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I am submitting herewith a thesis written by Clyde M. Davenport entitled "An Extension of the Complex Calculus to Four Real Dimensions, with an Application to Special Relativity." I recommend that it be accepted in partial fulfillment of the requirements for the degree of Master of Science, with a major in Mathematics.

David E. Dobbs

David E. Dobbs, Major Professor

We have read this thesis
and recommend its acceptance:

Philip W. Schaefer

W. E. Deeds

Accepted for the Council:

L Evans Roth

Vice Chancellor
Graduate Studies and Research

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AN EXTENSION OF THE COMPLEX CALCULUS TO FOUR REAL DIMENSIONS, WITH AN APPLICATION TO SPECIAL RELATIVITY

A Thesis

Presented for the

Master of Science

Degree

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Clyde M. Davenport

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ABSTRACT

A new system of analysis of one four-dimensional variable has been constructed. It includes a differential calculus, an integral calculus, and a theory of functions, and parallels the system of complex analysis in form, content, and notation. Moreover, it includes the system of complex analysis as a subsystem. The new system is based on a commutative, four-dimensional real algebra.

An application to the theory of Special Relativity of physics is investigated. A group of four-dimensional operators, containing the Lorentz group (in one space coordinate and time) as a subgroup, is constructed and analyzed. The conjecture is made that the larger group can serve as a model for predicting relativistic effects in four dimensions, with relative motion in an arbitrary direction. Transformations from the larger group may be viewed as linear, nonorthogonal transformations in four dimensions, or as analytic functions of a four-dimensional variable.

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LIST OF SYMBOLS

α, β, γ	Four-dimensional elements, numbers, vectors, variables; or real, ordered quadruples (a,b,c,d).
η, ξ	Complex quantities of the form $\eta = (a+d) + i(b-c)$, $\xi = (a-d) + i(b+c)$.
θ	The four-dimensional zero element, (0,0,0,0).
λ	An eigenvalue of a four-dimensional element (in matrix form) of $\mathbb{D}$.
$\emptyset$	The empty set.
ρ	The metric on $\mathbb{C} \times \mathbb{C}$.
ψ	A real-valued, analytic function of four real, independent variables.
a, b, c, d	Real quantities.
A	A 2 × 2 real basis matrix for the real algebra of complex numbers.
B_{ij}	Signed minors of a four-dimensional element in matrix form.
$\mathbb{C}$	The complex field.
$\mathbb{C}^2$	The product ring, $\mathbb{C} \times \mathbb{C}$.
$\mathbb{D}$	The set of all real, ordered quadruples (a,b,c,d); the set of all four-dimensional numbers $a + bi + cj + dk$; the real algebra of all four-dimensional numbers.
$\frac{df}{d\beta}$	The first derivative of f with respect to β.
$\mathbb{E}$	The multiplicative group of 4 × 4 real (basis) matrices.
$E_1, \ldots, E_4$	Elements of $\mathbb{E}$.
e_1, e_2	The unity elements for the principal ideals O^+ and O^- in $\mathbb{D}$.

$EL_2(V)$	The Lorentz group in one space coordinate and time.
$EL_4(V)$	A group of linear transformations in four dimensions, containing $EL_2(V)$ as a subgroup.
$f(\beta)$	A function $f : \mathbb{D} \to \mathbb{D}$ or $f : \mathbb{C}^2 \to \mathbb{C}^2$.
$f'(\beta)$	The first derivative of f with respect to β.
$F_1, \ldots, F_8$	Real 8×8 matrices; basis elements for an eight-dimensional system.
$\mathbb{F}$	The complement of $(0^+ \cup 0^-)$ in $\mathbb{D}$.
$\mathbb{G}$	The abelian multiplicative group generated by the 1, i, j, k basis elements.
$\mathbb{G}_1, \ldots, \mathbb{G}_8$	Cyclic subgroups of $\mathbb{G}$.
$\mathbb{H}$	The space of all analytic functions from $\mathbb{D}$ to $\mathbb{D}$.
i, j, k, 1	Basis elements for the real algebra on $\mathbb{D}$.
I	The 2×2 real identity matrix.
L, L_0	Lengths of physical objects.
$0^+, 0^-$	Principal/maximal ideals of $\mathbb{D}$.
	$0^- = \{a+bi+cj+dk \mid a = d;\ b = -c;\ a, b, c, d \in \mathbb{R}\}$
	$0^+ = \{a+bi+cj+dk \mid a = -d,\ b = c;\ a, b, c, d \in \mathbb{R}\}$.
0	$0^+ \cup 0^-$.
p, q	Eigenvalues of the relative motion vector (or operator), V. $p = (v_1-v_4) + i(v_2+v_3)$, $q = (v_1+v_4) + i(v_2-v_3)$.
$\mathbb{R}$	The real number field.
S	A distance in Euclidean three-space.
S_1, S_2	Subsets of $\mathbb{D}$.
T, T_1, T_2	Linear operators, or linear transformations from $\mathbb{D}$ to $\mathbb{D}$.

u, v, w, s	Real-valued analytic functions; conjugate functions of $f : \mathbb{D} \to \mathbb{D}$.
$u_x, \frac{\partial u}{\partial x}$	The first partial derivative of an analytic function u with respect to the real variable x.
U	Relative motion operator: $U = -v_1 - iv_2 + jv_3 + kv_4$.
$v, v_1, \ldots, v_4$	Real quantities. In the proper context, real components of V.
V	Relative motion vector: $V = v_1 + iv_2 + jv_3 + kv_4$.
x, y, z, ct	Coordinates in real Euclidean four-space.
x', y', z', ct'	Relativistically transformed coordinates in real Euclidean four-space.
X, X'	Four dimensional position vectors: $X = x + iy + jz + kct$, $X' = x' + iy' + jz' + kct'$.
$\square$	The gradient operator in four dimensions: $\square = \frac{\partial}{\partial x} + i\frac{\partial}{\partial y} + j\frac{\partial}{\partial z} + k\frac{\partial}{\partial t}$, or, in certain contexts, $\square = \frac{\partial}{\partial x} + i\frac{\partial}{\partial y} + j\frac{\partial}{\partial z} + k\frac{\partial}{\partial ct}$.
$\lvert\beta\rvert$	Euclidean norm of an element β of $\mathbb{D}$, stated in vector form. $\lvert\beta\rvert^2 = a^2 + b^2 + c^2 + d^2$, or $\lvert X\rvert^2 = x^2 + y^2 + z^2 + c^2t^2$.

I. INTRODUCTION

The objective of this thesis is to present a four-dimensional analogue of the complex calculus. Note carefully the word analogue, because by Frobenius' theorem [1] no commutative four-dimensional division algebra can exist.

In the mid-nineteenth century, the physicist, W. R. Hamilton, devoted fifteen years to studying this area [2, p. 124], seeking to develop a mathematical system in which ordered quadruples of real numbers could be treated as conventional numbers and variables. At the time, the vector- and matrix analysis systems did not exist, nor did any unified treatment of classical mechanics in three space coordinates and time. Hamilton constructed a noncommutative four-dimensional division algebra based on the quaternions [3]. Although, as Sheffers [4] later showed, one cannot construct a function theory and derivatives (satisfying the conventional properties) on a noncommutative algebra, Hamilton and his successors did succeed in stating classical mechanics in terms of quaternions. They avoided the problems of constructing a functional calculus (with its notions of derivatives, etc.) by using the quaternions as vector operators, just as vector basis elements are used today. This treatment, augmented by a growing body of matrix theory applications, remained the most important mode of expression in classical mechanics throughout most of the nineteenth century. Any further searching for an uncomplicated commutative system largely ceased after Frobenius' theorem in 1877.

By 1880, the physicist, J. W. Gibbs, had extracted certain features of the quaternion system and had developed what amounts to a shorthand quaternion notation [5]. By virtue of its ease and clarity, Gibbs' new system (which he named vector calculus) quickly supplanted the quaternion notation. The nature of the change is seen most clearly by noting that the modern vector dot products and cross products are components of a more complicated quaternion product. (In the same way, computations on complex numbers will be subsumed under computations on the elements of the hypercomplex system developed in this thesis.)

In the early twentieth century, there was significant German and Italian interest in the study of analytic functions defined on commutative associative algebras, as indicated in the extensive bibliography of Ward [6]. Perhaps the most familiar name among the authors of that period was Hausdorff [7], who developed function theory, including the Cauchy-Riemann conditions, for commutative, higher-order systems.

Ward's paper [6] in 1940 presented a theory of analytic functions in linear associative algebras, and summarized the state of development of the subject. Although the main results of complex calculus and function theory had been extended into higher dimensions, the form and notation were not. Ward, for example, considered only general expansions of functions over the basis. Ketchum [8] used an infinite-dimensional basis, with variables, functions, and derivatives represented as infinite series. Two subsequent application papers [9, 10] point up the notational complexities.

Of the results of complex analysis which were extended to higher dimensions, we cite Cauchy's integral theorem and Weierstrass' theorem on analytic functions. Such extensions apply to the system to be described in this thesis, because this system's multiplication basis was observed by Ketchum to be a special case of his more general construction [8, p. 662]. He did not pursue its development along the same lines as in this thesis, however.

Since 1940, work in this area has continued at a reduced rate. Agmon and Bers [11] and Douglis [12] presented results on analytic functions. Vekua [13] authored a large volume on generalized analytic functions, and Sloan [14] and McMurray [15] expanded upon the work of Ketchum.

Recent interest has been shown in the quaternion system. Deavours [16] presents a quaternion calculus; Buff [17] characterizes an analytic function of a quaternion variable; Obolašvili [18] presents a three-dimensional analogue of generalized analytic functions; Balabaev [19] discusses a quaternionic analogue of the Cauchy-Riemann system; and Melńičenko [20] derives the Cauchy-Riemann conditions in four dimensions, based on the quaternions. The above results involving quaternions are, of course, limited by Scheffers' result that commutativity is required if properties of a classical sort are to be expected. On the other hand, the system presented below <u>is</u> commutative and <u>does</u> extend classical properties.

II. CONSTRUCTION AND ANALYSIS OF A COMMUTATIVE FOUR-DIMENSIONAL REAL ALGEBRA

This section will develop the algebraic properties of a commutative, hypercomplex system, which will be seen later to admit a calculus with certain classical properties. The algebraic structure will be presented as an analogue of a four-dimensional division algebra over the real numbers.

2.1 Representations and Group Structure of the Basis

Definition 2.1. A four-dimensional, hypercomplex element (or number) β will be denoted by

$$\beta = a + bi + cj + dk, \tag{2.1.1}$$

where a, b, c, d are real and 1, i, j, k are the basis elements, whose multiplicative properties are given by Table 1.

TABLE 1

MULTIPLICATION TABLE FOR THE BASIS

	1	i	j	k
1	1	i	j	k
i	i	-1	k	-j
j	j	k	-1	-i
k	k	-j	-i	1

From the symmetry of Table 1 it is evident that multiplication is commutative; hence 1, i, j, k are not quaternion elements. Table 1 defines a group on the set of basis elements and their negatives.

Theorem 2.2. Let

$$\mathbb{G} = \{1, i, j, k, -1, -i, -j, -k\};$$

then Table 1 defines an abelian multiplicative group of order eight on $\mathbb{G}$, with exactly three cyclic subgroups of order two and two cyclic subgroups of order four.

Proof. As noted previously, multiplication on $\mathbb{G}$ is commutative. The following group properties are also satisfied:

(2.1.2) Existence of an identity element. If $\ell \in \mathbb{G}$, then

$$1 \cdot \ell = \ell \cdot 1 = \ell,$$

hence 1 is the identity element.

(2.1.3) Associative law obeyed. Associativity need be verified only among the three non-identity, unsigned elements i, j, k. By use of Table 1,

$$i(jk) = i(-i) = -i^2 = -(-1) = 1;$$
$$(ij)k = (k)k = k^2 = 1 = i(jk).$$

$$i(kj) = i(-i) = 1;$$
$$(ik)j = (-j)j = -j^2 = -(-1) = 1 = i(kj).$$

$$j(ik) = j(-j) = 1;$$
$$(ji)k = (k)k = 1 = j(ik).$$

$$j(ki) = j(-j) = 1;$$
$$(jk)i = (-i)i = 1 = j(ki).$$

$$k(ij) = k(k) = 1;$$
$$(ki)j = (-j)j = 1 = k(ij).$$

$$k(ji) = k(k) = 1;$$
$$(kj)i = (-i)i = 1 = k(ji).$$

(2.1.4) Existence of inverses.

$$1 \cdot 1 = 1 \cdot 1 = 1$$
$$i(-i) = (-i)i = 1$$
$$j(-j) = (-j)j = 1$$
$$k \cdot k = k \cdot k = 1,$$

hence the respective inverses of 1, i, j, k are 1, -i, -j, k. The cyclic subgroups are:

$$\mathbb{G}_1 = \{(-1), (-1)^2\} = \{-1,1\}$$

$$\mathbb{G}_2 = \{(-i), (-i)^2, (-i)^3, (-i)^4\} = \{-i, -1, i, 1\}$$

$$\mathbb{G}_3 = \{(-j), (-j)^2, (-j)^3, (-j)^4\} = \{-j, -1, j, 1\}$$

$$\mathbb{G}_4 = \{(-k), (-k)^2\} = \{-k,1\}$$

$$\mathbb{G}_5 = \{1, 1^2\} = \{1,1\}$$

$$\mathbb{G}_6 = \{i, i^2, i^3, i^4\} = \{i, -1, -i, 1\}$$

$$\mathbb{G}_7 = \{j, j^2, j^3, j^4\} = \{j, -1, -j, 1\}$$

$$\mathbb{G}_8 = \{k, k^2\} = \{k, 1\}.$$

There are three of order two and two of order four. The proof is concluded.

The next theorem is of central importance in the subsequent developments, because it provides a matrix representation of the basis.

Theorem 2.3. Let

$$E_1 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \quad E_3 = \begin{bmatrix} 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}$$

(2.1.5)

$$E_2 = \begin{bmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{bmatrix} \quad E_4 = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \\ 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix} ;$$

then $\mathbb{E} = \{E_1, E_2, E_3, E_4, -E_1, -E_2, -E_3, -E_4\}$ is an abelian multiplicative group which is isomorphic to $\mathbb{G}$.

Proof. The proof is accomplished by showing that $\mathbb{E}$ and $\mathbb{G}$ have equivalent multiplication tables. The necessary multiplication can be done in block matrix form. Let

$$(2.1.6) \qquad I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad A = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}, \quad 0 = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} ;$$

then

$$(2.1.7) \qquad E_1 = \begin{bmatrix} I & 0 \\ 0 & I \end{bmatrix} \quad E_3 = \begin{bmatrix} 0 & -I \\ I & 0 \end{bmatrix}$$

$$E_2 = \begin{bmatrix} A & 0 \\ 0 & A \end{bmatrix} \quad E_4 = \begin{bmatrix} 0 & -A \\ A & 0 \end{bmatrix} .$$

Note that

$$IA = AI = A$$

(2.1.8)
$$I^2 = I$$

$$A^2 = -I.$$

Now, for example,

$$E_1E_2 = \begin{bmatrix} I & 0 \\ 0 & I \end{bmatrix}\begin{bmatrix} A & 0 \\ 0 & A \end{bmatrix} = \begin{bmatrix} IA & 0 \\ 0 & IA \end{bmatrix} = \begin{bmatrix} A & 0 \\ 0 & A \end{bmatrix} = E_2$$

(2.1.9)
$$E_2E_3 = \begin{bmatrix} A & 0 \\ 0 & A \end{bmatrix}\begin{bmatrix} 0 & -I \\ I & 0 \end{bmatrix} = \begin{bmatrix} 0 & -AI \\ AI & 0 \end{bmatrix} = \begin{bmatrix} 0 & -A \\ A & 0 \end{bmatrix} = E_4$$

$$E_3E_4 = \begin{bmatrix} 0 & -I \\ I & 0 \end{bmatrix}\begin{bmatrix} 0 & -A \\ A & 0 \end{bmatrix} = \begin{bmatrix} -IA & 0 \\ 0 & -IA \end{bmatrix} = -\begin{bmatrix} A & 0 \\ 0 & A \end{bmatrix} = -E_2,$$

and the reader may similarly verify that the remaining entries in Table 2 are correct.

TABLE 2

MULTIPLICATION TABLE FOR THE MATRIX REPRESENTATION

	E_1	E_2	E_3	E_4
E_1	E_1	E_2	E_3	E_4
E_2	E_2	$-E_1$	E_4	$-E_3$
E_3	E_3	E_4	$-E_1$	$-E_2$
E_4	E_4	$-E_3$	$-E_2$	E_1

If the correspondences $1 \longleftrightarrow E_1$, $i \longleftrightarrow E_2$, $j \longleftrightarrow E_3$, $k \longleftrightarrow E_4$ are made, then Table 2 is equivalent to Table 1, page 4. The proof is concluded.

Note that Equations (2.1.6) and (2.1.8) define a 2×2 matrix basis for the real division algebra of complex numbers, and the 4×4 matrix basis is constructed of its elements.

2.2 Generalization of the Basis Group to Orders 2^n

Theorem 2.4. The basis group $\mathbb{G}$ may be generalized to orders 2^n, where $n = 0, 1, 2, \ldots$.

Proof. The simplest demonstration centers about the form of the basis matrices. In the second-order case, they are given by Equations (2.1.6). The fourth-order case, Equations (2.1.7), can be written in terms of the second-order case; hence the fourth-order multiplicative properties are completely fixed by the second-order properties. There

is also a two-for-one correspondence of elements: $I \longleftrightarrow \{E_1, E_4\}$ and $A \longleftrightarrow \{E_2, E_3\}$. The first correspondence is between group elements of orders one or two, and the second correspondence is between those of order four.

The generalization merely repeats the process with successive 2^n-orders. For example, the eighth-order basis matrices in block form are

$$F_1 = \begin{bmatrix} E_1 & 0 \\ 0 & E_1 \end{bmatrix} \qquad F_5 = \begin{bmatrix} 0 & -E_1 \\ E_1 & 0 \end{bmatrix}$$

$$F_2 = \begin{bmatrix} E_2 & 0 \\ 0 & E_2 \end{bmatrix} \qquad F_6 = \begin{bmatrix} 0 & -E_2 \\ E_2 & 0 \end{bmatrix}$$

(2.2.1)

$$F_3 = \begin{bmatrix} E_3 & 0 \\ 0 & E_3 \end{bmatrix} \qquad F_7 = \begin{bmatrix} 0 & -E_3 \\ E_3 & 0 \end{bmatrix}$$

$$F_4 = \begin{bmatrix} E_4 & 0 \\ 0 & E_4 \end{bmatrix} \qquad F_8 = \begin{bmatrix} 0 & -E_4 \\ E_4 & 0 \end{bmatrix} ,$$

where now 0 is a 4×4 zero matrix. The two-for-one correspondences are:

$$\{E_1, E_4\} \longleftrightarrow \{F_1, F_4, F_6, F_7\}$$

(2.2.2)

$$\{E_2, E_3\} \longleftrightarrow \{F_2, F_3, F_5, F_8\}.$$

The proof is concluded.

It is apparent that the entire sequence of 2^n-order groups is increasing, in the sense that the 2^n-order group is constructed exclusively from the elements of the 2^{n-1}-order group, and has the same multiplicative properties as the 2^{n-1}-order group as a result. This being so, all subsequent developments in this thesis can be generalized to spaces of dimension 2^n, where n is a positive integer.

2.3 Algebraic Operations and Ring Structure

Definitions of algebraic operations.

Definition 2.5. Let $\mathbb{D}$ be the set of all real, ordered quadruples (a,b,c,d), and make the correspondence

$$(a,b,c,d) \longleftrightarrow a + bi + cj + dk.$$

Then $\mathbb{D}$ may also be considered as the set of all four-dimensional hypercomplex numbers.

