

**Two-Dimensional Formulation and Quasi-One-Dimensional
Approximation to Inverse Heat Conduction by the
Calibration Integral Equation Method (CIEM)**

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Abstract

The recently devised calibration integral equation method developed at the University of Tennessee for resolving transient inverse heat conduction in one-dimensional applications is extended and studied in the context of two-dimensional linear inverse heat conduction. This study investigates a simplified plate geometry possessing three known boundary conditions and one unknown boundary condition. This plate contains a series of temperature sensors located on a fixed plane below the surface of interest. To begin the investigation, a quasi-one-dimensional formulation is proposed for predicting the surface heat flux (W/m^2) based on a zonal formulation where each zone contains a single thermocouple. In this way, a locally one-dimensional view is proposed for predicting the local or zonal surface heat flux. The thermocouple data set is composed of physically two-dimensional information; however, each surface projection only considers one-dimensional heat flow based on its zone. In this concept, each zone produces a spatial constant heat flux that can temporally vary from zone-to-zone. Each zonal surface heat flux is mathematically described in terms of a Volterra integral equation of the first kind. Being ill posed, regularization based on a local future time method is introduced for stabilization. A new metric is proposed and demonstrated for extracting the optimal regularization parameter. This zonal approximation for materials composed of a low thermal conductivity is shown to yield favorable results. The second study presented in this thesis considers the development of a total heat transfer (W) calibration integral equation based on a fully two-dimensional analysis. In this form, the total surface heat transfer (i.e., the spatially integrated value along the entire surface of interest), is directly derived and implement bypassing the need to determine the local surface heat flux (W/m^2). This formulation yields a Volterra integral equation of the first kind similar to the mathematical structure previously described. In many applications, the total surface heat transfer is more important than the local surface heat flux. As such, this new formulation appears highly pertinent. This formulation is shown to produce favorable results over a large range of thermal conductivities and thermal diffusivities.

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Nomenclature

a	length of two-dimensional plate, cm
A, B	coefficient of Gaussian function
b	height of two-dimensional plate, cm
C_γ	constant defined in Eq. (3.20d), °Cs
$C_{Q,\gamma}$	constant defined in Eq. (5.20d), °Cms
$C(s), D(s)$	coefficient in frequency domain, Eqs. (3.7), °Cs,
d	distance between thermocouple and origin for one-dimensional IHCP
d	distance between thermocouples and x axis for two-dimensional IHCP
f	function, $f(t)$
f_{sampling}	sampling frequency, Hz
$\hat{f}$	Laplace transform of function $f(t)$
g	function, $g(t)$
$\hat{g}$	Laplace transform of function $g(t)$
h	convective heat transfer coefficient, W/m ² K
k	thermal conductivity, W/mK
$K_{tc,cal}$	kernel defined in Eq. (5.14b), °Cm
$K_{tc,run}$	kernel defined in Eq. (5.14c), °Cm
L	length of one-dimensional slab, cm
$L_{\text{Stainless Steel}}$	length of stainless steel in electrical heating experiment, mm
L_{Mica}	length of mica in electrical heating experiment, mm
L_{Paint}	length of thermal paint in electrical heating experiment, mm
L_{Heater}	length of heater in electrical heating experiment, mm
M_f	multiplying factor
$\hat{M}$	transformed function, Eq. (5.8b)
N	total number of time increments
$N(\beta_m)$	normalization integrals, m
$N(\eta_n)$	normalization integrals, m
$\hat{N}$	transformed function, Eq. (3.9c), J/m ² K
q''	heat flux, W/cm ²
q''_c	surface heat flux in “calibration” run, W/cm ²
$q''_{c,max}$	maximum value of “calibration” run surface heat flux, W/cm ²
$q''_{c,original}$	original “calibration” run surface heat flux in Section 3.5, W/cm ²
q''_r	surface heat flux in unknown “run”, W/cm ²

$q''_{r,max}$	maximum value of unknown “run” surface heat flux, W/cm ²
$q''_{r,original}$	original unknown “run” surface heat flux in Section 3.5, W/cm ²
$q''_{r,\gamma}$	predicted unknown “run” surface heat flux, W/cm ²
q''_{r,γ_m}	predicted unknown “run” surface heat flux, W/cm ²
$q''_{r,\gamma_{opt}}$	predicted unknown “run” surface heat flux using optimal γ , W/cm ²
q''_s	net surface heat flux, W/cm ²
q''_x	heat flux in x direction, W/cm ²
q''_y	heat flux in y direction, W/cm ²
$\hat{q}''$	transformed heat flux, W/m ²
$\hat{q}''_c$	transformed “calibration” run surface heat flux, W/cm ²
$\hat{q}''_r$	transformed unknown “run” surface heat flux, W/cm ²
Q	total heat transfer, W
Q_{cal}	“calibration” run total heat transfer in Section 5.1, W
Q_r	exact unknown “run” total heat transfer in Section 4.1, W
Q_{run}	exact unknown “run” total heat transfer in Section 5.1, W
$Q_{run,\gamma}$	predicted unknown “run” total heat transfer in Section 5.2, W
$Q_{r,\gamma_{opt}}$	predicted total heat transfer using optimal γ in Section 4.1, W
$\hat{Q}$	transformed total heat transfer, Ws
r_j	random number in the interval [-1,1]
r_γ	residual function defined in Eq. (3.22), J°C/cm ²
$\bar{r}_\gamma$	running average of residual function defined in Eq. (3.23), J°C/cm ²
R_γ	residual function defined in Eq. (5.22), J°Cm
$\bar{R}_\gamma$	running average of residual function defined in Eq. (5.23), J°Cm
t	time, s
t_{max}	maximum time range, s
u, v	dummy time variables, s
T	temperature, °C
T_c, T_{cal}	“calibration” run positional temperature at thermocouple site, °C
T_r, T_{run}	unknown “run” positional temperature at thermocouple site, °C
$T_{tc,c}, T_{tc,cal}$	measured thermocouple temperature in “calibration” run, °C
$T_{tc,c,original}$	original “calibration” run thermocouple temperature in Section 3.5, °C
$T_{tc,r}, T_{tc,run}$	measured thermocouple temperature in unknown “run”, °C
$T_{tc,r,original}$	original unknown “run” thermocouple temperature in Section 3.5, °C

$\hat{T}$	transformed temperature, °Cs
$\hat{T}_c$	transformed “calibration” run position temperature, °C
$\hat{T}_r$	transformed unknown “run” position temperature, °C
W	width of stainless steel in electrical heat experiment, mm
x, y	spatial variables, m
x', y'	dummy spatial variables, m

Greek

α	thermal diffusivity, m ² /s
β_m, η_n	eigenvalues, m ⁻¹
ε	relative error
ε_q	noise factor of heat flux
ε_Q	noise factor of total heat transfer
ε_T	noise factor of temperature
Δt	time increment, s
Δx	space increment, m
γ	future time parameter, s
γ_m	future time parameter, s
γ_{opt}	optimal future time parameter, s
θ	angle, °
$\theta_{tc,c}$	thermocouple temperature difference in “calibration” run, Eq. (3.15b)
$\theta_{tc,r}$	thermocouple temperature difference in unknown “run”, Eq. (3.15c)
σ_γ	square root of time-running variance of residual defined in Eq. (3.24)
σ_γ^*	normalized square root of time-running variance of residual defined in Eq. (3.25)
$\sigma_{R,\gamma}$	square root of time-running variance of residual defined in Eq. (5.24), J°Cm
$\sigma_{R,\gamma}^*$	normalized square root of time-running variance of residual defined in Eq. (5.25)

Chapter 1 Introduction

1.1 Problem Statement

Inverse heat conduction problems (IHCP's) involve predicting the surface thermal conditions based on in-depth temperature measurements [1-3]. In conventional heat conduction problems, the interior temperature distribution of a solid body is calculated when the boundary and initial conditions are known. These are direct problems that are well studied and well understood [4,5]. In many real-world applications, sensors can not be placed on the boundaries due to harsh surface thermal environment. Thus, a technological void is introduced when a direct measurement is not available. In-depth probe placement is necessary for practical problems associated numerous aerospace applications involving reentry, combustors, solid rockets, nozzles and fire research [1]. With in-depth measurement data, surface physics can be indirectly reconstructed.

This slight variation in probe placement (surface to interior) abruptly alters the mathematical framework by producing an ill-posed problem. An ill-posed problem does not satisfy Hadamard's three conditions for well posedness [1-3,6]. These three conditions involve existence, uniqueness and stability of the solution. In particular, the discrete noisy temperature data collected at the probe site cause instability in the prediction. Alternatively stated, a small change in the collected data can cause substantial changes in the prediction. This phenomenon can also be explained by the physics of heat diffusion. Heat diffusion damps high frequency oscillation as heat passes through the conducting body. In the opposite direction, high frequency oscillations associated with in-depth measurements are magnified when projected to the surface. Therefore, some sort of regularization [1-3,7] is necessary for all inverse heat conduction problems.

1.2 Purpose and Scope of Thesis

This thesis will study the two-dimensional plate geometry as shown in Fig. 1.1. Figure 1.1 displays a two-dimensional plate of length a , height b and unit depth. It has adiabatic conditions on three sides. The top of the plate is exposed to some time varying spatially distributed source, given as $q''_s(x, t)$.

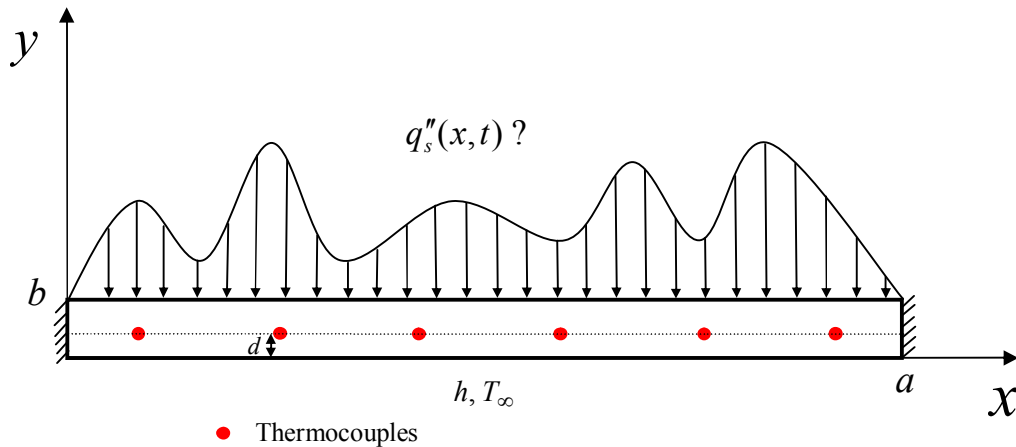


Figure 1.1: Schematic for two-dimensional plate geometry with time-varying and spatially distributed source. Thermocouples are indicated as solid circles.

The plate itself does not have any sources or sinks. As shown, six thermocouples are placed equidistant apart so that common width zones can be defined. The thermocouples are placed at $y=d$. The direct analysis would require the determination of either or both the temperature field $T(x,y,t)$ and heat flux given by $q''(x,y,t) = q''_x(x,y,t)\vec{i} + q''_y(x,y,t)\vec{j}$ when provided the boundary and initial conditions; and, appropriate thermophysical properties of material. In contrast, the inverse problem would require the reconstruction of either or both the net surface heat flux $q''_y(x,b,t)$ and surface temperature $T(x,b,t)$ using a finite number of in-depth temperature sensors potentially placed at $y=d$. Resolution of physical surface details rely on many factors such as the numbers of probes, location and depth of probes, accuracy of measurements and thermophysical properties, etc. [1-3].

Compared to one-dimensional IHCP's, two-dimensional IHCP's are not well studied due to their complexity and difficulty. A transformative calibration methodology has been recently proposed [8-10] based on the integration of mathematical reasoning and experimental design. This produces an alternative framework for studying inverse problems. The basis of this thesis involves extending the calibration methodology described in [8-10] to two-dimensional plate geometries. The finite two-dimensional domain serves several practical physical problems as well as serving as a benchmark for developing a quasi-one-dimensional approach for the geometry shown in Fig. 1.2. Owing to practical considerations in low conductivity materials, such as required in most thermal protection systems (TPS), a quasi-one-dimensional study is necessitated and studied in detail. This represents the first analysis performed in the two-dimensional body. The total heat transfer is a quantity of significant interest. As such, we revisit Fig. 1.2 and view the system in a different manner. That is, we derive a total heat transfer calibration integral equation that bypasses the need to obtain the local heat flux. The calibration integral equation approach reduces systematic errors through carefully integrating of calibration and analysis. Key to all inverse studies is the ability to extract the optimal regularization parameter. The calibration integral equation method will be shown to possess a fundamental measure that permits its estimation.

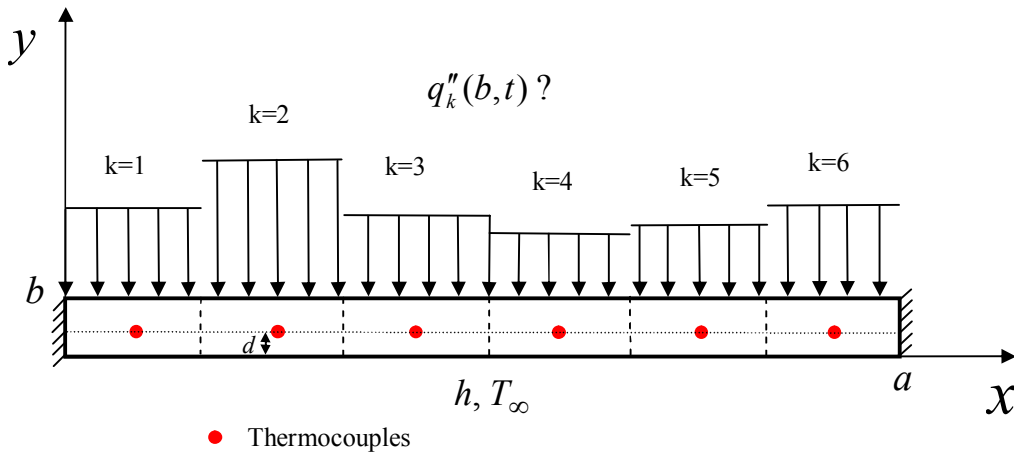


Figure 1.2: Schematic of quasi-one-dimensional approximation of two-dimensional problem. (Here $k=1,2,\dots,6$ represent local one-dimensional heat flux approximations corresponding to fully two-dimensional surface heat flux displayed in Fig. 1.1)

Finally, the methodology is highly robust and leads to a simple numerical implementation. In Chapter 3, the calibration methodology for IHCP's is introduced and described in detail, in the context of one-dimensional IHCP's. This reviews and sets the mathematical framework for this thesis. Chapter 4 presents a quasi-one-dimensional approximation to a two-dimensional IHCP as graphically depicted in Fig. 1.2. This idealization represents the first step toward investigating a fully two-dimensional inverse calibration study. A locally one-dimensional view, based on two-dimensionally gathered transient temperature data, is investigated for forming the surface heat flux approximation. This approximation is shown accurate for low conductivity materials that have practical engineering value in the study of thermal protection systems (TPS's). In Chapter 5, a two-dimensional total heat transfer calibration integral equation method is developed and numerical results display the merit and novelty of the concept. It is shown to produce favorable results for a large set of engineering materials under the assumption of constant thermo-physical properties. Finally, Chapter 6 provides some conclusions and recommendation for future research.

Chapter 2 Literature Review

The advent of space program and aerospace industry in 1950's gave considerable impetus to the studies of inverse heat conduction problems [1,7-10]. A Russian paper by Shumakov [11] on IHCP's was translated in 1957. This is one of the earliest research publications concerned on this topic. The applications therein were related to nose cones of missiles, rocket nozzles and other devices. Another early paper published on the IHCP's was published by Stolz [12] in 1960. This paper addressed calculating the heat transfer rate in a quenching process. Stolz's method involved the numerical solution of a first kind Volterra integral equation. However, the time-marching algorithm required relatively large time steps to retain stability. Beck et al. [1] developed several IHCP techniques that are used in practice today. The work of Beck and his colleagues strongly influenced the future time and function specification methods. Beck published a landmark paper [13] in 1970, that became the basis for many early computer programs. Though IHCP's were initially used to estimate the surface heat flux of reentry space vehicles, it also has numerous applications in industry. For example, studies have been presented involving nuclear reactor components [14-20], periodic heating in combustion chambers of internal combustion engines [21], solidification of glass [22], indirect calorimetry for laboratory use [23], and transient boiling curve studies [24]. Furthermore, the techniques employed in IHCP's can be applied to many other types of inverse analyses including oil exploration, remote sensing, x-ray tomography [25], nondestructive evaluation of materials and determination of the Earth's interior structure [1].

A variety of methods have been exploited to resolve the inverse heat conduction problem, among them are Tikhonov regularization [26], function specification [27-29], space marching and finite difference [30-36], global time method [37-40], non-integer system identification [41-43], exact solution [44], digital filtering [45-47], conjugate gradient method [48-52], singular-value decomposition (SVD) [53- 55], iteration method [56], boundary-element method [57,58], and neural networks [59-61].

The classical Tikhonov regularization stabilizes the ill-posed problem by adding the product of a "regularization parameter" with a semi-norm involving some function. Often this semi-norm involves the heat flux [26]. However, determining the value of the "regularization parameter" is often problematic as it does not have a clear physical interpretation. There are several approaches available for estimating the optimal Tikhonov regularization parameter. These include Morozov's discrepancy principle [62,63], L-curve method [62,64-66], and maximum likelihood method [62]. These methods have their limitations. Thus, determining the suitable Tikhonov regularization parameter is still under intensive research.

In the function specification method [1], the transient surface heat flux with time is assumed to be of a functional form. The regularization parameter in this approach involves specifying the number of future time steps required for stabilizing the approximation. A present, no a priori rule exists for estimating this regularization parameter. The function specification method is computationally efficient since it is sequential in time. The difficulty of this method lies defining the number of future time steps since it depends on the unknown surface heat flux.

In the space-marching method, the spatial and temporal domains are discretized, and the partial derivatives with respect to time and space are represented by finite difference approximations [32-36]. Temperature data from thermocouple sensors are commonly used and imposed as

known boundary conditions. The implicit finite difference scheme is sensitive to high frequency measurement error [30]. Various methods have been proposed for damping out the noise in measurements. Al-Khalidy [45] used a least squares fit based on a polynomial representation for filtering the noise in the measured temperatures. Notwithstanding, least-squares fitting cannot guarantee the optimal representation of the time derivatives of the filtered function [67].

Elkins et al. [38] presented a global time and discrete space formulation of an IHCP. In contrast with conventional IHCP techniques, the heating rate and higher time derivatives of temperature data are directly measured by a rate-based sensor [68]. This is done in lieu of using a finite difference representation for the time derivatives of the measured temperature data. The rate-based sensor concept involves analog filtering prior to entering as voltage rate circuit. This concept uses the filter cut-off frequency as the regularization parameter. In Ref. [47] a Gauss low-pass filter with a physically based cut-off frequency is used for regularization in resolving the null point equation associated with arcjet testing. The Gauss filter removes high frequency noise from the collected temperature data as a processor. Data interrogation by discrete Fourier transform produces a power spectrum that leads itself to weight filtering concepts designed for estimating the cut-off frequency. Robustness has been shown using this principle. The Gauss filter maintains smoothness in higher time derivatives unlike most low pass digital filters. The robustness of global time method [38] lies in its accuracy to predict the surface heat flux as the sampling rate increases. This contrasts many traditional inverse methods.

The conjugate gradient method with the adjoint problem has also been widely used to resolve IHCP's. Zhou et al. [52] studied the one-sided inverse heat conduction problem where both the temperature and heat flux are specified at the back boundary. The temperature data are used as back surface boundary condition and the heat flux is adopted as the objective function to be minimized. The IHCP formulation was shown to possess good stability in the parameter range considered in that study. However, the conjugate gradient method is computationally expensive and requires a large amount of memory.

Singular-value decomposition is another approach used in resolving IHCP's based on matrix manipulations. The dependency of the surface heat flux and temperature response at the thermocouple site can be obtained by Duhamel's principle [53-55]. The ill-conditioned matrix is decomposed into two orthonormal matrices and a diagonal matrix that contains its singular values in descending order. Elements in the diagonal matrix after a certain row number can be set to zero in order to remove noise. Singular-value decomposition can also be viewed as a digital filter. The row number serves as the regularization parameter in this method. Again, key to this method is the determination of optimal row number.

Calibration is a novel approach to resolve the inverse heat conduction problems. The Non-Integer System Identification (NISI) method [41-43] is a calibration method that requires an accurate extraction of the impulse function based on the fractional derivative formulation of the heat equation. This approach has been used in null-point calorimetry. A known net surface source is first used as a calibration source to get the relationship between net surface heat flux and temperature response at the sensor site. The sensor characteristics, depth of sensor, and thermophysical properties of the host material are accounted in the calibration coefficients that are determined by a least squares method. The unknown surface heat flux can be estimated by the corresponding sensor response and the calibration coefficients. Nevertheless, the NISI method is limited to one-dimensional, semi-infinite cases involving isotropic materials with constant properties.

To date, less work has been reported on two-dimensional and three-dimensional inverse heat conduction problems due to the additional complexity associated with dimensionality and probe placement strategies [69-79]. Chen and Lin [69] formulated a numerical scheme involving the Laplace transformation and control volume method for the problem. A shape function is used in the control volume formulation as regularization. Pourgholi et al. [71] resolved a two-dimensional IHCP via a finite difference method and least-squares scheme in the presence of noisy data. Tikhonov regularization method is then applied to obtain the stable numerical approximation to the solution.

Osman et al. [72] combined the function specification and Tikhonov regularization methods to resolve a two-dimensional IHCP. A piecewise polynomial function is used for the parameterization of the spatial distribution for the unknown surface heat flux while “stair-wise” steps on time approximation are used at each discretized spatial location. A sequential-in-time procedure is used for the prediction of the surface heat flux. The objective function combines the function specification and regularization which is minimized in the least-squares sense.

Busby and Trujillo [75] used the numerical method of dynamic programming to resolve two-dimensional inverse heat conduction problems. The heat equation is expressed as a vector-matrix differential equation with corresponding initial conditions. A finite element method is used to generate the discretization. It is noted that the predicted results are sensitive to the diagonal value of the weighting matrix. A filtering technique such as Butterworth filter [80] is suggested to precondition the data when the data are extremely noisy.

Guo and Murio [76] introduced a fully explicit and stable space marching finite difference scheme to resolve a two-dimensional IHCP. The noisy data are filtered by a discrete mollification [81] against a suitable averaging kernel. The method uses a direct discretization of the differential equation. A finite difference space marching scheme is used to resolve the problem. However, this method is presently limited to two-dimensions in a semi-infinite slab geometry [76]. The implementation of the method to a general but finite region two-dimensional problem requires future investigation.

A prediction method based on the frequency domain was implemented by Lutich et al. [77] to resolve multidimensional inverse heat conduction problems. This method extends the work of Blum and Marquardt [78]. Determination of an optimal boundary heat flux parametrization and placement of the sensors is based on maximizing the energy content of the input-output operator of the heat conduction model.

Garcia et al. [55] considered a sequential SVD method for the two-dimensional nonlinear IHCP in irregular-shaped bodies. The nonlinearity is due to temperature dependent thermophysical properties. The finite-element method is applied to solve the direct problem. Test cases were presented to verify the stability of the method. An overall error estimation was been defined in order to find the optimal estimation of the two-dimensional IHCP. In addition, the accuracy of the method was evaluated by comparison with the function specification method [1]. The sequential SVD method provides slightly more accurate results than the function specification method in most cases.

An alternative to purely numerical treatments of IHCP's involves analytical methods. Compared to numerical methods, analytical methods can provide additional insight into the physics of inverse heat conduction problem as well as saving computational time. Monde et al. [79], and Monde and Mitsutake [82] developed an analytical method for both one-dimensional and

two-dimensional inverse heat conduction problems using the Laplace transform technique. The temperature changes measured in the body are approximated by combining a polynomial power series in time with Fourier series expansion in space. The spatial resolution of the surface heat flux predicted using this method is limited by the sensor spacing, probe depth placement and the accuracy of the sensor.

In all previous methods noted in this chapter, with exception to the NISI method, thermophysical properties require specification and should be accurately known. Probe positioning must be accurately portrayed and measured by some means such as x-ray, CT scans, etc. In addition, issues associated with the sensor attachment to the host material must be quantified in order to account for potential delay and attenuation effects. The calibration methodology proposed by Frankel et al. [8], Elkins et al. [9], Frankel and Keyhani [10] inherently contains sensor positioning, sensor characteristics and thermophysical properties of the host material in the final mathematical expression that relates the in-depth measured temperature data to the surface heat flux. The final mathematical expression is presented in terms of a Volterra integral equation of the first kind [83,84] which is inherently ill-posed. Therefore, regularization is required to stabilize the prediction [85]. The goal of this thesis is to extend the calibration method to finite two-dimensional domains.

Chapter 3 Background

In this Chapter, background material is presented describing the physics-based calibration methodology. This methodology was initially proposed for resolving one-dimensional IHCP's. To illustrate the approach, the one-dimensional, one-probe calibration equation is derived in Section 3.1. Section 3.2 describes the implementation of a local future time method. This approach is employed as the regularization scheme for stabilizing the prediction. Section 3.3 describes several metrics used for extracting the optimal prediction. Section 3.4 describes the numerical implementation and results using numerically simulated data. Section 3.5 illustrates experimental validation of the inverse methodology [9]. Section 3.6 presents some concluding remarks pertaining to the contents of this chapter.

3.1 Derivation of The One-Dimensional, One-Probe Calibration Equation

The one-dimensional calibration equation is derived based on frequency domain analysis of the linear heat equation [8]. The one-region, one-dimensional geometry is shown in Fig. 3.1. The width of the slab is denoted by L and the thermocouple depth is given as d . The front surface is exposed to transient heat flux $q_s''(t)$ while the back surface is exposed to a convection condition having constant convective heat transfer coefficient h and ambient temperature T_∞ .

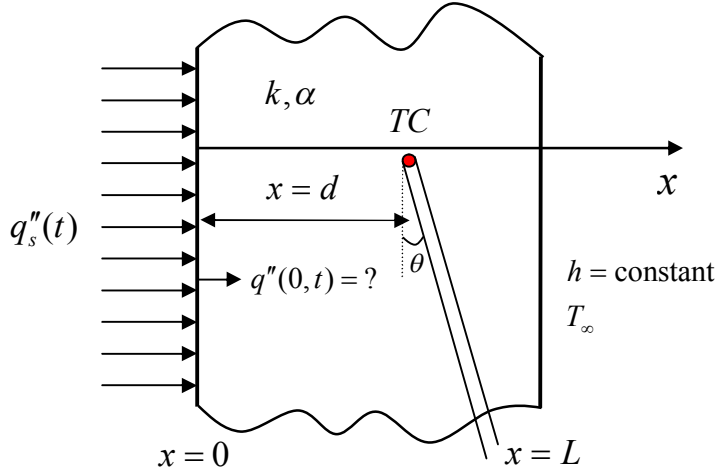


Figure 3.1: Schematic of one-region, one-dimensional IHCP in finite slab possessing constant thermophysical properties.

The initial condition of the slab is given as T_0 prior to the boundary condition exposures. The thermophysical properties involved in the linear transient heat equation are thermal conductivity k , and thermal diffusivity α . Finally, Fig. 3.1 shows a thermocouple (TC) oriented at angle θ . If the leads have orientation $\theta = 0^\circ$ then the leads are oriented along the isotherms, and thereby minimizing conductive lead losses. It should be noted that in general the positional temperature required by the heat equation may not be equal to the thermocouple temperature. The TC temperature is voltage converted temperature based on the thermocouple calibration curve.

The transient one-dimensional linear heat equation [4,5] is given as

$$\frac{1}{\alpha} \frac{\partial T}{\partial t}(x, t) = \frac{\partial^2 T}{\partial x^2}(x, t), \quad x \in [0, L], \quad t \geq 0, \quad (3.1a)$$

where T denotes temperature, x denotes spatial position and t denotes time. For the geometry presented in Fig. 3.1, the boundary conditions are

$$q''(0, t) \equiv -k \frac{\partial T}{\partial x}(0, t), \quad t > 0, \quad (3.1b)$$

$$-k \frac{\partial T}{\partial x}(L, t) = h(T(L, t) - T_\infty), \quad t > 0, \quad (3.1c)$$

where $q''(0, t)$ is the net surface heat flux. The initial condition is

$$T(x, 0) = T_0, \quad x \in [0, L]. \quad (3.1d)$$

The convection heat transfer coefficient h in Eq. (3.1c) is assumed to be constant and identical among all test “runs”. The adiabatic back boundary condition is recovered as $h \rightarrow 0$ (W/m²K) while an isothermal back boundary condition $T(L, t) = T_\infty$ is recovered as $h \rightarrow \infty$ (W/m²K). For simplicity, the initial temperature T_0 and ambient temperature T_∞ are set as $T_0 = T_\infty = 0$ °C.

The basis for the calibration method lies in performing much of the analysis in the frequency domain. As such, utilization of the Laplace transform technique [86-90] provides a means to transform time variable, t into a frequency variable denoted as s . Recall the definition of the Laplace transform [86-90] as

$$\mathcal{L}[f(t)] = \hat{f}(s) = \int_{t=0}^{\infty} e^{-st} f(t) dt, \quad s \geq 0. \quad (3.2a)$$

The Laplace transform method possesses a convolution theorem that allows for the inversion or return to the time domain. The two-term convolution theorem that will later be utilized is given as [86-90]

$$\hat{f}(s) * \hat{g}(s) = \mathcal{L}\left[\int_{u=0}^t f(u)g(t-u)du\right] = \mathcal{L}\left[\int_{u=0}^t f(t-u)g(u)du\right], \quad s \geq 0. \quad (3.2b)$$

Taking the Laplace Transform of Eq. (3.1a) subject to the initial condition given in Eq. (3.1d) produces

$$\frac{1}{\alpha} (s\hat{T}(x, s) - T_0) = \frac{d^2 \hat{T}}{dx^2}(x, s), \quad x \in [0, L], \quad s \geq 0. \quad (3.3a)$$

Letting the initial condition $T_0 = 0$ °C, thereby reduces Eq. (3.3a) to

$$\frac{d^2 \hat{T}}{dx^2}(x, s) - \frac{s}{\alpha} \hat{T}(x, s) = 0, \quad x \in [0, L], \quad s \geq 0, \quad (3.3b)$$

whose exact solution is

$$\hat{T}(x, s) = C(s) \cosh\left(\sqrt{\frac{s}{\alpha}} x\right) + D(s) \sinh\left(\sqrt{\frac{s}{\alpha}} x\right), \quad x \in [0, L], \quad s \geq 0. \quad (3.3c)$$

The corresponding transformed heat flux based on Fourier’s law [4,5] $q''(x, t) = -k \frac{\partial T}{\partial x}(x, t)$ is given by

$$\hat{q}''(x, s) = -k \frac{d\hat{T}}{dx}(x, s), \quad x \in [0, L], \quad s \geq 0, \quad (3.4a)$$

which upon substituting Eq. (3.3c) for transformed temperature into Eq. (3.4a) produces

$$\hat{q}''(x, s) = -k\sqrt{\frac{s}{\alpha}} \left(C(s) \sinh\left(\sqrt{\frac{s}{\alpha}}x\right) + D(s) \cosh\left(\sqrt{\frac{s}{\alpha}}x\right) \right), \quad x \in [0, L], \quad s \geq 0. \quad (3.4b)$$

Unlike solving the transformed heat equation in terms of external boundary conditions at $x=0, L$, the calibration concept is based on an input-output viewpoint. Again, the inverse problem indicated in Fig. 3.1 involves predicting $q''(0, t)$ using temperature data collected at $x=d$ and known back boundary condition at $x=L$. With this said, our goal is to express $C(s), D(s)$ in terms of transformed conditions at $x=d, L$ instead of at $x=0, L$.

Equations (3.3c) and (3.4b) provide explicit expressions for the transformed temperature and heat flux in terms of two unknown coefficients, $C(s), D(s)$. As noted early, our objective now involves obtaining analytic expressions for $C(s), D(s)$ in terms of the boundary condition at $x=L$ and probe site data acquired at $x=d$. The transformed boundary condition at $x=L$ is obtained by taking the Laplace transform of Eq. (3.1c) to get

$$-k \frac{d\hat{T}}{dx}(L, s) - h \left(\hat{T}(L, s) - \frac{T_\infty}{s} \right) = 0, \quad s \geq 0. \quad (3.5)$$

While the transformed temperature data measured at $x=d$ are now denoted as $\hat{T}(d, s)$. Substituting Eq. (3.3c) into Eq. (3.5) and after performing some manipulations, we obtain

$$C(s) \left(k\sqrt{\frac{s}{\alpha}} \sinh\left(\sqrt{\frac{s}{\alpha}}L\right) + h \cosh\left(\sqrt{\frac{s}{\alpha}}L\right) \right) + D(s) \left(k\sqrt{\frac{s}{\alpha}} \cosh\left(\sqrt{\frac{s}{\alpha}}L\right) + h \sinh\left(\sqrt{\frac{s}{\alpha}}L\right) \right) = 0, \quad s \geq 0, \quad (3.6a)$$

under the condition $T_\infty = 0^\circ\text{C}$. Next, evaluate Eq. (3.3c) at the probe position $x=d$ to get

$$\hat{T}(d, s) = C(s) \cosh\left(\sqrt{\frac{s}{\alpha}}d\right) + D(s) \sinh\left(\sqrt{\frac{s}{\alpha}}d\right), \quad s \geq 0. \quad (3.6b)$$

Solving for $C(s), D(s)$ using Eq. (3.6a) and Eq. (3.6b), we obtain

$$C(s) = \frac{k\sqrt{\frac{s}{\alpha}} \cosh\left(\sqrt{\frac{s}{\alpha}}L\right) + h \sinh\left(\sqrt{\frac{s}{\alpha}}L\right)}{k\sqrt{\frac{s}{\alpha}} \cosh\left(\sqrt{\frac{s}{\alpha}}(L-d)\right) + h \sinh\left(\sqrt{\frac{s}{\alpha}}(L-d)\right)} \hat{T}(d, s), \quad s \geq 0, \quad (3.7a)$$

$$D(s) = -\frac{k\sqrt{\frac{s}{\alpha}} \sinh\left(\sqrt{\frac{s}{\alpha}}L\right) + h \cosh\left(\sqrt{\frac{s}{\alpha}}L\right)}{k\sqrt{\frac{s}{\alpha}} \cosh\left(\sqrt{\frac{s}{\alpha}}(L-d)\right) + h \sinh\left(\sqrt{\frac{s}{\alpha}}(L-d)\right)} \hat{T}(d, s), \quad s \geq 0. \quad (3.7b)$$

Next, we evaluate the transformed heat flux given in Eq. (3.4b) at the front surface $x=0$ to get

$$\hat{q}''(0, s) = -k\sqrt{\frac{s}{\alpha}} D(s), \quad s \geq 0. \quad (3.8)$$

Substituting Eq. (3.7b) into Eq. (3.8) yields

$$\hat{q}''(0, s) = \hat{N}(d, L, k, \alpha, h; s) \hat{T}(d, s), \quad s \geq 0, \quad (3.9a)$$

or

$$\frac{\hat{q}''(0, s)}{\hat{T}(d, s)} = \hat{N}(d, L, k, \alpha, h; s), \quad s \geq 0, \quad (3.9b)$$

where

$$\hat{N}(d, L, k, \alpha, h; s) = \frac{k^2 \frac{s}{\alpha} \sinh(\sqrt{\frac{s}{\alpha}} L) + kh \sqrt{\frac{s}{\alpha}} \cosh(\sqrt{\frac{s}{\alpha}} L)}{k \sqrt{\frac{s}{\alpha}} \cosh\left(\sqrt{\frac{s}{\alpha}}(L-d)\right) + h \sinh\left(\sqrt{\frac{s}{\alpha}}(L-d)\right)}, \quad s \geq 0. \quad (3.9c)$$

Observe that the function $\hat{N}(d, L, k, \alpha, h; s)$ is unchanging under our present set of assumptions and it carries diffusion physics, thermophysical properties and probe position. Under these conditions, and now thinking as an experimentalist, we can express the mathematics as a sequence of experiments. These experiments would involve two stages. The first stage involves a “calibration” run where we measure the output temperature at $x=d$ denoted as “ $T_c(d, t)$ ” due to a well quantified input net surface heat flux denoted as “ $q_c''(0, t)$ ”. With this notation, Eq. (3.9b) becomes

$$\frac{\hat{q}_c''(0, s)}{\hat{T}_c(d, s)} = \hat{N}(d, L, k, \alpha, h; s), \quad s \geq 0, \quad (3.10a)$$

the second stage involves reconstructing an unknown surface heat flux involving the “run” stage based on its measured response, given as “ $T_r(d, t)$ ”. Hence Eq. (3.9b) can be written as

$$\frac{\hat{q}_r''(0, s)}{\hat{T}_r(d, s)} = \hat{N}(d, L, k, \alpha, h; s), \quad s \geq 0. \quad (3.10b)$$

Since the function $\hat{N}(d, L, k, \alpha, h; s)$ is unchanging, Eq. (3.10a) and Eq. (3.10b) can be equated as

$$\frac{\hat{q}_c''(0, s)}{\hat{T}_c(d, s)} = \frac{\hat{q}_r''(0, s)}{\hat{T}_r(d, s)} = \hat{N}(d, L, k, \alpha, h; s), \quad s \geq 0. \quad (3.11)$$

Cross multiplying the first two terms in Eq. (3.11) yields

$$\hat{q}_r''(0, s) \hat{T}_c(d, s) = \hat{q}_c''(0, s) \hat{T}_r(d, s), \quad s \geq 0. \quad (3.12)$$

Applying the convolution theorem given in Eq. (3.2b) produces the time domain calibration integral equation as

$$\int_{u=0}^t q_r''(0, u) T_c(d, t-u) du = \int_{u=0}^t q_c''(0, u) T_r(d, t-u) du, \quad t \geq 0. \quad (3.13)$$

Based on the generalized linear thermocouple model (constant property) that accounts for the thermocouple time constant and lead losses and frequency domain analysis [8], we obtain

$$\int_{u=0}^t q_r''(0, u) T_{ic,c}(d, t-u) du = \int_{u=0}^t q_c''(0, u) T_{ic,r}(d, t-u) du, \quad t \geq 0, \quad (3.14)$$

which is the one-dimensional, one-probe calibration integral equation. In addition, if the initial temperature of “calibration” run denoted as “ $T_{0,c}$ ” and the initial temperature of second “run” denoted as “ $T_{0,r}$ ” are not identically zero, then the calibration integral equation can be written as

$$\int_{u=0}^t q_r''(0, u) \theta_{ic,c}(d, t-u) du = \int_{u=0}^t q_c''(0, u) \theta_{ic,r}(d, t-u) du, \quad t \geq 0, \quad (3.15a)$$

where $\theta_{ic,c}(d, t-u)$ and $\theta_{ic,r}(d, t-u)$ denote the temperature differences given as

$$\theta_{ic,c}(d, t-u) = T_{ic,c}(d, t-u) - T_{0,c}, \quad t \geq 0, \quad (3.15b)$$

$$\theta_{ic,r}(d, t-u) = T_{ic,r}(d, t-u) - T_{0,r}, \quad t \geq 0. \quad (3.15c)$$

3.2 Stabilization Through Regularization Based on the Local Future Time Method

Equations (3.13-3.15) are Volterra integral equations of the first kind which are ill-posed [83, 84]. Before implementing a numerical approximation, regularization is required. Otherwise, the numerical prediction will be unstable. Many methods are applicable to regularize Volterra integral equation of the first kind. In this study, we utilize a simplified variation of Lamm's local future time method [85] to recast the first kind Volterra integral equation into a second kind Volterra integral equation. Second kind Volterra integral equations are well-posed, but the original formulation is altered. Hence, this is an approximate model.

