mathbf{x}}(\mathfrak{p}_t^{d+1,d+1}) &= (d+1)\mathrm{e}^{-2\pi|\boldsymbol{\xi}|t} - \sum_{j=1}^d\left(\mathrm{e}^{-2\pi|\boldsymbol{\xi}|t} - (2\pi|\boldsymbol{\xi}|t)\mathrm{e}^{-2\pi|\boldsymbol{\xi}|t}\frac{\xi_j^2}{|\boldsymbol{\xi}|^2}\right) \\
&= (d+1)\mathrm{e}^{-2\pi|\boldsymbol{\xi}|t} - d\,\mathrm{e}^{-2\pi|\boldsymbol{\xi}|t} + (2\pi|\boldsymbol{\xi}|t)\mathrm{e}^{-2\pi|\boldsymbol{\xi}|t} \\
&= (1+2\pi|\boldsymbol{\xi}|t)\mathrm{e}^{-2\pi|\boldsymbol{\xi}'|t},
\end{aligned}$$

as desired. □

4.2 A New Characterization of Potential Spaces

4.2.1 Introduction

Recall that we have defined the function space $\mathcal{D}^p_s(\mathbb{R}^d)$ as the closure of the Schwarz space $\mathcal{S}(\mathbb{R}^d;\mathbb{R}^d)$ in the norm $\|\mathbf{u}\|_{L^p(\mathbb{R}^d)} + \|D^s(\mathbf{u})\|_{L^p(\mathbb{R}^d)}$, where

$$D^s(\mathbf{u})(\mathbf{x}) = \left(\int_{\mathbb{R}^d} \frac{\left|(\mathbf{u}(\mathbf{x}) - \mathbf{u}(\mathbf{y}))\cdot \frac{(\mathbf{x}-\mathbf{y})}{|\mathbf{x}-\mathbf{y}|}\right|^2}{|\mathbf{x}-\mathbf{y}|^{d+2s}} d\mathbf{y}\right)^{\frac{1}{2}}.$$

The function $D^s(\mathbf{u})$ is related to the energy space $W^{s,2}(\mathbb{R}^d;\mathbb{R}^d)$. A vector field $\mathbf{u} \in L^2(\mathbb{R}^d;\mathbb{R}^d)$ will belong to $W^{s,2}(\mathbb{R}^d;\mathbb{R}^d)$ if and only if $D^s(\mathbf{u}) \in L^2(\mathbb{R}^d)$. This can easily be seen using Theorem 4.1.

We recall also the space of Bessel potentials, $\mathcal{L}^q_s$. For $0 < s < 1$, $1 < q < \infty$

$$\mathcal{L}^q_s(\mathbb{R}^d) := \left\{ \mathbf{f} \in \mathcal{S}'(\mathbb{R}^d;\mathbb{R}^d) \,:\, \left(\left(1+4\pi^2|\boldsymbol{\xi}|^2\right)^{s/2}\widehat{\mathbf{f}}\right)^{\vee} \in L^q(\mathbb{R}^d;\mathbb{R}^d)\right\},$$

where $\mathcal{S}'(\mathbb{R}^d;\mathbb{R}^d)$ is the space of vector-valued tempered distributions. The norm in $\mathcal{L}^q_s(\mathbb{R}^d)$ is given by

$$\|\mathbf{f}\|_{\mathcal{L}^q_s(\mathbb{R}^d)} := \left\| \left(\left(1+4\pi^2|\boldsymbol{\xi}|^2\right)^{s/2}\widehat{\mathbf{f}}\right)^{\vee} \right\|_{L^q(\mathbb{R}^d)}. \tag{4.28}$$

Our second main result of the chapter is the characterization of functions in the potential space $\mathcal{L}^q_s$ in terms of the integrability of the function $D^s(\mathbf{f})$. Throughout this section we use the following standard notations for the fractional Sobolev exponents and its Hölder conjugate (denoted by the prime notation):

$$2^{*_s} = \frac{2d}{d-2s}, \quad \text{and} \quad (2^{*_s})' := 2_{*_s} = \frac{2d}{d+2s}.$$

Theorem 4.15. *Let $2_{*_s} \le p < \infty$ and $0 < s < 1$. Then $\mathbf{f} \in \mathcal{L}^p_s(\mathbb{R}^d)$ if and only if $\mathbf{f} \in \mathcal{D}^p_s(\mathbb{R}^d)$. Moreover, there exist positive constants C_1 and C_2 such that for any $\mathbf{f} \in L^p(\mathbb{R}^d;\mathbb{R}^d)$,*

$$C_1\|\mathbf{f}\|_{\mathcal{L}^p_s(\mathbb{R}^d)} \le \|\mathbf{f}\|_{L^p(\mathbb{R}^d)} + \|D^s(\mathbf{u})\|_{L^p(\mathbb{R}^d)} \le C_2\|\mathbf{f}\|_{\mathcal{L}^p_s(\mathbb{R}^d)}.$$

The main objective of this section is to prove Theorem 4.15. Our proof follows the steps presented in the proof of [85, Theorem 1]. We first develop necessary technical tools that allows us to relate the Marcinkiewicz-type integral $D^s(\mathbf{f})$ with the potential function of $\mathbf{f}$.

4.2.2 The Poisson-Type Kernel and Integral

We recall the standard Poisson kernel $p_t(\mathbf{y})$ and the modified Poisson-type kernel $\mathbb{P}_t(\mathbf{y})$ defined in Section 4.1. The properties that we will need are restated here for convenience. First, the Fourier transforms of both kernels are given by

$$\widehat{p_t}(\boldsymbol{\xi}) = \mathrm{e}^{-2\pi|\boldsymbol{\xi}|t}\,, \qquad \widehat{\mathbb{P}}_t(\boldsymbol{\xi}) = \mathrm{e}^{-2\pi|\boldsymbol{\xi}|t}\left(\mathbb{I}_{d+1} + (2\pi|\boldsymbol{\xi}|t)\begin{bmatrix} -\frac{\boldsymbol{\xi}\otimes\boldsymbol{\xi}}{|\boldsymbol{\xi}|^2} & -\imath\frac{\boldsymbol{\xi}}{|\boldsymbol{\xi}|} \\ -\imath\frac{\boldsymbol{\xi}}{|\boldsymbol{\xi}|} & 1\end{bmatrix}\right).$$

Notice that $\mathbb{P}_t(\mathbf{y})$ is a $(d+1)\times(d+1)$ matrix of functions which is explicitly given by the formula

$$\mathbb{P}_t(\mathbf{x}) = \frac{2(d+1)}{\omega_d}\frac{t}{(|\mathbf{x}|^2+t^2)^{\frac{d+3}{2}}}\begin{bmatrix}\mathbf{x}\otimes\mathbf{x} & t\mathbf{x} \\ t\mathbf{x} & t^2\end{bmatrix} \tag{4.29}$$

where $\mathbf{x}$ is considered both a column and row d-vector. Next, the matrix kernel $\mathbb{P}_t$ is in fact an approximation to the identity. For any $t>0$, if $\mathbb{I}_{d+1}$ denotes the $(d+1)\times(d+1)$ identity matrix, then

$$\int_{\mathbb{R}^d}\mathbb{P}_t(\mathbf{x})\,\mathrm{d}\mathbf{x} = \int_{\mathbb{R}^d}\mathbb{P}(\mathbf{x})\,\mathrm{d}\mathbf{x} = \mathbb{I}_{d+1}\,. \tag{4.30}$$

Moreover, for each j, k, and $\ell\in\{1,\dots,d,d+1\}$ and for every $t>0$ we have that $\partial_t\mathfrak{p}_t^{jk}(\mathbf{x})\in L^1(\mathbb{R}^d)$ and $\partial_{x_\ell}\mathfrak{p}_t^{jk}(\mathbf{x})\in L^1(\mathbb{R}^d)$. We also have the following pointwise estimates: there exists a constant $c=c(d)>0$ such that for any $j,k=1,2,\dots d+1$,

$$|\partial_t\mathfrak{p}_t^{jk}(\mathbf{x})|\le c\,|\mathbf{x}|^{-d-1}, \quad |\partial_t\mathfrak{p}_t^{jk}(\mathbf{x})|\le c\,t^{-d-1}, \quad \forall\,\mathbf{x}\in\mathbb{R}^d, \quad t>0.$$

In addition, one can easily establish $|\partial_{tt}\mathbb{P}_t(\mathbf{x})|\le\frac{C}{t^{d+2}}$ and $|\partial_{tt}\mathbb{P}_t(\mathbf{x})|\le\frac{C}{|\mathbf{x}|^{d+2}}$.

Throughout, functions $\mathbf{f}:\mathbb{R}^d\to\mathbb{R}^{d+1}$ are of the form $\mathbf{f}=(f_1,f_2,\dots,f_d,0)$. We treat $\mathbf{f}$ both as being a vector field in $\mathbb{R}^d$ and $\mathbb{R}^{d+1}$, as well as column and row vectors; it will be clear from context.

We recall the Poisson integral of $\mathbf{f}$ given by $\mathbf{u}^{\Delta}(\mathbf{x},t) := p_t * \mathbf{f}(\mathbf{x})$, where the convolution is component-wise. A Poisson-type integral of $\mathbf{f}$ can now be naturally defined using the Poisson-type kernel $\mathbb{P}_t = (\mathfrak{p}_t^{ij})$ as

$$\mathbf{U}(\mathbf{x},t) := \mathbb{P}_t * \mathbf{f}(\mathbf{x})\,.$$

The convolution in the above is taken in the sense of matrix multiplication. That is, the i^{th} entry component of $\mathbf{U}$ is given by $U_i = \sum_{j=1}^{d+1} \mathfrak{p}_t^{ij} * f_j$, where $\mathbb{P}_t = (\mathfrak{p}_t^{ij})$. Notice that taking the Fourier transform in $\mathbf{x}$ transforms the convolution into the matrix multiplication

$$\widehat{\mathbf{U}}(\xi,t) = \widehat{\mathbb{P}}_t(\boldsymbol{\xi})\widehat{\mathbf{f}}(\boldsymbol{\xi})\,.$$

The key connection of $\mathbb{P}_t$ with $D^s(\mathbf{f})$ is obtained through the following important relation that we will be using below. For any $\mathbf{z},\mathbf{x}\in\mathbb{R}^d$, we have

$$\mathbb{P}_t(\mathbf{x})\left(\begin{bmatrix}\mathbf{z}\\ 0\end{bmatrix}\right) = \overline{\mathbf{P}}(\mathbf{x},t)\left(\mathbf{z}\cdot\frac{\mathbf{x}}{|\mathbf{x}|}\right)\,, \qquad \mathbf{z}\in\mathbb{R}^d\,,$$

where the vector function $\overline{\mathbf{P}}(\mathbf{x},t)$ is given by

$$\overline{\mathbf{P}}(\mathbf{x},t) := \frac{2(d+1)}{\omega_d}\frac{t|\mathbf{x}|}{(|\mathbf{x}|^2+t^2)^{\frac{d+3}{2}}}\left(\begin{bmatrix}\mathbf{x}\\ t\end{bmatrix}\right)\,. \tag{4.31}$$

In particular, $\overline{\mathbf{P}}(\mathbf{x},t)$ and all its derivatives in t satisfy the same estimates as $\mathbb{P}_t(\mathbf{x})$ and corresponding derivatives. Using this relation and (4.30), we see that

$$\begin{aligned}\mathbf{U}(\mathbf{x},t) &= \mathbf{f}(\mathbf{x}) + \int_{\mathbb{R}^d}\mathbb{P}_t(\mathbf{y})(\mathbf{f}(\mathbf{x}+\mathbf{y})-\mathbf{f}(\mathbf{x}))\,\mathrm{d}\mathbf{y}\\ &= \mathbf{f}(\mathbf{x}) + \int_{\mathbb{R}^d}\overline{\mathbf{P}}(\mathbf{y},t)(\mathbf{f}(\mathbf{x}+\mathbf{y})-\mathbf{f}(\mathbf{x}))\cdot\frac{\mathbf{y}}{|\mathbf{y}|}\,\mathrm{d}\mathbf{y}\,.\end{aligned}$$

4.2.3 Littlewood-Paley-type g-function

We can define the analogue of the classical Littlewood-Paley g-function corresponding to the new Poisson-type integral $\mathbf{U}(\mathbf{x}, t)$. The following definition is natural:

$$\mathring{\mathfrak{g}}_1(\mathbf{f})(\mathbf{x}) := \left(\int_0^\infty t \left| \partial_t \mathbf{U}(\mathbf{x}, t) \right|^2 \, \mathrm{d}t \right)^{\frac{1}{2}} . \tag{4.32}$$

In what follows we denote $\nabla = (\nabla_{\mathbf{x}}, \partial_t)$, and $\left|\nabla \mathbf{U}(\mathbf{x}, t)\right|^2 = |\partial_t \mathbf{U}|^2 + \sum_{k=1}^{d+1} |\nabla_{\mathbf{x}} U_k|^2$. We can use these functions to characterize the L^p norm of a vector field. The following is a result similar to [86, Theorem 1, Chapter IV].

Theorem 4.16. *Suppose that $1 < p < \infty$. Then there are constants C_1, C_2 such that for any $\mathbf{f} \in L^p(\mathbb{R}^d)$*

$$C_1 \left\| \mathring{\mathfrak{g}}_1(\mathbf{f}) \right\|_{L^p(\mathbb{R}^d)} \le \left\| \mathbf{f} \right\|_{L^p(\mathbb{R}^d)} \le C_2 \left\| \mathring{\mathfrak{g}}_1(\mathbf{f}) \right\|_{L^p(\mathbb{R}^d)}$$

To prove the theorem we follow the steps and the approach given in [86] for the proof of [86, Theorem 1, Chapter IV]. We first prove the theorem for $p = 2$.

Lemma 4.17. *Let $\mathbf{f} \in L^2(\mathbb{R}^d)$. Then $\mathring{\mathfrak{g}}_1(\mathbf{f}) \in L^2(\mathbb{R}^d)$ with*

$$\left\| \mathring{\mathfrak{g}}_1(\mathbf{f}) \right\|_{L^2(\mathbb{R}^d)}^2 = \frac{1}{4} \int_{\mathbb{R}^d} \left| \left(\mathbb{I}_d + \frac{\boldsymbol{\xi} \otimes \boldsymbol{\xi}}{|\boldsymbol{\xi}|^2} \right) \widehat{\mathbf{f}}(\boldsymbol{\xi}) \right|^2 \, \mathrm{d}\boldsymbol{\xi} . \tag{4.33}$$

The proof is tedious but elementary. It is given in Section 4.2.5. In the next proposition we prove one of the inequalities in Theorem 4.16.

Proposition 4.18. *Let $1 < p < \infty$. If $\mathbf{f} \in L^p(\mathbb{R}^d)$, then $\mathring{\mathfrak{g}}_1(\mathbf{f}) \in L^p(\mathbb{R}^d)$. Moreover, there exists a positive constant depending only on p and d such that for all $\mathbf{f} \in L^p(\mathbb{R}^d)$,*

$$\left\| \mathring{\mathfrak{g}}_1(\mathbf{f}) \right\|_{L^p(\mathbb{R}^d)} \le C \left\| \mathbf{f} \right\|_{L^p(\mathbb{R}^d)} . \tag{4.34}$$

Proof. We use the theory of singular integrals for Hilbert space-valued functions outlined in [86, Chapter II, Section 5]. We use the notation of that section as well. Define the Hilbert space $\mathscr{H}$ to be the L^2 space on $(0, \infty)$ with the functions taking values in $\mathbb{R}^d$ with measure

$t\,\mathrm{d}t$, i.e.

$$\mathscr{H} := \left\{ \mathbf{h} : (0,\infty) \to \mathbb{R}^d \,\middle|\, \|\mathbf{h}\|_{\mathscr{H}}^2 := \int_0^\infty t\,|\mathbf{h}(t)|^2\,\mathrm{d}t < \infty \right\}.$$

The absolute value $|\mathbf{h}|$ is the norm in $\mathbb{R}^d$. Denote the Banach space of bounded linear operators from $\mathbb{R}^d$ to $\mathscr{H}$ by $B(\mathbb{R}^d, \mathscr{H})$. Let $\varepsilon > 0$ be fixed for now. For each $\mathbf{x}$ consider the matrix-valued function

$$\mathcal{K}_\varepsilon(\mathbf{x}, t) := \partial_t \mathbb{P}_{t+\varepsilon}(\mathbf{x}).$$

Then for any fixed $\mathbf{x} \in \mathbb{R}^d$, we identify the matrix function $\mathcal{K}_\varepsilon(\mathbf{x}, \cdot)$ by $\mathcal{K}_\varepsilon(\mathbf{x})$. Now we show that $\mathcal{K}_\varepsilon(\mathbf{x}) \in B(\mathbb{R}^d, \mathscr{H})$. This is equivalent to showing that the integral $\int_0^\infty t\,|\partial_t \mathbb{P}_{t+\varepsilon}(\mathbf{x})|^2\,\mathrm{d}t$ is finite. Indeed, from the formula (4.29) we have that $|\partial_t \mathbb{P}_t(\mathbf{x})| \le \frac{C}{(|\mathbf{x}|^2+t^2)^{\frac{d+1}{2}}}$, and therefore for each $\mathbf{x} \in \mathbb{R}^d$ we have the estimate after change of variables that

$$\begin{aligned}
\|\mathcal{K}_\varepsilon(\mathbf{x})\|_{B(\mathbb{R}^d,\mathscr{H})}^2 &= \sup_{|\mathbf{y}|\le 1} \|\langle \mathcal{K}_\varepsilon(\mathbf{x}), \mathbf{y}\rangle\|_{\mathscr{H}}^2 \\
&\le \sup_{|\mathbf{y}|\le 1} |\mathbf{y}| \int_0^\infty t|\partial_t \mathbb{P}_{t+\varepsilon}(\mathbf{x})|^2\,\mathrm{d}t \le C \int_0^\infty \frac{t}{(|\mathbf{x}|^2 + (t+\varepsilon)^2)^{d+1}} \mathrm{d}t \le C_\varepsilon
\end{aligned}$$

and

$$\|\mathcal{K}_\varepsilon(\mathbf{x})\|_{B(\mathbb{R}^d,\mathscr{H})}^2 \le C \int_0^\infty \frac{t}{(|\mathbf{x}|^2 + (t+\varepsilon)^2)^{d+1}}\,\mathrm{d}t \le \frac{1}{|\mathbf{x}|^{2d}} \int_0^\infty \frac{t}{(1+t^2)^{d+1}}\,\mathrm{d}t = \frac{C}{|\mathbf{x}|^{2d}}.$$

From the above two estimates we also conlcude that

$$\mathbf{x} \mapsto \|\mathcal{K}_\varepsilon(\mathbf{x})\|_{B(\mathbb{R}^d,\mathscr{H})} \in L^2(\mathbb{R}^d). \tag{4.35}$$

Similarly, for $1 \le j \le d$, again referring to (4.29) that

$$\begin{aligned}
\left\|\partial_{x_j} \mathcal{K}_\varepsilon(\mathbf{x})\right\|_{B(\mathbb{R}^d,\mathscr{H})}^2 &\le C \int_0^\infty \frac{t}{(|\mathbf{x}|^2 + (t+\varepsilon)^2)^{d+2}}\,\mathrm{d}t \\
&\le C \int_0^\infty \frac{t}{(|\mathbf{x}|^2 + t^2)^{d+2}}\,\mathrm{d}t \\
&= \frac{C}{|\mathbf{x}|^{2d+2}}.
\end{aligned}$$

Thus,

$$\left\|\partial_{x_j}\mathcal{K}_\varepsilon(\mathbf{x})\right\|_{B(\mathbb{R}^d,\mathscr{H})} \le \frac{C}{|\mathbf{x}|^{d+1}}\,, \qquad 1\le j\le d\,. \tag{4.36}$$

Now define the operator

$$T_\varepsilon(\mathbf{f})(\mathbf{x}) = \int_{\mathbb{R}^d} \mathcal{K}_\varepsilon(\mathbf{y})\mathbf{f}(\mathbf{x}-\mathbf{y})\,\mathrm{d}\mathbf{y}\,.$$

Notice that from the definition of $\mathcal{K}_\varepsilon(\mathbf{y})$, $T_\varepsilon(\mathbf{f})(\mathbf{x})$ in terms of the Poisson-type kernel $\mathbf{U}$ as $T_\varepsilon(\mathbf{f})(\mathbf{x}) = \partial_t\mathbf{U}(\mathbf{x},t+\varepsilon)$. It then follows that T_ε is a vector field since the integrand is a matrix multiplying a vector. In fact, $T_\varepsilon(\mathbf{f})(\mathbf{x})$ take their values in $\mathscr{H}$ for each $\mathbf{x}\in\mathbb{R}^d$. Moreover, we have

$$\begin{aligned}
\|T_\varepsilon(\mathbf{f})(\mathbf{x})\|^2_{\mathscr{H}} &= \int_0^\infty t\,|\partial_t\mathbf{U}(\mathbf{x},t+\varepsilon)|^2\,\mathrm{d}t \\
&= \int_0^\infty (t+\varepsilon)\,|\partial_t\mathbf{U}(\mathbf{x},t+\varepsilon)|^2\,\mathrm{d}t - \int_0^\infty \varepsilon\,|\partial_t\mathbf{U}(\mathbf{x},t+\varepsilon)|^2\,\mathrm{d}t \\
&= \int_\varepsilon^\infty t\,|\partial_t\mathbf{U}(\mathbf{x},t)|^2\,\mathrm{d}t - \varepsilon\int_\varepsilon^\infty |\partial_t\mathbf{U}(\mathbf{x},t)|^2\,\mathrm{d}t \\
&\le \int_0^\infty t\,|\partial_t\mathbf{U}(\mathbf{x},t)|^2\,\mathrm{d}t = \left[\mathring{\mathfrak{g}}_1(\mathbf{f})(\mathbf{x})\right]^2.
\end{aligned}$$

Therefore, by Lemma 4.17, $\mathbf{x}\mapsto\|T_\varepsilon(\mathbf{f})(\mathbf{x})\|_{\mathscr{H}}$ is square integrable and

$$\|T_\varepsilon(\mathbf{f})\|_{L^2_{\mathbf{x}}(\mathbb{R}^d)} \le \frac{1}{2}\left(\int_{\mathbb{R}^d}\left|\left(\mathbb{I}_d + \frac{\boldsymbol{\xi}\otimes\boldsymbol{\xi}}{|\boldsymbol{\xi}|^2}\right)\widehat{\mathbf{f}}(\boldsymbol{\xi})\right|^2\,\mathrm{d}\boldsymbol{\xi}\right)^{1/2} \le \|\mathbf{f}\|_{L^2(\mathbb{R}^d)}\,,$$

and thus, we obtain

$$\left\|\widehat{\mathcal{K}}_\varepsilon(\mathbf{x})\right\|_{B(\mathbb{R}^d;\mathscr{H})} \le 1\,. \tag{4.37}$$

Now using (4.35), (4.36) and (4.37) we can use the theory of singular integrals [86, Chapter II, Section 5] and conclude that

$$\|T_\varepsilon\mathbf{f}\|_{L^p(\mathbb{R}^d)} \le C\,\|\mathbf{f}\|_{L^p(\mathbb{R}^d)}\,, \qquad 1<p<\infty\,,$$

with C independent of ε. Notice from the above calculations that for each $\mathbf{x}$, the positive function $\|T_\varepsilon(\mathbf{f})(\mathbf{x})\|_{\mathscr{H}}$ increases to $\mathring{\mathfrak{g}}_1(\mathbf{f})(\mathbf{x})$ as $\varepsilon \to 0$ and therefore, we have that

$$\|\mathring{\mathfrak{g}}_1(\mathbf{f})\|_{L^p(\mathbb{R}^d)} \leq C \|\mathbf{f}\|_{L^p(\mathbb{R}^d)} , \qquad 1 < p < \infty .$$

That completes the proof. □

Next, we prove the reverse inequality in Theorem 4.16 by establishing a comparison of norms of operators with matrix symbols $\mathbb{I}_d$ and $\mathbb{I}_d + \frac{\boldsymbol{\xi}\otimes\boldsymbol{\xi}}{|\boldsymbol{\xi}|^2}$.

Lemma 4.19. *Let $\mathbf{f} \in L^p(\mathbb{R}^d;\mathbb{R}^d)$ for $1 < p < \infty$. For $\ell \in \{1, \dots, d\}$ denote the ℓ^{th} Riesz Transform by R_ℓ. Then the translation-invariant operator $L(\mathbf{f})$ defined by*

$$[L\mathbf{f}(\mathbf{x})]_k := f_k(\mathbf{x}) - 3R_k \left[\sum_{j=1}^d R_j f_j \right] (\mathbf{x}) , \qquad 1 \leq k \leq d ,$$

satisfies

$$\|\mathbf{f}\|_{L^p(\mathbb{R}^d)} \leq C \|L\mathbf{f}\|_{L^p(\mathbb{R}^d)} .$$

Proof. Clearly by the L^p boundedness of the Riesz transforms,

$$\|L\mathbf{f}\|_{L^p(\mathbb{R}^d)} \leq C \|\mathbf{f}\|_{L^p(\mathbb{R}^d)} ,$$

so the L^p norm of $L\mathbf{f}$ is finite. Applying the k^{th} Riesz transform to $(L\mathbf{f})_k$ and summing gives

$$\sum_{k=1}^d R_k (L\mathbf{f})_k = \sum_{k=1}^d \left(R_k f_k - 3R_k R_k \Big(\sum_{j=1}^d R_j f_j \Big) \right) = -2 \sum_{k=1}^d R_k f_k .$$

Thus, by the L^p boundedness of the Riesz transforms we have

$$\left\| \sum_{j=1}^d R_k f_k \right\|_{L^p(\mathbb{R}^d)} \leq C \|T\mathbf{f}\|_{L^p(\mathbb{R}^d)} . \tag{4.38}$$

Now write each component f_k as

$$f_k(\mathbf{x}) = f_k(\mathbf{x}) - 3R_k\left(\sum_{j=1}^{d} R_j f_j\right)(\mathbf{x}) + 3R_k\left(\sum_{j=1}^{d} R_j f_j\right)(\mathbf{x}),$$

and so taking the L^p norm on both sides and using the L^p boundedness of the Riesz transforms gives

$$\begin{aligned}
\|f_k\|_{L^p(\mathbb{R}^d)} &\le \left\| f_k - 3R_k\left(\sum_{j=1}^{d} R_j f_j\right)\right\|_{L^p(\mathbb{R}^d)} + \left\|3R_k\left(\sum_{j=1}^{d} R_j f_j\right)\right\|_{L^p(\mathbb{R}^d)} \\
&= \|(L\mathbf{f})_k\|_{L^p(\mathbb{R}^d)} + \left\|3R_k\left(\sum_{j=1}^{d} R_j f_j\right)\right\|_{L^p(\mathbb{R}^d)} \\
&\le C\,\|(L\mathbf{f})\|_{L^p(\mathbb{R}^d)} + C\left\|\sum_{j=1}^{d} R_j f_j\right\|_{L^p(\mathbb{R}^d)} \overset{(4.38)}{\le} C\,\|L\mathbf{f}\|_{L^p(\mathbb{R}^d)}\,.
\end{aligned}$$

Summing over k finishes the proof. □

Remark 4.20. The symbol associated to L is $\mathbb{I}_d + 3\frac{\boldsymbol{\xi}\otimes\boldsymbol{\xi}}{|\boldsymbol{\xi}|^2}$, which will appear in the proof of the reverse inequalities for $\mathring{\mathfrak{g}}_1$.

The next proposition proves the remaining inequality in Theorem 4.16.

Proposition 4.21. *Let $1 < p < \infty$. Then there exists a positive constant C such that for any $\mathbf{f} \in L^p(\mathbb{R}^d)$,*

$$\|\mathbf{f}\|_{L^p(\mathbb{R}^d)} \le C\,\|\mathring{\mathfrak{g}}_1(\mathbf{f})\|_{L^p(\mathbb{R}^d)}\,.$$

Proof. Let $\mathbf{f}_1$, $\mathbf{f}_2$ be in $L^2(\mathbb{R}^d;\mathbb{R}^d)$ with respective Poisson-type integrals $\mathbf{U}_1$, $\mathbf{U}_2$. Polarization of the identity (4.33) leads to

$$\begin{aligned}
&\int_0^\infty \int_{\mathbb{R}^d} t\,\langle \partial_t \mathbf{U}_1(\mathbf{x},t), \partial_t \mathbf{U}_2(\mathbf{x},t)\rangle d\mathbf{x}dt \\
&= \frac{1}{4}\int_{\mathbb{R}^d} \left\langle \left(\mathbb{I}_d + \frac{\boldsymbol{\xi}\otimes\boldsymbol{\xi}}{|\boldsymbol{\xi}|^2}\right)\widehat{\mathbf{f}_1}(\boldsymbol{\xi}), \left(\mathbb{I}_d + \frac{\boldsymbol{\xi}\otimes\boldsymbol{\xi}}{|\boldsymbol{\xi}|^2}\right)\widehat{\mathbf{f}_2}(\boldsymbol{\xi})\right\rangle \mathrm{d}\boldsymbol{\xi} \\
&= \frac{1}{4}\int_{\mathbb{R}^d} \left\langle \left(\mathbb{I}_d + 3\frac{\boldsymbol{\xi}\otimes\boldsymbol{\xi}}{|\boldsymbol{\xi}|^2}\right)\widehat{\mathbf{f}_1}(\boldsymbol{\xi}), \widehat{\mathbf{f}_2}(\boldsymbol{\xi})\right\rangle \mathrm{d}\boldsymbol{\xi} \\
&= \frac{1}{4}\int_{\mathbb{R}^d} \langle L\mathbf{f}_1(\mathbf{x}), \mathbf{f}_2(\mathbf{x})\rangle\, \mathrm{d}\mathbf{x}\,,
\end{aligned} \tag{4.39}$$

where the last inequality follows by Parseval's relation. Now suppose in addition that $\mathbf{f}_1 \in L^p(\mathbb{R}^d;\mathbb{R}^d)$ and $\mathbf{f}_2 \in L^{p'}(\mathbb{R}^d;\mathbb{R}^d)$ with $\|\mathbf{f}_2\|_{L^{p'}(\mathbb{R}^d)} \leq 1$. Then using the Cauchy-Schwarz inequality, Hölder's inequality and Proposition 4.18 we have

$$\begin{aligned}\left|\int_{\mathbb{R}^d} \langle L\mathbf{f}_1(\mathbf{x}), \mathbf{f}_2(\mathbf{x})\rangle \,\mathrm{d}\mathbf{x}\right| &\leq 4\int_{\mathbb{R}^d} \mathring{\mathfrak{g}}_1(\mathbf{f}_1)(\mathbf{x})\mathring{\mathfrak{g}}_1(\mathbf{f}_2)(\mathbf{x})\,\mathrm{d}\mathbf{x}\\ &\leq 4\left\|\mathring{\mathfrak{g}}_1(\mathbf{f}_1)\right\|_{L^p(\mathbb{R}^d)}\left\|\mathring{\mathfrak{g}}_1(\mathbf{f}_2)\right\|_{L^{p'}(\mathbb{R}^d)} \leq C\left\|\mathring{\mathfrak{g}}_1(\mathbf{f}_1)\right\|_{L^p(\mathbb{R}^d)}.\end{aligned}$$

Taking the supremum on both sides over all $\mathbf{f}_2 \in L^2(\mathbb{R}^d;\mathbb{R}^d)\cap L^{p'}(\mathbb{R}^d;\mathbb{R}^d)$ with $\|\mathbf{f}_2\|_{L^{p'}(\mathbb{R}^d)} \leq 1$ gives

$$\|L\mathbf{f}\|_{L^p(\mathbb{R}^d)} \leq C\left\|\mathring{\mathfrak{g}}_1(\mathbf{f})\right\|_{L^p(\mathbb{R}^d)} \tag{4.40}$$

for every $\mathbf{f} \in (L^2 \cap L^p)(\mathbb{R}^d)$. Using Lemma 4.19,

$$\|\mathbf{f}\|_{L^p(\mathbb{R}^d)} \leq C\left\|\mathring{\mathfrak{g}}_1(\mathbf{f})\right\|_{L^p(\mathbb{R}^d)} \tag{4.41}$$

for every $\mathbf{f} \in (L^2 \cap L^p)(\mathbb{R}^d)$. The passage to the general case $\mathbf{f} \in L^p$ follows by density. Let $\mathbf{f}_m$ be a sequence of functions in $(L^2 \cap L^p)(\mathbb{R}^d)$ which converges in L^p to an arbitrary function $\mathbf{f} \in L^p$. Then

$$\begin{aligned}\left|\mathring{\mathfrak{g}}_1(\mathbf{f}_m)(\mathbf{x}) - \mathring{\mathfrak{g}}_1(\mathbf{f})(\mathbf{x})\right|^2 &= \left|\left(\int_0^\infty t\,|\partial_t\mathbf{U}_m(\mathbf{x},t)|^2\,\mathrm{d}t\right)^{\frac{1}{2}} - \left(\int_0^\infty t\,|\partial_t\mathbf{U}(\mathbf{x},t)|^2\,\mathrm{d}t\right)^{\frac{1}{2}}\right|^2\\ &= \int_0^\infty t\,\left(|\partial_t\mathbf{U}_m(\mathbf{x},t)|^2 + |\partial_t\mathbf{U}(\mathbf{x},t)|^2\right)\,\mathrm{d}t\\ &\quad - 2\left(\int_0^\infty t\,|\partial_t\mathbf{U}_m(\mathbf{x},t)|^2\,\mathrm{d}t\right)^{\frac{1}{2}}\left(\int_0^\infty t\,|\partial_t\mathbf{U}(\mathbf{x},t)|^2\,\mathrm{d}t\right)^{\frac{1}{2}}.\end{aligned}$$

By Cauchy-Schwartz inequality the last expression cannot exceed

$$\int_0^\infty t\,\left(|\partial_t\mathbf{U}_m(\mathbf{x},t)|^2 + |\partial_t\mathbf{U}(\mathbf{x},t)|^2 - 2\left\langle\partial_t\mathbf{U}_m(\mathbf{x},t), \partial_t\mathbf{U}(\mathbf{x},t)\right\rangle\right)\mathrm{d}t\,,$$

and so

$$\left|\mathring{\mathfrak{g}}_1(\mathbf{f}_m)(\mathbf{x}) - \mathring{\mathfrak{g}}_1(\mathbf{f})(\mathbf{x})\right|^2 \leq \int_0^\infty t\,|\partial_t(\mathbf{U}_m - \mathbf{U})(\mathbf{x},t)|^2\,\mathrm{d}t = \left|\mathring{\mathfrak{g}}_1(\mathbf{f}_m - \mathbf{f})(\mathbf{x})\right|^2.$$

Therefore by Theorem 4.18, $\mathring{\mathfrak{g}}_1(\mathbf{f}_m)$ converges to $\mathring{\mathfrak{g}}_1(\mathbf{f})$ in $L^p(\mathbb{R}^d)$, so we obtain (4.41) for a general $\mathbf{f} \in L^p(\mathbb{R}^d)$ and the proof is complete. $\square$

Now we come to the final preliminary inequality that must be established before proving our main result. Recall that the Riesz potential $\mathcal{I}_s$ and the Bessel potential $\mathcal{J}_s$ acting on a function $\mathbf{f}$ are given by

$$\mathcal{I}_s(\mathbf{f})(\mathbf{x}) := c_{d,s} \int \frac{\mathbf{f}(\mathbf{y})}{|\mathbf{x}-\mathbf{y}|^{d-s}} \, \mathrm{d}\mathbf{y} \,, \tag{4.42}$$

$$\mathcal{J}_s(\mathbf{f})(\mathbf{x}) := (\widehat{\mathcal{G}_s}\, \widehat{\mathbf{f}})^{\vee} = \mathcal{G}_s * \mathbf{f} \,, \qquad \mathcal{G}_s(\mathbf{x}) := \big((1+4\pi^2|\boldsymbol{\xi}|^2)^{-s/2}\big)^{\vee}(\mathbf{x}) \,, \tag{4.43}$$

where $c_{d,s}$ is an appropriate normalizing constant.