The matrix representation of the basis may be used to define algebraic operations on $\mathbb{D}$. In terms of matrices (2.1.5), an arbitrary number $\beta \in \mathbb{D}$ may be represented as

$$\beta = aE_1 + bE_2 + cE_3 + dE_4, \tag{2.3.1}$$

or

$$\beta = a\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} + b\begin{bmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

(2.3.2)

$$c\begin{bmatrix} 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix} + d\begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \\ 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix},$$

or

(2.3.3) $$\beta = \begin{bmatrix} a & -b & -c & d \\ b & a & -d & -c \\ c & -d & a & -b \\ d & c & b & a \end{bmatrix}.$$

If $\beta_1, \beta_2 \in \mathbb{D}$, then form (2.3.1) and the common properties of matrices may be used to define

(2.3.4) $$\beta_1 + \beta_2 = (a_1+a_2)E_1 + (b_1+b_2)E_2 + (c_1+c_2)E_3 + (d_1+d_2)E_4$$

(2.3.5) $$\beta_1 - \beta_2 = (a_1-a_2)E_1 + (b_1-b_2)E_2 + (c_1-c_2)E_3 + (d_1-d_2)E_4.$$

Form (2.3.1) and Table 2 may be used to define

$$\beta_1\beta_2 = (a_1E_1 + b_1E_2 + c_1E_3 + d_1E_4)(a_2E_1 + b_2E_2 + c_2E_3 + d_2E_4)$$

$$= a_1a_2E_1E_1 + a_1b_2E_1E_2 + a_1c_2E_1E_3 + a_1d_2E_1E_4$$

$$+ b_1a_2E_2E_1 + b_1b_2E_2E_2 + b_1c_2E_2E_3 + b_1d_2E_2E_4$$

$$+ c_1a_2E_3E_1 + c_1b_2E_3E_2 + c_1c_2E_3E_3 + c_1d_2E_3E_4$$

$$+ d_1a_2E_4E_1 + d_1b_2E_4E_2 + d_1c_2E_4E_3 + d_1d_2E_4E_4,$$

$$= a_1a_2E_1 + a_1b_2E_2 + a_1c_2E_3 + a_1d_2E_4$$

$$+ b_1a_2E_2 - b_1b_2E_1 + b_1c_2E_4 - b_1d_2E_3$$

$$+ c_1a_2E_3 + c_1b_2E_4 - c_1c_2E_1 - c_1d_2E_2$$

$$+ d_1a_2E_4 - d_1b_2E_3 - d_1c_2E_2 + d_1d_2E_1,$$

$$\beta_1\beta_2 = (a_1a_2 - b_1b_2 - c_1c_2 + d_1d_2)E_1$$

$$+ (b_1a_2 + a_1b_2 - d_1c_2 - c_1d_2)E_2$$

(2.3.6)

$$+ (c_1a_2 - d_1b_2 + a_1c_2 - b_1d_2)E_3$$

$$+ (d_1a_2 + c_1b_2 + b_1c_2 + a_1d_2)E_4.$$

The commutativity of the basis assures that

$$\beta_2\beta_1 = \beta_1\beta_2.$$

Form (2.3.3) requires the following definition of the multiplicative inverse of a number β (wherever it exists):

$$\beta^{-1} = \begin{bmatrix} a & -b & -c & d \\ b & a & -d & -c \\ c & -d & a & -b \\ d & c & b & a \end{bmatrix}^{-1} . \tag{2.3.7}$$

It is instructive to expand the matrix inverse, as follows. In the matrix form,

$$\det(\beta) = \begin{vmatrix} a & -b & -c & d \\ b & a & -d & -c \\ c & -d & a & -b \\ d & c & b & a \end{vmatrix}$$

$$= [(a-d)^2 + (b+c)^2][(a+d)^2 + (b-c)^2], \tag{2.3.8}$$

after a great deal of algebraic rearrangement. Next, let B_{ij} denote the signed minor of the ith element in the jth column of β; then

$$(2.3.9) \qquad \text{adj } (\beta) = \begin{bmatrix} B_{11} & B_{21} & B_{31} & B_{41} \\ B_{12} & B_{22} & B_{32} & B_{42} \\ B_{13} & B_{23} & B_{33} & B_{43} \\ B_{14} & B_{24} & B_{34} & B_{44} \end{bmatrix} .$$

Direct calculation of the signed minors yields

$$(2.3.10) \qquad B_{11} = B_{22} = B_{33} = B_{44}$$

$$= a(a^2 + b^2 + c^2 + d^2) - 2d(ad - bc)$$

$$(2.3.11) \qquad B_{12} = - B_{21} = - B_{43} = B_{34}$$

$$= -b(a^2 + b^2 + c^2 + d^2) - 2c(ad - bc)$$

$$(2.3.12) \qquad B_{13} = - B_{31} = - B_{42} = B_{24}$$

$$= -c(a^2 + b^2 + c^2 + d^2) - 2b(ad - bc)$$

$$(2.3.13) \qquad B_{14} = -B_{23} = - B_{32} = B_{41}$$

$$= d(a^2 + b^2 + c^2 + d^2) - 2a(ad - bc).$$

Equation (2.3.9) may then be expanded into a simple form, using (2.3.10) through (2.3.13):

$$\text{adj}\ (\beta) = \begin{bmatrix} B_{11} & -B_{12} & -B_{13} & B_{14} \\ B_{12} & B_{11} & -B_{14} & -B_{13} \\ B_{13} & -B_{14} & B_{11} & -B_{12} \\ B_{14} & B_{13} & B_{12} & B_{11} \end{bmatrix}$$

$$= B_{11}\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} + B_{12}\begin{bmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

(2.3.14)

$$+ B_{13}\begin{bmatrix} 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix} + B_{14}\begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \\ 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}$$

$$= B_{11}E_1 + B_{12}E_2 + B_{13}E_3 + B_{14}E_4.$$

Note that β and adj (β) both have the same form (i.e., belong to the same subset of all real-valued 4×4 matrices). The inverse of β is thus

(2.3.15) $$\beta^{-1} = \text{adj}\ (\beta)/\det\ (\beta)$$

$$= (B_{11}E_1 + B_{12}E_2 + B_{13}E_3 + B_{14}E_4)/$$
$$\{[(a - d)^2 + (b + c)^2][(a + d)^2 + (b - c)^2]\}.$$

Noting the correspondences in Table 1, page 4, and Table 2, page 10, the following definitions are suggested:

Definition 2.6. If $\beta, \beta_1, \beta_2 \in \mathbb{D}$, and

$$\beta = a + bi + cj + dk$$

$$\beta_1 = a_1 + b_1 i + c_1 j + d_1 k$$

$$\beta_2 = a_2 + b_2 i + c_2 j + d_2 k,$$

then

$$\beta_1 + \beta_2 = (a_1 + a_2) + (b_1 + b_2)i + (c_1 + c_2)j + (d_1 + d_2)k \tag{2.3.16}$$

$$\beta_1 - \beta_2 = (a_1 - a_2) + (b_1 - b_2)i + (c_1 - c_2)j + (d_1 - d_2)k \tag{2.3.17}$$

$$\begin{aligned} \beta_1\beta_2 = {} & (a_1a_2 - b_1b_2 - c_1c_2 + d_1d_2) \\ & + i(b_1a_2 + a_1b_2 - d_1c_2 - c_1d_2) \\ & + j(c_1a_2 - d_1b_2 + a_1c_2 - b_1d_2) \\ & + k(d_1a_2 + c_1b_2 + b_1c_2 + a_1d_2) \end{aligned} \tag{2.3.18}$$

$$(2.3.19)\quad \beta^{-1} = \frac{a(a^2 + b^2 + c^2 + d^2) - 2d(ad - bc)}{[(a - d)^2 + (b + c)^2][(a + d)^2 + (b - c)^2]}$$

$$-i\,\frac{b(a^2 + b^2 + c^2 + d^2) + 2c(ad - bc)}{[(a - d)^2 + (b + c)^2][(a + d)^2 + (b - c)^2]}$$

$$-j\,\frac{c(a^2 + b^2 + c^2 + d^2) + 2b(ad - bc)}{[(a - d)^2 + (b + c)^2][(a + d)^2 + (b - c)^2]}$$

$$+k\,\frac{d(a^2 + b^2 + c^2 + d^2) - 2a(ad - bc)}{[(a - d)^2 + (b + c)^2][(a + d)^2 + (b - c)^2]}\,.$$

Up to Equation (2.3.19), the development thus far has been rather conventional; but (2.3.19) indicates that the multiplicative inverse of a number in four dimensions will not exist if either (a = d and b = -c) or (a = -d and b = c). However, it will be shown later that these conditions define separate, orthogonal, planar subspaces which play a part in the present algebra analogous to the part played by the multiplicative zero element in the real algebra of complex numbers.

Definitions 2.6 reduce to the ordinary complex variable definitions when c and d are set equal to zero:

$$\beta_1 + \beta_2 = (a_1 + a_2) + i(b_1 + b_2)$$
$$\beta_1 - \beta_2 = (a_1 - a_2) + i(b_1 - b_2)$$
$$\beta_1\beta_2 = (a_1a_2 - b_1b_2) + i(a_1b_2 + b_1a_2)$$
$$\beta^{-1} = (a - bi)/(a^2 + b^2).$$

Ring structure of the hypercomplex algebra.

Theorem 2.7. Definitions 2.6 produce a commutative ring with unity on $\mathbb{D}$.

Proof. Multiplication on $\mathbb{D}$ is commutative because of the symmetry of Table 1, page 4. If β, β_1, β_2, β_3 denote arbitrary elements of $\mathbb{D}$, then

(2.3.20) Addition is commutative.

$$\beta_1 + \beta_2 = (a_1 + a_2) + i(b_1 + b_2) + j(c_1 + c_2) + k(d_1 + d_2)$$

$$= (a_2 + a_1) + i(b_2 + b_1) + j(c_2 + c_1) + k(d_2 + d_1)$$

$$= \beta_2 + \beta_1.$$

(2.3.21) Addition is associative.

$$(\beta_1 + \beta_2) + \beta_3 = \{(a_1 + a_2) + i(b_1 + b_2) + j(c_1 + c_2) + k(d_1 + d_2)\}$$

$$+ \{a_3 + ib_3 + jc_3 + kd_3\}$$

$$= [(a_1 + a_2) + a_3] + i[(b_1 + b_2) + b_3]$$

$$+ j[(c_1 + c_2) + c_3] + k[(d_1 + d_2) + d_3]$$

$$= [a_1 + (a_2 + a_3)] + i[b_1 + (b_2 + b_3)]$$

$$+ j[c_1 + (c_2 + c_3)] + k[d_1 + (d_2 + d_3)]$$

$$= \{a_1 + ib_1 + jc_1 + kd_1\}$$

$$+ \{(a_2 + a_3) + i(b_2 + b_3) + j(c_2 + c_3) + k(d_2 + d_3)\}$$

$$= \beta_1 + (\beta_2 + \beta_3).$$

(2.3.22) The element $\theta = 0 + 0i + 0j + 0k$ plays the role of an additive zero, because

$$\beta + \theta = (a + 0) + i(b + 0) + j(c + 0) + k(d + 0)$$

$$= a + ib + jc + kd$$

$$= \beta.$$

(2.3.23) Additive inverses exist.

$$(-\beta) = (-a) + i(-b) + j(-c) + k(-d)$$

$$\beta + (-\beta) = [a + (-a)] + i[b + (-b)] + j[c + (-c)] + k[d + (-d)]$$

$$= 0 + i0 + j0 + k0$$

$$= \theta.$$

(2.3.24) Multiplication is associative.

Commutativity of addition and multiplication, and associativity of multiplication of the basis elements have been established. Using these properties freely, one may obtain

$$(\beta_1\beta_2)\beta_3 = \{(a_1a_2-b_1b_2-c_1c_2+d_1d_2) + i(b_1a_2+a_1b_2-d_1c_2-c_1d_2)$$

$$+ j(c_1a_2-d_1b_2+a_1c_2-b_1d_2) + k(d_1a_2+c_1b_2+b_1c_2+a_1d_2)\}$$

$$\cdot \{a_3 + ib_3 + jc_3 + kd_3\}$$

$$= [a_3(a_1a_2-b_1b_2-c_1c_2+d_1d_2) - b_3(b_1a_2+a_1b_2-d_1c_2-c_1d_2)$$

$$-c_3(c_1a_2-d_1b_2+a_1c_2-b_1d_2) + d_3(d_1a_2+c_1b_2+b_1c_2+a_1d_2)]$$

$$+ i[a_3(b_1a_2+a_1b_2-d_1c_2-c_1d_2) + b_3(a_1a_2-b_1b_2-c_1c_2+d_1d_2)$$

$$- c_3(d_1a_2+c_1b_2+b_1c_2+a_1d_2) - d_3(c_1a_2-d_1b_2+a_1c_2-b_1d_2)]$$

$$+ j[a_3(c_1a_2-d_1b_2+a_1c_2-b_1d_2) - b_3(d_1a_2+c_1b_2+b_1c_2+a_1d_2)$$

$$+ c_3(a_1a_2-b_1b_2-c_1c_2+d_1d_2) - d_3(b_1a_2+a_1b_2-d_1c_2-c_1d_2)]$$

$$+ k[a_3(d_1a_2+c_1b_2+b_1c_2+a_1d_2) + b_3(c_1a_2-d_1b_2+a_1c_2-b_1d_2)$$

$$+ c_3(b_1a_2+a_1b_2-d_1c_2-c_1d_2) + d_3(a_1a_2-b_1b_2-c_1c_2+d_1d_2)]$$

$$= [a_1(a_2a_3-b_2b_3-c_2c_3+d_2d_3) - b_1(b_2a_3+a_2b_3-d_2c_3-c_2d_3)$$

$$- c_1(c_2a_3-d_2b_3+a_2c_3-b_2d_3) + d_1(d_2a_3+c_2b_3+b_2c_3+a_2d_3)]$$

$$+ i[a_1(b_2a_3+a_2b_3-d_2c_3-c_2d_3) + b_1(a_2a_3-b_2b_3-c_2c_3+d_2d_3)$$

$$- c_1(d_2a_3+c_2b_3+b_2c_3+a_2d_3) - d_1(c_2a_3-d_2b_3+a_2c_3-b_2d_3)]$$

$$+ j[a_1(c_2a_3-d_2b_3+a_2c_3-b_2d_3) - b_1(d_2a_3+c_2b_3+b_2c_3+a_2d_3)$$

$$+ c_1(a_2a_3-b_2b_3-c_2c_3+d_2d_3) - d_1(b_2a_3+a_2b_3-d_2c_3-c_2d_3)]$$

$$+ k[a_1(d_2a_3+c_2b_3+b_2c_3+a_2d_3) + b_1(c_2a_3-d_2b_3+a_2c_3-b_2d_3)$$

$$+ c_1(b_2a_3 + a_2b_3-d_2c_3-c_2d_3) + d_1(a_2a_3-b_2b_3-c_2c_3+d_2d_3)]$$

$$= (\beta_2\beta_3)\beta_1 ;$$

and from commutativity,

$$(\beta_1\beta_2)\beta_3 = (\beta_2\beta_3)\beta_1 = \beta_1(\beta_2\beta_3).$$

(2.3.25) Distributive laws are obeyed.

$$\beta_2 + \beta_3 = (a_2+a_3) + (b_2+b_3)i + (c_2+c_3)j + (d_2+d_3)k$$

$$\begin{aligned}\beta_1(\beta_2 + \beta_3) &= [a_1(a_2+a_3)-b_1(b_2+b_3)-c_1(c_2+c_3)+d_1(d_2+d_3)] \\ &\quad + i[b_1(a_2+a_3)+a_1(b_2+b_3) - d_1(c_2+c_3) - c_1(d_2+d_3)] \\ &\quad + j[c_1(a_2+a_3)-d_1(b_2+b_3) + a_1(c_2+c_3) - b_1(d_2+d_3)] \\ &\quad + k[d_1(a_2+a_3)+c_1(b_2+b_3) + b_1(c_2+c_3) + a_1(d_2+d_3)] \\ &= [(a_1a_2-b_1b_2-c_1c_2+d_1d_2) + (a_1a_3-b_1b_3-c_1c_3+d_1d_3)] \\ &\quad + i[(b_1a_2+a_1b_2-d_1c_2-c_1d_2) + (b_1a_3+a_1b_3-d_1c_3-c_1d_3)] \\ &\quad + j[(c_1a_2-d_1b_2+a_1c_2-b_1d_2) + (c_1a_3-d_1b_3+a_1c_3-b_1d_3)] \\ &\quad + k[(d_1a_2+c_1b_2+b_1c_2+a_1d_2) + (d_1a_3+c_1b_3+b_1c_3+a_1d_3)] \\ &= \{(a_1a_2-b_1b_2-c_1c_2+d_1d_2) + i(b_1a_2+a_1b_2-d_1c_2-c_1d_2) \\ &\quad + j(c_1a_2-d_1b_2+a_1c_2-b_1d_2) + k(d_1a_2+c_1b_2+b_1c_2+a_1d_2)\}\end{aligned}$$

$$+ \{(a_1a_3-b_1b_3-c_1c_3+d_1d_3) + i(b_1a_3+a_1b_3-d_1c_3-c_1d_3)$$

$$+ j(c_1a_3-d_1b_3+a_1c_3-b_1d_3) + k(d_1a_3+c_1b_3+b_1c_3+a_1d_3)\}$$

$$= \beta_1\beta_2 + \beta_1\beta_3.$$

From commutativity of multiplication,

$$\beta_1(\beta_2 + \beta_3) = (\beta_2 + \beta_3)\beta_1.$$

The unity element is (1,0,0,0), because

$$(a + bi + cj + dk)(1 + 0i + 0j + 0k)$$

$$= (a \cdot 1 - b \cdot 0 - c \cdot 0 + d \cdot 0)$$

$$+ i(b \cdot 1 + a \cdot 0 - d \cdot 0 - c \cdot 0)$$

$$+ j(c \cdot 1 - d \cdot 0 + a \cdot 0 - b \cdot 0)$$

$$+ k(d \cdot 1 + c \cdot 0 + b \cdot 0 + a \cdot 0)$$

$$= a + bi + cj + dk.$$

The proof of Theorem 2.7 is concluded.

Scalar multiplication by real numbers on elements of $\mathbb{D}$ may be introduced as a special case of multiplication in $\mathbb{D}$. Then $\mathbb{D}$ is easily seen to be a real algebra, which contains as a subalgebra the real algebra of complex numbers. In subsequent sections, the symbol $\mathbb{D}$ may mean, in the proper context, "the real algebra of all four-dimensional numbers."

A new notational system will now be developed which will simplify all calculations involving elements of $\mathbb{D}$. Some preliminary results will be needed.

Lemma 2.8. If $\beta \in \mathbb{D}$, then

$$\begin{aligned} \beta &= a + bi + cj + dk \\ &= [(a-d) + i(b+c)](1-k)/2 + [(a+d) + i(b-c)](1+k)/2. \end{aligned} \tag{2.3.26}$$

Proof. Let $\beta = a + bi + cj + dk$, and separate it into two equal parts:

$$\beta = (a/2) + (b/2)i + (c/2)j + (d/2)k$$

$$+ (a/2) + (b/2)i + (c/2)j + (d/2)k.$$

Now add and subtract $-(d/2) + (c/2)i + (b/2)j - (a/2)k$ on the right-hand side and rearrange to get

$$\beta = (a/2) + (b/2)i + (c/2)j + (d/2)k - (d/2) + (c/2)i + (b/2)j - (a/2)k$$

$$+ (a/2) + (b/2)i + (c/2)j + (d/2)k + (d/2) - (c/2)i - (b/2)j + (a/2)k$$

$$= (a-d)/2 + i(b+c)/2 + j(c+b)/2 + k(d-a)/2$$

$$+ (a+d)/2 + i(b-c)/2 + j(c-b)/2 + k(d+a)/2$$

$$= (a-d)(1-k)/2 + (b+c)(i+j)/2$$

$$+ (a+d)(1+k)/2 + (b-c)(i-j)/2$$

$$= (a-d)(1-k)/2 + (b+c)i(1-k)/2$$

$$+(a+d)(1+k)/2 + (b-c)i(1+k)/2$$

$$\equiv [(a-d) + i(b+c)](1-k)/2$$
$$+ [(a+d) + i(b-c)](1+k)/2.$$

The proof is concluded.

The new notation is set down in the following definition:

Definition 2.9. Let $\beta \in \mathbb{D}$; then the canonical form of $\beta = a + bi + cj + dk$ is

$$\beta = \xi e_1 + \eta e_2, \tag{2.3.27}$$

where

$$\xi = (a-d) + i(b+c)$$

$$\eta = (a+d) + i(b-c),$$

and

$$e_1 = (1-k)/2$$

$$e_2 = (1+k)/2.$$

The factors ξ, η, e_1, e_2 are all elements of $\mathbb{D}$, and so may be manipulated as elements of a ring. The value of the canonical notation centers upon the properties of the elements e_1, e_2 as indicated in the following lemma.

Lemma 2.10. Each element of the set $\{1, e_1, e_2\}$ is its own multiplicative inverse in $\mathbb{D}$. Moreover, $e_1e_2 = \theta$, $e_1^n = e_1$, and $e_2^n = e_2$, where n is an integer.

Proof. Table 1, page 4, shows that

$$e_1 e_1 = [(1-k)/2][(1-k)/2]$$

$$= (1-2k+k^2)/4$$

$$= 2(1-k)/4$$

$$= e_1.$$

Similarly,

$$e_2 e_2 = e_2,$$

hence e_1 and e_2 are each their own multiplicative inverses. In the same way,

$$e_1 e_2 = [(1-k)/2][(1+k)/2]$$

$$= (1-k^2)/4$$

$$= \theta.$$

As shown above, $e_1^2 = e_1$; and by induction, $e_1^n = e_1$. Similar results hold for e_2. The proof is concluded.

The first useful consequence of Lemma 2.10 is that the multiplication operation that starts with two numbers in canonical form and

yields a product in canonical form, is notationally simple:

$$(\xi_1 e_1 + \eta_1 e_2)(\xi_2 e_1 + \eta_2 e_2)$$

$$= \xi_1\xi_2 e_1 e_1 + \xi_1\eta_2 e_1 e_2 + \eta_1\xi_2 e_2 e_1 + \eta_1\eta_2 e_2 e_2$$

(2.3.28)

$$= \xi_1\xi_2 e_1 + \xi_1\eta_2\,\theta + \eta_1\xi_2\,\theta + \eta_1\eta_2 e_2$$

$$= \xi_1\xi_2 e_1 + \eta_1\eta_2 e_2.$$

Secondly, the inverse of

$$\beta = \xi e_1 + \eta e_2,$$

wherever it exists, is

$$\beta^{-1} = \xi^{-1} e_1 + \eta^{-1} e_2,$$

where $\xi \neq 0$, $\eta \neq 0$, because by (2.3.28),

$$\beta\beta^{-1} = (\xi e_1 + \eta e_2)(\xi^{-1} e_1 + \eta^{-1} e_2)$$

$$= \xi\xi^{-1} e_1 + \eta\eta^{-1} e_2$$

(2.3.29) $$= e_1 + e_2$$

$$= (1-k)/2 + (1+k)/2$$

$$= 1.$$

Addition and subtraction rules follow naturally from the fact that e_1 and e_2 are elements of a ring. To summarize, if β, β_1, $\beta_2 \in \mathbb{D}$, then the four elementary algebraic operations in canonical form are:

$$\beta_1 + \beta_2 = (\xi_1 + \xi_2)e_1 + (\eta_1 + \eta_2)e_2 \tag{2.3.30}$$

$$\beta_1 - \beta_2 = (\xi_1 - \xi_2)e_1 + (\eta_1 - \eta_2)e_2 \tag{2.3.31}$$

$$\beta_1\beta_2 = \xi_1\xi_2 e_1 + \eta_1\eta_2 e_2 \tag{2.3.32}$$

$$\beta^{-1} = \xi^{-1}e_1 + \eta^{-1}e_2, \text{ if } \xi \neq \theta \text{ and } \eta \neq \theta. \tag{2.3.33}$$

From the point of view of clarity and notational simplicity, compare these formulas with the corresponding formulas (2.3.16) through (2.3.19) in the 1, i, j, k vector notation.