The linear calibration equation shown in Eq. (3.14) is alternatively expressed as

$$\int_{u=0}^t q_r''(0, u) T_{ic,c}(d, t-u) du = f(t), \quad t \geq 0, \quad (3.16a)$$

where

$$f(t) = \int_{u=0}^t q_c''(0, u) T_{ic,r}(d, t-u) du, \quad t \geq 0. \quad (3.16b)$$

We introduce the future time parameter γ , by advancing time through $t \rightarrow t + \gamma$ (γ has units of time), then Eq. (3.16a) becomes

$$\int_{u=0}^{t+\gamma} q_r''(0, u) T_{ic,c}(d, t+\gamma-u) du = f(t+\gamma), \quad t \in [0, t_{\max} - \gamma]. \quad (3.17)$$

Observe that the time domain is reduced by γ . The basic integral definition [91-93] allows us to express Eq. (3.17) as

$$\int_{u=0}^t q_r''(0, u) T_{ic,c}(d, t+\gamma-u) du + \int_{u=t}^{t+\gamma} q_r''(0, u) T_{ic,c}(d, t+\gamma-u) du = f(t+\gamma), \quad t \in [0, t_{\max} - \gamma]. \quad (3.18)$$

Define $v=u-t$, then the second integral on the left-hand side of Eq. (3.18) can be written as

$$\int_{u=0}^t q_r''(0, u) T_{ic,c}(d, t+\gamma-u) du + \int_{v=0}^{\gamma} q_r''(0, v+t) T_{ic,c}(d, \gamma-v) dv = f(t+\gamma), \quad t \in [0, t_{\max} - \gamma]. \quad (3.19)$$

If γ is small then we can approximate $q_r''(0, v+t) \approx q_r''(0, t)$ since $v \in [0, \gamma]$ for small γ . If this is the case, then we can form the approximation

$$\int_{u=0}^t q_r''(0, u) T_{ic,c}(d, t+\gamma-u) du + q_r''(0, t) \int_{v=0}^{\gamma} T_{ic,c}(d, \gamma-v) dv \approx f(t+\gamma), \quad t \in [0, t_{\max} - \gamma], \quad (3.20a)$$

or

$$\int_{u=0}^t q_{r,\gamma}''(0, u) T_{ic,c}(d, t+\gamma-u) du + q_{r,\gamma}''(0, t) \int_{v=0}^{\gamma} T_{ic,c}(d, \gamma-v) dv = f(t+\gamma), \quad t \in [0, t_{\max} - \gamma], \quad (3.20b)$$

or

$$\int_{u=0}^t q_{r,\gamma}''(0, u) T_{ic,c}(d, t+\gamma-u) du + q_{r,\gamma}''(0, t) C_{\gamma} = f(t+\gamma), \quad t \in [0, t_{\max} - \gamma], \quad (3.20c)$$

where

$$C_{\gamma} = \int_{v=0}^{\gamma} T_{ic,c}(d, \gamma-v) dv, \quad (3.20d)$$

which is now a second kind Volterra integral equation for the unknown surface heat flux denoted by $q_{r,\gamma}''(0, t)$. Here $q_{r,\gamma}''(0, t)$ is an approximation to $q_r''(0, t)$ and depends on γ . Data are collected up to some time defined as $t_{\max}$. Future information is incorporated into the numerical algorithm.

The predicted surface heat flux $q''_{r,\gamma}(0,t)$ can only be resolved in temporal domain $t \in [0, t_{\max} - \gamma]$ due to the inclusion of future data to stabilize the numerical implementation.

Discretization can be performed on Eqs. (3.20b-d) using a variety of low-order numerical integration rules, such as the rectangular rule, trapezoidal rule, product integration rules [84], etc. For convenience, a right-handed rectangular rule is employed in the present context. The discrete values for $q''_{r,\gamma}(0,t)$ can be obtained in the time-marching form where $\gamma \rightarrow \gamma_m = m M_f \Delta t$, $m = 1, 2, 3, \dots$, with Δt is given as $\Delta t = t_{\max}/N$, N is the number of segments (or samples) in the discretized temporal domain, and M_f is a multiplying factor. Discretizing Eq. (3.20b) produces

$$\sum_{j=1}^i q''_{r,\gamma_m}(0, t_j) \int_{u=t_{j-1}}^{t_j} T_{tc,c}(d, t_i + \gamma_m - u) du + q''_{r,\gamma_m}(0, t_i) \int_{u=0}^{\gamma_m} T_{tc,c}(d, \gamma_m - u) du = f(t_i + \gamma_m), \quad i = 1, 2, \dots, M, \quad (3.21a)$$

where $M = N - m M_f$, or upon extracting the desired heat flux at $t = t_i$ we obtain

$$q''_{r,\gamma_m}(0, t_i) = \frac{f(t_i + \gamma_m) - \sum_{j=1}^{i-1} q''_{r,\gamma_m}(0, t_j) \int_{u=t_{j-1}}^{t_j} T_{tc,c}(d, t_i + \gamma_m - u) du}{\int_{u=t_{i-1}}^{t_i} T_{tc,c}(d, t_i + \gamma_m - u) du + \int_{u=0}^{\gamma_m} T_{tc,c}(d, \gamma_m - u) du}, \quad i = 1, 2, \dots, M, \quad (3.21b)$$

where $t_j = j \Delta t$, $j=0, 1, \dots, M$. In this time-marching implementation, the future time parameter γ_m must be specified over a range of values and then interrogated until an optimal value is determined based on some measure.

3.3 Metrics for Determining the Optimal Future Time Parameter γ

In the present context, Eq. (3.14) is the exact formulation of the proposed inverse heat conduction problem described in Section 3.1. Equation (3.20b) is an approximation since a constant surface heat flux is assumed in the future time interval $[t, t + \gamma]$ in order to convert the first kind Volterra integral equation into well-posed second kind Volterra integral equation. There exist a well-known conflict between bias and variance [1]. If the future time parameter γ is too small then the prediction via Eq. (3.20b) is unstable. If the future time parameter γ is too large then over smoothing of the prediction will transpire as high frequencies are damped. Hence, significant surface physics can be lost. Compromise is required between stability and accuracy. To determine the optimal future time parameter γ , the residual function $r_\gamma(t)$ is defined as

$$r_\gamma(t) = \int_{u=0}^t q''_{r,\gamma}(0, u) T_{tc,c}(d, t - u) du - f(t), \quad t \in [0, t_{\max} - \gamma], \quad (3.22)$$

where $f(t)$ is defined in Eq. (3.16b). It can be mathematically shown that the behavior of shifted residual function $r_\gamma(t+\gamma)$ is approximately proportional to the first time derivative of predicted surface heat flux $q''_{r,\gamma}(0,t)$ when γ is close to its optimal value (see Appendix A, [8]). However, this knowledge is insufficient for determining the optimal future time parameter γ . As in the study of the least-squares method [67], the residuals should nearly approximate or replicate the random error under the optimal conditions. If this is the case, the running average of the residual should be about zero. The running average of residual is defined as

$$\bar{r}_\gamma(t) = \frac{1}{t} \int_{u=0}^t r_\gamma(u) du, \quad t \in [0, t_{\max} - \gamma]. \quad (3.23)$$

Both $r_\gamma(t)$ and $\bar{r}_\gamma(t)$, from Eqs. (3.22) and (3.23), respectively, can be discretized by a simple and

consistent integration rule. The right-handed rectangular rule is employed herein as with Eqs. (3.21a,b).

The time-running variance of residual function is defined through

$$\sigma_\gamma^2(t_i) = \frac{1}{i} \sum_{j=1}^i [r_\gamma(t_j) - \bar{r}_\gamma(t_i)]^2, \quad i=1, 2, \dots, M. \quad (3.24)$$

A second measure based on Eq. (3.24) can be defined to emphasis or enhance other effects. Eswein et al. [94] normalized the mass loss rate for different stagnation pressures. Following this concept, we defined the normalized square root of time-running variance of the residual as

$$\sigma_\gamma^*(t_i) = \frac{\sigma_\gamma(t_i)}{\|\sigma_\gamma(t_i)\|_\infty}, \quad i=1, 2, \dots, M. \quad (3.25)$$

where $\|\varphi\|_\infty$ is the maximum norm given as $\max_{1 \leq i \leq M} |\sigma_\gamma(t_i)|$. These four metrics assist in estimating the optimal future time parameter γ . These concepts are demonstrated in Section 3.4 in the context of a numerical simulation.

3.4 Numerical Simulation for Testing the One-Dimensional Calibration Integral Equation Method

In this section, a purely numerical simulation is employed to test the validity of the calibration integral equation displayed in Eq. (3.14) and the future time method described in Section 3.3. To form a computational test of the processes described in Sections 3.2 and 3.3, direct problems with known boundary and initial conditions must be solved for the “calibration” and unknown “run” cases. For the present study an implicit finite difference method [4,95] was employed for solving the complete spatial and temporal fields for temperature, $T_c(x, t)$, $T_r(x, t)$. With these, artificial thermocouple data (idealized) at $x=d$ can be obtained for representing the measured values $T_c(d, t)$, $T_r(d, t)$ required by the calibration integral equation described in Eq. (3.13). With these temperature data and known $q_c''(0, t)$, one can resolve $q_{r,\gamma}''(0, t)$ from Eq. (3.21b) over the future time, γ -spectrum. The direct solution provides uncontaminated (noiseless) data containing only numerical errors involving truncation and round-off errors. Convergence of the implicit finite difference method is verified by comparison to the exact solution.

In the numerical simulation, the host material of the slab is copper with thermal conductivity and thermal diffusivity values given in Table 3.1. All data used in the present simulation can be found in Table 3.1. The back surface boundary condition is convection with constant convective heat transfer coefficient h in both “calibration” run and test “run”. The sampling frequency is chosen as $f_{\text{sampling}}=50\text{Hz}$ in the inverse code by downsampling from the finite difference results in the time interval $[0, t_{\text{max}}]$. The “calibration” surface heat flux is a step that is displayed in Fig. 3.2. A Gauss heat source to be reconstructed is given by $q_r''(0, t) = q_{\text{max},r}'' e^{-(t-A)^2/B^2}$ (the constants are given in Table 3.1) and is used as “run” surface heat flux. This heat flux is displayed in Fig. 3.2.

Artificial temperature noise is added to the probe site thermocouple data in accordance to

$$T_{tc}(d, t_j) = T(d, t_j) + \varepsilon_T r_j \|T(d, t_j)\|_\infty, \quad j = 1, 2, \dots, N. \quad (3.26)$$

for both calibration “run” and test “run” stages. Here ε_T denotes noise factor, r_j denotes random numbers in the interval $[-1, 1]$ which are obtained by MATLAB command “2*(rand-0.5)”. For the calibration “run”, noise is also added to the net surface heat flux $q_c''(0, t)$ as

$$q''_c(0, t_j) = q''_{c, exact}(0, t_j) + \varepsilon_q r_j \parallel q''_{c, exact}(0, t_j) \parallel_\infty, \quad j = 1, 2, \dots, N. \quad (3.27)$$

Here N is the total number of discrete nodes in temporal domain. Figure 3.3a displays the positional surface ($x=0$) temperature $T_c(0, t)$, the positional in-depth ($x=d$) temperature $T_c(d, t)$, and the thermocouple temperature data with artificial noise $T_{tc,c}(d, t_j)$ that is given in Eq. (3.26) under the errorless “calibration” surface heat flux displayed in Fig. 3.2. Again we assume $T_{tc,c}(d, t_j) = T_c(d, t_j)$. That is a neglect of time constant and conductive lead losses of the thermocouple. Figure 3.3b contains the temperature responses to the Gaussian heat flux to be reconstructed.

Table 3.1: Properties and parameters for simulation

Property / Parameter	Value (units)
k	394 W/mK
α	$116 \times 10^{-6} \text{ m}^2/\text{s}$
h	100 W/m ² K
L	1 cm
d	0.7 cm
ε_T	0.01
ε_q	0.01
$q''_{\max, c}$	100 W/cm ²
$q''_{\max, r}$	100 W/cm ²
A	10 s
B	2 s
$t_{\max}$	20 s
N	1000
f_{sampling}	50 Hz
M_f	5

Figure 3.4 displays the local residual function, $r_\gamma(t)$, defined in Eq. (3.22) over time for increasing future time parameter γ . As the future time parameter increases, the residual’s magnitude is shown to increase. It is desirable to keep this function under control (low bias and low variance) over the entire time span. Figure 3.5 shows the corresponding running average of the residual. Clearly the smoothing operation of integration is seen in Fig. 3.5. Figure 3.6 displays the time-running variance of the residual function defined in Eq. (3.24) while Fig. 3.7 displays the normalized version of Fig. 3.6. In this form (Fig. 3.7), both instability and over-smoothing effects can be quickly identified. Small γ ’s produce unstable predictions (see for example $\gamma = 0.3\text{s}$) while large γ ’s produce over-smoothed predictions (see for example $\gamma = 1.2\text{s}$) of the net surface heat flux. The normalized plot possesses key physical effects. In fact, a bundling effect occurs for $\gamma = 0.5\text{-}0.8\text{s}$ for times, $t > 13\text{s}$. This range of γ -values describes the best prediction based on the balance between bias (data) and variance (model). This γ -range of the spectrum is where the optimal prediction for the heat flux will lie.

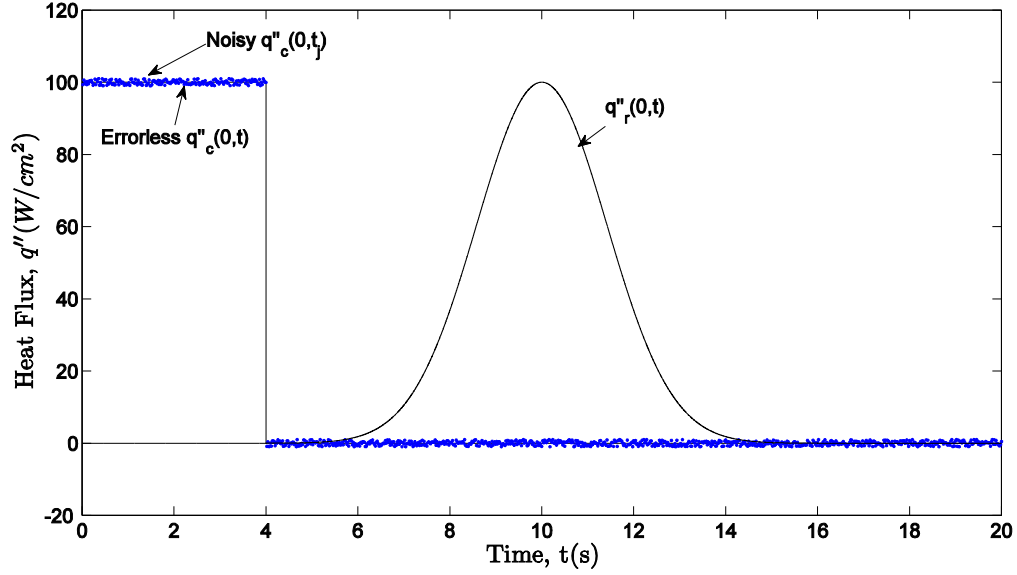


Figure 3.2: Errorless “calibration” net surface heat flux, $q''_c(0,t)$; noisy “calibration” net surface heat flux $q''_c(0,t_j)$ given in Eq. (3.27); and, exact “run” net surface heat flux $q''_r(0,t)$ to be reconstructed by the inverse analysis.

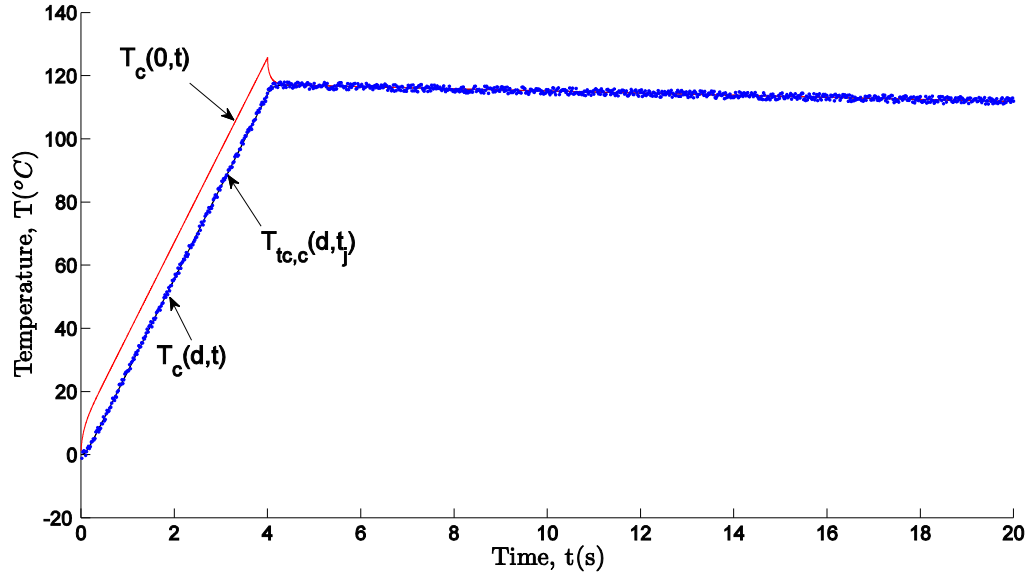


Figure 3.3a: Positional temperature responses $T_c(0,t)$, $T_c(d,t)$ at $x = 0, d$, respectively, under “calibration” surface heat flux, and contaminated “thermocouple” temperature $T_{tc,c}(d,t_j)$ using Eq. (3.26) and Table 3.1 data.

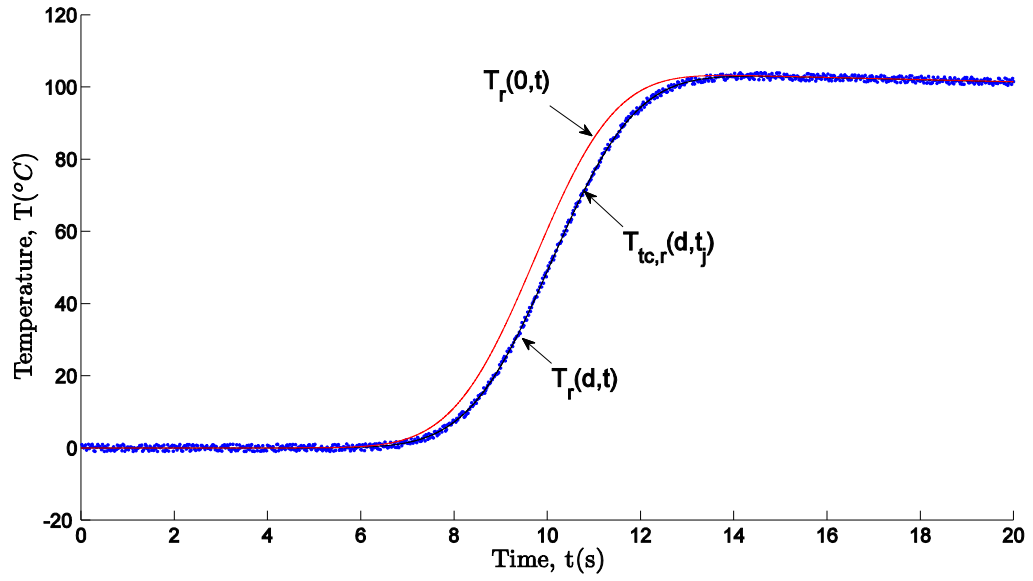


Figure 3.3b: Positional temperature responses $T_r(0,t)$, $T_r(d,t)$ at $x = 0, d$, respectively, under “run” surface heat flux, and contaminated “thermocouple” temperature $T_{tc,r}(d,t_j)$ using Eq. (3.26) and Table 3.1 data.

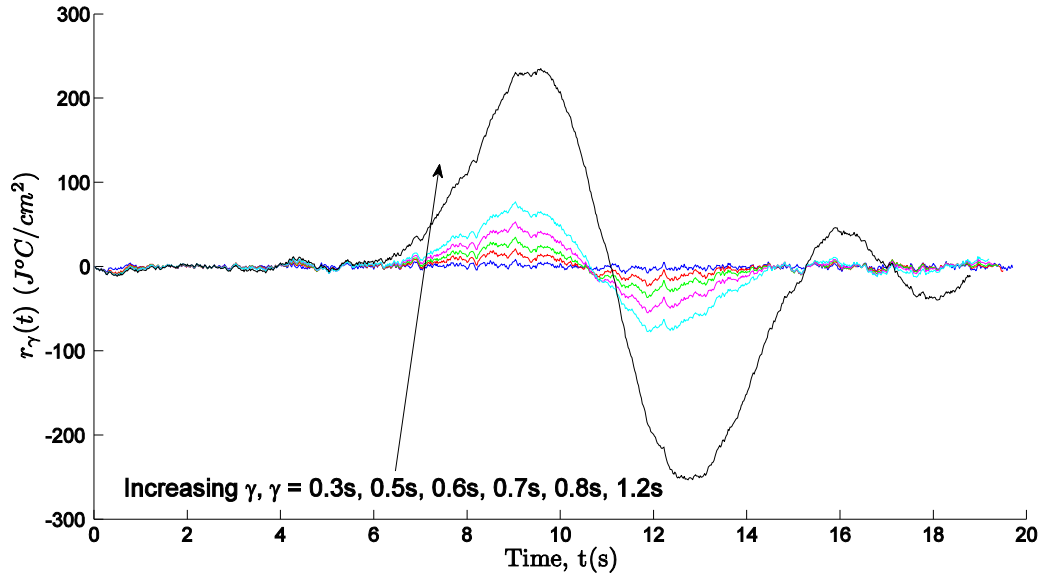


Figure 3.4: The local residual function, $r_\gamma(t)$, defined in Eq. (3.22) over the γ -spectrum.

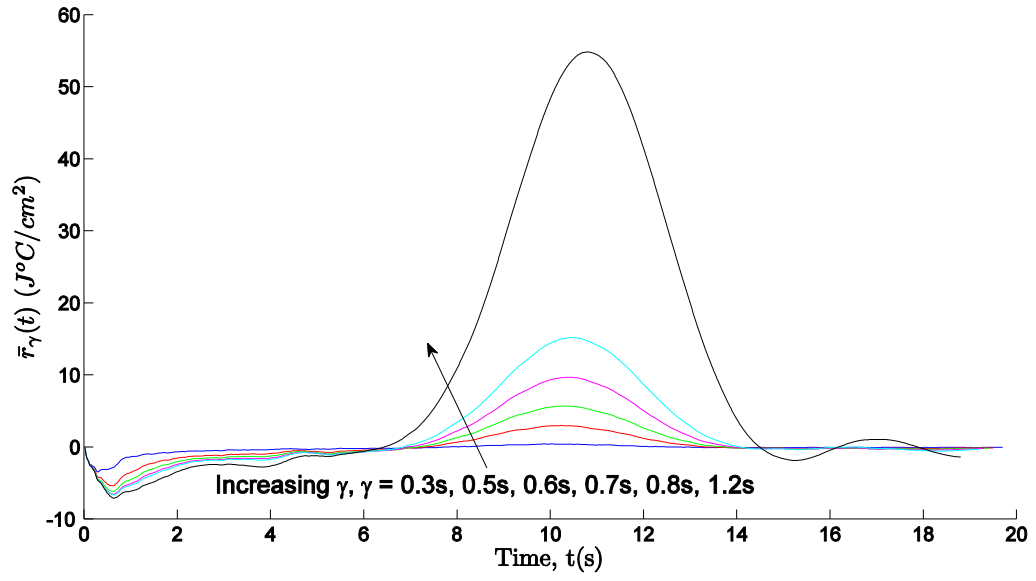


Figure 3.5: The running average of residual, $\bar{r}_\gamma(t)$, defined in Eq. (3.23) over the γ -spectrum.

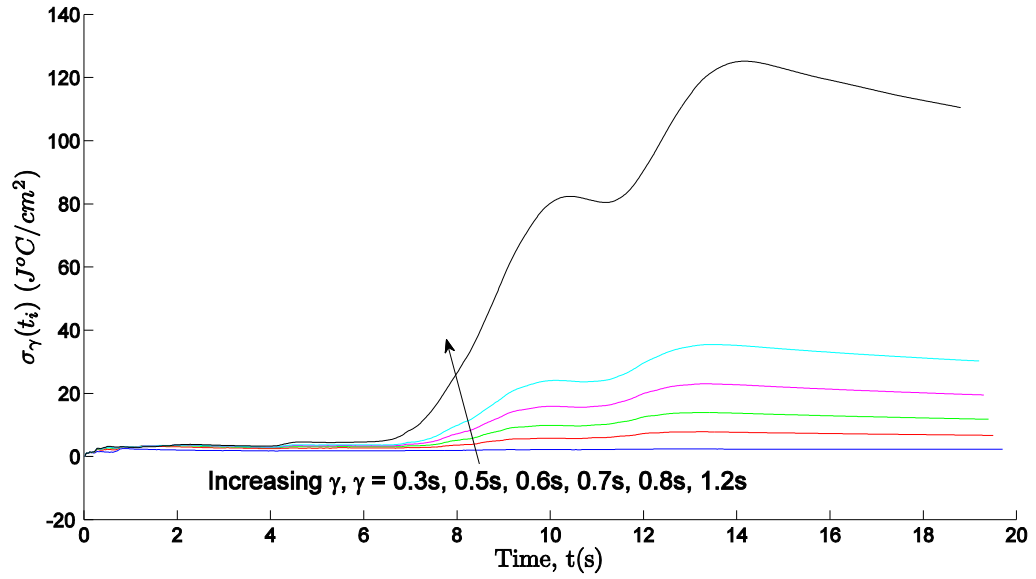


Figure 3.6: The square root of time-running variance of residual, $\sigma_\gamma(t_i)$, defined in Eq. (3.24) over the γ -spectrum.

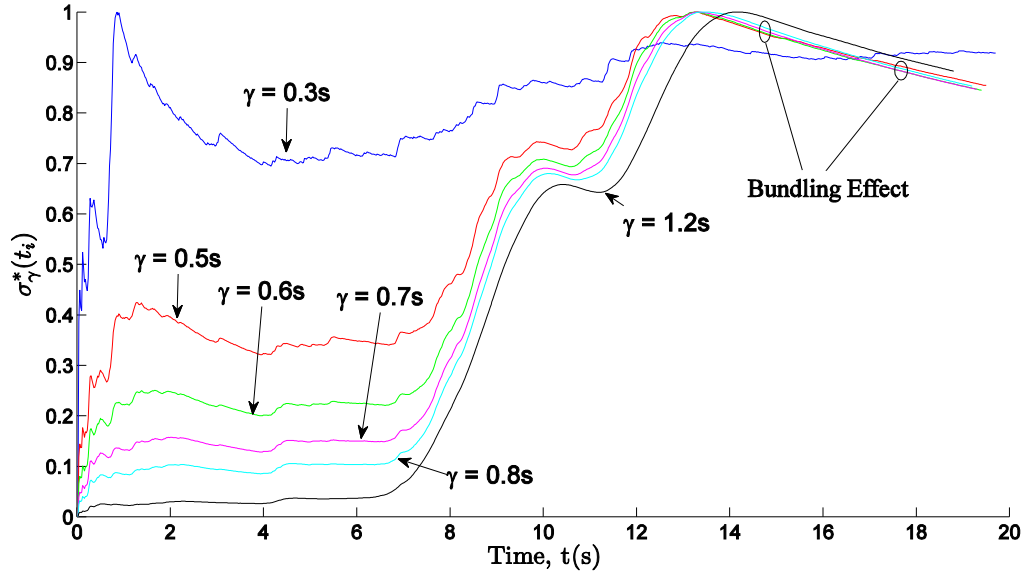


Figure 3.7: The normalized square root of time-running variance of residual, $\sigma_\gamma^*(t_i)$, defined in Eq. (3.25) over the γ -spectrum.

Figures 3.8a-f display a comparison between the exact “run” surface heat flux and predicted “run” surface heat flux corresponding to the γ values in Figs. 3.4-3.7. These results verify the effectiveness of employing the normalized square root of time-running variance of residual for determining optimal γ value. It is apparent that γ values between 0.5s to 0.8s are viable choices. In contrast, when $\gamma = 0.3s$ the prediction is unstable while $\gamma = 1.2s$ produces an over-smoothed prediction. In fact, it is desired to retain as much of the high frequency content in the signal for physics recovering. Hence, the optimal estimator is chosen as $\gamma_{opt} = 0.5s$.

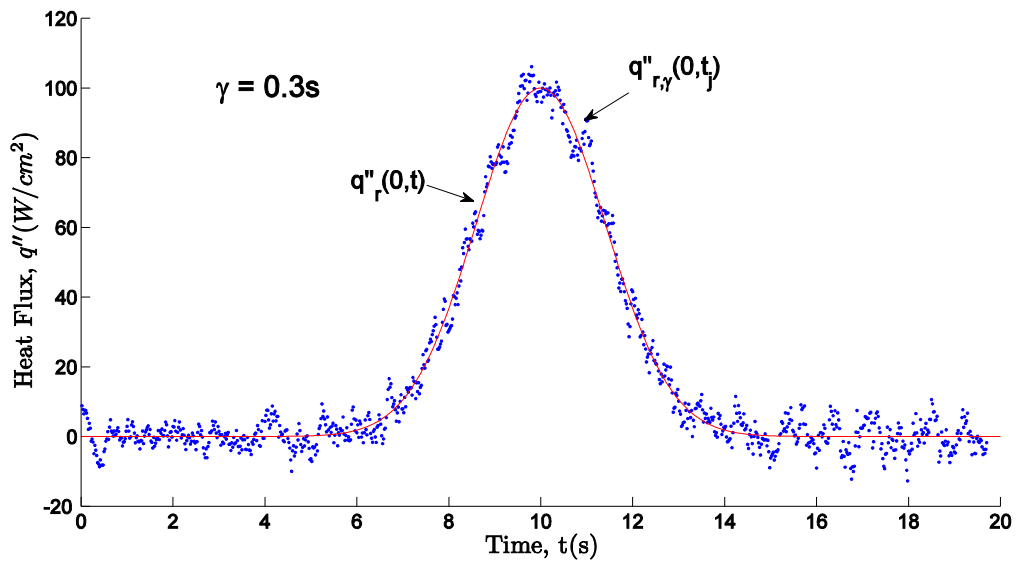


Figure 3.8a: Comparison between actual and predicted (“run”) surface heat flux when $\gamma = 0.3s$.

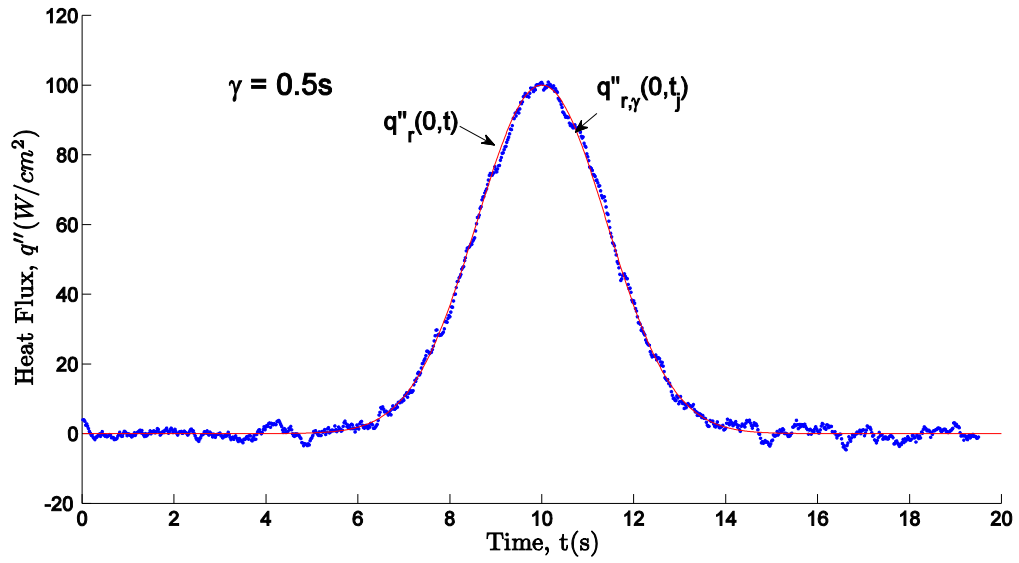


Figure 3.8b: Comparison between actual and predicted (“run”) surface heat flux when $\gamma = 0.5\text{s}$.

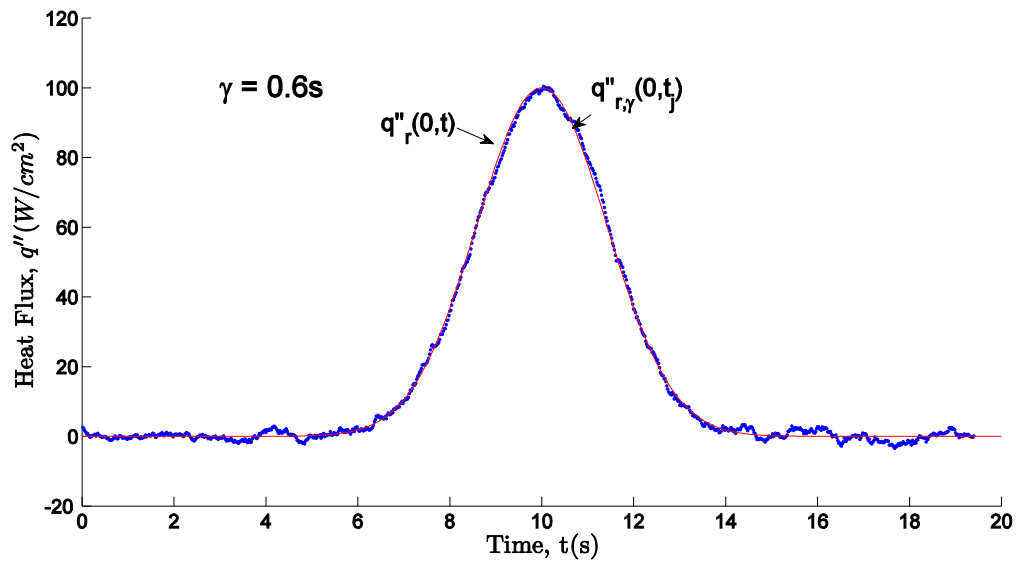


Figure 3.8c: Comparison between actual and predicted (“run”) surface heat flux when $\gamma = 0.6\text{s}$.

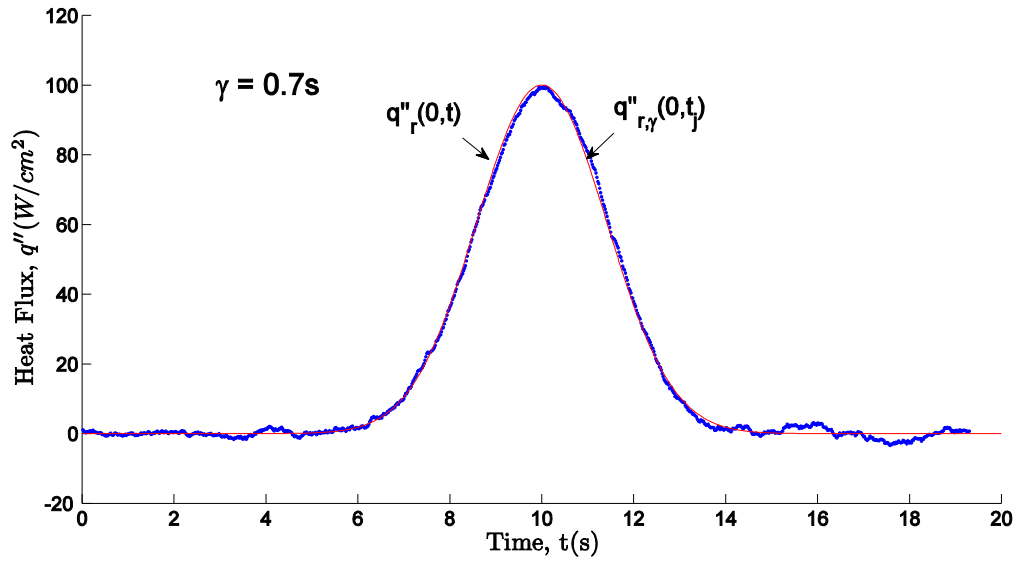


Figure 3.8d: Comparison between actual and predicted ("run") surface heat flux when $\gamma = 0.7\text{s}$.