Theorem 4.22. *Let $\mathbf{f} \in L^p(\mathbb{R}^d)$, $1 < p < \infty$ and let $0 < s < 1$. Denote $\mathbf{f}_s := \mathcal{I}_s(\mathbf{f})$. Let $\mathbf{U}$ and $\mathbf{U}_s$ be the Poisson-type integrals of $\mathbf{f}$, $\mathbf{f}_s$ respectively. Then for every $\mathbf{x} \in \mathbb{R}^d$ we have*

$$\mathring{\mathfrak{g}}_1(\mathbf{f})(\mathbf{x}) \le C\, D^s(\mathbf{f}_s)(\mathbf{x}) \,.$$

Proof. We first claim that

$$\partial_t \mathbf{U}(\mathbf{x},t) = \frac{-1}{\Gamma(1-s)} \int_0^\infty \partial_{tt} \mathbf{U}_s(\mathbf{x},t+r) r^{-s} \mathrm{d}r \,, \tag{4.44}$$

where Γ denotes the Gamma function. This identity can be established using the Fourier transform and will be done in Section 4.2.6. We will use this to estimate $\mathring{\mathfrak{g}}_1(\mathbf{f})$ pointwise. To that end, we write

$$\mathring{\mathfrak{g}}_1(\mathbf{f})(\mathbf{x}) = \int_0^\infty t\, |\partial_t \mathbf{U}(\mathbf{x},t)|^2 \, \mathrm{d}t = \int_0^\infty t \left| \frac{-1}{\Gamma(1-s)} \int_t^\infty \partial_{tt} \mathbf{U}_s(\mathbf{x},r)(r-t)^{-s} \, \mathrm{d}r \right|^2 \mathrm{d}t.$$

Dividing the intervals of integration in the inside integral, we see that

$$
\begin{aligned}
\mathring{\mathfrak{g}}_1(\mathbf{f})(\mathbf{x}) &\leq C \int_0^\infty t \left| \int_t^{2t} \partial_{tt} \mathbf{U}_s(\mathbf{x}, r)(r-t)^{-s} \, \mathrm{d}r \right|^2 \mathrm{d}t \\
&\quad + C \int_0^\infty t \left| \int_{2t}^\infty \partial_{tt} \mathbf{U}_s(\mathbf{x}, r)(r-t)^{\frac{1}{2}}(r-t)^{-\frac{1}{2}-s} \, \mathrm{d}r \right|^2 \mathrm{d}t \\
&\leq C \int_0^\infty t \left(\int_t^{2t} |\partial_{tt} \mathbf{U}_s(\mathbf{x}, r)|^2 (r-t)^{-s} \, \mathrm{d}r \right) \left(\int_t^{2t} \frac{1}{(r-t)^s} \, \mathrm{d}r \right) \mathrm{d}t \\
&\qquad + C \int_0^\infty t \left(\int_{2t}^\infty |\partial_{tt} \mathbf{U}_s(\mathbf{x}, r)|^2 (r-t) \, \mathrm{d}r \right) \left(\int_{2t}^\infty \frac{1}{(r-t)^{1+2s}} \, \mathrm{d}r \right) \mathrm{d}t,
\end{aligned}
$$

where we have used the Cauchy-Schwartz inequality in the last inequality. Simplification and interchanging of the integrals via Fubini implies that

$$
\begin{aligned}
\mathring{\mathfrak{g}}_1(\mathbf{f})(\mathbf{x}) &\leq C \int_0^\infty t^{2-s} \int_t^{2t} |\partial_{tt} \mathbf{U}_s(\mathbf{x}, r)|^2 (r-t)^{-s} \, \mathrm{d}r \, \mathrm{d}t \\
&+ C \int_0^\infty t^{1-2s} \int_{2t}^\infty |\partial_{tt} \mathbf{U}_s(\mathbf{x}, r)|^2 (r-t) \, \mathrm{d}r \, \mathrm{d}t \\
&= C \int_0^\infty |\partial_{tt} \mathbf{U}_s(\mathbf{x}, r)|^2 \left(\int_{r/2}^r t^{2-s}(r-t)^{-s} \, \mathrm{d}t + \int_0^{r/2} t^{1-2s}(r-t) \, \mathrm{d}t \right) \mathrm{d}r \\
&\leq C \int_0^\infty r^{3-2s} |\partial_{tt} \mathbf{U}_s(\mathbf{x}, r)|^2 \, \mathrm{d}r = C \int_0^\infty t^{3-2s} |\partial_{tt} \mathbf{U}_s(\mathbf{x}, t)|^2 \, \mathrm{d}t \, .
\end{aligned}
\tag{4.45}
$$

Notice also that using the fact that $\mathbb{P}_t$ integrates to $\mathbb{I}_{d+1}$ we see that

$$
\begin{aligned}
\partial_{tt} \mathbf{U}_s(\mathbf{x}, t) &= \int_{\mathbb{R}^d} \partial_{tt} \mathbb{P}_t(\mathbf{y})(\mathbf{f}_s(\mathbf{x}-\mathbf{y}) - \mathbf{f}_s(\mathbf{x})) \, \mathrm{d}\mathbf{y} \\
&= \int_{\mathbb{R}^d} \partial_{tt} \overline{\mathbf{P}}(\mathbf{y}, t)(\mathbf{f}_s(\mathbf{x}-\mathbf{y}) - \mathbf{f}_s(\mathbf{x})) \cdot \frac{\mathbf{y}}{|\mathbf{y}|} \, \mathrm{d}\mathbf{y} \, ,
\end{aligned}
$$

where we used the relation (4.31). By computation of $\partial_{tt}\mathbb{P}_t$ it is not difficult to show that

$$
|\partial_{tt} \mathbf{U}_s(\mathbf{x}, t)| \leq \int_{\mathbb{R}^d} |\partial_{tt} \overline{\mathbf{P}}(\mathbf{y}, t)| \left| (\mathbf{f}_s(\mathbf{x}+\mathbf{y}) - \mathbf{f}_s(\mathbf{x})) \cdot \frac{\mathbf{y}}{|\mathbf{y}|} \right| \mathrm{d}\mathbf{y} \, .
$$

We now divide the integration region in the right hand side and estimate using Hölder's inequality to obtain

$$\begin{aligned}
|\partial_{tt}\mathbf{U}_s(\mathbf{x},t)|^2 &\leq \left(\int_{|\mathbf{y}|\leq t} |\partial_{tt}\overline{\mathbf{P}}(\mathbf{y},t)| \left|\mathbf{f}_s(\mathbf{x}+\mathbf{y}) - \mathbf{f}_s(\mathbf{x})\cdot\frac{\mathbf{y}}{|\mathbf{y}|}\right| \mathrm{d}\mathbf{y} + \int_{t\leq|\mathbf{y}|} \cdots \mathrm{d}\mathbf{y}\right)^2 \\
&\leq \left(\int_{|\mathbf{y}|\leq t} |\partial_{tt}\overline{\mathbf{P}}(\mathbf{y},t)|\,\mathrm{d}\mathbf{y}\right)\left(\int_{|\mathbf{y}|\leq t} |\mathfrak{d}(\mathbf{f}_s)(\mathbf{x},\mathbf{y})|^2\,|\partial_{tt}\overline{\mathbf{P}}_t(\mathbf{y})|\,\mathrm{d}\mathbf{y}\right) \\
&\quad + \left(\int_{|\mathbf{y}|>t} |\partial_{tt}\overline{\mathbf{P}}(\mathbf{y},t)|\,\mathrm{d}\mathbf{y}\right)\left(\int_{|\mathbf{y}|>t} |\mathfrak{d}(\mathbf{f}_s)(\mathbf{x},\mathbf{y})|^2\,|\partial_{tt}\overline{\mathbf{P}}(\mathbf{y},t)|\,\mathrm{d}\mathbf{y}\right).
\end{aligned}$$

where we introduced the notation $\mathfrak{d}(\mathbf{f}_s)(\mathbf{x},\mathbf{y}) := \mathbf{f}_s(\mathbf{x}+\mathbf{y}) - \mathbf{f}_s(\mathbf{x})\cdot\frac{|\mathbf{y}|}{|\mathbf{y}|}$. We now use the estimates for $|\partial_{tt}\overline{\mathbf{P}}(\mathbf{y},t)|$ to get

$$\begin{aligned}
|\partial_{tt}\mathbf{U}_s(\mathbf{x},t)|^2 &\leq \left(\int_{|\mathbf{y}|\leq t} \frac{C}{t^{d+2}}\,\mathrm{d}\mathbf{y}\right)\left(\int_{|\mathbf{y}|\leq t} |\mathfrak{d}(\mathbf{f}_s)(\mathbf{x},\mathbf{y})|^2\,|\partial_{tt}\overline{\mathbf{P}}(\mathbf{y},t)|\,\mathrm{d}\mathbf{y}\right) \\
&\quad + \left(\int_{|\mathbf{y}|>t} \frac{C}{|\mathbf{y}|^{d+2}}\,\mathrm{d}\mathbf{y}\right)\left(\int_{|\mathbf{y}|>t} |\mathfrak{d}(\mathbf{f}_s)(\mathbf{x},\mathbf{y})|^2\,|\partial_{tt}\overline{\mathbf{P}}(\mathbf{y},t)|\,\mathrm{d}\mathbf{y}\right) \\
&\leq \frac{C}{t^2}\left(\int_{|\mathbf{y}|\leq t} |\mathfrak{d}(\mathbf{f}_s)(\mathbf{x},\mathbf{y})|^2\,|\partial_{tt}\overline{\mathbf{P}}(\mathbf{y},t)|\,\mathrm{d}\mathbf{y}\right) \\
&\quad + \frac{C}{t^2}\left(\int_{|\mathbf{y}|>t} |\mathfrak{d}(\mathbf{f}_s)(\mathbf{x},\mathbf{y})|^2\,|\partial_{tt}\overline{\mathbf{P}}(\mathbf{y},t)|\,\mathrm{d}\mathbf{y}\right).
\end{aligned}$$

As a consequence, combining the above with (4.45), and interchanging the integrals we have that

$$
\begin{aligned}
\left[\mathring{\mathfrak{g}}_1(\mathbf{f})(\mathbf{x})\right]^2 \leq C \int_0^\infty t^{3-2s}\left(\frac{1}{t^2}\right)\Bigg(\int_{|\mathbf{y}|\leq t} |\mathfrak{d}(\mathbf{f}_s)(\mathbf{x},\mathbf{y})|^2\,|\partial_{tt}\overline{\mathbf{P}}(\mathbf{y},t)|\,\mathrm{d}\mathbf{y} \\
+ \int_{|\mathbf{y}|>t} \ldots\,\mathrm{d}\mathbf{y}\Bigg)\,\mathrm{d}t \\
= C\int_{\mathbb{R}^d} |\mathfrak{d}(\mathbf{f}_s)(\mathbf{x},\mathbf{y})|^2\Bigg(\int_0^{|\mathbf{y}|} t^{1-2s}|\partial_{tt}\overline{\mathbf{P}}(\mathbf{y},t)|\,\mathrm{d}t \\
+ \int_{|\mathbf{y}|}^\infty t^{1-2s}|\partial_{tt}\overline{\mathbf{P}}(\mathbf{y},t)|\,\mathrm{d}t\Bigg)\,\mathrm{d}\mathbf{y} \\
\leq C\int_{\mathbb{R}^d} |\mathfrak{d}(\mathbf{f}_s)(\mathbf{x},\mathbf{y})|^2\left(\int_0^{|\mathbf{y}|}\frac{t^{1-2s}}{|\mathbf{y}|^{d+2}}\,\mathrm{d}t + \int_{|\mathbf{y}|}^\infty t^{-d-1-2s}\,\mathrm{d}t\right)\mathrm{d}\mathbf{y} \\
= C\int_{\mathbb{R}^d}\frac{|\mathfrak{d}(\mathbf{f}_s)(\mathbf{x},\mathbf{y})|^2}{|\mathbf{y}|^{d+2s}}\,\mathrm{d}\mathbf{y} = C\left[D^s(\mathbf{f}_s)(\mathbf{x})\right]^2.
\end{aligned}
$$

The proof is complete. □

4.2.4 Proof of Theorem 4.15

Finally we are ready to prove the second main result of the chapter, Theorem 4.15. As we have indicated, since for any $\mathbf{x}\in\mathbb{R}^d$ the pointwise estimate $D^s(\mathbf{f})(\mathbf{x}) \leq \Upsilon^s(\mathbf{f})(\mathbf{x})$ holds, the right-hand side inequality in Theorem 4.15 follows from the characterization in [85, Theorem 1]. What remains is to prove the left-hand side inequality in Theorem 4.15 which is stated in the following theorem.

Theorem 4.23. *Let $s\in(0,1)$ and $1<p<\infty$. If $\mathbf{f}\in\mathcal{D}^p_s(\mathbb{R}^d)$, then $\mathbf{f}\in\mathcal{L}^{s,p}(\mathbb{R}^d)$. Moreover, there exist positive constants C such that*

$$\|\mathbf{f}\|_{\mathcal{L}^p_s(\mathbb{R}^d)} \leq C\left(\|\mathbf{f}\|_{L^p(\mathbb{R}^d)} + \|D^s(\mathbf{f})\|_{L^p(\mathbb{R}^d)}\right)$$

Proof. It suffices to establish the inequality for vector fields in the Schwarz space $\mathcal{S}(\mathbb{R}^d;\mathbb{R}^d)$. To that end, we will show that for any $\mathbf{f}\in\mathcal{S}(\mathbb{R}^d;\mathbb{R}^d)$, the tempered distribution $\left((1+4\pi^2|\boldsymbol{\xi}|^2)^{s/2}\widehat{\mathbf{f}}\right)^\vee$ belongs to the space $L^p(\mathbb{R}^d;\mathbb{R}^d)$ with the appropriate estimate. In what follows we use a key result that relates the Riesz potentials $\mathcal{I}_s$ and the Bessel potentials $\mathcal{J}_s$. From [86, Lemma 2, Chapter V] there exists a pair of finite measures ν_s and λ_s on $\mathbb{R}^d$ with Fourier transforms $\widehat{\nu}_s(\boldsymbol{\xi})$ and $\widehat{\lambda}_s(\boldsymbol{\xi})$ respectively such that

$$\left(1+4\pi^2|\boldsymbol{\xi}|^2\right)^{s/2} = \widehat{\nu}_s(\boldsymbol{\xi}) + (2\pi|\boldsymbol{\xi}|)^s\widehat{\lambda}_s(\boldsymbol{\xi})\,.$$

Then for any $\mathbf{f} \in L^p(\mathbb{R}^d;\mathbb{R}^d)$, we have that in the sense of distributions

$$\left(\left(1+4\pi^2|\boldsymbol{\xi}|^2\right)^{s/2}\widehat{\mathbf{f}}\right)^\vee = \mathbf{f} * \nu_s + ((2\pi|\xi|)^s\widehat{\mathbf{f}})^\vee * \lambda_s$$

We now estimate the L^p norms of the terms in the right-hand side. We notice first that $\|\mathbf{f} * \nu_s\|_{L^p} \leq C\|\mathbf{f}\|_{L^p}$, which follows since ν_s is a finite measure. To estimate the second term we use again the fact that λ_s is a finite measure to get

$$\left\|((2\pi|\xi|)^s\widehat{\mathbf{f}})^\vee * \lambda_s\right\|_{L^p} \leq C\left\|((2\pi|\xi|)^s\widehat{\mathbf{f}})^\vee\right\|_{L^p} \leq C\left\|\mathring{\mathfrak{g}}_1\Big(\left((2\pi|\xi|)^s\widehat{\mathbf{f}}\right)^\vee\Big)\right\|_{L^p},$$

where in the last inequality we have applied Proposition 4.21. We now use Theorem 4.22 to estimate

$$\begin{aligned}\left\|((2\pi|\xi|)^s\widehat{\mathbf{f}})^\vee * \lambda_s\right\|_{L^p} &\leq C\left\|\mathring{\mathfrak{g}}_1\Big(\left((2\pi|\xi|)^s\widehat{\mathbf{f}}\right)^\vee\Big)\right\|_{L^p}\\ &\leq C\left\|D^s\Big(\mathcal{I}_s\Big(\left((2\pi|\xi|)^s\widehat{\mathbf{f}}\right)^\vee\Big)\Big)\right\|_{L^p}\\ &= C\|D^s(\mathbf{f})\|_{L^p}\,,\end{aligned}$$

where we used the identity $\mathcal{I}_s\Big(\left((2\pi|\xi|)^s\widehat{\mathbf{f}}\right)^\vee\Big) = \mathbf{f}$. □

4.2.5 Proof of Lemma 4.17

Proof. Using Fubini's Theorem and Plancherel's Theorem,

$$\|\mathring{g}_1(\mathbf{f})\|^2_{L^2(\mathbb{R}^d)} = \int_0^\infty t\int_{\mathbb{R}^d}|\partial_t\mathbf{U}(\mathbf{x},t)|^2\,\mathrm{d}\mathbf{x}\,\mathrm{d}t = \int_0^\infty t\int_{\mathbb{R}^d}|\widehat{\partial_t\mathbf{U}}(\boldsymbol{\xi},t)|^2\,\mathrm{d}\boldsymbol{\xi}\,\mathrm{d}t\,.$$

By a direct computation, for $1 \leq k \leq d$,

$$\begin{aligned}|\partial_t \widehat{U_k}(\boldsymbol{\xi}, t)|^2 &= \left|\partial_t \left(\mathrm{e}^{-2\pi|\boldsymbol{\xi}|t}\widehat{f_k} + (2\pi|\boldsymbol{\xi}|t)\mathrm{e}^{-2\pi|\boldsymbol{\xi}|t}\left(\frac{\boldsymbol{\xi}}{|\boldsymbol{\xi}|}\cdot\widehat{\mathbf{f}}(\boldsymbol{\xi})\right)\left(\frac{-\xi_k}{|\boldsymbol{\xi}|}\right)\right)\right|^2 \\ &= 4\pi^2|\boldsymbol{\xi}|^2\mathrm{e}^{-4\pi|\boldsymbol{\xi}|t}\left|A_k(\boldsymbol{\xi}, t)\right|^2 ,\end{aligned}$$

where

$$A_k(\boldsymbol{\xi}, t) := \widehat{f}_k(\boldsymbol{\xi}) + \left(\frac{\boldsymbol{\xi}}{|\boldsymbol{\xi}|}\cdot\widehat{\mathbf{f}}(\boldsymbol{\xi})\right)\left(\frac{\xi_k}{|\boldsymbol{\xi}|}\right) - (2\pi|\boldsymbol{\xi}|t)\left(\frac{\boldsymbol{\xi}}{|\boldsymbol{\xi}|}\cdot\widehat{\mathbf{f}}(\boldsymbol{\xi})\right)\left(\frac{\xi_k}{|\boldsymbol{\xi}|}\right).$$

(Note that here we are using the modulus $|\cdot|$ for complex numbers.) Next,

$$\begin{aligned}\left|\partial_t \widehat{U_{d+1}}(\boldsymbol{\xi}, t)\right|^2 &= \left|\partial_t \left((2\pi|\boldsymbol{\xi}|t)\mathrm{e}^{-2\pi|\boldsymbol{\xi}|t}\left(-\imath\frac{\boldsymbol{\xi}}{|\boldsymbol{\xi}|}\cdot\widehat{\mathbf{f}}(\boldsymbol{\xi})\right)\right)\right|^2 \\ &= 4\pi^2|\boldsymbol{\xi}|^2\mathrm{e}^{-4\pi|\boldsymbol{\xi}|t}\left|\frac{\boldsymbol{\xi}}{|\boldsymbol{\xi}|}\cdot\widehat{\mathbf{f}}(\boldsymbol{\xi})\right|^2(1 + 2\pi|\boldsymbol{\xi}|t)^2 ,\end{aligned}$$

Expanding the A_k terms,

$$\begin{aligned}\sum_{k=1}^{d}|A_k(\boldsymbol{\xi}, t)|^2 = |\widehat{\mathbf{f}}(\boldsymbol{\xi})|^2 + 3\left(\frac{\boldsymbol{\xi}}{|\boldsymbol{\xi}|}\cdot\widehat{\mathbf{f}}(\boldsymbol{\xi})\right)^2 - (8\pi|\boldsymbol{\xi}|t)\left(\frac{\boldsymbol{\xi}}{|\boldsymbol{\xi}|}\cdot\widehat{\mathbf{f}}(\boldsymbol{\xi})\right)^2 \\ + (4\pi^2|\boldsymbol{\xi}|^2t^2)\left(\frac{\boldsymbol{\xi}}{|\boldsymbol{\xi}|}\cdot\widehat{\mathbf{f}}(\boldsymbol{\xi})\right)^2 .\end{aligned}$$

Now we prove (4.33). By the above formulas,

$$\begin{aligned}&\left|\partial_t \widehat{\mathbf{U}}(\boldsymbol{\xi}, t)\right|^2 \\ &= 4\pi^2|\boldsymbol{\xi}|^2\mathrm{e}^{-4\pi|\boldsymbol{\xi}|t}\left(\sum_{k=1}^{d}|A_k(\boldsymbol{\xi}, t)|^2 + (1 + 2\pi|\boldsymbol{\xi}|t)^2\left|\frac{\boldsymbol{\xi}}{|\boldsymbol{\xi}|}\cdot\widehat{\mathbf{f}}(\boldsymbol{\xi})\right|^2\right) \\ &= 4\pi^2|\boldsymbol{\xi}|^2\mathrm{e}^{-4\pi|\boldsymbol{\xi}|t}\left(|\widehat{\mathbf{f}}(\boldsymbol{\xi})|^2 + \left(3 - 8\pi|\boldsymbol{\xi}|t + 4\pi^2|\boldsymbol{\xi}|^2t^2 + (1 + 2\pi|\boldsymbol{\xi}|t)^2\right)\left|\frac{\boldsymbol{\xi}}{|\boldsymbol{\xi}|}\cdot\widehat{\mathbf{f}}(\boldsymbol{\xi})\right|^2\right) \\ &= 4\pi^2|\boldsymbol{\xi}|^2\mathrm{e}^{-4\pi|\boldsymbol{\xi}|t}|\widehat{\mathbf{f}}(\boldsymbol{\xi})|^2 + 4\pi^2|\boldsymbol{\xi}|^2\mathrm{e}^{-4\pi|\boldsymbol{\xi}|t}\Big(4 - 4\pi|\boldsymbol{\xi}|t + 8\pi^2|\boldsymbol{\xi}|^2t^2\Big)\left|\frac{\boldsymbol{\xi}}{|\boldsymbol{\xi}|}\cdot\widehat{\mathbf{f}}(\boldsymbol{\xi})\right|^2\end{aligned}$$

Thus,

$$\int_0^\infty \int_{\mathbb{R}^d} t|\partial_t \widehat{\mathbf{U}}(\boldsymbol{\xi},t)|^2 \,\mathrm{d}\boldsymbol{\xi}\,\mathrm{d}t = \int_{\mathbb{R}^d} \frac{1}{4}|\widehat{\mathbf{f}}(\boldsymbol{\xi})|^2 + \left(\frac{1}{2} - \frac{1}{2} + \frac{3}{4}\right)\left|\frac{\boldsymbol{\xi}}{|\boldsymbol{\xi}|}\cdot \widehat{\mathbf{f}}(\boldsymbol{\xi})\right|^2 \,\mathrm{d}\boldsymbol{\xi}$$
$$= \frac{1}{4}\int_{\mathbb{R}^d} \left|\left(\mathbb{I}_d + \frac{\boldsymbol{\xi}\otimes\boldsymbol{\xi}}{|\boldsymbol{\xi}|^2}\right)\widehat{\mathbf{f}}(\boldsymbol{\xi})\right|^2 \,\mathrm{d}\boldsymbol{\xi}\,.$$

where we have used the formulas: by a change of variables $\eta = 4\pi|\boldsymbol{\xi}|t$,

$$\int_0^\infty 8\pi^2|\boldsymbol{\xi}|^2 t\mathrm{e}^{-4\pi|\boldsymbol{\xi}|t}\,\mathrm{d}t = \frac{1}{2}\int_0^\infty \eta\mathrm{e}^{-\eta}\,\mathrm{d}\eta = \frac{1}{2}\Gamma(1) = \frac{1}{2}\,,$$
$$\int_0^\infty 16\pi^2|\boldsymbol{\xi}|^2 t\mathrm{e}^{-4\pi|\boldsymbol{\xi}|t}\,\mathrm{d}t = \int_0^\infty \eta\mathrm{e}^{-\eta}\,\mathrm{d}\eta = 1\,,$$
$$-\int_0^\infty 32\pi^3|\boldsymbol{\xi}|^3 t^2\mathrm{e}^{-4\pi|\boldsymbol{\xi}|t}\,\mathrm{d}t = -\frac{1}{2}\int_0^\infty \eta^2\mathrm{e}^{\eta}\,\mathrm{d}\eta = -\frac{\Gamma(2)}{2} = -1\,,$$
$$\int_0^\infty 64\pi^4|\boldsymbol{\xi}|^4 t^3\mathrm{e}^{-4\pi|\boldsymbol{\xi}|t}\,\mathrm{d}t = \frac{1}{4}\int_0^\infty \eta^3\mathrm{e}^{-\eta}\,\mathrm{d}t = \frac{\Gamma(3)}{4} = \frac{3}{2}\,.$$

4.2.6 Proof of (4.44)

This can be done using the Fourier transform. Denote

$$\mathbb{A}(\boldsymbol{\xi}) =: \begin{bmatrix} -\frac{\boldsymbol{\xi}\otimes\boldsymbol{\xi}}{|\boldsymbol{\xi}|^2} & -i\frac{\boldsymbol{\xi}}{|\boldsymbol{\xi}|} \\ -i\frac{\boldsymbol{\xi}}{|\boldsymbol{\xi}|} & 1 \end{bmatrix}.$$

Then

$$
\begin{aligned}
&\frac{-1}{\Gamma(1-s)}\int_0^\infty \partial_{tt}\widehat{\mathbf{U}_s}(\boldsymbol{\xi},t+r)r^{-s}\,\mathrm{d}r \\
&= \frac{-1}{\Gamma(1-s)}\int_0^\infty 4\pi^2|\boldsymbol{\xi}|^2\mathrm{e}^{-2\pi|\boldsymbol{\xi}|(t+r)}\Big(\mathbb{I}_{d+1}-2\mathbb{A}(\boldsymbol{\xi})+2\pi|\boldsymbol{\xi}|(t+r)\mathbb{A}(\boldsymbol{\xi})\Big) \\
&\qquad\qquad\qquad\qquad\qquad\qquad \times (2\pi|\boldsymbol{\xi}|)^{-s}\widehat{\mathbf{f}}(\boldsymbol{\xi})r^{-s}\,\mathrm{d}r \\
&= \frac{-1}{\Gamma(1-s)}2\pi|\boldsymbol{\xi}|\mathrm{e}^{-2\pi|\boldsymbol{\xi}|t}\left(\int_0^\infty (2\pi|\boldsymbol{\xi}|r)^{-s}\mathrm{e}^{-2\pi|\boldsymbol{\xi}|r}(2\pi|\boldsymbol{\xi}|)\,\mathrm{d}r\right) \\
&\qquad\qquad\qquad\qquad\qquad\qquad \times \Big(\mathbb{I}_{d+1}-2\mathbb{A}(\boldsymbol{\xi})+2\pi|\boldsymbol{\xi}|t\mathbb{A}(\boldsymbol{\xi})\Big) \\
&\quad + \frac{-1}{\Gamma(1-s)}2\pi|\boldsymbol{\xi}|\mathrm{e}^{-2\pi|\boldsymbol{\xi}|t}\left(\int_0^\infty (2\pi|\boldsymbol{\xi}|r)^{1-s}\mathrm{e}^{-2\pi|\boldsymbol{\xi}|r}(2\pi|\boldsymbol{\xi}|)\,\mathrm{d}r\right)\mathbb{A}(\boldsymbol{\xi}) \\
&= \frac{-1}{\Gamma(1-s)}2\pi|\boldsymbol{\xi}|\mathrm{e}^{-2\pi|\boldsymbol{\xi}|t}\,\Gamma(1-s)\Big(\mathbb{I}_{d+1}-2\mathbb{A}(\boldsymbol{\xi})+2\pi|\boldsymbol{\xi}|t\mathbb{A}(\boldsymbol{\xi})\Big) \\
&\quad + \frac{-1}{\Gamma(1-s)}2\pi|\boldsymbol{\xi}|\mathrm{e}^{-2\pi|\boldsymbol{\xi}|t}\,\Gamma(2-s)\,\mathbb{A}(\boldsymbol{\xi}) \\
&= (-2\pi|\boldsymbol{\xi}|)\mathrm{e}^{-2\pi|\boldsymbol{\xi}|t}\left(\mathbb{I}_{d+1}-2\mathbb{A}(\boldsymbol{\xi})+(2\pi|\boldsymbol{\xi}|t)\mathbb{A}(\boldsymbol{\xi})+\frac{\Gamma(2-s)}{\Gamma(1-s)}\mathbb{A}(\boldsymbol{\xi})\right) \\
&= \partial_t\widehat{\mathbf{U}}(\boldsymbol{\xi},t)\,.
\end{aligned}
$$

In the last equality we used the identity $\Gamma(x+1)=x\Gamma(x)$ for every $x>0$. By a change of variables we have

$$
\partial_t\mathbf{U}(\mathbf{x},t)=\frac{-1}{\Gamma(1-s)}\int_t^\infty \partial_{tt}\mathbf{U}_s(\mathbf{x},r)(r-t)^{-s}\,\mathrm{d}r\,.
$$

□

Chapter 5

Applications: Regularity For Peridynamic-Type Operators

5.1 A Potential Space Estimate For Weak Solutions

5.1.1 Introduction and Statement of Main Results

This section focuses on a peridynamic-type system that uses interaction kernels $\rho(\mathbf{x}, \mathbf{y})$ which behave like $|\mathbf{x}-\mathbf{y}|^{-(d+2s)}$ for $|\mathbf{x}-\mathbf{y}|$ close to zero and infinity. To be precise, we study the system of nonlocal equations of the type formally given by

$$\mathbb{L}_{\mathfrak{h}}\mathbf{u}(\mathbf{x}) := c_{\mathfrak{h}} \int_{B_{\mathfrak{h}}(\mathbf{x})} \frac{A(\mathbf{x},\mathbf{y})}{|\mathbf{x}-\mathbf{y}|^{d+2s}} \frac{(\mathbf{x}-\mathbf{y})}{|\mathbf{x}-\mathbf{y}|} \otimes \frac{(\mathbf{x}-\mathbf{y})}{|\mathbf{x}-\mathbf{y}|} (\mathbf{u}(\mathbf{x}) - \mathbf{u}(\mathbf{y})) \, \mathrm{d}\mathbf{y} = \mathbf{F}(\mathbf{x}) \tag{5.1}$$

where $d \geq 2$, $0 < s < 1$ are fixed, and $A(\mathbf{x}, \mathbf{y})$ is a measurable function, which we refer to as the *coefficient*, that is elliptic and symmetric in the sense

$$\alpha_1 \leq A(\mathbf{x},\mathbf{y}) \leq \alpha_2, \qquad A(\mathbf{x},\mathbf{y}) = A(\mathbf{y},\mathbf{x}) \qquad \text{for all } \mathbf{x},\mathbf{y} \in \mathbb{R}^d.$$

In what follows we assume that the horizon $\mathfrak{h}$ is given and fixed. We will not track the dependence of generic constants on the horizon.