The canonical notation is used to advantage in the proof of the next theorem, for which the following definitions are required.

Definition 2.11. The subset $0^+ \subset \mathbb{D}$ is the set of all hypercomplex numbers of the form

$$\beta = a + bi + bj - ak = 2(a + bi)e_1, \tag{2.3.34}$$

and the subset $0^- \subset \mathbb{D}$ is the set of all hypercomplex numbers of the form

(2.3.35) $\beta = a + bi - bj + ak = 2(a + bi)e_2.$

The subset $\mathbb{F} \subset \mathbb{D}$ is the complement of $0^+ \cup 0^-$ in $\mathbb{D}$; then

$$\mathbb{D} = \mathbb{F} \cup 0^+ \cup 0^-$$

and

$$\mathbb{F} \cap (0^+ \cup 0^-) = \emptyset.$$

Definition 2.12. A trivial ideal of a ring $\mathbb{D}$ is either $\{\theta\}$ or $\mathbb{D}$ itself; a nontrivial ideal of $\mathbb{D}$ is one which is not trivial.

Lemma 2.13. Every element of $\mathbb{F}$ is invertible.

Proof. By Definition 2.11, $\mathbb{F}$ is the complement of $0^+ \cup 0^-$ in $\mathbb{D}$. By Definitions 2.6 and 2.11 the multiplicative inverse is defined everywhere except in 0^+ and 0^-. The desired result follows.

The next theorem could be shown (by more advanced methods) to follow from a later theorem, Theorem 2.15. A direct proof is given, nonetheless.

Theorem 2.14. $\mathbb{D}$ is a principal ideal ring having exactly two nontrivial ideals, both of which are maximal.

Proof. Consider the element e_1; multiply an arbitrary element $\beta = \xi e_1 + \eta e_2$ of $\mathbb{D}$ by e_1 and get

$$e_1\beta = \xi e_1,$$

an element of 0^+. Further, as β ranges over $\mathbb{D}$, ξ ranges over $\mathbb{C}$, and every element of 0^+ is generated. In set notation,

(2.3.36) $$e_1\mathbb{D} = 0^+.$$

In a similar fashion,

(2.3.37) $$e_2\mathbb{D} = 0^-.$$

Subsets 0^+ and 0^- are therefore principal ideals of $\mathbb{D}$, generated by e_1 and e_2, respectively. Next, consider any nonzero element $\beta_1 = \xi_1 e_1$ of 0^+: it also generates 0^+, because

$$\beta_1\beta = \xi_1\xi e_1,$$

and as β ranges over $\mathbb{D}$, $\xi_1\xi$ ranges over $\mathbb{C}$. In the notation of sets,

(2.3.38) $$\xi_1 e_1\mathbb{D} = \xi_1 0^+ = 0^+.$$

Similarly,

(2.3.39) $$\eta_1 e_2\mathbb{D} = \eta_1 0^- = 0^-.$$

In the same way, if $\beta_1 = \xi_1 e_1 + \eta_1 e_2$ is an arbitrary, fixed element of $\mathbb{F}$, with $\xi_1 \neq 0$, $\eta_1 \neq 0$, then by (2.3.38), (2.3.39), and the fact that

$$\beta_1\beta = \xi_1\xi e_1 + \eta_1\eta e_2,$$

one sees that β_1 generates the trivial ideal, $\mathbb{D}$:

$$\beta_1\mathbb{D} = \mathbb{D}. \tag{2.3.40}$$

At this point one may conclude that $\mathbb{D}$ has exactly two nontrivial principal ideals.

To show that 0^+ is maximal, consider the ideal S_1 generated by 0^+ and element β_2, where $\beta_2 \notin S_1$. Without loss of generality, assume $\beta_2 \notin \mathbb{F}$ because every element of $\mathbb{F}$ is invertible, by Lemma 2.13; thus $\beta_2 \in 0^-$. Now, $\beta_2 \neq \theta$ because $\theta \in 0^+$, hence by the arguments leading to (2.3.39), $0^- \subset S_1$. Consequently, $1 = e_1 + e_2$ is an element of $0^+ \cup 0^- \subset S_1$, forcing $S_1 = \mathbb{D}$. Similar arguments show that 0^- is also maximal.

Now suppose for contradiction that $\mathbb{D}$ has an ideal S_2 which is not principal. For arbitrary elements $\beta_1 \in S_2$ and $\beta \in \mathbb{D}$ the product $\beta_1\beta$ belongs to S_2. Further, for fixed β_1, the <u>set</u> of all such products, as β ranges over $\mathbb{D}$, belongs to S_2. But since $\mathbb{D} = \mathbb{F} \cup 0^+ \cup 0^-$, β_1 is in at least one of $\mathbb{F}$, 0^+, or 0^-, and equations (2.3.38) through (2.3.40) show that if $\beta_1 \neq \theta$, then S_2 must contain at least one of $\mathbb{F}$, 0^+, or 0^-. Since 0^+ and 0^- are maximal from above, S_2 is principal, in contradiction to the original assumption; hence no nontrivial, non-principal ideals of $\mathbb{D}$ exist. The proof is concluded.

The last theorem, below, of this section concerns an isomorphism of $\mathbb{D}$.

Theorem 2.15. The ring $\mathbb{D}$ is isomorphic to the product ring $\mathbb{C}^2$ of ordered pairs of complex numbers.

Proof. Let $\beta = \xi e_1 + \eta e_2$ be a typical element of $\mathbb{D}$, where ξ, η, e_1, e_2 are as above. Make the correspondence

$$(\xi,\eta) \longleftrightarrow \xi e_1 + \eta e_2. \tag{2.3.41}$$

Note that if β_1, $\beta_2 \in \mathbb{D}$, then the corresponding multiplication operation is

$$(\xi_1,\eta_1)(\xi_2,\eta_2) = (\xi_1\xi_2,\eta_1\eta_2),$$

as in $\mathbb{C}^2$. All the other elementary algebraic operations on $\mathbb{D}$ also carry over. Since the correspondence (2.3.41) is bijective, $\mathbb{D}$ and $\mathbb{C}^2$ are isomorphic as rings; indeed, as real algebras.

2.4 Axis Plane Subfields and Noninvertible Elements

This section will further characterize the ideals 0^+ and 0^-, and will present $\mathbb{D}$ as an analogue of a (hypothetical) commutative, four-dimensional, real division algebra.

Theorem 2.16. The ideals 0^+ and 0^- of $\mathbb{D}$ are fields, and $\mathbb{D}$ is a direct product of 0^+ and 0^-.

Proof. Consider the commutative subring 0^+. The typical element is $\beta = \xi e_1$, and the unity element is e_1, because

$$e_1\beta = e_1\xi e_1 = \xi e_1 = \beta.$$

Further, e_1 is its own inverse, by Lemma 2.10. The inverse of $\beta \in 0^+$ is

$$\begin{aligned} \beta^{-1} &= (\xi e_1)^{-1} \\ &= e_1^{-1}\xi^{-1} \\ &= e_1\xi^{-1} \\ &= \xi^{-1}e_1, \end{aligned} \tag{2.4.1}$$

wherever $\xi \neq 0$, because then

$$\begin{aligned} \beta^{-1}\beta &= \xi^{-1}e_1\xi e_1 \\ &= \xi^{-1}\xi e_1 e_1 \\ &= e_1 e_1 \\ &= e_1. \end{aligned} \tag{2.4.2}$$

Subring 0^+ is therefore a field and similarly for 0^-, with unity element e_2.

Now $\mathbb{D}$ is a direct product of 0^+ and 0^-, because $e_1e_2 = \theta$ and a typical element $\beta \in \mathbb{D}$ can be expressed uniquely as a sum of an element from 0^+ and an element from 0^-, by Lemma 2.8. The proof is concluded.

The preceding theorem could have been anticipated, in view of the fact that $\mathbb{C}^2$ is representable as a direct product of fields. In fact, $0^+ \cong \mathbb{C} \cong 0^-$.

The fields 0^+ and 0^- have a useful geometrical interpretation, as follows. Consider an ordered quadruple (x,y,z,t) belonging to $\mathbb{D}$, and associate each real component x, y, z, t with an orthogonal axis in four dimensions. The defining conditions for 0^+ and 0^-,

$$0^- : x = t,\ y = -z$$

(2.4.3)

$$0^+ : x = -t,\ y = z,$$

define two planes in four dimensions. (They are shown later to be orthogonal). Let e_1 be considered as an operator via multiplication (i.e., a projection operator, because $e_1^2 = e_1$), and let it operate on an arbitrary point $\beta \in \mathbb{D}$. Then β will be projected onto the plane containing 0^+ and similarly for e_2 and 0^-. Observe that the projection of an element in $\mathbb{D}$ onto 0^+ and 0^- uniquely characterizes that element, by Lemma 2.8; consequently, 0^+ and 0^- may be viewed as "coordinate axes" in four dimensions.

Definition 2.17. The plane containing 0^+ will be called the 0^+-axis, and that containing 0^- will be called the 0^--axis. Collectively, they will be called axis planes.

The above definition will be used to clarify further geometrical interpretations in Section 3.1, below.

The properties of the axis planes also admit an algebraic interpretation: Let $0 = 0^+ \cup 0^-$; then 0 is analogous in $\mathbb{D}$ to the multiplicative zero element in $\mathbb{C}$. For illustration, consider the properties of the zero element of $\mathbb{C}$: If 0 is the zero element,

(i) No inverse of 0 exists in $\mathbb{C}$.

(ii) If $\beta \in \mathbb{C}$, then $0 \cdot \beta = 0$.

(iii) If $\alpha, \beta \in \mathbb{C}$, then $\alpha\beta = 0$ implies either $\alpha = 0$ or $\beta = 0$, or both.

These relations can be stated in set notation. Let $\theta = (0,0)$ and $0 = \{\theta\}$. Then

(2.4.4) Elements of 0 have no inverses in $\mathbb{C}$.

(2.4.5) If $\gamma \in 0$ and $\beta \in \mathbb{C}$, then $\gamma\beta \in 0$.

(2.4.6) If $\alpha, \beta \in \mathbb{C}$, then $\alpha\beta \in 0$ implies either $\alpha \in 0$ or $\beta \in 0$, or both.

The four-dimensional analogues of statements (2.4.4) through (2.4.6) are given in the following lemma.

Lemma 2.18. Let $0 = 0^+ \cup 0^-$. Then

(2.4.7) Elements of 0 have no inverses in $\mathbb{D}$.

(2.4.8) If $\gamma \in 0$ and $\beta \in \mathbb{D}$, then $\gamma\beta \in 0$.

(2.4.9) If $\alpha, \beta \in \mathbb{D}$, then $\alpha\beta \in 0$ implies either $\alpha \in 0$ or $\beta \in 0$, or both.

Proof. For the first statement, Equation (2.3.19) along with Definition 2.11, which defines 0^+ and 0^-, show that multiplicative inverses are not defined in 0.

For the second statement, let $\gamma \in 0^+$ and $\beta \in \mathbb{D}$; then $\gamma\beta \in 0^+$ since 0^+ is an ideal (Theorem 2.14). Similarly, if $\gamma \in 0^-$, then $\gamma\beta \in 0^-$.

For the last statement, (2.4.9), it is enough to note that 0^+ and 0^-, being maximal ideals of $\mathbb{D}$, are necessarily prime ideals. The proof is concluded.

A direct proof using canonical forms is also readily obtained.

The analogy of $0 = 0^+ \cup 0^-$ in $\mathbb{D}$ with the multiplicative zero element in $\mathbb{C}$ is most clear when stated in the notation of sets: if $\beta \in \mathbb{D}$, then

(2.4.10) $$0\beta = \beta 0 = 0.$$

This analogy will be profoundly evident throughout the rest of the development in this thesis. For example, analytic functions which are undefined at (0,0) in the two-dimensional case (i.e., in $\mathbb{C}$) will be observed to be undefined in $0 = 0^+ \cup 0^-$ in the four-dimensional case (i.e., in $\mathbb{D}$).

Lemmas 2.13 and 2.18 constitute the basis for the previous claim that $\mathbb{D}$ is analogous to a field. In $\mathbb{D}$, the multiplicative zero has been distinguished from the additive zero. The multiplicative zero has been extended into two planes in four dimensions, which planes have as their intersection solely the additive zero element, (0,0,0,0).

III. RUDIMENTS OF A FOUR-DIMENSIONAL CALCULUS

The objective of this section is to develop a calculus of functions (of the "four-dimensional" variable β, introduced earlier), which parallels the complex calculus in form, notation, and content. Its development will depend on the algebra which was constructed on $\mathbb{D}$ in Section II. This algebra differs from a field, but allows certain compensations which permit the development of a calculus along notationally conventional lines. Among the unconventional aspects, one will note that the planes 0^+ and 0^- are fields, although their union possesses zero divisors in $\mathbb{D}$ (their union is precisely the subset of $\mathbb{D}$ on which the multiplicative inverse is not defined). But 0^+ and 0^- may be used to advantage, inasmuch as the four-space can be mapped bijectively onto their cartesian product. More will be said on this subject in the following subsections.

3.1 Some Geometrical Interpretations

$\mathbb{D}$ may be viewed, i.a., as a real vector space of dimension four, for which $\{1, i, j, k\}$ is a basis. Then the typical vector operations hold:

(3.1.1) Euclidean norm: If $\beta \in \mathbb{D}$,

$$|\beta|^2 = |x + iy + jz + kt|^2$$

$$= x^2 + y^2 + z^2 + t^2.$$

In the canonical form (Definition 2.9),

$$\beta = \xi e_1 + \eta e_2,$$

where

$$\xi = (x-t) + i(y+z)$$

$$\eta = (x+t) + i(y-z),$$

and

$$e_1 = (1-k)/2$$

$$e_2 = (1+k)/2;$$

then

$$|\beta|^2 = |\xi e_1 + \eta e_2|^2$$

$$= |\xi e_1|^2 + |\eta e_2|^2$$

$$= |\xi|^2|e_1|^2 + |\eta|^2|e_2|^2$$

$$= |\xi|^2/2 + |\eta|^2/2.$$

More generally,

(3.1.2) Euclidean inner product: If $\beta_1, \beta_2 \in \mathbb{D}$,

$$\beta_1 \cdot \beta_2 = (x_1 + iy_1 + jz_1 + kt_1) \cdot (x_2 + iy_2 + jz_2 + kt_2)$$

$$= x_1x_2 + y_1y_2 + z_1z_2 + t_1t_2.$$

In the canonical form,

$$\beta_1 \cdot \beta_2 = (\xi_1 e_1 + \eta_1 e_2) \cdot (\xi_2 e_1 + \eta_2 e_2)$$

$$(3.1.3) \qquad = (\xi_1 \cdot \xi_2)(e_1 \cdot e_1) + (\eta_1 \cdot \eta_2)(e_2 \cdot e_2)$$

$$= (\xi_1 \cdot \xi_2)/2 + (\eta_1 \cdot \eta_2)/2.$$

As a consequence, if $\beta_1 \varepsilon 0^+$ and $\beta_2 \varepsilon 0^-$, then

$$\beta_1 \cdot \beta_2 = (\xi_1 e_1) \cdot (\eta_2 e_2) = \theta,$$

indicating that the axis planes are orthogonal.

One might inquire at this point as to the topological relationship of spaces $\mathbb{C}^2$ and $\mathbb{D}$. That they are topologically equivalent (homeomorphic) may be seen from the following considerations. Let

$$\alpha_1 = (\xi_1, \eta_1)$$

$$\alpha_2 = (\xi_2, \eta_2);$$

then the usual Euclidean metric ρ on $\mathbb{C}^2$ is

$$\rho(\alpha_1,\alpha_2) = [(\alpha_1 - \alpha_2) \cdot (\alpha_1 - \alpha_2)]^{1/2}$$

(3.1.4)

$$= [|\xi_1 - \xi_2|^2 + |\eta_1 - \eta_2|^2]^{1/2}.$$

If one now makes the correspondence

$$(\xi_1,\eta_1) \longleftrightarrow \xi_1 e_1 + \eta_1 e_2$$

$$(\xi_2,\eta_2) \longleftrightarrow \xi_2 e_1 + \eta_2 e_2,$$

then the inner product of $\alpha_1 - \alpha_2$ with itself in $\mathbb{D}$ is

$$(\alpha_1 - \alpha_2) \cdot (\alpha_1 - \alpha_2) = |\alpha_1 - \alpha_2|^2$$

(3.1.5)

$$= |\xi_1 - \xi_2|^2/2 + |\eta_1 - \eta_2|^2/2;$$

that is,

(3.1.6) $$\rho(\alpha_1,\alpha_2) = \sqrt{2}\ |\alpha_1 - \alpha_2|,$$

or

$$\text{metric } (\mathbb{C}^2) = \sqrt{2}\ \text{metric } (\mathbb{D}).$$

The next theorem reveals several notational forms for ring $\mathbb{D}$, some of which will yield useful interpretations later on.

Theorem 3.1. The ring operations on $\mathbb{D}$ may be formulated solely in terms of vectors (respectively, solely in terms of 4×4 real matrices; respectively, solely in terms of ordered pairs of complex numbers in $\mathbb{C}^2$; respectively, via a mixed vector-matrix notation).

Proof. The vector representation is given in Definition 2.6. The matrix representation follows as a result of the isomorphism of $\mathbb{G}$ and $\mathbb{E}$, which was shown in Theorem 2.3. The formulation in $\mathbb{C}^2$ notation follows from the isomorphism of $\mathbb{D}$ and $\mathbb{C}^2$, which was shown in Theorem 2.15. The "mixed" notation uses the vector notation, except in the case of multiplication; there, it uses a vector-matrix notation which will be given below. First, consider $\beta_1, \beta_2, \beta_3 \in \mathbb{D}$ such that

$$\beta_1 = \beta_2\beta_3 = \beta_3\beta_2.$$

In the matrix form,

$$\begin{bmatrix} \overline{x_1} & -y_1 & -z_1 & \overline{t_1} \\ y_1 & x_1 & -t_1 & -z_1 \\ z_1 & -t_1 & x_1 & -y_1 \\ \underline{t_1} & z_1 & y_1 & x_1 \end{bmatrix} = \begin{bmatrix} \overline{x_2} & -y_2 & -z_2 & \overline{t_2} \\ y_2 & x_2 & -t_2 & -z_2 \\ z_2 & -t_2 & x_2 & -y_2 \\ \underline{t_2} & z_2 & y_2 & x_2 \end{bmatrix} \begin{bmatrix} \overline{x_3} & -y_3 & -z_3 & \overline{t_3} \\ y_3 & x_3 & -t_3 & -z_3 \\ z_3 & -t_3 & x_3 & -y_3 \\ \underline{t_3} & z_3 & y_3 & x_3 \end{bmatrix}$$

$$(3.1.7) \qquad = \begin{bmatrix} \overline{x_3} & -y_3 & -z_3 & \overline{t_3} \\ y_3 & x_3 & -t_3 & -z_3 \\ z_3 & -t_3 & x_3 & -y_3 \\ \underline{t_3} & z_3 & y_3 & x_3 \end{bmatrix} \begin{bmatrix} \overline{x_2} & -y_2 & -z_2 & \overline{t_2} \\ y_2 & x_2 & -t_2 & -z_2 \\ z_2 & -t_2 & x_2 & -y_2 \\ \underline{t_2} & z_2 & y_2 & x_2 \end{bmatrix}.$$

Note that the first column vector in a matrix representation of a vector is just the vector itself. Moreover, the first column vector in the matrix product of the matrix representations of two given vectors is the vector formulation of the product of the given vectors. Because of this, the reader may verify that the following "mixed" equation,

$$(3.1.8)\qquad \begin{bmatrix} x_1 \\ y_1 \\ z_1 \\ t_1 \end{bmatrix} = \begin{bmatrix} x_2 & -y_2 & -z_2 & t_2 \\ y_2 & x_2 & -t_2 & -z_2 \\ z_2 & -t_2 & x_2 & -y_2 \\ t_2 & z_2 & y_2 & x_2 \end{bmatrix} \begin{bmatrix} x_3 \\ y_3 \\ z_3 \\ t_3 \end{bmatrix} = \begin{bmatrix} x_3 & -y_3 & -z_3 & t_3 \\ y_3 & x_3 & -t_3 & -z_3 \\ z_3 & -t_3 & x_3 & -y_3 \\ t_3 & z_3 & y_3 & x_3 \end{bmatrix} \begin{bmatrix} x_2 \\ y_2 \\ z_2 \\ t_2 \end{bmatrix},$$

holds, and has the same import as the all-matrix form, (3.1.7). The proof is concluded.

The vector-matrix formulation is a shorthand notation for the pure matrix forms. It is not particularly useful as such, because the formulation in $\mathbb{C}^2$ and the canonical formulation are both more concise. However, it is important in that it provides a link to linear algebra theory: Equation (3.1.8) shows vector β_3 being operated on by matrix β_2, and vice-versa. The properties of such transformations are investigated below.