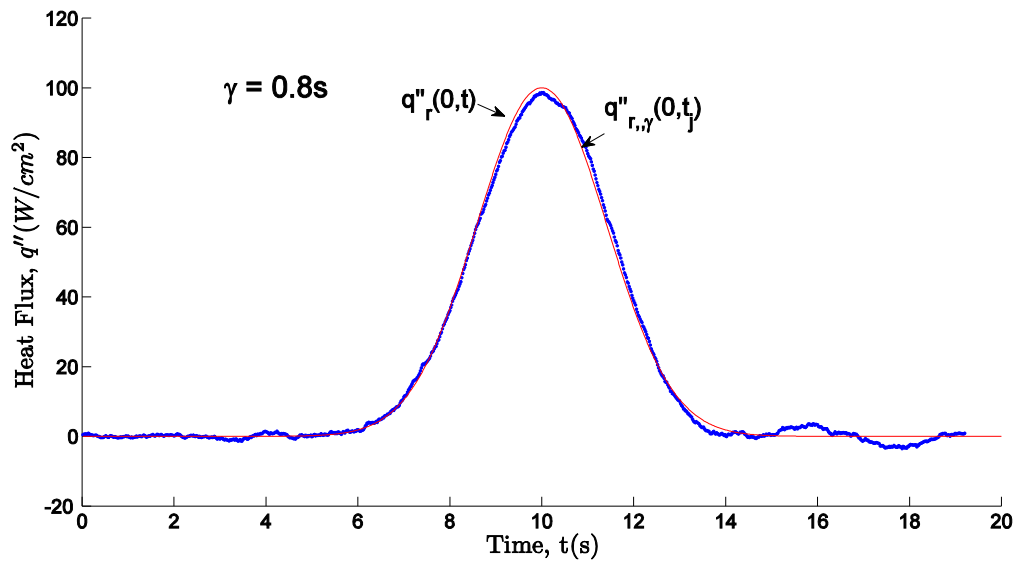


Figure 3.8e: Comparison between actual and predicted ("run") surface heat flux when $\gamma = 0.8\text{s}$.

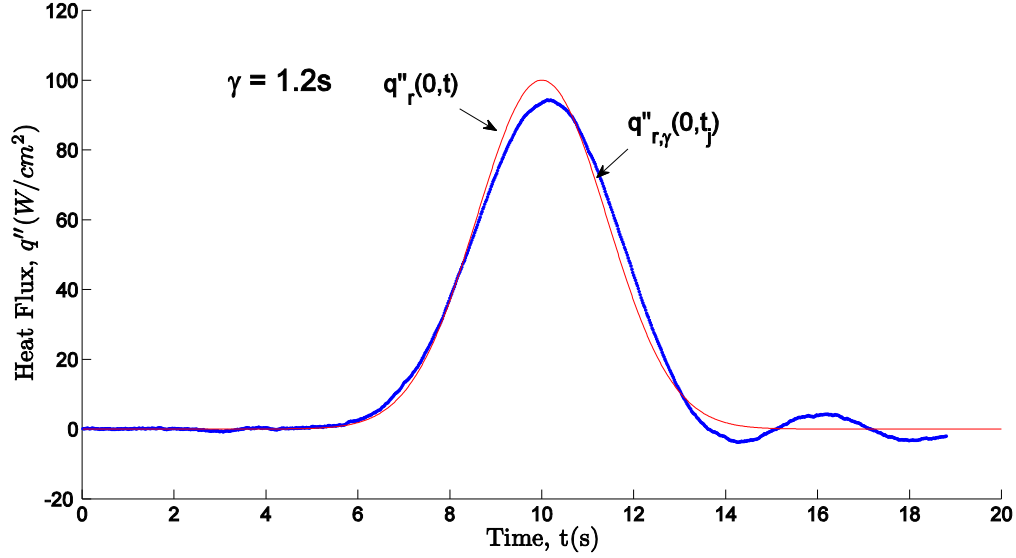


Figure 3.8f: Comparison between actual and predicted (“run”) surface heat flux when $\gamma = 1.2s$.

3.5 Experimental Validation of the One-Dimensional Calibration Integral Equation Method

Electrical heating experiments [9,39,40,96] performed at the University of Tennessee under the guidance of Prof. Keyhani have been used to validate the one-dimensional calibration integral equation method. Figure 3.9 is a schematic illustrating the electrical heating experimental setup. A custom nichrome element heater is sandwiched between two identical bronze plates [9,96] or two identical stainless steel plates [39,40] to create symmetrical thermal condition. For the present analysis, two identical stainless steel plates with thickness $L=25.79mm$ are exploited as host material. Thermocouples are embedded into one of the stainless steel plate, and their distance from the heated-side surface of the stainless steel plate is $d=6.57mm$. Two identical stainless steel plates are coated with a thin layer of Omegatherm 201 thermal paint on their heated-side surface to reduce contact resistance. Two thin mica sheets are placed between the heater and stainless steel plates to avoid leakage of electricity as well as providing good heat conduction from the heater to the stainless steel plates. Thermophysical and electrical properties of the materials used in the experimental setup are shown in Table 3.2, while material dimensions are shown in Table 3.3. Details to these experimental studies can be found in Ref. [9,39,40,96]. The rationale for revisiting these experiments lies in using the newly proposed metrics, Eq. (3.24) and Eq. (3.25), for determining the optimal future time parameter γ . The metrics were investigated in Section 3.4 using MATLAB’s random number generator. This section uses experimental data that contain noises from the data acquisition system and environment.

The time interval of the electrical heating experiments is $t \in [0,60s]$. The thermocouple responses are sampled at 200Hz via a DT9824 data acquisition board (DAQ) [9]. Figure 3.10 displays the “calibration” surface heat flux $q''_{c,original}(0,t)$ and “run” surface heat flux $q''_{r,original}(0,t)$ used in the electrical heating experiment. Figure 3.11 displays the thermocouple temperature responses $T_{tc,c,original}(d,t_j)$, $T_{tc,r,original}(d,t_j)$ ($j=1,2,3,\dots, 12000$) corresponding to

“calibration” surface heat flux $q''_{c,original}(0,t)$ and “run” surface heat flux $q''_{r,original}(0,t)$, respectively.

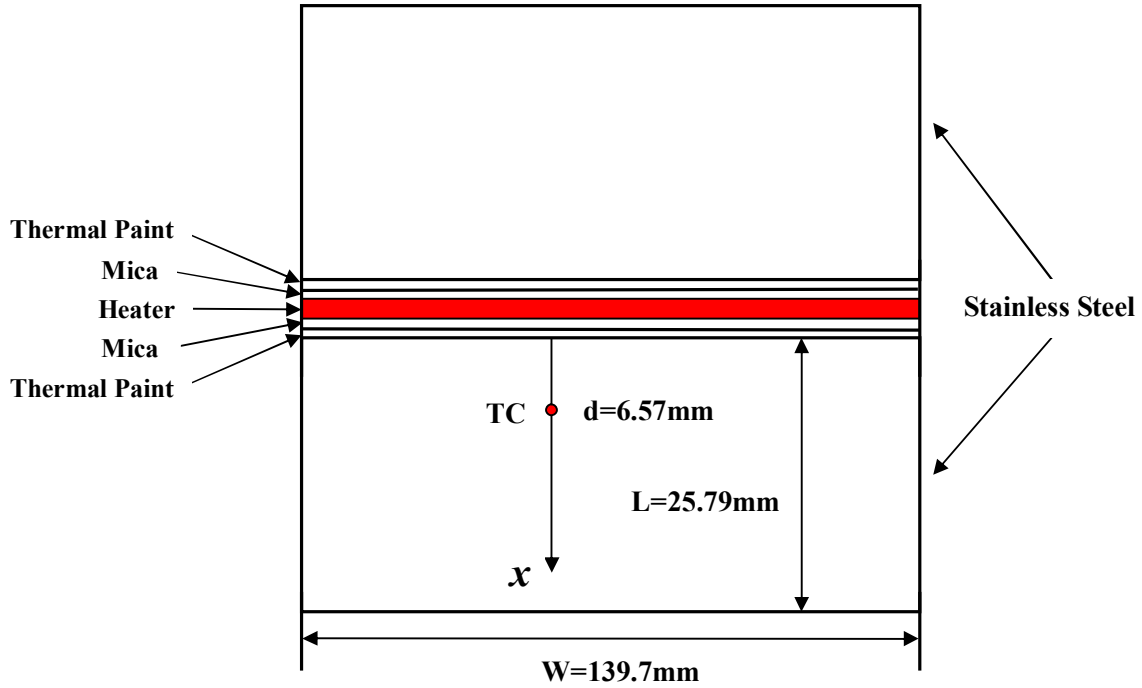


Figure 3.9: Electrical heating experimental setup, the centerline of the heater is the symmetry axis [39] (this is a representative schematic only indicating one thermocouple).

Table 3.2: Thermophysical and electrical properties of materials used in the electrical heating experiment [39]

Property	Value (units)
Stainless Steel	
Thermal diffusivity, α	$3.95 \times 10^{-6} \text{ m}^2/\text{s}$
Thermal conductivity, k	$14.9 \text{ W}/\text{m}^2\text{K}$
Mica	
Thermal diffusivity	$4.73 \times 10^{-3} \text{ m}^2/\text{s}$
Density	$300 \text{ kg}/\text{m}^3$
Specific heat	$0.5 \text{ J}/(\text{kgK})$
Heater (nichrome)	
Thermal diffusivity	$7.75 \times 10^{-5} \text{ m}^2/\text{s}$
Density	$1420 \text{ kg}/\text{m}^3$
Specific heat	$1.09 \text{ J}/(\text{kgK})$
Heater resistance	$4.237 \text{ } \Omega$
Potting compound (Cotronics 989F)	
Thermal conductivity	$1.7 \text{ W}/(\text{mK})$

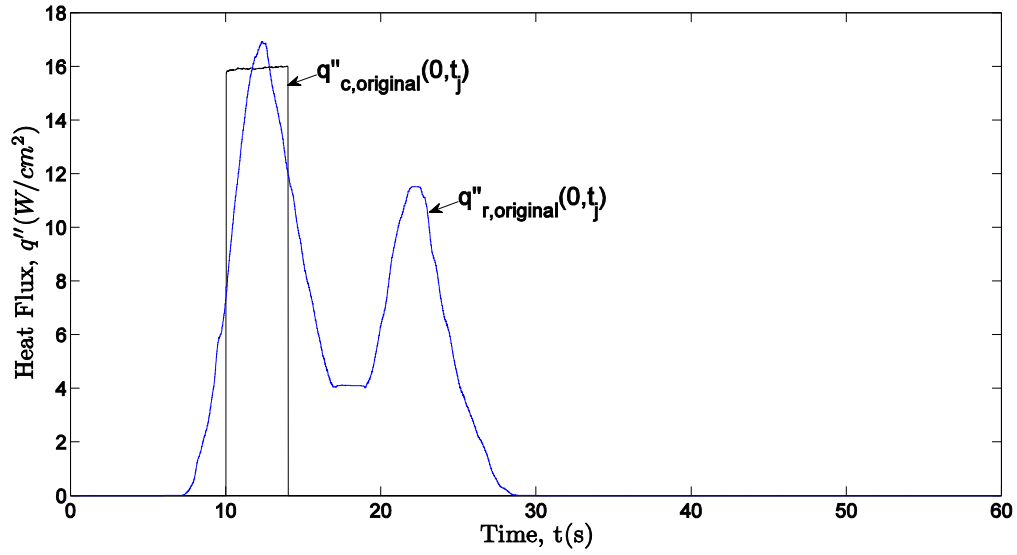


Figure 3.10: “Calibration” surface heat flux $q''_{c,original}(0,t_j)$, and actual “run” surface heat flux $q''_{r,original}(0,t_j)$ used in the electrical heating experiment.

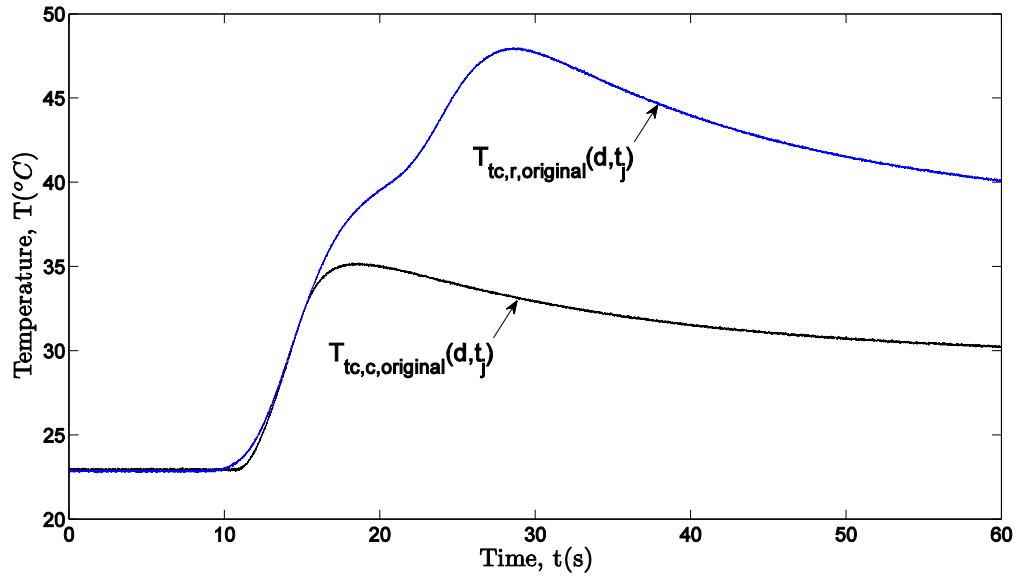


Figure 3.11: Measured thermocouple temperatures $T_{tc,c,original}(d,t_j)$, $T_{tc,r,original}(d,t_j)$ under “calibration” surface heat flux and “run” surface heat flux, respectively, in the electrical heating experiment.

In the inverse analysis, both “calibration” and “run” thermocouple temperatures are subtracted by the average of lead temperature data ($\approx$ room temperature) to satisfy the one-dimensional calibration equation (Eq. 3.15a) for the situation when initial temperature is not 0°C . In the calibration “run”, time interval $t \in [10, 60\text{s}]$ is employed in the inverse code, while time interval $t \in [0, 50\text{s}]$ of the second “run” is employed in the inverse code for numerical convenience. In addition, both thermocouple temperature data and “calibration” surface heat flux data are downsampled from 200Hz to 20Hz to save computational time of the inverse code. Figure 3.12 displays the “calibration” surface heat flux $q_c''(0, t)$ and “run” surface heat flux $q_r''(0, t)$ used in the inverse code. Figure 3.13 displays the thermocouple temperature responses $T_{tc,c}(d, t_j)$, $T_{tc,r}(d, t_j)$ ($j=1, 2, 3, \dots, 1000$) corresponding to “calibration” surface heat flux $q_c''(0, t)$ and “run” surface heat flux $q_r''(0, t)$, respectively.

Table 3.3: Measured Distances for the electrical heating experiment [39].

Parameter	Value (units)
$L_{\text{Stainless Steel}}$	27.9 mm
L_{Mica}	0.102 mm
L_{Paint}	≤ 0.03 mm
L_{Heater}	0.125 mm
d	6.57 mm

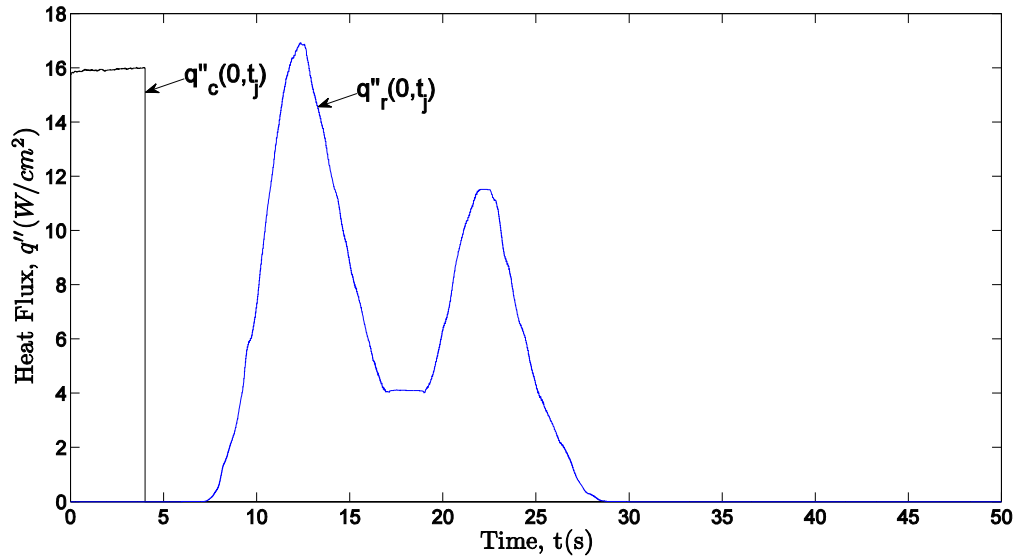


Figure 3.12: Experimental “calibration” net surface heat flux $q_c''(0, t_j)$ and actual “run” net surface heat flux $q_r''(0, t_j)$ used in the inverse code (calibration data are shifted to begin at time $t=0\text{s}$).

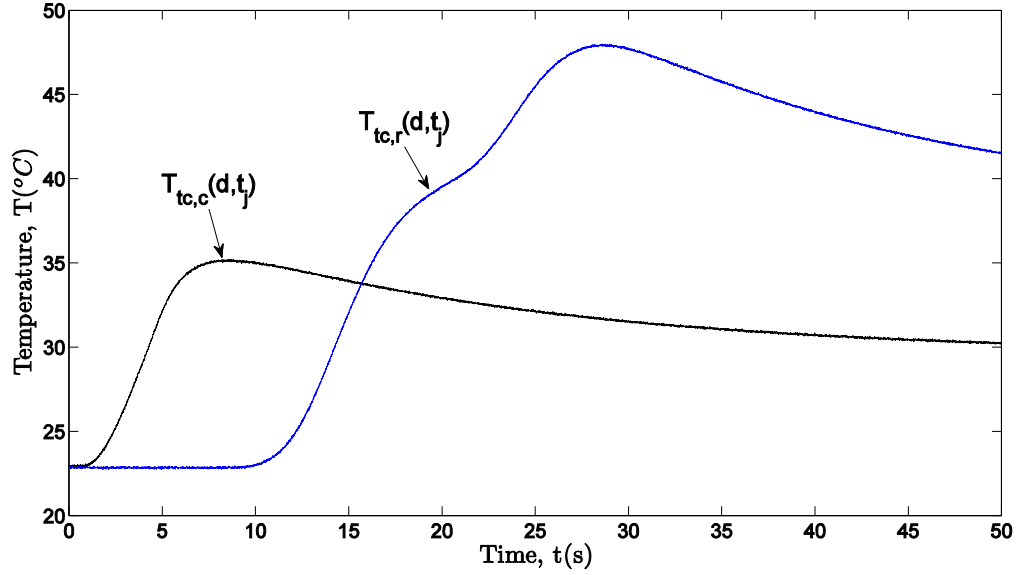


Figure 3.13: Experimental thermocouple temperatures $T_{tc,c}(d,t_j)$, $T_{tc,r}(d,t_j)$ under “calibration” surface heat flux and “run” surface heat flux, respectively, which are employed in the inverse code (calibration data are shifted to begin at time $t=0$ s).

Figure 3.14 displays the local residual function, $r_\gamma(t)$, defined in Eq. (3.22) over time for increasing future time parameter γ . The residual’s magnitude is shown to increase with as future time parameter γ increases. Figure 3.15 shows the corresponding running average of the residual, $\bar{r}_\gamma(t)$. As in Section 3.4, low bias and low variance of local residual function $r_\gamma(t)$ are desired. Next, time-running variance of the residual function $\sigma_\gamma(t_i)$ (Eq. (3.24) and its normalized format $\sigma_\gamma^*(t_i)$) are plotted. Figure 3.16 displays the time-running variance of the residual function while Fig. 3.17 displays the normalized version of Fig. 3.16. In Fig. 3.17, it is obvious that small γ ’s produce unstable predictions (see $\gamma = 1.5$ s) while large γ ’s produce over-smoothed predictions (see $\gamma = 3.5$ s) of the net surface heat flux. The normalized plot possesses key physical effects. In fact, a bundling effect occurs for $\gamma = 2.0$ - 2.75 s for times, $t > 27$ s. This range of γ -values describes the best prediction based on the balance between bias (data) and variance (model). This γ -range of the spectrum is where the optimal prediction for the heat flux will lie.

Figures 3.18a-f display a comparison between the exact “run” surface heat flux and predicted “run” surface heat flux corresponding to the γ values in Figs. 3.14-3.17. Again, these heat flux predictions are presented to verify the effectiveness of the normalized square root of time-running variance as a metric for extracting the optimal γ value. It is apparent that γ values between 2.0 s to 2.75 s are viable choices. In contrast, when $\gamma = 1.5$ s the prediction is unstable, when $\gamma = 3.5$ s the prediction is over-smoothed. Hence, the optimal estimator is chosen as $\gamma_{opt} = 2.0$ s.

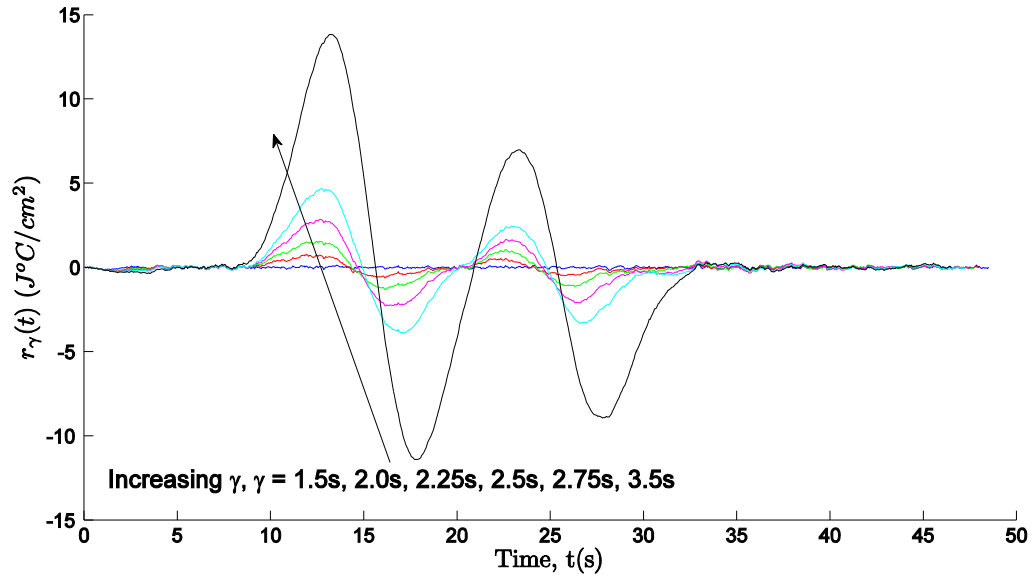


Figure 3.14: The local residual function, $r_\gamma(t)$, defined in Eq. (3.22) over the γ -spectrum (using experimental data).

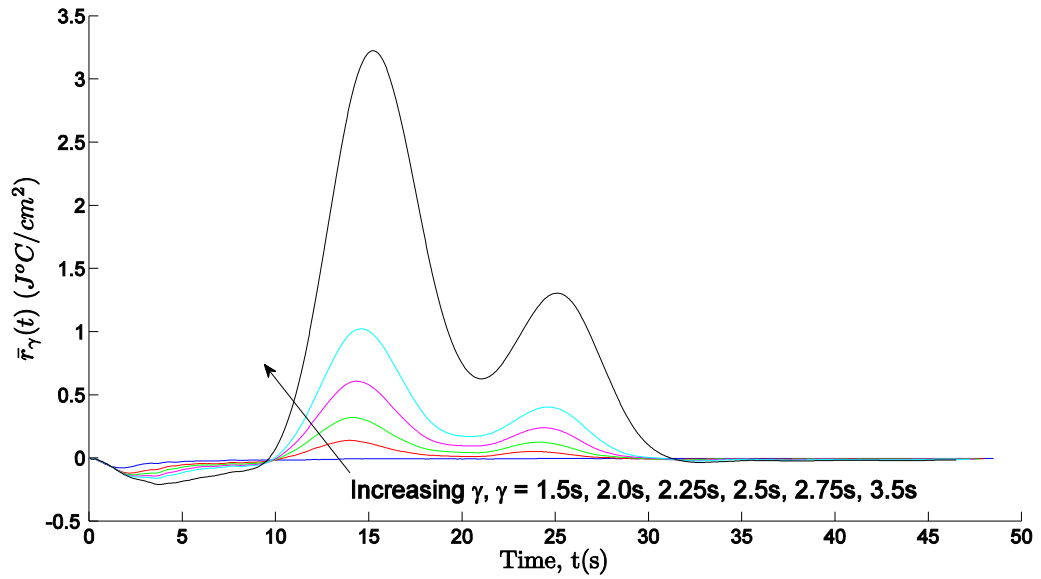


Figure 3.15: The running average of residual, $\bar{r}_\gamma(t)$, defined in Eq. (3.23) over the γ -spectrum (using experimental data).

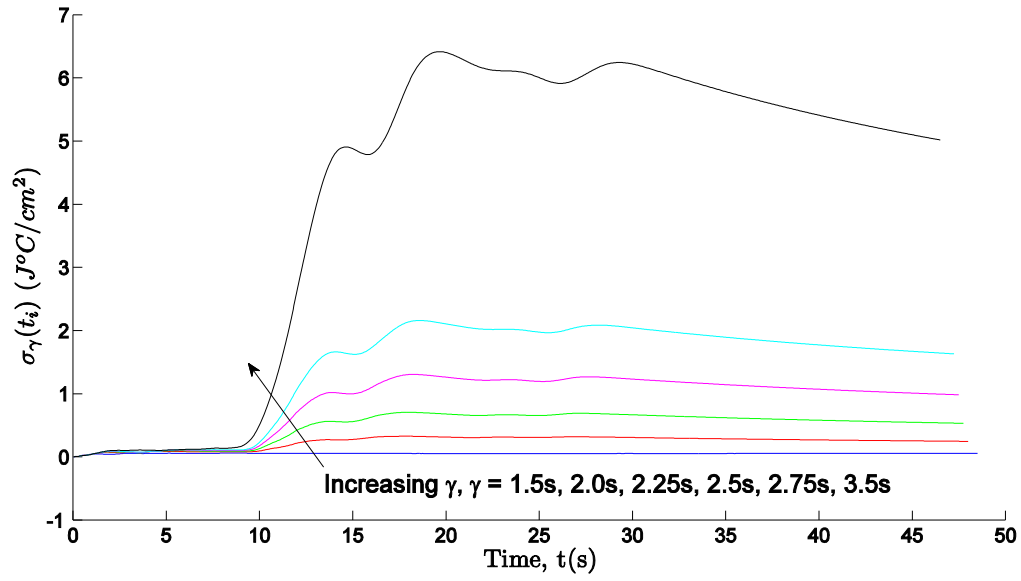


Figure 3.16: The square root of time-running variance of residual, $\sigma_\gamma(t_i)$, defined in Eq. (3.24) over the γ -spectrum (using experimental data).

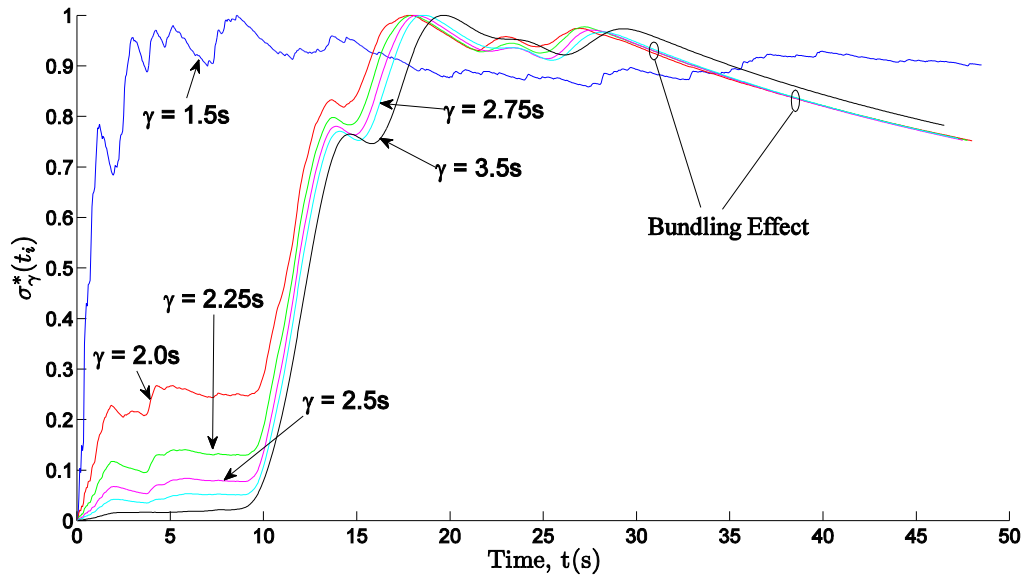


Figure 3.17: The normalized square root of time-running variance of residual, $\sigma_\gamma^*(t_i)$, defined in Eq. (3.25) over the γ -spectrum (using experimental data).

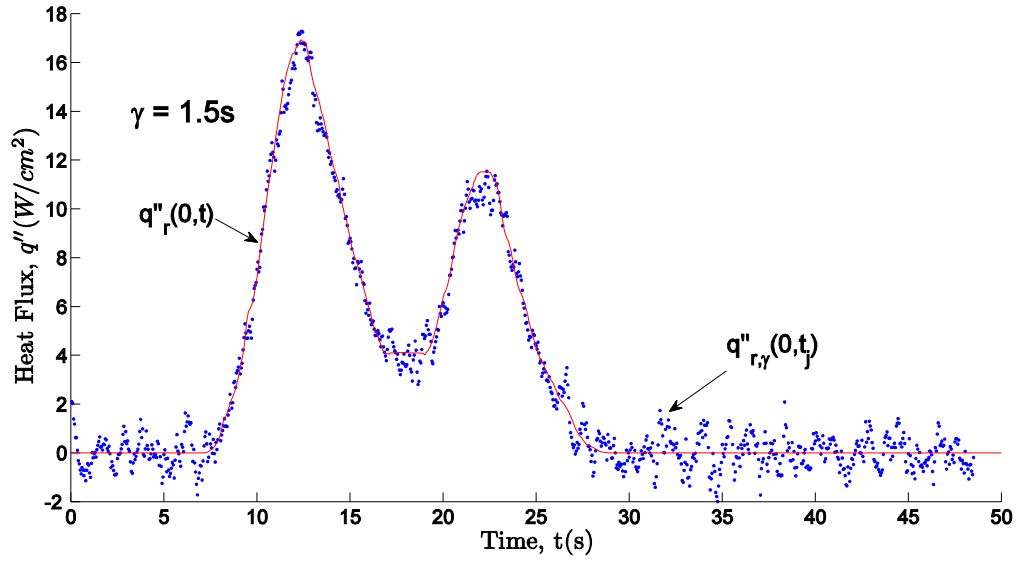


Figure 3.18a: Comparison between actual and predicted (“run”) surface heat flux when $\gamma = 1.5\text{s}$ (using experimental data).

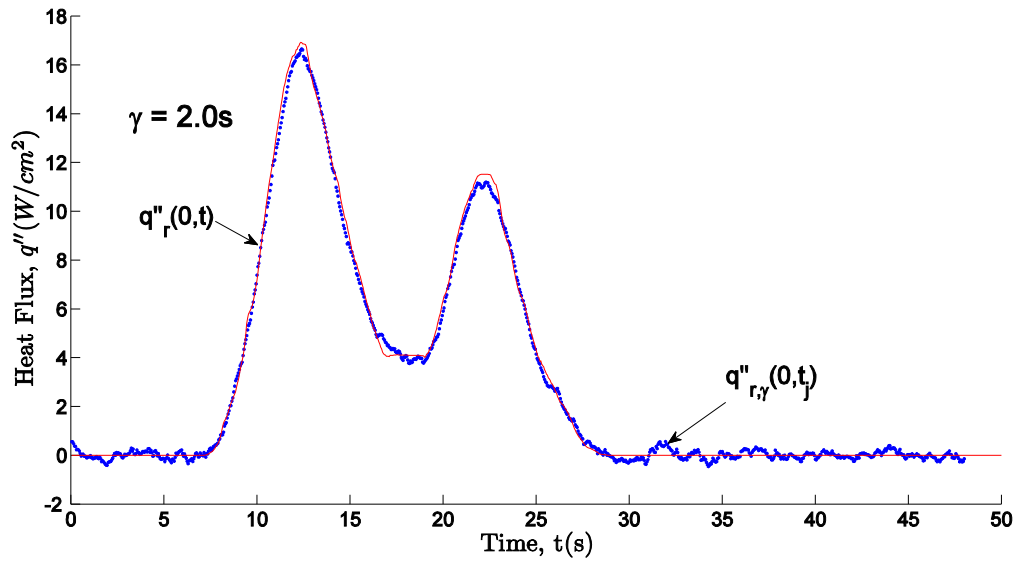


Figure 3.18b: Comparison between actual and predicted (“run”) surface heat flux when $\gamma = 2.0\text{s}$ (using experimental data).

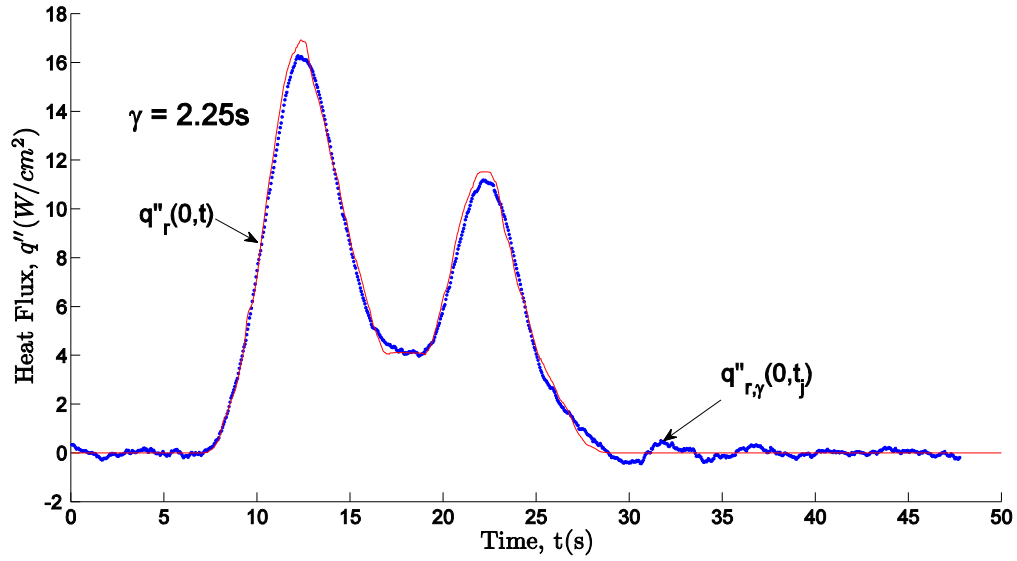


Figure 3.18c: Comparison between actual and predicted (“run”) surface heat flux when $\gamma = 2.25\text{s}$ (using experimental data).

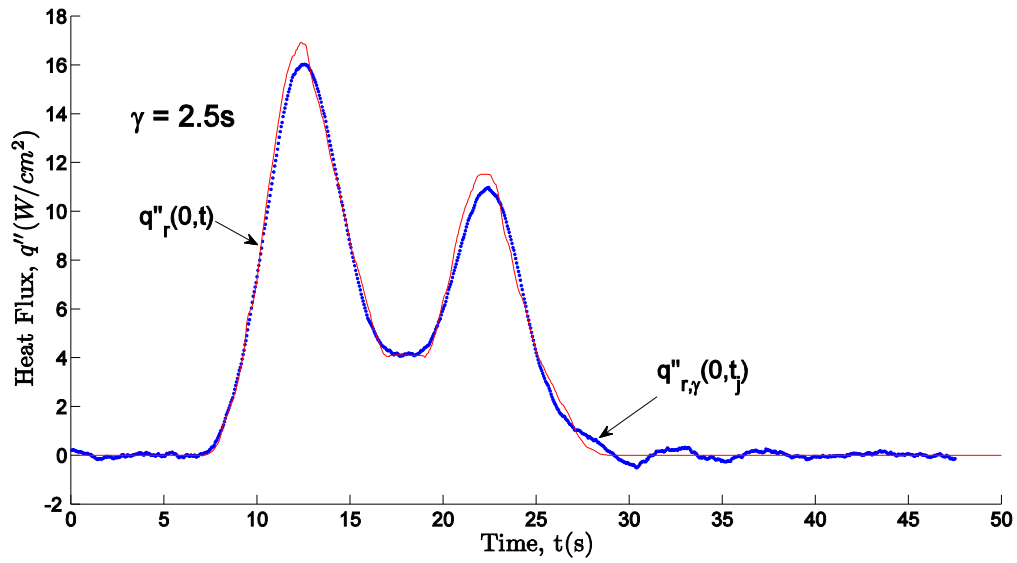


Figure 3.18d: Comparison between actual and predicted (“run”) surface heat flux when $\gamma = 2.5\text{s}$ (using experimental data).