To state the higher integrability result we first introduce the quadratic bilinear form associated with the operator $\mathbb{L}_{\mathfrak{h}}$ in (5.1):

$$\mathcal{E}_{\mathfrak{h}}(\mathbf{u},\mathbf{v}) = \frac{c_{\mathfrak{h}}}{2}\int_{\mathbb{R}^d}\int_{B_{\mathfrak{h}}(\mathbf{x})} \frac{A(\mathbf{x},\mathbf{y})}{|\mathbf{x}-\mathbf{y}|^{d+2s}}(\mathbf{u}(\mathbf{x})-\mathbf{u}(\mathbf{y}))\cdot\frac{(\mathbf{x}-\mathbf{y})}{|\mathbf{x}-\mathbf{y}|}(\mathbf{v}(\mathbf{x})-\mathbf{v}(\mathbf{y}))\cdot\frac{(\mathbf{x}-\mathbf{y})}{|\mathbf{x}-\mathbf{y}|}\,\mathrm{d}\mathbf{y}\,\mathrm{d}\mathbf{x}\,.$$

We use the notation $\mathcal{E}$ to represent the bilinear form when both integrals are on $\mathbb{R}^d$, that is formally, the horizon $\mathfrak{h}=\infty$. In this case we take $c_\infty = 1$, and write the operator as $\mathbb{L}$. For $\mathbf{u}\in L^2(\mathbb{R}^d;\mathbb{R}^d)$, the vector valued map $\mathbb{L}_{\mathfrak{h}}\mathbf{u}$ is a vector of distributions acting on test functions $\phi\in C_c^\infty(\mathbb{R}^d;\mathbb{R}^d)$ as

$$\langle \mathbb{L}_{\mathfrak{h}}\mathbf{u},\phi\rangle := \mathcal{E}_{\mathfrak{h}}(\mathbf{u},\phi).$$

It is now clear that if $\mathbf{u}$ is in the energy space $\{\mathbf{u}\in L^2 : \mathcal{E}_{\mathfrak{h}}(\mathbf{u},\mathbf{u})<\infty\}$, then $\mathbb{L}_{\mathfrak{h}}\mathbf{u}$ is in its dual space. We will shortly characterize that the energy $\mathcal{E}_{\mathfrak{h}}(\mathbf{u},\mathbf{u})$ is finite if and only if $\mathbf{u}\in W^{s,2}(\mathbb{R}^d;\mathbb{R}^d)$. This assertion follows from the lower and upper bounds of the coefficient $A(\mathbf{x},\mathbf{y})$ and using the equivalence of spaces that is recently proved in [57]. The notion of weak solution to (5.1) is standard and is given in terms of the bilinear form $\mathcal{E}$.

Definition 5.1. Given $\mathbf{F}\in W^{-s,2}(\mathbb{R}^d;\mathbb{R}^d)$, we say that $\mathbf{u}\in W^{s,2}(\mathbb{R}^d;\mathbb{R}^d)$ is a *weak solution to* (5.1) if

$$\mathcal{E}_{\mathfrak{h}}(\mathbf{u},\phi) = \langle \mathbf{F},\phi\rangle \tag{5.2}$$

for any $\phi\in W^{s,2}(\mathbb{R}^d;\mathbb{R}^d)$. In the above $\langle\cdot,\cdot\rangle$ denotes the duality pairing.

Existence and uniqueness of weak solutions to this particular system will be demonstrated shortly, see also Chapter 3 for when complementary boundary conditions are imposed on the solution. However, the focus of this section will be the issue of regularity of solutions. We seek to address the following question: If the data is regular, how regular is the solution? In particular, we are interested in data $\mathbf{F}$ coming from a class of low integrability. Just as in Chapter 4 we use the following standard notations for the fractional Sobolev exponents and its Hölder conjugate (denoted by the prime notation):

$$2^{*_s} = \frac{2d}{d-2s}\,, \quad \text{and} \quad (2^{*_s})' := 2_{*_s} = \frac{2d}{d+2s}\,.$$

The main result of this section states that any weak solution $\mathbf{u}$ to the system (5.1) with Poisson data $\mathbf{F}$ in the described class in fact belongs to the Bessel potential space $\mathcal{L}^p_s(\mathbb{R}^d)$ for some $p>2$. Note that when $p>2$, the Bessel potential space is finer than the Sobolev space, i.e. $\mathcal{L}^p_s(\mathbb{R}^d) \subset W^{s,p}(\mathbb{R}^d;\mathbb{R}^d)$. For precision, we state the result in the following theorem.

Theorem 5.2. *Suppose that $0<\alpha_1 \leq \alpha_2 < \infty$, $0<s<1$, $d \geq 2$, $\delta_0>0$ and $0<\mathfrak{h} \leq \infty$ are all given. Let $A(\mathbf{x},\mathbf{y})$ be symmetric and measurable with $\alpha_1 \leq A(\mathbf{x},\mathbf{y}) \leq \alpha_2$ and $\mathbf{u} \in W^{s,2}(\mathbb{R}^d;\mathbb{R}^d)$ be any weak solution in the sense of (5.2) corresponding to a given $\mathbf{F} \in L^{2_{*s}+\delta_0}(\mathbb{R}^d;\mathbb{R}^d) \cap L^{2_{*s}}(\mathbb{R}^d;\mathbb{R}^d)$. Then there exists $\varepsilon>0$ depending only on α_1, α_2, d and s with the following property: If $p \in [2,\infty)$ satisfies*

$$p \in [2, 2^{*_s})\,, \qquad p-2<\varepsilon\,, \qquad \frac{dp}{(2-\frac{p}{2})d+sp} - \frac{2d}{d+2s} < \delta_0\,, \tag{5.3}$$

then $\mathbf{u} \in \mathcal{L}^s_p(\mathbb{R}^d;\mathbb{R}^d)$. Moreover, there exists a constant $C>0$ such that

$$\|\mathbf{u}\|_{\mathcal{L}^s_p(\mathbb{R}^d)} \leq C\Big(\|\mathbf{F}\|_{L^{2_{*s}+\delta_0}(\mathbb{R}^d)} + \|\mathbf{F}\|_{L^{2_{*s}}(\mathbb{R}^d)} + \|\mathbf{u}\|_{W^{s,2}(\mathbb{R}^d)} \Big)\,. \tag{5.4}$$

The constant C depends only on α_1, α_2, d, p, δ_0 and $\mathfrak{h}$.

The main implication of the regularity result is that, with no additional smoothness condition on the coefficient $A(\mathbf{x},\mathbf{y})$, a weak solution satisfying (5.2) has a higher integrable fractional "s-derivative". For scalar equations, this type of nonlocal analogue of Meyers inequality was obtained in the recent work [7][1]. It has been shown in several subsequent works that solutions to fractional equations surprisingly also improve in *differentiability*, again with no additional smoothness conditions on the coefficients. Some of these approaches use reverse Hölder inequalities and nonlocal Gehring-type lemmas as in [49], or prove the result via a commutator estimate as in [76] and in [4] using a functional analytic approach. The approaches in [49, 76] are local in nature and application of appropriate embedding estimates in their work show improved local differentiability will lead to improved local integrability. The approach in [4] is rather robust and implies local and global regularity

[1] The argument presented in the paper appears to have a gap that we are unable to fix as of the writing of this dissertation.

results. We should mention that the results in [49, 76] hold for vectorial systems of fractional equations, even for coupled systems of fractional equations such as the vectorial fractional p-Laplacian, which are uniformly elliptic in their natural energy space. However, none of these works apply directly to the system of coupled nonlocal equations under consideration here, as the system (5.1) has a weaker ellipticity property in the spirit of Legendre-Hadamard. The proof of Theorem 5.2 we present in this section follows the argument presented in [4] for nonlocal scalar equations extending its applicability to the system of coupled nonlocal equations and for rough data.

We also remark that for δ_0 small, the exponent $2_{*_s} + \delta_0$ can be smaller than 2, and the higher potential space estimate holds true for rough data $\mathbf{F} \in L^r(\mathbb{R}^d, \mathbb{R}^d)$ and $r < 2$. We note that there are vector functions $\mathbf{F}$ that are in $L^{2_{*_s}+\delta_0}(\mathbb{R}^d; \mathbb{R}^d) \cap L^{2_{*_s}}(\mathbb{R}^d; \mathbb{R}^d)$ but not in $L^2(\mathbb{R}^d; \mathbb{R}^d)$. Indeed, for δ_0 sufficiently small and $\tau = \frac{d\delta_0}{2_{*_s}(2_{*_s}+\delta_0)}$ it is straightforward to verify via Sobolev embedding that vector fields in $W^{\tau, 2_{*_s}}(\mathbb{R}^d; \mathbb{R}^d)$ will belong to the former class but not necessarily the latter. While the higher integrability result remains true in the event $\mathbf{F} \in L^2(\mathbb{R}^d, \mathbb{R}^d)$, this case must be treated differently. In fact for nonlocal scalar equations an argument is presented in [4, Corollary 4.9] that is also applicable to our case.

In Sections 5.1.3 and 5.1.4 we prove the higher integrability result when $\mathfrak{h} = \infty$ and $\mathfrak{h} < \infty$, respectively. We first establish invertibility over a range of Sobolev spaces for the operator $\mathbb{I}_d + \mathbb{L}$. For a given weak solution $\mathbf{u}$ to the nonlocal elliptic system (5.1), we will use this invertibility to prove higher integrability of $D^s(\mathbf{u})$. We state this higher integrability result precisely in the following theorem.

Theorem 5.3. *Suppose that $0 < \alpha_1 \leq \alpha_2 < \infty$, $0 < s < 1$, $d \geq 2$, $0 < \mathfrak{h} \leq \infty$, and $\delta_0 > 0$ are given. Then there exists $\varepsilon > 0$ depending only on α_1, α_2, d and s with the following property: if $p > 2$ satisfies the conditions (5.3), then there exists a constant $C > 0$ such that for any $A(\mathbf{x}, \mathbf{y})$ symmetric, measurable with $\alpha_1 \leq A(\mathbf{x}, \mathbf{y}) \leq \alpha_2$ and any weak solution $\mathbf{u} \in W^{s,2}(\mathbb{R}^d; \mathbb{R}^d)$ in the sense of (5.2) corresponding to $\mathbf{F} \in L^{2_{*_s}+\delta_0}(\mathbb{R}^d; \mathbb{R}^d) \cap L^{2_{*_s}}(\mathbb{R}^d; \mathbb{R}^d)$, we have*

$$\|\mathbf{u}\|_{L^p(\mathbb{R}^d)} + \|\Upsilon^s(\mathbf{u})\|_{L^p(\mathbb{R}^d)} \leq C \left(\|\mathbf{F}\|_{L^{2_{*_s}+\delta_0}(\mathbb{R}^d)} + \|\mathbf{F}\|_{L^{2_{*_s}}(\mathbb{R}^d)} + \|\mathbf{u}\|_{W^{s,2}(\mathbb{R}^d)} \right) .$$

The constant C depends only on α_1, α_2, d, p, δ_0 and $\mathfrak{h}$.

By the log-convexity of L^p norms the data $\mathbf{F}$ belongs to $L^r(\mathbb{R}^d;\mathbb{R}^d)$ for any $r \in (2_{*_s}, 2_{*_s}+\delta_0)$. In fact, we will see that one can replace the two L^p norms of $\mathbf{F}$ on the right-hand side of the regularity estimate with the single term $\|\mathbf{F}\|_{L^r}$, where $r = \frac{dp}{(2-\frac{p}{2})d+sp}$.

We notice that Theorem 5.2 now follows as a corollary of this theorem. Indeed, applying the Bessel potential space characterization theorem (Theorem 4.15) we see that $\mathbf{u} \in \mathcal{L}^p_s(\mathbb{R}^d)$. Moreover, by applying the well-known embedding [86, Chapter V] of the spaces $\mathcal{L}^p_s(\mathbb{R}^d) \subset W^{s,p}(\mathbb{R}^d;\mathbb{R}^d)$ for $p > 2$, Theorem 5.2 implies a higher Sobolev space regularity result for solutions.

5.1.2 Key Inequalities

Before we prove Theorem 5.3, we first state two inequalities. The first inequality is the classical fractional Sobolev embedding. The second inequality is the Fractional Korn's Inequality proved in Chapter 4, stated here for convenience. This inequality will be used to prove an invertibility result for the operator $\varpi\mathbb{I}_d + \mathbb{L}$ for some $\varpi > 0$.

Lemma 5.4 ([4], Lemma 4.2)**.** *Suppose $s \in (0,1)$ and $q \in [1,\infty)$. Suppose also that $sq < d$. Then*

$$\|\mathbf{u}\|_{L^{q^{*s}}(\mathbb{R}^d)} \leq C[\mathbf{u}]_{W^{s,q}(\mathbb{R}^d)}\,. \tag{5.5}$$

In particular, $W^{s,q}(\mathbb{R}^d;\mathbb{R}^d) \subset L^{q^{*s}}(\mathbb{R}^d;\mathbb{R}^d)$ and $W^{s,q_{*s}}(\mathbb{R}^d;\mathbb{R}^d) \subset L^q(\mathbb{R}^d;\mathbb{R}^d)$.

Theorem 5.5 (Chapter 4, Theorem 4.1. Fractional Korn's inequality)**.** *For any $s \in (0,1)$ and $1 < q < \infty$,*

$$\mathcal{X}^s_q(\mathbb{R}^d) = W^{s,q}(\mathbb{R}^d;\mathbb{R}^d)\,.$$

Moreover, there exists a universal constant $\kappa = \kappa(d,q,s)$ such that for all vector fields $\mathbf{f} \in W^{s,q}(\mathbb{R}^d;\mathbb{R}^d)$ we have

$$[\mathbf{f}]_{\mathcal{X}^s_q(\mathbb{R}^d)} \leq [\mathbf{f}]_{W^{s,q}(\mathbb{R}^d)} \leq \kappa[\mathbf{f}]_{\mathcal{X}^s_q(\mathbb{R}^d)}\,. \tag{5.6}$$

For a given bounded domain Ω, the function space $\mathcal{X}_q^s(\Omega)$ is defined in the same way as $\mathcal{X}_q^s(\mathbb{R}^d)$ but the functions are defined in Ω and the integrations in the semi-norm are on Ω.

5.1.3 Higher Integrability When the Horizon is Infinite

In this subsection we will prove Theorem 5.3 when the horizon is infinite. In this case we recall that the bilinear form we use is $\mathcal{E}$, the operator is written $\mathbb{L}$, and $c_\infty = 1$. We will show that there exists a constant $\varpi > 0$ such that the operator $\varpi\mathbb{I}_d + \mathbb{L}$ from $W^{s,2}(\mathbb{R}^d;\mathbb{R}^d)$ to $[W^{s,2}(\mathbb{R}^d;\mathbb{R}^d)]^*$ is invertible, where we use the notation Y^* for the dual space of Y. We then use a perturbation lemma of Schneiberg reproduced in [4] to deduce that in fact $\varpi\mathbb{I}_d + \mathbb{L}$ is invertible on a range of "nearby" fractional Sobolev spaces forming a complex interpolation scale. This is the key step in proving the higher potential space regularity result for weak solutions in the sense of (5.2).

Lemma 5.6. *Let $0 < \alpha_1 \le A(\mathbf{x},\mathbf{y}) \le \alpha_2 < \infty$, $0 < s < 1$ and $d \ge 2$. Let t, $t' \in (0,1)$ and p, $p' \in (1,\infty)$ satisfy $t + t' = 2s$ and $p' = \frac{p}{p-1}$. Then there exists $\varepsilon > 0$ such that if $\frac{1}{2} - \frac{1}{p} < \varepsilon$ and $t - s < \varepsilon$, then*

$$\varpi\mathbb{I}_d + \mathbb{L} : W^{t,p}(\mathbb{R}^d;\mathbb{R}^d) \to [W^{t',p'}(\mathbb{R}^d;\mathbb{R}^d)]^*$$

is invertible, where $\varpi = \frac{\alpha_1}{\kappa}$ and κ is as in (5.6) corresponding to $q = 2$. Any inverse agrees with the inverse obtained for $t = s$ and $p = 2$ on their common domains of definition. Both ε and the norms of the inverses depend only on d, s, and the ellipticity constants α_1 and α_2.

Proof. The objective here is to show that the perturbation lemma of Schneiberg (see [4, Theorem A.1]) can be applied to the operator $\varpi\mathbb{I}_d + \mathbb{L}$. We will prove the theorem in three steps.

Step I: We show that $\varpi\mathbb{I}_d + \mathbb{L} : W^{s,2}(\mathbb{R}^d;\mathbb{R}^d) \to [W^{s,2}(\mathbb{R}^d;\mathbb{R}^d)]^*$ is bounded and invertible. To verify the boundedness of the operator, it suffices to check if $\mathbb{L}$ is a bounded operator $W^{s,2}(\mathbb{R}^d;\mathbb{R}^d)$ to its dual space. But this is a consequence of the definition of $\mathbb{L}$ and Hölder's inequality. Indeed, it follows from the fact that for any $\mathbf{u},\mathbf{v} \in W^{s,2}(\mathbb{R}^d;\mathbb{R}^d)$

$$|\langle \mathbb{L}\mathbf{u},\mathbf{v}\rangle| = |\mathcal{E}(\mathbf{u},\mathbf{v})| \le \alpha_2[\mathbf{u}]_{\mathcal{X}^{s,2}(\mathbb{R}^d)}[\mathbf{v}]_{\mathcal{X}^{s,2}(\mathbb{R}^d)} \le \alpha_2 \|\mathbf{u}\|_{W^{s,2}(\mathbb{R}^d)} \|\mathbf{v}\|_{W^{s,2}(\mathbb{R}^d)}.$$

To see that $\varpi\mathbb{I}_d + \mathbb{L}$ is invertible, note that by the ellipticity assumptions on A and by the fractional Korn's inequality (Theorem 5.5) applied to $q = 2$, we have and

$$|\langle \mathbb{L}\mathbf{u}, \mathbf{u}\rangle| \geq \alpha_1 [\mathbf{u}]^2_{\mathcal{X}^{s,2}(\mathbb{R}^d)} \geq \frac{\alpha_1}{\kappa} [\mathbf{u}]^2_{W^{s,2}(\mathbb{R}^d)} = \varpi [\mathbf{u}]^2_{W^{s,2}(\mathbb{R}^d)},$$

implying that $\langle \varpi\mathbf{u} + \mathbb{L}\mathbf{u}, \mathbf{u}\rangle \geq \varpi \|\mathbf{u}\|^2_{W^{s,2}(\mathbb{R}^d)}$. Invertibility of the operator $\varpi\mathbf{u} + \mathbb{L}$ is now a consequence of the well-known Lax-Milgram theorem.

Step II: The operator $\varpi\mathbb{I}_d + \mathbb{L}$ extends from $C^\infty_c(\mathbb{R}^d;\mathbb{R}^d)$ by density to a bounded operator $W^{t,p}(\mathbb{R}^d;\mathbb{R}^d) \to [W^{t',p'}(\mathbb{R}^d;\mathbb{R}^d)]^*$, also denoted by $\varpi\mathbb{I}_d + \mathbb{L}$. To see this for any $\mathbf{u} \in W^{t,p}(\mathbb{R}^d;\mathbb{R}^d)$ and any $\mathbf{v} \in W^{t',p'}(\mathbb{R}^d;\mathbb{R}^d)$, by Hölder's inequality

$$\begin{aligned}
|\langle \varpi\mathbf{u} + \mathbb{L}\mathbf{u}, \mathbf{v}\rangle| &\leq \varpi \|\mathbf{u}\|_{L^p(\mathbb{R}^d)} \|\mathbf{v}\|_{L^{p'}(\mathbb{R}^d)} \\
&\quad + \left| \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} A(\mathbf{x},\mathbf{y}) \frac{\big(\mathbf{u}(\mathbf{x}) - \mathbf{u}(\mathbf{y})\big)\cdot \frac{\mathbf{x}-\mathbf{y}}{|\mathbf{x}-\mathbf{y}|}}{|\mathbf{x}-\mathbf{y}|^{\frac{d}{p}+t}} \cdot \frac{\big(\mathbf{v}(\mathbf{x}) - \mathbf{v}(\mathbf{y})\big)\cdot \frac{\mathbf{x}-\mathbf{y}}{|\mathbf{x}-\mathbf{y}|}}{|\mathbf{x}-\mathbf{y}|^{\frac{d}{p'}+t'}} \, \mathrm{d}\mathbf{y}\, \mathrm{d}\mathbf{x} \right| \\
&\leq \varpi \|\mathbf{u}\|_{L^p(\mathbb{R}^d)} \|\mathbf{v}\|_{L^{p'}(\mathbb{R}^d)} + \alpha_2 [\mathbf{u}]_{W^{t,p}(\mathbb{R}^d)} [\mathbf{v}]_{W^{t',p'}(\mathbb{R}^d)} \,.
\end{aligned}$$

Since $C^\infty_c(\mathbb{R}^d;\mathbb{R}^d)$ is dense in all fractional Sobolev spaces involved, it follows then that the operator $\varpi\mathbb{I}_d + \mathbb{L} : W^{s,2}(\mathbb{R}^d;\mathbb{R}^d) \to [W^{s,2}(\mathbb{R}^d;\mathbb{R}^d)]^*$ extends to a bounded operator $W^{t,p}(\mathbb{R}^d;\mathbb{R}^d) \to [W^{t',p'}(\mathbb{R}^d;\mathbb{R}^d)]^*$.

Step III: The remaining argument is essentially the same as the one presented in [4] with the modification that it now applies to fractional Sobolev spaces of vector fields. We include it here for completeness. We begin by displaying the complex interpolation scale of the fractional Sobolev spaces of vector fields, and then use the two steps we proved above to verify the assumptions of Schneiberg's lemma [4, Theorem A.1]. Denoting the scale of complex interpolation spaces between two Banach spaces X_0 and X_1 by $[X_0, X_1]_\theta$ for $\theta \in (0,1)$, it is well-known that

$$\big[W^{t_0,p_0}(\mathbb{R}^d;\mathbb{R}^d), W^{t_1,p_1}(\mathbb{R}^d;\mathbb{R}^d)\big]_\theta = W^{t,p}(\mathbb{R}^d;\mathbb{R}^d)$$

for t_0, $t_1 \in (0,1)$ and p_0, $p_1 \in (1,\infty)$ with t and p given by

$$\frac{1}{p} = \frac{1-\theta}{p_0} + \frac{\theta}{p_1}, \qquad t = (1-\theta)t_0 + \theta t_1 \,.$$

Now consider the spaces $W^{t,p}(\mathbb{R}^d;\mathbb{R}^d)$ as points in the $(t,1/p)$-plane. Choose $(t_0,\frac{1}{p_0})$, $(t_1,1/p_1)$ so that $(s,1/2)$ lies in the line segment joining them, i.e. so that there exists $\theta^*\in(0,1)$ such that

$$\big[W^{t_0,p_0}(\mathbb{R}^d;\mathbb{R}^d),W^{t_1,p_1}(\mathbb{R}^d;\mathbb{R}^d)\big]_{\theta^*}=W^{s,2}(\mathbb{R}^d;\mathbb{R}^d)\,.$$

By definition of the (anti)dual exponents t_0', t_1', p_0', p_1',

$$\Big[[W^{t_0',p_0'}(\mathbb{R}^d;\mathbb{R}^d)]^*,[W^{t_1',p_1'}(\mathbb{R}^d;\mathbb{R}^d)]^*\Big]_{\theta^*}=[W^{s,2}(\mathbb{R}^d;\mathbb{R}^d)]^*$$

for the same θ^*. Note also that the spaces $W^{t_0',p_0'}(\mathbb{R}^d;\mathbb{R}^d)$ and $W^{t_1',p_1'}(\mathbb{R}^d;\mathbb{R}^d)$ lie on the same line segment defined above. Thus, Steps I and II in tandem state that $\varpi\mathbb{I}_d+\mathbb{L}$ is a bounded linear operator from the complex interpolation scale $\big[W^{t_0,p_0}(\mathbb{R}^d;\mathbb{R}^d),W^{t_1,p_1}(\mathbb{R}^d;\mathbb{R}^d)\big]_{\theta}$ to $\Big[[W^{t_0',p_0'}(\mathbb{R}^d;\mathbb{R}^d)]^*,[W^{t_1',p_1'}(\mathbb{R}^d;\mathbb{R}^d)]^*\Big]_{\theta^*}$ for any $\theta\in(0,1)$, and it is invertible for some $\theta^*\in(0,1)$. We are now in a position to invoke the quantitative Schneiberg lemma, which states that invertibility at the interior point $(s,1/2)$ of this line segment implies invertibility on some open interval of the line segment containing the point $(s,1/2)$. The width of the line segment depends on the coercivity bound obtained in Step I and the continuity bounds in Step II. The inverses obtained coincide with the inverse obtained at the point $(s,1/2)$ on their common domains of definition. Finally, on every line segment through $(s,1/2)$ we can choose an open interval of the same width containing the point, and they sum up to a two-dimensional ε-neighborhood of $(s,1/2)$ in the $(t,1/p)$-plane, as desired. $\square$

Proof of Theorem 5.3 when $\mathfrak{h}=\infty$. Let $\mathbf{u}\in W^{s,2}(\mathbb{R}^d;\mathbb{R}^d)$ be a weak solution in the sense of (5.2) corresponding to $\mathbf{F}\in L^{2_{*s}+\delta_0}(\mathbb{R}^d;\mathbb{R}^d)\cap L^{2_{*s}}(\mathbb{R}^d;\mathbb{R}^d)$. Let $\varepsilon>0$ be as in Lemma 5.6, and let p satisfy (5.3). Define $t=\frac{d}{2}-\frac{d}{p}+s$, and define t', p' as in Lemma 5.6. We prove the theorem by showing that the solution $\mathbf{u}$ lives in the space $\mathbf{u}\in W^{t,p}(\mathbb{R}^d;\mathbb{R}^d)$ with appropriate bound, and that the function $\|\Upsilon^s(\mathbf{u})\|_{L^p(\mathbb{R}^d)}\leq C\,\|\mathbf{u}\|_{W^{t,p}(\mathbb{R}^d)}$, for some universal constant C. The latter is proved for scalar functions in [4] and its extension to vector fields follows easily.

To demonstrate the former we will use Lemma 5.6 and prove the estimate

$$\|\mathbf{u}\|_{W^{t,p}(\mathbb{R}^d)} \leq C\Big(\|\mathbf{F}\|_{L^{2_{*s}+\delta_0}(\mathbb{R}^d)} + \|\mathbf{F}\|_{L^{2_{*s}}(\mathbb{R}^d)} + \|\mathbf{u}\|_{W^{s,2}(\mathbb{R}^d)} \Big) . \tag{5.7}$$

To that end, since $\mathbf{u}$ is a weak solution to the coupled system in the sense of (5.2), we write (5.1) in the form

$$(\varpi \mathbb{I}_d + \mathbb{L})\, \mathbf{u} = \mathbf{F} + \varpi \mathbf{u} . \tag{5.8}$$

in the dual space where ϖ is as in Lemma 5.6 . Note that $t - s \in (0, \varepsilon)$ by choice of t and by (5.3). Thus by Lemma 5.6, the operator $\varpi \mathbb{I}_d + \mathbb{L} : W^{t,p}(\mathbb{R}^d;\mathbb{R}^d) \to [W^{t',p'}(\mathbb{R}^d;\mathbb{R}^d)]^*$ is invertible, and by (5.8)

$$\|\mathbf{u}\|_{W^{t,p}(\mathbb{R}^d)} \leq C \, \|\mathbf{F} + \varpi \mathbf{u}\|_{[W^{t',p'}(\mathbb{R}^d)]^*} \ , \tag{5.9}$$

provided the right-hand side is finite. We now show that this is in fact the case. Note that $t'p' < 2s < 2 \leq d$, and therefore by the fractional Sobolev embedding theorem and by choice of t,

$$\|\mathbf{F}\|_{W^{t',p'}(\mathbb{R}^d)]^*} \leq C \, \|\mathbf{F}\|_{L^r(\mathbb{R}^d)} \ , \qquad r = \frac{dp}{(2-\frac{p}{2})d + sp} .$$

The exponent r satisfies $r > 2_{*_s}$, with $r \to 2_{*_s}$ as $p \to 2$. Hence, for p satisfying (5.3) we have $r \in (2_{*_s}, 2_{*_s} + \delta_0)$. By the log-convexity of L^p norms there exists $\beta \in (0,1)$ such that $\|\mathbf{F}\|_{L^r(\mathbb{R}^d)} \leq \|\mathbf{F}\|^{\beta}_{L^{2_{*s}+\delta_0}(\mathbb{R}^d)} \|\mathbf{F}\|^{1-\beta}_{L^{2_{*s}}(\mathbb{R}^d)}$, and using the inequality $a^\beta b^{1-\beta} \leq a + b$,

$$\|\mathbf{F}\|_{[W^{t',p'}(\mathbb{R}^d)]^*} \leq C\Big(\|\mathbf{F}\|_{L^{2_{*s}+\delta_0}(\mathbb{R}^d)} + \|\mathbf{F}\|_{L^{2_{*s}}(\mathbb{R}^d)} \Big) .$$

To control the $(W^{t',p'})^*$ norm of $\mathbf{u}$ we follow a similar program. Treating $\mathbf{u}$ as a distribution acting on functions in $W^{t',p'}(\mathbb{R}^d;\mathbb{R}^d)$ via the L^2 inner product, by Hölder's inequality we easily have $\|\mathbf{u}\|_{[W^{t',p'}(\mathbb{R}^d)]^*} \leq \|\mathbf{u}\|_{L^p(\mathbb{R}^d)}$. Since $p \in [2, 2^{*_s})$ we have by log-convexity of L^p norms

$$\|\mathbf{u}\|_{[W^{t',p'}(\mathbb{R}^d)]^*} \leq C \, \|\mathbf{u}\|_{L^p(\mathbb{R}^d)} \leq C\Big(\|\mathbf{u}\|_{L^2(\mathbb{R}^d)} + \|\mathbf{u}\|_{L^{2^{*s}}(\mathbb{R}^d)} \Big) \leq C \, \|\mathbf{u}\|_{W^{s,2}(\mathbb{R}^d)} \ ,$$

where the last inequality follows from Sobolev embedding. Combining the two estimates for $\mathbf{F}$ and $\mathbf{u}$ gives

$$\|\mathbf{F} + \varpi \mathbf{u}\|_{[W^{t',p'}(\mathbb{R}^d)]^*} \leq C\Big(\|\mathbf{F}\|_{L^{2*_s+\delta_0}(\mathbb{R}^d)} + \|\mathbf{F}\|_{L^{2*_s}(\mathbb{R}^d)} + \|\mathbf{u}\|_{W^{s,2}(\mathbb{R}^d)} \Big). \tag{5.10}$$

Thus, the bound (5.7) follows from (5.9) and (5.10). Putting together these inequalities we obtain that

$$\|\Upsilon^s(\mathbf{u})\|_{L^p(\mathbb{R}^d)} \leq \|\mathbf{u}\|_{W^{t,p}(\mathbb{R}^d)} \leq C\Big(\|\mathbf{F}\|_{L^{2*_s+\delta_0}(\mathbb{R}^d)} + \|\mathbf{F}\|_{L^{2*_s}(\mathbb{R}^d)} + \|\mathbf{u}\|_{W^{s,2}(\mathbb{R}^d)} \Big), \tag{5.11}$$

as desired. □

Remark 5.7. One notices that in the course of the proof of Theorem 5.3 we actually proved that weak solutions $\mathbf{u}$ of the coupled nonlocal system satisfy *both higher integrability and higher differentiability* on a Sobolev scale. This self-improvement of solutions is of independent interest that has been studied in [4, 49] for nonlocal elliptic scalar equations. We restrict our attention to regularity of solutions in the scale of the Bessel potential spaces.