Consider the transformation of multiplication by a typical $\beta \in \mathbb{D}$, given by $\beta_1 = \beta\beta_2$. In the vector-matrix form,

$$(3.1.9) \qquad \begin{bmatrix} x_1 \\ y_1 \\ z_1 \\ t_1 \end{bmatrix} = \begin{bmatrix} x & -y & -z & t \\ y & x & -t & -z \\ z & -t & x & -y \\ t & z & y & x \end{bmatrix} \begin{bmatrix} x_2 \\ y_2 \\ z_2 \\ t_2 \end{bmatrix} .$$

This is not an orthogonal transformation in general, because

$$\det(\beta) = [(x - t)^2 + (y + z)^2][(x + t)^2 + (y - z)^2],$$

which only coincides with ± 1 in special cases. The nature of the nonorthogonality may be seen by inspection of the Euclidean norm of a product (in the vector notation):

$$|\beta\beta_2|^2 = |(\xi e_1 + \eta e_2)(\xi_2 e_1 + \eta_2 e_2)|^2$$

$$(3.1.10) \qquad = |\xi\xi_2 e_1 + \eta\eta_2 e_2|^2$$

$$= |\xi\xi_2|^2/2 + |\eta\eta_2|^2/2;$$

but, using the notation $\beta_2' = \eta_2 e_1 + \xi_2 e_2$, one obtains

$$|\beta|^2|\beta_2|^2 = (|\xi|^2/2 + |\eta|^2/2)(|\xi_2|^2/2 + |\eta_2|^2/2)$$

$$(3.1.11) \qquad = (|\xi\xi_2|^2/4 + |\eta\eta_2|^2/4) + (|\xi\eta_2|^2/4 + |\eta\xi_2|^2/4)$$

$$= |\beta\beta_2|^2/2 + |\beta\beta_2'|^2/2.$$

If (3.1.9) represented an orthogonal transformation, then the length of β_2 would be unchanged by it; that is, one would have

$$|\beta\beta_2|^2 = |\beta_2|^2 = |\xi_2|^2/2 + |\eta_2|^2/2,$$

which is frequently contrary to the results of (3.1.10). Also, from (3.1.11),

$$(3.1.12) \quad |\beta\beta_2|^2 = |\beta|^2|\beta_2|^2 + \{|\beta|^2|\beta_2|^2 - |\beta\beta_2'|^2\}.$$

The term in brackets on the right is a measure of the nonorthogonality.

The characteristic equation of transformation (3.1.9) is

$$(3.1.13) \quad [(x - \lambda - t)^2 + (y + z)^2][(x - \lambda + t)^2 + (y - z)^2] = 0,$$

with roots (eigenvalues)

$$(3.1.14) \qquad \begin{aligned} \lambda_1 &= (x - t) + i(y + z) \\ \lambda_2 &= (x + t) + i(y - z) \\ \lambda_3 &= (x - t) - i(y + z) = \lambda_1^* \\ \lambda_4 &= (x + t) - i(y - z) = \lambda_2^*. \end{aligned}$$

Note that, by (3.1.14) and Definition 2.9,

$$\beta = \lambda_1 e_1 + \lambda_2 e_2;$$

and since the following shows that e_1 and e_2 are eigenvectors, i.e.,

$$\beta e_1 = \lambda_1 e_1$$

(3.1.15)

$$\beta e_2 = \lambda_2 e_2,$$

the canonical form, Definition 2.9, is a statement of an arbitrary vector $\beta \in \mathbb{D}$ in terms of eigenvalues and eigenvectors. Since the axis planes are orthogonal, the perpendicular distance from the point (x,y,z,t) to the 0^--plane is $|\lambda_1|$, and the corresponding distance to the 0^+-plane is $|\lambda_2|$.

Further, one may compare the expression for the determinant,

$$\det(\beta) = [(x - t)^2 + (y + z)^2][(x + t)^2 + (y - z)^2],$$

with the expressions for the eigenvalues, (3.1.14), to see that

$$\det(\beta) = \lambda_1 \lambda_1^* \lambda_2 \lambda_2^* = |\lambda_1 \lambda_2|^2, \tag{3.1.16}$$

in accordance with standard matrix theory. Since the canonical form displays λ_1 and λ_2 explicitly, it is easy to calculate a determinant from a canonical form.

The preceding also lends some sensibility to calling even the matrix form of an element $\beta \in \mathbb{D}$ a "canonical form;" although it is not in any conventional matrix canonical form, its eigenvalues may be read off by inspection.

3.2 Definitions of Elementary Functions

Any definition of an analytic function $f : \mathbb{D} \to \mathbb{D}$ should be compatible with the matrix formulation of the algebra on $\mathbb{D}$. For motivational purposes, recall that analytic functions are defined for real or complex diagonizable matrices in MacDuffee [21, p. 97] and discussed by Bellman [22, p. 89], as follows. If $f : \mathbb{C} \to \mathbb{C}$ is an analytic function, if A is a 4×4 diagonizable matrix, and if T is an invertible matrix such that TAT^{-1} is diagonal, then the definition of f may be extended to allow matrix arguments, as follows

$$f(A,T) = T^{-1} \begin{bmatrix} f(\lambda_1) & 0 & 0 & 0 \\ 0 & f(\lambda_2) & 0 & 0 \\ 0 & 0 & f(\lambda_3) & 0 \\ 0 & 0 & 0 & f(\lambda_4) \end{bmatrix} T, \tag{3.2.1}$$

where the λ_i are the eigenvalues of A with the order appearing in TAT^{-1}. Whatever notational system is used (see Theorem 3.1), a function f of $\beta \in \mathbb{D}$ apparently should operate on those factors of β which correspond to the eigenvalues of β in the 4×4 matrix form. Recall that the elements $\beta \in \mathbb{D}$ are expressible in terms of eigenvalues and eigenvectors (the canonical form, Definition 2.9), or purely in terms of eigenvalues (as an element of the isomorphic ring $\mathbb{C}^2$; see Theorem 2.15). This suggests how to infer a functional structure on $\mathbb{C}^2$, as follows.

Definition 3.2. Let $f : \mathbb{C} \to \mathbb{C}$ be a function and let $\beta = (\xi,\eta)$ be an element of $\mathbb{C}^2$. Then

$$f(\beta) = (f(\xi), f(\eta))$$

defines a function $f : \mathbb{C}^2 \to \mathbb{C}^2$. The new function $f(\beta)$ is said to be analytic at a point $(\xi,\eta) \in \mathbb{C}^2$ if $f : \mathbb{C} \to \mathbb{C}$ is analytic separately at points ξ and η in $\mathbb{C}$.

Since $\mathbb{D} \overset{\sim}{=} \mathbb{C}^2$ by an explicit isomorphism (see Theorem 2.15), Definition 3.2 provides an unambiguous means of "extending" $f : \mathbb{C} \to \mathbb{C}$ to a function $f : \mathbb{D} \to \mathbb{D}$. The corresponding definition in the canonical form is

$$f(\beta) = f(\xi)e_1 + f(\eta)e_2, \tag{3.2.2}$$

which is compatible with a previous definition, (2.3.33), of the inverse of an element $\beta \in \mathbb{D}$.

An advantage of Definition 3.2 is that it always reduces to the complex variable case when the j and k components of β are zero, because

$$\begin{aligned} f(\beta)\Big|_{z=t=0} &= f((x-0) + i(y+0))e_1 \\ &\quad + f((x+0) + i(y-0))e_2 \\ &= f(x+iy)e_1 + f(x+iy)e_2 \\ &= f(x+iy)(e_1+e_2) \\ &= f(x+iy). \end{aligned} \tag{3.2.3}$$

Remark 3.3. If f is an analytic function, if

$$\beta = x + iy + jz + kt = \xi e_1 + \eta e_2,$$

and if

$$f(\beta) = f(\xi)e_1 + f(\eta)e_2, \tag{3.2.4}$$

then $f(\beta)$ can be represented as

$$\begin{aligned} f(\beta) = \; & u + iv + jw + ks \\ = \; & [(u-s) + i(v+w)]e_1 \\ & + [(u+s) + i(v-w)]e_2, \end{aligned} \tag{3.2.5}$$

where u, v, w, s are explicit functions of x, y z, t (and f).

Indeed, the two forms on the right-hand side of Equation (3.2.5) are equal by Lemma 2.8. One may expand the complex-valued functions on the right-hand side of Equation (3.2.4) to obtain

$$\begin{aligned} f(\beta) = \; & \{\mathrm{Re}[f(\xi)] + i\ \mathrm{Im}[f(\xi)]\}e_1 \\ & + \{\mathrm{Re}[f(\eta)] + i\ \mathrm{Im}[f(\eta)]\}e_2. \end{aligned} \tag{3.2.6}$$

A comparison of terms between Equations (3.2.5) and (3.2.6) yields

$$
\begin{aligned}
u - s &= \mathrm{Re}[f(\xi)] \\
u + s &= \mathrm{Re}[f(\eta)] \\
v - w &= \mathrm{Im}[f(\eta)] \\
v + w &= \mathrm{Im}[f(\xi)],
\end{aligned}
\tag{3.2.7}
$$

whose solution is easily seen to be the asserted "explicit" formulas:

$$
\begin{aligned}
u &= \{\mathrm{Re}[f(\xi)] + \mathrm{Re}[f(\eta)]\}/2 \\
v &= \{\mathrm{Im}[f(\xi)] + \mathrm{Im}[f(\eta)]\}/2 \\
w &= \{\mathrm{Im}[f(\xi)] - \mathrm{Im}[f(\eta)]\}/2 \\
s &= \{-\mathrm{Re}[f(\xi)] + \mathrm{Re}[f(\eta)]\}/2.
\end{aligned}
\tag{3.2.8}
$$

Following Ketchum [8], we call the above functions u, v, w, s <u>conjugate functions</u> of $f : \mathbb{D} \to \mathbb{D}$, if f is analytic.

Some examples using Definition 3.2 and Remark 3.3 are given below.

The <u>Exponential</u>.

$$
\begin{aligned}
f(\beta) &= \exp(\beta) \\
&= \exp(\xi)e_1 + \exp(\eta)e_2 \\
&= \{\exp(x-t)[\cos(y+z) + i\sin(y+z)]\}e_1 \\
&\quad + \{\exp(x+t)[\cos(y-z) + i\sin(y-z)]\}e_2 \\
&= \{\exp(x-t)\cos(y+z) + \exp(x+t)\cos(y-z)\}/2 \\
&\quad + i\{\exp(x-t)\sin(y+z) + \exp(x+t)\sin(y-z)\}/2 \\
&\quad + j\{\exp(x-t)\sin(y+z) - \exp(x+t)\sin(y-z)\}/2 \\
&\quad + k\{-\exp(x-t)\cos(y+z) + \exp(x+t)\cos(y-z)\}/2.
\end{aligned}
\tag{3.2.9}
$$

The Logarithm.

$$
\begin{aligned}
f(\beta) &= \log(\beta) \\
&= \log(\xi)e_1 + \log(\eta)e_2 \\
&= \{\log|\xi\eta|\}/2 \\
&\quad + i\{\arg\xi + \arg\eta\}/2 \\
&\quad + j\{\arg\xi - \arg\eta\}/2 \\
&\quad + k\{\log|\eta/\xi|\}/2,
\end{aligned}
\tag{3.2.10}
$$

where

$$
\begin{aligned}
\xi &= (x-t) + i(y+z) \\
\eta &= (x+t) + i(y-z)
\end{aligned}
$$

$$
\begin{aligned}
\arg\xi &= \arctan[(y+z)/(x-t)] \\
\arg\eta &= \arctan[(y-z)/(x+t)].
\end{aligned}
$$

Since $\arg\xi$ and $\arg\eta$ are not defined in the axis planes, neither is $\log(\beta)$.

The Sine.

$$
\begin{aligned}
f(\beta) &= \sin(\beta) \\
&= \sin(\xi)e_1 + \sin(\eta)e_2 \\
&= \{\sin(x-t)\cosh(y+z) + \sin(x+t)\cosh(y-z)\}/2 \\
&\quad + i\{\cos(x-t)\sinh(y+z) + \cos(x+t)\sinh(y-z)\}/2 \\
&\quad + j\{\cos(x-t)\sinh(y+z) - \cos(x+t)\sinh(y-z)\}/2 \\
&\quad + k\{-\sin(x-t)\cosh(y+z) + \sin(x+t)\cosh(y-z)\}/2.
\end{aligned}
\tag{3.2.11}
$$

The Cosine.

$$
\begin{aligned}
f(\beta) &= \cos(\beta) \\
&= \cos(\xi)e_1 + \cos(\eta)e_2 \\
&= \{\cos(x-t)\cosh(y+z) + \cos(x+t)\cosh(y-z)\}/2 \\
&\quad + i\{-\sin(x-t)\sinh(y+z) - \sin(x+t)\sinh(y-z)\}/2 \\
&\quad + j\{-\sin(x-t)\sinh(y+z) + \sin(x+t)\sinh(y-z)\}/2 \\
&\quad + k\{-\cos(x-t)\cosh(y+z) + \cos(x+t)\cosh(y-z)\}/2.
\end{aligned} \tag{3.2.12}
$$

The Square Root.

$$
\begin{aligned}
f(\beta) &= \beta^{1/2} \\
&= \xi^{1/2}e_1 + \eta^{1/2}e_2 \\
&= \{|\xi|\cos[\arg(\xi)/2] + |\eta|\cos[\arg(\eta)/2]\}/2 \\
&\quad + i\{|\xi|\sin[\arg(\xi)/2] + |\eta|\sin[\arg(\eta)/2]\}/2 \\
&\quad + j\{|\xi|\sin[\arg(\xi)/2] - |\eta|\sin[\arg(\eta)/2]\}/2 \\
&\quad + k\{-|\xi|\cos[\arg(\xi)/2] + |\eta|\cos[\arg(\eta)/2]\}/2.
\end{aligned} \tag{3.2.13}
$$

The square root is not defined in the axis planes.

The Hyperbolic Sine.

$$
\begin{aligned}
f(\beta) &= \sinh(\beta) \\
&= \sinh(\xi)e_1 + \sinh(\eta)e_2 \\
&= \{\sinh(x-t)\cos(y+z) + \sinh(x+t)\cos(y-z)\}/2 \\
&\quad + i\{\cosh(x-t)\sin(y+z) + \cosh(x+t)\sin(y-z)\}/2 \\
&\quad + j\{\cosh(x-t)\sin(y+z) - \cosh(x+t)\sin(y-z)\}/2 \\
&\quad + k\{-\sinh(x-t)\cos(y+z) + \sinh(x+t)\cos(y-z)\}/2.
\end{aligned} \tag{3.2.14}
$$

The Hyperbolic Cosine.

$$
\begin{aligned}
f(\beta) &= \cosh(\beta) \\
&= \cosh(\xi)e_1 + \cosh(\eta)e_2 \\
&= \{\cosh(x-t)\cos(y+z) + \cosh(x+t)\cos(y-z)\}/2 \\
&\quad + i\{\sinh(x-t)\sin(y+z) + \sinh(x+t)\sin(y-z)\}/2 \\
&\quad + j\{\sinh(x-t)\sin(y+z) - \sinh(x+t)\sin(y-z)\}/2 \\
&\quad + k\{-\cosh(x-t)\cos(y+z) + \cosh(x+t)\cos(y-z)\}/2
\end{aligned} \tag{3.2.15}
$$

In the opening statement of Section III, it was proposed to construct a hypercomplex system which would parallel the complex system in form, content, and notation. As an illustration that this objective is being met, it is easy to verify that the extended functions $f : \mathbb{D} \to \mathbb{D}$ satisfy some of the well-known identities from the complex variable theory. For instance:

$$
\begin{aligned}
\log(\exp(\beta)) &= \log(\exp(\xi))e_1 + \log(\exp(\eta))e_2 \\
&= \xi e_1 + \eta e_2 \\
&= \beta.
\end{aligned} \tag{3.2.16}
$$

$$
\begin{aligned}
\cos^2(\beta) + \sin^2(\beta) &= [\cos^2(\xi)e_1 + \cos^2(\eta)e_2] \\
&\quad + [\sin^2(\xi)e_1 + \sin^2(\eta)e_2] \\
&= [\cos^2(\xi) + \sin^2(\xi)]e_1 \\
&\quad + [\cos^2(\eta) + \sin^2(\eta)]e_2 \\
&= 1 \cdot e_1 + 1 \cdot e_2 \\
&= 1.
\end{aligned} \tag{3.2.17}
$$

$$\begin{aligned}\cosh^2(\beta) - \sinh^2(\beta) &= [\cosh^2(\xi)e_1 + \cosh^2(\eta)e_2] \\ &\quad - [\sinh^2(\xi)e_1 + \sinh^2(\eta)e_2] \\ &= [\cosh^2(\xi) - \sinh^2(\xi)]e_1 \\ &\quad + [\cosh^2(\eta) - \sinh^2(\eta)]e_2 \\ &= 1 \cdot e_1 + 1 \cdot e_2 \\ &= 1.\end{aligned} \tag{3.2.18}$$

$$\begin{aligned}\beta^{1/2}\beta^{1/2} &= (\xi^{1/2}e_1 + \eta^{1/2}e_2)(\xi^{1/2}e_1 + \eta^{1/2}e_2) \\ &= \xi^{1/2}\xi^{1/2}e_1 + \eta^{1/2}\eta^{1/2}e_2 \\ &= \xi e_1 + \eta e_2 \\ &= \beta.\end{aligned} \tag{3.2.19}$$

$$\begin{aligned}\exp(\beta_1)\exp(\beta_2) &= [\exp(\xi_1)e_1 + \exp(\eta_1)e_2] \cdot [\exp(\xi_2)e_1 + \exp(\eta_2)e_2] \\ &= \exp(\xi_1)\exp(\xi_2)e_1 + \exp(\eta_1)\exp(\eta_2)e_2 \\ &= \exp(\xi_1 + \xi_2)e_1 + \exp(\eta_1 + \eta_2)e_2 \\ &= \exp(\beta_1 + \beta_2).\end{aligned} \tag{3.2.20}$$

The next two theorems provide further extensions to $\mathbb{D}$ of themes familiar from the context of $\mathbb{C}$.

Theorem 3.4. DeMoivre's Theorem in Four Dimensions: If n is a positive integer and $\beta \in \mathbb{D}$, then

$$(\cos \beta + i \sin \beta)^n = [\exp(i\beta)]^n = \cos(n\beta) + i \sin(n\beta).$$

Proof. Recall that since $\xi \in \mathbb{C}$,

$$[\exp(i\xi) + \exp(-i\xi)]/2 = \cos \xi$$
$$[\exp(i\xi) - \exp(-i\xi)]/(2i) = \sin \xi;$$

consequently,

$$\exp(i\xi) = \cos \xi + i \sin \xi.$$

Then

$$\begin{aligned} \exp(i\beta) &= \exp(i\xi)e_1 + \exp(i\eta)e_2 \\ &= (\cos \xi + i \sin \xi)e_1 + (\cos \eta + i \sin \eta)e_2 \\ &= [\cos(\xi)e_1 + \cos(\eta)e_2] + i[\sin(\xi)e_1 + \sin(\eta)e_2] \\ &= \cos \beta + i \sin \beta. \end{aligned} \tag{3.2.21}$$

We prove that

$$[\exp(i\beta)]^n = \exp(in\beta) \tag{3.2.22}$$

by induction, as follows. Certainly, (3.2.22) is true for n = 1. Suppose that it is also true for some fixed positive integer n; then

$$[\exp(i\beta)]^n = \exp(in\beta),$$

and

$$[\exp(i\beta)]^n \exp(i\beta) = \exp(in\beta)\exp(i\beta).$$

By (3.2.20),

$$[\exp(i\beta)]^{n+1} = \exp(in\beta + i\beta)$$

$$= \exp[i(n+1)\beta].$$

We may conclude by induction that (3.2.22) holds for all positive integers.

As a special case of (3.2.21),

(3.2.23) $$\exp(in\beta) = \cos(n\beta) + i \sin(n\beta).$$

The proof is concluded.

Theorem 3.5. DeMoivre's Theorem for Hyperbolic Functions: If t is real, if n is a positive integer, and if k is the basis element such that $k^2 = 1$, then

$$(\cosh t + k \sinh t)^n = [\exp(kt)]^n = \cosh(nt) + k \sinh(nt).$$

Proof. As a special case of the exponential function, (3.2.9), with $x = y = z = 0$, one has

$$\begin{aligned} \exp(kt) &= \exp(-t)e_1 + \exp(t)e_2 \\ &= \exp(-t)(1-k)/2 + \exp(t)(1+k)/2 \\ &= [\exp(t) + \exp(-t)]/2 \\ &\quad + k[\exp(t) - \exp(-t)]/2 \\ &= \cosh t + k \sinh t. \end{aligned} \tag{3.2.24}$$

One may notice that

$$[\exp(kt)]^n = \exp(nkt) \tag{3.2.25}$$

is a special case of (3.2.22), with $i\beta = kt$. Lastly, as a special case of (3.2.24),

$$\exp(nkt) = \cosh(nt) + k \sinh(nt). \tag{3.2.26}$$

The proof is concluded.

The following definition is needed for the next theorem.

Definition 3.6. The modulus of an element $\beta \in \mathbb{D}$ is given by

$$\begin{aligned} \mathrm{mod}(\beta) &= \mathrm{mod}(\xi)e_1 + \mathrm{mod}(\eta)e_2 \\ &= |\xi|e_1 + |\eta|e_2. \end{aligned}$$

The argument of an invertible element $\beta \in \mathbb{D}$ is given by

$$\begin{aligned} \arg(\beta) &= \arg(\xi)e_1 + \arg(\eta)e_2 \\ &= \arctan[(y+z)/(x-t)]e_1 \\ &\quad + \arctan[(y-z)/(x+t)]e_2. \end{aligned}$$

These definitions reduce to the complex variable case when $z = t = 0$:

$$\begin{aligned} \text{mod}(\beta)\Big|_{z=t=0} &= |x + iy|e_1 + |x + iy|e_2 \\ &= |x + iy|(e_1 + e_2) \\ &= |x + iy|. \end{aligned} \tag{3.2.27}$$

$$\begin{aligned} \arg(\beta)\Big|_{z=t=0} &= \arctan(y/x)e_1 + \arctan(y/x)e_2 \\ &= \arctan(y/x)(e_1 + e_2) \\ &= \arctan(y/x). \end{aligned} \tag{3.2.28}$$

In general, since the coefficients of e_1 and e_2 are real for both $\text{mod}(\beta)$ and $\arg(\beta)$, the latter both reduce to the form $u + ks$, with real u and s:

$$\text{mod}(\beta) = |\xi|(1-k)/2 + |\eta|(1+k)/2$$

$$= (|\xi| + |\eta|)/2 + k(-|\xi| + |\eta|)/2. \tag{3.2.29}$$

$$\begin{aligned}\arg(\beta) = &\ \arctan[(y+z)/(x-t)](1-k)/2 \\ &+ \arctan[(y-z)/(x+t)](1+k)/2 \\ = &\ \{\arctan[(y+z)/(x-t)] + \arctan[(y-z)/(x+t)]\}/2 \\ &+ k\{-\arctan[(y+z)/(x-t) + \arctan[(y-z)/(x+t)]\}/2.\end{aligned} \tag{3.2.30}$$

This fact will be used later.