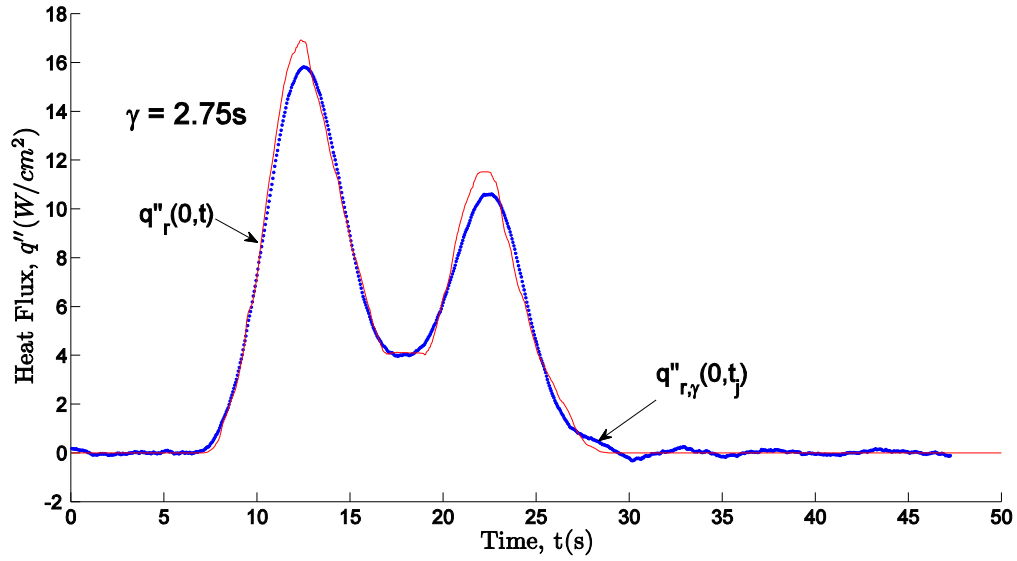


Figure 3.18e: Comparison between actual and predicted (“run”) surface heat flux when $\gamma = 2.75s$ (using experimental data).

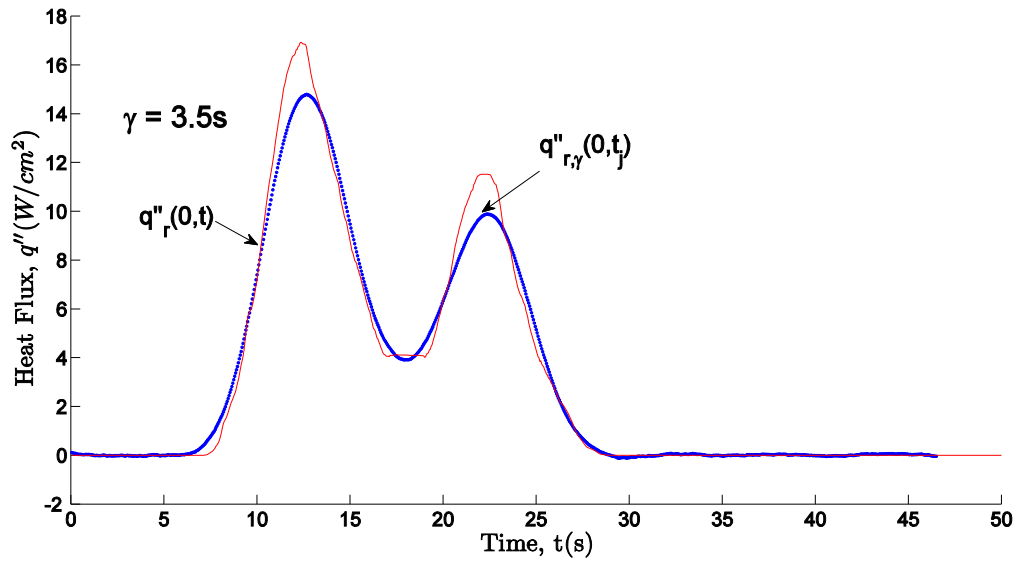


Figure 3.18f: Comparison between actual and predicted (“run”) surface heat flux when $\gamma = 3.5s$ (using experimental data).

3.6 Chapter Summary

This Chapter reviewed the calibration integral equation method in the context of linear analysis. In Section 3.5, the results based on experimental data confirmed the proposed metric described in Section 3.3. These metrics were first investigated in Section 3.4 using purely numerically simulated data. The one-dimensional calibration integral equation method can be applied to various materials, and it can produce excellent results at different thermocouple depths by changing the future time parameter. In next chapter, the calibration integral equation method is extended to two-dimensional domain by quasi- one-dimensional approximation.

Chapter 4 Quasi-One-Dimensional Approximation to Two-Dimensional IHCP

In this Chapter, a quasi-one-dimensional approximation for two-dimensional IHCP will be formulated. A locally one-dimensional view, based on two-dimensionally gathered transient temperature data, is numerically investigated for forming the surface heat flux prediction via one-dimensional calibration integral equation method. This approximation is shown to be accurate for low conductivity materials that have practical engineering value in the study of thermal protection systems (TPS's).

4.1 Quasi-One-Dimensional Formulation and Numerical Simulation of Two-Dimensional IHCP

A graphical depiction of a quasi-one-dimensional approximation for the two-dimensional IHCP is displayed in Fig. 4.1. The two-dimensional geometry described in Fig. 4.1 shows adiabatic boundary conditions at the side walls and a convective boundary condition at the bottom wall. The top surface has a both space and time varying heat flux boundary condition. With the heat fluxes prescribed, the two-dimensional linear heat equation can be solved by an explicit finite difference method for producing the entire temperature field $T(x,y,t)$ for both the “calibration” and “run” stages. Evaluating the resulting temperature field at the probe sites provides the in-depth temperature data. These temperature responses are numerically “exact” in the sense that the data are not initially contaminated with random noise. The only errors in the data, at this stage, are due to discretization (truncation) errors from the numerical method. These errors can be indirectly monitored by space and time splitting until numerical convergence is obtained to some desired accuracy.

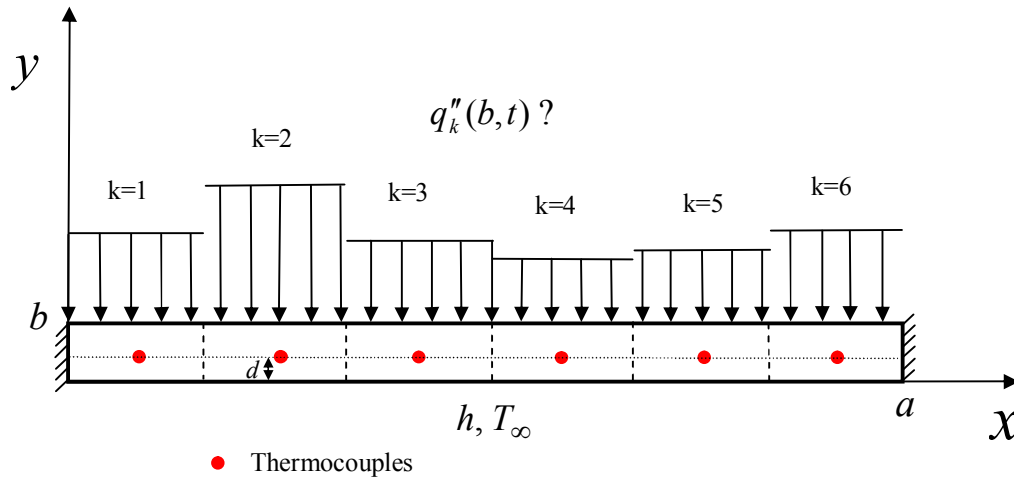


Figure 4.1: Schematic of quasi-one-dimensional approximation of two-dimensional problem (Here $k=1,2,3,4,5,6$ represent local one-dimensional heat flux approximations corresponding to fully two-dimensional surface heat flux displayed in Fig. 1.1).

To generate noisy data at the defined in-depth probe sites, random noise is added to the “numerically exact” data obtained from the direct problems. The quasi-one-dimensional formulation is based on a locally one-dimensional view at each thermocouple position. In terms of numerical simulation in this context, the thermocouple temperature data are obtained by numerically solving two-dimensional direct transient heat conduction problem. At each thermocouple position, the “calibration” and “run” thermocouple data denoted as $T_{ic,c}(x_k, d, t)$, $T_{ic,r}(x_k, d, t)$, respectively. The local “calibration” surface heat flux data as a function of time denoted as $q_c''(x_k, b, t)$. These data are used in the inverse code based on one-dimensional calibration integral equation method discussed in Chapter 3. The local “run” surface heat flux $q_r''(x_k, b, t)$ resulting from each thermocouple site is shown in Fig. 4.1. In each region, the “run” surface heat flux $q_r''(x_k, b, t)$ is assumed to be constant in spatial domain.

In this chapter, the quasi-one-dimensional approximation of the two-dimensional problem is based on purely numerical simulation. In this context, the host material of slab in the simulation is stainless steel with thermal conductivity $k=14.7$ W/mK, and thermal diffusivity $\alpha=3.75 \times 10^{-6}$ m²/s. The length of slab is $a=12$ cm, the height of slab is $b=1$ cm and possesses unit depth (1cm). The convective heat transfer coefficient used in the simulation is $h=100$ W/m²K. The thermocouples are located at $d=5$ mm. The sampling frequency $f_{sampling}$ used in the simulation is 50Hz while the total simulation time is $t_{max}=40$ s.

The “calibration” surface heat flux used in the slab is uniform in the spatial domain and constant in the temporal domain. That is, we assign $q_c''(x, b, t) = 10$ W / cm², $x \in [0, a]$ as shown in Fig. 4.2. In contrast, the “run” surface heat flux is given as a parabolic function in the spatial domain and triangular function in the temporal domain. It is mathematically expressed as

$$q_r''(x, b, t) = \begin{cases} 0 \text{ W / cm}^2, & 0 \leq t \leq 4s \\ 100\sqrt{x}\left(\frac{1}{12}t - \frac{1}{3}\right) \text{ W / cm}^2, & 4s < t \leq 16s \\ 100\sqrt{x}\left(-\frac{1}{12}t + \frac{7}{3}\right) \text{ W / cm}^2, & 16s < t \leq 28s \\ 0 \text{ W / cm}^2, & 28s < t \leq 40s \end{cases} \quad (4.1)$$

The schematic of “run” surface heat flux is displayed in Fig. 4.3. Artificial temperature noise is added to the probe site thermocouple data in accordance to

$$T_{ic}(x_k, d, t_j) = T(x_k, d, t_j) + \varepsilon_T r_j \|T(x_k, d, t_j)\|_{\infty}, \quad k = 1, 2, 3, 4, 5, 6, \quad j = 1, 2, \dots, N. \quad (4.2)$$

for both “calibration” and test “run” stages. Here ε_T denotes a noise factor, r_j denotes random numbers in the interval $[-1, 1]$ which are obtained by MATLAB command “2*(rand-0.5)”. For the “calibration” run, noise is also added to the net surface heat flux as

$$q_c''(x_k, b, t_j) = q_{c, exact}''(x_k, b, t_j) + \varepsilon_q r_j \|q_{c, exact}''(x_k, b, t_j)\|_{\infty}, \quad k = 1, 2, 3, 4, 5, 6, \quad j = 1, 2, \dots, N. \quad (4.3)$$

Here, N is the total number of time increments beyond the initial condition and given as $N=f_{sampling} t_{max}$. The parameters and thermophysical properties for quasi-one-dimensional simulation are collected in Table 4.1. Observe that the calibration stage uses a purely one-dimensional heating profile. This is highly convenient for the study of quasi-one-dimensional calibration integral equation method.

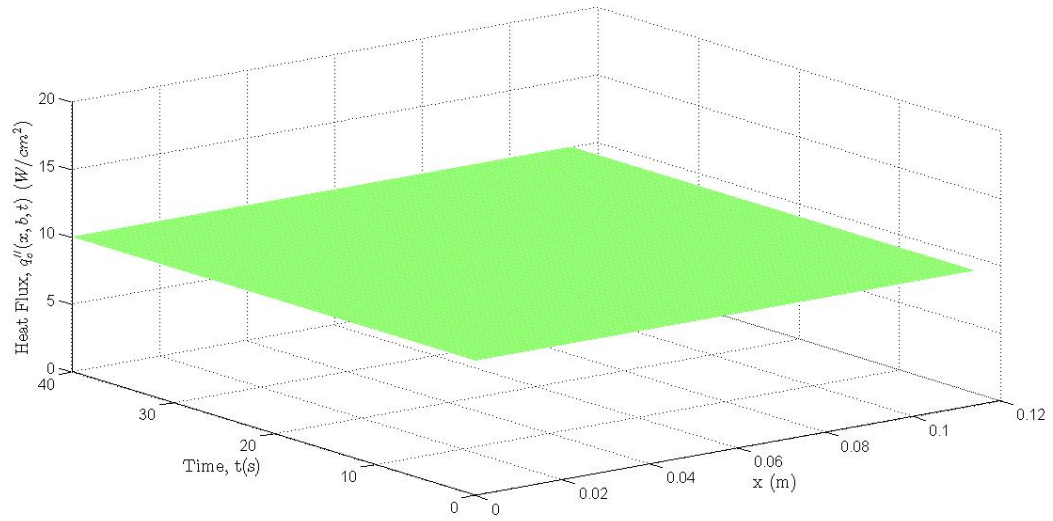


Figure 4.2: The “calibration” surface heat flux $q_c''(x, b, t)$.

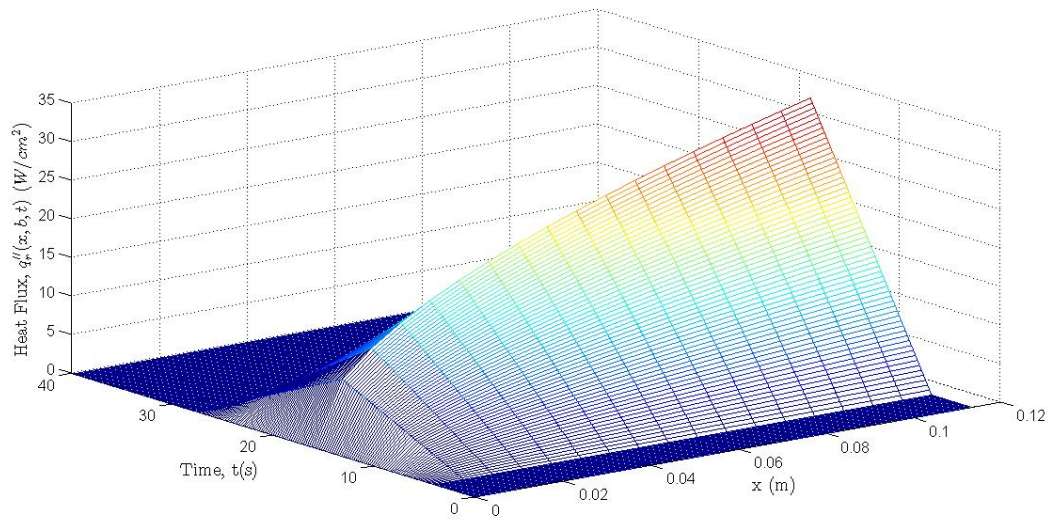


Figure 4.3: The exact “run” surface heat flux $q_r''(x, b, t)$.

Table 4.1: Properties and parameters for quasi-one-dimensional simulation

Property / Parameter	Value (units)
k	14.7 W/mK
α	3.75×10^{-6} m ² /s
h	100 W/m ² K
a	12 cm
b	1 cm
d	0.5cm
ε_T	0.01
ε_q	0.01
$f_{sampling}$	50 Hz
t_{max}	40 s

Figure 4.4 displays both exact and noisy “calibration” surface heat fluxes as a function of time at the third thermocouple site. The “calibration” surface heat flux is uniform in the spatial domain, thus in any x position, the “calibration” surface heat flux is identical. Figures 4.5 and 4.6 display sample cuts of the exact “run” surface heat fluxes. These are the exact “run” surface heat flux as a function of space at time instant $t=20$ s and exact “run” surface heat flux as a function of time at the third thermocouple site, respectively. Figure 4.7 displays the temperature distribution as a function of time under the spatially uniform “calibration” surface heat flux. The resulting temperature distribution is uniform in the x -direction owing to the imposed boundary and initial conditions; and, is only dependent on the y -direction and time. Figure 4.8 displays the “run” (two-dimensional) temperature histories as a function of time for the six probe locations.

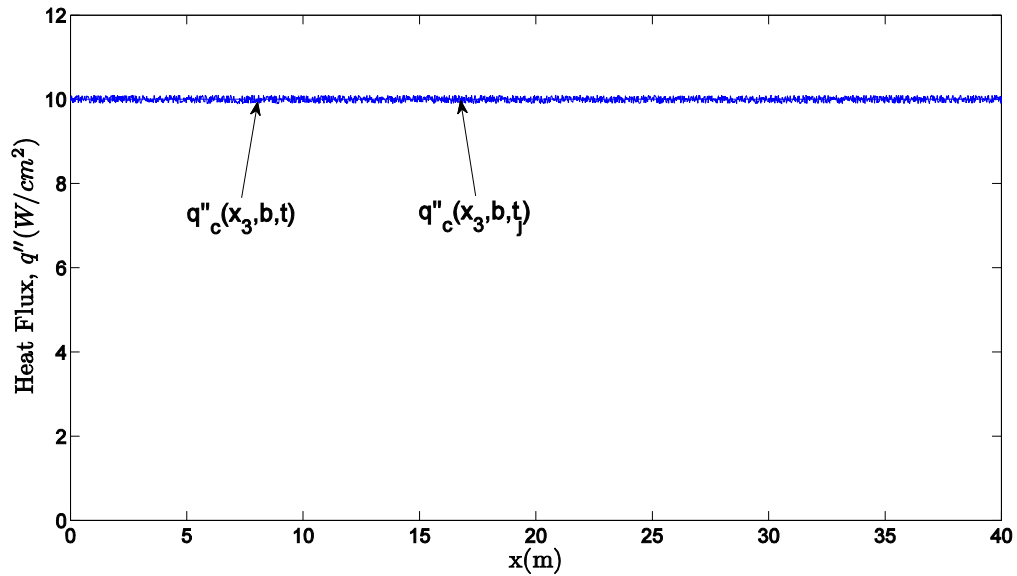


Figure 4.4: The exact “calibration” surface heat flux $q''_c(x_3, b, t)$ and the noise added “calibration” $q''_c(x_3, b, t_j)$ as a function of time at the third thermocouple site.

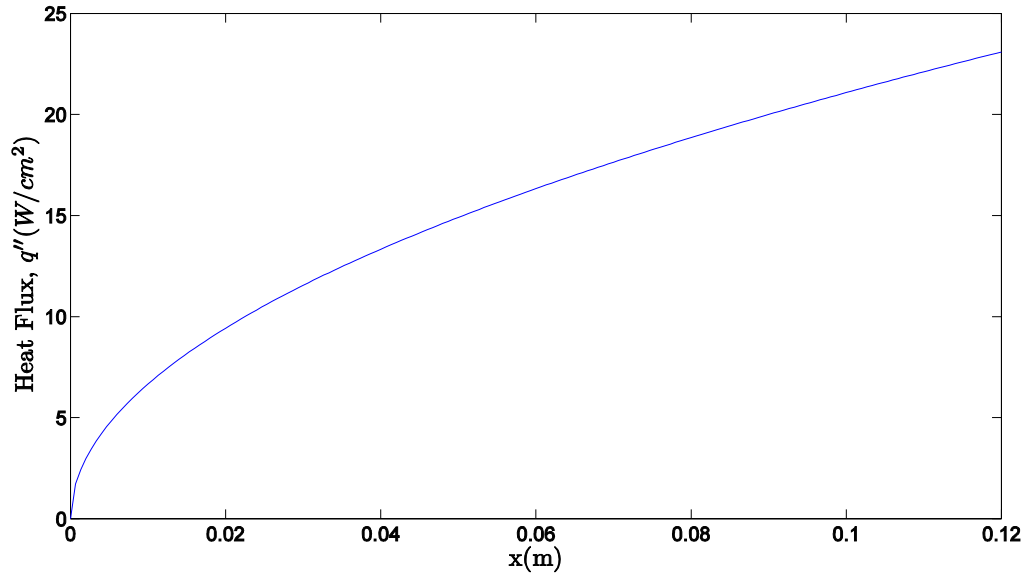


Figure 4.5: The exact “run” surface heat flux in spatial domain at $t = 20\text{s}$

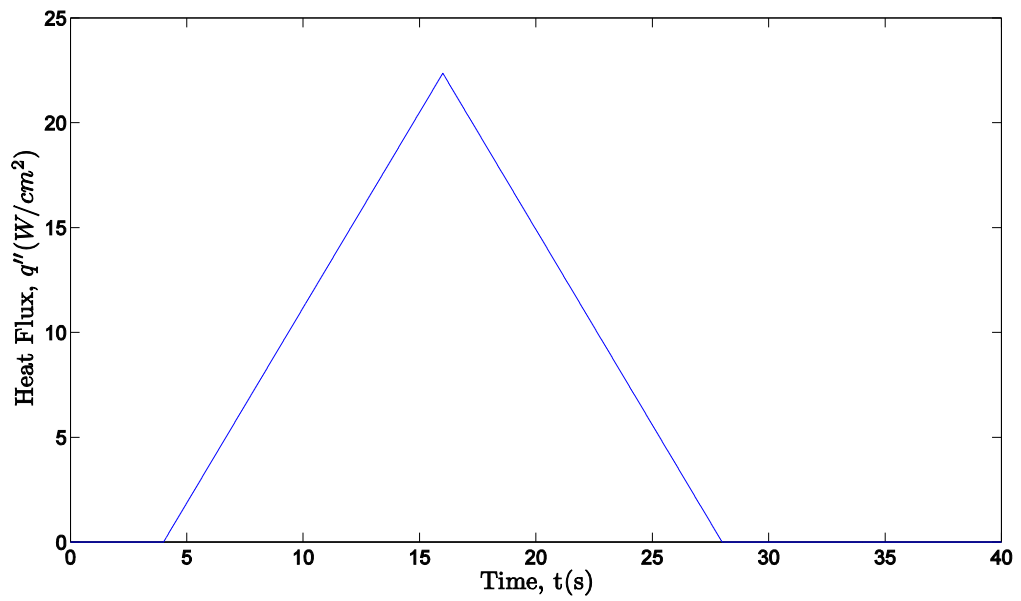


Figure 4.6: The exact “run” surface heat flux in temporal domain at the third thermocouple site.

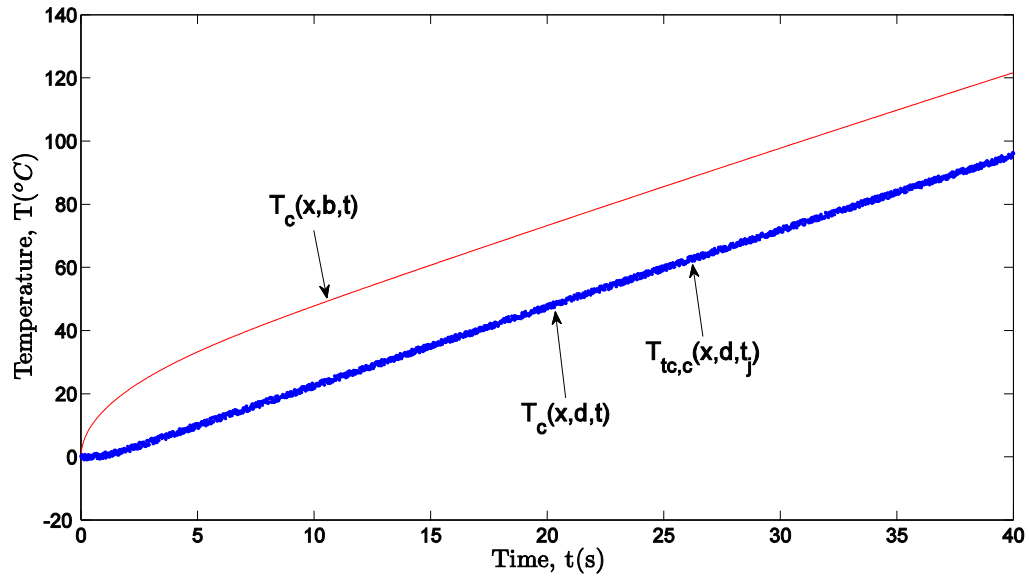


Figure 4.7: The temperature distribution using spatially uniform “calibration” surface heat flux $q_c''(x,b,t)$ as a function of time at any x position.

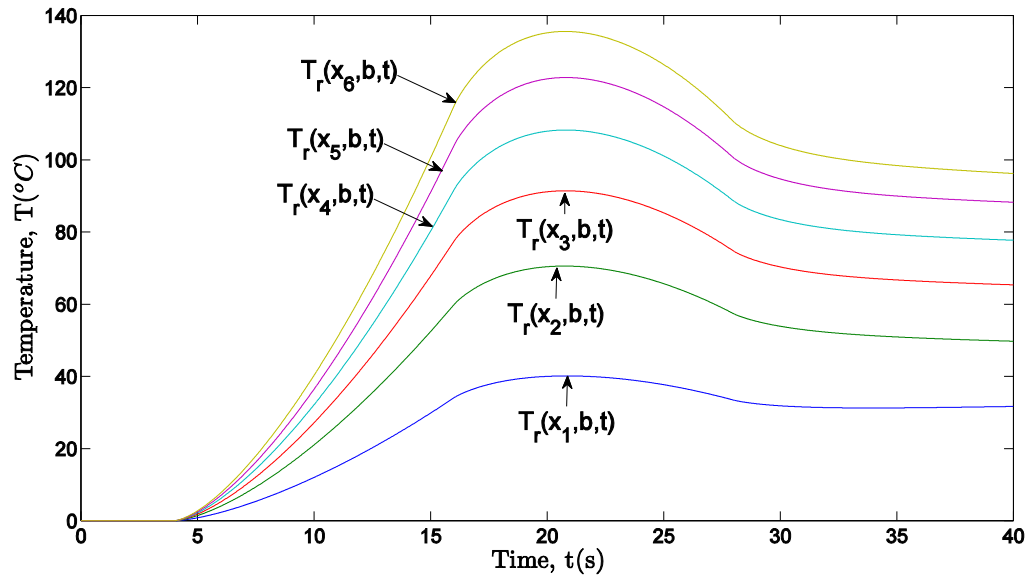


Figure 4.8a: The positional surface temperature distributions under “run” surface heat flux $q_r''(x,b,t)$ at each thermocouple position.

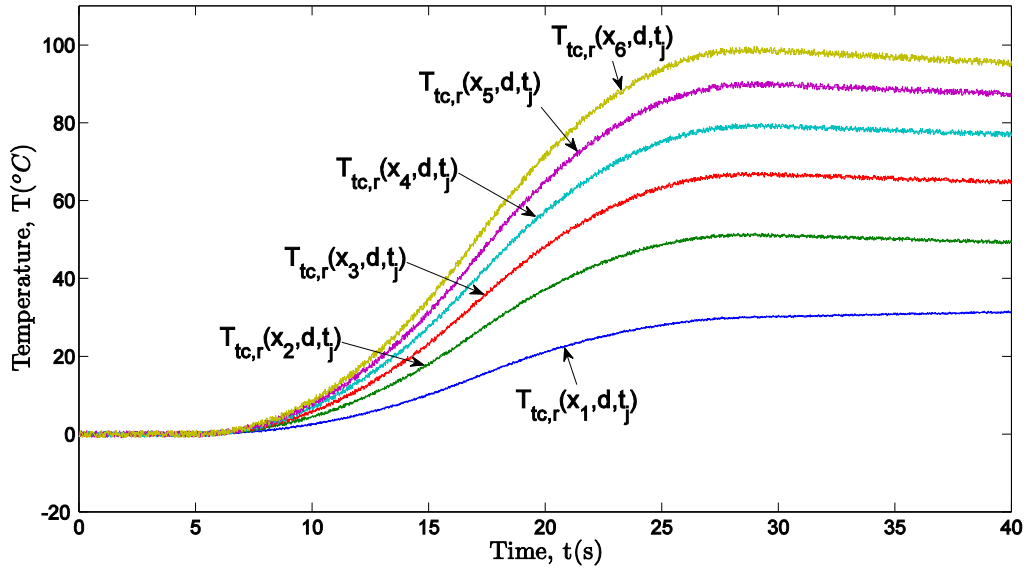


Figure 4.8b: The temperature distributions under “run” surface heat flux $q''_r(x,b,t)$ at each thermocouple site.

The regularization method used to stabilize the IHCP is based on a variation of Lamm’s local future-time method [8,85]. For brevity, we only state the results obtained by this methodology based on probe number 3 located on $x=2.5\text{cm}$, $y=0.5\text{cm}$. It should be noted again that the data created for the “run” case is generated from the fully two-dimensional solution of the heat equation. It is our objective to predict the surface heat flux associated with Eq. (4.1) but based on the quasi-one-dimensional simplification. Hence, a piecewise constant predicted surface heat flux is produced that is then compared with the physically imposed boundary condition.

Again, the Greek symbol γ is used to denote the future time parameter. The metrics defined in Section 3.3 are employed for determining optimal future time parameter γ . Figure 4.9 displays the local residual function, $r_\gamma(t)$, defined in Eq. (3.22) over time for increasing future time parameter γ based on the “calibration” and “run” thermocouple temperature as well as the local “calibration” surface heat flux and the local predicted “run” surface heat flux at the third thermocouple site. As the future time parameter increases, the residual’s magnitude is shown to increase. It is desirable to keep this function under control (low bias and low variance) over the entire time span. Figure 4.10 shows the corresponding running average of the residual. Clearly the smoothing operation of integration is seen in Fig. 4.10. Figure 4.11 displays the time-running variance of the residual function defined in Eq. (3.24) while Fig. 4.12 displays the normalized version of Fig. 4.11. In this form (Fig. 4.12), both instability and over-smoothing effects can be quickly identified. Small γ ’s produce unstable predictions (see for example $\gamma = 1.4\text{s}$) while large γ ’s produce over-smoothed predictions (see for example $\gamma = 4.0\text{s}$) of the net surface heat flux. The normalized plot possesses key physical effects. In fact, a bundling effect occurs for $\gamma = 1.8\text{--}2.2\text{s}$ for times, $t > 27\text{s}$. This range of γ -values describes the best prediction based on the balance between bias (data) and variance (model). This γ -range of the spectrum is where the optimal prediction for the heat flux will lie. In fact, it is desired to retain as much of

the high frequency content in the signal for physics recovering. Hence, the optimal estimator is chosen as $\gamma_{opt} = 1.8s$.

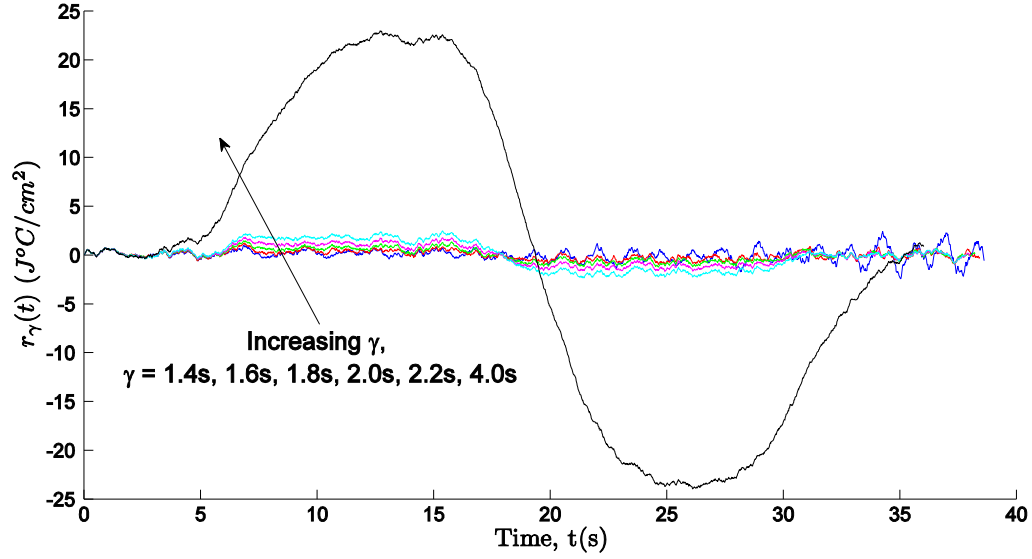


Figure 4.9: The local residual function, $r_\gamma(t)$, defined in Eq. (3.22) over the γ -spectrum based on the data of the third thermocouple site.

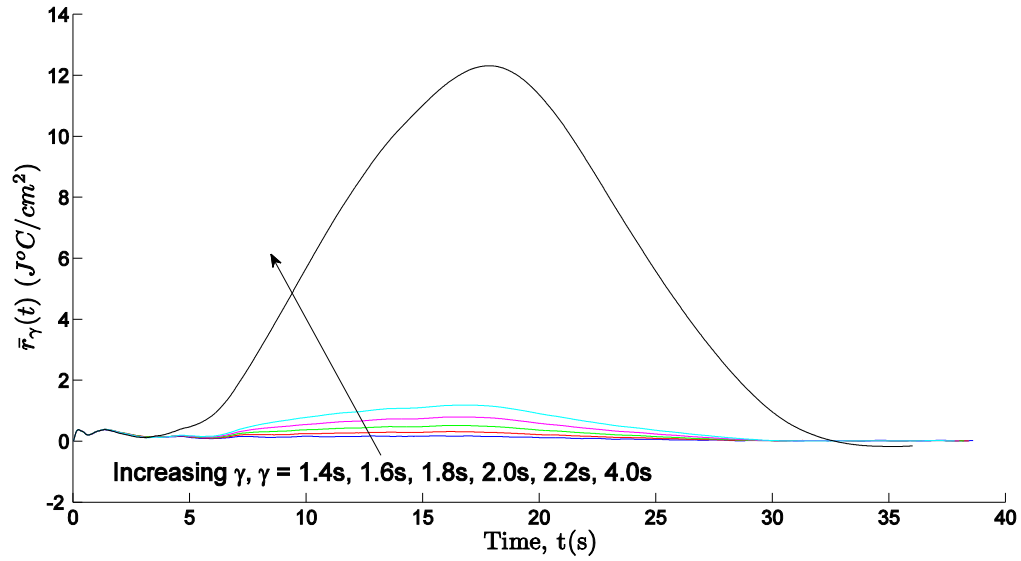


Figure 4.10: The running average of residual, $\bar{r}_\gamma(t)$, defined in Eq. (3.23) over the γ -spectrum based on the data of the third thermocouple site.

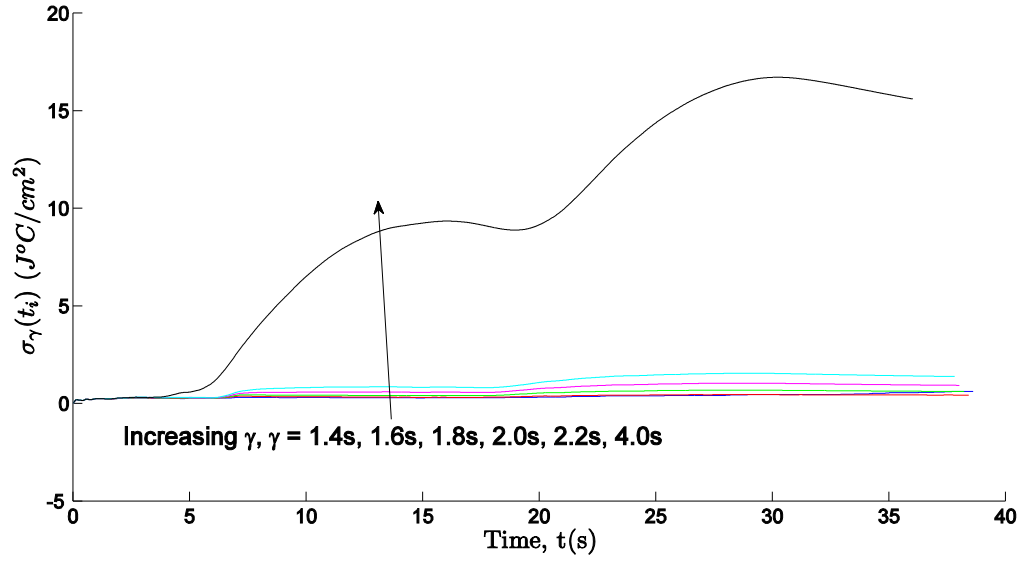


Figure 4.11: The square root of time-running variance of residual function, $\sigma_\gamma(t_i)$, defined in Eq. (3.24) over the γ -spectrum based on the data of the third thermocouple site.

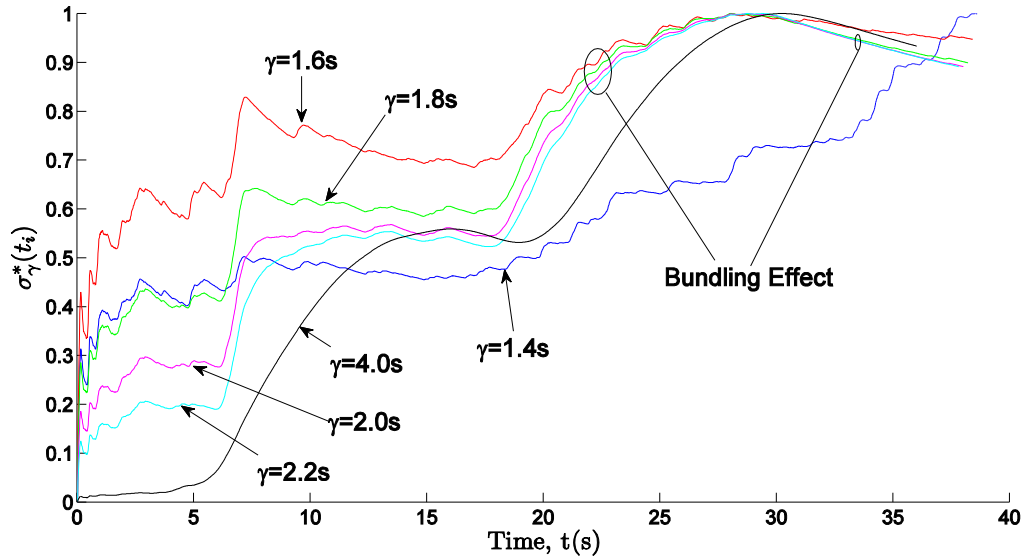


Figure 4.12: The normalized square root of time-running variance of residual, $\sigma_\gamma^*(t_i)$, defined in Eq. (3.25) over the γ -spectrum based on the data of the third thermocouple site.