5.1.4 Higher Integrability When the Horizon is Finite

We now address the remaining case of the proof of Theorem 5.3 where the horizon $0 < \mathfrak{h} < \infty$, and $c_\mathfrak{h} \in (0, \infty)$. We begin with the following simple observation relating the bilinear forms $\mathcal{E}_\mathfrak{h}$ and $\mathcal{E}$.

Proposition 5.8. *For $p \in [1, \infty)$ there exists a bounded linear operator $\mathcal{P}_\mathfrak{h}$ from $L^p(\mathbb{R}^d; \mathbb{R}^d)$ to itself such that for any $\mathbf{u}, \mathbf{v} \in W^{s,2}(\mathbb{R}^d, \mathbb{R}^d)$, we have*

$$\mathcal{E}_\mathfrak{h}(\mathbf{u}, \mathbf{v}) = c_\mathfrak{h}\, \mathcal{E}(\mathbf{u}, \mathbf{v}) + \int_{\mathbb{R}^d} \langle \mathcal{P}_\mathfrak{h} \mathbf{u}(\mathbf{x}), \mathbf{v}(\mathbf{x}) \rangle \, d\mathbf{x}.$$

Proof. We write

$$
\begin{aligned}
\mathcal{E}_{\mathfrak{h}}(\mathbf{u},\mathbf{v}) &= c_{\mathfrak{h}}\, \mathcal{E}(\mathbf{u},\mathbf{v}) \\
&\quad + \frac{1}{2}\int_{\mathbb{R}^d}\int_{\mathbb{R}^d} k(\mathbf{x},\mathbf{y})(\mathbf{u}(\mathbf{x})-\mathbf{u}(\mathbf{y}))\cdot\frac{(\mathbf{x}-\mathbf{y})}{|\mathbf{x}-\mathbf{y}|}(\mathbf{v}(\mathbf{x})-\mathbf{v}(\mathbf{y}))\cdot\frac{(\mathbf{x}-\mathbf{y})}{|\mathbf{x}-\mathbf{y}|}\,\mathrm{d}\mathbf{x}\,\mathrm{d}\mathbf{y}\,,
\end{aligned}
$$

where $k(\mathbf{x},\mathbf{y}) = c_{\mathfrak{h}}\big(1-\chi_{B_{\mathfrak{h}}(\mathbf{0})}(|\mathbf{x}-\mathbf{y}|)\big)\frac{A(\mathbf{x},\mathbf{y})}{|\mathbf{x}-\mathbf{y}|^{d+2s}}$ is symmetric. Notice also that k is bounded and has no singularity along the diagonal $\mathbf{x}=\mathbf{y}$. Moreover, it decays fast enough at infinity that for a fixed $\mathbf{x}$, the function $k(\mathbf{x},\mathbf{y})$ is integrable in $\mathbf{y}$. We may then apply Fubini's theorem, iterate the integrals and get

$$
\begin{aligned}
&\frac{1}{2}\int_{\mathbb{R}^d}\int_{\mathbb{R}^d} k(\mathbf{x},\mathbf{y})(\mathbf{u}(\mathbf{x})-\mathbf{u}(\mathbf{y}))\cdot\frac{(\mathbf{x}-\mathbf{y})}{|\mathbf{x}-\mathbf{y}|}(\mathbf{v}(\mathbf{x})-\mathbf{v}(\mathbf{y}))\cdot\frac{(\mathbf{x}-\mathbf{y})}{|\mathbf{x}-\mathbf{y}|}\,\mathrm{d}\mathbf{x}\,\mathrm{d}\mathbf{y} \\
&= \int_{\mathbb{R}^d}\left(-\int_{\mathbb{R}^d} k(\mathbf{x},\mathbf{y})\frac{(\mathbf{x}-\mathbf{y})\otimes(\mathbf{x}-\mathbf{y})}{|\mathbf{x}-\mathbf{y}|^2}(\mathbf{u}(\mathbf{x})-\mathbf{u}(\mathbf{y}))\,\mathrm{d}\mathbf{y}\right)\mathbf{v}(\mathbf{x})\,\mathrm{d}\mathbf{x}.
\end{aligned}
$$

We now define the operator $\mathcal{P}_{\mathfrak{h}} : L^p(\mathbb{R}^d;\mathbb{R}^d)\to L^p(\mathbb{R}^d;\mathbb{R}^d)$ as

$$
\mathcal{P}_{\mathfrak{h}}\mathbf{u}(\mathbf{x}) = -\int_{\mathbb{R}^d} k(\mathbf{x},\mathbf{y})\frac{(\mathbf{x}-\mathbf{y})\otimes(\mathbf{x}-\mathbf{y})}{|\mathbf{x}-\mathbf{y}|^2}(\mathbf{u}(\mathbf{x})-\mathbf{u}(\mathbf{y}))\,\mathrm{d}\mathbf{y}, \quad \mathbf{x}\in\mathbb{R}^d.
$$

It is clear that the operator is linear. To show that it is bounded, we notice that

$$
\mathcal{P}_{\mathfrak{h}}\mathbf{u}(\mathbf{x}) = \int_{\mathbb{R}^d} k(\mathbf{x},\mathbf{y})\frac{(\mathbf{x}-\mathbf{y})\otimes(\mathbf{x}-\mathbf{y})}{|\mathbf{x}-\mathbf{y}|^2}\mathbf{u}(\mathbf{y})\,\mathrm{d}\mathbf{y} - \mathbb{K}(\mathbf{x})\mathbf{u}(\mathbf{x})
$$

where easy estimates show that $\mathbb{K}(\mathbf{x}) = \int_{\mathbb{R}^d} k(\mathbf{x},\mathbf{y})\frac{(\mathbf{x}-\mathbf{y})\otimes(\mathbf{x}-\mathbf{y})}{|\mathbf{x}-\mathbf{y}|^2}\,\mathrm{d}\mathbf{y}$ is a uniformly bounded matrix-valued function. The first term, on the other hand, is a convolution type operator and its magnitude is bounded from above by the function

$$
\alpha_2\gamma * |\mathbf{u}|(\mathbf{x}), \quad \text{where} \quad \gamma(\boldsymbol{\xi}) = c_{\mathfrak{h}}\Big(1-\chi_{B_{\mathfrak{h}}(\mathbf{0})}(|\boldsymbol{\xi}|)\Big)\frac{1}{|\boldsymbol{\xi}|^{d+2s}} \in L^1(\mathbb{R}^d).
$$

The boundedness of the operator $\mathcal{P}_{\mathfrak{h}}$ on $L^p(\mathbb{R}^d;\mathbb{R}^d)$ now follows from Young's inequality. $\square$

Note that the same identity is true for the corresponding operators, i.e. $\mathbb{L}_{\mathfrak{h}} = c_{\mathfrak{h}}\mathbb{L} + \mathcal{P}_{\mathfrak{h}}$.

Remark 5.9. It is now an easy corollary of the above proposition and Theorem 5.5 to state that for $0 < \mathfrak{h} \le \infty$, $\mathbf{u} \in L^2(\mathbb{R}^d;\mathbb{R}^d)$, $\mathcal{E}_{\mathfrak{h}}(\mathbf{u},\mathbf{u}) < \infty$ if and only if $\mathbf{u} \in W^{s,2}(\mathbb{R}^d;\mathbb{R}^d)$.

Proof of Theorem 5.3 when $0 < \mathfrak{h} < \infty$. We begin with a solution $\mathbf{u}$ to the system of nonlocal equations in the sense of (5.2) corresponding to $\mathbf{F} \in L^{2_{*s}+\delta_0}(\mathbb{R}^d,\mathbb{R}^d) \cap L^{2_{*s}}(\mathbb{R}^d,\mathbb{R}^d)$. Using Proposition 5.8, we can conclude that $\mathbf{u}$ satisfies

$$\mathcal{E}(\mathbf{u},\mathbf{v}) = \frac{1}{c_{\mathfrak{h}}}\left(\mathcal{E}_{\mathfrak{h}}(\mathbf{u},\mathbf{v}) - \int_{\mathbb{R}^d} \langle \mathcal{P}_{\mathfrak{h}}\mathbf{u}(\mathbf{x}), \mathbf{v}(\mathbf{x})\rangle \, \mathrm{d}\mathbf{x}\right) = \langle \mathbf{F}_{\mathfrak{h}}, \mathbf{v}\rangle$$

for any $\mathbf{v} \in W^{s,2}(\mathbb{R}^d,\mathbb{R}^d)$, where we have defined $\mathbf{F}_{\mathfrak{h}} = \frac{1}{c_{\mathfrak{h}}}(\mathbf{F} + \mathcal{P}_{\mathfrak{h}}\mathbf{u})$. That is, $\mathbf{u}$ solves a nonlocal system corresponding to infinite horizon in the sense of (5.2) with modified right-hand side data $\mathbf{F}_{\mathfrak{h}}$. We can therefore apply all of the arguments in the previous subsection so long as $\mathcal{P}_{\mathfrak{h}}(\mathbf{u})$ belongs to the space $[W^{t',p'}(\mathbb{R}^d;\mathbb{R}^d)]^*$. Since $p \in (2, 2^{*_s})$ and $\mathbf{u} \in L^{2^{*s}}$ from the fractional Sobolev embedding theorem, it follows from Proposition 5.8 that $\mathcal{P}_{\mathfrak{h}}(\mathbf{u}) \in L^p(\mathbb{R}^d;\mathbb{R}^d)$. Moreover, by our choice of t and p, we have the estimate

$$\begin{aligned}
\|\mathcal{P}_{\mathfrak{h}}(\mathbf{u})\|_{[W^{t',p'}(\mathbb{R}^d)]^*} &\le C\,\|\mathcal{P}_{\mathfrak{h}}(\mathbf{u})\|_{L^p(\mathbb{R}^d)} \\
&\le C\Big(\|\mathcal{P}_{\mathfrak{h}}(\mathbf{u})\|_{L^2(\mathbb{R}^d)} + \|\mathcal{P}_{\mathfrak{h}}(\mathbf{u})\|_{L^{2^{*s}}(\mathbb{R}^d)}\Big) \\
&\le C\left(\|\mathbf{u}\|_{L^2(\mathbb{R}^d)} + \|\mathbf{u}\|_{L^{2^{*s}}(\mathbb{R}^d)}\right) \\
&\le C\|\mathbf{u}\|_{W^{s,2}(\mathbb{R}^d)}\,.
\end{aligned}$$

We can now apply the higher integrability result in case of infinite horizon to conclude that $\Upsilon^s(\mathbf{u}) \in L^p(\mathbb{R}^d)$ with the appropriate estimates. That concludes the proof. □

5.2 Self Improving Properties of a Nonlocal Nonlinear System

5.2.1 Introduction

As an application of the Fractional Korn's Inequality (Theorem 4.1) we show improved Sobolev regularity of weak solutions to a system of nonlinear nonlocal equations formally given as

$$\mathbb{L}^s_{p,\Omega}(\mathbf{u})(\mathbf{x}) = \mathbf{F}(\mathbf{x})\,, \tag{5.12}$$

where the operator $\mathbb{L}^s_{p,\Omega}$ is defined as

$$\mathbb{L}^s_{p,\Omega}(\mathbf{u})(\mathbf{x}) := \int_\Omega \frac{A(\mathbf{x},\mathbf{y})}{|\mathbf{x}-\mathbf{y}|^{d+2s}}|\mathcal{D}(\mathbf{u})(\mathbf{x},\mathbf{y})|^{p-2}\mathcal{D}(\mathbf{u})(\mathbf{x},\mathbf{y})\frac{\mathbf{x}-\mathbf{y}}{|\mathbf{x}-\mathbf{y}|}\,\mathrm{d}\mathbf{y}\,.$$

In the above Ω is a bounded subset of $\mathbb{R}^d$ for $d \geq 2$, the functions $\mathbf{F}$, $\mathbf{u} : \mathbb{R}^d \to \mathbb{R}^d$, and the quantity $\mathcal{D}(\mathbf{u})$ is given by

$$\mathcal{D}(\mathbf{u})(\mathbf{x},\mathbf{y}) := (\mathbf{u}(\mathbf{x}) - \mathbf{u}(\mathbf{y}))\cdot\frac{\mathbf{x}-\mathbf{y}}{|\mathbf{x}-\mathbf{y}|}\,.$$

We also assume that $s \in (0,1)$, $p \geq 2$, and that $A(\mathbf{x},\mathbf{y})$ is a measurable function such that $\alpha_1 \leq A(\mathbf{x},\mathbf{y}) \leq \alpha_2$ and symmetric in the sense that $A(\mathbf{x},\mathbf{y}) = A(\mathbf{y},\mathbf{x})$ for any $\mathbf{x}$, $\mathbf{y} \in \mathbb{R}^d$. Properly speaking, for a given vector field $\mathbf{u} \in \mathcal{X}^s_p(\Omega)$, the operator $\mathbb{L}^s_{p,\Omega}(\mathbf{u})$ is a vector of distributions acting on test functions $\varphi \in C^\infty_0(\mathbb{R}^d;\mathbb{R}^d)$ via

$$\langle \mathbb{L}^s_{p,\Omega}(\mathbf{u}), \varphi\rangle = \int_\Omega\int_\Omega |\mathcal{D}(\mathbf{u})(\mathbf{x},\mathbf{y})|^{p-2}\,\mathcal{D}(\mathbf{u})(\mathbf{x},\mathbf{y})\cdot\mathcal{D}(\varphi)(\mathbf{x},\mathbf{y})\frac{A(\mathbf{x},\mathbf{y})}{|\mathbf{x}-\mathbf{y}|^{d+sp}}\,\mathrm{d}\mathbf{y}\,\mathrm{d}\mathbf{x}\,. \tag{5.13}$$

Let $\mathbf{F} \in [\mathcal{X}^s_p(\Omega)]^*$, the dual space of $\mathcal{X}^s_p(\Omega)$, be given. We say $\mathbf{u} \in \mathcal{X}^s_p(\Omega)$ is a *weak solution* to the nonlocal system 5.12 if for all $\varphi \in C^\infty_0(\mathbb{R}^d;\mathbb{R}^d)$,

$$\langle \mathbb{L}^s_{p,\Omega}(\mathbf{u}), \varphi\rangle = \langle \mathbf{F}, \varphi\rangle$$

where $\langle\cdot,\cdot\rangle$ is the duality pairing between $[\mathcal{X}^s_p(\Omega)]^*$ and $\mathcal{X}^s_p(\Omega)$.

The second main result of this chapter is an *a priori* estimate for the self-improving properties of solutions to the nonlocal nonlinear coupled system 5.12. The following is the precise statement we will prove.

Theorem 5.10. *Let $s \in (0,1)$, $p \geq 2$, and $\Omega \subset \mathbb{R}^d$ be a bounded domain. Let $\mathbf{F} \in [\mathcal{X}^{s-\varepsilon(p-1)}_p(\Omega)]^*$, and let $\mathbf{u} \in C^1_{loc}(\Omega;\mathbb{R}^d) \cap \mathcal{X}^s_p(\Omega)$ be a weak solution to the coupled system of nonlocal equations 5.12 corresponding to $\mathbf{F}$. Then there exists a positive constant ε_0 such that for all $\varepsilon \in (0,\varepsilon_0)$ the weak solution $\mathbf{u}$ belongs to $W^{s+\varepsilon,p}_{loc}(\Omega)$. Moreover, for any $\eta \in C^\infty_c(\Omega)$,* *there exists a positive constant C such that*

$$[\eta \mathbf{u}]_{W^{s+\varepsilon,p}(\mathbb{R}^d)} \leq C\|\mathbf{F}\|^{\frac{1}{p-1}}_{\left[\mathcal{X}_p^{s-\varepsilon(p-1)}(\Omega)\right]^*} + C\|\mathbf{u}\|_{\mathcal{X}_p^s(\Omega)} .$$

To prove Theorem 5.10, we follow the approach in [76] and is close in spirit with the technique of "differentiating the equation," and finding relations between higher derivatives of solutions and test functions in order to estimate derivatives of the solution. This is possible for classical linear equations via integration by parts and transferring derivatives to test functions. For a special case of the nonlocal system at hand, for $p = 2$, $A = \kappa_{d,s}$ (see Chapter 2), and $\Omega = \mathbb{R}^d$ we can demonstrate this easily. First notice that we can write the operator $\mathbb{L}^s_{2,\mathbb{R}^d}$ in Fourier symbols as

$$\mathcal{F}(\mathbb{L}^s_{2,\mathbb{R}^d}\mathbf{u})(\boldsymbol{\xi}) = \left(2\pi|\boldsymbol{\xi}|\right)^{2s}\left(\mathbb{I} + 2s\frac{\boldsymbol{\xi}\otimes\boldsymbol{\xi}}{|\boldsymbol{\xi}|^2}\right) ;$$

see [29, 57]. Then for $\varepsilon > 0$ small, via Plancherel's theorem the equality $\langle \mathbb{L}^{s+\varepsilon}_{2,\mathbb{R}^d}\mathbf{u}, \varphi\rangle = \langle \mathbb{L}^s_{2,\mathbb{R}^d}\mathbf{u}, (-\Delta)^\varepsilon\varphi\rangle + \langle(-\Delta)^s\mathbf{u}, \mathbb{L}^\varepsilon_{2,\mathbb{R}^d}\varphi\rangle - \langle(-\Delta)^s\mathbf{u}, (-\Delta)^\varepsilon\varphi\rangle$ holds, where for any α the operator $(-\Delta)^\alpha\varphi$ is the α-fractional Laplacian. When working with the nonlinear "regional" operator $\mathbb{L}^s_{p,\Omega}(\mathbf{u})$, such a clean transfer of derivatives to the test function is not possible. However, as has been done in [76] one can measure the price of transferring the derivatives by estimating the resulting commutator. Unlike [76], the estimates we establish are based on the smaller $[\cdot]_{\mathcal{X}_p^s}$ norm, leading us to write some arguments closely resembling those in [76]. Afterward, we use the fractional Korn-type inequality (Theorem 4.1) to conclude that the estimates are also valid using the larger semi-norm $[\cdot]_{W^{s,p}}$.

5.2.2 Preliminaries

Given a ball $B \subset \mathbb{R}^d$ with radius r, κB represents a ball with the same center but with radius κr. Note that for a given $\mathbf{u} \in \mathcal{X}_p^s(2B)$ and $\eta \in C_c^\infty(2B)$ with $\eta \equiv 1$ in B, the function $\eta\mathbf{u} \in \mathcal{X}_p^s(\mathbb{R}^d)$. Moreover,

$$[\eta\mathbf{u}]_{\mathcal{X}_p^s(\mathbb{R}^d)} \leq C\|\mathbf{u}\|_{\mathcal{X}_p^s(2B)}$$

for a constant C independent of $\mathbf{u}$.

We also recall that for $\mathbf{u} \in \mathcal{X}_p^s(\Omega)$, we have that $\mathbb{L}_{p,\Omega}^s(\mathbf{u}) \in [\mathcal{X}_p^s(\Omega)]^*$. Indeed, by (5.13) and Hölder's inequality we have

$$|\langle \mathbb{L}_{p,\Omega}^s(\mathbf{u}), \phi \rangle| \leq C[\mathbf{u}]_{\mathcal{X}_p^s(\Omega)}[\phi]_{\mathcal{X}_p^s(\Omega)}, \quad \text{for all } \phi \in \mathcal{X}_p^s(\Omega)\,.$$

More generally, for $t \in (0,1)$, we define the dual norm of $\mathbb{L}_{p,\Omega}^s(\mathbf{u})$ in $\mathring{\mathcal{X}}_p^t(\Omega)$ by

$$\|\mathbb{L}_{p,\Omega}^s(\mathbf{u})\|_{[\mathring{\mathcal{X}}_p^t(\Omega)]^*} := \sup_{\phi} |\langle \mathbb{L}_{p,\Omega}^s(\mathbf{u}), \phi \rangle|\,,$$

where the supremum is taken over all $\phi \in C_c^\infty(\Omega; \mathbb{R}^d)$ such that $[\phi]_{\mathcal{X}_p^t(\mathbb{R}^d)} \leq 1$. The fractional Laplacian $(-\Delta)^\beta \mathbf{u}$ is defined as

$$(-\Delta)^\beta \mathbf{u}(\mathbf{x}) = c_{s,d}\, \mathcal{F}^{-1}(|\boldsymbol{\xi}|^{2\beta} \mathcal{F}(\mathbf{u}))\,,$$

where $c_{s,d} > 0$ is a normalizing constant.

The fractional Laplacian has a quasi-local behavior that has been quantified in the following estimates: Let $p, q \in [1, \infty]$ and $s \in (0,1)$. For any Ω_1 and Ω_2 be bounded, disjoint open sets such that $\rho := \mathrm{dist}(\Omega_1, \Omega_2) > 0$ and for any $\varphi \in C_c^\infty(\mathbb{R}^d)$,

$$\|(-\Delta)^s(\varphi \chi_{\Omega_2})\|_{L^p(\Omega_1)} \leq \rho^{-d-2s} |\Omega_1|^{1/p} |\Omega_2|^{1-1/q} \|\varphi\|_{L^q(\Omega_2)}\,. \tag{5.14}$$

This is established in [74, Lemma A.1]. We also state the following technical lemma – in a form needed for our purposes – that summarizes the action of the fractional Laplacian as a potential. The result in this lemma is contained in an inequality in [76], but we will give the proof here for the sake of completeness.

Lemma 5.11. *Let $p, q \in [1, \infty]$ and $s \in (0,1)$. Let $\varepsilon \in (0, 1-s)$. Suppose that $B \subset \mathbb{R}^d$ is a ball, $\varphi \in C_c^\infty(4B; \mathbb{R}^d)$ with $[\varphi]_{\mathcal{X}_p^{s+\varepsilon}(\mathbb{R}^d)} \leq 1$, and $\eta \in C_c^\infty(6B)$ and $\eta \equiv 1$ on $5B$. Then the function $\eta(-\Delta)^{\frac{\varepsilon p}{2}} \varphi \in C_c^\infty(6B; \mathbb{R}^d)$, and there exists a constant $C > 0$, independent of φ such that*

$$[\eta(-\Delta)^{\frac{\varepsilon p}{2}} \varphi]_{\mathcal{X}_p^{s-\varepsilon(p-1)}(\mathbb{R}^d)} \leq C[\varphi]_{\mathcal{X}_p^{s+\varepsilon}(\mathbb{R}^d)}\,. \tag{5.15}$$

Proof. Adding and subtracting $\eta(\mathbf{y})(-\Delta)^{\frac{\varepsilon p}{2}}\varphi(\mathbf{x})$ we have

$$
\begin{aligned}
&[\eta(-\Delta)^{\frac{\varepsilon p}{2}}\varphi]^p_{\mathcal{X}_p^{s-\varepsilon(p-1)}(\mathbb{R}^d)} \\
&= \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} \frac{\left|\left(\eta(\mathbf{x})(-\Delta)^{\frac{\varepsilon p}{2}}\varphi(\mathbf{x}) - \eta(\mathbf{y})(-\Delta)^{\frac{\varepsilon p}{2}}\varphi(\mathbf{y})\right)\cdot \frac{\mathbf{y}-\mathbf{x}}{|\mathbf{y}-\mathbf{x}|}\right|^p}{|\mathbf{x}-\mathbf{y}|^{d+(s-\varepsilon(p-1))p}}\,\mathrm{d}\mathbf{x}\,\mathrm{d}\mathbf{y} \\
&\leq C[(-\Delta)^{\frac{\varepsilon p}{2}}\varphi]^p_{W^{s-\varepsilon(p-1),p}(\mathbb{R}^d)} + C\int_{\mathbb{R}^d}\int_{\mathbb{R}^d} \frac{|\eta(\mathbf{x})-\eta(\mathbf{y})|^p|(-\Delta)^{\frac{\varepsilon p}{2}}\varphi(\mathbf{x})|^p}{|\mathbf{x}-\mathbf{y}|^{d+(s-\varepsilon(p-1))p}}\,\mathrm{d}\mathbf{x}\,\mathrm{d}\mathbf{y} \\
&= I + II\,.
\end{aligned}
$$

We estimate the first term I first. We will use the following identity that relates Riesz and Bessel potentials, which can be found in [86, Lemma 2, Chapter V], that there exists a finite measure μ that depends on ε and p such that

$$
(-\Delta)^{\frac{\varepsilon p}{2}}\varphi = (1-\Delta)^{\frac{\varepsilon p}{2}}(\varphi * \mu)\,.
$$

Using the fact that the operator

$$
(1-\Delta)^{\frac{\varepsilon p}{2}} : W^{s+\varepsilon,p}(\mathbb{R}^d) \;=\; W^{s-\varepsilon(p-1)+\varepsilon p,p}(\mathbb{R}^d) \to W^{s-\varepsilon(p-1),p}(\mathbb{R}^d)
$$

is an isomorphism [86, Theorem 4', Chapter V], we have

$$
I \leq C\,\|\varphi * \mu\|^p_{W^{s+\varepsilon,p}(\mathbb{R}^d)}\ .
$$

Now by Jensen's inequality and Fubini's theorem,

$$
\begin{aligned}
&\int_{\mathbb{R}^d}\int_{\mathbb{R}^d} \frac{|(\varphi * \mu)(\mathbf{x}) - (\varphi * \mu)(\mathbf{y})|^p}{|\mathbf{x}-\mathbf{y}|^{d+(s+\varepsilon)p}}\,\mathrm{d}\mathbf{x}\,\mathrm{d}\mathbf{y} \\
&= \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} \left|\int_{\mathbb{R}^d} \big(\varphi(\mathbf{z}-\mathbf{x}) - \varphi(\mathbf{z}-\mathbf{y})\big)\,\mathrm{d}\mu(\mathbf{z})\right|^p \frac{1}{|\mathbf{x}-\mathbf{y}|^{d+(s+\varepsilon)p}}\,\mathrm{d}\mathbf{x}\,\mathrm{d}\mathbf{y} \\
&\leq \mu(\mathbb{R}^d)^{p-1}\int_{\mathbb{R}^d}\int_{\mathbb{R}^d}\int_{\mathbb{R}^d} \frac{|\varphi(\mathbf{z}-\mathbf{x}) - \varphi(\mathbf{z}-\mathbf{y})|^p}{|\mathbf{x}-\mathbf{y}|^{d+(s+\varepsilon)p}}\,\mathrm{d}\mu(\mathbf{z})\,\mathrm{d}\mathbf{x}\,\mathrm{d}\mathbf{y} \\
&= \mu(\mathbb{R}^d)^p[\varphi]^p_{W^{s+\varepsilon,p}(\mathbb{R}^d)}\,.
\end{aligned}
$$

Similarly,

$$\|\varphi * \mu\|_{L^p}^p \leq \mu(\mathbb{R}^d)^p \|\varphi\|_{L^p}^p \leq C\mu(\mathbb{R}^d)^p [\varphi]_{W^{s+\varepsilon,p}(\mathbb{R}^d)}^p$$

where the last inequality follows from a Poincaré-Korn type inequality, Corollary 4.10, where C depends on the support set $4B$, p, and ε. Combining the above two inequalities and using the fractional Korn's inequality we proved we have that,

$$I \leq C\, \mu(\mathbb{R}^d)^p [\varphi]_{\mathcal{X}_p^{s+\varepsilon}(\mathbb{R}^d)}^p .$$

To estimate the second term II we proceed as follows.

$$\begin{aligned}
II &\leq C \int_{\mathbb{R}^d\setminus 7B}\int_{7B} \frac{|\eta(\mathbf{x})-\eta(\mathbf{y})|^p |(-\Delta)^{\frac{\varepsilon p}{2}}\varphi(\mathbf{x})|^p}{|\mathbf{x}-\mathbf{y}|^{d+(s-\varepsilon(p-1))p}}\,\mathrm{d}\mathbf{x}\,\mathrm{d}\mathbf{y} + C\int_{7B}\int_{7B}\cdots\,\mathrm{d}\mathbf{x}\,\mathrm{d}\mathbf{y}\\
&\leq C\,\|\eta\|_{L^\infty}\int_{\mathbb{R}^d\setminus 7B}\int_{7B} \frac{|(-\Delta)^{\frac{\varepsilon p}{2}}\varphi(\mathbf{x})|^p}{|\mathbf{x}-\mathbf{y}|^{d+(s-\varepsilon(p-1))p}}\,\mathrm{d}\mathbf{x}\,\mathrm{d}\mathbf{y}\\
&\qquad + C\,\|\nabla\eta\|_{L^\infty}\int_{7B}\int_{7B} \frac{|(-\Delta)^{\frac{\varepsilon p}{2}}\varphi(\mathbf{x})|^p}{|\mathbf{x}-\mathbf{y}|^{d+(s-\varepsilon(p-1))p}}\,\mathrm{d}\mathbf{x}\,\mathrm{d}\mathbf{y}\\
&\leq C\left\|(-\Delta)^{\frac{\varepsilon p}{2}}\varphi\right\|_{L^p(\mathbb{R}^d)}^p \leq C\left\|(-\Delta)^{\frac{\varepsilon p}{2}}\varphi\right\|_{W^{s-\varepsilon(p-1),p}(\mathbb{R}^d)}^p .
\end{aligned}$$

We repeat the argument used to bound I and get that

$$II \leq C\,\|\varphi\|_{W^{s+\varepsilon,p}(\mathbb{R}^d)}^p \leq C\mu(\mathbb{R}^d)^p [\varphi]_{\mathcal{X}_p^{s+\varepsilon}(\mathbb{R}^d)}^p$$

where again we have applied the Poincaré-Korn and fractional Korn inequalities. Thus 5.15 is proved. $\square$

Lemma 5.12. *Let $\Omega \subset \mathbb{R}^d$ be an open set, and let $s \in (0,1)$, $p \geq 2$. Suppose $\mathbf{u} \in \mathcal{X}_p^s(\mathbb{R}^d)$ such that $\operatorname{supp}\mathbf{u} \subset\subset \Omega$. Then there exists a sequence $(\mathbf{u}_n) \in C_c^\infty(\Omega;\mathbb{R}^d)$ with the property that*

$$\mathbf{u}_n \to \mathbf{u} \quad \textit{in } \mathcal{X}_p^s(\mathbb{R}^d)$$

as $n \to \infty$.

Proof. The sequence $\mathbf{u}_n$ will be obtained via mollification. Let $\phi \in C_c^\infty(\mathbb{R}^d)$ be a standard mollifier, i.e.

$$\phi \geq 0\,, \quad \operatorname{supp}\phi \subset B_1(\mathbf{0})\,, \quad \int_{\mathbb{R}^d} \phi(\mathbf{x})\,\mathrm{d}\mathbf{x} = 1\,.$$

For any $\delta < \frac{1}{4}\mathrm{dist}(\operatorname{supp}\mathbf{u}, \mathbb{R}^d \setminus \Omega)$, introduce $\phi_\delta \in C_c^\infty(\mathbb{R}^d)$ by $\phi_\delta(\mathbf{x}) := \frac{1}{\delta^d}\phi\left(\frac{\mathbf{x}}{\delta}\right)$. Then take

$$\mathbf{u}_\delta := \mathbf{u} * \phi_\delta \in C_c^\infty(\Omega;\mathbb{R}^d)\,.$$

Extending the vector fields with value zero outside of Ω, we can easily show via basic estimates that $\|\mathbf{u}_n - \mathbf{u}\|_{\mathcal{X}_p^s(\mathbb{R}^d)} \to 0$ as $\delta \to 0$. □

The following is an adaptation to our setting of the interpolation lemma proved in [76].

Lemma 5.13. *Let $B \subset \mathbb{R}^d$ be a ball . Then for any $\delta > 0$ and any $\mathbf{u} \in \mathcal{X}_p^s(4B)$, we have that*

$$[\mathbf{u}]^p_{\mathcal{X}_p^s(B)} \leq \delta^p [\mathbf{u}]^p_{\mathcal{X}_p^s(4B)} + \frac{C}{\delta^{p'}}\left(\sup_\phi \langle \mathbb{L}^s_{p,4B}\mathbf{u}, \phi\rangle\right)^{p'} + C\frac{\mathrm{diam}(B)^{-sp}}{\delta^{p(p-1)}}\int_{4B} |\mathbf{u}(\mathbf{x})|^p\,\mathrm{d}\mathbf{x}\,,$$

where the supremum is over all $\phi \in C_c^\infty(2B;\mathbb{R}^d)$ and $[\phi]_{\mathcal{X}_p^s(\mathbb{R}^d)} \leq 1$. The constant $C > 0$ depends only on d, s, p, and the ellipticity constants α_1 and α_2. Moreover if $s_0 > 0$, and $s \in (s_0, 1)$, then the constant C can be made dependent on s_0 instead of s.