The function mod(β), with proper interpretation, has all the properties of the corresponding complex variable entity. For example, $|\text{mod}(\beta)| = |\beta|$ since

$$|\text{mod}(\beta)|^2 = |\xi|^2/2 + |\eta|^2/2$$

$$= |\beta|^2. \tag{3.2.31}$$

Also,

$$\begin{aligned}\text{mod}(\beta_1)\text{mod}(\beta_2) &= (|\xi_1|e_1 + |\eta_1|e_2)(|\xi_2|e_1 + |\eta_2|e_2) \\ &= |\xi_1||\xi_2|e_1 + |\eta_1||\eta_2|e_2 \\ &= |\xi_1\xi_2|e_1 + |\eta_1\eta_2|e_2 \\ &= \text{mod}(\beta_1\beta_2),\end{aligned} \tag{3.2.32}$$

$$\begin{aligned}\mathrm{mod}(\beta_1)/\mathrm{mod}(\beta_2) &= (|\xi_1|e_1 + |\eta_1|e_2)/(|\xi_2|e_1 + |\eta_2|e_2) \\ &= (|\xi_1|/|\xi_2|)e_1 + (|\eta_1|/|\eta_2|)e_2 \\ &= |\xi_1/\xi_2|e_1 + |\eta_1/\eta_2|e_2 \\ &= \mathrm{mod}(\beta_1/\beta_2).\end{aligned} \tag{3.2.33}$$

Theorem 3.7. Argand's Representation in Four Dimensions: Every invertible element $\beta \in \mathbb{D}$ can be represented in the form

$$\beta = \mathrm{mod}(\beta)\exp[i\ \arg(\beta)].$$

Proof.

$$\begin{aligned}\beta &= \xi e_1 + \eta e_2 \\ &= \xi(|\xi|/|\xi|)e_1 + \eta(|\eta|/|\eta|)e_2 \\ &= |\xi|(\xi/|\xi|)e_1 + |\eta|(\eta/|\eta|)e_2 \\ &= (|\xi|e_1 + |\eta|e_2)[(\xi/|\xi|)e_1 + (\eta/|\eta|)e_2] \\ &= \mathrm{mod}(\beta)\{[(x-t)/|\xi| + i(y+z)/|\xi|]e_1 + [(x+t)/|\eta| + i(y-z)/|\eta|]e_2\} \\ &= \mathrm{mod}(\beta)\{(\cos[\arg(\xi)] + i\sin[\arg(\xi)])e_1 \\ &\qquad + (\cos[\arg(\eta)] + i\sin[\arg(\eta)])e_2\} \\ &= \mathrm{mod}(\beta)\{\exp[i\ \arg(\xi)]e_1 + \exp[i\ \arg(\eta)]e_2\} \\ &= \mathrm{mod}(\beta)\exp[i\ \arg(\beta)].\end{aligned}$$

The proof is concluded.

Note that since $\arg(\beta)$ is of the form $u + ks$, the factor $\exp [i \arg(\beta)]$ reduces to the form $\exp (iu) \exp (-js)$, with u and s real. It follows that

$$|\exp [i \arg(\beta)]| = 1. \tag{3.2.34}$$

From Definitions 3.6 and Theorem 3.7, it is evident that a non-axis point $\beta \in \mathbb{D}$ is determined by two moduli and two arguments, in the usual complex polar representation sense, of the axis-plane component elements ξ and η corresponding to β.

3.3 Notation For Derivatives, Integrals, And Other Linear Operators

The standard definitions for derivatives and linear operators (or transformations) will be used. Some preliminary work will need to be done before they can be stated, however.

The notion of the limit of a function

$$f(\beta) = f(\xi)e_1 + f(\eta)e_2$$

as β approaches some fixed value $\beta_0 \in \mathbb{D}$ will be needed. (Henceforth, all "functions" will be assumed "analytic," in the sense defined earlier.) The space $\mathbb{D}$ will admit such a notion because, as we observed earlier, $\mathbb{D}$ is a metric space with metric $|\alpha_1 - \alpha_2|$, where $\alpha_1, \alpha_2 \in \mathbb{D}$. It is obvious that such a limit for $f(\beta)$ will exist only if the corresponding

limits exist for both $f(\xi)$ and $f(\eta)$. Conversely, suppose the latter limits exist, and that the domain of f contains a ball about β_0. Then given $\varepsilon > 0$ we can find δ_1 and δ_2 such that

$$|f(\xi) - f(\xi_0)| < \varepsilon \text{ whenever } 0 < |\xi - \xi_0| < \delta_1,$$

and

$$|f(\eta) - f(\eta_0)| < \varepsilon \text{ whenever } 0 < |\eta - \eta_0| < \delta_2.$$

If we choose $\delta = \min(\delta_1, \delta_2)$, then

$$\begin{aligned} |\beta - \beta_0|^2 &= |\xi - \xi_0|^2/2 + |\eta - \eta_0|^2/2 \\ &< \delta^2/2 + \delta^2/2 \\ &= \delta^2, \end{aligned} \tag{3.3.1}$$

and

$$\begin{aligned} |f(\beta) - f(\beta_0)|^2 &= |[f(\xi) - f(\xi_0)]e_1 + [f(\eta) - f(\eta_0)]e_2|^2 \\ &= |f(\xi) - f(\xi_0)|^2/2 + |f(\eta) - f(\eta_0)|^2/2 \\ &< \varepsilon^2/2 + \varepsilon^2/2 \\ &= \varepsilon^2. \end{aligned} \tag{3.3.2}$$

By (3.3.1) and (3.3.2), if we are given $\varepsilon > 0$, we can find a $\delta > 0$ such that

$$|f(\beta) - f(\beta_0)| < \varepsilon \text{ whenever } 0 < |\beta - \beta_0| < \delta.$$

Accordingly, we make the following definition.

Definition 3.8. The limit of a function $f(\beta)$ as β approaches some fixed value β_0 is given by

$$\lim_{\beta \to \beta_0} f(\beta) = \left(\lim_{\xi \to \xi_0} f(\xi), \lim_{\eta \to \eta_0} f(\eta) \right),$$

where the limits for $f(\xi)$ and $f(\eta)$ are defined as in the complex variable sense. The limit of $f(\beta)$ as β approaches β_0 exists only if the corresponding limits exist for both $f(\xi)$ and $f(\eta)$.

We now proceed to define a function space $\mathbb{IH}$ and the notion of a linear operator on $\mathbb{IH}$.

Definition 3.9. $\mathbb{IH}$ is the set of all analytic functions from $\mathbb{D}$ to $\mathbb{D}$.

Observe that $\mathbb{IH}$ is a left $\mathbb{D}$-module, with addition of functions taken to be pointwise addition and a scalar multiplication operation defined by $(af)(\beta) = a(f(\beta))$ for $a, \beta \in \mathbb{D}$, $f \in \mathbb{IH}$. It now makes sense to talk of linear operators on $\mathbb{IH}$. The standard definition from vector algebra is used.

Definition 3.10. A linear operator $T : \mathbb{H} \to \mathbb{H}$ is a function $\mathbb{H}$ to $\mathbb{H}$ satisfying

$$T(af + bg) = aT(f) + bT(g),$$

for all $a, b \in \mathbb{D}$ and all $f, g \in \mathbb{H}$.

If T_1 and T_2 are linear operators from $\mathbb{H}$ to $\mathbb{H}$, then the product (composite) of T_1 and T_2 is clearly a linear operator, T_1T_2.

We next introduce the derivative operator, which may be viewed as a linear operator from $\mathbb{H}$ to $\mathbb{H}$. Notation for it is aided by Definitions 3.8, 3.9, and 3.10.

Definition 3.11. If β is an independent variable in $\mathbb{D}$ and if $f(\beta)$ is a function from $\mathbb{H}$ to $\mathbb{H}$, then the derivative of $f(\beta)$ with respect to β at a point β_0 is given by

$$\begin{aligned} f'(\beta_0) &= \lim_{\beta \to \beta_0} \frac{f(\beta) - f(\beta_0)}{\beta - \beta_0} \\ &= \lim_{\xi \to \xi_0} \frac{f(\xi) - f(\xi_0)}{\xi - \xi_0} e_1 + \lim_{\eta \to \eta_0} \frac{f(\eta) - f(\eta_0)}{\eta - \eta_0} e_2 \\ &= f'(\xi_0)e_1 + f'(\eta_0)e_2. \end{aligned}$$

In $\mathbb{C}^2$ notation, it is $f'(\beta) = (f'(\xi), f'(\eta))$. Once again, possible questions related to some operation in four space have been reduced to the corresponding questions in the complex plane. For

example, the derivative exists at a point in four space if and only if it exists at both image points in the axis planes.

For emphasis that the same form, content, and notation as the complex system is being achieved, the following examples are offered.

$$\frac{d}{d\beta}\exp(\beta) = \frac{d}{d\xi}\exp(\xi)e_1 + \frac{d}{d\eta}\exp(\eta)e_2$$

$$= \exp(\xi)e_1 + \exp(\eta)e_2 \qquad (3.3.3)$$

$$= \exp(\beta).$$

$$\frac{d}{d\beta}\log(\beta) = \frac{d}{d\xi}\log(\xi)e_1 + \frac{d}{d\eta}\log(\eta)e_2$$

$$= (1/\xi)e_1 + (1/\eta)e_2 \qquad (3.3.4)$$

$$= 1/\beta.$$

$$\frac{d}{d\beta}\cos(\beta) = \frac{d}{d\xi}\cos(\xi)e_1 + \frac{d}{d\eta}\cos(\eta)e_2$$

$$= -\sin(\xi)e_1 - \sin(\eta)e_2 \qquad (3.3.5)$$

$$= -\sin(\beta).$$

$$\frac{d}{d\beta}\beta^{1/2} = \frac{d}{d\xi}\xi^{1/2}e_1 + \frac{d}{d\eta}\eta^{1/2}e_2$$

$$= (1/2)\xi^{-1/2}e_1 + (1/2)\eta^{-1/2}e_2 \qquad (3.3.6)$$

$$= (1/2)\beta^{-1/2}.$$

The integral operator also has the same form as in the complex calculus.

Definition 3.12. If β is an independent variable in $\mathbb{D}$ and if $f(\beta)$ is a function from IH to IH, then the integral of $f(\beta)$ over any path from point β_0 to point β_1 is given by

$$\int_{\beta_0}^{\beta_1} f(\beta)d\beta = e_1 \int_{\xi_0}^{\xi_1} f(\xi)d\xi + e_2 \int_{\eta_0}^{\eta_1} f(\eta)d\eta.$$

The two integrals on the right-hand side are the usual complex integrals. Of course, the integral of $f(\beta)$ exists only if both the component integrals on the right exist. The notion of a contour integral in four dimensions will not be treated here.

3.4 The Cauchy-Riemann Conditions in the Four-Dimensional, Commutative System

The four-dimensional Cauchy-Riemann conditions follow easily from the fact that an arbitrary analytic function $f(\beta)$ can be described via a sum involving complex-valued functions $f(\xi)$ and $f(\eta)$. Consider the forms

$$\beta = x + iy + jz + kt$$

(3.4.1)
$$= [(x-t) + i(y+z)]e_1 + [(x+t) + i(y-z)]e_2$$

$$= \xi e_1 + \eta e_2,$$

and

$$f(\beta) = u + iv + jw + ks$$

(3.4.2)
$$= [(u-s) + i(v+w)]e_1 + [(u+s) + i(v-w)]e_2$$

$$= f(\xi)e_1 + f(\eta)e_2.$$

Functions $f(\xi)$ and $f(\eta)$ each have a complex independent variable, hence each have associated Cauchy-Riemann conditions. Combining their statements leads, as follows, to the corresponding conditions in four dimensions.

Definition 3.13. The Cauchy-Riemann Conditions in Four Dimensions. If β, $f(\beta)$ are as above, then

(3.4.3)
$$\frac{\partial(u-s)}{\partial(x-t)} = \frac{\partial(v+w)}{\partial(y+z)},$$

(3.4.4)
$$\frac{\partial(u-s)}{\partial(y+z)} = -\frac{\partial(v+w)}{\partial(x-t)},$$

(3.4.5)
$$\frac{\partial(u+s)}{\partial(x+t)} = \frac{\partial(v-w)}{\partial(y-z)},$$

(3.4.6)
$$\frac{\partial(u+s)}{\partial(y-z)} = -\frac{\partial(v-w)}{\partial(x+t)}.$$

A more conventional-looking representation may be obtained as follows. (Underlying the derivation will be the implicit requirement that the derivative or partial derivative at a point be independent of the direction of approach to the point). In (3.4.3), if the partial derivative is taken in a plane in which z and t are constant, one obtains

$$\frac{\partial}{\partial x}(u-s) = \frac{\partial}{\partial y}(v+w).$$

Similarly, in a plane in which y and t are constant,

$$\frac{\partial}{\partial x}(u-s) = \frac{\partial}{\partial z}(v+w);$$

hence it is a consequence of (3.4.3) that

(3.4.7) $$u_x - s_x = v_y + w_y = v_z + w_z.$$

From similar manipulations on (3.4.3) in planes in which either (x and y are constant) or (x and z are constant), one obtains

(3.4.8) $$-u_t + s_t = v_y + w_y = v_z + w_z.$$

The above operations may be repeated with each of the other three equations of Definition 3.13, with the result

(3.4.9) $$u_y - s_y = -v_x - w_x = v_t + w_t,$$

(3.4.10) $$u_z - s_z = -v_x - w_x = v_t + w_t,$$

(3.4.11) $$u_x + s_x = v_y - w_y = -v_z + w_z,$$

(3.4.12) $$u_t + s_t = v_y - w_y = -v_z + w_z,$$

(3.4.13) $$u_y + s_y = -v_x + w_x = -v_t + w_t,$$

(3.4.14) $$-u_z - s_z = -v_x + w_x = -v_t + w_t.$$

Equations (3.4.7) and (3.4.11) may be added to obtain

$$2u_x = 2v_y = 2w_z,$$

or

$$u_x = v_y = w_z.$$

By similar additions or subtractions of appropriate pairs of Equations (3.4.7) through (3.4.14), one obtains an alternate form of the Cauchy-Riemann conditions in four dimensions:

(3.4.15)
$$\begin{aligned} u_x &= v_y = w_z = s_t \\ u_y &= -v_x = w_t = -s_z \\ u_z &= v_t = -w_x = -s_y \\ u_t &= -v_z = -w_y = s_x. \end{aligned}$$

If the variable β is restricted so that $\beta = x + iy$, then $f(\beta)$ is a function of a complex variable, for which

$$u_x = v_y$$
$$u_y = -v_x;$$

i.e., the Cauchy-Riemann conditions in four dimensions reduce to the usual Cauchy-Riemann conditions. Similar reduction is possible also for the two-dimensional restricted variables $(x + jz)$, $(iy + kt)$, and $(jz + kt)$; note that the latter two factor into $i(y + jt)$ and $j(z + it)$, respectively. However, similar reduction is not possible for the restricted variable $\beta = x + kt$, because in that case $u_x = s_t$ and $u_t = s_x$, which are <u>not</u> the usual Cauchy-Riemann conditions in two dimensions. A similar result holds for the restricted variable $\beta = iy + jz = i(y - kz)$.

Equations (3.4.15) may be used to provide alternate expressions for the derivative of an analytic function, as the next theorem shows.

<u>Theorem 3.14</u>. If f is an analytic function of the variable $\beta = x + iy + jz + kt$, then

$$\frac{df}{d\beta} = \frac{\partial f}{\partial x} = -i\frac{\partial f}{\partial y} = -j\frac{\partial f}{\partial z} = k\frac{\partial f}{\partial t}.$$

<u>Proof</u>. From the definitions of the canonical forms,

$$\xi = (x-t) + i(y+z)$$

$$\eta = (x+t) + i(y-z)$$

$$f(\beta) = f(\xi)e_1 + f(\eta)e_2$$

$$= [(u-s) + i(v+w)]e_1 + [(u+s) + (v-w)]e_2$$

$$f'(\beta) = f'(\xi)e_1 + f'(\eta)e_2$$

$$= \frac{d}{d\xi}[(u-s) + i(v+w)]e_1 + \frac{d}{d\eta}[(u+s) + i(v-w)]e_2.$$

Consider the lines along which y, z, t are constant (i.e., parallel to the x axis); along such lines, the operators $\frac{d}{d\xi}$ and $\frac{d}{d\eta}$ both reduce to $\frac{\partial}{\partial x}$, and

$$f'(\beta) = \frac{\partial}{\partial x}[(u-s) + i(v+w)]e_1 + \frac{\partial}{\partial x}[(u+s) + i(v-w)]e_2, \tag{3.4.16}$$

hence

$$\frac{df}{d\beta} = \frac{\partial f}{\partial x} = [(u_x-s_x) + i(v_x+w_x)]e_1 + [u_x+s_x) + i(v_x-w_x)]e_2$$

$$= u_x + iv_x + jw_x + ks_x. \tag{3.4.17}$$

The Cauchy-Riemann conditions (3.4.15) may be used to get the right-hand side of (3.4.17) into the form

$$\frac{df}{d\beta} = v_y + i(-u_y) + js_y + k(-w_y)$$

$$= -i(u_y + iv_y + jw_y + ks_y) \tag{3.4.18}$$

$$= -i\frac{\partial f}{\partial y}.$$

In a similar way,

$$\frac{df}{d\beta} = -j\frac{\partial f}{\partial z} = k\frac{\partial f}{\partial t},$$

and the proof is concluded.

The conjugate functions of an analytic function of a hypercomplex variable obey several important vector relationships, as the next five theorems show.

Definition 3.15. The gradient operator for an analytic function of real variables x, y, z, t is denoted by

$$\square = \frac{\partial}{\partial x} + i\frac{\partial}{\partial y} + j\frac{\partial}{\partial z} + k\frac{\partial}{\partial t}. \tag{3.4.19}$$

Theorem 3.16. If u, v, w, s are the conjugate functions of an analytic function of a hypercomplex variable, then

$$\square u = -i\square v = -j\square w = k\square s.$$

Proof. The Cauchy-Riemann conditions in the form (3.4.15) may be interpreted in such a way as to give the desired result. If they are written in vector form, it is easily seen that they expand into a more revealing form involving the gradient vectors:

$$(3.4.20)\qquad \begin{bmatrix} u_x \\ u_y \\ u_z \\ u_t \end{bmatrix} = \begin{bmatrix} v_y \\ -v_x \\ v_t \\ -v_z \end{bmatrix} = \begin{bmatrix} w_z \\ w_t \\ -w_x \\ -w_y \end{bmatrix} = \begin{bmatrix} s_t \\ -s_z \\ -s_y \\ s_x \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} u_x \\ u_y \\ u_z \\ u_t \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{bmatrix} \begin{bmatrix} v_x \\ v_y \\ v_z \\ v_t \end{bmatrix}$$

$$(3.4.21)\qquad = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{bmatrix} \begin{bmatrix} w_x \\ w_y \\ w_z \\ w_t \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \\ 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} s_x \\ s_y \\ s_z \\ s_t \end{bmatrix} .$$

By the pure matrix/vector-matrix dualism which was shown in Theorem 3.1, and by the fact that the 4×4 matrices in (3.4.21) are the basis matrices E_1, E_2, E_3, E_4, Equation (3.4.21) can be written

$$\begin{aligned} 1 \cdot (u_x + iu_y + ju_z + ku_t) &= -\, i(v_x + iv_y + jv_z + kv_t) \\ &= -\, j(w_x + iw_y + jw_z + kw_t) \\ &= k(s_x + is_y + js_z + ks_t). \end{aligned} \tag{3.4.22}$$

By Definition 3.14, this is just

$$\Box u = -\, i \Box v = -\, j \Box w = k \Box s,$$

and the proof is concluded.

Theorem 3.17. If u, v, w, s are conjugate functions of an analytic function, then

$$|\Box u|^2 = |\Box v|^2 = |\Box w|^2 = |\Box s|^2.$$

Proof. Since multiplication by $\pm i$, $\pm j$ or $\pm k$ merely permutes the components (up to sign) of any element of $\mathbb{D}$, the result follows from Theorem 3.16.

Theorem 3.18. If u, v, w, s are conjugate functions of an analytic function, and if $\square^2$ denotes the "Laplacian" operator

$$\square \cdot \square = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} + \frac{\partial^2}{\partial t^2},$$

then

$$\square^2 u = \square^2 v = \square^2 w = \square^2 s = 0.$$

Proof.

$$\square^2 u = \square \cdot \square u = u_{xx} + u_{yy} + v_{zz} + s_{tt},$$

and by use of (3.4.15),

$$\begin{aligned} u_{xx} &= v_{yx} \\ u_{yy} &= -v_{xy} \\ u_{zz} &= v_{tz} \\ u_{tt} &= -v_{zt}. \end{aligned} \tag{3.4.23}$$

Recall that the components ρ, σ of an analytic function of a complex variable ξ, i.e., $f(\xi) = \rho + \sigma i$, are themselves analytic functions, and that an analytic function composed with another analytic function is analytic. It is evident from (3.2.8) that u, v, w, s are sums and

differences of components of complex analytic functions, whose arguments are analytic functions of x, y, z, t; hence u, v, w, s are analytic functions of x, y, z, t. Returning to (3.4.23), since u and v are analytic, the order of taking mixed partial differentials is immaterial, hence the summation of Equations (3.4.23) is zero:

$$\Box^2 u = u_{xx} + u_{yy} + u_{zz} + u_{tt} = 0.$$

By similar operations, the same result is obtained for each of the other conjugate functions, and the proof is concluded.

Theorem 3.19. If u, v, w, s are conjugate functions of an analytic function, then

$$\Box u \cdot \Box v = \Box u \cdot \Box w = \Box v \cdot \Box s = \Box w \cdot \Box s = 0,$$

and

$$\Box u \cdot \Box s = - \Box v \cdot \Box w = 2(s_x s_t - s_y s_z).$$

Proof. By use of the Cauchy-Riemann conditions, (3.4.15),

$$\Box u \cdot \Box v = u_x v_x + u_y v_y + u_z v_z + u_t v_t$$

$$= u_x v_x + (-v_x)u_x + u_z v_z + (-v_z)u_z = 0;$$

and similarly for the other parts of the same collection of equations.