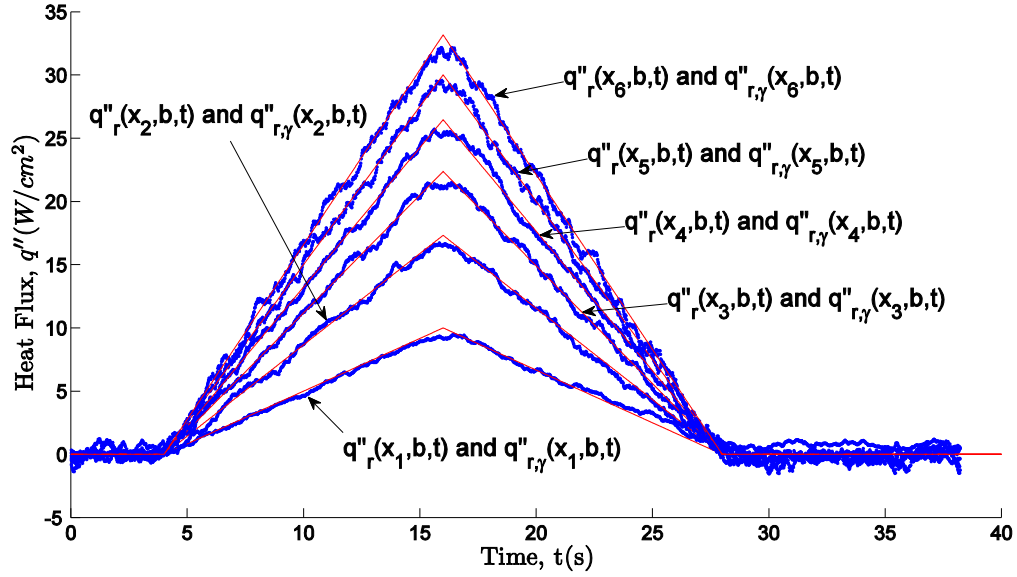


Figure 4.13: Comparison between the predicted surface heat flux using $\gamma_{opt}=1.8s$ (blue) and the exact surface heat flux (red) at six probe positions over time.

The predicted time histories of the “run” surface heat fluxes for all six probe positions are presented in Fig. 4.13. These predictions are based on $\gamma_{opt}=1.8s$. Figure 4.14 displays the surface heat flux as a function of space at three specified times. These results are highly favorable when compared to the exact input described by Eq. (4.1).

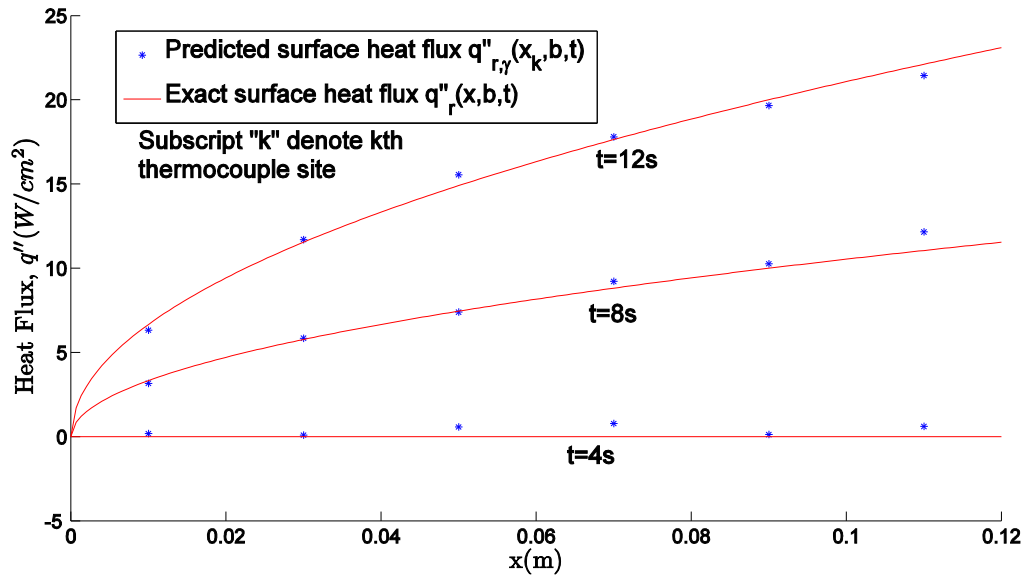


Figure 4.14: Comparison between the predicted surface heat flux using $\gamma_{opt}=1.8s$ (star symbol) and the exact surface heat flux (red line) over space at the three specified time instants.

The total “run” surface heat transfer (Watts) can be approximated by numerical integration over the piecewise constant values of the acquired “run” surface heat flux as geometrically described in Fig. 4.1. Thus, we form this approximation as

$$Q_{r,\gamma_{opt}}(b,t) = \int_{x=0}^a q''_{r,\gamma_{opt}}(x,b,t)dx \times 1cm \approx \sum_{k=1}^6 q''_{r,\gamma_{opt}}(x_k,b,t)\Delta x \times \frac{1}{100} (W). \quad (4.2)$$

where $\Delta x=0.01m$. The “run” total heat transfer after numerically integrating the spatial domain, is displayed in Fig. 4.15. The approximation per Eq. (4.2) is compared with the exact solution. Again, highly favorable results are produced.

Finally, energy can be calculated and compared to the exact energy input. This is obtained by numerically integrating the “run” surface heat flux over both the temporal and spatial domains. Performing these approximations, we find, $Q_{r,\gamma} = 3.3246 kJ$, $Q_r = 3.3390 kJ$, and the relative error is found as $\varepsilon = \frac{|Q_{r,\gamma} - Q_r|}{Q_r} \times 100\% = 0.43\%$.

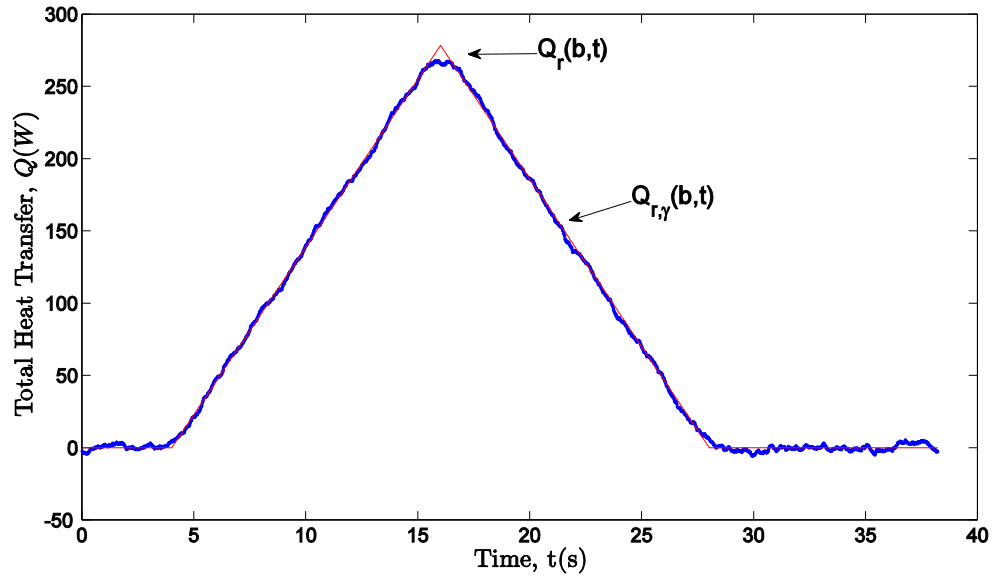


Figure 4.15: Comparison between the predicted total “run” surface heat transfer $Q_{r,\gamma}(b,t)$ using $\gamma_{opt}=1.8s$ and the exact total “run” surface heat transfer $Q_r(b,t)$.

4.2 Chapter Summary

A quasi-one-dimensional view of the two-dimensional IHCP is formulated and numerically studied and applied to a low thermal conductivity material stainless steel. Further numerical studies are underway directed toward actual TPS materials, such as Carbon-Carbon, SiC and ZrO_2 . It is recommended to develop an experimental test verifying these simulated results. Nevertheless, the quasi-one-dimensional approximation of the two-dimensional IHCP has limitation to low thermal conductivity materials. In the next chapter, a two-dimensional total heat transfer rate calibration equation will be derived that performs well for both low thermal conductivity materials and high thermal conductivity materials.

Chapter 5 Two-Dimensional Total Heat Transfer Calibration Integral Equation Method

In this Chapter, a two-dimensional total surface heat transfer calibration equation is derived based on the analytical solution of two-dimensional transient heat conduction. No assumptions are employed in the derivation except for the constant thermophysical properties hypothesis. Results from numerical simulations illustrate that the two-dimensional total surface heat transfer calibration integral equation method works well for a large set of engineering materials under the assumption of constant thermophysical properties.

5.1 Derivation of the Two-Dimensional Total Heat Transfer Calibration Equation

The two-dimensional total surface heat transfer calibration equation is derived by placing the analytical solution of the two-dimensional transient heat conduction into an input-output relationship. The schematic of the two-dimensional transient heat conduction geometry to be considered is displayed in Fig. 5.1.

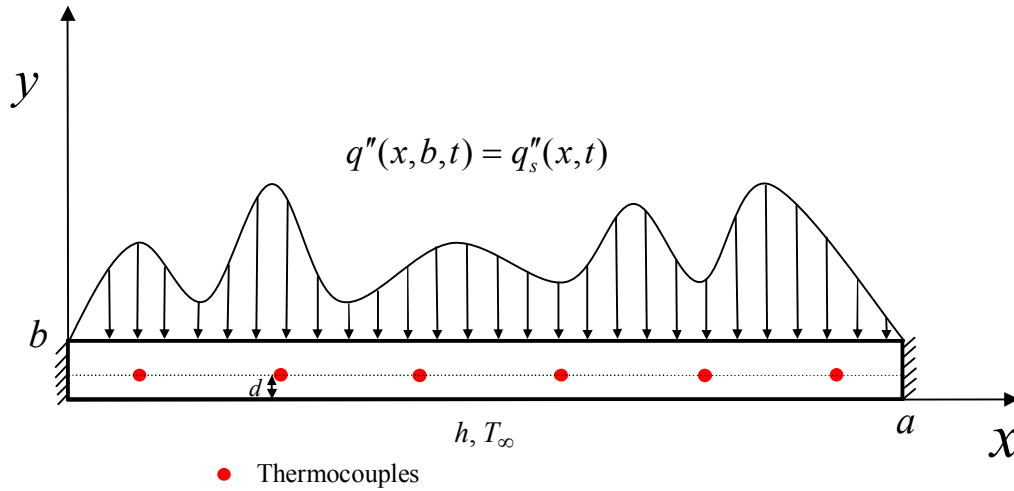


Figure 5.1: Schematic for two-dimensional plate geometry with time-varying and spatially distributed source. Thermocouples are indicated as solid circles.

The analytical solution for the two-dimensional transient heat conduction (see Appendix B for derivation) is given as

$$T(x, y, t) = \frac{\alpha}{k} \sum_{m=0}^{\infty} \sum_{n=1}^{\infty} \frac{\cos(\beta_m x) \cos \eta_n (b-y)}{N(\beta_m) N(\eta_n)} \int_{u=0}^t e^{-\alpha(\beta_m^2 + \eta_n^2)(t-u)} \int_{x'=0}^a \cos(\beta_m x') q''(x', b, u) dx' du, \quad x \in [0, a], y \in [0, b], t \geq 0, \quad (5.1a)$$

where the eigenvalues and normalization integrals are

$$\beta_m = m \frac{\pi}{a}, \quad m = 0, 1, 2, 3, \dots \quad (5.1b)$$

$$N(\beta_m) = \begin{cases} a, & m = 0, \\ \frac{a}{2}, & m = 1, 2, 3, \dots \end{cases} \quad (5.1c)$$

$$\eta_n \text{'s are positive roots of } \eta_n \tan(\eta_n b) = \frac{h}{k}, n = 1, 2, \dots \quad (5.1d)$$

$$N(\eta_n) = \frac{b(\eta_n^2 + \frac{h^2}{k^2}) + \frac{h}{k}}{2(\eta_n^2 + \frac{h^2}{k^2})}, n = 1, 2, \dots \quad (5.1e)$$

Equation (5.1a) assumes the trivial initial condition. Evaluating Eq. (5.1a) at $y=d$ yields

$$T(x, d, t) = \frac{\alpha}{k} \sum_{m=0}^{\infty} \sum_{n=1}^{\infty} \frac{\cos(\beta_m x) \cos \eta_n (b-d)}{N(\beta_m) N(\eta_n)} \int_{u=0}^t e^{-\alpha(\beta_m^2 + \eta_n^2)(t-u)} \int_{x'=0}^a \cos(\beta_m x') q''(x', b, u) dx' du. \quad (5.2)$$

Next, integrating Eq. (5.2) over x produces

$$\int_{x=0}^a T(x, d, t) dx = \frac{\alpha}{k} \sum_{m=0}^{\infty} \sum_{n=1}^{\infty} \frac{\int_{x=0}^a \cos(\beta_m x) dx \cos \eta_n (b-d)}{N(\beta_m) N(\eta_n)} \int_{u=0}^t e^{-\alpha(\beta_m^2 + \eta_n^2)(t-u)} \int_{x'=0}^a \cos(\beta_m x') q''(x', b, u) dx' du, \quad (5.3a)$$

where

$$\int_{x=0}^a \cos(\beta_m x) dx = \int_{x=0}^a \cos\left(\frac{m\pi}{a}x\right) dx = \begin{cases} a, & m = 0, \\ \frac{a}{m\pi} \sin\left(\frac{m\pi}{a}x\right) \Big|_{x=0}^a = 0, & m = 1, 2, 3, \dots \end{cases} \quad (5.3b)$$

Substituting Eq. (5.3b) into Eq. (5.3a) yields

$$\begin{aligned} \int_{x=0}^a T(x, d, t) dx &= \frac{\alpha}{k} \sum_{n=1}^{\infty} \frac{a \cos \eta_n (b-d)}{a N(\eta_n)} \int_{u=0}^t e^{-\alpha \eta_n^2 (t-u)} \int_{x'=0}^a 1 \times q''(x', b, u) dx' du \\ &= \frac{\alpha}{k} \sum_{n=1}^{\infty} \frac{\cos \eta_n (b-d)}{N(\eta_n)} \int_{u=0}^t e^{-\alpha \eta_n^2 (t-u)} \int_{x'=0}^a q''(x', b, u) dx' du. \end{aligned} \quad (5.4)$$

Taking the Laplace transform [86-90] of Eq. (5.4), we get

$$\mathcal{L}\left[\int_{x=0}^a T(x, d, t) dx\right] = \frac{\alpha}{k} \sum_{n=1}^{\infty} \frac{\cos \eta_n (b-d)}{N(\eta_n)} \mathcal{L}\left[\int_{u=0}^t e^{-\alpha \eta_n^2 (t-u)} \int_{x'=0}^a q''(x', b, u) dx' du\right], \quad (5.5)$$

which upon using the convolution theorem [86-90] produces

$$\int_{x=0}^a \hat{T}(x, d, s) dx = \frac{\alpha}{k} \sum_{n=1}^{\infty} \frac{\cos \eta_n (b-d)}{N(\eta_n)} \mathcal{L}\left[e^{-\alpha \eta_n^2 t}\right] \int_{x'=0}^a \hat{q}''(x', b, s) dx', \quad (5.6a)$$

where [97, p.1022, (29.3.8)]

$$\mathcal{L}\left[e^{-\alpha \eta_n^2 t}\right] = \frac{1}{s + \alpha \eta_n^2}. \quad (5.6b)$$

Substituting Eq. (5.1e) and Eq. (5.6b) into Eq. (5.6a) produces

$$\int_{x=0}^a \hat{T}(x, d, s) dx = \frac{\alpha}{k} \sum_{n=1}^{\infty} \frac{2(\eta_n^2 + \frac{h^2}{k^2}) \cos \eta_n(b-d)}{b(\eta_n^2 + \frac{h^2}{k^2}) + \frac{h}{k}} \frac{1}{s + \alpha \eta_n^2} \int_{x'=0}^a \hat{q}''(x', b, s) dx', \quad (5.7)$$

which allow us to write Eq. (5.7) in the input-output format as

$$\frac{\int_{x=0}^a \hat{T}(x, d, s) dx}{\int_{x'=0}^a \hat{q}''(x', b, s) dx'} = \hat{M}(b, d, \alpha, k, n; s), \quad (5.8a)$$

where

$$\hat{M}(b, d, \alpha, k, n; s) = \frac{\alpha}{k} \sum_{n=1}^{\infty} \frac{2(\eta_n^2 + \frac{h^2}{k^2}) \cos \eta_n(b-d)}{b(\eta_n^2 + \frac{h^2}{k^2}) + \frac{h}{k}} \frac{1}{s + \alpha \eta_n^2}. \quad (5.8b)$$

Suppose that the total surface heat transfer at the surface $y=b$ is the only requirement of estimation, then, we can define total heat transfer rate as

$$Q(b, t) = \int_{x=0}^a q''(x, b, t) dx \times 1m, \quad (5.9a)$$

or in the frequency domain as

$$\hat{Q}(b, s) = \int_{x=0}^a \hat{q}''(x, b, s) dx \times 1m. \quad (5.9b)$$

Multiplying $\frac{1}{1m}$ on both sides of Eq. (5.8a), we get

$$\frac{\int_{x=0}^a \hat{T}(x, d, s) dx}{\hat{Q}(b, s)} = \frac{\hat{M}(b, d, \alpha, k, n; s)}{1m}. \quad (5.10)$$

Now, we can use the concept of calibration-run as

$$\left. \frac{\int_{x=0}^a \hat{T}(x, d, s) dx}{\hat{Q}(b, s)} \right|_{cal} = \left. \frac{\int_{x=0}^a \hat{T}(x, d, s) dx}{\hat{Q}(b, s)} \right|_{run}. \quad (5.11)$$

Cross multiplying and employing the convolution theorem [86-90] yields

$$\int_{u=0}^t Q_{run}(b, u) \int_{x'=0}^a T_{cal}(x', d, t-u) dx' du = \int_{u=0}^t Q_{cal}(b, u) \int_{x'=0}^a T_{run}(x', d, t-u) dx' du, \quad t \geq 0. \quad (5.12)$$

Equation (5.12) can be written as

$$\int_{u=0}^t Q_{run}(b, u) \int_{x'=0}^a T_{tc, cal}(x', d, t-u) dx' du = \int_{u=0}^t Q_{cal}(b, u) \int_{x'=0}^a T_{tc, run}(x', d, t-u) dx' du, \quad t \geq 0, \quad (5.13)$$

which follows the logic discussed in Section 3.1. Equation (5.13) can be written in compact format as

$$\int_{u=0}^t Q_{run}(b, u) K_{tc, cal}(t-u) du = \int_{u=0}^t Q_{cal}(b, u) K_{tc, run}(t-u) du, \quad (5.14a)$$

where

$$K_{tc, cal}(t-u) = \int_{x'=0}^a T_{tc, cal}(x', d, t-u) dx', \quad (5.14b)$$

$$K_{tc, run}(t-u) = \int_{x'=0}^a T_{tc, run}(x', d, t-u) dx'. \quad (5.14c)$$

Here, $K_{tc,cal}$ and $K_{tc,run}$ are obtained by numerical integration. The mid-point rule is applied in this context for discretizing space as

$$K_{tc,cal}(t-u) = \int_{x'=0}^a T_{tc,cal}(x', d, t-u) dx' = \sum_{k=1}^{N_x} T_{tc,cal}(x_k, d, t-u) \Delta x, \quad (5.15b)$$

$$K_{tc,run}(t-u) = \int_{x'=0}^a T_{tc,run}(x', d, t-u) dx' = \sum_{k=1}^{N_x} T_{tc,run}(x_k, d, t-u) \Delta x. \quad (5.15c)$$

where x_k denotes the k^{th} thermocouple position, Δx denotes the distance between each thermocouples as indicated in Figs. 4.1 or 5.1. Here, N_x denotes the total number of thermocouples placed along x at $y=d$.

5.2 Regularization Based on Local Future Time Method and the Metrics to Determine optimal γ

Eq. (5.14a) is Volterra integral equation of the first kind [83,84]. As in Section 3.2, Lamm's local future time method [8,85] is applied to stabilize the equation. The algorithm and inverse code for the two-dimensional total surface heat transfer calibration integral equation method are the same as the one-dimensional surface heat flux calibration integral equation method described in Chapter 3 except for the kernels of the integral equations are different.

The two-dimensional total surface heat transfer calibration integral equation in Eq. (5.14a) is alternatively express as

$$\int_{u=0}^t Q_{run}(b, u) K_{tc,cal}(t-u) du = F(t), \quad t \geq 0, \quad (5.16a)$$

where

$$F(t) = \int_{u=0}^t Q_{cal}(b, u) K_{tc,run}(t-u) du, \quad t \geq 0. \quad (5.16b)$$

We introduce the future time parameter γ , by advancing time through $t \rightarrow t + \gamma$ (γ has units of time), then Eq. (5.16a) becomes

$$\int_{u=0}^{t+\gamma} Q_{run}(b, u) K_{tc,cal}(t+\gamma-u) du = F(t+\gamma), \quad t \in [0, t_{\max} - \gamma]. \quad (5.17)$$

Observe that the time domain is reduced by γ . The basic integral definition [91-93] allows us to express Eq. (5.17) as

$$\int_{u=0}^t Q_{run}(b, u) K_{tc,cal}(t+\gamma-u) du + \int_{u=t}^{t+\gamma} Q_{run}(b, u) K_{tc,cal}(t+\gamma-u) du = F(t+\gamma), \quad t \in [0, t_{\max} - \gamma]. \quad (5.18)$$

Define $v=u-t$, then the second integral on the left-hand side of Eq. (5.18) can be written as

$$\int_{u=0}^t Q_{run}(b, u) K_{tc,cal}(t+\gamma-u) du + \int_{v=0}^{\gamma} Q_{run}(b, v+t) K_{tc,cal}(\gamma-v) dv = F(t+\gamma), \quad t \in [0, t_{\max} - \gamma]. \quad (5.19)$$

If γ is small then we can approximate $Q_{run}(b, v+t) \approx Q_{run}(b, t)$ since $v \in [0, \gamma]$ for small γ . If this is the case, then we can form the approximation

$$\int_{u=0}^t Q_{run}(b, u) K_{tc,cal}(t+\gamma-u) du + Q_{run}(b, t) \int_{v=0}^{\gamma} K_{tc,cal}(\gamma-v) dv \approx F(t+\gamma), \quad t \in [0, t_{\max} - \gamma], \quad (5.20a)$$

or

$$\int_{u=0}^t Q_{run,\gamma}(b,u)K_{tc,cal}(t+\gamma-u)du + Q_{run,\gamma}(b,t)\int_{v=0}^{\gamma} K_{tc,cal}(\gamma-v)dv = F(t+\gamma), \quad t \in [0, t_{\max} - \gamma], \quad (5.20b)$$

or

$$\int_{u=0}^t Q_{run,\gamma}(b,u)K_{tc,cal}(t+\gamma-u)du + Q_{run,\gamma}(b,t)C_{Q,\gamma} = F(t+\gamma), \quad t \in [0, t_{\max} - \gamma], \quad (5.20c)$$

where

$$C_{Q,\gamma} = \int_{v=0}^{\gamma} K_{tc,cal}(\gamma-v)dv. \quad (5.20d)$$

Equation (5.20a) is now a second kind Volterra integral equation of the second kind for the unknown total surface heat transfer denoted by $Q_{run,\gamma}(b,u)$. Here $Q_{run,\gamma}(b,u)$ is an approximation to $Q_{run}(b,u)$ and depends on γ . Data are collected up to some time defined as $t_{\max}$. Future information is incorporated into the numerical algorithm. The predicted total surface heat transfer rate $Q_{run,\gamma}(b,u)$ can only be resolved in temporal domain $t \in [0, t_{\max} - \gamma]$ due to the inclusion of future data to stabilize the numerical implementation.

Discretization of Eqs. (5.20b-d) can be performed using a variety of low-order numerical integration rules, such as the rectangular rule, trapezoidal rule, product integration rules [84], etc. Again, a right-handed rectangular rule is employed in the present context. The discrete values for $Q_{run,\gamma}(b,u)$ can be obtained in the time-marching form where $\gamma \rightarrow \gamma_m = mM_f\Delta t$, $m = 1, 2, 3, \dots$, where Δt is given as $\Delta t = t_{\max}/N$, N is the number of segments (or samples) in the discretized temporal domain, and M_f is a multiplying factor. Discretizing Eq. (5.20b) produces

$$\sum_{j=1}^i Q_{run,\gamma_m}(b,t_j) \int_{u=t_{j-1}}^{t_j} K_{tc,cal}(t_i + \gamma_m - u)du + Q_{run,\gamma_m}(b,t_i) \int_{u=0}^{\gamma_m} K_{tc,cal}(\gamma_m - u)du = F(t_i + \gamma_m), \quad i = 1, 2, \dots, M, \quad (5.21a)$$

where $M = N - mM_f$, or upon extracting the desired total heat transfer at $t = t_i$ we obtain

$$Q_{run,\gamma}(b,t_i) = \frac{F(t_i + \gamma_m) - \sum_{j=1}^{i-1} Q_{run,\gamma}(b,t_j) \int_{u=t_{j-1}}^{t_j} K_{tc,cal}(t_i + \gamma_m - u)du}{\int_{u=t_{i-1}}^{t_i} K_{tc,cal}(t_i + \gamma_m - u)du + \int_{u=0}^{\gamma_m} K_{tc,cal}(\gamma_m - u)du}, \quad i = 1, 2, \dots, M. \quad (5.21b)$$

where $t_j = j\Delta t$, $j=0, 1, \dots, M$. In this time-marching implementation, the future time parameter γ_m must be specified over a range of values and then interrogated until an optimal value is determined based on some measure.

In the present context, Eq. (5.16) is the exact formulation of the proposed total surface heat transfer rate calibration integral equation. Equation (5.20b) is an approximation since a constant surface heat transfer rate is assumed in the future time interval $[t, t + \gamma]$ in order to convert the first kind Volterra integral equation into well-posed second kind Volterra integral equation. If the future time parameter γ is too small then the prediction via Eq. (5.20b) is unstable. If the future time parameter γ is too large then over smoothing of the prediction will transpire as high frequencies are damped. Hence, significant surface physics can be lost. Compromise is required between stability and accuracy. To determine the optimal future time parameter γ , the residual function $R_\gamma(t)$ is defined as

$$R_\gamma(t) = \int_{u=0}^t Q_{run,\gamma}(b,u)K_{tc,cal}(t-u)du - F(t), \quad t \in [0, t_{\max} - \gamma]. \quad (5.22)$$

As with the previous chapter, we define the running average of residual is defined as

$$\bar{R}_\gamma(t) = \frac{1}{t} \int_{u=0}^t R_\gamma(u) du, \quad t \in [0, t_{\max} - \gamma]. \quad (5.23)$$

Both $R_\gamma(t)$ and $\bar{R}_\gamma(t)$, from Eqs. (5.22) and (5.23), respectively, can be discretized by a simple and consistent integration rule. The right-handed rectangular rule is employed herein as with Eqs. (5.21a,b).

The time-running variance of residual function is defined through

$$\sigma_{R,\gamma}^2(t_i) = \frac{1}{i} \sum_{j=1}^i [R_\gamma(t_j) - \bar{R}_\gamma(t_i)]^2, \quad i=1, 2, \dots, M. \quad (5.24)$$

A second measure based on normalized version of Eq. (5.24) can be defined to emphasis or enhance other effects. Following this logic, we defined the normalized square root of time-running variance of the residual as

$$\sigma_{R,\gamma}^*(t_i) = \frac{\sigma_{R,\gamma}(t_i)}{\|\sigma_{R,\gamma}(t_i)\|_\infty}, \quad i=1, 2, \dots, M. \quad (5.25)$$

where $\|\varphi\|_\infty$ is the maximum norm given as $\max_{1 \leq i \leq M} |\sigma_\gamma(t_i)|$. These four metrics assist in estimating the optimal future time parameter γ .

5.3 Numerical Simulation using Copper as the Host Material

In this section, the host material of slab in the simulation for the two-dimensional total surface heat transfer calibration integral equation method is copper with thermal conductivity $k=401$ W/mK and thermal diffusivity $\alpha=117 \times 10^{-6}$ m²/s. The length of slab is $a=12$ cm, the height of slab is $b=1$ cm and possesses unit depth (1 cm). The convective heat transfer coefficient used in the simulation is $h=100$ W/m²K. The thermocouple is positioned at $d=5$ mm. The sampling frequency f_{sampling} used in the simulation is 200Hz while the total simulation time is $t_{\max}=10$ s.

The subjected “calibration” surface heat flux used in the slab is uniform in the spatial domain and constant in the temporal domain. That is, we assign $q_{\text{cal}}''(x, b, t) = 100$ W/cm² as shown in Fig. 5.2. The “calibration” total surface heat transfer that is the spatially integrated value of “calibration” surface heat flux is $Q_{\text{cal}}(b, t) = 100$ W/cm² $\times$ 12 cm $\times$ 1 cm = 1200 W. In contrast, the “run” surface heat flux is given as a parabolic function in the spatial domain and Gaussian function in the temporal domain. It is mathematically expressed as

$$q_{\text{run}}''(x, b, t) = 1000 x^{0.5} e^{-\frac{(t-A)^2}{B^2}} \quad (\text{W/cm}^2), \quad (5.26a)$$

where

$$A = t_{\max}/2, \quad (5.26b)$$

$$B = t_{\max}/8, \quad (5.26c)$$

and shown in Fig. 5.3. The exact total surface heat transfer in the “run” stage is the spatially integrated value of the “run” surface heat flux along the entire surface. Performing this integration yields

$$Q_{\text{run}, \text{exact}}(b, t) = \int_{x=0}^a q_{\text{run}}''(x, b, t) dx \times 1 \text{ cm} = \int_{x=0}^a 1000 x^{0.5} e^{-\frac{(t-A)^2}{B^2}} dx \times 1 \text{ cm} = \frac{2000}{3} x^{3/2} e^{-\frac{(t-A)^2}{B^2}} (W). \quad (5.26d)$$

Artificial temperature noise is added to the probe site thermocouple data in accordance to

$$T_{ic}(x_k, d, t_j) = T(x_k, d, t_j) + \varepsilon_T r_j \| T(x_k, d, t_j) \|_{\infty}, \quad k = 1, 2, \dots, N_x, \quad j = 1, 2, \dots, N, \quad (5.27)$$

for both calibration “run” and test “run” stages. Here ε_T denotes noise factor, r_j denotes random numbers in the interval $[-1, 1]$ which are obtained by MATLAB command “ $2*(rand-0.5)$ ”. For the calibration “run”, in terms of numerical simulation in this context, the total surface heat transfer is obtained by numerical integration. Noise is also added to the total surface heat transfer as

$$Q_{cal}(b, t_j) = Q_{cal, exact}(b, t_j) + \varepsilon_Q r_j \| Q_{cal, exact}(b, t_j) \|_{\infty}, \quad k = 1, 2, \dots, N_x, \quad j = 1, 2, \dots, N. \quad (5.28)$$

The parameters and properties for the two-dimensional total heat transfer calibration integral equation method for copper are collected in Table 5.1.

Figure 5.4 displays the exact total surface heat transfer $Q_{cal, exact}(b, t)$ and noise introduced total surface heat transfer $Q_{cal}(b, t_j)$ in the “calibration” stage. Figure 5.5 displays the surface temperature in the “calibration” stage and is denoted as $T_{cal}(x, b, t)$. The positional temperature at the thermocouple site is denoted as $T_{cal}(x, d, t)$ and the thermocouple temperature is denoted as $T_{ic, cal}(x, d, t_j)$, in the “calibration” stage. These are displayed in Fig. 5.6. Temperature in the “calibration” stage is uniform in the spatial domain due to the uniform “calibration” surface heat flux. Figure 5.7 shows the kernel $K_{ic, cal}(t)$ in the “calibration” stage defined in Eq. (5.15b).

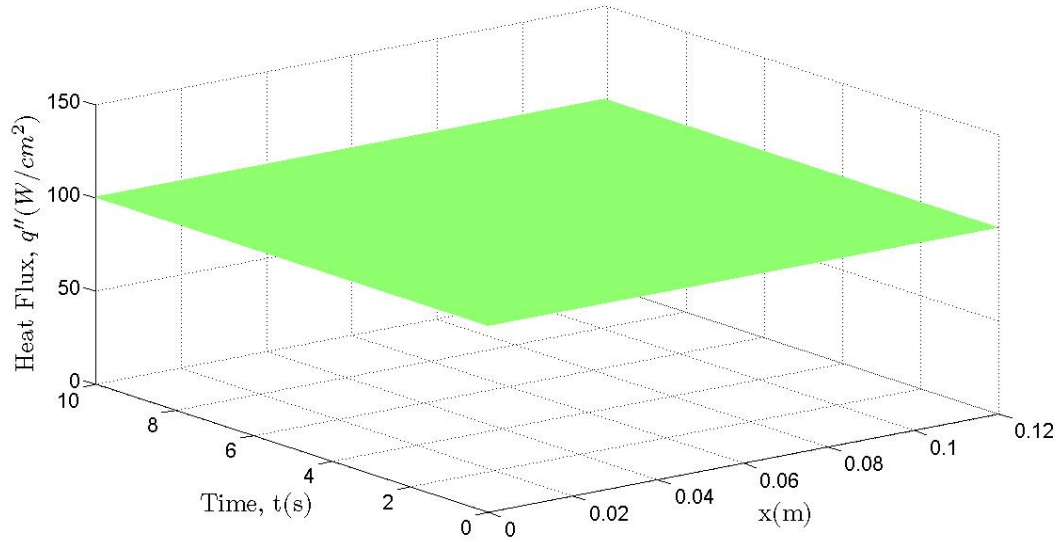


Figure 5.2: The surface heat flux in the “calibration” stage, $q''_{cal}(x, b, t)$ (using copper as the host material for the numerical simulation).

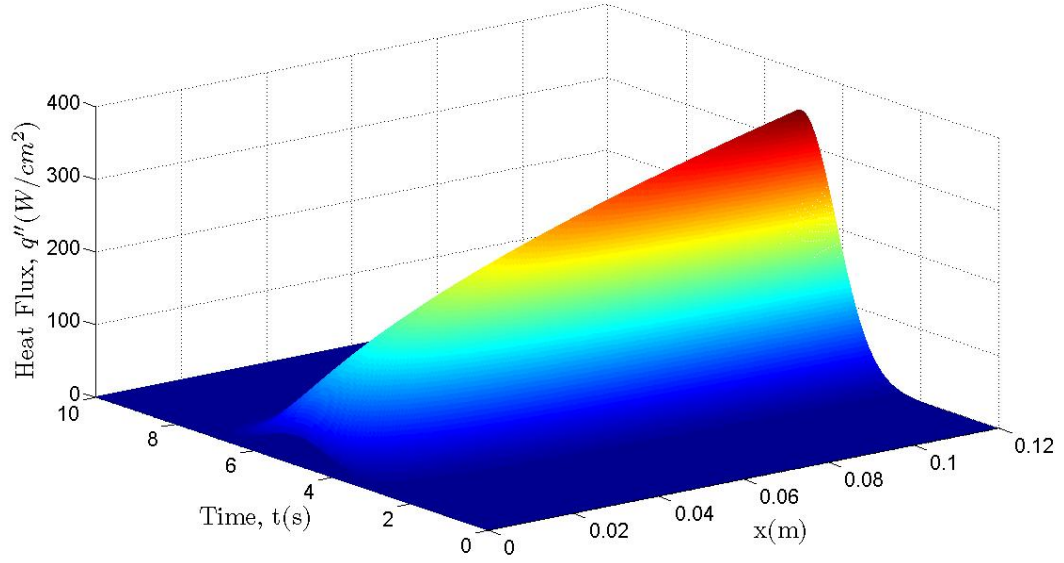


Figure 5.3: The surface heat flux in the “run” stage, $q''_{run}(x, b, t)$ (using copper as the host material for the numerical simulation).

Table 5.1: Properties and parameters for two-dimensional total heat transfer calibration integral equation method simulation using copper as the host material

Property / Parameter	Value (units)
k	401 W/mK
α	$117 \times 10^{-6} \text{ m}^2/\text{s}$
h	100 W/m ² K
a	12 cm
b	1 cm
d	0.5cm
ε_T	0.01
ε_Q	0.01
$f_{sampling}$	200 Hz
N	2000
t_{max}	10 s
Δt	0.005
M_f	10
N_x	6

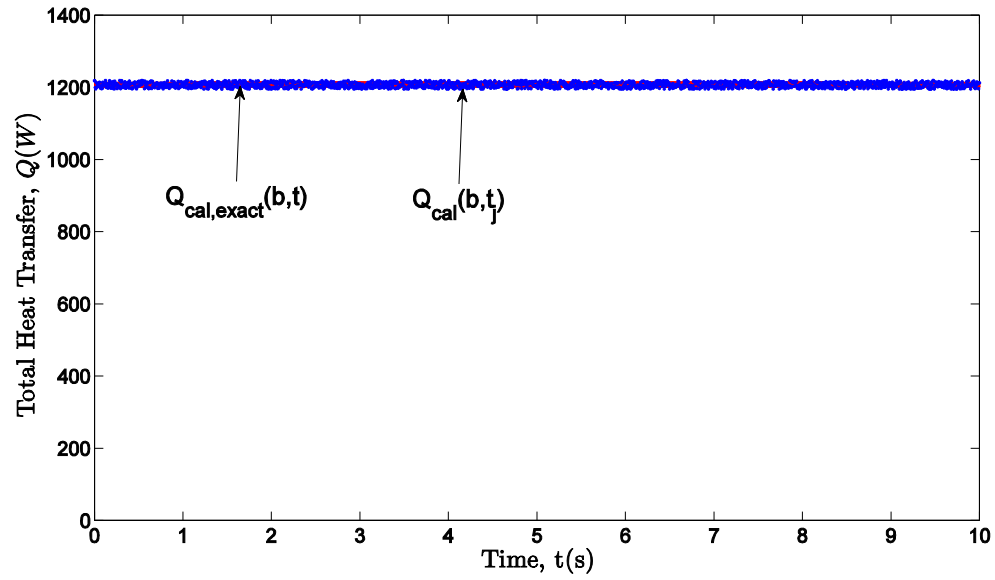


Figure 5.4: The exact total surface heat transfer $Q_{cal,exact}(b,t)$ and the noise added total surface heat transfer $Q_{cal}(b,t_j)$ in the “calibration” stage (using copper as the host material for the numerical simulation).