Proof. Let $\eta \in C_c^\infty(2B)$, $\eta \equiv 1$ in B be the usual cutoff function in $2B$.

Define

$$\psi(\mathbf{x}) := \eta(\mathbf{x})\mathbf{u}(\mathbf{x})\,, \qquad \varphi(\mathbf{x}) := \eta^2(\mathbf{x})\mathbf{u}(\mathbf{x})\,.$$

Then

$$[\psi]_{\mathcal{X}_p^s(\mathbb{R}^d)} + [\varphi]_{\mathcal{X}_p^s(\mathbb{R}^d)} \leq C\,\|\mathbf{u}\|_{\mathcal{X}_p^s(2B)}\,. \tag{5.16}$$

By definition of ψ and using the lower bound on K we have

$$\begin{aligned}[\mathbf{u}]^p_{\mathcal{X}_p^s(B)} &\leq \alpha_1 \int_B\int_B |\mathcal{D}(\mathbf{u})(\mathbf{x},\mathbf{y})|^p \frac{K(\mathbf{x},\mathbf{y})}{|\mathbf{x}-\mathbf{y}|^{d+sp}}\,\mathrm{d}\mathbf{x}\,\mathrm{d}\mathbf{y} \\ &\leq \alpha_2 \int_{4B}\int_{4B} |\mathcal{D}(\mathbf{u})(\mathbf{x},\mathbf{y})|^{p-2}\left(\mathcal{D}(\psi)(\mathbf{x},\mathbf{y})\right)^2 \frac{K(\mathbf{x},\mathbf{y})}{|\mathbf{x}-\mathbf{y}|^{d+sp}}\,\mathrm{d}\mathbf{x}\,\mathrm{d}\mathbf{y}\,.\end{aligned}$$

We now use the following algebraic identity: for a, b, c, d real numbers

$$(ab-cd)^2 = (ab-cd)(a-c)b + ab(c-a)(b-d) + (a^2b-c^2d)(b-d)\,.$$

we can thus expand $(\mathcal{D}(\psi)(\mathbf{x},\mathbf{y}))^2$ as

$$\begin{aligned}(\mathcal{D}(\psi)(\mathbf{x},\mathbf{y}))^2 = &(\eta(\mathbf{x})-\eta(\mathbf{y}))(\mathcal{D}(\psi)(\mathbf{x},\mathbf{y}))\mathbf{u}(\mathbf{x})\cdot\frac{(\mathbf{y}-\mathbf{x})}{|\mathbf{y}-\mathbf{x}|}\\ &+(\eta(\mathbf{y})-\eta(\mathbf{x}))\mathcal{D}(\mathbf{u})(\mathbf{x},\mathbf{y})\psi(\mathbf{x})\cdot\frac{(\mathbf{y}-\mathbf{x})}{|\mathbf{y}-\mathbf{x}|}\\ &+(\mathcal{D}(\varphi)(\mathbf{x},\mathbf{y}))(\mathcal{D}(\mathbf{u})(\mathbf{x},\mathbf{y}))\,.\end{aligned}$$

Now, by adding and subtracting the appropriate quantities and splitting the integral accordingly, we have

$$[\mathbf{u}]^p_{\mathcal{X}_p^s(B)} \leq \langle \mathbb{L}^s_{p,4B}\mathbf{u},\varphi\rangle + I_1 + I_2\,,$$

where

$$I_1 := \Lambda\int_{4B}\int_{4B}\frac{|\mathcal{D}(\mathbf{u})(\mathbf{x},\mathbf{y})|^{p-2}\,|\eta(\mathbf{x})-\eta(\mathbf{y})|\,|\mathcal{D}(\psi)(\mathbf{x},\mathbf{y})|\left|\mathbf{u}(\mathbf{x})\cdot\frac{\mathbf{x}-\mathbf{y}}{|\mathbf{x}-\mathbf{y}|}\right|}{|\mathbf{x}-\mathbf{y}|^{d+sp}}\,\mathrm{d}\mathbf{x}\,\mathrm{d}\mathbf{y}\,,$$

$$I_2 := \Lambda\int_{4B}\int_{4B}\frac{|\mathcal{D}(\mathbf{u})(\mathbf{x},\mathbf{y})|^{p-1}\,|\eta(\mathbf{y})-\eta(\mathbf{x})|\,\left|\psi(\mathbf{x})\cdot\frac{\mathbf{x}-\mathbf{y}}{|\mathbf{x}-\mathbf{y}|}\right|}{|\mathbf{x}-\mathbf{y}|^{d+sp}}\,\mathrm{d}\mathbf{x}\,\mathrm{d}\mathbf{y}\,.$$

Note that since the vector field $\varphi \in \mathcal{X}^{s,p}(\mathbb{R}^d)$ and since $\operatorname{supp}\varphi \subset\subset 2B$, by the density of smooth functions proved in Lemma 5.12

$$\langle \mathbb{L}^s_{p,4B}\mathbf{u},\varphi\rangle \leq \sup_{\phi}\langle \mathbb{L}^s_{p,4B}\mathbf{u},\phi\rangle\|\varphi\|_{\mathcal{X}_p^s(\mathbb{R}^d)} \leq C\,\sup_{\phi}\langle \mathbb{L}^s_{p,4B}\mathbf{u},\phi\rangle\|\mathbf{u}\|_{\mathcal{X}_p^s(4B)}\,,$$

where the supremum is over all $\phi \in C_c^\infty(2B;\mathbb{R}^d)$ and $[\phi]_{\mathcal{X}_p^s(\mathbb{R}^d)} \leq 1$. As for I_1, using the estimate that $\|\nabla\eta\|_{L^\infty} \leq C\frac{1}{\operatorname{diam}(B)}$,

$$I_1 \leq C\operatorname{diam}(B)^{-1}\int_{4B}\int_{4B}|\mathcal{D}(\mathbf{u})(\mathbf{x},\mathbf{y})|^{p-2}\,\frac{|\mathcal{D}(\psi)(\mathbf{x},\mathbf{y})|\,|\mathbf{u}(\mathbf{x})|}{|\mathbf{x}-\mathbf{y}|^{d+sp-1}}\,\mathrm{d}\mathbf{x}\,\mathrm{d}\mathbf{y}\,.$$

Let $t_2 = 1 - s$. Then $d + sp - 1 = d + s(p-2) + s - t_2$, and Hölder's inequality with $q = \frac{p}{p-2}$, $q' = \frac{p}{2}$ implies that

$$
\begin{aligned}
I_1 &\leq C\,\mathrm{diam}(B)^{-1} \int_{4B}\int_{4B} \frac{|\mathcal{D}(\mathbf{u})(\mathbf{x},\mathbf{y})|^{p-2}}{|\mathbf{x}-\mathbf{y}|^{d(\frac{p-2}{p})+s(p-2)}} \cdot \frac{|\mathcal{D}(\psi)(\mathbf{x},\mathbf{y})||\mathbf{u}(\mathbf{x})|}{|\mathbf{x}-\mathbf{y}|^{d(\frac{2}{p})+s-t_2}}\,\mathrm{d}\mathbf{x}\,\mathrm{d}\mathbf{y} \\
&\leq C\,\mathrm{diam}(B)^{-1}[\mathbf{u}]^{p-2}_{\mathcal{X}_p^s(4B)} \left(\int_{4B}\int_{4B} \frac{|\mathcal{D}(\psi)(\mathbf{x},\mathbf{y})|^{p/2}|\mathbf{u}(\mathbf{x})|^{p/2}}{|\mathbf{x}-\mathbf{y}|^{d+\frac{sp}{2}-\frac{t_2 p}{2}}}\,\mathrm{d}\mathbf{x}\,\mathrm{d}\mathbf{y} \right)^{2/p}.
\end{aligned}
$$

Applying Cauchy-Schwarz to the last integral,

$$
I_1 \leq C\,\mathrm{diam}(B)^{-1}[\mathbf{u}]^{p-2}_{\mathcal{X}_p^s(4B)}[\psi]_{\mathcal{X}_p^s(4B)} \left(\int_{4B}\int_{4B} \frac{|\mathbf{u}(\mathbf{x})|^p}{|\mathbf{x}-\mathbf{y}|^{d-t_2 p}}\,\mathrm{d}\mathbf{x}\,\mathrm{d}\mathbf{y} \right)^{1/p}.
$$

Thus, since $t_2 > 0$,

$$
\int_{4B}\int_{4B} \frac{|\mathbf{u}(\mathbf{x})|^p}{|\mathbf{x}-\mathbf{y}|^{d-t_2 p}}\,\mathrm{d}\mathbf{x}\,\mathrm{d}\mathbf{y} \leq C\,(\mathrm{diam}(B))^{t_2 p}\,\|\mathbf{u}\|^p_{L^p(4B)}.
$$

Using again 5.16, the final estimate of I_1 is

$$
I_1 \leq C\,\mathrm{diam}(B)^{-s}\,\|\mathbf{u}\|^{p-1}_{\mathcal{X}_p^s(4B)}\,\|\mathbf{u}\|_{L^p(4B)}.
$$

The integral I_2 can be estimated the same way as I_1. Therefore,

$$
[\mathbf{u}]^p_{\mathcal{X}_p^s(B)} \leq C \left(\sup_{\phi} \langle \mathbb{L}^s_{p,4B}\mathbf{u}, \phi\rangle \|\mathbf{u}\|_{\mathcal{X}_p^s(4B)} + \mathrm{diam}(B)^{-s}\,\|\mathbf{u}\|^{p-1}_{\mathcal{X}_p^s(4B)}\,\|\mathbf{u}\|_{L^p(4B)} \right),
$$

where the supremum is over all $\phi \in C_c^\infty(2B;\mathbb{R}^d)$ and $[\phi]_{\mathcal{X}_p^s(\mathbb{R}^d)} \leq 1$. From the last estimate we apply Young's inequality to obtain the result and conclude the proof. □

5.2.3 Higher Differentiability of Solutions

In this section we prove the main result of Section 5.2. Before presenting the proof, we state a commutator estimate that is an adaptation of the commutator estimate established in [76]. The proof of the theorem is essentially identical to the result given in [76], and we omit it here.

Theorem 5.14. *Let $s \in (0,1)$, $\varepsilon \in [0, 1-s)$. Take $B \subset \mathbb{R}^d$ a ball or all of $\mathbb{R}^d$. Let $\mathbf{u} \in \mathcal{X}_p^s(B)$ and $\varphi \in C_c^\infty(B;\mathbb{R}^d)$. For a certain normalizing constant c depending on s, p, and ε denote the commutator*

$$R_\varepsilon(\mathbf{u}, \phi) := \langle \mathbb{L}_{p,B}^{s+\varepsilon}\mathbf{u}, \varphi\rangle - c\,\langle \mathbb{L}_{p,B}^{s}\mathbf{u}, (-\Delta)^{\frac{\varepsilon p}{2}}\varphi\rangle\,.$$

Then there exists a constant $C_\varepsilon = C(s,p,\varepsilon,n,\Lambda) > 0$ such that

$$|R_\varepsilon(\mathbf{u},\varphi)| \leq C_\varepsilon\,\varepsilon\,[\mathbf{u}]^{p-1}_{\mathcal{X}_p^{s+\varepsilon}(B)}[\varphi]_{\mathcal{X}_p^{s+\varepsilon}(\mathbb{R}^d)}\,.$$

Moreover, C_ε is monotone increasing in ε. That is $C_\varepsilon \leq C_{\varepsilon_0}$ for any $0 < \varepsilon < \varepsilon_0$.

Proof of Theorem 5.10. Let $\Omega_0 \subset\subset \Omega_1 \subset\subset \Omega_2 \subset\subset \Omega$ be given, and $\eta \in C_c^\infty(\Omega_1)$ such that $\eta = 1$ in Ω_0. Let $\tilde{\mathbf{u}} = \eta\mathbf{u}$. The proof of the theorem will be done in two steps.

Step 1. In this step we establish that there exists $\varepsilon_0 \in (0, 1-s)$ such that for any $0 < \varepsilon < \varepsilon_0$, we have

$$\|\tilde{\mathbf{u}}\|^{p-1}_{\mathcal{X}_p^{s+\varepsilon}(\mathbb{R}^d)} \leq C\left(\|\tilde{\mathbf{u}}\|^{p-1}_{\mathcal{X}_p^{s}(\mathbb{R}^d)} + \|\mathbb{L}^s_{p,\Omega_2}\tilde{\mathbf{u}}\|_{[\mathring{\mathcal{X}}_p^{s-\varepsilon(p-1)}(\Omega_2)]^*}\right). \tag{5.17}$$

We apply the technique and the argument in [76]. First notice that since the support of $\tilde{\mathbf{u}}$ is contained in Ω_1, we have that $\|\tilde{\mathbf{u}}\|^{p-1}_{\mathcal{X}_p^{s+\varepsilon}(\mathbb{R}^d)} = \|\tilde{\mathbf{u}}\|^{p-1}_{\mathcal{X}_p^{s+\varepsilon}(\Omega)}$. Next, find finitely many balls $(B_k)_{k=1}^N \subset \Omega_2$ so that $\bigcup_{k=1}^N B_k \supset \Omega_1$. We also may assume that $\bigcup_{k=1}^N 10B_k \subset \Omega_2$. Then we have

$$\begin{aligned}
&\|\tilde{\mathbf{u}}\|^p_{\mathcal{X}_p^{s+\varepsilon}(\Omega)}\\
&= \int_\Omega \int_{\Omega_1} \frac{|(\tilde{\mathbf{u}}(\mathbf{y}) - \tilde{\mathbf{u}}(\mathbf{x})) \cdot \frac{\mathbf{y}-\mathbf{x}}{|\mathbf{y}-\mathbf{x}|}|^p}{|\mathbf{y}-\mathbf{x}|^{d+(s+\varepsilon)p}} \,\mathrm{d}\mathbf{x}\,\mathrm{d}\mathbf{y} + \int_\Omega \int_{\Omega\setminus\Omega_1} \frac{|(\tilde{\mathbf{u}}(\mathbf{y}) - \tilde{\mathbf{u}}(\mathbf{x})) \cdot \frac{\mathbf{y}-\mathbf{x}}{|\mathbf{y}-\mathbf{x}|}|^p}{|\mathbf{y}-\mathbf{x}|^{d+(s+\varepsilon)p}} \,\mathrm{d}\mathbf{x}\,\mathrm{d}\mathbf{y}\\
&\leq \sum_{k=1}^N \int_\Omega \int_{B_k} \frac{|(\tilde{\mathbf{u}}(\mathbf{y}) - \tilde{\mathbf{u}}(\mathbf{x})) \cdot \frac{\mathbf{y}-\mathbf{x}}{|\mathbf{y}-\mathbf{x}|}|^p}{|\mathbf{y}-\mathbf{x}|^{d+(s+\varepsilon)p}} \,\mathrm{d}\mathbf{x}\,\mathrm{d}\mathbf{y} + \int_{\Omega_0} \int_{\Omega\setminus\Omega_1} \frac{|\tilde{\mathbf{u}}(\mathbf{y})|^p}{|\mathbf{x}-\mathbf{y}|^{d+(s+\varepsilon)p}} \,\mathrm{d}\mathbf{x}\,\mathrm{d}\mathbf{y}\\
&\leq \sum_{k=1}^N \int_{2B_k} \int_{B_k} \frac{|(\tilde{\mathbf{u}}(\mathbf{y}) - \tilde{\mathbf{u}}(\mathbf{x})) \cdot \frac{\mathbf{y}-\mathbf{x}}{|\mathbf{y}-\mathbf{x}|}|^p}{|\mathbf{y}-\mathbf{x}|^{d+(s+\varepsilon)p}} \,\mathrm{d}\mathbf{x}\,\mathrm{d}\mathbf{y}\\
&\qquad\qquad + \sum_{k=1}^N \int_{\Omega\setminus 2B_k} \int_{B_k} \frac{|(\tilde{\mathbf{u}}(\mathbf{y}) - \tilde{\mathbf{u}}(\mathbf{x})) \cdot \frac{\mathbf{y}-\mathbf{x}}{|\mathbf{y}-\mathbf{x}|}|^p}{|\mathbf{y}-\mathbf{x}|^{d+(s+\varepsilon)p}} \,\mathrm{d}\mathbf{x}\,\mathrm{d}\mathbf{y} + C_\varepsilon \|\mathbf{u}\|^p_{L^p(\Omega)}\\
&\leq \sum_{k=1}^N [\tilde{\mathbf{u}}]^p_{\mathcal{X}_p^{s+\varepsilon}(2B_k)} + C(\varepsilon)\|\mathbf{u}\|^p_{L^p(\Omega)}\ .
\end{aligned}$$

This is because the second term on the second line and the second term on the third line have disjoint support in the integrals. When using the constant $C(\varepsilon)$ we are emphasizing that the constant depends on ε, and it may also depend on other quantities.

Using Lemma 5.13 and the fact that the union of the finite number of balls B_k cover no more than Ω, we get for any $\delta > 0$

$$\|\tilde{\mathbf{u}}\|^p_{\mathcal{X}_p^{s+\varepsilon}(\Omega)} \leq \delta^p \|\tilde{\mathbf{u}}\|^p_{\mathcal{X}_p^{s+\varepsilon}(\Omega)} + C(\delta,\varepsilon)\|\mathbf{u}\|^p_{L^p(\Omega)} + C\sum_{k=1}^N \delta^{-p'} \left(\sup_\phi \langle \mathbb{L}^{s+\varepsilon}_{p,8B_k} \tilde{\mathbf{u}}, \phi \rangle \right)^{\frac{p}{p-1}}$$

where the supremum is over all $\phi \in C_c^\infty(4B_k;\mathbb{R}^d)$ with $[\phi]_{\mathcal{X}_p^{s+\varepsilon}(\mathbb{R}^d)} \leq 1$. Choosing $\delta^p = \frac{1}{2}$ and by assumption on $\mathbf{u}$ we can absorb the first term in the right hand side towards the left hand side to obtain the estimate

$$\|\tilde{\mathbf{u}}\|^p_{\mathcal{X}_p^{s+\varepsilon}(\Omega)} \leq C(\varepsilon)\|\mathbf{u}\|^p_{L^p(\Omega)} + C\sum_{k=1}^N \left(\sup_\phi \langle \mathbb{L}^{s+\varepsilon}_{p,8B_k} \tilde{\mathbf{u}}, \phi \rangle \right)^{\frac{p}{p-1}}\ ,$$

where the supremum is over all $\phi \in C_c^\infty(4B_k;\mathbb{R}^d)$ with $[\phi]_{\mathcal{X}_p^{s+\varepsilon}(\mathbb{R}^d)} \leq 1$. With Theorem 5.14, adding and subtracting $\langle \mathbb{L}^s_{p,8B_k}\tilde{\mathbf{u}}, (-\Delta)^{\frac{\varepsilon p}{2}}\phi\rangle$ we can estimate by

$$\|\tilde{\mathbf{u}}\|^p_{\mathcal{X}_p^{s+\varepsilon}(\Omega)} \leq C_1(\varepsilon)\,\|\mathbf{u}\|^p_{L^p(\Omega)} + \varepsilon^{\frac{p}{p-1}}\,C_2(\varepsilon)\,\|\tilde{\mathbf{u}}\|^p_{\mathcal{X}_p^{s+\varepsilon}(\Omega)}$$

$$+\,C\sum_{k=1}^N\left(\sup_\phi\left\{\left|\langle \mathbb{L}^s_{p,8B_k}\tilde{\mathbf{u}}, (-\Delta)^{\frac{\varepsilon p}{2}}\phi\rangle\right|\right\}\right)^{\frac{p}{p-1}},$$

where the supremum is over all $\phi \in C_c^\infty(4B_k;\mathbb{R}^d)$ with $[\phi]_{\mathcal{X}_p^{s+\varepsilon}(\mathbb{R}^d)} \leq 1$. Notice from Theorem 5.14 that for ε_0 small enough, we have that $C_2(\varepsilon) \leq C_2(\varepsilon_0)$ for all $\varepsilon \in (0,\varepsilon_0)$ and therefore we can absorb the right-hand side term involving $\|\tilde{\mathbf{u}}\|^p_{\mathcal{X}_p^{s+\varepsilon}(\Omega)}$ on the left-hand side for $\varepsilon \in (0,\varepsilon_0)$. The estimate becomes

$$\|\tilde{\mathbf{u}}\|^p_{\mathcal{X}_p^{s+\varepsilon}(\Omega)} \leq C(\varepsilon)\,\|\mathbf{u}\|^p_{L^p(\Omega)} + \sum_{k=1}^N\left(\sup_\phi\left\{\left|\langle \mathbb{L}^s_{p,8B_k}\tilde{\mathbf{u}}, (-\Delta)^{\frac{\varepsilon p}{2}}\phi\rangle\right|\right\}\right)^{\frac{p}{p-1}},$$

where the supremum is over all $\phi \in C_c^\infty(4B_k;\mathbb{R}^d)$ with $[\phi]_{\mathcal{X}_p^{s+\varepsilon}(\mathbb{R}^d)} \leq 1$. Now for a given $\phi \in C_c^\infty(4B_k;\mathbb{R}^d)$ such that $[\phi]_{\mathcal{X}_p^{s+\varepsilon}(\mathbb{R}^d)} \leq 1$ and for each $1 \leq k \leq N$ define

$$\Phi = \rho_k(-\Delta)^{\frac{\varepsilon p}{2}}\phi\,,$$

where $\rho_k \in C_c^\infty(6B_k)$ and $\rho_k \equiv 1$ on $5B_k$. Then by Lemma 5.11 we have that $\Phi \in C_c^\infty(6B_k)$, and

$$[\Phi]_{\mathcal{X}_p^{s-\varepsilon(p-1)}(\mathbb{R}^d)} \leq C_k[\phi]_{\mathcal{X}_p^{s+\varepsilon}(\mathbb{R}^d)} \leq C_k\,. \tag{5.18}$$

Now, since the support of $(1-\rho_k)$ and ϕ are disjoint we can apply the quasi-local behavior of the fractional Laplacian described in 5.14 to obtain

$$
\begin{aligned}
&\left\|\nabla\left((1-\rho_k)(-\Delta)^{\frac{\varepsilon p}{2}}\phi\right)\right\|_{L^\infty(8B_k)} \\
&\leq \left\|-\nabla\rho_k\cdot(-\Delta)^{\frac{\varepsilon p}{2}}\phi\right\|_{L^\infty(8B_k)} + \left\|(1-\rho_k)(-\Delta)^{\frac{1+\varepsilon p}{2}}\phi\right\|_{L^\infty(8B_k)} \\
&\leq C_k\left\|(-\Delta)^{\frac{\varepsilon p}{2}}\phi\right\|_{L^\infty(6B_k\setminus 5B_k)} + C_k\left\|(-\Delta)^{\frac{1+\varepsilon p}{2}}\phi\right\|_{L^\infty(8B_k\setminus 5B_k)} \\
&\leq C\mathrm{diam}(B_k)^{-d-\varepsilon p}|4B_k|^{\frac{p-1}{p}}\|\phi\|_{L^p(4B_k)} + \mathrm{diam}(B_k)^{-d-1-\varepsilon p}|4B_k|^{\frac{p-1}{p}}\|\phi\|_{L^p(4B_k)} \\
&\leq C_k[\phi]_{\mathcal{X}_p^{s+\varepsilon}(\mathbb{R}^d)},
\end{aligned}
\tag{5.19}
$$

where we have used the Poincaré-Korn and fractional Korn inequalities. As a consequence of 5.18 and 5.19, an application of Hölder's inequality gives us

$$
\begin{aligned}
&\left|\langle\mathbb{L}^s_{p,8B_k}\tilde{\mathbf{u}},(-\Delta)^{\frac{\varepsilon p}{2}}\phi-\Phi\rangle\right| \\
&\leq \left\|\nabla\left((1-\rho_k)(-\Delta)^{\frac{\varepsilon p}{2}}\phi\right)\right\|_{L^\infty(8B_k)}\int_{8B_k}\int_{8B_k}\frac{|\mathcal{D}(\mathbf{u})(\mathbf{x},\mathbf{y})|^{p-1}}{|\mathbf{x}-\mathbf{y}|^{d+sp-1}}\,\mathrm{d}\mathbf{x}\,\mathrm{d}\mathbf{y} \\
&\leq C[\tilde{\mathbf{u}}]^{p-1}_{\mathcal{X}_p^s(8B_k)}\left(\int_{8B_k}\int_{8B_k}\frac{1}{|\mathbf{x}-\mathbf{y}|^{d+p(s-1)}}\,\mathrm{d}\mathbf{x}\,\mathrm{d}\mathbf{y}\right)^{1/p}\leq C[\tilde{\mathbf{u}}]^{p-1}_{\mathcal{X}_p^s(\mathbb{R}^d)}.
\end{aligned}
$$

With the above, adding and subtracting $\langle\mathbb{L}^s_{p,8B_k}\tilde{\mathbf{u}},\Phi\rangle$ and using the properties of Φ shown above, our estimate becomes

$$
\begin{aligned}
\|\tilde{\mathbf{u}}\|^p_{\mathcal{X}_p^{s+\varepsilon}(\Omega)} &\leq C\|\tilde{\mathbf{u}}\|^p_{\mathcal{X}_p^s(\mathbb{R}^d)} \\
&\quad+\sum_{k=1}^N\left(\sup\left\{\left|\langle\mathbb{L}^s_{p,8B_k}\tilde{\mathbf{u}},\psi\rangle\right| : \psi\in C_c^\infty(6B_k;\mathbb{R}^d),\ [\psi]_{\mathcal{X}_p^{s-\varepsilon(p-1)}(\mathbb{R}^d)}\leq C_k\right\}\right)^{\frac{p}{p-1}}.
\end{aligned}
$$

Lastly, we need to transform the support of the operator $\mathcal{L}$ from $8B_k$ to Ω_2. Since $\operatorname{supp}\psi\subset 6B_k$, the disjoint support of the integrals gives (using Hölder's inequality and then the

Poincaré and Korn inequalities on the ψ integral)

$$|\langle \mathbb{L}^s_{p,8B_k}\tilde{\mathbf{u}}, \psi\rangle - \langle \mathbb{L}^s_{p,\Omega_2}\tilde{\mathbf{u}}, \psi\rangle| \leq 2\int_{\Omega\setminus 8B_k}\int_{6B_k} \frac{|\mathcal{D}(\mathbf{u})(\mathbf{x},\mathbf{y})|^{p-1}\,|\mathcal{D}(\psi)(\mathbf{x},\mathbf{y})|}{|\mathbf{x}-\mathbf{y}|^{d+sp}}\,\mathrm{d}\mathbf{x}\,\mathrm{d}\mathbf{y}$$
$$\leq C_k[\tilde{\mathbf{u}}]^{p-1}_{\mathcal{X}^s_p(\mathbb{R}^d)}[\psi]_{\mathcal{X}^{s-\varepsilon(p-1)}_p(\mathbb{R}^d)}\,.$$

Therefore,

$$\begin{aligned}
\|\tilde{\mathbf{u}}\|^p_{\mathcal{X}^{s+\varepsilon}_p(\Omega)} &\leq C\,\|\tilde{\mathbf{u}}\|^p_{\mathcal{X}^s_p(\mathbb{R}^d)} \\
&+C\sum_{k=1}^N\left(\sup\left\{\left|\langle \mathbb{L}^s_{p,\Omega_2}\tilde{\mathbf{u}}, \psi\rangle\right|\,:\,\psi\in C^\infty_c(6B_k;\mathbb{R}^d),\ [\psi]_{\mathcal{X}^{s-\varepsilon(p-1)}_p(\mathbb{R}^d)}\leq C_k\right\}\right)^{p'} \\
&\leq C\left(\|\tilde{\mathbf{u}}\|^p_{\mathcal{X}^s_p(\mathbb{R}^d)} + \|\mathbb{L}^s_{p,\Omega_2}\tilde{\mathbf{u}}\|^{p'}_{[\mathring{\mathcal{X}}^{s-\varepsilon(p-1)}_p(\Omega_2)]^*}\right)\,.
\end{aligned}$$

Step 2. In this step we estimate the right-hand side of 5.17 in terms of the dual norm of $\mathbf{F}$ and $\|\mathbf{u}\|_{\mathcal{X}^s_p(\Omega)}$. By computation it is not difficult to show that $[\tilde{\mathbf{u}}]_{\mathcal{X}^s_p(\mathbb{R}^d)}\leq C\|\mathbf{u}\|_{\mathcal{X}^s_p(\Omega)}$. Moreover, one can also prove that

$$\|\mathbb{L}^s_{p,\Omega_2}\tilde{\mathbf{u}}\|^{\frac{p}{p-1}}_{[\mathring{\mathcal{X}}^{s-\varepsilon(p-1)}_p(\Omega_2)]^*} \leq \|\mathbf{F}\|^{\frac{p}{p-1}}_{[\mathring{\mathcal{X}}^{s-\varepsilon(p-1)}_p(\Omega)]^*} + \|\mathbf{u}\|^p_{\mathcal{X}^s_p(\Omega)}.$$

Indeed, this is possible to show by using the same argument as in [76, Localization Lemma] and in fact prove that for any $t\in(2s-1,s)$ we have

$$\|\mathbb{L}^s_{p,\Omega_2}\tilde{\mathbf{u}}\|_{[\mathring{\mathcal{X}}^t_p(\Omega_2)]^*}\leq C\left(\|\mathbb{L}^s_{p,\Omega}\mathbf{u}\|_{[\mathring{\mathcal{X}}^t_p(\Omega)]^*}+\|\mathbf{u}\|^{p-1}_{\mathcal{X}^s_p(\Omega)}\right).$$

The proof is complete. $\square$

Chapter 6

Strong Solvability for Peridynamic-Type Equations

6.1 Introduction

In this chapter we report a solvability result for the strongly-coupled system of linear equations

$$\begin{cases} \partial_{tt}\mathbf{u}(\mathbf{x},t) + \mathbb{L}\mathbf{u}(\mathbf{x},t) = \mathbf{f}(\mathbf{x},t)\,, & (\mathbf{x},t) \text{ in } \mathbb{R}^d \times [0,T]\,, \\ \mathbf{u}(\mathbf{x},0) = \mathbf{u}_0(\mathbf{x})\,, & \mathbf{x} \text{ in } \mathbb{R}^d\,, \\ \partial_t \mathbf{u}(\mathbf{x},0) = \mathbf{v}_0(\mathbf{x})\,, & \mathbf{x} \text{ in } \mathbb{R}^d\,, \end{cases} \tag{6.1}$$

where the vector valued operator $\mathbb{L}$ is given by

$$-\mathbb{L}\mathbf{u}(\mathbf{x},t) = \text{P.V.} \int_{\mathbb{R}^d} \left(\frac{\mathbf{y}\otimes\mathbf{y}}{|\mathbf{y}|^2} \right) \big(\mathbf{u}(\mathbf{x}+\mathbf{y},t) - \mathbf{u}(\mathbf{x},t) - \varepsilon(\mathbf{u})(\mathbf{x},t)\mathbf{y}\,\chi^{(s)}(\mathbf{y}) \big)\ \rho(\mathbf{y})\,\mathrm{d}\mathbf{y}\,. \tag{6.2}$$

The function $\chi^{(s)}$ is defined as

$$\chi^{(s)}(\mathbf{y}) = \begin{cases} 0 & \text{if } s \in (0,1/2) \\ \mathbb{1}_{B_1}(\mathbf{y}) & \text{if } s = 1/2 \\ 1 & \text{if } s \in (1/2,1)\,, \end{cases}$$

and $\varepsilon(\mathbf{u})(\mathbf{x})$ denotes the symmetric part of the gradient matrix given by

$$\varepsilon(\mathbf{u})(\mathbf{x}) = \frac{1}{2}(\nabla\mathbf{u}(\mathbf{x}) + \nabla\mathbf{u}(\mathbf{x})^{\intercal}).$$

We also recall the operator defined in (2.1) in Chapter 2, denoted here as

$$-(-\mathring{\Delta})^s\mathbf{u}(\mathbf{x}) := \text{P.V.}\int_{\mathbb{R}^d} \frac{\kappa_{d,s}}{|\mathbf{y}|^{d+2s}}\left(\frac{\mathbf{y}\otimes\mathbf{y}}{|\mathbf{y}|^2}\right)\big(\mathbf{u}(\mathbf{x}+\mathbf{y}) - \mathbf{u}(\mathbf{x})\big)\,\mathrm{d}\mathbf{y}\,. \tag{6.3}$$

The kernel ρ that we will consider comes from two classes.