In the second collection of equations, again by use of (3.4.15),

$$\Box u \cdot \Box s = u_x s_x + u_y s_y + u_z s_z + u_t s_t$$

$$= s_t s_x + (-s_z) s_y + (-s_y) s_z + s_x s_t$$

$$= 2(s_x s_t - s_y s_z);$$

and similarly for $-\Box v \cdot \Box w$. The proof is concluded.

The preceding theorems have some potentially useful geometrical interpretations, which are extensions of similar interpretations for complex variables. Theorems 3.16 and 3.17 indicate that the gradients of the conjugate functions are each the "same" vector quantity, except rotated in space by the 1, i, j, k operators considered as 4×4 real orthogonal matrices. They also force the gradient of a conjugate function to be the same regardless of the direction of approach to the point, provided that $f(\beta)$ is analytic at the point.

Theorem 3.18 indicates that the conjugate functions of an analytic function are harmonic; that is, they are solutions of Laplace's equation in four dimensions. Physical applications are conceivable since the harmonic property, in the two-dimensional case (complex variables), has been widely used in computing electric and magnetic field configurations of certain geometrically-suitable cases.

Theorem 3.19 also extends a result of complex variable theory, with a slight difference. If c_1, c_2, c_3, c_4 are real constants, and

if the equations

$$
\begin{aligned}
u(x,y,z,t) &= c_1 \\
v(x,y,z,t) &= c_2 \\
w(x,y,z,t) &= c_3 \\
s(x,y,z,t) &= c_4
\end{aligned}
\tag{3.4.24}
$$

are considered as families of four-dimensional hypersurfaces, then Theorem 3.19 indicates that those pairs of families associated with basis elements of differing order in $\mathbb{G}$ intersect orthogonally, but the remainder do not necessarily do so. Of course, orthogonality in the complex variable case always results since the single pair u, v is always associated with basis elements 1, i of differing order.

Theorem 3.20. Analytic Transformation of the Laplacian Operator. If $\psi(x,y,z,t)$ is a real-valued, analytic function of real variables, and if conjugate functions u, v, w, s of some analytic function are used to make a change of independent variables

$$
\begin{aligned}
x &\to u(x,y,z,t) \\
y &\to v(x,y,z,t) \\
z &\to w(x,y,z,t) \\
t &\to s(x,y,z,t)\,,
\end{aligned}
\tag{3.4.25}
$$

and if

$$\square_1 = \frac{\partial}{\partial u} + i\,\frac{\partial}{\partial v} + j\,\frac{\partial}{\partial w} + k\,\frac{\partial}{\partial s}\,,$$

then the Laplacian operator is transformed as follows:

$$\square^2\psi = |\square u|^2(\square_1)^2\psi + (\square u\cdot\square s)\,\left(\frac{\partial^2\psi}{\partial u\partial s} - \frac{\partial^2\psi}{\partial v\partial w}\right).$$

Proof. First, assume that the transformation (3.4.25) involves analytic functions u, v, w, s which are not necessarily conjugate; then the reader may verify by the chain rule that the Laplacian operator transforms as follows:

$$\begin{aligned}\square^2\psi = {} & |\square u|^2\frac{\partial^2\psi}{\partial u^2} + |\square v|^2\frac{\partial^2\psi}{\partial v^2} + |\square w|^2\frac{\partial^2\psi}{\partial w^2} + |\square s|^2\frac{\partial^2\psi}{\partial s^2} \\ & + (\square^2 u)\frac{\partial\psi}{\partial u} + (\square^2 v)\frac{\partial\psi}{\partial v} + (\square^2 w)\frac{\partial\psi}{\partial w} + (\square^2 s)\frac{\partial\psi}{\partial s} \\ & + 2(\square u\cdot\square v)\frac{\partial^2\psi}{\partial u\partial v} + 2(\square u\cdot\square w)\frac{\partial^2\psi}{\partial u\partial w} \\ & + 2(\square u\cdot\square s)\frac{\partial^2\psi}{\partial u\partial s} + 2(\square v\cdot\square w)\frac{\partial^2\psi}{\partial v\partial w} \\ & + 2(\square v\cdot\square s)\frac{\partial^2\psi}{\partial v\partial s} + 2(\square w\cdot\square s)\frac{\partial^2\psi}{\partial w\partial s}\,,\end{aligned} \tag{3.4.26}$$

where the $(\cdot)$ indicates the vector dot product; for example

$$\square u\cdot\square v = u_x v_x + u_y v_y + u_z v_z + u_t v_t.$$

Now assume that u, v, w, s are conjugate functions. By application of

Theorems 3.17 through 3.19 to the coefficient expressions on the right-hand side of (3.4.26), the desired result follows. The proof is concluded.

The gradient expression $\square\psi$ may also be transformed as in Theorem 3.20, as shown below:

Lemma 3.21. If $\psi(x,y,z,t)$ is a real-valued, analytic function of real variables, and if conjugate functions u, v, w, s are used to make a change of independent variables

$$x \to u(x,y,z,t)$$
$$y \to v(x,y,z,t)$$
$$z \to w(x,y,z,t)$$
$$t \to s(x,y,z,t),$$

then the gradient expression for ψ transforms as follows:

$$\square\psi = \square u \square_1 \psi = -i \square v \square_1 \psi,$$

$$= -j \square w \square_1 \psi = k \square s \square_1 \psi,$$

where

$$\square_1 = \frac{\partial}{\partial u} + i \frac{\partial}{\partial v} + j \frac{\partial}{\partial w} + k \frac{\partial}{\partial s} .$$

Proof. By the chain rule,

$$\psi_x = \psi_u u_x + \psi_v v_x + \psi_w w_x + \psi_s s_x$$

$$\psi_y = \psi_u u_y + \psi_v v_y + \psi_w w_y + \psi_s s_y$$

$$\psi_z = \psi_u u_z + \psi_v v_z + \psi_w w_z + \psi_s s_z$$

$$\psi_t = \psi_u u_t + \psi_v v_t + \psi_w w_t + \psi_s s_t,$$

which can be written in the form

$$(3.4.27) \quad \begin{bmatrix} \psi_x \\ \psi_y \\ \psi_z \\ \psi_t \end{bmatrix} = \begin{bmatrix} u_x & v_x & w_x & s_x \\ u_y & v_y & w_y & s_y \\ u_z & v_z & w_z & s_z \\ u_t & v_t & w_t & s_t \end{bmatrix} \begin{bmatrix} \psi_u \\ \psi_v \\ \psi_w \\ \psi_s \end{bmatrix}.$$

Now if the Cauchy-Riemann conditions (3.4.15) are appropriately substituted into (3.4.27), one can obtain

$$(3.4.28) \quad \begin{bmatrix} \psi_x \\ \psi_y \\ \psi_z \\ \psi_t \end{bmatrix} = \begin{bmatrix} u_x & -u_y & -u_z & u_t \\ u_y & u_x & -u_t & -u_z \\ u_z & -u_t & u_x & -u_y \\ u_t & u_z & u_y & u_x \end{bmatrix} \begin{bmatrix} \psi_u \\ \psi_v \\ \psi_w \\ \psi_s \end{bmatrix}.$$

The 4×4 matrix in (3.4.28) may be seen to be the matrix representation of the gradient of u; hence by Theorem 3.1,

$$\square\psi = \square u \; \square_1 \psi. \tag{3.4.29}$$

The proof is completed by making successive substitutions for $\square u$ from the results of Theorem 3.16.

The Cauchy-Riemann conditions will next be used to establish local uniqueness of conjugate function coordinate transformations as discussed in Theorem 3.20 and Lemma 3.21.

Lemma 3.22. If u, v, w, s are conjugate functions, then the coordinate transformation

$$\begin{aligned} x &\to u(x,y,z,t) \\ y &\to v(x,y,z,t) \\ z &\to w(x,y,z,t) \\ t &\to s(x,y,z,t) \end{aligned}$$

is locally unique everywhere away from the axis planes.

Proof. The nonvanishing of the Jacobian,

$$\begin{vmatrix} u_x & v_x & w_x & s_x \\ u_y & v_y & w_y & s_y \\ u_z & v_z & w_z & s_z \\ u_t & v_t & w_t & s_t \end{vmatrix} \neq 0, \tag{3.4.30}$$

will establish local uniqueness; i.e., each point will have a neighborhood on which an inverse function exists. If the Cauchy-Riemann conditions are appropriately substituted into (3.4.30), one may obtain

$$\begin{vmatrix} u_x & -u_y & -u_z & u_t \\ u_y & u_x & -u_t & -u_z \\ u_z & -u_t & u_x & -u_y \\ u_t & u_z & u_y & u_x \end{vmatrix} \neq 0,$$

which is easily seen to be

$$\det(\Box u) \neq 0. \tag{3.4.31}$$

But the gradient vector is just a quantity in $\mathbb{D}$, hence by Lemma 2.13 is invertible (i.e., has nonzero determinant) everywhere except in the axis planes.

Thus far, all operations involving the gradient operator have dealt only with real-valued functions. The next theorem investigates the gradient for vector-valued functions in IH.

Theorem 3.23. The gradient of a vector-valued function $f \in \mathrm{IH}$ of a four-dimensional variable $\beta \in \mathbb{D}$ is zero, everywhere: $\Box f(\beta) \equiv \theta$.

Proof. By Remark 3.3, $f(\beta)$ can always be written in the form $u + iv + jw + ks$; then

$$\square f(\beta) = \left(\frac{\partial}{\partial x} + i\,\frac{\partial}{\partial y} + j\,\frac{\partial}{\partial z} + k\,\frac{\partial}{\partial t}\right)(u + iv + jw + ks)$$

$$= (u_x - v_y - w_z + s_t)$$

$$+ i(u_y + v_x - w_t - s_z) \tag{3.4.32}$$

$$+ j(u_z - v_t + w_x - s_y)$$

$$+ k(u_t + v_z + w_y + s_x).$$

When the Cauchy-Riemann conditions (3.4.15) are substituted into (3.4.32), one obtains

$$\square f(\beta) \equiv \theta,$$

the desired result.

The same condition holds for a function of a complex variable (two dimensions).

IV. AN APPLICATION IN SPECIAL RELATIVITY: A FOUR-DIMENSIONAL MODEL FOR LORENTZIAN CONTRACTION

Let us, at the outset, acknowledge the profound correctness of Einstein's Theory of Relativity [23]. The passage of time continues to bring new and novel verifications of its validity. It will not be the purpose here to detract from it or to contradict any of its results.

The purpose of this Section will be to develop a group of transformations in four dimensions which extends the properties of the Lorentz group in one space coordinate and time to four dimensions and allows relative motion in an arbitrary direction. The conjecture will be made that the larger group provides a formulation of Einstein's Theory of Special Relativity in the new mathematical system which has been developed earlier in this thesis. (The word "conjecture" is used because several new phenomena, which are not anticipated by the existing Relativity Theory, will be predicted below.)

Indeed, the algebra on $\mathbb{D}$ seems to have relativity built in. Let c denote the speed of light in a vacuum, and consider the particular vector form

$$V = v + kc,$$

with $|v| < c$. In the space $\mathbb{D}$, these vectors have inverses of the form

$$V^{-1} = (-v + kc)/\{c^2[1 - (v/c)^2]\}.$$

The Lorentz-Fitzgerald contraction factor, $[1 - (v/c)^2]^{1/2}$, occurs as an intrinsic feature of the algebraic system. The inverse does not exist under the conditions $v = \pm c$. More generally, the inverse of the "typical" number $\beta = x + iy + jz + kct$ in $\mathbb{D}$ does not exist under the conditions ($x = ct$ and $y = -z$) or ($x = -ct$ and $y = z$), or both, but exists everywhere else. Consequently, if β is considered as a multiplication operator (linear transformation) on $\mathbb{D}$, then the operation has no inverse under the conditions given above.

4.1 $EL_4(V)$: A Group Of Operators In The Four-Dimensional Real Space, $\mathbb{D}$

We first reexamine the group of Lorentz transformations in one space coordinate and time, and recast it in a form which will facilitate its generalization to the space $\mathbb{D}$. Consider two Euclidean coordinate systems, one taken to be at rest and the other taken to be in motion with a constant relative velocity v with respect to the rest system. Further suppose that the direction of motion is parallel to the x-axis of the rest system. Let c denote the speed of light in a vacuum, and let $X = x + iy + jz + kct$ denote a position vector in the real four-space. We will use the following notation, selected for reasons which will become more clear later on.

Definition 4.1. $V = v + kc$ will be called the relative motion vector for two Euclidean coordinate systems in one space coordinate and time, and $EL_2(V)$ will denote the group of Lorentz transformations between such systems. The typical element of $EL_2(V)$ will be denoted by T, and can be represented in a 2×2 real matrix form, as follows:

$$(4.1.1) \qquad T = \frac{1}{[1 - (v/c)^2]^{1/2}} \begin{bmatrix} 1 & -v/c \\ -v/c & 1 \end{bmatrix},$$

with $|v| < c$. The inverse of the typical element is

$$(4.1.2) \qquad T^{-1} = \frac{1}{[1 - (v/c)^2]^{1/2}} \begin{bmatrix} 1 & v/c \\ v/c & 1 \end{bmatrix},$$

and the group multiplication is given by

$$(4.1.3) \qquad T_1 T_2 = T_3 = \frac{1}{[1 - (v_3/c)^2]^{1/2}} \begin{bmatrix} 1 & -v_3/c \\ -v_3/c & 1 \end{bmatrix},$$

with

$$(4.1.4) \qquad v_3 = \frac{v_1 + v_2}{1 + \frac{v_1 v_2}{c^2}} .$$

Calculations with the Lorentz group predict the following phenomenon, which is borne out by experiment. Given a vector (x, ct) in a coordinate system which is assumed to be at rest, and given a second coordinate system moving with constant relative velocity v along the x axis of the first system, then an observer in the moving system will measure the vector (x, ct) as

$$(4.1.5) \qquad X' = TX,$$

or

$$\begin{bmatrix} x' \\ ct' \end{bmatrix} = \frac{1}{[1 - (v/c)^2]^{1/2}} \begin{bmatrix} 1 & -v/c \\ -v/c & 1 \end{bmatrix} \begin{bmatrix} x \\ ct \end{bmatrix}$$

(4.1.6)

$$= \frac{1}{[1 - (v/c)^2]^{1/2}} \begin{bmatrix} x & -(v/c)ct \\ ct & -(v/c)x \end{bmatrix} .$$

In other words, the components x and ct transform as

$$x' = [x - (v/c)ct]/[1 - (v/c)^2]^{1/2}$$

(4.1.7)

$$ct' = [ct - (v/c)x]/[1 - (v/c)^2]^{1/2}.$$

As a direct consequence of (4.1.7), two observers in relative motion will disagree on the apparent length of an object in the system which is designated as the moving frame of reference (because of the Lorentz-Fitgerald contraction effect) [24, p. 357]. If an observer who is moving along with the object in question measures its length as

$$L_0 = x_2' - x_1',$$

where x_1' and x_2' are the coordinates of the end points of the object in the moving system, then the at rest observer will measure it as

$$L = x_2 - x_1$$

$$(4.1.8) \qquad = (x_2' - x_1')[1 - (v/c)^2]^{1/2}$$

$$= L_0[1 - (v/c)^2]^{1/2},$$

where x_1 and x_2 are the apparent instantaneous coordinates in the rest system of the end points of the moving object, and are assumed to be measured at the same time t.

The time dilation effect is another direct consequence of (4.1.7) Suppose the origins of the two coordinate systems coincide at $t' = t = 0$, and suppose that the origin of the moving system is observed from the rest system. Its coordinate in the rest system will be $x = vt$; and from (4.1.7), it follows that

$$ct' = [ct - (v/c)vt]/[1 - (v/c)^2]^{1/2}$$

$$(4.1.9) \qquad = ct[1 - (v/c)^2]/[1 - (v/c)^2]^{1/2}$$

$$= ct[1 - (v/c)^2]^{1/2}.$$

Now suppose that some process taking place in the moving system has a lifetime of $\Delta t'$. By (4.1.9), the same process observed from the rest system appears to have a lifetime

$$(4.1.10) \qquad \Delta t = \Delta t'/[1 - (v/c)^2]^{1/2},$$

a larger value.

Returning to the purely mathematical aspects of $EL_2(V)$, we observe that multiplication is commutative, as is immediate from (4.1.4). We also observe that $EL_2(V)$ has a formulation in the space $\mathbb{H}$ of hypercomplex functions which has been constructed in this thesis. Consider the multiplication operator

$$T = [1 - k(v/c)]/[1 - (v/c)^2]^{1/2}, \tag{4.1.11}$$

where k is the basis element in $\mathbb{D}$ such that $k^2 = 1$. If T operates on a vector $X = x + kct$, the result is

$$\begin{aligned} TX &= [1 - k(v/c)][x + kct]/[1 - (v/c)^2]^{1/2} \\ &= \{[x - (v/c)ct] + k[ct - (v/c)x]\}/[1 - (v/c)^2]^{1/2} \qquad (4.1.12) \\ &= x_1 + kct_1. \end{aligned}$$

Equating coefficients of corresponding basis vectors on opposite sides of (4.1.12), we obtain the Lorentz transformation in the form (4.1.7), hence (4.1.11) is a representation in $\mathbb{D}$ of the typical element of $EL_2(V)$.

In the function space $\mathbb{H}$, (4.1.11) is an <u>analytic function</u> of the variable $X = x + kct$, which may be shown as follows. For simplicity, let $a = v/c$ and $b = [1 - (v/c)^2]^{1/2}$; then the components of the transformation (4.1.12) are

$$x_1 = (x - act)/b$$

(4.1.13)

$$ct_1 = (ct - ax)/b.$$

Now,

$$\frac{\partial x_1}{\partial x} = \frac{\partial(ct_1)}{\partial(ct)} = 1/b, \text{ and}$$

(4.1.14)

$$\frac{\partial x_1}{\partial(ct)} = \frac{\partial(ct_1)}{\partial x} = - a/b.$$

These are the Cauchy-Riemann conditions for T in the function space IH, hence T is analytic in IH.

The typical element T of $EL_2(V)$ has a simple expression in terms of the relative motion vector V = v + kc. To show this, we need the following preliminary result. If β_1 and β_2 are elements in the space $\mathbb{D}$, then

$$\begin{aligned}(\beta_1\beta_2)^{1/2} &= [(\xi_1 e_1 + \eta_1 e_2)(\xi_2 e_1 + \eta_2 e_2)]^{1/2} \\ &= (\xi_1\xi_2 e_1 + \eta_1\eta_2 e_2)^{1/2} \\ &= (\xi_1\xi_2)^{1/2} e_1 + (\eta_1\eta_2)^{1/2} e_2 \\ &= \xi_1^{1/2}\ \xi_2^{1/2} e_1 + \eta_1^{1/2}\ \eta_2^{1/2} e_2 \\ &= (\xi_1^{1/2} e_1 + \eta_1^{1/2} e_2)(\xi_2^{1/2} e_1 + \eta_2^{1/2} e_2) \\ &= \beta_1^{1/2}\ \beta_2^{1/2}.\end{aligned} \tag{4.1.15}$$

provided that the proper interpretation is made of $(\xi_1\xi_2)^{1/2}$ and $(\eta_1\eta_2)^{1/2}$ in the complex variable sense. Care must be taken that all complex square root operations take place on the same sheet of the Riemann surface. Now, starting from (4.1.11) and using (4.1.15) freely, we can obtain

$$
\begin{aligned}
T &= [1 - k(v/c)]/[1 - (v/c)^2]^{1/2} \\
&= [1 - k(v/c)]/\{[1 - k(v/c)][1 + k(v/c)]\}^{1/2} \\
&= [1 - k(v/c)]^{1/2}/[1 + k(v/c)]^{1/2} \\
&= [c - kv]^{1/2}/[c + kv]^{1/2} \qquad (4.1.16) \\
&= [k\,(-v + kc)]^{1/2}/[k(v + kc)]^{1/2} \\
&= [(-v + kc)/(v + kc)]^{1/2} \\
&= [(-v + kc)/V]^{1/2}.
\end{aligned}
$$

From an earlier observation that

$$V^{-1} = (-v + kc)/\{c^2[1 - (v/c)^2]\},$$

we can see that

$$(-v + kc) = c^2[1 - (v/c)^2]/V,$$

hence

$$T = c[1 - (v/c)^2]^{1/2}/V. \tag{4.1.17}$$

The functional form of a relativistic transformation in the space IH, in one space coordinate and time, is therefore

$$\begin{aligned} X' &= TX \\ &= c[1 - (v/c)^2]^{1/2}(X/V). \end{aligned} \tag{4.1.18}$$

We make one last observation about the elements of $EL_2(V)$ in the form (4.1.1). The typical transformation matrix T, although it has determinant (+1), is not an orthogonal matrix, since

$$T^{-1} = \frac{1}{[1 - (v/c)^2]^{1/2}} \begin{bmatrix} 1 & v/c \\ v/c & 1 \end{bmatrix}, \tag{4.1.19}$$

which is not equal to the transpose of T. The generalization of $EL_2(V)$, below, will produce 4×4 real matrix transformations which also are nonorthogonal, and have determinant +1.

We are now in a position to extend $EL_2(V)$ to four dimensions. We first construct the group $EL_4(V)$ in four dimensions, and then show that it extends the properties of $EL_2(V)$.

Definition 4.2. Let (v_1, v_2, v_3, v_4) be a quadruple of real numbers such that $[(v_1 - v_4)^2 + (v_2 + v_3)^2][(v_1 + v_4)^2 + (v_2 - v_3)^2] \neq 0$, and let

$$V = v_1 + iv_2 + jv_3 + kv_4$$

denote the generalized relative motion vector. Also, let

$$U = -v_1 - iv_2 + jv_3 + kv_4;$$

then $EL_4(V)$ will denote the set of all multiplicative operators (transformations) of the form $T = (U/V)^{1/2}$.