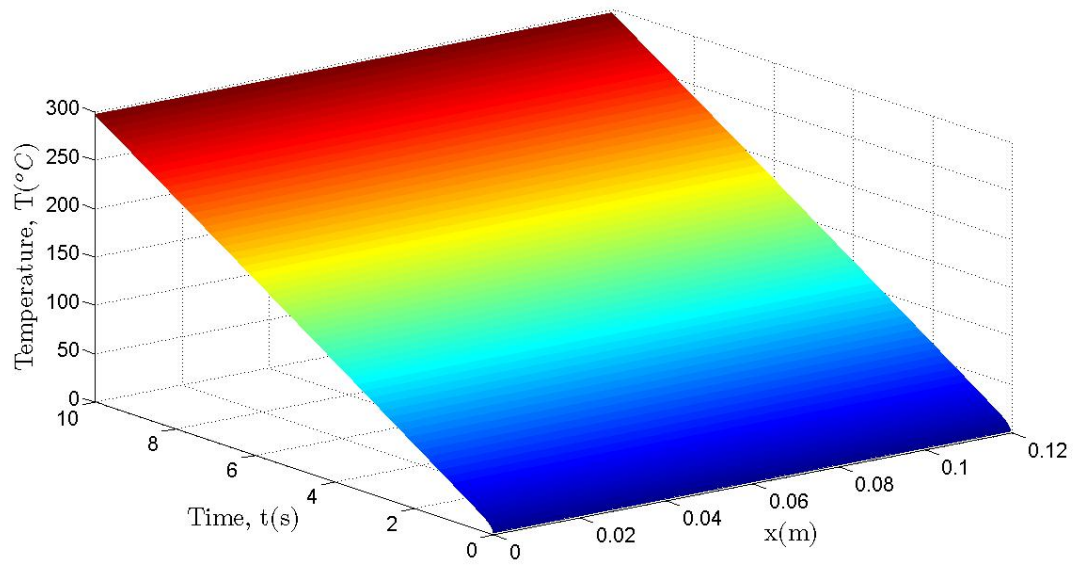


Figure 5.5: The surface temperature in the “calibration” stage, $T_{cal}(x,b,t)$ (using copper as the host material for the numerical simulation).

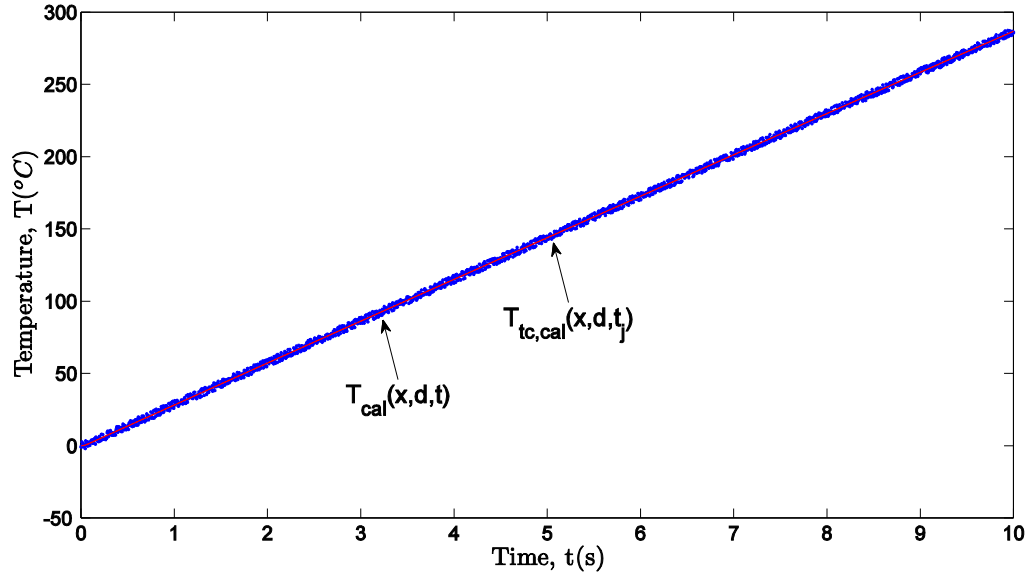


Figure 5.6: The positional temperature at thermocouple site $T_{cal}(x,d,t)$ and the thermocouple temperature $T_{tc,cal}(x,d,t_j)$, respectively, in the “calibration” stage. Temperature in the “calibration” stage is uniform in spatial domain (when using copper as the host material for the numerical simulation).

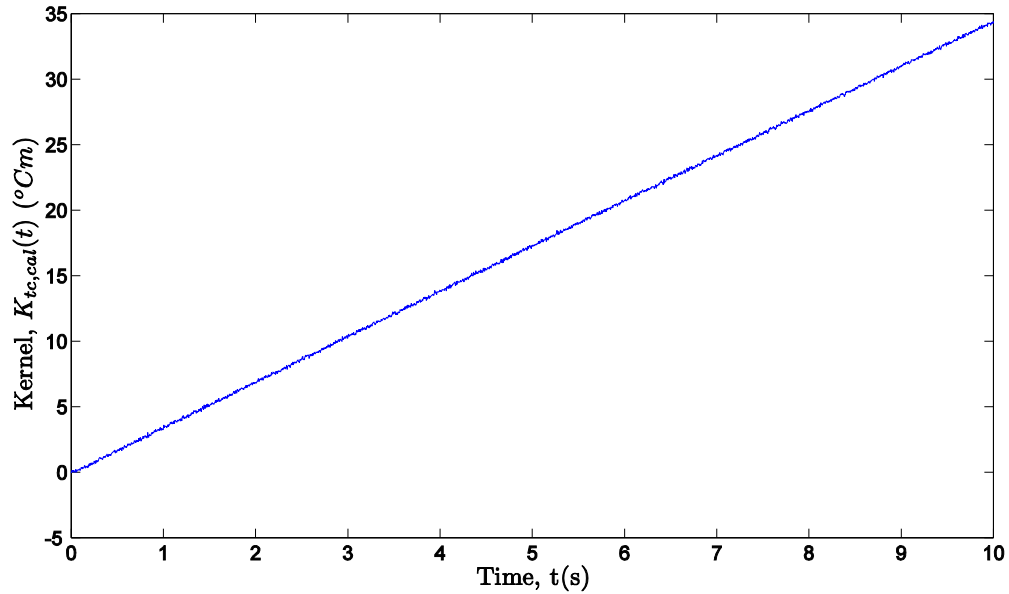


Figure 5.7: The kernel $K_{tc,cal}(t)$ in the “calibration” stage defined by Eq. (5.15b) (using copper as the host material for the numerical simulation).

Figure 5.8 displays the exact total surface heat transfer $Q_{run,exact}(b,t)$ in the “run” stage to be reconstructed that is mathematically denoted in Eq. (5.26d). Figure 5.9 displays the surface temperature in the “run” stage which is denoted as $T_{run}(x,b,t)$. The positional temperature at thermo-

couple site is denoted as $T_{cal}(x,d,t)$ and the thermocouple temperature is denoted as $T_{tc,run}(x,d,t_j)$, in the “run” stage. Both are displayed in Fig. 5.10. Figure 5.11 shows the kernel $K_{tc,run}(t)$ in the “run” stage defined in Eq. (5.15c).

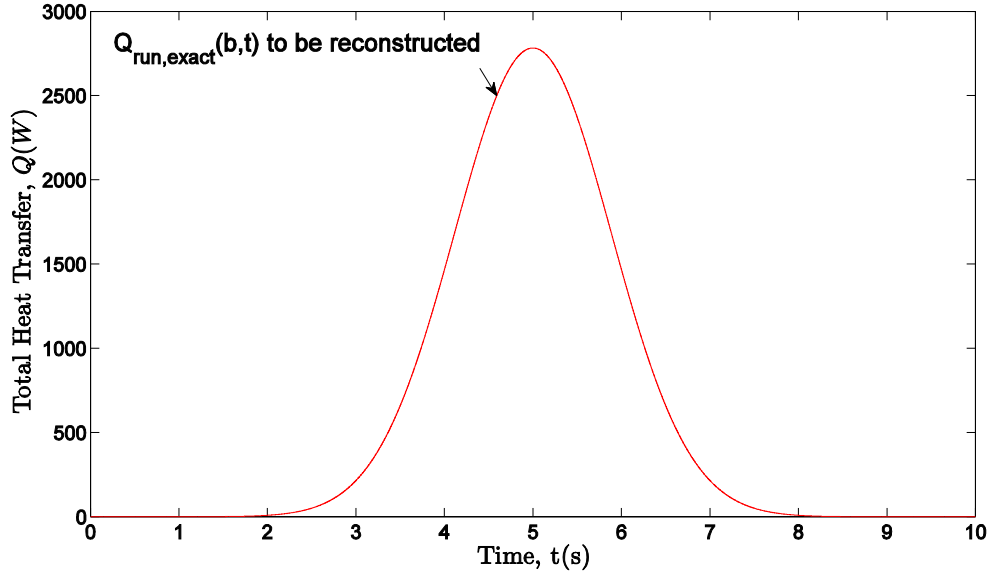


Figure 5.8: The exact “run” total surface heat transfer rate $Q_{run,exact}(b,t)$ to be reconstructed (using copper as the host material for the numerical simulation).

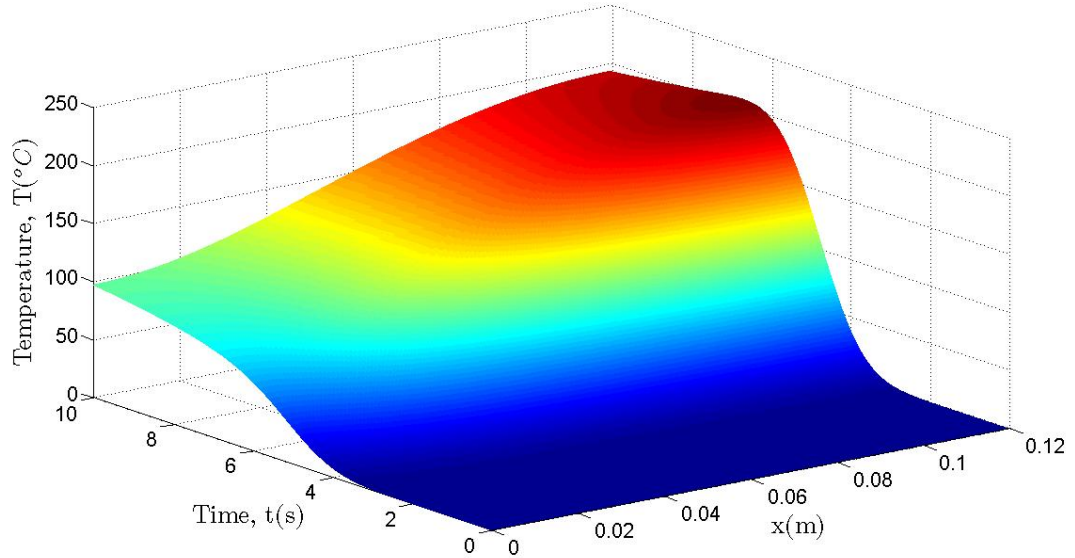


Figure 5.9: The surface temperature in the “run” stage, $T_{run}(x,b,t)$ (when using copper as the host material for the numerical simulation).

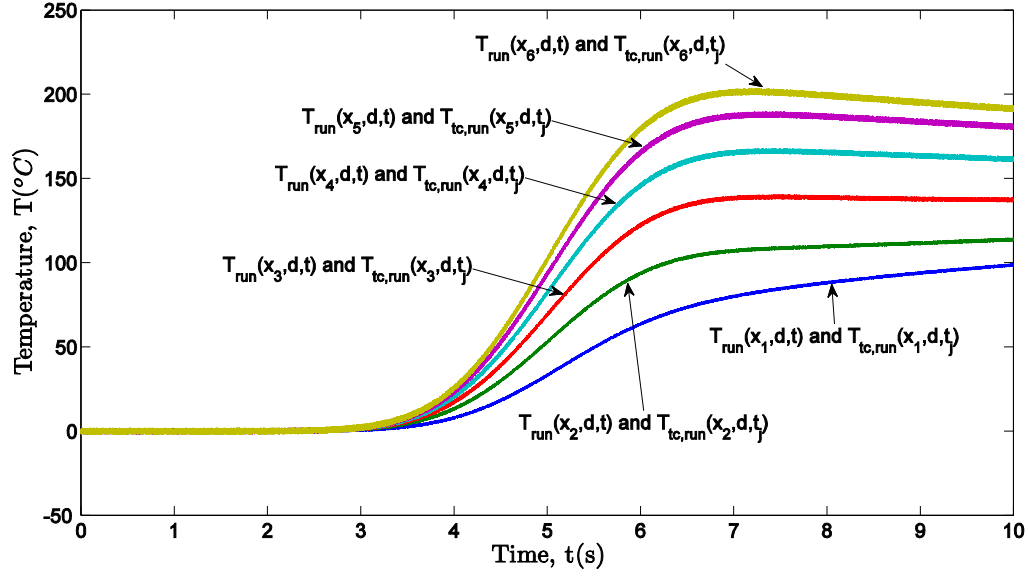


Figure 5.10: The positional temperature at various thermocouple sites $T_{run}(x_k, d, t)$ ($k=1, 2, \dots, 6$) and the thermocouple temperature $T_{tc,run}(x_k, d, t_j)$, respectively, in the “run” stage (when using copper as the host material for the numerical simulation).

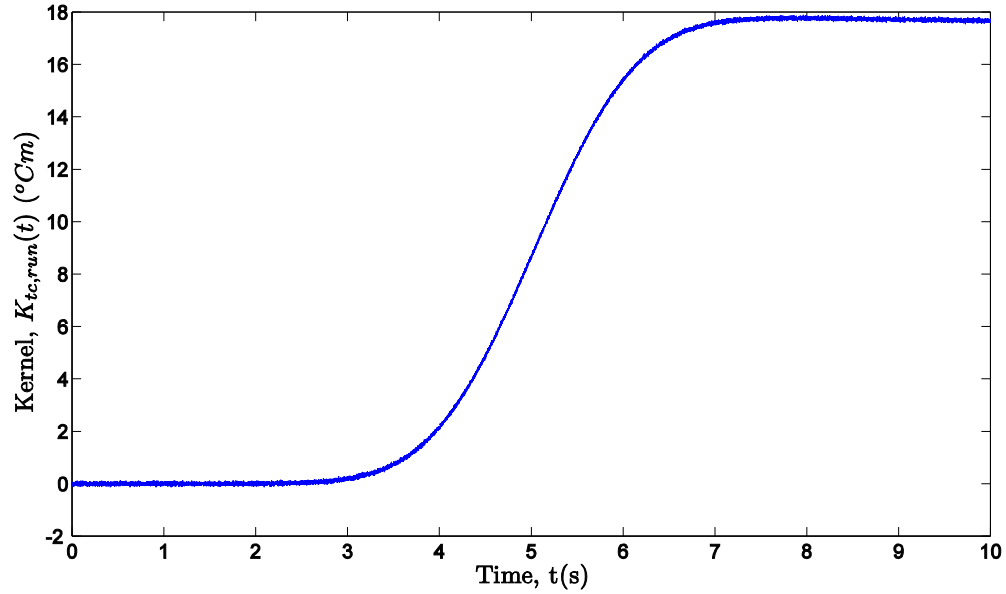


Figure 5.11: The kernel $K_{tc,run}(t)$ in the “run” stage defined by Eq. (5.15c) (using copper as the host material for the numerical simulation).

The metrics defined in Section 5.2 are employed here for determining the optimal future time parameter γ . Figure 5.12 displays the local residual function, $R_\gamma(t)$, defined in Eq. (5.22) over time for increasing future time parameter γ based on the “calibration” and “run” thermocouple temperature measurement as well as the local “calibration” total heat transfer and the predicted “run” total heat transfer. As the future time parameter increases, the residual’s magnitude is shown to increase. Figure 5.13 shows the corresponding running average of the residual. Clearly the smoothing operation of integration is seen in Fig. 5.13. Figure 5.14 displays the time-running variance of the residual function defined in Eq. (5.24) while Fig. 5.15 displays the normalized version of Fig. 5.14. In this form (Fig. 5.15), both instability and over-smoothing effects can be quickly identified. Small γ ’s produce unstable predictions (see for example $\gamma = 0.10s$) while large γ ’s produce over-smoothed predictions (see for example $\gamma = 0.5s$) of the total heat transfer. The normalized plot possesses key physical effects. In fact, a bundling effect occurs for $\gamma = 0.20-0.30s$ for times, $t > 7.5s$. This range of γ -values describes the best prediction based on the balance between bias (data) and variance (model). This γ -range of the spectrum is where the optimal prediction for the total surface heat transfer will lie. In fact, it is desired to retain as much of the high frequency content in the signal for physics recovering. Hence, the optimal estimator is chosen as $\gamma_{opt} = 0.20s$. The highly favorable predicted time history of the “run” total surface heat transfer based on $\gamma_{opt} = 0.20s$ is presented in Fig. 5.16.

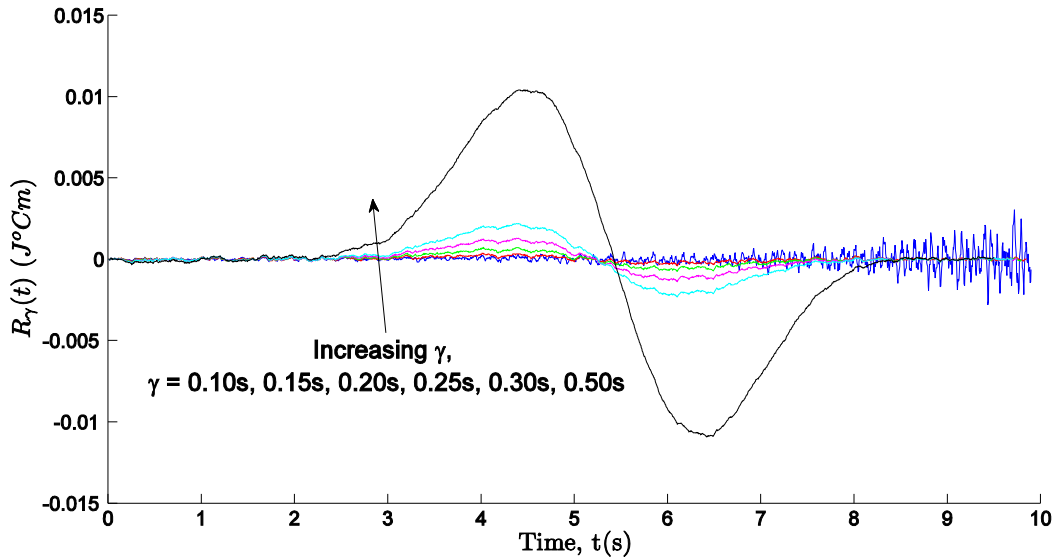


Figure 5.12: The local residual function, $R_\gamma(t)$ defined in Eq. (5.22) over the γ -spectrum (using copper as the host material for the numerical simulation).

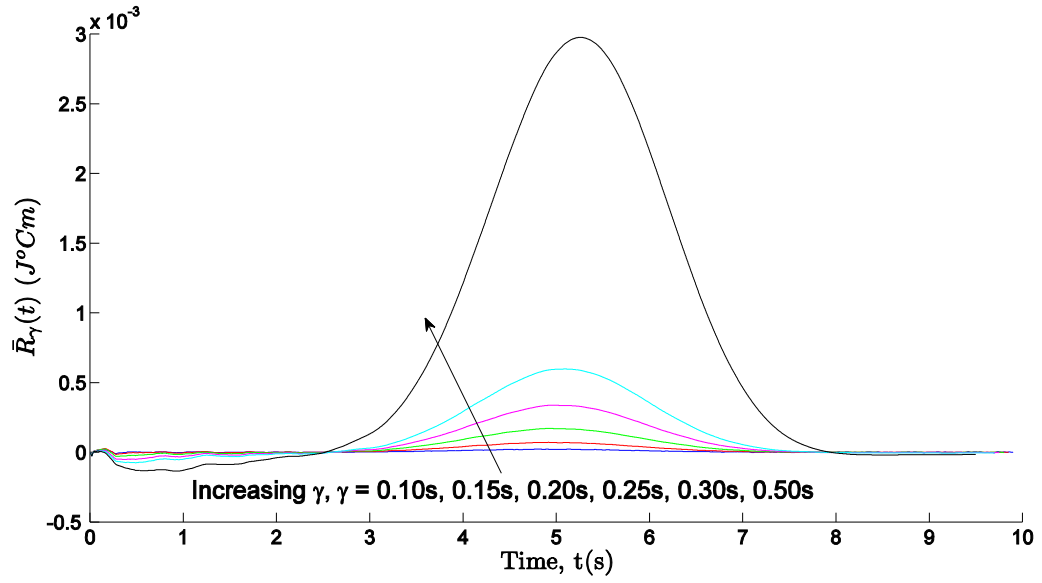


Figure 5.13: The running average of residual function, $\bar{R}_\gamma(t)$ defined in Eq. (5.23) over the γ -spectrum (using copper as the host material for the numerical simulation).

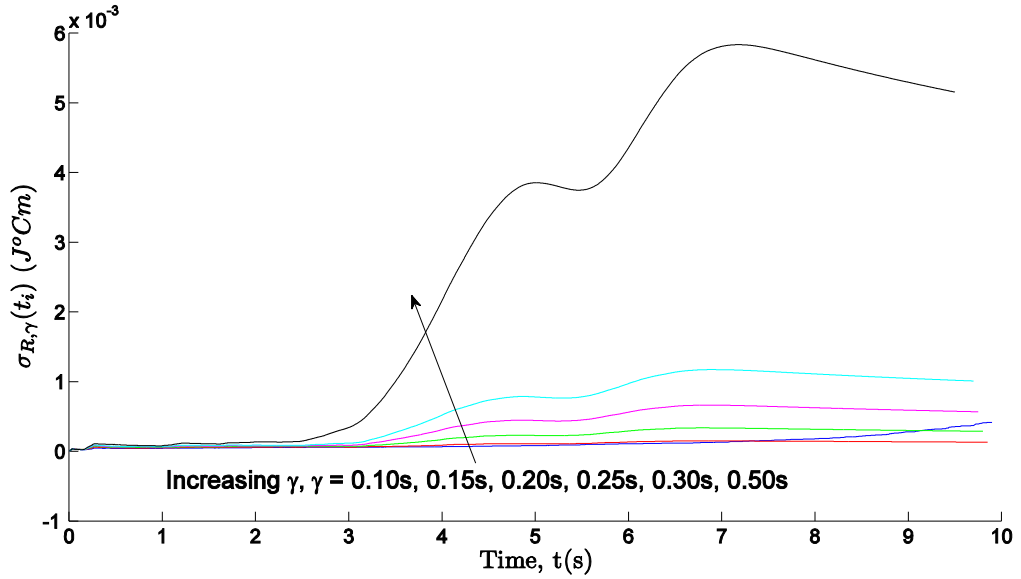


Figure 5.14: The square root of time-running variance of residual, $\sigma_\gamma(t_i)$ defined in Eq. (5.24) over the γ -spectrum (using copper as the host material for the numerical simulation).

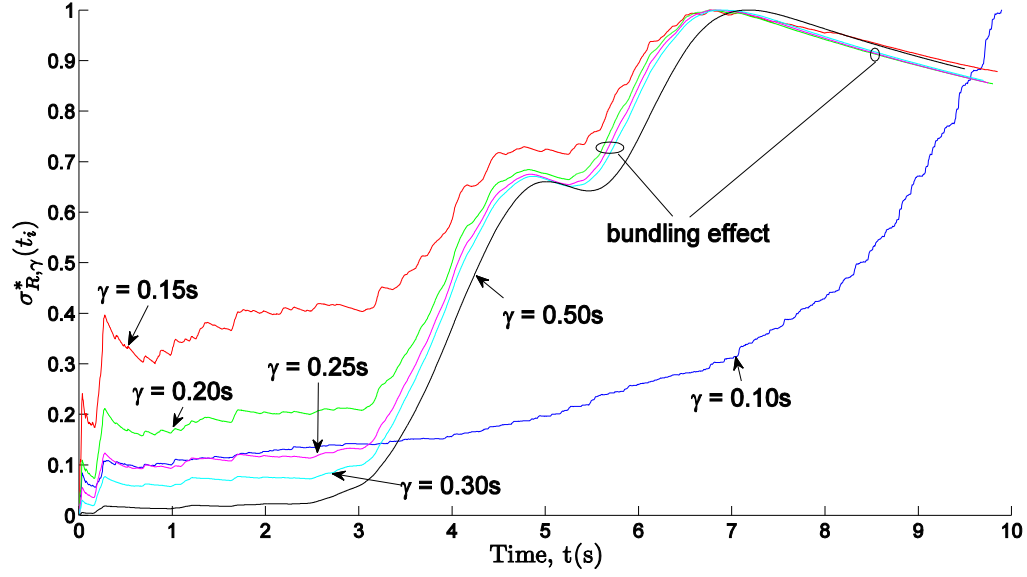


Figure 5.15: The normalized square root of time-running variance of residual, $\sigma_{\gamma}^*(t_i)$ defined in Eq. (5.23) over the γ -spectrum (using copper as the host material for the numerical simulation).

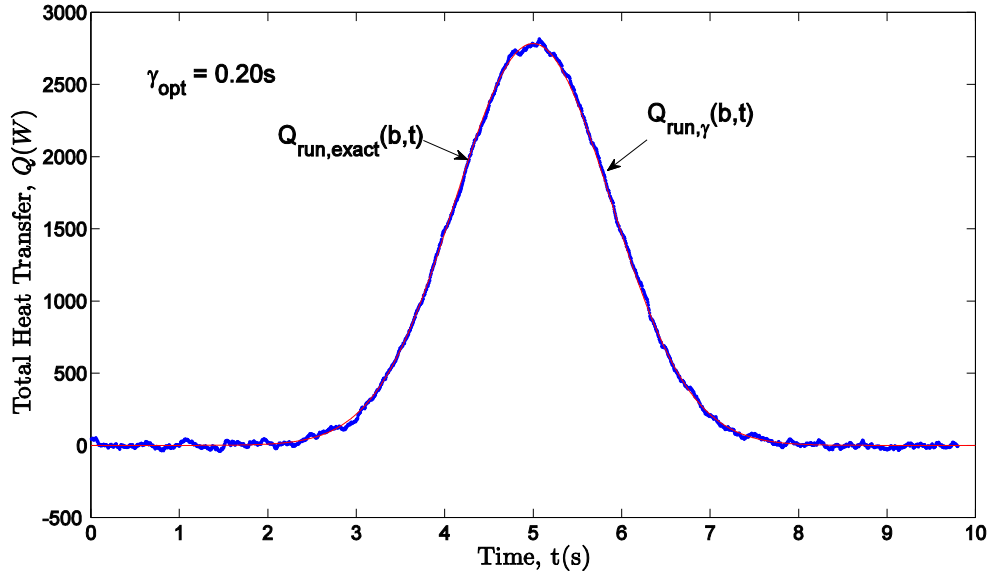


Figure 5.16: Comparison between the predicted total “run” surface heat transfer $Q_{run,\gamma}(b,t)$ using $\gamma_{opt} = 0.20s$ and the exact total “run” surface heat transfer $Q_{run,exact}(b,t)$ (using copper as the host material for the numerical simulation).

5.4 Numerical Simulation Using Stainless Steel as the Host Material

In this section, the host material of slab in the simulation for the two-dimensional total surface heat transfer calibration integral equation method is stainless steel with thermal conductivity $k=14.7$ W/mK, thermal diffusivity $\alpha=3.75 \times 10^{-6}$ m²/s. The length of slab is $a=12$ cm, the height of slab is $b=1$ cm and possesses unit depth (1 cm). The convective heat transfer coefficient used in the simulation is $h=100$ W/m²K. The thermocouple is located at $d=5$ mm. The sampling frequency f_{sampling} used in the simulation is 50Hz while the total simulation time is $t_{\text{max}}=40$ s.

The “calibration” surface heat flux used in the slab is uniform in the spatial domain and constant in the temporal domain. That is, we assign $q''_{\text{cal}}(x, b, t) = 10$ W/cm² as shown in Fig. 5.17. The “calibration” total surface heat transfer that is the spatially integrated value of “calibration” surface heat flux is $Q_{\text{cal}}(b, t) = 10$ W/cm² $\times$ 12 cm $\times$ 1 cm = 120 W. In contrast, the “run” surface heat flux is given as a parabolic function in the spatial domain and Gaussian function in the temporal domain. It is mathematically expressed as

$$q''_{\text{run}}(x, b, t) = 100 x^{0.5} e^{-\frac{(t-A)^2}{B^2}} \text{ (W/cm}^2\text{)}, \quad (5.29a)$$

where

$$A = t_{\text{max}}/2, \quad (5.29b)$$

$$B = t_{\text{max}}/8, \quad (5.29c)$$

and shown in Fig. 5.18. The exact total surface heat transfer in the “run” stage is the spatially integrated value of the “run” surface heat flux along the entire surface. Performing this integration produces

$$Q_{\text{run, exact}}(b, t) = \int_{x=0}^a q''_{\text{run}}(x, b, t) dx \times 1 \text{ cm} = \int_{x=0}^a 100 x^{0.5} e^{-\frac{(t-A)^2}{B^2}} dx \times 1 \text{ cm} = \frac{200}{3} x^{3/2} e^{-\frac{(t-A)^2}{B^2}} (W). \quad (5.29d)$$

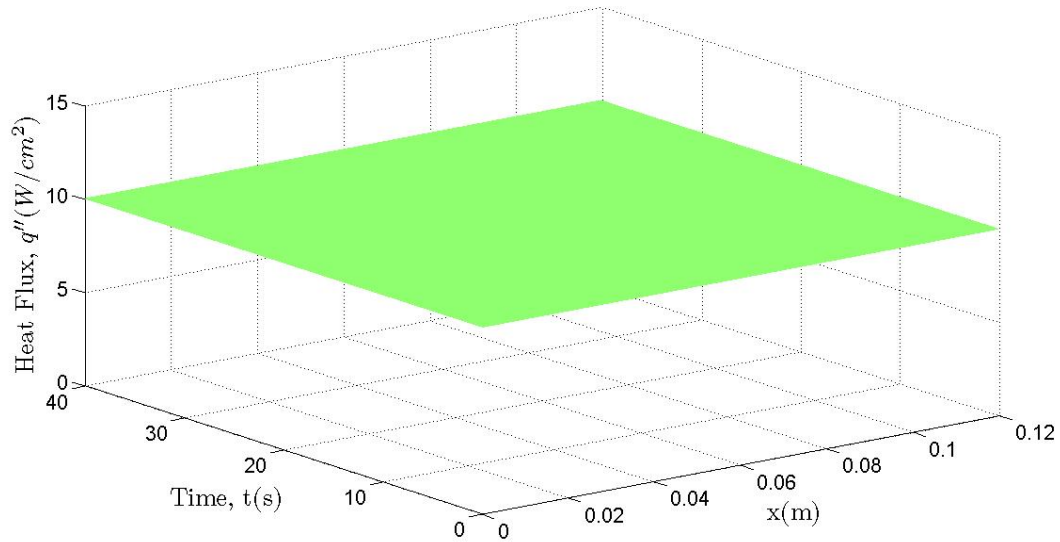


Figure 5.17: The surface heat flux in the “calibration” stage, $q''_{\text{cal}}(x, b, t)$ (using stainless steel as the host material for the numerical simulation).

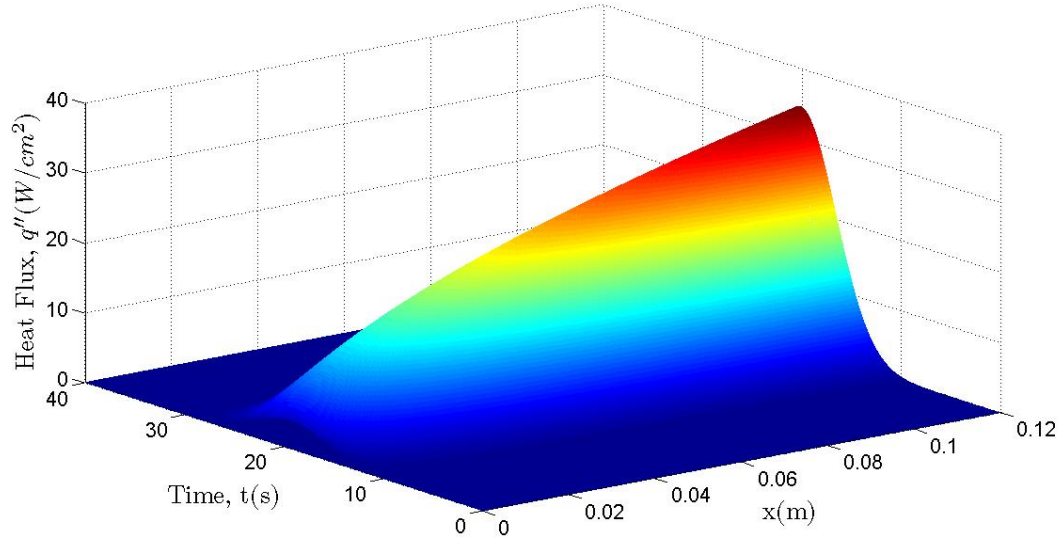


Figure 5.18: The surface heat flux in the “run” stage, $q''_{run}(x, b, t)$ (using stainless steel as the host material for the numerical simulation).

As in Section 5.3, noise is added to thermocouple temperature in both “calibration” and “run” stages as well as the total heat transfer in the “calibration” stage. The parameters and properties for the two-dimensional total heat transfer calibration integral equation method investigation using stainless steel are collected in Table 5.2.

Table 5.2: Properties and parameters for two-dimensional total heat transfer calibration integral equation method study using stainless steel as the host material

Property / Parameter	Value (units)
k	14.7 W/mK
α	$3.75 \times 10^{-6} \text{ m}^2/\text{s}$
h	100 W/m ² K
a	12 cm
b	1 cm
d	0.5cm
ε_T	0.01
ε_Q	0.01
$f_{sampling}$	50 Hz
N	2000
t_{max}	40 s
Δt	0.02
M_f	10
N_x	6

As previous presented, in the context of copper, we display a similar series of plots describing various feathers of temperature, heat flux, kernels and predictions using stainless steel as the host material. As before, Fig. 5.19-5.26 physically conform to the copper results previously organized and discussed.

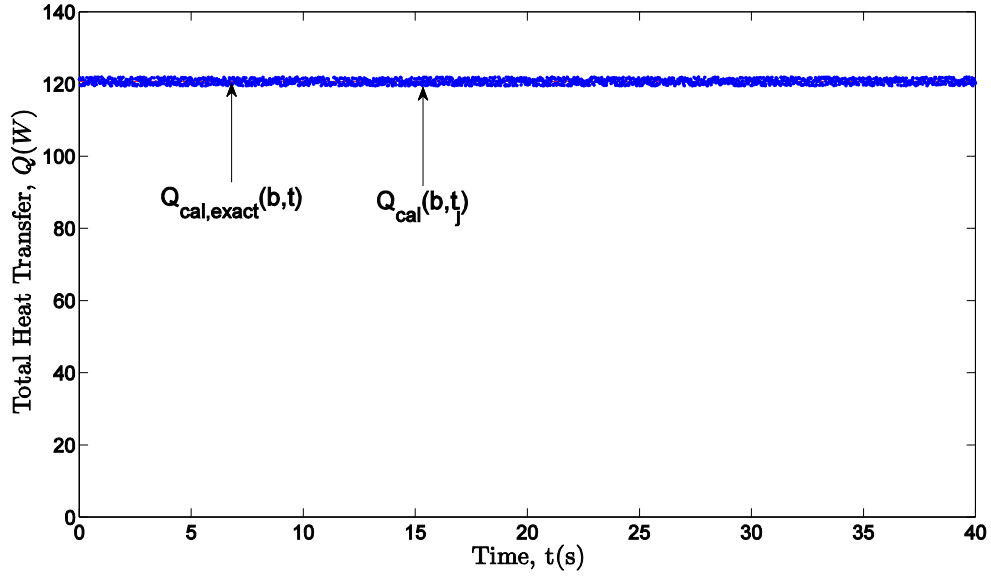


Figure 5.19: The exact total surface heat transfer $Q_{cal,exact}(b,t)$ and the noise added total surface heat transfer $Q_{cal}(b,t_j)$ in the “calibration” stage (using stainless steel as the host material for the numerical simulation).

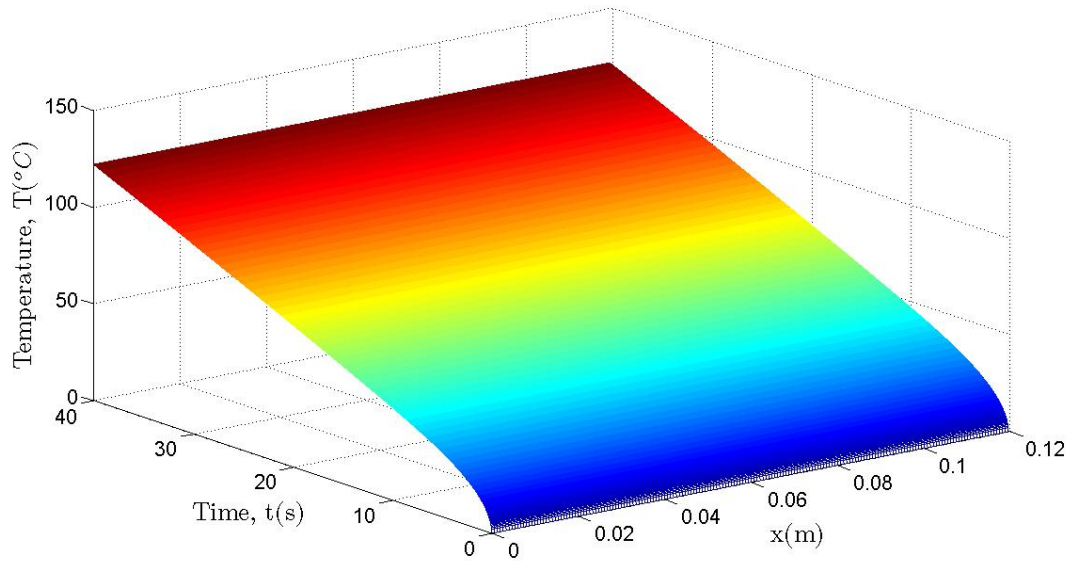


Figure 5.20: The surface temperature in the “calibration” stage, $T_{cal}(x,b,t)$ (using stainless steel as the host materials for the numerical simulation).