Class A: Integrable kernels: ρ is a nonnegative integrable function in $\mathbb{R}^d$. That is, $\rho(\mathbf{y}) \in L^1(\mathbb{R}^d)$. In this case, in the definition of $\mathbb{L}$ in (6.2), we take $\chi^{(s)}(\mathbf{y}) \equiv 0$. Integrable kernels commonly used in applications have compact support.

Class B: Nonintegrable kernels: ρ is a nonnegative singular kernel that is comparable to $|\mathbf{y}|^{-d-2s}$. More precisely, we assume that the kernel ρ is of the form

$$\rho(\mathbf{y}) := \frac{m(\mathbf{y})}{|\mathbf{y}|^{d+2s}}\,\mathbb{1}_{\Lambda_r}(\mathbf{y})\,,$$

where for some $0 < r \leq \infty$ the set $\Lambda_r := \Lambda \cap B_r$ is a truncated double cone $\Lambda := \{\mathbf{x} \in \mathbb{R}^d \,\big|\, \frac{\mathbf{x}}{|\mathbf{x}|} \in \Gamma \cup -\Gamma\}$ corresponding to a given measurable subset $\Gamma \subseteq \mathbb{S}^{d-1}$ that has positive Hausdorff measure, and the positive measurable function $m : \mathbb{R}^d \to [0,\infty)$ satisfies

$$0 < \alpha_1 \leq m(\mathbf{y}) \leq \alpha_2 < \infty\,,$$

for positive constants α_1 and α_2. We also assume that when $s = 1/2$, ρ satisfies the cancellation condition

$$\int_{\partial B_\mu} y_i y_j y_k \rho(\mathbf{y})\,\mathrm{d}\sigma(\mathbf{y}) = 0\,, \quad \forall\, i,j,k \in \{1,2,\dots,d\}\,, \quad \forall \mu > 0\,. \tag{6.4}$$

Note that (6.4) is always satisfied when $m(\mathbf{y})$ is an even function.

To show well posedness of the time dependent problem, we establish the solvability of the linear system

$$\mathbb{L}\mathbf{u}(\mathbf{x}) + \lambda\mathbf{u}(\mathbf{x}) = \mathbf{f}(\mathbf{x}) \quad \text{in } \mathbb{R}^d\,, \tag{6.5}$$

which is the first main goal of this chapter. The constant λ is nonnegative and the operator $\mathbb{L}$ is assumed to possess a kernel from one of the above classes of kernels. For operators that use kernels in the nonintegrable class, we prove the existence and uniqueness of a strong solution in the Bessel potential space $\mathbf{u} \in H^{2s,2}$ corresponding to data $\mathbf{f} \in L^2$ satisfying (6.5) almost everywhere. Further, when $\mathbb{L} = (-\mathring{\Delta})$, for each $\mathbf{f} \in L^p$ there exists a unique $\mathbf{u} \in H^{2s,p}$ solving (6.5). These and other results will be precisely stated in Section 2. The proof of the strong L^2 solvability result for the equation (6.5) is proved in Section 3. Section 4 contains the proofs leading to the strong L^p solvability for $1 < p < \infty$ of the equation (6.5) when the choice of kernel $\rho(\mathbf{y}) = |\mathbf{y}|^{-d-2s}$ is used. In the last section we show well posedness of a wave equation closely resembling (6.1).

6.2 Statement of Main Results

Before we state the main results, let us establish notation and define the relevant function spaces. Euclidean balls of radius r centered at $\mathbf{x}_0$ are denoted $B_r(\mathbf{x}_0) := \{\mathbf{x} \in \mathbb{R}^d \,\big|\, |\mathbf{x}-\mathbf{x}_0| < r\}$. If the center $\mathbf{x}_0$ is $\mathbf{0}$, or if the center is clear from context, we omit it and write B_r. We denote the Fourier transform $\mathcal{F}$ by

$$\mathcal{F}f(\boldsymbol{\xi}) = \widehat{f}(\boldsymbol{\xi}) = \int_{\mathbb{R}^d} e^{-2\pi\imath \mathbf{x}\cdot\boldsymbol{\xi}} f(\mathbf{x}) \, \mathrm{d}\mathbf{x} \, .$$

We write the $L^p(\Omega)$-norm of a function as the standard $\|\cdot\|_{L^p(\Omega)}$, with abbreviation $\|\cdot\|_{L^p}$ whenever the domain of integration is all of $\mathbb{R}^d$. Let $\mathcal{S}(\mathbb{R}^d)$ and $\mathcal{S}'(\mathbb{R}^d)$ be the space of Schwartz functions and tempered distributions, respectively. Denote the space of $\mathbb{R}^d$-valued Schwartz vector fields by $\left[\mathcal{S}(\mathbb{R}^d)\right]^d$, and its dual by $\left[\mathcal{S}'(\mathbb{R}^d)\right]^d$. For $p \in (1,\infty)$ and $s \in (0,1)$, the Bessel potential space is given by

$$\left[H^{2s,p}(\mathbb{R}^d)\right]^d := \left\{ \mathbf{u} \in \left[\mathcal{S}'(\mathbb{R}^d)\right]^d \,\middle|\, \left(\left(1 + 4\pi^2|\boldsymbol{\xi}|^2\right)^s \widehat{\mathbf{u}} \right)^{\vee} \in \left[L^p(\mathbb{R}^d)\right]^d \right\} ,$$

with norm

$$\|\mathbf{u}\|_{H^{2s,p}} = \left\| \left(\left(1 + 4\pi^s|\boldsymbol{\xi}|^2\right)^s \widehat{\mathbf{u}} \right)^{\vee} \right\|_{L^p} .$$

Define the homogeneous space

$$\left[\dot{H}^{2s,p}(\mathbb{R}^d)\right]^d = \left\{ \mathbf{u} \in \left[\mathcal{S}'(\mathbb{R}^d)\right]^d \,\middle|\, \left(\left(4\pi|\boldsymbol{\xi}|^2\right)^s \widehat{\mathbf{u}}\right)^{\vee} := (-\Delta)^s \mathbf{u} \in \left[L^p(\mathbb{R}^d)\right]^d \right\},$$

and denote the semi-norm $[\mathbf{u}]_{H^{2s,p}(\mathbb{R}^d)} = \|(-\Delta)^s \mathbf{u}\|_{L^p(\mathbb{R}^d)}$. Since $1 + \left(4\pi^2|\boldsymbol{\xi}|^2\right)^s$ can be controlled by $\left(1 + 4\pi^2|\boldsymbol{\xi}|^2\right)^s$ and vice versa, we have

$$\|\mathbf{u}\|_{H^{2s,p}} \approx \|\mathbf{u}\|_{L^p} + [\mathbf{u}]_{H^{2s,p}}.$$

With these notations and definition at hand, we can now state the first result of the chapter.

Theorem 6.1 (L^2 Solvability for general nonintegrable kernels)**.** *Suppose that ρ is in* **Class B**. *There exists a constant $\lambda_0 = \lambda_0(d, s, \Lambda, r, \alpha_1) > 0$ such that the following holds: For $\lambda > \lambda_0$ and for any $\mathbf{f} \in \left[L^2(\mathbb{R}^d)\right]^d$ there exists a unique strong solution $\mathbf{u} \in \left[H^{2s,2}(\mathbb{R}^d)\right]^d$ to the equation* (6.5) *satisfying the estimate*

$$\|(-\Delta)^s \mathbf{u}\|_{L^2} + \sqrt{\lambda}\left\|(-\Delta)^{s/2}\mathbf{u}\right\|_{L^2} + \left(\lambda - \lambda_0\right)\|\mathbf{u}\|_{L^2} \leq C\|\mathbf{f}\|_{L^2}, \tag{6.6}$$

where $C = C(d, s, \Lambda, \alpha_1) > 0$.

For the solvability of the system corresponding to kernels in **Class A**, we have the following result.

Theorem 6.2 (L^2 Solvability for general integrable kernels)**.** *Suppose that ρ is in* **Class A**. *Then for any $\lambda > 0$ and for any $\mathbf{f} \in \left[L^2(\mathbb{R}^d)\right]^d$ there exists a unique strong solution $\mathbf{u} \in \left[L^2(\mathbb{R}^d)\right]^d$ to the equation* (6.5) *satisfying the estimate*

$$\lambda\|\mathbf{u}\|_{L^2} \leq C\|\mathbf{f}\|_{L^2}, \tag{6.7}$$

where $C = C(d, \rho) > 0$.

L^2 solvability of problems of the type (6.5) have been considered in [29]. There, the $H^{2s,2}$ regularity of appropriately-defined weak solutions of (6.5) with $\lambda = 1$ and ρ positive and radially symmetric was obtained via the Fourier transform and inverting the positive definite operator $\mathbb{L} + \mathbb{I}$, where $\mathbb{I}$ is the $d \times d$ identity matrix. Without much difficulty, the same technique could be used to prove existence and uniqueness of strong solutions of (6.5) with ρ positive and radially symmetric. Here, we prove this result of well posedness – but for a wider class of kernels – using the Fourier transform in a slightly different way, via *a priori* estimates and the celebrated method of continuity [48]. The novelty here is that no symmetry assumptions of any kind are made on the kernel. We also assume that the kernel can vanish on a substantial set, specifically outside of a double cone whose apex is at the origin. This work may begin the mathematical analysis for a system of equations that potentially model anisotropic interactions in materials, generalizing the model equations in [29].

The L^p theory, on the other hand, for systems of equations of the type (6.5) is relatively unstudied. In Chapter 3, the Dirichlet problem for equations resembling (6.5) was studied, where we used Hilbert space techniques to prove existence and uniqueness of weak solutions satisfying a complementary condition (a "volume constraint problem" in peridynamics). In this chapter, in place of weak solutions, we consider strong solutions solving an equation almost everywhere. The equation is also posed on all of $\mathbb{R}^d$ rather than on a bounded domain. The Fourier matrix symbol associated to the operator $\mathbb{L}$ is

$$\mathbb{M}(\boldsymbol{\xi}) := \int_{\mathbb{R}^d} \left(\frac{\mathbf{y} \otimes \mathbf{y}}{|\mathbf{y}|^2} \right) \left(\mathrm{e}^{2\pi\iota \mathbf{y}\cdot\boldsymbol{\xi}} - 1 - 2\pi\iota \boldsymbol{\xi} \cdot \mathbf{y}\, \chi^{(s)}(\mathbf{y}) \right) \rho(\mathbf{y})\, \mathrm{d}\mathbf{y}\,, \tag{6.8}$$

which in general lacks the differentiability necessary to apply classical multiplier theorems. The L^p results obtained here are for the specific kernel $|\mathbf{y}|^{-d-2s}$. This choice of kernel allows for the use of Fourier multiplier theorems, specifically the Marcinkiewicz multiplier theorem. The following theorem states the second main result of the chapter.

Theorem 6.3 (L^p Solvability for Specific Kernels)**.** *For $1 < p < \infty$ and corresponding to any $\mathbf{f} \in L^p$ and for any $\lambda > 0$ the equation*

$$(-\mathring{\Delta})^s \mathbf{u} + \lambda \mathbf{u} = \mathbf{f} \qquad \textit{in } \mathbb{R}^d \tag{6.9}$$

has a unique strong solution $\mathbf{u} \in \left[H^{2s,p}(\mathbb{R}^d)\right]^d$ *satisfying the estimate*

$$\|\mathbf{u}\|_{H^{2s,p}} \leq C \|\mathbf{f}\|_{L^p} \,, \tag{6.10}$$

where $C = C(d, s, p, \lambda) > 0$*.*

We finally remark on the case of scalar operators related to $\mathbb{L}$, for example

$$\mathcal{L}u(\mathbf{x}) := \text{P.V.} \int_{\mathbb{R}^d} \Big(u(\mathbf{x}+\mathbf{y}) - u(\mathbf{x}) - \nabla u(\mathbf{x})\,\mathbf{y}\chi^{(s)}(\mathbf{y}) \Big) \rho(\mathbf{y}) \, \mathrm{d}\mathbf{y} \,.$$

The operator $\mathcal{L}$ is a nonlocal elliptic operator associated to a stochastic process called a Markov jump process, or specifically a Lévy process. For instance, see [5] for some earlier work. The solvability in Sobolev and Hölder spaces for parabolic equations associated to the elliptic problem $\mathcal{L}u + \lambda u$ has been considered in [64] via a probabilistic approach. Lévy processes were studied using Fourier multipliers in [5]. The Fourier multipliers studied in [5] are quotients of symbols consisting of integrals. Because of their particular form, *a priori* estimates for solutions to scalar-valued equations are easily obtained as a corollary. However, the results of [5] do not directly apply to the matrix symbols necessary for the analogous *a priori* estimates of solutions to (6.5), as matrix inverses take the place of quotients in the symbols.

The work in this chapter is inspired by the paper [25] where it was shown that the equation $\mathcal{L}u + \lambda u = f$ is L^p-solvable, with ρ merely measurable and satisfying a certain cancellation condition. The result was obtained via *mean oscillation estimates* of u, which are used effectively in the method developed in [48] to treat second-order linear elliptic and parabolic equations. These mean oscillation estimates of u were obtained using maximum principle techniques coming from the study of viscosity solutions to fully nonlinear nonlocal equations; see [6]. For the system (6.5) an analogous L^p solvability result remains unclear. The maximum principle techniques in [25] do not apply to systems of the type (6.5).

With Theorem 6.1 and Theorem 6.2, we show the well posedness of the nonlocal hyperbolic system of wave equations

$$\begin{cases} \partial_{tt}\mathbf{u} + \mathbb{L}\mathbf{u} + \lambda\mathbf{u} = \mathbf{f}\,, & (\mathbf{x},t) \text{ in } \mathbb{R}^d \times [0,T]\,, \\ \mathbf{u}(\mathbf{x},0) = \mathbf{u}_0(\mathbf{x})\,, & \mathbf{x} \text{ in } \mathbb{R}^d\,, \\ \partial_t \mathbf{u}(\mathbf{x},0) = \mathbf{v}_0(\mathbf{x})\,, & \mathbf{x} \text{ in } \mathbb{R}^d\,, \end{cases} \tag{6.11}$$

Well-posedness of systems of the type (6.11) have been established via techniques of differential equations as initial value problems [17, 28, 31, 32, 35, 51, 52], but here we use the approach of semigroups and the solvability of the associated elliptic steady-state system (6.5). The main existence and uniqueness result we prove is the following.

Theorem 6.4. *Let $T > 0$, and let $\mathbf{f} \in C^1\left([0,T];\left[L^2(\mathbb{R}^d)\right]^d\right)$, $\mathbf{u}_0 \in \left[H^{2s,2}(\mathbb{R}^d)\right]^d$, and $\mathbf{v}_0 \in \left[H^{s,2}(\mathbb{R}^d)\right]^d$. Suppose that ρ is an even function that is in* **Class B**. *Suppose that λ is a nonnegative constant satisfying $\lambda > \lambda_0 - 1$, where λ_0 is the constant appearing in Theorem 6.1. Then there exists a unique solution*

$$\mathbf{u} \in C\left([0,T];\left[H^{2s,2}(\mathbb{R}^d)\right]^d\right) \cap C^1\left([0,T];\left[H^{s,2}(\mathbb{R}^d)\right]^d\right) \cap C^2\left([0,T];\left[L^2(\mathbb{R}^d)\right]^d\right)$$

to (6.11). In the case that $\mathbf{f}(\mathbf{x},t) = \mathbf{f}_0(\mathbf{x})$, then the solution $\mathbf{u}$ additionally satisfies the conservation law

$$\begin{aligned} &\|\partial_t \mathbf{u}(t)\|^2_{L^2(\mathbb{R}^d)} + \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} \rho(\mathbf{x}-\mathbf{y}) \left(\big(\mathbf{u}(\mathbf{x},t) - \mathbf{u}(\mathbf{y},t)\big) \cdot \frac{\mathbf{x}-\mathbf{y}}{|\mathbf{x}-\mathbf{y}|} \right)^2 \mathrm{d}\mathbf{y}\,\mathrm{d}\mathbf{x} \\ &\qquad + \lambda \|\mathbf{u}(t)\|^2_{L^2(\mathbb{R}^d)} + 2\int_{\mathbb{R}^d} \langle \mathbf{f}_0, \mathbf{u}(t)\rangle \,\mathrm{d}\mathbf{x} \\ &= \|\partial_t \mathbf{u}_0\|^2_{L^2(\mathbb{R}^d)} + \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} \rho(\mathbf{x}-\mathbf{y}) \left(\big(\mathbf{u}_0(\mathbf{x}) - \mathbf{u}_0(\mathbf{y})\big) \cdot \frac{\mathbf{x}-\mathbf{y}}{|\mathbf{x}-\mathbf{y}|} \right)^2 \mathrm{d}\mathbf{y}\,\mathrm{d}\mathbf{x} \\ &\qquad + \lambda \|\mathbf{u}_0\|^2_{L^2(\mathbb{R}^d)} + 2\int_{\mathbb{R}^d} \langle \mathbf{f}_0, \mathbf{u}_0\rangle \,\mathrm{d}\mathbf{x}\,. \end{aligned} \tag{6.12}$$

If $\lambda_0 < 1$, then λ can be taken to be 0.

An analogous result when ρ is radial and integrable can be found in [29].

6.3 L^2 solvability

In this section we will use the method of continuity to prove the solvability of the coupled system of nonlocal equations. The general result of the method of continuity is stated as follows.

Proposition 6.5 (The method of continuity)**.** *Suppose that X is a Banach space and V is a normed space. Suppose also $T_0, T_1 : X \to V$ are bounded linear operators. Assume that there exists $C > 0$ such that for $T_t = (1-t)T_0 + tT_1$ we have*

$$\|x\|_X \leq C\|T_t x\|_V, \quad \forall t \in [0,1], \forall x \in X.$$

Then T_0 is onto if and only if T_1 is onto.

In our case, we take the operator T_1 to be $\mathbb{L} + \lambda\mathbb{I}$, where $\mathbb{L}$ is as defined in (6.2). In the case of nonintegrable kernels, T_0 will be $\alpha_1(-\mathring{\Delta})^s + \lambda$ for some appropriately chosen $\lambda > 0$, since the solvability of the system $\alpha_1(-\mathring{\Delta})^s\mathbf{u} + \lambda\mathbf{u} = \mathbf{f}$ in $\left[H^{2s,2}(\mathbb{R}^d)\right]^d$ is well-known, see [29] or Theorem 6.3 for $p = 2$. For the case of integrable kernels, we take the operator T_0 to be the same operator $\mathbb{L} + \lambda\mathbb{I}$ but one with radial kernel where invertibility of the operator will be proved. In both cases, in order to apply the method of continuity we need to show boundedness of the operators and obtain *a priori* estimates for the corresponding operator T_t.

6.3.1 The Nonintegrable Case

Let us introduce the parametrized linear operators

$$-\mathbb{L}_t\mathbf{u}(\mathbf{x}) := \text{P.V.} \int_{\mathbb{R}^d} \left(\frac{\mathbf{y} \otimes \mathbf{y}}{|\mathbf{y}|^2}\right) \big(\mathbf{u}(\mathbf{x}+\mathbf{y}) - \mathbf{u}(\mathbf{x}) - \varepsilon(\mathbf{u})(\mathbf{x})\mathbf{y}\,\chi^{(s)}(\mathbf{y})\big)\ \rho_t(\mathbf{y})\,\mathrm{d}\mathbf{y}\,,$$

and the kernel ρ_t is defined as

$$\rho_t(\mathbf{y}) := t\,\rho(\mathbf{y}) + (1-t)\frac{\alpha_1}{|\mathbf{y}|^{d+2s}}\,.$$

Notice that $-\mathbb{L}_0 = \alpha_1(-\mathring{\Delta})^s$ and $\mathbb{L}_1 = \mathbb{L}$. In order to use the method of continuity, we need to show that the operators $\mathbb{L}_0$, and $\mathbb{L}_1$ are bounded from $\left[H^{2s,2}(\mathbb{R}^d)\right]^d$ to $\left[L^2(\mathbb{R}^d)\right]^d$ and establish the *a priori* estimate for solutions of $\mathbb{L}_t\mathbf{u} + \lambda\mathbf{u} = \mathbf{f}$ for $t \in [0,1]$. First let us prove that the operators are continuous.

Lemma 6.6 (Continuity of the parametrized operator: nonintegrable case)**.** *Suppose that ρ is in* **Class B**. *Then for each $t \in [0,1]$ the operator $\mathbb{L}_t : \left[H^{2s,2}(\mathbb{R}^d)\right]^d \to \left[L^2(\mathbb{R}^d)\right]^d$ is continuous. More precisely, there exists a constant $C > 0$ such that for all $t \in [0,1]$, and for all $\mathbf{u} \in \left[\mathcal{S}(\mathbb{R}^d)\right]^d$ we have the estimate*

$$\left\|\mathbb{L}_t\mathbf{u}\right\|_{L^2} \leq C \left\|\mathbf{u}\right\|_{H^{2s,2}} . \tag{6.13}$$

Proof. Recalling that the symbol of $\mathbb{L}$ is the matrix symbol $\mathbb{M}(\boldsymbol{\xi})$ given by (6.8), to prove continuity it suffices to show that there exists a constant $C = C(d, s, \alpha_2)$ such that

$$|\mathbb{M}_t(\boldsymbol{\xi})| \leq C(2\pi|\boldsymbol{\xi}|)^{2s}\,, \tag{6.14}$$

where $\mathbb{M}_t(\boldsymbol{\xi})$ is the matrix symbol associated to $\mathbb{L}_t$. To that end, if $s \in (0, \frac{1}{2})$, by definition $\mathbb{M}_t(\boldsymbol{\xi}) = \displaystyle\int_{\mathbb{R}^d} \left(\frac{\mathbf{y}\otimes\mathbf{y}}{|\mathbf{y}|^2}\right)\left(e^{2\pi\imath\mathbf{y}\cdot\boldsymbol{\xi}} - 1\right)\rho_t(\mathbf{y})\,\mathrm{d}\mathbf{y}$. By using the upper bound $\rho_t(\mathbf{y}) \leq \alpha_2|\mathbf{y}|^{-d-2s}$, we obtain

$$|\mathbb{M}_t(\boldsymbol{\xi})| \leq \alpha_2 \int_{\mathbb{R}^d} \frac{1}{|\mathbf{y}|^{d+2s}}\left|e^{2\pi\imath\mathbf{y}\cdot\boldsymbol{\xi}} - 1\right|\mathrm{d}\mathbf{y}.$$

Now a simple calculation shows that $|e^{2\pi\imath\mathbf{y}\cdot\boldsymbol{\xi}} - 1| = \sqrt{2(1-\cos(2\pi\mathbf{y}\cdot\boldsymbol{\xi}))}$. Using this and by making the substitution $\mathbf{h} = 2\pi|\boldsymbol{\xi}|\mathbf{y}$, we obtain

$$\begin{aligned}|\mathbb{M}_t(\boldsymbol{\xi})| &\leq \alpha_2 \int_{\mathbb{R}^d} \frac{1}{|\mathbf{y}|^{d+2s}}\sqrt{2(1-\cos(2\pi\mathbf{y}\cdot\boldsymbol{\xi}))}\,\mathrm{d}\mathbf{y} \\ &\leq \alpha_2(2\pi|\boldsymbol{\xi}|)^{2s}\int_{\mathbb{R}^d}\frac{\sqrt{2}}{|\mathbf{h}|^{d+2s}}\sqrt{1-\cos\left(\mathbf{h}\cdot\frac{\boldsymbol{\xi}}{|\boldsymbol{\xi}|}\right)}\,\mathrm{d}\mathbf{h}\,.\end{aligned}$$

Notice that the last integral is uniformly bounded in $\boldsymbol{\xi}$, since $1-\cos(a) \approx |a|^2$ for a near 0 and $s < 1/2$.

When $s = 1/2$, we have by the cancellation condition (6.4) on ρ that

$$\mathbb{M}_t(\boldsymbol{\xi}) = \int_{\mathbb{R}^d} \left(\frac{\mathbf{y} \otimes \mathbf{y}}{|\mathbf{y}|^2} \right) \left(\mathrm{e}^{2\pi\iota \mathbf{y}\cdot\boldsymbol{\xi}} - 1 - 2\pi\iota\boldsymbol{\xi} \cdot \mathbf{y}\, \mathbb{1}_{B_{1/(2\pi|\boldsymbol{\xi}|)}}(\mathbf{y}) \right) \rho_t(\mathbf{y})\, \mathrm{d}\mathbf{y}\,.$$

Then by the same substitution $\mathbf{h} = 2\pi|\boldsymbol{\xi}|\mathbf{y}$ we have

$$\begin{aligned}
|\mathbb{M}_t(\boldsymbol{\xi})| &\leq \alpha_2 \int_{\mathbb{R}^d} \frac{1}{|\mathbf{y}|^{d+1}} \Big(|1 - \cos(2\pi\mathbf{y}\cdot\boldsymbol{\xi})| + |\sin(2\pi\mathbf{y}\cdot\boldsymbol{\xi}) - 2\pi\boldsymbol{\xi}\cdot\mathbf{y}\, \mathbb{1}_{B_{1/(2\pi|\boldsymbol{\xi}|)}}(\mathbf{y})| \Big)\, \mathrm{d}\mathbf{y} \\
&\leq \alpha_2 (2\pi|\boldsymbol{\xi}|)^{2s} \int_{\mathbb{R}^d} \frac{1}{|\mathbf{h}|^{d+1}} \left(\left| 1 - \cos\left(\mathbf{h} \cdot \frac{\boldsymbol{\xi}}{|\boldsymbol{\xi}|} \right) \right| + \left| \sin\left(\mathbf{h} \cdot \frac{\boldsymbol{\xi}}{|\boldsymbol{\xi}|} \right) - \frac{\boldsymbol{\xi}}{|\boldsymbol{\xi}|} \cdot \mathbf{h}\, \mathbb{1}_{B_1}(\mathbf{h}) \right| \right) \mathrm{d}\mathbf{h} \\
&\leq C(2\pi|\boldsymbol{\xi}|)^{2s}\,.
\end{aligned}$$

The integral is again uniformly bounded in $\boldsymbol{\xi}$ since $|\sin(a) - a| \approx |a|^3$ for a near 0 and since the term $\frac{\boldsymbol{\xi}}{|\boldsymbol{\xi}|} \cdot \mathbf{h}\mathbb{1}_{B_1}(\mathbf{h})$ vanishes for $|\mathbf{h}| \geq 1$.

When $s \in (\frac{1}{2}, 1)$, again by using the upper bound on ρ and then making the substitution $\mathbf{h} = 2\pi|\boldsymbol{\xi}|\mathbf{y}$,

$$\begin{aligned}
|\mathbb{M}_t(\boldsymbol{\xi})| &\leq \alpha_2 \int_{\mathbb{R}^d} \frac{1}{|\mathbf{y}|^{d+2s}} \Big(\big|1 - \cos(2\pi\mathbf{y}\cdot\boldsymbol{\xi})\big| + |\sin(2\pi\mathbf{y}\cdot\boldsymbol{\xi}) - 2\pi\mathbf{y}\cdot\boldsymbol{\xi}| \Big)\, \mathrm{d}\mathbf{y} \\
&= \alpha_2 (2\pi|\boldsymbol{\xi}|)^{2s} \int_{\mathbb{R}^d} \frac{1}{|\mathbf{h}|^{d+2s}} \left(\left| 1 - \cos\left(\mathbf{h} \cdot \frac{\boldsymbol{\xi}}{|\boldsymbol{\xi}|} \right) \right| + \left| \sin\left(\mathbf{h} \cdot \frac{\boldsymbol{\xi}}{|\boldsymbol{\xi}|} \right) - \mathbf{h} \cdot \frac{\boldsymbol{\xi}}{|\boldsymbol{\xi}|} \right| \right) \mathrm{d}\mathbf{h} \\
&\leq C(2\pi|\boldsymbol{\xi}|)^{2s}\,,
\end{aligned}$$

since the integral is once again uniformly bounded in $\boldsymbol{\xi}$. $\square$

The next theorem gives us *a priori* estimates for solutions of the system that imply uniqueness of strong solutions. It is also the key estimate to implement the method of continuity to prove existence of a solution.