Equation (4.1.16) provides motivation for investigating such transformations, inasmuch as Definition 4.2 provides a plausible generalization of the former.

Theorem 4.3. The algebraic rules on the space $\mathbb{D}$ induce a group structure on $EL_4(V)$.

Proof. This is best established by use of the canonical forms given by Definitions 2.9 and 3.2. Let

$$p = (v_1 - v_4) + i(v_2 + v_3)$$

(4.1.20)

$$q = (v_1 + v_4) + i(v_2 - v_3);$$

then

$$U = -qe_1 - pe_2$$

(4.1.21)

$$V = pe_1 + qe_2,$$

and

$$T = (U/V)^{1/2}$$

$$= (UV^{-1})^{1/2}$$

(4.1.22) $$= \{[-qe_1 - pe_2][p^{-1}e_1 + q^{-1}e_2]\}^{1/2}$$

$$= \{(-q/p)e_1 + (-p/q)e_2\}^{1/2}$$

$$= (-q/p)^{1/2}e_1 + (-p/q)^{1/2}e_2.$$

We now show that the group properties are obeyed.

(4.1.23) Closure. If $T_1, T_2 \in EL_4(V)$, then

$$T_1T_2 = [(-q_1/p_1)^{1/2}e_1 + (-p_1/q_1)^{1/2}e_2][(-q_2/p_2)^{1/2}e_1 + (-p_2/q_2)^{1/2}e_2]$$

$$= (-q_1/p_1)^{1/2}(-q_2/p_2)^{1/2}e_1 + (-p_1/q_1)^{1/2}(-p_2/q_2)^{1/2}e_2$$

$$= [(q_1q_2)/(p_1p_2)]^{1/2}e_1 + [(p_1p_2)/(q_1q_2)]^{1/2}e_2.$$

By Definition 4.2,

$$|p_1|^2 = [(v_1 - v_4)^2 + (v_2 + v_3)^2] \neq 0$$

$$|q_1|^2 = [(v_1 + v_4)^2 + (v_2 - v_3)^2] \neq 0;$$

similarly, $|p_2|^2 \neq 0$ and $|q_2|^2 \neq 0$. It follows that T_1T_2 exists and is an element of $EL_4(V)$, because p_1p_2 and q_1q_2 are complex quantities.

(4.1.24) Existence of identity. $T_0 = 1$ is the identity, because $T_0T = TT_0 = T$ for all $T \in EL_4(V)$.

(4.1.25) Inverses. As observed above, $|p|^2 \neq 0$ and $|q|^2 \neq 0$, hence by Definition 3.2,

$$T^{-1} = (-q/p)^{-1/2}e_1 + (-p/q)^{-1/2}e_2$$

$$= (-p/q)^{1/2}e_1 + (-q/p)^{1/2}e_2,$$

which is an element of $EL_4(V)$. The conditions on (v_1, v_2, v_3, v_4) in Definition 4.2 are precisely the axis plane conditions; hence the domain of definition of $EL_4(V)$ excludes the axis planes. A similar situation arises in $EL_2(V)$, for $v = \pm c$.

(4.1.26) Associativity. This follows from associativity of the underlying algebra on $\mathbb{D}$.

The proof of Theorem 4.3 is concluded.

We now show that $EL_4(V)$ extends the properties of $EL_2(V)$, beginning with the observation that $EL_2(V)$ is contained in $EL_4(V)$.

Theorem 4.4. $EL_2(V)$ is a subgroup in $EL_4(V)$.

Proof. $EL_2(V)$ is already known to be a group, and has the same identity element as $EL_4(V)$. It is sufficient to show that the typical element of $EL_2(V)$ is a special case of the typical element of $EL_4(V)$. From (4.1.16) and Definition 4.2, we see that

$$EL_2(V) : T = [(-v + kc)/(v + kc)]^{1/2}$$

$$EL_4(V) : T = [(-v_1 - iv_2 + jv_3 + kv_4)/(v_1 + iv_2 + jv_3 + kv_4)]^{1/2}.$$

Apparently, if we set $v_1 = v$, $v_2 = v_3 = 0$, and $v_4 = c$ in $EL_4(V)$, we obtain elements of $EL_2(V)$. The proof is concluded.

By Theorem 3.1, the group $EL_4(V)$ has a 4×4 real matrix representation. Its elements may be found as follows. Given the canonical form

$$T = (-q/p)^{1/2}e_1 + (-p/q)^{1/2}e_2,$$

Remark 3.3 asserts that it has the expanded form

$$T = u + ir + jw + ks, \tag{4.1.27}$$

where

$$u = \{\mathrm{Re}[(-q/p)^{1/2}] + \mathrm{Re}[(-p/q)^{1/2}]\}/2$$

$$r = \{\mathrm{Im}[(-q/p)^{1/2}] + \mathrm{Im}[(-p/q)^{1/2}]\}/2$$

$$w = \{\mathrm{Im}[(-q/p)^{1/2}] - \mathrm{Im}[(-p/q)^{1/2}]\}/2$$

$$s = \{-\mathrm{Re}[(-q/p)^{1/2}] + \mathrm{Re}[(-p/q)^{1/2}]\}/2,$$

or

$$u = \mathrm{Re}[(-q/p)^{1/2} + (-p/q)^{1/2}]/2$$

$$r = \mathrm{Im}[(-q/p)^{1/2} + (-p/q)^{1/2}]/2$$

(4.1.28)

$$w = \mathrm{Im}[(-q/p)^{1/2} - (-p/q)^{1/2}]/2$$

$$s = \mathrm{Re}[-(q/p)^{1/2} + (-p/q)^{1/2}]/2.$$

Now, by the explicit isomorphism of $\mathbb{G}$ (the 1, i, j, k basis) and $\mathbb{E}$ (the matrix basis) as indicated in Theorem 2.3, the matrix form of (4.1.28) is

$$T = \begin{bmatrix} u & -r & -w & s \\ r & u & -s & -w \\ w & -s & u & -r \\ s & w & r & u \end{bmatrix} . \tag{4.1.29}$$

One could use the elements of $EL_4(V)$ in the form (4.1.29), as 4×4 real matrix transformations on a four-dimensional real space. However, their use in the functional form in the space IH is computationally much simpler, as subsequent developments will show. We start by showing that the groups $EL_2(V)$ and $EL_4(V)$ have some important computational properties in common.

(4.1.30) The elements of $EL_4(V)$ in matrix form are nonorthogonal, with determinant + 1.

Elements T of $EL_4(V)$ are nonorthogonal because in the matrix form the inverse of T is not simply its transpose. The value of the determinant is found as follows. Recall from (3.1.16) that the determinant of T is the squared absolute value of the product of the eigenvalues of T, as displayed in the canonical form, (4.1.22). Hence

$$\begin{aligned} \det(T) &= \left|(-q/p)^{1/2}(-p/q)^{1/2}\right|^2 \\ &= \left|[(qp)/(pq)]^{1/2}\right|^2 \\ &= \left|[1]^{1/2}\right|^2 = +1. \end{aligned} \tag{4.1.31}$$

(4.1.32) A time reversal has no effect on a relativistic transformation.

Consider a vector X_c in the rest system which connects the origins of the moving and rest systems. The relative motion vector V is the time derivative of X_c; and, as in traditional treatments, V has constant components. Then the components x_c, y_c, z_c must be linear functions of the time, t, and

$$V = \frac{dX_c}{dt} = \frac{dx_c}{dt} + i\frac{dy_c}{dt} + j\frac{dz_c}{dt} + kc$$

$$= v_1 + iv_2 + jv_3 + kc.$$

If the sign is changed on the t-parameter (time reversal), and if for the immediate discussion we let primes (') denote the time reversed condition, then

$$X_c'(t) = X_c(-t) = -x_c - iy_c - jz_c - kct = -X_c,$$

and

$$V' = \frac{dX_c'}{dt} = -V.$$

A spatial velocity sign change has been produced as a consequence of time reversal. Also,

$$U' = v_1 + iv_2 - jv_3 - kc = -U,$$

hence

$$T' = (U'/V')^{1/2}$$

$$= [(-U)/(-V)]^{1/2}$$

$$= (U/V)^{1/2} \equiv T,$$

which is the desired result.

(4.1.33) $EL_2(V)$ and $EL_4(V)$ are abelian groups.

Commutativity of $EL_2(V)$ is obvious from the multiplication rule (4.1.4), or from its formulation in the space $\mathbb{H}$, which is a commutative system. Commutativity of $EL_4(V)$ follows similarly from its formulation in the space $\mathbb{H}$. A consequence is that a finite number of transformations applied successively yields the same effect regardless of the order of application.

(4.1.34) Zero spatial relative velocity produces an identity operator.

In the relative motion vector V, if we set the spatial components (relative velocity) equal to zero, $v_1 = v_2 = v_3 = 0$, we get

$$T = [(kc)/(kc)]^{1/2} = 1.$$

Other conditions which produce the same effect will be discussed later.

4.2 A Conjecture on the Application of $EL_4(V)$ to Special Relativity.

Up to this point, we have discussed the group $EL_4(V)$ from a mainly mathematical viewpoint, showing that it contains $EL_2(V)$ as a subgroup and that it has several transformational properties in common with $EL_2(V)$. We now speculate upon the physical applications of $EL_4(V)$.

Conjecture 4.5. The group $EL_4(V)$ provides a mathematical formulation in the space IH for special relativistic effects involving relative velocities in arbitrary directions.

In order to provide some basis for this conjecture, we must show that Einstein's two relativity postulates [24, p. 353] are not contradicted under a transformation belonging to $EL_4(V)$.

The first postulate is that the laws of physics and the results of all experiments performed in a given coordinate system are independent of the translational motion of the system as a whole. This is usually shown for the Lorentz transformations by noting that such transformations are covariant, and by writing the equations of physics in forms which remain invariant in form under a covariant transformation of the independent variables. ("Covariant" will be taken to mean that the Euclidean coordinates transform in the same way as the basis vectors.) We refer the reader to Goldstein [25, pp. 194-205] or Jackson [24, pp. 377-386] for a more complete discussion.

In the present case, we note that a transformation belonging to $EL_4(V)$ is covariant. In the matrix form, it is

$$(4.2.1) \qquad \begin{bmatrix} x' \\ y' \\ z' \\ ct' \end{bmatrix} = \begin{bmatrix} u & -r & -w & s \\ r & u & -s & -w \\ w & -s & u & -r \\ s & w & r & u \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ ct \end{bmatrix},$$

where u, r, w, s are functions of the relative velocity and are given by (4.1.28) and (4.1.20). We now appeal to the same covariance arguments which are used in the existing Lorentz transformation theories in order to conclude that the equations of physics are invariant in form under a transformation belonging to $EL_4(V)$.

The new mathematical system which has been developed in this thesis allows an additional formulation of the above covariance arguments. Recall that we have developed a calculus and a system of analysis for four-dimensional numbers (vectors) which subsumes the calculus for real or complex systems. We have defined functions, derivatives, integrals, and so on. Now we show that if the relative motion vector is assumed fixed with respect to time, and if the position vector X is taken to be the independent variable, then a transformation operator T in $EL_4(V)$ acts like a multiplicative constant which commutes with the differential and integral operators. Indeed, by (4.2.1) and the rules for real functions.

$$\begin{aligned} dx' &= udx - rdy - wdz + sdct \\ dy' &= rdx + udy - sdz - wdct \\ dz' &= wdx - sdy + udz - rdct \\ dct' &= sdx + wdy + rdz + udct, \end{aligned}$$

which can be written in the form

$$(4.2.2) \qquad \begin{bmatrix} dx' \\ dy' \\ dz' \\ dct' \end{bmatrix} = \begin{bmatrix} u & -r & -w & s \\ r & u & -s & -w \\ w & -s & u & -r \\ s & w & r & u \end{bmatrix} \begin{bmatrix} dx \\ dy \\ dz \\ dct \end{bmatrix} .$$

Designating the two vectors in (4.2.2) as <u>differentials</u>, by Theorem 3.1 we can write

$$(4.2.3) \qquad dX' = d(TX) = TdX.$$

In a similar way, we can prove a chain rule for derivatives which will show that a derivative operation transforms in the same way as the differential. Let $X' = TX$, as before, and let

$$f(X') = E + iF + jG + kH$$

be an analytic function, with conjugate functions $E(x', y', z', ct')$, $F(x', y', z', ct')$, etc. By the usual chain rule for real functions, letting subscripts denote partial differentiation, we get

$$E_x = E_{x'}(x')_x + E_{y'}(y')_x + E_{z'}(z')_x + E_{ct'}(ct')_x$$

$$F_x = F_{x'}(x')_x + F_{y'}(y')_x + F_{z'}(z')_x + F_{ct'}(ct')_x$$

$$G_x = G_{x'}(x')_x + G_{y'}(y')_x + G_{z'}(z')_x + G_{ct'}(ct')_x$$

$$H_x = H_{x'}(x')_x + H_{y'}(y')_x + H_{z'}(z')_x + H_{ct'}(ct')_x;$$

which can be written in the form

$$(4.2.4) \qquad \begin{bmatrix} E_x \\ F_x \\ G_x \\ H_x \end{bmatrix} = \begin{bmatrix} E_{x'} & E_{y'} & E_{z'} & E_{ct'} \\ F_{x'} & F_{y'} & F_{z'} & F_{ct'} \\ G_{x'} & G_{y'} & G_{z'} & G_{ct'} \\ H_{x'} & H_{y'} & H_{z'} & H_{ct'} \end{bmatrix} \begin{bmatrix} (x')_x \\ (y')_x \\ (z')_x \\ (ct')_x \end{bmatrix}.$$

Since f is analytic in X', we can apply the Cauchy-Riemann condition to the partial derivatives in the matrix in (4.2.4) to get

$$(4.2.5) \qquad \begin{bmatrix} E_x \\ F_x \\ G_x \\ H_x \end{bmatrix} = \begin{bmatrix} E_{x'} & -F_{x'} & -G_{x'} & H_{x'} \\ F_{x'} & E_{x'} & -H_{x'} & -G_{x'} \\ G_{x'} & -H_{x'} & E_{x'} & -F_{x'} \\ H_{x'} & G_{x'} & F_{x'} & E_{x'} \end{bmatrix} \begin{bmatrix} (x')_x \\ (y')_x \\ (z')_x \\ (ct')_x \end{bmatrix},$$

which, by Theorem 3.1, we recognize as

(4.2.6) $$\frac{\partial f}{\partial x} = \frac{\partial f}{\partial x'} \frac{\partial X'}{\partial x} .$$

Now we apply Theorem 3.14 to get

(4.2.7) $$\frac{df}{dX} = \frac{df}{dX'} \frac{dX'}{dX};$$

and since $X' = TX$, we have

(4.2.8) $$\frac{df}{dX} = T \frac{df}{dX'}.$$

In other words, derivative operations transform covariantly in the space IH.

In passing, the arguments leading up to (4.2.7) may be used to assert that

(4.2.9) $$\frac{df}{dX} = \frac{df}{dg} \frac{dg}{dX},$$

where g is any analytic function of X. We point this out in order to continue to emphasize that the new system exhibits the same notation and follows the same rules as for the complex calculus.

Returning to invariance properties, we observe as follows that integral operations transform covariantly under a transformation in $EL_4(V)$:

(4.2.10) $$\int_{X_1'}^{X_2'} f(X')dX' = \int_{X_1}^{X_2} f(X')TdX = T \int_{X_1}^{X_2} f(TX)dX.$$

Next, we show that the volume element $d\tau = dx\, dy\, dz\, dct$ is invariant in form under a transformation T in $EL_4(V)$. The Jacobian of the transformation is, referring to (4.2.1),

$$\frac{\partial(x', y', z', ct')}{\partial(x, y, z, ct)} = \begin{vmatrix} u & -r & -w & s \\ r & u & -s & -w \\ w & -s & u & -r \\ s & w & r & u \end{vmatrix} \tag{4.2.11}$$

$$= \det(T) = +1,$$

hence $dx'\, dy'\, dz'\, dct' = dx\, dy\, dz\, dct$.

We may also observe via Lemma 3.21 that the gradient operator transforms as

$$\square\psi = (\square x')\square_1\psi \tag{4.2.12}$$

under a transformation T in $EL_4(V)$, where

$$\square_1\psi = \frac{\partial\psi}{\partial x'} + i\frac{\partial\psi}{\partial y'} + j\frac{\partial\psi}{\partial z'} + k\frac{\partial\psi}{\partial ct'}$$

and

$$\square x' = \frac{\partial x'}{\partial x} + i\frac{\partial x'}{\partial y} + j\frac{\partial x'}{\partial z} + k\frac{\partial x'}{\partial ct} \tag{4.2.13}$$

$$= u - ir - jw + ks,$$

with u, r, w, s being the vector components of T, given by (4.1.28) and (4.1.20). Note that (4.2.13) is "almost T," because it has the same components as T, up to sign. As a consequence, the untransformed gradient is just a "constant" vector times the transformed gradient. Additionally, when (4.2.12) is written in matrix form, it becomes

$$(4.2.14)\qquad \begin{bmatrix} \psi_x \\ \psi_y \\ \psi_z \\ \psi_{ct} \end{bmatrix} = \begin{bmatrix} u & r & w & s \\ -r & u & -s & w \\ -w & -s & u & r \\ s & -w & -r & u \end{bmatrix} \begin{bmatrix} \psi_{x'} \\ \psi_{y'} \\ \psi_{z'} \\ \psi_{ct'} \end{bmatrix} .$$

Comparing the transforming matrix in (4.2.14) with the matrix representation of T in (4.2.1), one sees that the former is the transpose of the latter. Denoting transpose by (†), we thus have

$$(4.2.15)\qquad \Box x' = T^{\dagger},$$

and (4.2.12) becomes

$$(4.2.16)\qquad \Box\psi = T^{\dagger}\Box_1\psi.$$

The Laplacian operator transforms as shown in Theorem 3.20, for an arbitrary, real $\psi(x,y,z,ct)$. If the coordinate transformation belongs to $EL_4(V)$, the the Laplacian is

$$
\begin{aligned}
\Box^2\psi &= |\Box x'|^2(\Box_1)^2\psi + 2(\Box x'\cdot\Box ct')\left(\frac{\partial^2\psi}{\partial x'\partial ct'} - \frac{\partial^2\psi}{\partial y'\partial z'}\right) \\
&= |\Box x'|^2(\Box_1)^2\psi + 2(\Box x'\cdot k\Box x')\left(\frac{\partial^2\psi}{\partial x'\partial ct'} - \frac{\partial^2\psi}{\partial y'\partial z'}\right) \\
&= |T^\dagger|^2(\Box_1)^2\psi + 2(\Box x'\cdot k\Box x')\left(\frac{\partial^2\psi}{\partial x'\partial ct'} - \frac{\partial^2\psi}{\partial y'\partial z'}\right) \\
&= |T|^2(\Box_1)^2\psi + 2(T^\dagger\cdot kT^\dagger)\left(\frac{\partial^2\psi}{\partial x'\partial ct'} - \frac{\partial^2\psi}{\partial y'\partial z'}\right) ,
\end{aligned}
\tag{4.2.17}
$$

where the $(\cdot)$ indicates vector dot product. If we further assume that ψ is a conjugate function of some analytic function of X', then by the Cauchy-Riemann conditions (3.4.15) there exists a complementary conjugate function $\Phi(x', y', z', ct')$ for which

$$
\frac{\partial^2\psi}{\partial x'\partial ct'} - \frac{\partial^2\psi}{\partial y'\partial z'} = \frac{\partial^2\Phi}{\partial(ct')^2} + \frac{\partial^2\Phi}{\partial(z')^2} . \tag{4.2.18}
$$

But we have seen in the discussion following (3.4.15) that the restricted variable $jz' + kct'$ acts as a complex variable; hence the right-hand side of (4.2.18) is zero, as a consequence of the usual two-dimensional Cauchy-Riemann conditions. Returning to (4.2.17), we see that we are left with

$$
\Box^2\psi = |T|^2(\Box_1)^2\psi. \tag{4.2.19}
$$

From the above, it is clear that $EL_4(V)$ preserves the forms of the equations of physics, hence is in agreement with Einstein's first postulate.

Einstein's second postulate is that the speed of light is the same in all frames of reference. Consider two Euclidean coordinate systems in relative motion. Suppose that at time $t = t' = 0$ the origins of the two systems coincide, and further suppose that at coincidence a spherical wavefront of light is emitted from the coincident origins. Then Einstein's second postulate dictates that an observer in either the moving or the rest system will observe that the wavefront expands spherically outward from the observer's origin at the speed of light.

To show that transformations in $EL_4(V)$ preserve the speed of light, we must make some careful interpretations of the transformation process in the new system. First it is not solely the form $x^2 + y^2 + z^2 - c^2t^2$ which is invariant, as in the traditional development, but the form $[(x - ct)^2 + (y + z)^2][(x + ct)^2 + (y - z)^2]$, as well. To see this, note that by Theorem 3.1 we can write the typical transformation, (4.2.1), in an equivalent full matrix form, as follows:

$$(4.2.20)\qquad \begin{bmatrix} x' & -y' & -z' & ct' \\ y' & x' & -ct' & -z' \\ z' & -ct' & x' & -y' \\ ct' & z' & y' & x' \end{bmatrix} = \begin{bmatrix} u & -r & -w & s \\ r & u & -s & -w \\ w & -s & u & -r \\ s & w & r & u \end{bmatrix} \begin{bmatrix} x & -y & -z & ct \\ y & x & -ct & -z \\ z & -ct & x & -y \\ ct & z & y & x \end{bmatrix}.$$

We have previously determined that if $X = x + iy + jz + kct$, then in the matrix form, $\det(X) = [(x - ct)^2 + (y + z)^2][(x + ct)^2 + (y - z)^2]$. We have also found that the determinant of a transformation matrix belonging to $EL_4(V)$ is $+1$. If we apply the product rule of determinants to

(4.2.20), we get

$$[(x' - ct')^2 + (y' + z')^2][(x' + ct')^2 + (y' - z')^2]$$

(4.2.21)

$$= [(x - ct)^2 + (y + z)^2][(x + ct)^2 + (y - z)^2].$$

We must now show that (4.2.21) admits the invariance of the speed of light, c. First, consider the simple case wherein the two observers measure the speed of light in their respective systems by measuring the time t (or t') taken by the wavefront described earlier to traverse a distance x (or x') along the x (or x') axis from the origin. By (4.2.21), with all space coordinates except x and x' equal to zero, we have $[(x')^2 - (ct')^2]^2 = [(x)^2 - (ct)^2]^2$, one solution of which is contained in

(4.2.22) $$(x')^2 - (ct')^2 = (x)^2 - (ct)^2.$$

Since t and t' are not zero, on physical grounds, then we can rearrange (4.2.22) as follows:

(4.2.23) $$(t')^2[(x'/t')^2 - c^2] = (t)^2[(x/t)^2 - c^2].$$

Now if the rest system observer designates

(4.2.24) $$c = (x/t),$$

the moving system observer must have

$$c = (x'/t'), \tag{4.2.25}$$

within a sign on c.