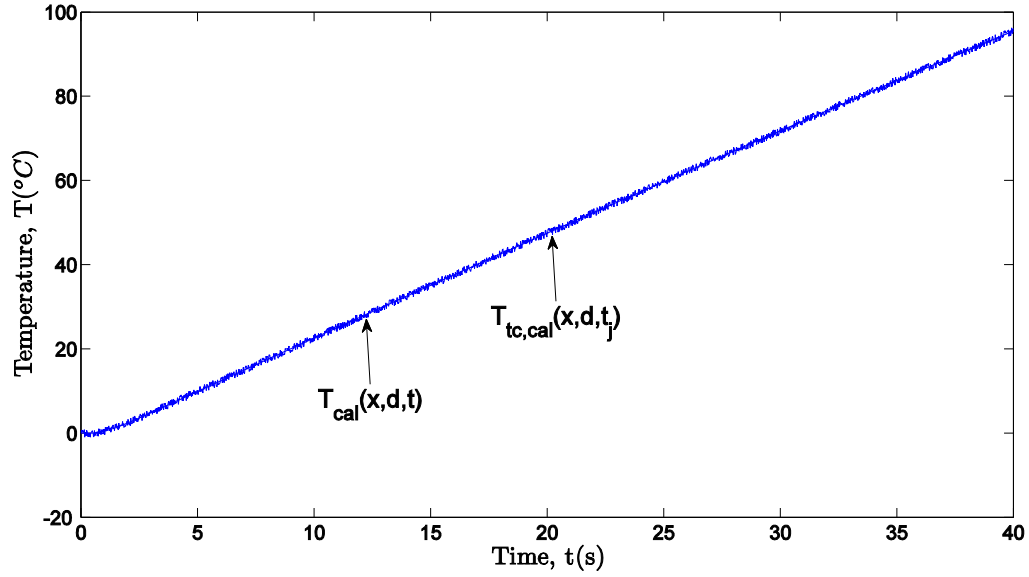


Figure 5.21: The positional temperature at thermocouple site $T_{cal}(x,d,t)$ and the thermocouple temperature $T_{tc,cal}(x,d,t_j)$, respectively, in the “calibration” stage. Temperature in the “calibration” stage is uniform in spatial domain (using stainless steel as the host material for the numerical simulation).

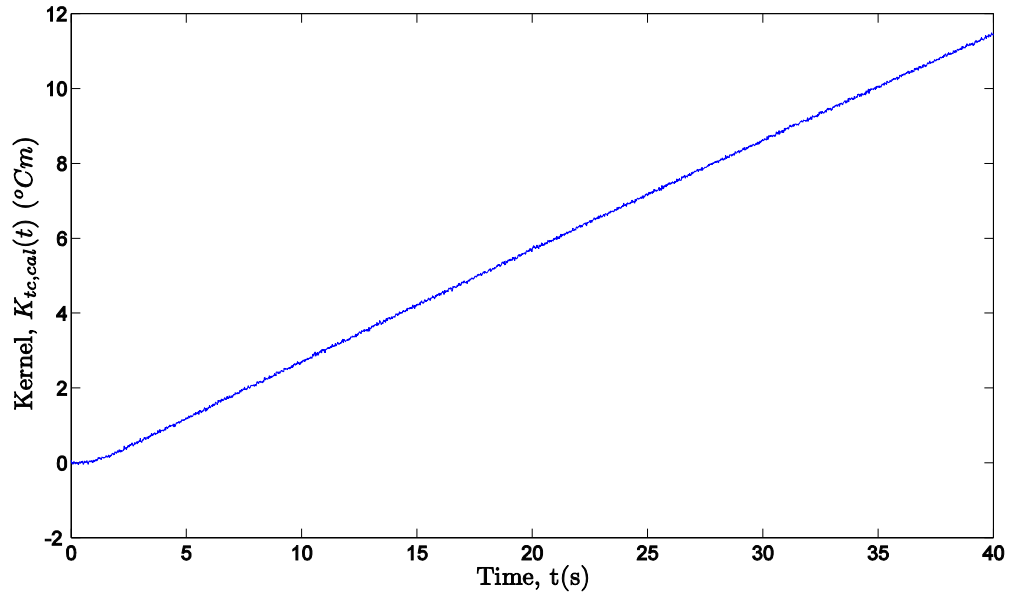


Figure 5.22: The kernel $K_{tc,cal}(t)$ in the “calibration” stage, defined in Eq. (5.15b) (using stainless steel as the host material for the numerical simulation).

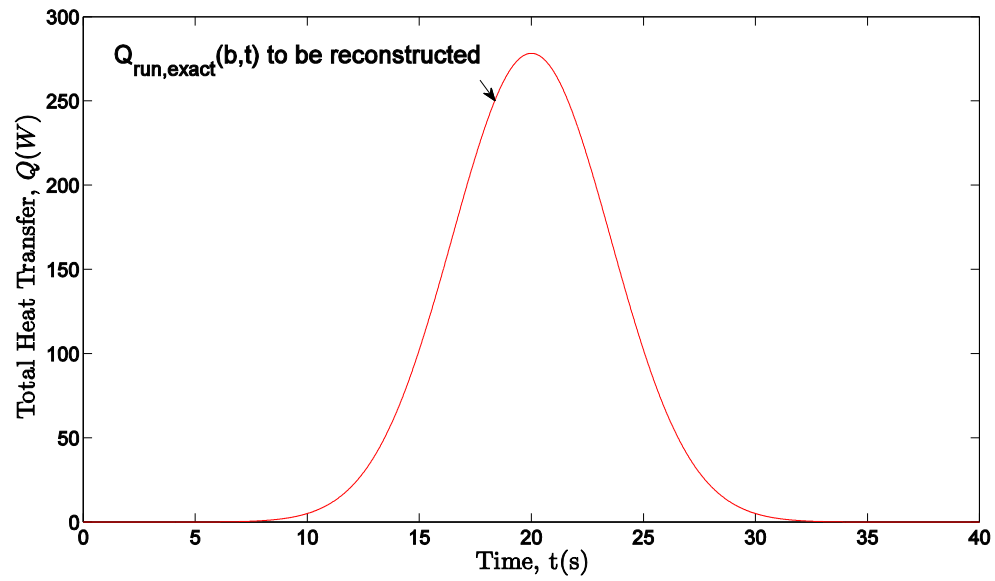


Figure 5.23: The exact “run” total surface heat transfer rate $Q_{run,exact}(b,t)$ to be reconstructed (when using stainless steel as the host material for the numerical simulation).

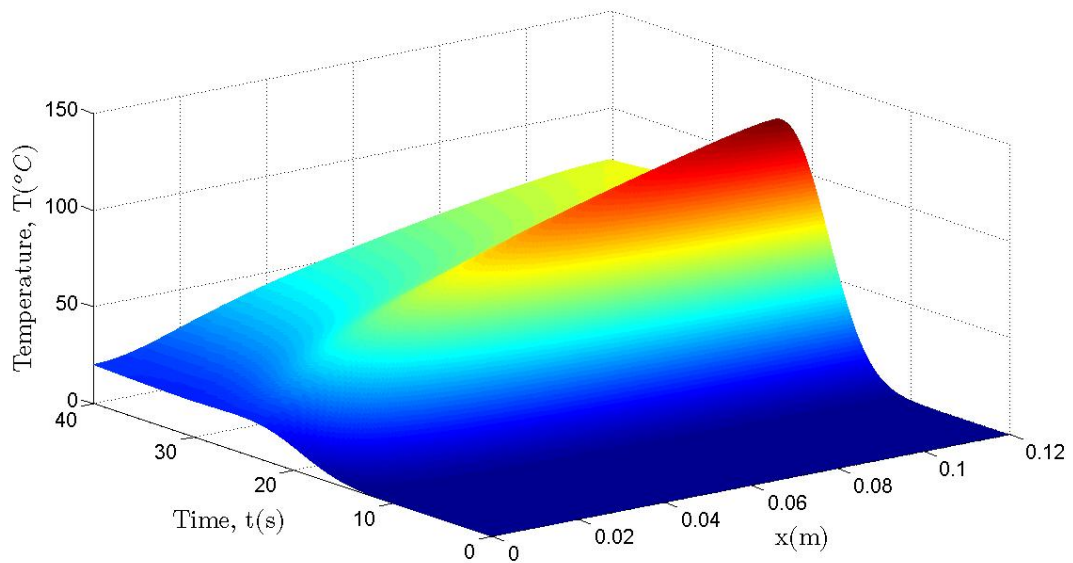


Figure 5.24: The surface temperature in the “run” stage, $T_{run}(x,b,t)$ (using stainless steel as the host material for the numerical simulation).

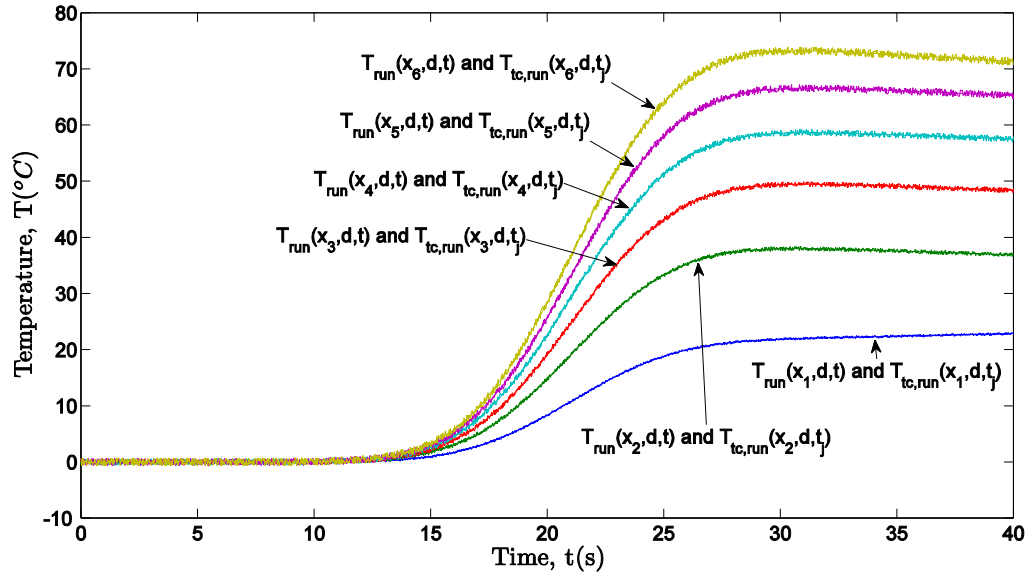


Figure 5.25: The positional temperature at various thermocouple sites $T_{run}(x_k, d, t)$ ($k=1, 2, \dots, 6$) and the thermocouple temperature $T_{tc,run}(x_k, d, t_j)$, respectively, in the “run” stage (using stainless steel as the host material for the numerical simulation).

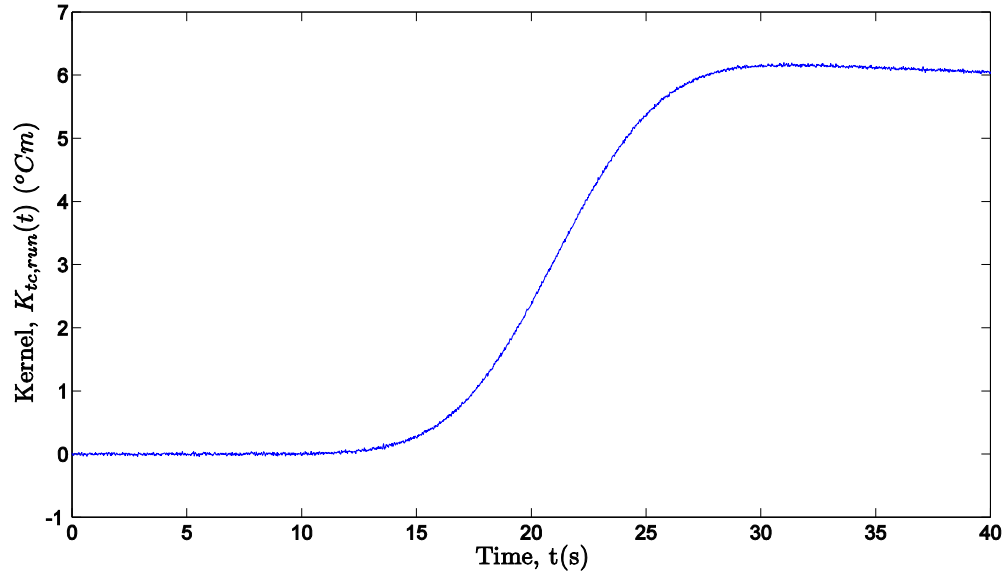


Figure 5.26: The kernel $K_{tc,run}(t)$ in the “run” stage, defined in Eq. (5.15c) (using stainless steel as the host materials for the numerical simulation).

Again, the metrics defined in Section 5.2 are employed for determining optimal future time parameter γ . Figures 5.27 display the local residual function, $R_\gamma(t)$, defined in Eq. (5.22) over time for increasing future time parameter γ based on the “calibration” and “run” thermocouple temperature measurement as well as the local “calibration” total heat transfer and the predicted “run” total heat transfer. As the future time parameter increases, the residual’s magnitude is shown to increase. From Fig. 5.27b, it is very clear that when γ is small (see for example $\gamma = 1.0\text{s}$) the local residual function is unstable. Figures 5.28 show the corresponding running average of the residual. Again, clearly the smoothing operation of integration is seen in Figs. 5.28. Figures 5.29 display the time-running variance of the residual function defined in Eq. (5.24) while Fig. 5.30 displays the normalized version of Figs. 5.29. In the normalized form (Fig. 5.30), both instability and over-smoothing effects can be quickly identified. Small γ ’s produce unstable predictions (see for example $\gamma = 1.0\text{s}$) while large γ ’s produce over-smoothed predictions (see for example $\gamma = 4.0\text{s}$) of the total surface heat transfer. The normalized plot possesses key physical effects. In fact, a bundling effect occurs as labeled in Fig. 5.30. Here, we choose $\gamma = 2.0\text{s}$ as the value of optimal future time parameter. Figures 5.31 display the comparisons between the actual total “run” surface heat transfer and the predicted total “run” surface heat transfer. Figure 5.31a displays the predicted total “run” surface heat transfer by $\gamma = 1.0\text{s}$. It is obvious that oscillation of the predicted results occur from right to left. Figure 5.31e shows the predicted total “run” surface heat transfer by $\gamma = \gamma_{opt} = 2.0\text{s}$, it is obvious that the prediction are very accurate. When future time parameter is too large, see Fig. 5.31, the attenuation of the prediction to the exact total “run” surface heat transfer is observed.

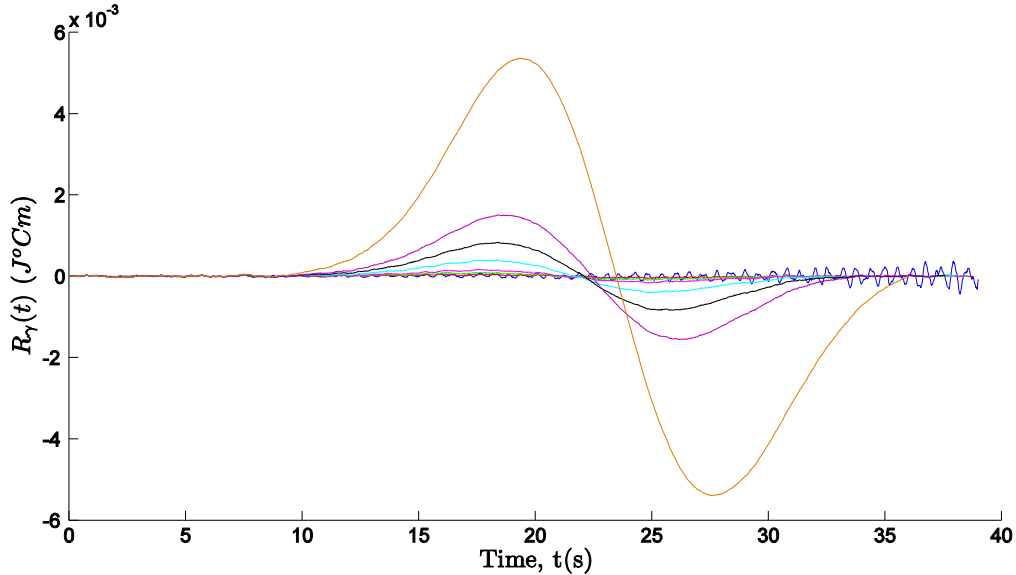


Figure 5.27a: The local residual function, $R_\gamma(t)$, defined in Eq. (5.22) over the γ -spectrum from 1.0s to 4.0s (using stainless steel as the host material for the numerical simulation).

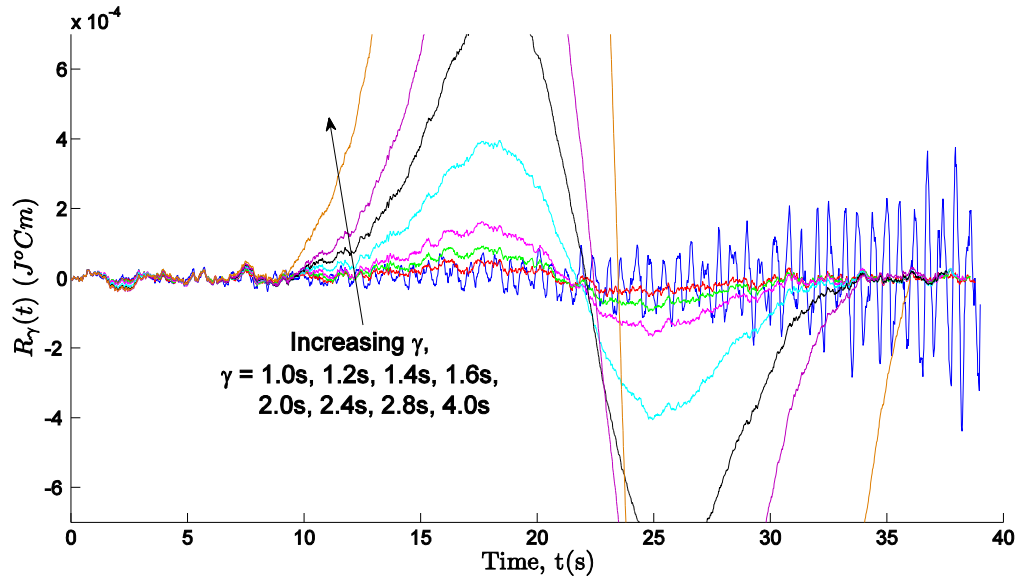


Figure 5.27b: Zoomed in view of the local residual function, $R_\gamma(t)$, defined in Eq. (5.22) over the γ -spectrum from 1.0s to 4.0s (using stainless steel as the host material for the numerical simulation).

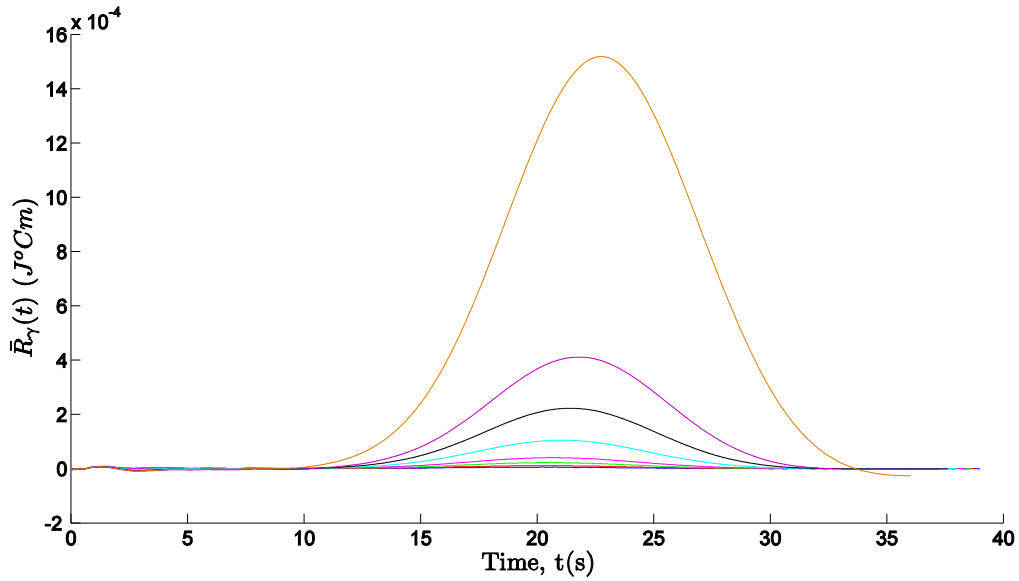


Figure 5.28a: The running average of residual function, $\bar{R}_\gamma(t)$, defined in Eq. (5.23) over the γ -spectrum from 1.0s to 4.0s (using stainless steel as the host material for the numerical simulation).

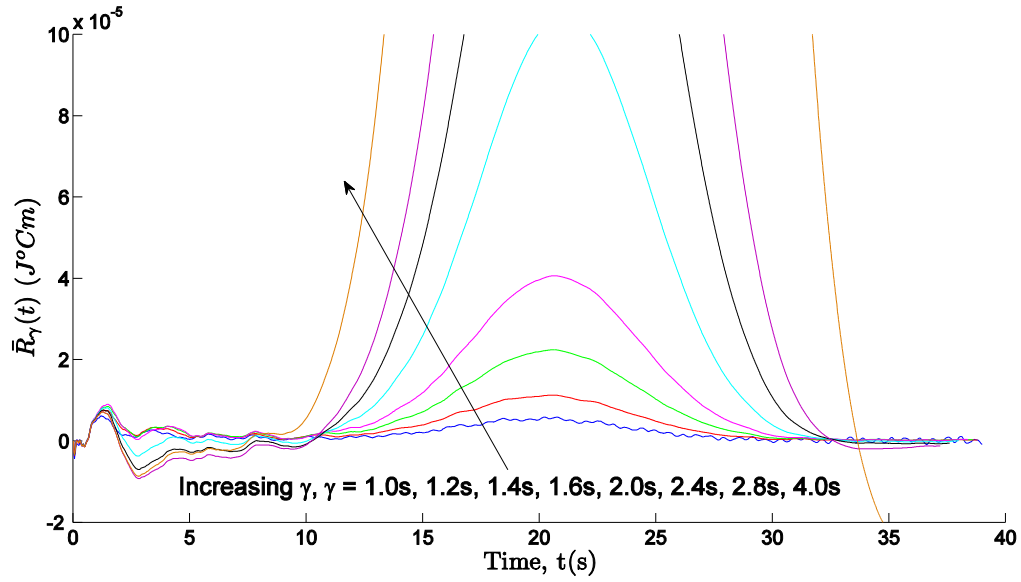


Figure 5.28b: Zoomed in view of the running average of residual function, $\bar{R}_\gamma(t)$, defined in Eq. (5.23) over the γ -spectrum from 1.0s to 4.0s (using stainless steel as the host material for the numerical simulation).

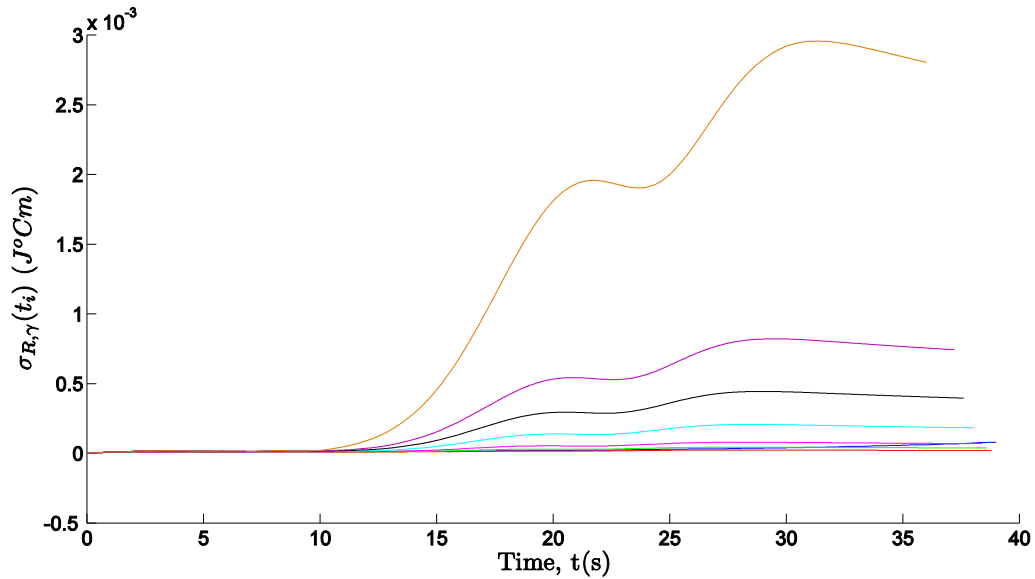


Figure 5.29a: The square root of time-running variance of residual, $\sigma_\gamma(t_i)$, defined in Eq. (5.24) over the γ -spectrum from 1.0s to 4.0s (using stainless steel as the host material for the numerical simulation).

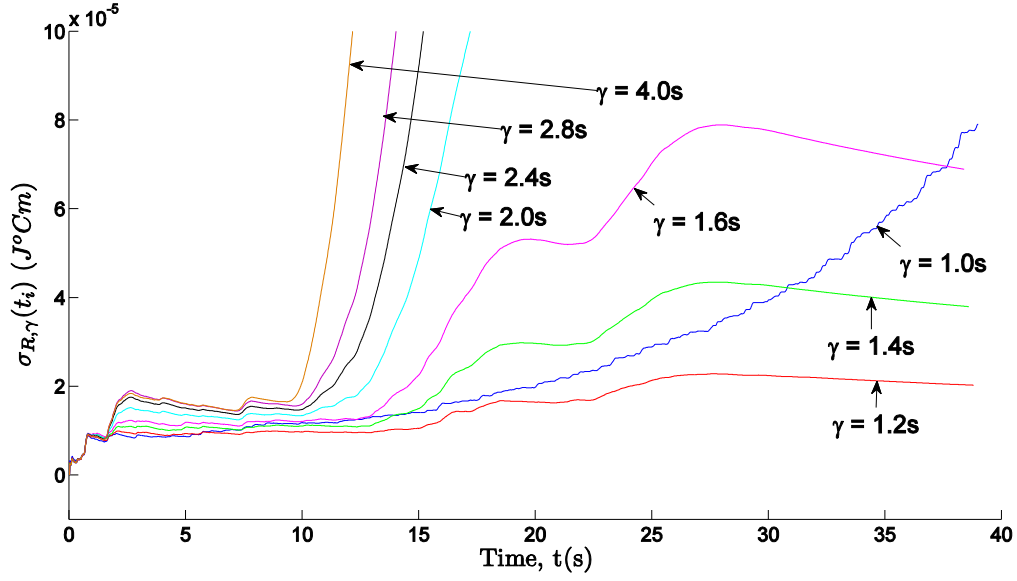


Figure 5.29b: Zoomed in view of the square root of time-running variance of residual, $\sigma_{\gamma}(t_i)$, defined in Eq. (5.24) over the γ -spectrum from 1.0s to 4.0s (using stainless steel as the host material for the numerical simulation).

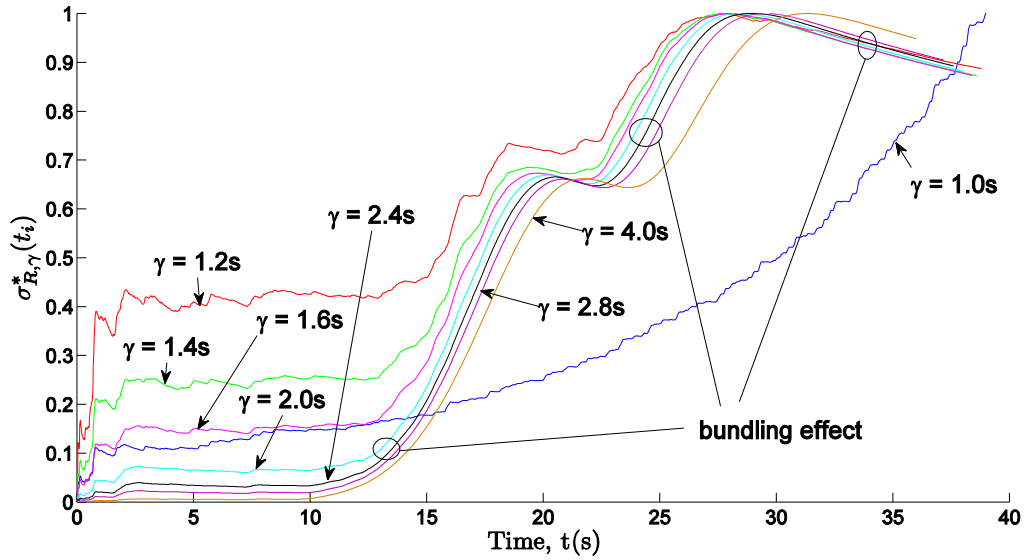


Figure 5.30: The normalized square root of time-running variance of residual, $\sigma_{\gamma}^*(t_i)$, defined in Eq. (5.23) over the γ -spectrum (using stainless steel as the host material for the numerical simulation).

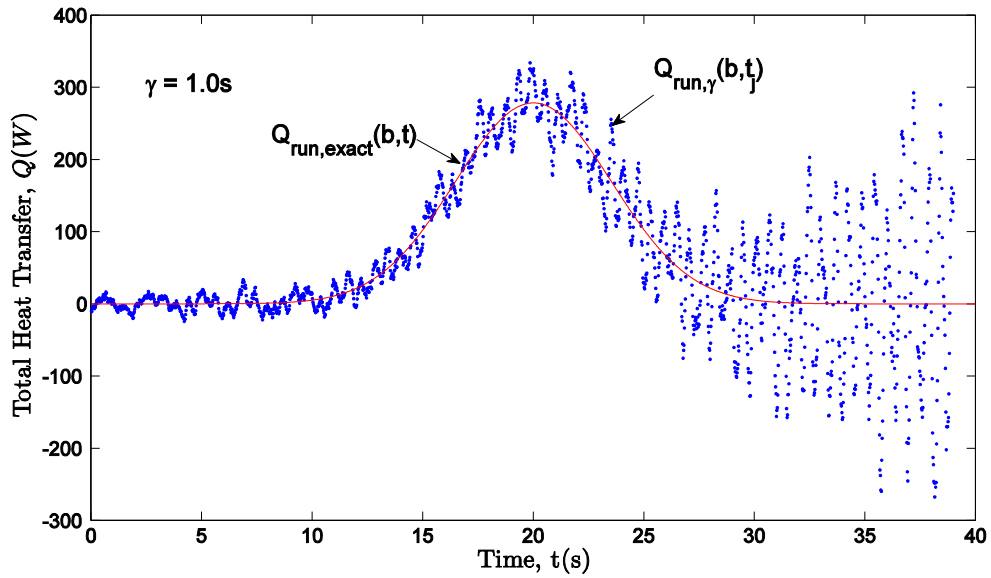


Figure 5.31a: Comparison between the predicted total “run” surface heat transfer $Q_{run,\gamma}(b,t)$ using $\gamma=1.0s$ and the exact total “run” surface heat transfer $Q_{run,exact}(b,t)$. Observe the serious oscillation from right to left (using stainless steel as the host material for the numerical simulation).

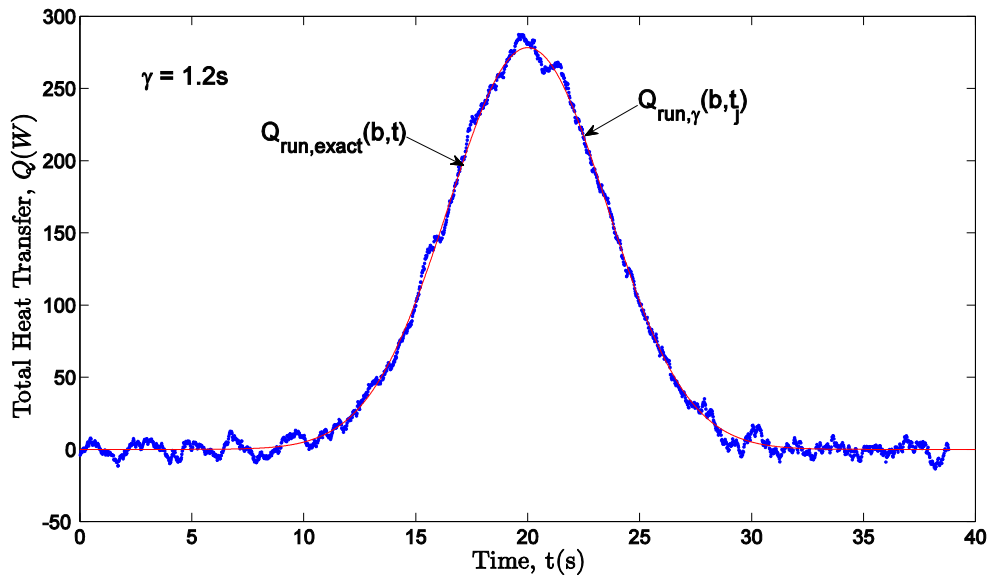


Figure 5.31b: Comparison between the predicted total “run” surface heat transfer $Q_{run,\gamma}(b,t)$ using $\gamma=1.2s$ and the exact total “run” surface heat transfer $Q_{run,exact}(b,t)$ (using stainless steel as the host material for the numerical simulation).

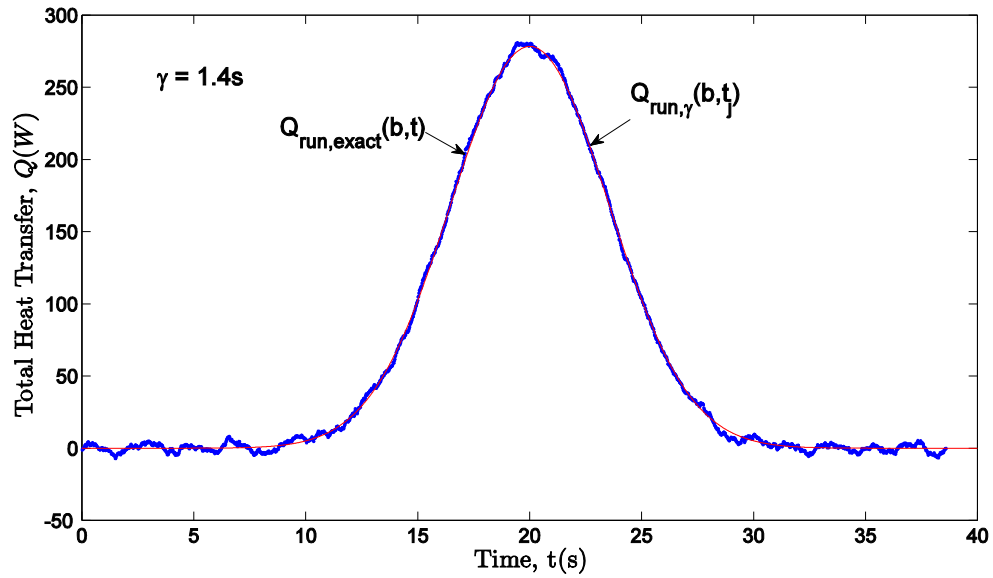


Figure 5.31c: Comparison between the predicted total “run” surface heat transfer $Q_{run,\gamma}(b,t)$ using $\gamma=1.4s$ and the exact total “run” surface heat transfer $Q_{run,exact}(b,t)$ (using stainless steel as the host material for the numerical simulation).

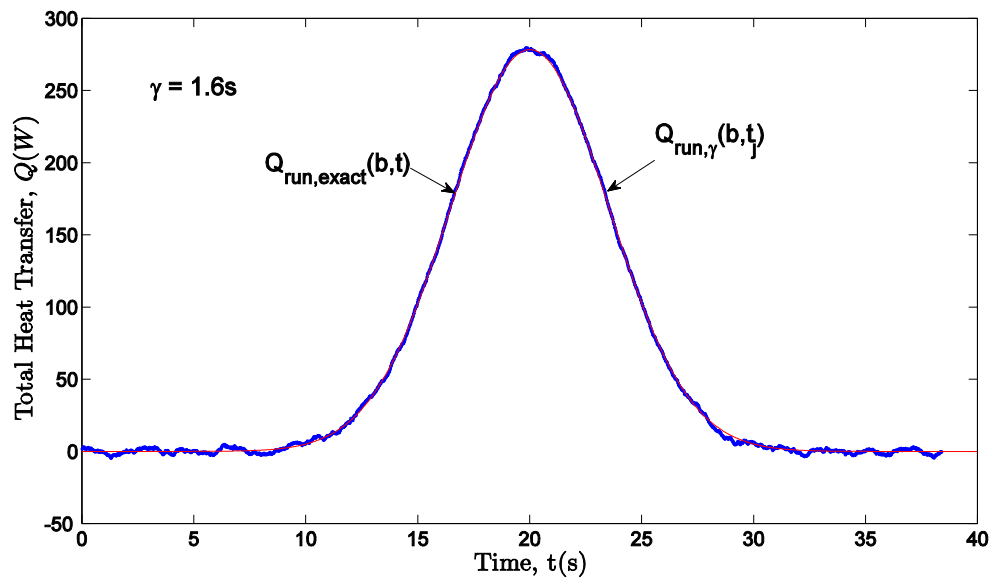


Figure 5.31d: Comparison between the predicted total “run” surface heat transfer $Q_{run,\gamma}(b,t)$ using $\gamma=1.6s$ and the exact total “run” surface heat transfer $Q_{run,exact}(b,t)$ (using stainless steel as the host material for the numerical simulation).