Lemma 6.7 (*A priori* estimates for parametrized operator: nonintegrable case)**.** *Suppose that ρ is in* **Class B**. *Let $\lambda > 0$ be a constant and let $\mathbf{f} \in \left[L^2(\mathbb{R}^d)\right]^d$. Suppose $\mathbf{u} \in \left[H^{2s,2}(\mathbb{R}^d)\right]^d$ satisfies the equation $\mathbb{L}_t\mathbf{u} + \lambda\mathbf{u} = \mathbf{f}$ in $\mathbb{R}^d$. Then there exist constants $N_1 = N_1(d, s, \Lambda, \alpha_1) > 0$ and $N_2 = N_2(\alpha_1, \Lambda, r) > 0$ such that*

$$\|(-\Delta)^s\mathbf{u}\|_{L^2} + \sqrt{\lambda}\, \big\|(-\Delta)^{s/2}\mathbf{u}\big\|_{L^2} + \lambda\, \|\mathbf{u}\|_{L^2} \leq N_1 \Big(\|\mathbf{f}\|_{L^2} + N_2\, \|\mathbf{u}\|_{L^2} \Big)\,. \tag{6.15}$$

As a consequence,

$$\|(-\Delta)^s \mathbf{u}\|_{L^2} + \sqrt{\lambda}\left\|(-\Delta)^{s/2}\mathbf{u}\right\|_{L^2} + \left(\lambda - N_1 N_2\right)\|\mathbf{u}\|_{L^2} \leq N_1 \|\mathbf{f}\|_{L^2} \,. \tag{6.16}$$

Proof. Using the continuity of $\mathbb{L}_t$, Lemma 6.6 and the fact that $\left[C_c^\infty(\mathbb{R}^d)\right]^d$ is dense in $\left[H^{2s,2}(\mathbb{R}^d)\right]^d$, it suffices to show (6.15) for $\mathbf{u} \in \left[C_c^\infty(\mathbb{R}^d)\right]^d$. To accomplish this, we introduce the modified kernel

$$\widetilde{\rho}_t(\mathbf{y}) := \rho_t(\mathbf{y}) + \frac{t\alpha_1}{|\mathbf{y}|^{d+2s}}\chi_{\Lambda\cap \complement B_r}(\mathbf{y}) = \begin{cases} \frac{t\alpha_1}{|\mathbf{y}|^{d+2s}} + \rho_t(\mathbf{y}) & \mathbf{y} \in \Lambda \cap \complement B_r\,, \\ \rho_t(\mathbf{y}) & \text{otherwise}\,, \end{cases}$$

and the associated operator $\widetilde{\mathbb{L}}_t$; i. e. $\mathbb{L}_t$ with $\widetilde{\rho}_t$ in place of ρ_t. Note that

$$\widetilde{\rho}_t(\mathbf{y}) \geq 0 \text{ on } \mathbb{R}^d\,, \qquad \frac{\alpha_1}{|\mathbf{y}|^{d+2s}} \leq \widetilde{\rho}_t(\mathbf{y}) \leq \frac{\alpha_2}{|\mathbf{y}|^{d+2s}} \quad \text{for } \mathbf{y} \in \Lambda\,, t \in [0,1]\,. \tag{6.17}$$

Then we have that

$$\begin{aligned}
\widetilde{\mathbb{L}}_t\mathbf{u}(\mathbf{x}) &= \mathbb{L}_t\mathbf{u}(\mathbf{x}) + \int_{\mathbb{R}^d}\left(\frac{\mathbf{y}\otimes\mathbf{y}}{|\mathbf{y}|^2}\right)\left(\mathbf{u}(\mathbf{x}+\mathbf{y}) - \mathbf{u}(\mathbf{x}) - \varepsilon(\mathbf{u})(\mathbf{x})\mathbf{y}\,\chi^{(s)}(\mathbf{y})\right)\frac{t\alpha_1\,\mathbb{1}_{\Lambda\cap\complement B_r}(\mathbf{y})}{|\mathbf{y}|^{d+2s}}\,\mathrm{d}\mathbf{y} \\
&= \mathbb{L}_t\mathbf{u}(\mathbf{x}) + \int_{\mathbb{R}^d}\left(\frac{\mathbf{y}\otimes\mathbf{y}}{|\mathbf{y}|^2}\right)\left(\mathbf{u}(\mathbf{x}+\mathbf{y}) - \mathbf{u}(\mathbf{x})\right)\frac{t\alpha_1\,\mathbb{1}_{\Lambda\cap\complement B_r}(\mathbf{y})}{|\mathbf{y}|^{d+2s}}\,\mathrm{d}\mathbf{y} \\
&:= \mathbb{L}_t\mathbf{u}(\mathbf{x}) + \mathbb{L}_t^c\mathbf{u}(\mathbf{x})\,.
\end{aligned}$$

The second equality follows from the definition of the function $\chi^s(\mathbf{y})$, the cancellation condition (6.4), and the fact that the function $\mathbb{1}_{\Lambda\cap\complement B_r}(\mathbf{y})$ is an even function. It follows by the triangle inequality that

$$\left\|\widetilde{\mathbb{L}}_t\mathbf{u} + \lambda\mathbf{u}\right\|_{L^2} \leq \|\mathbb{L}_t\mathbf{u} + \lambda\mathbf{u}\|_{L^2} + \|\mathbb{L}_t^c\mathbf{u}\|_{L^2} \,. \tag{6.18}$$

The operator $\mathbb{L}_t^c\mathbf{u}(\mathbf{x})$ has a Fourier matrix symbol that belongs to $\left[L^\infty(\mathbb{R}^d)\right]^{d\times d}$, and is therefore bounded from $\left[L^2(\mathbb{R}^d)\right]^d$ to $\left[L^2(\mathbb{R}^d)\right]^d$. To be precise,

$$\begin{aligned}\left|\widehat{\mathbb{L}_t^c\mathbf{u}}(\boldsymbol{\xi})\right| &= \left|t\alpha_1\int_{\Lambda\cap\complement B_r}\left(\frac{\mathbf{y}\otimes\mathbf{y}}{|\mathbf{y}|^2}\right)\frac{\mathrm{e}^{2\pi\iota\mathbf{y}\cdot\boldsymbol{\xi}}-1}{|\mathbf{y}|^{d+2s}}\,\mathrm{d}\mathbf{y}\;\widehat{\mathbf{u}}(\boldsymbol{\xi})\right| \\ &\le 2\alpha_1\int_{(\Gamma\cup-\Gamma)}d\sigma(\nu)\int_r^\infty\tau^{-1-2s}\,\mathrm{d}\tau\;|\widehat{\mathbf{u}}(\boldsymbol{\xi})| \\ &:= N_2|\widehat{\mathbf{u}}(\boldsymbol{\xi})|\,,\end{aligned}$$

where $d\sigma$ is the surface measure on $\mathbb{S}^{d-1}$. Thus, $\|\mathbb{L}_t^c\mathbf{u}\|_{L^2}\le N_3\|\mathbf{u}\|_{L^2}$. Next we need to show that

$$N_1\left\|\widetilde{\mathbb{L}}_t\mathbf{u}+\lambda\mathbf{u}\right\|_{L^2}\ge\|(-\Delta)^s\mathbf{u}\|_{L^2}+\sqrt{\lambda}\left\|(-\Delta)^{s/2}\mathbf{u}\right\|_{L^2}+\lambda\left\|\mathbf{u}\right\|_{L^2}\,. \tag{6.19}$$

Note that

$$\left\|\widetilde{\mathbb{L}}_t\mathbf{u}+\lambda\mathbf{u}\right\|_{L^2}^2=\int_{\mathbb{R}^d}|\widetilde{\mathbb{L}}_t\mathbf{u}|^2\,\mathrm{d}\mathbf{x}+\lambda\int_{\mathbb{R}^d}\left\langle\widetilde{\mathbb{L}}_t\mathbf{u},\mathbf{u}\right\rangle\,\mathrm{d}\mathbf{x}+\lambda^2\|\mathbf{u}\|_{L^2}^2\,,$$

and thus to show (6.19) it suffices to estimate the first two terms. To that end, we begin writing $\widetilde{\rho}_t$ as a sum of its even and odd parts:

$$\widetilde{\rho}_t=\widetilde{\rho}_t^e+\widetilde{\rho}_t^o,\quad\widetilde{\rho}_t^e(\mathbf{y})=\frac{1}{2}(\widetilde{\rho}_t(\mathbf{y})+\widetilde{\rho}_t(-\mathbf{y})),\quad\widetilde{\rho}_t^o(\mathbf{y})=\frac{1}{2}(\widetilde{\rho}_t(\mathbf{y})-\widetilde{\rho}_t(-\mathbf{y}))\,.$$

$\widetilde{\rho}_t^e$ satisfies the same assumptions as $\widetilde{\rho}_t$. Let $\widetilde{\mathbb{L}}_t^e$ and $\widetilde{\mathbb{L}}_t^o$ be the operators with kernels $\widetilde{\rho}_t^e$ and $\widetilde{\rho}_t^o$ respectively. Observe that $\widetilde{\mathbb{L}}_t=\widetilde{\mathbb{L}}_t^e+\widetilde{\mathbb{L}}_t^o$ and that for any $\mathbf{u}\in\left[C_c^\infty(\mathbb{R}^d)\right]^d$ we have

$$\int_{\mathbb{R}^d}\left\langle\widetilde{\mathbb{L}}_t^e\mathbf{u},\widetilde{\mathbb{L}}_t^o\mathbf{u}\right\rangle\,\mathrm{d}\mathbf{x}=0\,.$$

Thus,

$$\int_{\mathbb{R}^d}|\widetilde{\mathbb{L}}_t\mathbf{u}|^2\,\mathrm{d}\mathbf{x}\ge\int_{\mathbb{R}^d}|\widetilde{\mathbb{L}}_t^e\mathbf{u}(\mathbf{x})|^2\,\mathrm{d}\mathbf{x}=\int_{\mathbb{R}^d}|\widehat{\widetilde{\mathbb{L}}_t^e\mathbf{u}}(\boldsymbol{\xi})|^2\,\mathrm{d}\boldsymbol{\xi}=\int_{\mathbb{R}^d}|\widetilde{\mathbb{M}}_t^e(\boldsymbol{\xi})\widehat{\mathbf{u}}(\boldsymbol{\xi})|^2\,\mathrm{d}\boldsymbol{\xi}\,,$$

where

$$\widetilde{\mathbb{M}}_t^e(\boldsymbol{\xi}):=\int_{\mathbb{R}^d}\left(\frac{\mathbf{y}\otimes\mathbf{y}}{|\mathbf{y}|^2}\right)\left(1-\cos(2\pi\mathbf{y}\cdot\boldsymbol{\xi})\right)\widetilde{\rho}_t^e(\mathbf{y})\,\mathrm{d}\mathbf{y}\,.$$

For every $\boldsymbol{\xi} \in \mathbb{R}^d$, the matrix $\widetilde{\mathbb{M}}_t^e(\boldsymbol{\xi})$ is symmetric with real entries, and therefore, the least of the eigenvalues of $\widetilde{\mathbb{M}}_t^e(\boldsymbol{\xi})$ is given by $\min_{\mathbf{v}\in\mathbb{S}^{d-1}} \mathbf{v}^\intercal \widetilde{\mathbb{M}}_t^e(\boldsymbol{\xi})\mathbf{v}$. We will estimate the smallest eigenvalue from below as as function of $\boldsymbol{\xi}$. By making the substitution $\mathbf{h} = 2\pi|\boldsymbol{\xi}|\mathbf{y}$, we see that

$$\begin{aligned}
\min_{\mathbf{v}\in\mathbb{S}^{d-1}} \mathbf{v}^\intercal \widetilde{\mathbb{M}}_t^e(\boldsymbol{\xi})\mathbf{v} &= \min_{\mathbf{v}\in\mathbb{S}^{d-1}} \int_{\mathbb{R}^d} \big(1-\cos(2\pi\mathbf{y}\cdot\boldsymbol{\xi})\big) \left|\mathbf{v}\cdot\frac{\mathbf{y}}{|\mathbf{y}|}\right|^2 \widetilde{\rho}_t^e(\mathbf{y})\,\mathrm{d}\mathbf{y} \\
&\geq \min_{\mathbf{v}\in\mathbb{S}^{d-1}} \alpha_1 \int_\Lambda \frac{\big(1-\cos(2\pi\mathbf{y}\cdot\boldsymbol{\xi})\big)}{|\mathbf{y}|^{d+2s}} \left|\mathbf{v}\cdot\frac{\mathbf{y}}{|\mathbf{y}|}\right|^2 \mathrm{d}\mathbf{y} \\
&= \min_{\mathbf{v}\in\mathbb{S}^{d-1}} \alpha_1 \big(2\pi|\boldsymbol{\xi}|\big)^{2s} \int_\Lambda \frac{1-\cos\left(2\pi\mathbf{h}\cdot\frac{\boldsymbol{\xi}}{|\boldsymbol{\xi}|}\right)}{|\mathbf{h}|^{d+2s}} \left|\mathbf{v}\cdot\frac{\mathbf{h}}{|\mathbf{h}|}\right|^2 \mathrm{d}\mathbf{h} \\
&= \min_{\mathbf{v}\in\mathbb{S}^{d-1}} \alpha_1 \big(2\pi|\boldsymbol{\xi}|\big)^{2s} \Psi\left(\frac{\boldsymbol{\xi}}{|\boldsymbol{\xi}|},\mathbf{v}\right),
\end{aligned}$$

where $\Psi : \mathbb{S}^{d-1}\times\mathbb{S}^{d-1} \to \mathbb{R}$ defined by $\Psi(\boldsymbol{\eta},\mathbf{v}) := \displaystyle\int_\Lambda \frac{1-\cos(2\pi\mathbf{h}\cdot\boldsymbol{\eta})}{|\mathbf{h}|^{d+2s}} \left|\mathbf{v}\cdot\frac{\mathbf{h}}{|\mathbf{h}|}\right|^2 \mathrm{d}\mathbf{h}$. The lower bound $\widetilde{\rho}_t^e(\mathbf{y}) \geq \alpha_1$ can be used since the integrand is nonnegative. Since Λ is a double cone with apex at the origin, Ψ is a positive function. Ψ is also clearly continuous on the compact set $\mathbb{S}^{d-1}\times\mathbb{S}^{d-1}$ so there exists a positive constant $C = C(d,s,\Lambda)$ such that $\displaystyle\min_{\boldsymbol{\eta},\mathbf{v}\in\mathbb{S}^{d-1}} \Psi(\boldsymbol{\eta},\mathbf{v}) = C > 0$. We use this to estimate the least eigenvalue of the matrix. We conclude that for any $\boldsymbol{\xi}\neq 0$, we have $\displaystyle\min_{\mathbf{v}\in\mathbb{S}^{d-1}} \mathbf{v}^\intercal \widetilde{\mathbb{M}}_t^e(\boldsymbol{\xi})\mathbf{v} \geq C\big(2\pi|\boldsymbol{\xi}|\big)^{2s}$. We also use this lower bound to estimate the smallest eigenvalue of $(\widetilde{\mathbb{M}}_t^e)^\intercal(\boldsymbol{\xi})\,\widetilde{\mathbb{M}}_t^e(\boldsymbol{\xi})$ from below. Since $\widetilde{\mathbb{M}}_t^e(\boldsymbol{\xi})$ is symmetric,

$$\begin{aligned}
\min_{\mathbf{v}\in\mathbb{S}^{d-1}} \mathbf{v}^\intercal (\widetilde{\mathbb{M}}_t^e)^\intercal(\boldsymbol{\xi})\,\widetilde{\mathbb{M}}_t^e(\boldsymbol{\xi})\mathbf{v} &= \min\{\beta^2 : \beta \text{ is an eigenvalue of } \widetilde{\mathbb{M}}_t^e(\boldsymbol{\xi})\} \\
&\geq C(d,s,\Lambda,\alpha_1)\left|\big(2\pi|\boldsymbol{\xi}|\big)^{2s}\right|^2.
\end{aligned}$$

Therefore,

$$\begin{aligned}
\int_{\mathbb{R}^d} |\widetilde{\mathbb{L}}_t\mathbf{u}|^2\,\mathrm{d}\mathbf{x} &= \int_{\mathbb{R}^d} \left\langle \widetilde{\mathbb{M}}_t^e(\boldsymbol{\xi})\widehat{\mathbf{u}}(\boldsymbol{\xi}), \widetilde{\mathbb{M}}_t^e(\boldsymbol{\xi})\widehat{\mathbf{u}}(\boldsymbol{\xi})\right\rangle \mathrm{d}\boldsymbol{\xi} \\
&\geq C\int_{\mathbb{R}^d} \left|\big(2\pi|\boldsymbol{\xi}|\big)^{2s}\widehat{\mathbf{u}}(\boldsymbol{\xi})\right|^2 \mathrm{d}\boldsymbol{\xi} = C\,\|(-\Delta)^s\mathbf{u}\|_{L^2}^2 .
\end{aligned} \tag{6.20}$$

Similarly, by applying Plancherel's theorem and noting that the matrix

$$\widetilde{\mathbb{M}}_t(\boldsymbol{\xi}) := \int_{\mathbb{R}^d} \left(\frac{\mathbf{y}\otimes\mathbf{y}}{|\mathbf{y}|^2}\right) \big(\cos(2\pi\boldsymbol{\xi}\cdot\mathbf{y}) - 1\big)\widetilde{\rho}_t(\mathbf{y})\,\mathrm{d}\mathbf{y}$$

is symmetric,

$$\begin{aligned}
\int_{\mathbb{R}^d} \left\langle \widetilde{\mathbb{L}}_t\mathbf{u}, \mathbf{u}\right\rangle \mathrm{d}\mathbf{x} &= \int_{\mathbb{R}^d} \left\langle \widehat{\widetilde{\mathbb{L}}_t\mathbf{u}}, \overline{\widehat{\mathbf{u}}}\right\rangle \mathrm{d}\boldsymbol{\xi} \\
&= \int_{\mathbb{R}^d} \Re\left\langle \widehat{\widetilde{\mathbb{L}}_t\mathbf{u}}, \overline{\widehat{\mathbf{u}}}\right\rangle \mathrm{d}\boldsymbol{\xi} \\
&= \int_{\mathbb{R}^d} \left\langle \widetilde{\mathbb{M}}_t(\boldsymbol{\xi})\widehat{\mathbf{u}}(\boldsymbol{\xi}), \widehat{\mathbf{u}}(\boldsymbol{\xi})\right\rangle \mathrm{d}\boldsymbol{\xi} \\
&\geq \int_{\mathbb{R}^d} |\widehat{\mathbf{u}}(\boldsymbol{\xi})|^2 \min_{\mathbf{v}\in\mathbb{S}^{d-1}} \mathbf{v}^\intercal\widetilde{\mathbb{M}}_t(\boldsymbol{\xi})\mathbf{v}\,\mathrm{d}\boldsymbol{\xi} \\
&\geq C\int_{\mathbb{R}^d} \big(2\pi|\boldsymbol{\xi}|\big)^{2s}|\widehat{\mathbf{u}}(\boldsymbol{\xi})|^2\,\mathrm{d}\boldsymbol{\xi}\,.
\end{aligned}$$

It then follows that

$$\int_{\mathbb{R}^d} \left\langle \widetilde{\mathbb{L}}_t\mathbf{u}, \mathbf{u}\right\rangle \mathrm{d}\mathbf{x} \geq C\int_{\mathbb{R}^d} |(-\Delta)^{s/2}\mathbf{u}|^2\,\mathrm{d}\mathbf{x}\,. \tag{6.21}$$

The result (6.19) follows from the equality

$$\int_{\mathbb{R}^d} \left|\widetilde{\mathbb{L}}_t\mathbf{u} + \lambda\mathbf{u}\right|^2 \mathrm{d}\mathbf{x} = \int_{\mathbb{R}^d} |\mathbf{f}|^2\,\mathrm{d}\mathbf{x}$$

and the estimates (6.20) and (6.21) above. Then (6.15) follows from (6.18) and (6.19). □

Remark 6.8. Constants N_1 and N_2 satisfying (6.15) can be found explicitly. For example, the constant N_2 can be chosen to be

$$N_2 = \frac{1+\alpha_1}{s}\,\sigma(\Gamma\cup-\Gamma)\,r^{-2s}\,. \tag{6.22}$$

It is clear that $N_2 \to 0$ as $r\to\infty$.

The proof of Theorem 6.1 now follows.

Proof of Theorem 6.1. We will apply Proposition 6.5. Pick $\lambda_0 = N_1N_2$. Then for any $\lambda > \lambda_0$, take $T_0 = \alpha_1(-\mathring{\Delta})+\lambda\mathbb{I}$, $T_1 = \mathbb{L}+\lambda\mathbb{I}$, and therefore for $t\in[0,1]$, $T_t = (1-t)T_0+tT_1 = \mathbb{L}_t+\lambda\mathbb{I}$, where $\mathbb{L}_t$ is as defined at the beginning of this subsection. Now the continuity and *a priori* estimates assumptions of Proposition 6.5 are satisfied by Lemma 6.6 and Lemma 6.7. As a consequence $\mathbb{L}+\lambda\mathbb{I}$ is onto if and only if $\alpha_1(-\mathring{\Delta})+\lambda\mathbb{I}$ is onto. The latter is proved to be the case in Theorem 6.3 for $p=2$ or see [29] . $\square$

6.3.2 The Integrable Case

For integrable kernels, we begin with a special case by assuming that ρ is radially symmetric.

Theorem 6.9. *Suppose ρ is a nonnegative, integrable and radially symmetric function that is not identically 0. For every $\lambda>0$ and for any $\mathbf{f}\in\left[L^2(\mathbb{R}^d)\right]^d$ there exists a unique strong solution $\mathbf{u}\in\left[L^2(\mathbb{R}^d)\right]^d$ to $\mathbb{L}\mathbf{u}+\lambda\mathbf{u}=\mathbf{f}$ satisfying the estimate*

$$\|\mathbf{u}\|_{L^2}\leq\lambda^{-1}\|\mathbf{f}\|_{L^2}\,. \tag{6.23}$$

Proof. To show existence, we prove that the Fourier matrix multiplier $\mathbb{M}(\boldsymbol{\xi})+\lambda\mathbb{I}$ is invertible, with inverse bounded in L^∞. Since ρ is radially symmetric, the Fourier matrix associated to $\mathbb{L}$ is

$$\mathbb{M}(\boldsymbol{\xi})=\int_{\mathbb{R}^d}\left(\frac{\mathbf{y}\otimes\mathbf{y}}{|\mathbf{y}|^2}\right)\big(1-\cos(2\pi\mathbf{y}\cdot\boldsymbol{\xi})\big)\rho(|\mathbf{y}|)\,\mathrm{d}\mathbf{y}\,.$$

Let $R(\boldsymbol{\xi})$ be a rotation such that $R(\boldsymbol{\xi})^\intercal\mathbf{e}_1=\frac{\boldsymbol{\xi}}{|\boldsymbol{\xi}|}$. Then setting $\mathbf{h}=|\boldsymbol{\xi}|R(\boldsymbol{\xi})\mathbf{y}$,

$$\begin{aligned}\mathbb{M}(\boldsymbol{\xi})&=\int_{\mathbb{R}^d}\frac{\rho\left(\frac{|\mathbf{h}|}{|\boldsymbol{\xi}|}\right)}{|\boldsymbol{\xi}|^d}\frac{R(\boldsymbol{\xi})^\intercal\mathbf{h}\otimes R(\boldsymbol{\xi})^\intercal\mathbf{h}}{|\mathbf{h}|^2}\big(1-\cos(2\pi h_1)\big)\,\mathrm{d}\mathbf{h}\\&=R(\boldsymbol{\xi})^\intercal\left(\int_{\mathbb{R}^d}\frac{\rho\left(\frac{|\mathbf{h}|}{|\boldsymbol{\xi}|}\right)}{|\boldsymbol{\xi}|^d}\left(\frac{\mathbf{h}\otimes\mathbf{h}}{|\mathbf{h}|^2}\right)\big(1-\cos(2\pi h_1)\big)\,\mathrm{d}\mathbf{h}\right)R(\boldsymbol{\xi})\,.\end{aligned}$$

Using rotations, we see that the off-diagonal terms of $R\mathbb{M}R^\intercal$ are equal to 0 for every $\boldsymbol{\xi}\in\mathbb{R}^d$. Thus,

$$\mathbb{M}(\boldsymbol{\xi})=R(\boldsymbol{\xi})^\intercal\operatorname{diag}(\ell_1(\boldsymbol{\xi}),\ell_2(\boldsymbol{\xi}),\dots,\ell_d(\boldsymbol{\xi}))R(\boldsymbol{\xi})\,,$$

where $\ell_i(\boldsymbol{\xi}) = \displaystyle\int_{\mathbb{R}^d} \frac{\rho\left(\frac{|\mathbf{h}|}{|\boldsymbol{\xi}|}\right)}{|\boldsymbol{\xi}|^d} \frac{h_i^2}{|\mathbf{h}|^2}\big(1-\cos(2\pi h_1)\big)\,\mathrm{d}\mathbf{h}$. Again using rotations, $\ell_2 = \ell_3 = \ldots = \ell_d$. Thus,

$$\mathbb{M}(\boldsymbol{\xi}) + \lambda\mathbb{I} = R(\boldsymbol{\xi})^{\intercal}\,\mathrm{diag}\,\big(\ell_1(\boldsymbol{\xi})+\lambda, \ell_2(\boldsymbol{\xi})+\lambda, \ldots, \ell_2(\boldsymbol{\xi})+\lambda\big) R(\boldsymbol{\xi})\,.$$

Therefore,

$$\big(\mathbb{M}(\boldsymbol{\xi}) + \lambda\mathbb{I}\big)^{-1} = R(\boldsymbol{\xi})^{\intercal}\,\mathrm{diag}\left(\frac{1}{\ell_1(\boldsymbol{\xi})+\lambda}, \frac{1}{\ell_2(\boldsymbol{\xi})+\lambda}, \ldots, \frac{1}{\ell_2(\boldsymbol{\xi})+\lambda}\right) R(\boldsymbol{\xi})\,.$$

Now, it is clear that $0 \leq \ell_i(\boldsymbol{\xi}) \leq 2\left\|\rho\right\|_{L^1(\mathbb{R}^d)}$ for almost every $\boldsymbol{\xi} \in \mathbb{R}^d$. Thus, $\big(\mathbb{M}(\boldsymbol{\xi}) + \lambda\mathbb{I}\big)^{-1} \in \left[L^\infty(\mathbb{R}^d)\right]^{d\times d}$, so the linear operator $T : \left[L^2(\mathbb{R}^d)\right]^d \to \left[L^2(\mathbb{R}^d)\right]^d$ defined by $T\mathbf{f} := \left(\big(\mathbb{M}(\boldsymbol{\xi}) + \lambda\mathbb{I}\big)^{-1}\widehat{\mathbf{f}}\right)^{\vee}$ is bounded, with $\left\|T\mathbf{f}\right\|_{L^2} \leq \lambda^{-1}\left\|\mathbf{f}\right\|_{L^2}$. Further, $\mathbf{u} := T\mathbf{f}$ solves the equation (6.5).

For uniqueness, we show that any solution $\mathbf{u} \in L^2$ of (6.5) with data $\mathbf{f} \equiv \mathbf{0}$ is identically $\mathbf{0}$. This follows from the fact that if $\mathbb{L}\mathbf{u} + \lambda\mathbf{u} = 0$, then $\langle \mathbb{L}\mathbf{u} + \lambda\mathbf{u}, \mathbf{u}\rangle_{L^2(\mathbb{R}^d)} = 0$ and as proved earlier $\langle \mathbb{L}\mathbf{u}, \mathbf{u}\rangle_{L^2(\mathbb{R}^d)} \geq 0$. $\square$

Lemma 6.10 (Continuity of the operator: integrable case)**.** *Suppose that ρ is in* **Class A***. Let $1 \leq p < \infty$. The operator $\mathbb{L} : \left[L^p(\mathbb{R}^d)\right]^d \to \left[L^p(\mathbb{R}^d)\right]^d$ is continuous. More precisely, we have for every $\mathbf{u} \in \left[L^p(\mathbb{R}^d)\right]^d$ the estimate*

$$\left\|\mathbb{L}\mathbf{u}\right\|_{L^p} \leq C\left\|\mathbf{u}\right\|_{L^p}\,, \tag{6.24}$$

where $C = C(\rho) > 0$.

Proof. Since $\rho \in L^1(\mathbb{R}^d)$,

$$\begin{aligned}\mathbb{L}\mathbf{u}(\mathbf{x}) &= \left(\int_{\mathbb{R}^d} \rho(\mathbf{y})\left(\frac{\mathbf{y}\otimes\mathbf{y}}{|\mathbf{y}|^2}\right)\mathrm{d}\mathbf{y}\right)\mathbf{u}(\mathbf{x}) - \int_{\mathbb{R}^d} \rho(\mathbf{y})\left(\frac{\mathbf{y}\otimes\mathbf{y}}{|\mathbf{y}|^2}\right)\mathbf{u}(\mathbf{x}-\mathbf{y})\,\mathrm{d}\mathbf{y} \\ &= \mathbb{A}\mathbf{u}(\mathbf{x}) + \mathbb{K} * \mathbf{u}(\mathbf{x})\,,\end{aligned}$$

where

$$\mathbb{A} := \int_{\mathbb{R}^d} \rho(\mathbf{y})\left(\frac{\mathbf{y}\otimes\mathbf{y}}{|\mathbf{y}|^2}\right)\mathrm{d}\mathbf{y}\,, \qquad \mathbb{K}(\mathbf{y}) := \rho(\mathbf{y})\left(\frac{\mathbf{y}\otimes\mathbf{y}}{|\mathbf{y}|^2}\right)\,.$$

Using the bound $|\mathbb{A}| \leq \|\rho\|_{L^1}$ and Young's inequality for integrals,

$$\|\mathbb{L}\mathbf{u}\|_{L^p} \leq \|\mathbb{A}\mathbf{u}\|_{L^p} + \|\mathbb{K} * \mathbf{u}\|_{L^p} \leq 2\,\|\rho\|_{L^1}\,\|\mathbf{u}\|_{L^p}\ .$$

□

Lemma 6.11 (*A priori* estimate: integrable case)**.** *Suppose that ρ is in* **Class A***. Let $\lambda > 0$ be a constant and let $\mathbf{f} \in \left[L^2(\mathbb{R}^d)\right]^d$. Suppose $\mathbf{u} \in \left[L^2(\mathbb{R}^d)\right]^d$ satisfies the equation (6.5) in $\mathbb{R}^d$. Then there exists a constant $C = C(\lambda) > 0$ such that*

$$\|\mathbf{u}\|_{L^2} \leq \lambda^{-1}\,\|\mathbf{f}\|_{L^2}\ . \tag{6.25}$$

Proof. Since $\left[C_c^\infty(\mathbb{R}^d)\right]^d$ is dense in $\left[L^2(\mathbb{R}^d)\right]^d$, by Lemma 6.10 it suffices to show (6.25) for $\mathbf{u} \in \left[C_c^\infty(\mathbb{R}^d)\right]^d$. Since

$$\int_{\mathbb{R}^d} |\mathbf{f}|^2 \,\mathrm{d}\mathbf{x} = \int_{\mathbb{R}^d} |\mathbb{L}\mathbf{u} + \lambda\mathbf{u}|^2 \,\mathrm{d}\mathbf{x} = \int_{\mathbb{R}^d} |\mathbb{L}\mathbf{u}|^2 \,\mathrm{d}\mathbf{x} + 2\lambda \int_{\mathbb{R}^d} \langle \mathbb{L}\mathbf{u}, \mathbf{u}\rangle \,\mathrm{d}\mathbf{x} + \lambda^2 \int_{\mathbb{R}^d} |\mathbf{u}|^2 \,\mathrm{d}\mathbf{x}\,, \tag{6.26}$$

it suffices to show only that

$$\int_{\mathbb{R}^d} \langle \mathbb{L}\mathbf{u}, \mathbf{u}\rangle \,\mathrm{d}\mathbf{x} \geq 0\,. \tag{6.27}$$

Then (6.25) follows by dropping the first two terms on the right-hand side of (6.26). To prove (6.27), note that

$$\begin{aligned}
\int_{\mathbb{R}^d} \langle \mathbb{L}\mathbf{u}, \mathbf{u}\rangle \,\mathrm{d}\mathbf{x} = \int_{\mathbb{R}^d} \left\langle \widehat{\mathbb{L}\mathbf{u}}, \overline{\widehat{\mathbf{u}}} \right\rangle \mathrm{d}\boldsymbol{\xi} &= \int_{\mathbb{R}^d} \left\langle \mathbb{M}(\boldsymbol{\xi})\widehat{\mathbf{u}}(\boldsymbol{\xi}), \overline{\widehat{\mathbf{u}}(\boldsymbol{\xi})} \right\rangle \mathrm{d}\boldsymbol{\xi} \\
&= \int_{\mathbb{R}^d} |\widehat{\mathbf{u}}(\boldsymbol{\xi})|^2 \left\langle \mathbb{M}(\boldsymbol{\xi}) \frac{\widehat{\mathbf{u}}(\boldsymbol{\xi})}{|\widehat{\mathbf{u}}(\boldsymbol{\xi})|}, \frac{\overline{\widehat{\mathbf{u}}(\boldsymbol{\xi})}}{|\widehat{\mathbf{u}}(\boldsymbol{\xi})|} \right\rangle \mathrm{d}\boldsymbol{\xi}\,.
\end{aligned}$$

Notice that $\langle \mathbb{M}(\boldsymbol{\xi})\rangle$ is in general a matrix with complex entries. However, for every $\mathbf{v} \in \mathbb{C}^d$ with $|\mathbf{v}| = 1$, a simple computation shows that

$$\langle \mathbb{M}(\boldsymbol{\xi})\mathbf{v}, \overline{\mathbf{v}}\rangle = \int_{\mathbb{R}^d} \rho(\mathbf{y})\big(1 - \mathrm{e}^{2\pi\imath \mathbf{y}\cdot\boldsymbol{\xi}}\big) \left(\frac{(\Re\mathbf{v}\cdot\mathbf{y})^2 + (\Im\mathbf{v}\cdot\mathbf{y})^2}{|\mathbf{y}|^2} \right) \mathrm{d}\mathbf{y}\,.$$

Since $\int_{\mathbb{R}^d} \langle \mathbb{L}\mathbf{u}, \mathbf{u} \rangle \, \mathrm{d}\mathbf{x}$ is real-valued, it follows that

$$\begin{aligned}\int_{\mathbb{R}^d} \langle \mathbb{L}\mathbf{u}, \mathbf{u} \rangle \, \mathrm{d}\mathbf{x} &= \int_{\mathbb{R}^d} |\widehat{\mathbf{u}}(\boldsymbol{\xi})|^2 \left\langle \mathbb{M}(\boldsymbol{\xi}) \frac{\widehat{\mathbf{u}}(\boldsymbol{\xi})}{|\widehat{\mathbf{u}}(\boldsymbol{\xi})|}, \frac{\overline{\widehat{\mathbf{u}}(\boldsymbol{\xi})}}{|\widehat{\mathbf{u}}(\boldsymbol{\xi})|} \right\rangle \, \mathrm{d}\boldsymbol{\xi} \\ &= \int_{\mathbb{R}^d} |\widehat{\mathbf{u}}(\boldsymbol{\xi})|^2 \left\langle \widetilde{\mathbb{M}}(\boldsymbol{\xi}) \frac{\widehat{\mathbf{u}}(\boldsymbol{\xi})}{|\widehat{\mathbf{u}}(\boldsymbol{\xi})|}, \frac{\overline{\widehat{\mathbf{u}}(\boldsymbol{\xi})}}{|\widehat{\mathbf{u}}(\boldsymbol{\xi})|} \right\rangle \, \mathrm{d}\boldsymbol{\xi} \,,\end{aligned}$$

where

$$\widetilde{\mathbb{M}}(\boldsymbol{\xi}) := \Re \mathbb{M}(\boldsymbol{\xi}) = \int_{\mathbb{R}^d} \left(\frac{\mathbf{y} \otimes \mathbf{y}}{|\mathbf{y}|^2} \right) \big(1 - \cos(2\pi \mathbf{y} \cdot \boldsymbol{\xi})\big) \rho(\mathbf{y}) \, \mathrm{d}\mathbf{y} \,.$$

Clearly, $\left\langle \widetilde{\mathbb{M}}(\boldsymbol{\xi}) \mathbf{v}, \overline{\mathbf{v}} \right\rangle \geq 0$ for all $|\mathbf{v}| = 1$, and so

$$\int_{\mathbb{R}^d} \langle \mathbb{L}\mathbf{u}, \mathbf{u} \rangle \, \mathrm{d}\mathbf{x} = \int_{\mathbb{R}^d} |\widehat{\mathbf{u}}(\boldsymbol{\xi})|^2 \left\langle \widetilde{\mathbb{M}}(\boldsymbol{\xi}) \frac{\widehat{\mathbf{u}}(\boldsymbol{\xi})}{|\widehat{\mathbf{u}}(\boldsymbol{\xi})|}, \frac{\overline{\widehat{\mathbf{u}}(\boldsymbol{\xi})}}{|\widehat{\mathbf{u}}(\boldsymbol{\xi})|} \right\rangle \, \mathrm{d}\boldsymbol{\xi} \geq 0 \,,$$

which is (6.27). □

The proof of Theorem 6.2 for general ρ in **Class A** now follows by applying the method of continuity, treating the operator $\mathbb{L}_t \mathbf{u} := t(\mathbb{L}_0 \mathbf{u} + \lambda \mathbf{u}) + (1-t)(\mathbb{L}\mathbf{u} + \lambda \mathbf{u})$. Here, $\mathbb{L}\mathbf{u}$ denotes the operator with a general kernel ρ belonging to **Class A**, and $\mathbb{L}_0 \mathbf{u}$ denotes an operator with a kernel ρ_0 of the type considered in Theorem 6.9, i.e. belonging to **Class A** and radially symmetric. Thus the kernel of the operator $\mathbb{L}_t$ has the expression $t\rho(\mathbf{y}) + (1-t)\rho_0(|\mathbf{y}|)$, and satisfies the criteria of **Class A** with bounds independent of t. Therefore the *a priori* estimates of Theorem 6.11 can be leveraged for the operator $\mathbb{L}_t + \lambda$ in the method of continuity program, and the results of Theorem 6.2 follow.