More generally, if the two observers each measure off a distance S in a specified direction in their respective systems, using some standard procedure, and then arbitrarily agree that the measured distances are the same, then no contradiction will arise if they also agree that the respective times taken by light to traverse the distances S are the same. For proof, suppose that a spherical wavefront centered at the origin is generated at $t = t' = 0$, as before. Further suppose that the specified distances and directions are obtained by measuring coordinates x_0, y_0, z_0 which yield the desired direction, and with $S^2 = x_0^2 + y_0^2 + z_0^2$ in either system. By (4.2.21),

$$[(x_0 - ct')^2 + (y_0 + z_0)^2][x_0 + ct')^2 + (y_0 - z_0)^2]$$

$$= [(x_0 - ct)^2 + (y_0 + z_0)^2][(x_0 + ct)^2 + (y_0 - z_0)^2],$$

or

$$\frac{[(x_0 - ct')^2 + (y_0 + z_0)^2]}{[(x_0 - ct)^2 + (y_0 + z_0)^2]} = \frac{[(x_0 + ct)^2 + (y_0 - z_0)^2]}{[(x_0 + ct')^2 + (y_0 - z_0)^2]}. \tag{4.2.26}$$

By inspection, $t' = t$ is a solution of (4.2.26); hence

$$c = S/t = S/t'$$

(invariance of the speed of light) is admissible under transformations belonging to $EL_4(V)$. We conclude that the $EL_4(V)$ group of transformations, with the proper interpretation, does not contradict Einstein's postulates.

4.3 Unusual Relativistic Effects Predicted Via $EL_4(V)$.

A simple calculation with an element T of $EL_4(V)$ indicates that the Lorentz-Fitzgerald contraction may be accompanied by an equal transverse expansion, in four dimensions. (We have already seen, in (4.1.8) and (4.1.9), that this is true in one space dimension and time). Consider the canonical forms

$$V = pe_1 + qe_2,$$

$$T = (-q/p)^{1/2}e_1 + (-p/q)^{1/2}e_2,$$

$$X = \xi e_1 + \eta e_2,$$

and

$$X' = \xi' e_1 + \eta' e_2.$$

If $X' = TX$, then

$$|X'|^2 = (|\xi'|^2 + |\eta'|^2)/2, \tag{4.3.1}$$

and

$$|TX|^2 = |[(-q/p)^{1/2}e_1 + (-p/q)^{1/2}e_2][\xi e_1 + \eta e_2]|^2$$

$$= |\xi(-q/p)^{1/2}e_1 + \eta(-p/q)^{1/2}e_2|^2$$

(4.3.2)

$$= \{|\xi(-q/p)^{1/2}|^2 + |\eta(-p/q)^{1/2}|^2\}/2$$

$$= \{|\xi|^2|q|/|p| + |\eta|^2|p|/|q|\}/2.$$

Suppose that the observer in the primed system measures a four-dimensional hypersphere of radius $1/(2)^{1/2}$ centered at his origin. It can be described by

$$|X'|^2 = 1/2,$$

or

$$|\xi'|^2 + |\eta'|^2 = 1. \tag{4.3.3}$$

In order to make it clear that (4.3.3) represents a sphere in real four-space, we can make the (invertible) change of variables

$$\begin{aligned} e' &= x' - ct' \\ f' &= y' + z' \\ g' &= x' + ct' \\ h' &= y' - z', \end{aligned} \tag{4.3.4}$$

and get

$$\xi' = (x' - ct') + i(y' + z') = e' + if'$$

$$\eta' = (x' + ct') + i(y' - z') = g' + ih'.$$

Then

$$(4.3.5) \qquad |\xi'|^2 + |\eta'|^2 = (e')^2 + (f')^2 + (g')^2 + (h')^2 = 1.$$

The observer in the rest system, meanwhile, will measure the same surface as

$$|TX|^2 = 1/2,$$

or

$$(4.3.6) \qquad \frac{|\xi|^2}{\left(\frac{p}{q}\right)} + \frac{|\eta|^2}{\left(\frac{q}{p}\right)} = 1.$$

By means of a change of variables similar to (4.3.4), we can get (4.3.6) into the form

$$(4.3.7) \qquad \frac{e^2}{\left(\frac{P}{Q}\right)} + \frac{f^2}{\left(\frac{P}{Q}\right)} + \frac{g^2}{\left(\frac{Q}{P}\right)} + \frac{h^2}{\left(\frac{Q}{P}\right)} = 1,$$

which we recognize as a hyperellipsoid in the real four-space coordinatized by (e, f, g, h). (The change of variables has taken $p \to P$ and

$q \to Q$.) The semiaxes are aligned with the coordinate axes in the (e, f, g, h)-space, but in the (x, y, z, ct)-space they have an orientation which depends upon the relative motion vector, $V = pe_1 + qe_2$.

We now show that a transverse expansion occurs. Recall that ξe_1 and ηe_2 are the projections of a vector $X = \xi e_1 + \eta e_2$ onto the axis planes. It follows that the plane which contains ξe_1 and ηe_2 also contains X. Further, recall that the axis planes are perpendicular. It is now not too difficult to see that if X is constrained to any fixed plane which goes through the origin of the (x, y, z, ct)-space, equation (4.3.6) describes an ellipse, in the given plane, whose semiaxis lengths are reciprocals of each other. We conclude that the ellipsoid described by (4.3.7) exhibits elongation in a direction transverse (orthogonal) to the direction of contraction, in the exact amount of the contraction.

The preceding conjecture is amenable to experimental verification. Suppose that the hypersurface in question is an equal-intensity surface of the electric field about a charged particle, in a multi-slot diffraction experiment. (The diffraction apparatus plays the role of the rest system.) As the particle's kinetic energy is increased, the intensity of the diffraction lines away from the beam axis should vary in a fashion consistent with a transverse expansion factor which is exactly the reciprocal of the longitudinal contraction factor. Existing theory [26, p. 346] predicts that the transverse expansion factor is the reciprocal of the square root of the longitudinal contraction factor.

A second unusual effect predicted via the group $EL_4(V)$ is that zero relative velocity may not be the only condition that produces an identity transformation. We can see from

$$T = (U/V)^{1/2} = [(-qe_1 - pe_2)/(pe_1 + qe_2)]^{1/2}$$

that we have an identity transformation whenever $p = -q$, which is the same as

$$(v_1 - v_4) + i(v_2 + v_3) = -(v_1 + v_4) - i(v_2 - v_3),$$

or

$$v_1 - v_4 = -v_1 - v_4$$

$$v_2 + v_3 = -v_2 + v_3.$$

The preceding equations imply $v_1 = v_2 = 0$, with v_3 and v_4 arbitrary, which implies that there is a particular plane containing $jv_3 + kv_4$ in which no relativistic effects are produced. We only speculate that there must be some physical grounds to exclude, a priori, such cases from consideration.

Lastly, the generalization of Einstein's addition law for velocities also has unusual implications. The generalization is found as follows. We want to find a resultant relative motion vector $V_3 = p_3e_1 + q_3e_2$ whose associated transformation $T_3 \in EL_4(V)$ is the same as the product of two given transformations $T_1, T_2 \in EL_4(V)$. From

(4.1.23), we can see that we must have

$$\frac{-p_3}{q_3} = \frac{p_1 p_2}{q_1 q_2} \tag{4.3.8}$$

or

$$\frac{-(p_3/c)}{(q_3/c)} = \frac{(p_1/c)(p_2/c)}{(q_1/c)(q_2/c)},$$

one solution of which is

$$-(p_3/c) = (p_1/c)(p_2/c) = p_1 p_2/c^2$$

$$(q_3/c) = (q_1/c)(q_2/c) = q_1 q_2/c^2,$$

or

$$p_3 = -p_1 p_2/c$$

$$q_3 = q_1 q_2/c. \tag{4.3.9}$$

Using the notation $V_n = v_{n1} + iv_{n2} + jv_{n3} + kv_{n4}$, and setting the real and imaginary parts equal in (4.3.9), we get

$$v_{31} - v_{34} = -[(v_{11} - v_{14})(v_{21} - v_{24}) - (v_{12} + v_{13})(v_{22} + v_{23})]/c$$

$$v_{31} + v_{34} = [(v_{11} + v_{14})(v_{21} + v_{24}) - (v_{12} - v_{13})(v_{22} - v_{23})]/c$$

(4.3.10)

$$v_{32} + v_{33} = -[(v_{11} - v_{14})(v_{22} + v_{23}) + (v_{12} + v_{13})(v_{21} - v_{24})]/c$$

$$v_{32} - v_{33} = [(v_{11} + v_{14})(v_{22} - v_{23}) + (v_{12} - v_{13})(v_{21} + v_{24})]/c.$$

The preceding is easily solved by elimination, yielding

$$v_{31} = (v_{11}v_{24} + v_{12}v_{23} + v_{13}v_{22} + v_{14}v_{21})/c$$

$$v_{32} = (-v_{11}v_{23} + v_{12}v_{24} - v_{13}v_{21} + v_{14}v_{22})/c$$

(4.3.11)

$$v_{33} = (-v_{11}v_{22} - v_{12}v_{21} + v_{13}v_{24} + v_{14}v_{23})/c$$

$$v_{34} = (v_{11}v_{21} - v_{12}v_{22} - v_{13}v_{23} + v_{14}v_{24})/c.$$

It follows that a relative motion $V_3 = v_{31} + iv_{32} + jv_{33} + kv_{34}$ given by (4.3.11) produces the same effect as the combination of motions $V_1 = v_{11} + iv_{12} + jv_{13} + kv_{14}$ and $V_2 = v_{21} + iv_{22} + jv_{23} + kv_{24}$; but (4.3.11) does not have the appearance of a generalization of Einstein's addition law, nor does it reduce to Einstein's law. The reconciliation is achieved by noting that Einstein's law results from a renormalization of the resultant relative motion vector so as to have the k-component

always equal to c, as may be seen from the following. If $T_1, T_2 \in EL_2(V)$, then

$$T_1T_2 = \frac{1}{[1 - (v_1/c)^2]^{1/2}} \begin{bmatrix} 1 & -v_1/c \\ -v_1/c & 1 \end{bmatrix} \frac{1}{[1 - (v_2/c)^2]^{1/2}} \begin{bmatrix} 1 & -v_2/c \\ -v_2/c & 1 \end{bmatrix}$$

(4.3.12)

$$= \frac{1}{\{[1 - (v_1/c)^2][1 - (v_2/c)^2]\}^{1/2}} \begin{bmatrix} 1 + v_1v_2/c^2 & -(v_1+v_2)/c \\ -(v_1+v_2)/c & 1 + v_1v_2/c^2 \end{bmatrix}$$

$$= \frac{1 + v_1v_2/c^2}{[1+(v_1v_2)^2/c^4-(v_1^2+v_2^2)/c^2]^{1/2}} \begin{bmatrix} 1 & -\left(\frac{v_1 + v_2}{1 + \frac{v_1v_2}{c^2}}\right)\Big/c \\ -\left(\frac{v_1 + v_2}{1 + \frac{v_1v_2}{c^2}}\right)\Big/c & 1 \end{bmatrix}$$

$$= \left\{\frac{[1 + v_1v_2/c^2]^2}{[1+v_1v_2/c^2]^2-(v_1+v_2)^2/c^2}\right\}^{1/2} \begin{bmatrix} 1 & -\left(\frac{v_1 + v_2}{1 + \frac{v_1v_2}{c^2}}\right)\Big/c \\ -\left(\frac{v_1 + v_2}{1 + \frac{v_1v_2}{c^2}}\right)\Big/c & 1 \end{bmatrix}$$

(4.3.13)

$$= \frac{1}{\left[1 - \left(\frac{v_1 + v_2}{1 + \frac{v_1v_2}{c^2}}\right)^2\Big/c^2\right]^{1/2}} \begin{bmatrix} 1 & -\left(\frac{v_1 + v_2}{1 + \frac{v_1v_2}{c^2}}\right)\Big/c \\ -\left(\frac{v_1 + v_2}{1 + \frac{v_1v_2}{c^2}}\right)\Big/c & 1 \end{bmatrix}.$$

Note that, starting in (4.3.12), both the numerator and the denominator have been divided through by $(1 + v_1v_2/c^2)$. This produces a product transformation which is of the same form as each of the original transformations, and leaves us with the implication that $(v_1 + v_2)/(1 + v_1v_2/c^2)$ is the effective relative velocity. The preceding being so, the associated relative motion vector is

$$V_3 = \left(\frac{v_1 + v_2}{1 + \frac{v_1v_2}{c^2}} \right) + kc, \tag{4.3.14}$$

always with c as the k-component.

It is instructive to see the same process carried out by use of the vector notation. We have

$$\begin{aligned} V_1V_2 &= (v_1 + kc)(v_2 + kc) \\ &= (v_1v_2 + c^2) + kc(v_1 + v_2) \\ &= kc[(v_1 + v_2) + k(c + v_1v_2/c)], \end{aligned}$$

$$\begin{aligned} U_1U_2 &= (-v_1 + kc)(-v_2 + kc) \\ &= (v_1v_2 + c^2) - kc(v_1 + v_2) \\ &= kc[-(v_1 + v_2) + k(c + v_1v_2/c)], \end{aligned}$$

and

$$T_1T_2 = (U_1/V_1)^{1/2}(U_2/V_2)^{1/2}$$

$$= [(U_1U_2)/V_1V_2)]^{1/2} \tag{4.3.15}$$

$$= \{[-(v_1+v_2) + k(c+v_1v_2/c)]/[(v_1+v_2) + k(c+v_1v_2/c)]\}^{1/2}.$$

We might at this point compare the form $T = (U/V)^{1/2}$ with (4.3.15) and interpret $[(v_1 + v_2) + k(c + v_1v_2/c)]$ as the effective relative motion vector (as we might have done in (4.3.12)), but instead we divide through both the numerator and denominator inside the braces in (4.3.15) by $(c + v_1v_2/c)/c$ to get

$$T_1T_2 = \left\{ \left[-\left(\frac{v_1 + v_2}{1 + \frac{v_1v_2}{c^2}} \right) + kc \right] \Big/ \left[\left(\frac{v_1 + v_2}{1 + \frac{v_1v_2}{c^2}} \right) + kc \right] \right\}^{1/2}. \tag{4.3.16}$$

Interpreting $(v_1 + v_2)/(1 + v_1v_2/c^2)$ as the effective relative motion vector has left us with the inference that the k-component of the resultant relative motion vector is always c.

We now return to the reconciliation of the four-dimensional addition law for velocities, (4.3.11), with Einstein's addition law. We can perform the renormalization process, which we have seen above, on the four-dimensional law (4.3.11) by dividing all components v_{3i}

by the k-component over c, v_{34}/c, to get

$$v_{31}\Big|_{\text{norm}} = c\left(\frac{v_{11}v_{24} + v_{12}v_{23} + v_{13}v_{22} + v_{14}v_{21}}{v_{11}v_{21} - v_{12}v_{22} - v_{13}v_{23} + v_{14}v_{24}}\right),$$

$$v_{32}\Big|_{\text{norm}} = c\left(\frac{-v_{11}v_{23} + v_{12}v_{24} - v_{13}v_{21} + v_{14}v_{22}}{v_{11}v_{21} - v_{12}v_{22} - v_{13}v_{23} + v_{14}v_{24}}\right),$$

(4.3.17)

$$v_{33}\Big|_{\text{norm}} = c\left(\frac{-v_{11}v_{22} - v_{12}v_{21} + v_{13}v_{24} + v_{14}v_{23}}{v_{11}v_{21} - v_{12}v_{22} - v_{13}v_{23} + v_{14}v_{24}}\right),$$

$$v_{34}\Big|_{\text{norm}} = c.$$

If we always consider only the normalized forms, we can drop the "norm" subscript and assume $v_{14} = v_{24} = v_{i4} = c$; then (4.3.17) becomes

$$v_{31} = c\left(\frac{v_{11}c + v_{12}v_{23} + v_{13}v_{22} + cv_{21}}{v_{11}v_{21} - v_{12}v_{22} - v_{13}v_{23} + c^2}\right),$$

$$v_{32} = c\left(\frac{-v_{11}v_{23} + v_{12}c - v_{13}v_{21} + cv_{22}}{v_{11}v_{21} - v_{12}v_{22} - v_{13}v_{23} + c^2}\right),$$

(4.3.18)

$$v_{33} = c\left(\frac{-v_{11}v_{22} - v_{12}v_{21} + v_{13}c + cv_{23}}{v_{11}v_{21} - v_{12}v_{22} - v_{13}v_{23} + c^2}\right),$$

$$v_{34} = c.$$

The preceding set of equations reduces to Einstein's law when all spatial components of velocity except v_{11} and v_{21} are equal to zero, because then

$$v_{31} = c\left(\frac{v_{11}c + cv_{21}}{v_{11}v_{21} + c^2}\right) = \frac{v_{11} + v_{21}}{1 + \frac{v_{11}v_{21}}{c^2}},$$

$$v_{32} = v_{33} = 0.$$

We conclude that equations (4.3.18) represent the generalized Einstein's addition law for relative velocities.

Equations (4.3.18) imply that for certain combinations of relative velocities, the speed of light c is no longer an upper bound on the resultant relative velocity. Even if c is imposed as an upper bound on the individual relative velocities, a resultant velocity greater than c can still occur. For example, suppose $v_{11} = v_{12} = v_{13} = v_{22} = v_{23} = c/(3)^{1/}$ and $v_{21} = -v_{11}$; then $v_{11}^2 + v_{12}^2 + v_{13}^2 = c^2$ and $v_{21}^2 + v_{22}^2 + v_{23}^2 = c^2$. However, $v_{11}v_{21} - v_{12}v_{22} - v_{13}v_{23} + c^2 = 0$, but $v_{11}c + v_{12}v_{23} + v_{13}v_{22} + cv_{21} = 2c^2/3$, leaving $v_{31} = \infty$.

More rigorously, if we select ε to satisfy $0 < \varepsilon < c^2/3$, and let $v_{12} = v_{13} = v_{22} = v_{23} = c/(3)^{1/2}$ and $v_{11} = -v_{21} = (c^2/3 - \varepsilon)^{1/2}$; then both V_1 and V_2 have the magnitude of their spatial relative velocities less than c, but the resultant of V_1 and V_2 has a first component $v_{31} = 2c^3/(3\varepsilon)$. In particular, choosing $\varepsilon = 2c/3$ leads to $v_{31} = c^2$, so that the magnitude of the spatial resultant exceeds c.

Such a condition may not be experimentally observable, because all experiments which relate to special relativity indicate that c is an upper bound for the speed of transmission of observable data from one point to another. If electromagnetic waves are the medium of transmission, for example, then a source moving away from an observer with a speed greater than c would have its wavelength extended beyond conventional detection limits. Perhaps certain combinations of relative motions in four dimensions can be eliminated from consideration on physical grounds, so as to leave c as an upper bound on the resultant velocities predicted by (4.3.18). These and other questions will be pursued at a later date.

We conclude this thesis with a brief summary. In Sections II and III, we have seen the development of a calculus and a system of analysis for a four-dimensional variable, which contains and extends the complex system of analysis. We have seen that the new system has several notational systems; among them being a 4×4 real matrix formulation, and also the traditional notation for the complex system of analysis, with only minor exceptions.

In Section IV, we have seen that the new system intrinsically exhibits relativistic behavior when the four-dimensional space-time position is taken as the independent variable. Further, we have seen the development of a group of transformations in four dimensions which appears to be a generalization of the Lorentz group in one space coordinate and time. The larger group allows relative motion in an arbitrary direction; and though Einstein's postulates are obeyed (with the proper interpretations of mathematical operations), some nontraditional results

are predicted. For example, a transverse expansion (in four dimensions) is predicted which is equal to the contraction in the direction of motion. The relationship of the substance of Section IV to the real world is only conjectured, with specific experiment being suggested. However, if the results of experiment confirm the above conjecture, then we will be left with the implication that points in the physical space-time must be considered as ordered pairs of complex numbers, rather than ordered quadruples of real numbers.

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VITA

Clyde McCall Davenport was born in Sevier County, Tennessee, on December 15, 1938. He was graduated from Gatlinburg High School in 1956. Following three years of various employments, he served four years in the U.S. Air Force. Immediately afterwards, he enrolled at the University of Tennessee and in 1967 he received the Bachelor of Science degree in Engineering Physics. He held memberships in Sigma Pi Sigma, Phi Kappa Phi, and Tau Beta Pi scholastic honor societies.

He entered the Graduate School at the University of Tennessee in September, 1967, and received the Master of Science degree with a major in Physics in August, 1969. He is employed by the Union Carbide Corporation in Oak Ridge, Tennessee, doing developmental work in mathematical analysis, computer simulation, and real-time computer applications. Background work toward this thesis was carried out on a spare-time basis from May, 1967 to September, 1976, when a leave of absence was obtained in order to attack the problem in earnest and commit the results to paper. It is hoped that the reader has found something of interest in these pages.