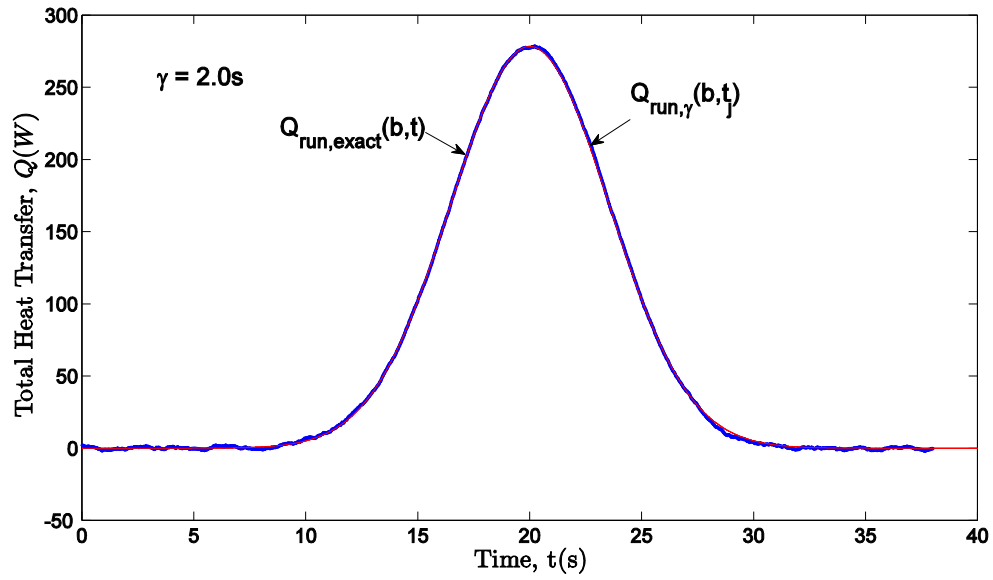


Figure 5.31e: Comparison between the predicted total “run” surface heat transfer $Q_{run,\gamma}(b,t)$ using $\gamma=2.0s$ and the exact total “run” surface heat transfer $Q_{run,exact}(b,t)$ (using stainless steel as the host material for the numerical simulation).

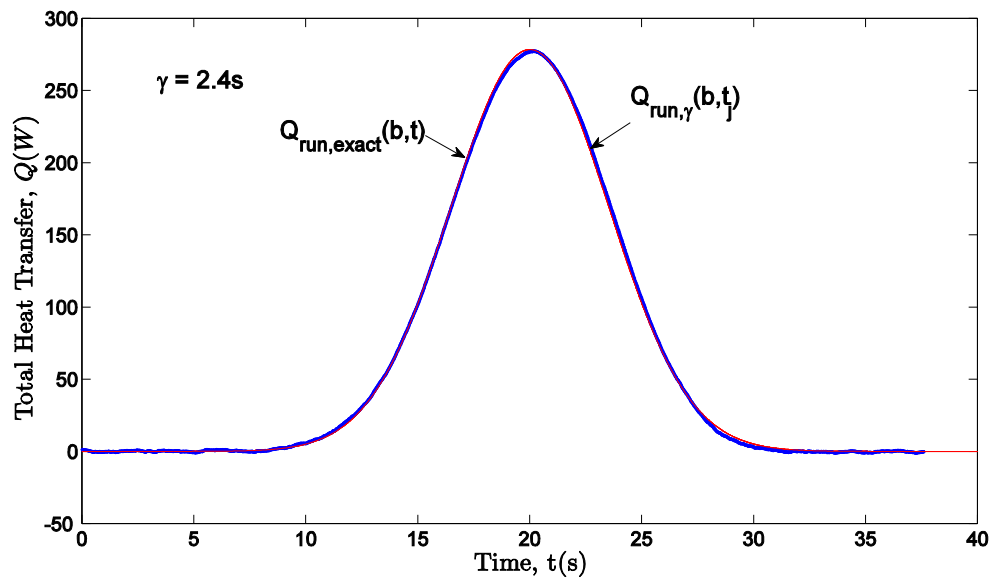


Figure 5.31f: Comparison between the predicted total “run” surface heat transfer $Q_{run,\gamma}(b,t)$ using $\gamma=2.4s$ and the exact total “run” surface heat transfer $Q_{run,exact}(b,t)$ (using stainless steel as the host material for the numerical simulation).

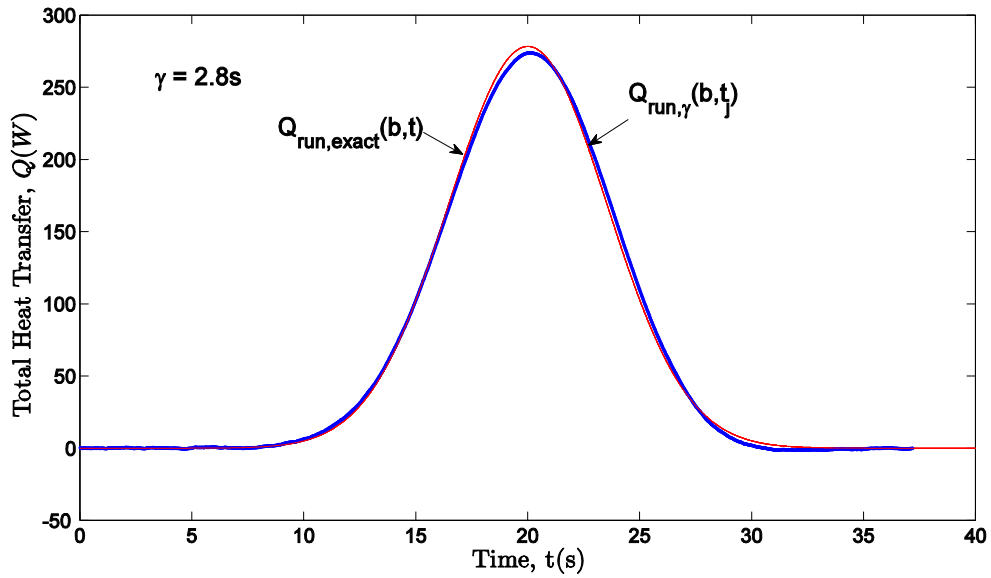


Figure 5.31g: Comparison between the predicted total “run” surface heat transfer $Q_{run,\gamma}(b,t)$ using $\gamma=2.8s$ and the exact total “run” surface heat transfer $Q_{run,exact}(b,t)$ (using stainless steel as the host material for the numerical simulation).

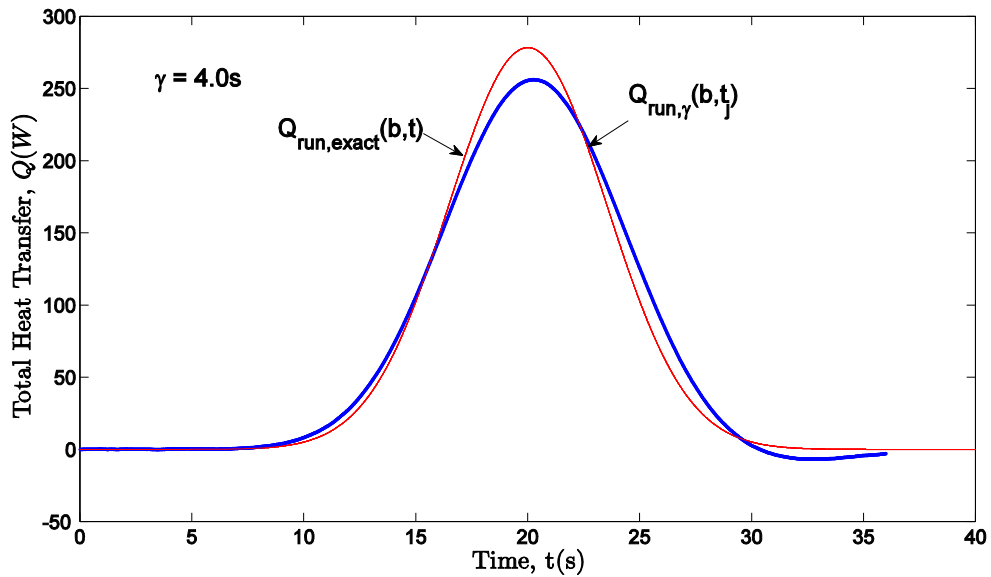


Figure 5.31h: Comparison between the predicted total “run” surface heat transfer $Q_{run,\gamma}(b,t)$ using $\gamma=4.0s$ and the exact total “run” surface heat transfer $Q_{run,exact}(b,t)$. Observe the attenuation on the top (using stainless steel as the host material for the numerical simulation).

The metric for verifying the visual comparison of prediction “run” surface heat transfer and the exact “run” surface heat transfer is the root-mean-square (RMS) of “run” surface heat transfer defined as

$$\sigma_{Q,\gamma} = \sqrt{\frac{(Q_{run,\gamma}(b,t_i) - Q_{run,exact}(b,t_i))^2}{M}}, \quad i = 1, 2, \dots, M, \quad (5.30)$$

where $M = \frac{t_{\max} - \gamma}{\Delta t}$.

Table 5.3 collects RMS values for the “run” total surface heat transfer over a set of γ 's. It shown when $\gamma = \gamma_{opt} = 2.0\text{s}$, RMS value of the “run” total surface heat transfer is the smallest. This proves the effectiveness of using the normalized square root of time-running variance of residual for estimating optimal γ .

Table 5.3: RMS of the “run” total surface heat transfer defined in Eq. (5.30) over several regularization parameters, γ

γ (s)	$\sigma_{Q,\gamma}$ (W)
1.0	61.18
1.2	4.72
1.4	2.57
1.6	1.70
1.8	1.23
2.0	1.03
2.2	1.15
2.4	1.57
2.6	2.23
2.8	3.06
4.0	10.48

5.5 Chapter Summary

A two-dimensional total heat transfer calibration integral equation method is presented in this chapter. Numerical results show the effectiveness and robustness of the method. The methodology developed in this chapter, under the constant thermophysical properties assumption, can be applied to a large set of materials involving low thermal conductivities to high thermal conductivities. Extending the two-dimensional total heat transfer calibration integral equation method to the three-dimensional domain should be considered in the near future.

Chapter 6 Conclusions and Future Work

6.1 Conclusions

A transformative calibration methodology for resolving inverse heat conduction problems has been presented. The calibration methodology is derived based on mathematical reasoning infused with experimental insight. The methodology, initially proposed for one-dimensional, single-region, isotropic materials case, has been extended to two-dimensional domain. The primary advantage of the calibration methodology over the other inverse methodologies lies in the removal of explicitly requiring the sensor positioning, sensor characteristics and thermophysical properties. These parameters are implicitly contained in the final mathematical expression that relates the in-depth measured temperature data to the surface boundary condition.

The primary focus of this thesis was to extend the one-dimensional concept to multidimensional systems. To this end, a preliminary two-dimensional study was undertaken to investigate this extension in terms of local heat flux (W/m^2) and total heat transfer (W). As noted within this thesis, some applications are mainly concerned with total heat transfer.

The first study was based on a practical quasi-one-dimensional outlook to the two-dimensional problem possessing constant thermophysical properties. It was demonstrated that favorable results occur under the restriction of low conductivity materials. However, this restriction can be removed by studying the two-dimensional calibration integral equation method involving total surface heat transfer. Excellent results were generated as noted in Chapter 5 using copper and stainless steel as the host materials.

6.2 Future work

New theories require experimental validation in today's engineering environment. Hence optimally designed experiments to validate the quasi-one-dimensional approximation and the linear two-dimensional total heat transfer calibration integral equation method are necessary.

In the near future, the extension of the linear two-dimensional total heat transfer calibration integral equation method to three-dimensional domains should be considered. In addition, multidimensional total heat transfer calibration integral equations with temperature dependent thermophysical properties should be investigated. Furthermore, deliberation should be initiated on using the calibration methodology for resolving the two-dimensional inverse heat conduction problem that quantifies local surface heat flux.

Extending the calibration methodology to the inverse heat conduction problems involving cylindrical coordinates would have numerous aerospace and industrial applications. These applications include nozzles and engine cylinder walls.

Overall, this new methodology is still expanding and possesses significant advantages over purely numerical inverse methods.

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Appendices

Appendix A

Residual Behavior Near Optimal Future Time Parameter γ

This appendix develops a mathematical explanation to the comment [8] put forth on Section 3.3 stating “It can be shown mathematically that the behavior of shifted residual function $r_\gamma(t+\gamma)$ is approximately proportional to the first derivative of predicted surface heat flux $q''_{r,\gamma}(0,t)$ when γ is close to its optimal value [see Appendix A]”.

Recall that the linear calibration equation from Eq. (3.16) is

$$\int_{u=0}^t q''_r(0,u)T_{ic,c}(d,t-u)du = f(t), \quad t \geq 0, \quad (\text{A.1a})$$

where

$$f(t) = \int_{u=0}^t q''_c(0,u)T_{ic,r}(d,t-u)du, \quad t \geq 0. \quad (\text{A.1b})$$

Introducing the future time parameter γ , and advancing time by $t = t + \gamma$, we get

$$\int_{u=0}^{t+\gamma} q''_r(0,u)T_{ic,c}(d,t+\gamma-u)du = f(t+\gamma). \quad (\text{A.2})$$

Using the basic integral definition, we get

$$\int_{u=0}^t q''_r(0,u)T_{ic,c}(d,t+\gamma-u)du + \int_{u=t}^{t+\gamma} q''_r(0,u)T_{ic,c}(d,t+\gamma-u)du = f(t+\gamma). \quad (\text{A.3a})$$

Let $v = u - t$ in the second integral on the left-hand side, we obtain

$$\int_{u=0}^t q''_r(0,u)T_{ic,c}(d,t+\gamma-u)du + \int_{v=0}^{\gamma} q''_r(0,v+t)T_{ic,c}(d,\gamma-v)dv = f(t+\gamma). \quad (\text{A.3b})$$

Using Taylor series expansion on $q''_r(0,v+t)$, we get

$$q''_r(0,v+t) = q''_r(0,t) + \frac{\partial q''_r}{\partial t}(0,t)v + \frac{\partial^2 q''_r}{\partial t^2}(0,t)\frac{v^2}{2!} + \frac{\partial^3 q''_r}{\partial t^3}(0,t)\frac{v^3}{3!} + \dots H.O.T. \quad (\text{A.4})$$

Substituting Eq. (A.4) into Eq. (A.3b) yields

$$\begin{aligned} & \int_{u=0}^t q''_r(0,u)T_{ic,c}(d,t+\gamma-u)du + \int_{v=0}^{\gamma} \left[q''_r(0,t) + \frac{\partial q''_r}{\partial t}(0,t)v + \frac{\partial^2 q''_r}{\partial t^2}(0,t)\frac{v^2}{2!} + \dots \right] T_{ic,c}(d,\gamma-v)dv \\ & = f(t+\gamma). \end{aligned} \quad (\text{A.5})$$

Recall Eq. (3.22),

$$r_\gamma(t) = \int_{u=0}^t q''_{r,\gamma}(0,u)T_{ic,c}(d,t-u)du - f(t). \quad (\text{A.6})$$

Upon advancing time by $t = t + \gamma$, we get

$$\begin{aligned} r_\gamma(t+\gamma) &= \int_{u=0}^{t+\gamma} q''_{r,\gamma}(0,u)T_{ic,c}(d,t+\gamma-u)du - f(t+\gamma) \\ &= \int_{u=0}^t q''_{r,\gamma}(0,u)T_{ic,c}(d,t+\gamma-u)du + \int_{u=t}^{t+\gamma} q''_{r,\gamma}(0,u)T_{ic,c}(d,t+\gamma-u)du - f(t+\gamma) \\ &= \int_{u=0}^t q''_{r,\gamma}(0,u)T_{ic,c}(d,t+\gamma-u)du + \int_{v=0}^{\gamma} q''_{r,\gamma}(0,v+t)T_{ic,c}(d,\gamma-v)dv - f(t+\gamma) \\ &= \int_{u=0}^t q''_{r,\gamma}(0,u)T_{ic,c}(d,t+\gamma-u)du - f(t+\gamma) \end{aligned}$$

$$+ \int_{v=0}^{\gamma} \left[q''_{r,\gamma}(0,t) + \frac{\partial q''_{r,\gamma}}{\partial t}(0,t)v + \frac{\partial^2 q''_{r,\gamma}}{\partial t^2}(0,t) \frac{v^2}{2!} \dots \right] T_{ic,c}(d, \gamma - v) dv, \quad (\text{A.7})$$

which can be written as

$$\begin{aligned} r_{\gamma}(t + \gamma) &= \int_{u=0}^t q''_{r,\gamma}(0,u) T_{ic,c}(d, t + \gamma - u) du + q''_{r,\gamma}(0,t) \int_{v=0}^{\gamma} T_{ic,c}(d, \gamma - v) dv \\ &\quad + \frac{\partial q''_{r,\gamma}}{\partial t}(0,t) \int_{v=0}^{\gamma} v T_{ic,c}(d, \gamma - v) dv + \frac{\partial^2 q''_{r,\gamma}}{\partial t^2}(0,t) \int_{v=0}^{\gamma} \frac{v^2}{2!} T_{ic,c}(d, \gamma - v) dv \\ &\quad + \frac{\partial^3 q''_{r,\gamma}}{\partial t^3}(0,t) \int_{v=0}^{\gamma} \frac{v^3}{3!} T_{ic,c}(d, \gamma - v) dv + \dots H.O.T. - f(t + \gamma). \end{aligned} \quad (\text{A.8})$$

Recall Eq. (3.20a)

$$\int_{u=0}^t q''_r(0,u) T_{ic,c}(d, t + \gamma - u) du + q''_r(0,t) \int_{v=0}^{\gamma} T_{ic,c}(d, \gamma - v) dv \approx f(t + \gamma), \quad t \in [0, t_{\max} - \gamma]. \quad (\text{A.9})$$

Substituting Eq. (A.9) into Eq. (A.8) and rearrange the resulting equation yields

$$\begin{aligned} r_{\gamma}(t + \gamma) &\approx \int_{u=0}^t [q''_{r,\gamma}(0,u) - q''_r(0,u)] T_{ic,c}(d, t + \gamma - u) du + [q''_{r,\gamma}(0,t) - q''_r(0,t)] \int_{v=0}^{\gamma} T_{ic,c}(d, \gamma - v) dv \\ &\quad + \frac{\partial q''_{r,\gamma}}{\partial t}(0,t) \int_{v=0}^{\gamma} v T_{ic,c}(d, \gamma - v) dv + \frac{\partial^2 q''_{r,\gamma}}{\partial t^2}(0,t) \int_{v=0}^{\gamma} \frac{v^2}{2!} T_{ic,c}(d, \gamma - v) dv \\ &\quad + \frac{\partial^3 q''_{r,\gamma}}{\partial t^3}(0,t) \int_{v=0}^{\gamma} \frac{v^3}{3!} T_{ic,c}(d, \gamma - v) dv + \dots H.O.T. \end{aligned} \quad (\text{A.10})$$

When γ is approximately γ_{opt} , then $q''_{r,\gamma}(0,t) \approx q''_r(0,t)$, so the first two terms on the right-hand side is approximately equal to zero. In addition, the summation of terms after second term of right-hand-side of Eq. (A.10) is smaller than $\frac{\partial q''_{r,\gamma}}{\partial t}(0,t) \int_{v=0}^{\gamma} v T_{ic,c}(d, \gamma - v) dv$. Therefore, after

neglecting these terms, $r_{\gamma}(t + \gamma)$ is approximately proportional to $\frac{\partial q''_{r,\gamma}}{\partial t}(0,t)$ when γ is close to γ_{opt} .

Appendix B

Derivation of Analytical Solution for Two-Dimensional Transient Heat Conduction

This appendix develops the exact solution displayed in Eq. (5.1a) based on finite integral transforms. Figure 1.1 displays the two-dimensional geometry of interest.

The transient two-dimensional linear heat equation [4] is

$$\frac{1}{\alpha} \frac{\partial T}{\partial t}(x, y, t) = \frac{\partial^2 T}{\partial x^2}(x, y, t) + \frac{\partial^2 T}{\partial y^2}(x, y, t), \quad x \in [0, a], \quad y \in [0, b], \quad t \geq 0, \quad (\text{B.1a})$$

where T denotes temperature, x, y denote spatial coordinates and t denotes time. The boundary conditions are

$$\frac{\partial T}{\partial x}(0, y, t) = \frac{\partial T}{\partial x}(a, y, t) = 0, \quad y \in [0, b], \quad t \geq 0, \quad (\text{B.1b})$$

$$q''(x, b, t) = k \frac{\partial T}{\partial y}(x, b, t) = q_s''(x, t), \quad x \in [0, a], \quad t \geq 0, \quad (\text{B.1c})$$

$$k \frac{\partial T}{\partial y}(x, 0, t) = h(T(x, 0, t) - T_\infty), \quad x \in [0, a], \quad t \geq 0, \quad (\text{B.1d})$$

where $q''(x, b, t)$ is the net surface heat flux at the top surface shown in Fig. 1.1. The initial condition is

$$T(x, y, 0) = T_0, \quad x \in [0, a], \quad y \in [0, b]. \quad (\text{B.1e})$$

In this context, the finite Fourier transform technique [4,98-100] is employed twice to algebratize the spatial variables x and y , leaving a first order ordinary differential equation in the time variable t . The first order equation is solved for the twice transformed temperature. Next, the result is inverted successively by the inversion formulas to obtain the analytical solution of temperature. For simplicity, the initial temperature T_0 and ambient temperature T_∞ are set as $T_0 = T_\infty = 0^\circ\text{C}$.

The integral transform pair for $T(x, y, t)$ with respect to the x variable is defined as

$$\text{Integral transform: } \bar{T}(\beta_m, y, t) = \int_{x'=0}^a X(\beta_m, x') T(x', y, t) dx', \quad (\text{B.2a})$$

$$\text{Inversion formula: } T(x, y, t) = \sum_{m=0}^{\infty} \frac{X(\beta_m, x)}{N(\beta_m)} \bar{T}(\beta_m, y, t), \quad (\text{B.2b})$$

and the integral transform pair for $\bar{T}(\beta_m, y, t)$ with respect to the y variable is defined as

$$\text{Integral transform: } \tilde{\bar{T}}(\beta_m, \eta_n, t) = \int_{y'=0}^a Y(\eta_n, y') \bar{T}(\beta_m, y', t) dy', \quad (\text{B.3a})$$

$$\text{Inversion formula: } \bar{T}(\beta_m, y, t) = \sum_{n=1}^{\infty} \frac{Y(\eta_n, y)}{N(\eta_n)} \tilde{\bar{T}}(\beta_m, \eta_n, t), \quad (\text{B.3b})$$

where the bar denotes the integral transform with respect to x variable and the tilde denotes the integral transform with respect to the y variable. The eigensets required in Eqs. (B.2) and (B.3) can be derived by the classical separation of variables method. Ozisik [4] collected eigensets with different combinations of boundary conditions.

We begin by letting $x \rightarrow x'$ in Eq. (B.1a) and then operating on the result with

$\int_{x'=0}^a X(\beta_m, x') dx'$ to get

$$\frac{1}{\alpha} \int_{x'=0}^a X(\beta_m, x') \frac{\partial T}{\partial t}(x', y, t) dx' = \int_{x'=0}^a X(\beta_m, x') \frac{\partial^2 T}{\partial x'^2}(x', y, t) dx' + \int_{x'=0}^a X(\beta_m, x') \frac{\partial^2 T}{\partial y^2}(x', y, t) dx', \quad (\text{B.4a})$$

where the eigenset can be obtained from case 5 of Table 2-2 in Ref.[4] as

$$X(\beta_m, x) = \cos \beta_m x, \quad m = 0, 1, 2, 3, \dots \quad (\text{B.4b})$$

$$\beta_m = m \frac{\pi}{a}, \quad m = 0, 1, 2, 3, \dots \quad (\text{B.4c})$$

$$N(\beta_m) = \begin{cases} a, & m = 0, \\ \frac{a}{2}, & m = 1, 2, 3, \dots \end{cases} \quad (\text{B.4d})$$

Integrating by parts twice on the first term of the right-hand-side of Eq. (B.4a) yields

$$\begin{aligned} & \int_{x'=0}^a X(\beta_m, x') \frac{\partial^2 T}{\partial x'^2}(x', y, t) dx' \\ &= X(\beta_m, x') \frac{\partial T}{\partial x'}(x', y, t) \Big|_{x'=0}^a - \int_{x'=0}^a X'(\beta_m, x') \frac{\partial T}{\partial x'}(x', y, t) dx' \\ &= X(\beta_m, x') \frac{\partial T}{\partial x'}(x', y, t) \Big|_{x'=0}^a - X'(\beta_m, x') T(x', y, t) \Big|_{x'=0}^a + \int_{x'=0}^a X''(\beta_m, x') T(x', y, t) dx', \end{aligned} \quad (\text{B.5a})$$

or upon using the transformed boundary conditions (B.1b), eigenfunction (B.4b) and eigenvalues (B.4c) and the definition of the eigenvalue problem, $X'' + \beta_m^2 X = 0$, and the definition of the integral transform (B.2a), we get

$$\begin{aligned} \int_{x'=0}^a X(\beta_m, x') \frac{\partial^2 T}{\partial x'^2}(x', y, t) dx' &= 0 + \frac{m\pi}{a} \sin\left(\frac{m\pi}{a} x'\right) T(x', y, t) \Big|_{x'=0}^a - \beta_m^2 \int_{x'=0}^a X(\beta_m, x') T(x', y, t) dx' \\ &= 0 + 0 - \beta_m^2 \bar{T}(\beta_m, y, t). \end{aligned} \quad (\text{B.5b})$$

Upon substituting Eq. (B.5b) into Eq. (B.4a) and making use of Leibnitz's rule (on left-hand-side) and the definition of the integral transform (B.2a), we obtain

$$\frac{1}{\alpha} \frac{\bar{T}(\beta_m, y, t)}{\partial t} = -\beta_m^2 \bar{T}(\beta_m, y, t) + \frac{\partial^2 \bar{T}}{\partial y^2}(\beta_m, y, t). \quad (\text{B.6})$$

By letting $y \rightarrow y'$ and then operating on the result with $\int_{y'=0}^b Y(\eta_n, y') dy'$ to get

$$\frac{1}{\alpha} \int_{y'=0}^b Y(\eta_n, y') \frac{\partial \bar{T}}{\partial t}(\beta_m, y', t) dy' = -\beta_m^2 \int_{y'=0}^b Y(\eta_n, y') \bar{T}(\beta_m, y', t) dy' + \int_{y'=0}^b Y(\eta_n, y') \frac{\partial^2 \bar{T}}{\partial y'^2}(\beta_m, y', t) dy', \quad (\text{B.7a})$$

where the eigenset can be obtained from case 2 of Table 2-2 in Ref.[4] as

$$Y(\eta_n, y) = \cos \eta_n (b - y), \quad n = 1, 2, 3, \dots \quad (\text{B.7b})$$

$$\eta_n \text{'s are positive roots of } \eta_n \tan(\eta_n b) = \frac{h}{k}, \quad n = 1, 2, \dots \quad (\text{B.7c})$$

$$N(\eta_n) = \frac{b(\eta_n^2 + \frac{h^2}{k^2}) + \frac{h}{k}}{2(\eta_n^2 + \frac{h^2}{k^2})}, n = 1, 2, \dots \quad (\text{B.7d})$$

Integrating by parts twice on the second term of the right-hand-side of Eq. (B.7a) yields

$$\begin{aligned} & \int_{y'=0}^b Y(\eta_n, y') \frac{\partial^2 \bar{T}}{\partial y'^2}(\beta_m, y', t) dy' \\ &= Y(\eta_n, y') \frac{\partial \bar{T}}{\partial y'}(\beta_m, y', t) \Big|_{y'=0}^b - Y'(\eta_n, y') \bar{T}(\beta_m, y', t) \Big|_{y'=0}^b + \int_{y'=0}^b Y''(\eta_n, y') \bar{T}(\beta_m, y', t) dy'. \end{aligned} \quad (\text{B.8})$$

The transformed boundary conditions Eq. (B.1c) and Eq. (B.1d) become

$$\bar{q}''(\beta_m, b, t) = \int_{x'=0}^a X(\beta_m, x') q''(x', b, t) dx' = \int_{x'=0}^a X(\beta_m, x') k \frac{\partial T}{\partial y}(x', b, t) dx' = k \frac{\partial \bar{T}}{\partial y}(\beta_m, b, t), \quad (\text{B.9a})$$

and

$$\int_{x'=0}^a X(\beta_m, x') k \frac{\partial T}{\partial y}(x', 0, t) dx' = \int_{x'=0}^a X(\beta_m, x') h(T(x', 0, t) - T_\infty) dx', \quad (\text{B.9b})$$

respectively. Using the condition $T_\infty = 0^\circ\text{C}$ reduces Eq. (B.9b) to

$$k \frac{\partial \bar{T}}{\partial y}(\beta_m, 0, t) = h \bar{T}(\beta_m, 0, t). \quad (\text{B.9c})$$

Taking first and second derivative with respect to y of eigenfunction defined in Eq. (B.7b) yield

$$Y'(\eta_n, y) = \eta_n \sin \eta_n (b - y), \quad (\text{B.10a})$$

$$Y''(\eta_n, y) = -\eta_n^2 \cos \eta_n (b - y) = -\eta_n^2 Y(\eta_n, y). \quad (\text{B.10b})$$

Substituting Eqs. (B.9) and (B.10) into Eq. (B.8) and after performing some manipulations, we obtain

$$\begin{aligned} & \int_{y'=0}^b Y(\eta_n, y') \frac{\partial^2 \bar{T}}{\partial y'^2}(\beta_m, y', t) dy' \\ &= \frac{\bar{q}''(\beta_m, b, t)}{k} - \cos(\eta_n b) \frac{h \bar{T}(\beta_m, 0, t)}{k} + \eta_n \sin(\eta_n b) \bar{T}(\beta_m, 0, t) - \eta_n^2 \tilde{\bar{T}}(\beta_m, \eta_n, t) \\ &= \frac{\bar{q}''(\beta_m, b, t)}{k} + \bar{T}(\beta_m, 0, t) \left(\eta_n \sin(\eta_n b) - \frac{h}{k} \cos(\eta_n b) \right) - \eta_n^2 \tilde{\bar{T}}(\beta_m, \eta_n, t). \end{aligned} \quad (\text{B.11})$$

From Eq. (B.7c), we get

$$\sin(\eta_n b) = \frac{h}{\sqrt{h^2 + k^2 \eta_n^2}}, \quad (\text{B.12a})$$

$$\cos(\eta_n b) = \frac{k \eta_n}{\sqrt{h^2 + k^2 \eta_n^2}}. \quad (\text{B.12b})$$

Substituting Eqs. (B.12) into Eq. (B.11) yields

$$\begin{aligned}
& \int_{y'=0}^b Y(\eta_n, y') \frac{\partial^2 \bar{T}}{\partial y'^2}(\beta_m, y', t) dy' \\
&= \frac{\bar{q}''(\beta_m, b, t)}{k} + \bar{T}(\beta_m, 0, t) \left(\frac{h\eta_n}{\sqrt{h^2 + k^2\eta_n^2}} - \frac{h\eta_n}{\sqrt{h^2 + k^2\eta_n^2}} \right) - \eta_n^2 \tilde{\bar{T}}(\beta_m, \eta_n, t) \\
&= \frac{\bar{q}''(\beta_m, b, t)}{k} - \eta_n^2 \tilde{\bar{T}}(\beta_m, \eta_n, t).
\end{aligned} \tag{B.13}$$

Substituting Eq. (B.13) into Eq. (B.7a) and after performing some manipulation, we obtain

$$\frac{1}{\alpha} \frac{d\tilde{\bar{T}}}{dt}(\beta_m, \eta_n, t) = -\beta_m^2 \tilde{\bar{T}}(\beta_m, \eta_n, t) + \frac{\bar{q}''(\beta_m, b, t)}{k} - \eta_n^2 \tilde{\bar{T}}(\beta_m, \eta_n, t), \tag{B.14a}$$

or

$$\frac{d\tilde{\bar{T}}}{dt}(\beta_m, \eta_n, t) + \alpha(\beta_m^2 + \eta_n^2) \tilde{\bar{T}}(\beta_m, \eta_n, t) = \frac{\bar{q}''(\beta_m, b, t)}{k} \alpha, \tag{B.14b}$$

subject to the double transformed initial condition

$$\tilde{\bar{T}}(\beta_m, \eta_n, 0) = 0. \tag{B.14c}$$

Equation (B.14b) can be written as

$$\frac{d}{dt} \left(e^{\alpha(\beta_m^2 + \eta_n^2)t} \tilde{\bar{T}}(\beta_m, \eta_n, t) \right) = e^{\alpha(\beta_m^2 + \eta_n^2)t} \frac{\bar{q}''(\beta_m, b, t)}{k} \alpha. \tag{B.15}$$

Let $t \rightarrow u$, we get

$$\frac{d}{dt} \left(e^{\alpha(\beta_m^2 + \eta_n^2)u} \tilde{\bar{T}}(\beta_m, \eta_n, u) \right) = e^{\alpha(\beta_m^2 + \eta_n^2)u} \frac{\bar{q}''(\beta_m, b, u)}{k} \alpha. \tag{B.16}$$

Integrating from $u=0$ to $u=t$ on both sides of Eq. (B.16) produces

$$\int_{u=0}^t \frac{d}{du} \left(e^{\alpha(\beta_m^2 + \eta_n^2)u} \tilde{\bar{T}}(\beta_m, \eta_n, u) \right) du = \frac{\alpha}{k} \int_{u=0}^t e^{\alpha(\beta_m^2 + \eta_n^2)u} \bar{q}''(\beta_m, b, u) du, \tag{B.17a}$$

or

$$\begin{aligned}
\tilde{\bar{T}}(\beta_m, \eta_n, t) &= \frac{\alpha}{k} \int_{u=0}^t e^{-\alpha(\beta_m^2 + \eta_n^2)(t-u)} \bar{q}''(\beta_m, b, u) du \\
&= \frac{\alpha}{k} \int_{u=0}^t e^{-\alpha(\beta_m^2 + \eta_n^2)(t-u)} \int_{x'=0}^a X(\beta_m, x') q''(x', b, u) dx' du.
\end{aligned} \tag{B.17b}$$

Inverting Eq. (B.17c) by the inversion formula (B.3b), and then again using (B.2b) to obtain the analytical solution given as

$$\begin{aligned}
T(x, y, t) &= \sum_{m=0}^{\infty} \frac{X(\beta_m, x)}{N(\beta_m)} \sum_{n=1}^{\infty} \frac{Y(\eta_n, y)}{N(\eta_n)} \frac{\alpha}{k} \int_{u=0}^t e^{-\alpha(\beta_m^2 + \eta_n^2)(t-u)} \int_{x'=0}^a X(\beta_m, x') q''(x', b, u) dx' du, \\
&= \frac{\alpha}{k} \sum_{m=0}^{\infty} \sum_{n=1}^{\infty} \frac{X(\beta_m, x) Y(\eta_n, y)}{N(\beta_m) N(\eta_n)} \int_{u=0}^t e^{-\alpha(\beta_m^2 + \eta_n^2)(t-u)} \int_{x'=0}^a X(\beta_m, x') q''(x', b, u) dx' du,
\end{aligned} \tag{B.18}$$

where the two eigensets are given in Eqs. (B.4b-B.4d) and Eqs. (B.7b-B.7d), respectively.

Substituting the eigenfunctions given in Eq. (B.4b) and Eq. (B.7b) into Eq. (B.18) yields

$$T(x, y, t) = \frac{\alpha}{k} \sum_{m=0}^{\infty} \sum_{n=1}^{\infty} \frac{\cos(\beta_m x) \cos \eta_n (b - y)}{N(\beta_m) N(\eta_n)} \int_{u=0}^t e^{-\alpha(\beta_m^2 + \eta_n^2)(t-u)} \int_{x'=0}^a \cos(\beta_m x') q''(x', b, u) dx' du, \tag{B.19a}$$

where the eigenvalues and normalization integrals are

$$\beta_m = m \frac{\pi}{a}, \quad m = 0, 1, 2, 3, \dots \quad (\text{B.19b})$$

$$N(\beta_m) = \begin{cases} a, & m = 0, \\ \frac{a}{2}, & m = 1, 2, 3, \dots \end{cases} \quad (\text{B.19c})$$

$$\eta_n \text{'s are positive roots of } \eta_n \tan(\eta_n b) = \frac{h}{k}, \quad (\text{B.19d})$$

$$N(\eta_n) = \frac{b(\eta_n^2 + \frac{h^2}{k^2}) + \frac{h}{k}}{2(\eta_n^2 + \frac{h^2}{k^2})}, \quad n = 1, 2, \dots \quad (\text{B.19e})$$

which produces Eq. (5.1).

Vita

Hongchu Chen was born in Pujiang, Zhejiang province, China on August 24, 1989. He attended Pujiang Middle School between August, 2004 and June, 2007. He began his undergraduate study at Central South University in Changsha, Hunan province in September, 2007 and received a Bachelor of Engineering degree with a major in Thermal Energy and Dynamic Engineering in June, 2011. Afterward he came to the University of Tennessee, Knoxville for graduate study with a major in mechanical engineering. He was a graduate research assistant in the heat transfer lab of the Department of Mechanical, Aerospace and Biomedical Engineering under Prof. Frankel and Prof. Keyhani. His research in the heat transfer lab was concerned with inverse heat conduction problems. He received his Master of Science degree in August, 2013.