6.4 L^p solvability

We use traditional Fourier multiplier techniques. The analogous result for the fractional Laplacian in the scalar case seems to be known, but we are unable to find a proof in the literature. We provide a proof of Theorem 6.3 using the Marcinkiewicz multiplier theorem [42].

Let $s \in (0,1)$ and let $\kappa_{d,s}$ be the constant defined in (2.2). Consider the operator

$$-(-\mathring{\Delta})^s \mathbf{u}(\mathbf{x}) := \int_{\mathbb{R}^d} \frac{\kappa_{d,s}}{|\mathbf{y}|^{d+2s}} \left(\frac{\mathbf{y} \otimes \mathbf{y}}{|\mathbf{y}|^2} \right) (\mathbf{u}(\mathbf{x}+\mathbf{y}) - \mathbf{u}(\mathbf{x}))\, \mathrm{d}\mathbf{y}\,.$$

In the following calculation, we assume that $\mathbf{u} \in \left[\mathcal{S}(\mathbb{R}^d)\right]^d$. Using the Fourier transform we can write $-\widehat{(-\mathring{\Delta})^s \mathbf{u}}$ as follows:

$$-\widehat{(-\mathring{\Delta})^s \mathbf{u}}(\boldsymbol{\xi}) = \left(\kappa_{d,s} \int_{\mathbb{R}^d} \frac{e^{2\pi \imath \mathbf{h}\cdot\boldsymbol{\xi}} - 1}{|\mathbf{h}|^{d+2s}} \left(\frac{\mathbf{h} \otimes \mathbf{h}}{|\mathbf{h}|^2} \right) \mathrm{d}\mathbf{h} \right) \widehat{\mathbf{u}}(\boldsymbol{\xi}) := -\mathbb{M}_s^{\Delta}(\boldsymbol{\xi}) \widehat{\mathbf{u}}(\boldsymbol{\xi})\,.$$

From Theorem 2.3,

$$\mathbb{M}_s^{\Delta}(\boldsymbol{\xi}) = \big(2\pi|\boldsymbol{\xi}|\big)^{2s} \left(\mathbb{I} + 2s \left(\frac{\boldsymbol{\xi} \otimes \boldsymbol{\xi}}{|\boldsymbol{\xi}|^2} \right) \right)\,.$$

Theorem 6.3 hinges on the following lemma concerning Fourier multipliers.

Lemma 6.12. *For $s \in (0,1)$, $1 < p < \infty$ and $\lambda > 0$. Define the matrix of symbols*

$$\mathfrak{M}(\boldsymbol{\xi}) := \big(1 + 4\pi^2 |\boldsymbol{\xi}|^2\big)^{-s} \left(\mathbb{M}_s^{\Delta}(\boldsymbol{\xi}) + \lambda \mathbb{I} \right)\,.$$

Then both $\mathfrak{M}(\boldsymbol{\xi})$ and $\mathfrak{M}^{-1}(\boldsymbol{\xi})$ L^p-multipliers; that is, there exists a constant $C = C(d,s,p,\lambda)$ such that for every $\mathbf{u} \in \left[\mathcal{S}(\mathbb{R}^d)\right]^d$

$$\left\| \big(\mathfrak{M}\widehat{\mathbf{u}}\big)^{\vee} \right\|_{L^p} \leq C \left\| \mathbf{u} \right\|_{L^p}\,, \quad \textit{and,}\ \left\| \big(\mathfrak{M}^{-1}\widehat{\mathbf{u}}\big)^{\vee} \right\|_{L^p} \leq C \left\| \mathbf{u} \right\|_{L^p}\,. \tag{6.28}$$

We postpone the proof of the lemma for the moment, and prove Theorem 6.3.

Proof of Theorem 6.3. First we show that the *a priori* estimate (6.10) holds for every $\mathbf{u} \in \left[H^{2s,p}(\mathbb{R}^d)\right]^d$ solving (6.9). By Lemma 6.12 the operator $(-\mathring{\Delta})^s + \lambda : \left[H^{2s,p}(\mathbb{R}^d)\right]^d \to \left[L^p(\mathbb{R}^d)\right]^d$ is continuous. Therefore, we need only show that (6.10) holds for $\mathbf{u} \in \left[\mathcal{S}(\mathbb{R}^d)\right]^d$.

Then by definition and by Lemma 6.12 we have that

$$\begin{aligned}
\|\mathbf{u}\|_{H^{2s,p}(\mathbb{R}^d)} &= \left\| \left(1+4\pi^s|\boldsymbol{\xi}|^2\right)^s \widehat{\mathbf{u}}\right)^\vee \right\|_{L^p(\mathbb{R}^d)} \\
&= \left\| \left(\mathfrak{M}^{-1}(\mathbb{M}_s^\Delta + \lambda\mathbb{I})\widehat{\mathbf{u}}\right)^\vee \right\|_{L^p(\mathbb{R}^d)} \\
&\leq C \left\| \left((\mathbb{M}_s^\Delta + \lambda\mathbb{I})\widehat{\mathbf{u}}\right)^\vee \right\|_{L^p(\mathbb{R}^d)} = C \left\| (-\mathring{\Delta})^s \mathbf{u} + \lambda \mathbf{u} \right\|_{L^p(\mathbb{R}^d)} = C \left\|\mathbf{f}\right\|_{L^p(\mathbb{R}^d)} ,
\end{aligned}$$

which is (6.10). Uniqueness clearly follows from the a priori estimate as well.

To see that a solution to (6.9) exists, simply note that for every $\mathbf{f} \in \left[L^p(\mathbb{R}^d)\right]^d$ the distribution

$$\mathbf{u}(\mathbf{x}) = \left((\mathbb{M}_s^\Delta + \lambda\mathbb{I})^{-1} \widehat{\mathbf{f}} \right)^\vee (\mathbf{x})$$

belongs to $\left[\mathcal{S}'(\mathbb{R}^d)\right]^d$, and that

$$\left((1+4\pi^2|\boldsymbol{\xi}|^2)^s \widehat{\mathbf{u}}\right)^\vee = \left(\mathfrak{M}^{-1}\widehat{\mathbf{f}}\right)^\vee \in \left[L^p(\mathbb{R}^d)\right]^d \tag{6.29}$$

by Lemma 6.12. Thus $\mathbf{u}$ is a function in $\left[H^{2s,p}(\mathbb{R}^d)\right]^d$ that solves (6.9). The proof is complete.

□

We use the Marcinkiewicz multiplier theorem to prove the lemma; see [42].

Proof of Lemma 6.12. We begin by showing that $\mathfrak{M}(\boldsymbol{\xi})$ is an L^p multiplier. By a direct computation, the function $\mathfrak{M}(\boldsymbol{\xi})$ has the explicit expression

$$\mathfrak{M}(\boldsymbol{\xi}) = \left(\frac{4\pi^2|\boldsymbol{\xi}|^2}{1+4\pi^2|\boldsymbol{\xi}|^2} \right)^s \left(\mathbb{I} + 2s \left(\frac{\boldsymbol{\xi}\otimes\boldsymbol{\xi}}{|\boldsymbol{\xi}|^2} \right) \right) + \frac{\lambda}{\left(1+4\pi^2|\boldsymbol{\xi}|^2\right)^s} .$$

The symbol $\frac{\boldsymbol{\xi}\otimes\boldsymbol{\xi}}{|\boldsymbol{\xi}|^2}$ is a matrix symbol whose ij-th entry is the symbol of the composition of Riesz transforms $-R_iR_j$. Thus the matrix symbol is an L^p-multiplier. The second expression in the sum defining $\mathfrak{M}$ is the Bessel potential of order $-2s$, and is also an L^p-multiplier; see [86, Chapter 5, Section 3.3]. Finally, it is established in [86, Chapter V, Section 3.2., Lemma 2] that the symbol $\left(\frac{4\pi^2|\boldsymbol{\xi}|^2}{1+4\pi^2|\boldsymbol{\xi}|^2}\right)^s$ is a finite measure. Putting all this together gives us the result.

Next, we show that $\mathfrak{M}^{-1}(\boldsymbol{\xi})$ an L^p multiplier. Again by direct computation, the inverse $\mathfrak{M}^{-1}(\boldsymbol{\xi})$ has the expression

$$\mathfrak{M}^{-1}(\boldsymbol{\xi}) = \frac{\big(1+4\pi^2|\boldsymbol{\xi}|^2\big)^s}{\big(4\pi^2|\boldsymbol{\xi}|^2\big)^s+\lambda}\left(\mathbb{I} - \frac{2s\big(4\pi^2|\boldsymbol{\xi}|^2\big)^s}{(1+2s)\big(4\pi^2|\boldsymbol{\xi}|^2\big)^s+\lambda}\left(\frac{\boldsymbol{\xi}\otimes\boldsymbol{\xi}}{|\boldsymbol{\xi}|^2}\right)\right).$$

It suffices to show that the symbols

$$m_1(\boldsymbol{\xi}) := \frac{\big(1+4\pi^2|\boldsymbol{\xi}|^2\big)^s}{\big(4\pi^2|\boldsymbol{\xi}|^2\big)^s+\lambda}, \qquad m_2(\boldsymbol{\xi}) := \frac{2s\big(4\pi^2|\boldsymbol{\xi}|^2\big)^s}{(1+2s)\big(4\pi^2|\boldsymbol{\xi}|^2\big)^s+\lambda}$$

are L^p-multipliers. Using this, we then observe that $\mathfrak{M}^{-1}$ is a matrix whose entries consist of sums and products of L^p-multipliers, and the lemma is proved.

We use the Marcinkiewicz multiplier theorem; specifically, we show that m_1 and m_2 verify condition (6.2.9) in [42, Corollary 6.2.5]. We introduce the auxiliary function $F : \mathbb{R}^{d+2} \to \mathbb{R}$ defined by

$$F(\boldsymbol{\eta}) := \frac{\big(|\eta_1|^2+|\boldsymbol{\eta}'|^2\big)^s}{|\boldsymbol{\eta}'|^{2s}+|\eta_2|^2}, \quad \boldsymbol{\eta} = (\eta_1,\eta_2,\dots,\eta_{d+1},\eta_{d+2}), \quad \boldsymbol{\eta}' := (\eta_3,\eta_4,\dots,\eta_{d+1},\eta_{d+2}).$$

By inspection it is clear that F is bounded on $\mathbb{R}^{d+2}$ and belongs to the class C^{d+2} away from the coordinate axes on $\mathbb{R}^{d+2}$. Also by inspection, for any $\eta_1 \neq 0$ and $\eta_2 \neq 0$ the function $F(\eta_1,\eta_2,\cdot)$ is bounded on $\mathbb{R}^d$ and belongs to the class C^d on $\mathbb{R}^d \setminus \{\mathbf{0}\}$. The function F is invariant under weighted dilation: for any $\mu > 0$, and $s > 0$

$$F(\mu\eta_1,\mu^s\eta_2,\mu\boldsymbol{\eta}') = F(\eta_1,\eta_2,\boldsymbol{\eta}').$$

As a consequence, for any multi-index $\beta = (\beta_1,\beta_2,\dots,\beta_{d+1},\beta_{d+2})$ differentiation yields

$$\mu^{|\beta|+(s-1)\beta_2}(\partial_\beta F)(\mu\eta_1,\mu^s\eta_2,\mu\boldsymbol{\eta}') = \partial_\beta F(\eta_1,\eta_2,\boldsymbol{\eta}'). \tag{6.30}$$

Now for fixed $\eta_1 \neq 0$ and $\eta_2 \neq 0$ and every vector of the form $\boldsymbol{\eta} = (\eta_1,\eta_2,\boldsymbol{\eta}') \in \mathbb{R}^d$, we will choose $\mu = \mu_{\boldsymbol{\eta}}$ such that $(\mu\eta_1,\mu^s\eta_2,\mu\boldsymbol{\eta}')$ has unit length in $\mathbb{R}^{d+2}$. Such a $\mu > 0$ satisfies the equation

$$\mu^2\eta_1^2 + \mu^{2s}\eta_2^2 + \mu^2|\boldsymbol{\eta}'|^2 = 1,$$

and its existence can be shown as an intersection point of the quadratic function $\mu \mapsto \mu^2(\eta_1^2 + |\boldsymbol{\eta}'|^2)$ and the curve $\mu \mapsto 1 - \mu^{2s}\eta_2^2$. Note that as $|\boldsymbol{\eta}'| \to 0$, $\mu_{\boldsymbol{\eta}} \to \mu_0 > 0$, the intersection of the quadratic function $\mu \mapsto \mu^2\eta_1^2$ and the curve $\mu \mapsto 1 - \mu^{2s}\eta_2^2$. Also as $|\boldsymbol{\eta}'| \to \infty$, $\mu_{\boldsymbol{\eta}} \to 1$. Moreover, for any $\boldsymbol{\eta}' \neq 0$, $\mu_{\boldsymbol{\eta}} < |\boldsymbol{\eta}'|^{-1}$. As a consequence the set of unit vector

$$\mathbb{V} = \{(\mu_{\boldsymbol{\eta}}\eta_1, \mu_{\boldsymbol{\eta}}^s\eta_2, \mu_{\boldsymbol{\eta}}\boldsymbol{\eta}') : \boldsymbol{\eta}' \in \mathbb{R}^d\} \subset \mathbb{S}^{d+1}$$

avoids all points of singularity of F and its derivatives. Now, if β is a multi-index with $\beta_1 = 0 = \beta_2$, then by construction $\mu_{\boldsymbol{\eta}}^{\beta_j} \leq |\eta_j|^{-\beta_j}$ for $j \in \{3, 4, \ldots, d+1, d+2\}$. Plugging this $\mu_{\boldsymbol{\eta}}$ in (6.30), we have that for any $\boldsymbol{\eta}' \neq 0$,

$$|\partial_\beta F(\eta_1, \eta_2, \boldsymbol{\eta}')| \leq \sup_{\mathbf{v}\in\mathbb{V}} |\partial_\beta F(\mathbf{v})|\mu^{\beta_3+\beta_4+\ldots+\beta_{d+2}} \leq C|\eta_3|^{-\beta_3} \cdots |\eta_{d+2}|^{-\beta_{d+2}}$$

for some constant C that depends on η_1, η_2, β and s. Therefore, for any multi-index $\gamma = (\gamma_1, \ldots, \gamma_d)$, $\gamma_i \in \{1, 2, \cdots, d\}$, we set $\beta = (0, 0, \gamma_1, \ldots, \gamma_d)$ and obtain

$$|\partial_\gamma m_1(\boldsymbol{\xi})| = \left|\partial_\beta F\Big(1, \sqrt{\lambda}, 2\pi|\boldsymbol{\xi}|\Big)\right| \leq C|\xi_1|^{-\gamma_1} \cdots |\xi_d|^{-\gamma_d}\,.$$

Thus the hypotheses of [42, Corollary 6.2.5] are satisfied for $m_1(\boldsymbol{\xi})$.

A similar strategy fails for $m_2(\boldsymbol{\xi})$ because the numerator fails to be bounded away from zero. Specifically, $m_2(\boldsymbol{\xi})$ is C^d on $\mathbb{R}^d$ away from the coordinate axes. We must check the derivatives directly. Introduce the auxiliary function $G_a : (0, \infty) \to \mathbb{R}$ defined by

$$G_a(\eta) := \frac{\eta^s}{\eta^s + a} = \frac{1}{1 + a\eta^{-s}}\,, \qquad a > 0\,.$$

Then it is clear that for every $n \in \mathbb{N}$ the function G_a satisfies

$$\left|G_a^{(n)}(\eta)\right| \leq C\eta^{-n}\,.$$

Let $a = \frac{\lambda}{(2s+1)4\pi^2}$, and let β be a multi-index with $\beta_j \in \{0,1\}$. Then

$$\partial_\beta m_2(\boldsymbol{\xi}) = \frac{2s}{2s+1} G_a^{(|\beta|)}\big(|\boldsymbol{\xi}|^2\big)\ 2^{|\beta|} \prod_{j=1}^{d} \xi_j^{\beta_j}$$

and so

$$\big|\partial_\beta m_2(\boldsymbol{\xi})\big| \le C|\boldsymbol{\xi}|^{-2|\beta|} \prod_{j=1}^{d} |\xi_j|^{\beta_j} \le C \prod_{j=1}^{d} |\xi_j|^{-\beta_j}\,.$$

Thus m_2 also satisfies the hypotheses of of [42, Corollary 6.2.5], and the proof is complete.

□

Remark 6.13. It is also possible to prove that $m_2(\boldsymbol{\xi})$ is an L^p-multiplier without using the Marcinkiewicz multiplier theorem. In fact, one can show using the proof of [86, Chapter 5, Section 3.2, Lemma 2] that $m_2(\boldsymbol{\xi})$ is in fact the Fourier transform of a finite measure, hence an L^p-multiplier.

6.5 The Peridynamic-Type Wave Equation

In this section we turn our attention to the system of time dependent equations given in (6.11) and prove Theorem 6.4. We recall that we are interested in the system

$$\begin{cases} \partial_{tt}\mathbf{u} + \mathbb{L}\mathbf{u} + \lambda\mathbf{u} = \mathbf{f}\,, & (\mathbf{x},t) \text{ in } \mathbb{R}^d \times [0,T]\,, \\ \mathbf{u}(\mathbf{x},0) = \mathbf{u}_0(\mathbf{x})\,, & \mathbf{x} \text{ in } \mathbb{R}^d\,, \\ \partial_t\mathbf{u}(\mathbf{x},0) = \mathbf{v}_0(\mathbf{x})\,, & \mathbf{x} \text{ in } \mathbb{R}^d\,, \end{cases} \tag{6.31}$$

where $\mathbf{f} \in C^1\left([0,T]; \left[L^2(\mathbb{R}^d)\right]^d\right)$, $\mathbf{u}_0 \in \left[H^{2s,2}(\mathbb{R}^d)\right]^d$, and $\mathbf{v}_0 \in \left[H^{s,2}(\mathbb{R}^d)\right]^d$.

We use the semigroup approach described in [13, Chapter 10]. To this end, we assume for the rest of this section that ρ is in **Class B** and that $m(\mathbf{y})$ is an even function. In this case $\mathbb{L}$ becomes

$$-\mathbb{L}\mathbf{u}(\mathbf{x}) = \text{P.V.} \int_{\mathbb{R}^d} \left(\frac{\mathbf{y}\otimes\mathbf{y}}{|\mathbf{y}|^2}\right) \big(\mathbf{u}(\mathbf{x}+\mathbf{y}) - \mathbf{u}(\mathbf{x})\big)\rho(\mathbf{y})\, \mathrm{d}\mathbf{y}\,. \tag{6.32}$$

Let λ_0 be the quantity defined in Theorem 6.1, and for $\lambda > \lambda_0 - 1$ define the operator

$$\mathbb{L}_\lambda := \mathbb{L} + \lambda \mathbb{I} : \left[H^{2s,2}(\mathbb{R}^d)\right]^d \to \left[L^2(\mathbb{R}^d)\right]^d. \tag{6.33}$$

Note that by the symmetry assumption on $\mathbb{L}$,

$$\langle \mathbb{L}\mathbf{u}, \mathbf{u} \rangle_{L^2(\mathbb{R}^d)} = \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \rho(\mathbf{x}-\mathbf{y}) \left| \big(\mathbf{u}(\mathbf{x}) - \mathbf{u}(\mathbf{y})\big) \cdot \frac{\mathbf{x}-\mathbf{y}}{|\mathbf{x}-\mathbf{y}|} \right|^2 \mathrm{d}\mathbf{y}\, \mathrm{d}\mathbf{x} \geq 0$$

for every $\mathbf{u} \in \left[H^{2s,2}(\mathbb{R}^d)\right]^d$. Therefore for any $\lambda \geq 0$ we have

$$\langle \mathbb{L}\mathbf{u} + \lambda \mathbf{u}, \mathbf{u} \rangle_{L^2(\mathbb{R}^d)} \geq 0 \quad \forall \mathbf{u} \in \left[H^{2s,2}(\mathbb{R}^d)\right]^d. \tag{6.34}$$

Further, Theorem 6.1 implies that:

$$\begin{gathered} \text{for every } \mathbf{g} \in \left[L^2(\mathbb{R}^d)\right]^d \text{ there exists a unique } \mathbf{u} \in \left[H^{2s,2}(\mathbb{R}^d)\right]^d \\ \text{satisfying } \mathbb{L}_\lambda \mathbf{u} + \mathbf{u} = \mathbf{g} \text{ in } \mathbb{R}^d, \end{gathered} \tag{6.35}$$

since $\lambda > \lambda_0 - 1$. Equations (6.34) and (6.35) define $\mathbb{L}_\lambda$ as a maximal monotone operator for any nonnegative λ satisfying $\lambda > \lambda_0 - 1$. Since the unbounded operator $\mathbb{L}_\lambda$ is symmetric, by (6.34) and (6.35) it is self-adjoint on $\left[H^{2s,2}(\mathbb{R}^d)\right]^d$ as well, see [13, Proposition 7.6]. Therefore the positive square root of $\mathbb{L}_\lambda$, denoted $\mathbb{L}_\lambda^{1/2}$, is well-defined. Therefore, by the estimate in Theorem 6.1 we have that for every $\mathbf{u} \in \left[H^{2s,2}(\mathbb{R}^d)\right]^d$ the norm

$$\left\langle \mathbb{L}_\lambda^{1/2} \mathbf{u}, \mathbb{L}_\lambda^{1/2} \mathbf{u} \right\rangle_{L^2(\mathbb{R}^d)} = \langle \mathbb{L}\mathbf{u}, \mathbf{u} \rangle_{L^2(\mathbb{R}^d)} + \lambda \|\mathbf{u}\|_{L^2}^2 \tag{6.36}$$

is equivalent to the norm $\left\|(-\Delta)^{s/2}\mathbf{u}\right\|_{L^2(\mathbb{R}^d)}^2 + \|\mathbf{u}\|_{L^2(\mathbb{R}^d)}^2$. Thus $\left\langle \mathbb{L}_\lambda^{1/2}(\,\cdot\,), \mathbb{L}_\lambda^{1/2}(\,\cdot\,) \right\rangle$ defines an equivalent inner product on $\left[H^{s,2}(\mathbb{R}^d)\right]^d$.

Now, we rewrite the system (6.31) as a larger system of equations

$$\begin{cases} \mathbf{v} = \partial_t \mathbf{u}\,, & (\mathbf{x},t) \text{ in } \mathbb{R}^d \times [0,T]\,, \\ \partial_t \mathbf{v} + \mathbb{L}\mathbf{u} + \lambda \mathbf{u} = \mathbf{f}\,, & (\mathbf{x},t) \text{ in } \mathbb{R}^d \times [0,T]\,, \\ \mathbf{u}(\mathbf{x},0) = \mathbf{u}_0(\mathbf{x})\,, & \mathbf{x} \text{ in } \mathbb{R}^d\,, \\ \partial_t \mathbf{u}(\mathbf{x},0) = \mathbf{v}_0(\mathbf{x})\,, & \mathbf{x} \text{ in } \mathbb{R}^d\,, \end{cases} \tag{6.37}$$

or equivalently

$$\begin{cases} \partial_t \mathbf{U} + \mathfrak{A}_\lambda \mathbf{U} = \mathbf{F}\,, & (\mathbf{x},t) \text{ in } \mathbb{R}^d \times [0,T]\,, \\ \mathbf{U}(\mathbf{x},0) = \mathbf{U}_0(\mathbf{x})\,, & \mathbf{x} \text{ in } \mathbb{R}^d\,, \end{cases} \tag{6.38}$$

where $\mathbf{U} = (\mathbf{u},\mathbf{v})^\intercal$, $\mathbf{F} = (\mathbf{0},\mathbf{f})^\intercal$, $\mathbf{U}_0 = (\mathbf{u}_0,\mathbf{v}_0)^\intercal$ and the operator $\mathfrak{A}_\lambda$ is defined as

$$\mathfrak{A}_\lambda := \begin{pmatrix} 0 & -\mathbb{I} \\ \mathbb{L}_\lambda & 0 \end{pmatrix}.$$

We denote the Hilbert space $\mathcal{H} := \left[H^{s,2}(\mathbb{R}^d)\right]^d \times \left[L^2(\mathbb{R}^d)\right]^d$, and we define the domain of the operator $\mathfrak{A}_\lambda$ by $D(\mathfrak{A}_\lambda) := \left[H^{2s,2}(\mathbb{R}^d)\right]^d \times \left[H^{s,2}(\mathbb{R}^d)\right]^d \subset \mathcal{H}$. By (6.36), the inner product on $\mathcal{H}$ can be defined by

$$\langle \mathbf{U},\mathbf{V}\rangle_{\mathcal{H}} := \int_{\mathbb{R}^d} \left\langle \mathbb{L}_\lambda^{1/2}\mathbf{u}_1, \mathbb{L}_\lambda^{1/2}\mathbf{v}_1 \right\rangle \,\mathrm{d}\mathbf{x} + \int_{\mathbb{R}^d} \langle \mathbf{u}_1,\mathbf{v}_1\rangle \,\mathrm{d}\mathbf{x} + \int_{\mathbb{R}^d} \langle \mathbf{u}_2,\mathbf{v}_2\rangle \,\mathrm{d}\mathbf{x} \tag{6.39}$$

where $\mathbf{U} = (\mathbf{u}_1,\mathbf{u}_2)$, $\mathbf{V} = (\mathbf{v}_1,\mathbf{v}_2)$.

Denoting the identity operator on $\mathcal{H}$ by $\mathfrak{I}$, we note that the operator $\mathfrak{A}_\lambda + \mathfrak{I}$ is nonnegative. Indeed, for any $\mathbf{U} = (\mathbf{u}, \mathbf{v})^\intercal \in D(\mathfrak{A}_\lambda)$ we have using (6.39) that

$$
\begin{aligned}
&\langle (\mathfrak{A}_\lambda + \mathfrak{I})\mathbf{U}, \mathbf{U} \rangle_{\mathcal{H}} \\
&= \langle \mathfrak{A}_\lambda \mathbf{U}, \mathbf{U} \rangle_{\mathcal{H}} + \langle \mathbf{U}, \mathbf{U} \rangle_{\mathcal{H}} \\
&= - \int_{\mathbb{R}^d} \left\langle \mathbb{L}_\lambda^{1/2} \mathbf{u}, \mathbb{L}_\lambda^{1/2} \mathbf{v} \right\rangle \, \mathrm{d}\mathbf{x} - \int_{\mathbb{R}^d} \langle \mathbf{u}, \mathbf{v} \rangle \, \mathrm{d}\mathbf{x} + \int_{\mathbb{R}^d} \langle \mathbb{L}_\lambda \mathbf{u}, \mathbf{v} \rangle \, \mathrm{d}\mathbf{x} \\
&\quad + \int_{\mathbb{R}^d} |\mathbb{L}_\lambda^{1/2} \mathbf{u}|^2 \, \mathrm{d}\mathbf{x} + \int_{\mathbb{R}^d} |\mathbf{u}|^2 \, \mathrm{d}\mathbf{x} + \int_{\mathbb{R}^d} |\mathbf{v}|^2 \, \mathrm{d}\mathbf{x} \\
&= - \int_{\mathbb{R}^d} \langle \mathbb{L}_\lambda \mathbf{u}, \mathbf{v} \rangle \, \mathrm{d}\mathbf{x} + \int_{\mathbb{R}^d} \langle \mathbb{L}_\lambda \mathbf{u}, \mathbf{v} \rangle \, \mathrm{d}\mathbf{x} + \int_{\mathbb{R}^d} |\mathbb{L}_\lambda^{1/2} \mathbf{u}|^2 \, \mathrm{d}\mathbf{x} \\
&\quad + \int_{\mathbb{R}^d} |\mathbf{u}|^2 \, \mathrm{d}\mathbf{x} - \int_{\mathbb{R}^d} \langle \mathbf{u}, \mathbf{v} \rangle \, \mathrm{d}\mathbf{x} + \int_{\mathbb{R}^d} |\mathbf{v}|^2 \, \mathrm{d}\mathbf{x} \\
&= \int_{\mathbb{R}^d} |\mathbb{L}_\lambda^{1/2} \mathbf{u}|^2 \, \mathrm{d}\mathbf{x} + \frac{1}{2} \int_{\mathbb{R}^d} (|\mathbf{u}|^2 + |\mathbf{v}|^2) \, \mathrm{d}\mathbf{x} + \frac{1}{2} \int_{\mathbb{R}^d} |\mathbf{u} - \mathbf{v}|^2 \, \mathrm{d}\mathbf{x} \geq 0 \,.
\end{aligned}
\tag{6.40}
$$

In addition, the operator $\mathfrak{A}_\lambda + 2\mathfrak{I}$ is invertible. To see this, note that inverting $\mathfrak{A}_\lambda$ is equivalent to solving the system

$$
\begin{cases} -\mathbf{v} + 2\mathbf{u} = \mathbf{g}_1 \,, \quad \mathbb{L}\mathbf{u} + \lambda \mathbf{u} + 2\mathbf{v} = \mathbf{g}_2 \,, \qquad \mathbf{x} \in \mathbb{R}^d \,, \end{cases}
\tag{6.41}
$$

for any $(\mathbf{g}_1, \mathbf{g}_2)^\intercal \in \mathcal{H}$, which is equivalent to solving

$$
\mathbb{L}\mathbf{u} + (\lambda + 4)\mathbf{u} = 2\mathbf{g}_1 + \mathbf{g}_2 \,, \quad \mathbf{x} \in \mathbb{R}^d \,.
\tag{6.42}
$$

Since $\lambda + 4 > \lambda_0$ and since $2\mathbf{g}_1 + \mathbf{g}_2 \in \left[L^2(\mathbb{R}^d)\right]^d$, (6.42) has a unique solution $\mathbf{u} \in \left[H^{2s,2}(\mathbb{R}^d)\right]^d$. Then $\mathbf{v} = 2\mathbf{u} - \mathbf{g}_1 \in \left[H^{s,2}(\mathbb{R}^d)\right]^d$ and so $\mathbf{U} = (\mathbf{u}, \mathbf{v})^\intercal \in D(\mathfrak{A}_\lambda)$ solves (6.41).

Therefore, since $\mathfrak{A}_\lambda + \mathfrak{I}$ is *maximal monotone* for $\lambda > \lambda_0 - 1$ nonnegative, by the Hille-Yosida theorem [22, Theorem 1, Chapter XVII, section 3], there is a continuous contraction semigroup $S_{\mathfrak{A}_\lambda + \mathfrak{I}}(t) : L^2 \to L^2$ such that for any $\mathbf{G} \in [L^2(\mathbb{R}^d)]^{2d}$ the solution $\mathbf{U}$ of

$$
\begin{cases} \partial_t \mathbf{U} + \mathfrak{A}_\lambda \mathbf{U} + \mathfrak{I}\mathbf{u} = \mathbf{G} \,, & (\mathbf{x}, t) \text{ in } \mathbb{R}^d \times [0, T] \,, \\ \mathbf{U}(\mathbf{x}, 0) = \mathbf{U}_0(\mathbf{x}) \,, & \mathbf{x} \text{ in } \mathbb{R}^d \,, \end{cases}
\tag{6.43}
$$

can be given by the formula [13, Theorem 7.10]

$$\mathbf{U}(\mathbf{x},t) = S_{\mathfrak{A}_\lambda+\mathfrak{I}}(t)\mathbf{U}_0(\mathbf{x}) + \int_0^t S_{\mathfrak{A}_\lambda+\mathfrak{I}}(t-s)\mathbf{G}(\mathbf{x},s)\,\mathrm{d}s\,.$$

Taking $\mathbf{G} = \mathrm{e}^t\mathbf{F}$ and making the substitution $\mathbf{W} = \mathrm{e}^t\mathbf{U}$, we see that $\mathbf{W}$ is the unique solution to (6.38) with source data $\mathbf{F}$. Thus the first component of $\mathbf{W}$ is the unique solution to (6.31).

The conservation law (6.12) follows from multiplying the equation by $\partial_t\mathbf{u}$ and integrating by parts.

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Vita

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