

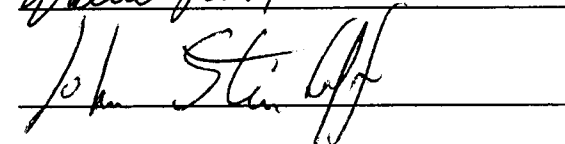
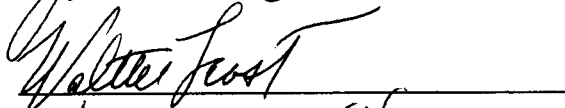
To the Graduate Council:

I am submitting herewith a dissertation written by Michael Lee Stokes entitled "A Solution of the Euler Equations using an Unstructured Grid System." I have examined the final copy of this dissertation for form and content and recommend that it be accepted in partial fulfillment of the requirements for the degree of Doctor of Philosophy, with a major in Engineering Science.



K.C. Reddy, Major Professor

We have read this dissertation
and recommend its acceptance:



Accepted for the Council:



Vice Provost and Dean of
The Graduate School

**A SOLUTION OF THE EULER EQUATIONS
ON AN UNSTRUCTURED GRID SYSTEM**

**A Dissertation
Presented for the
Doctor of Philosophy
Degree
The University of Tennessee, Knoxville**

**Michael Lee Stokes
May 1990**

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ABSTRACT

A numerical algorithm is presented for the solution of the one and two-dimensional Euler Equations based on a conservative formulation. The objective of the study is to combine the theory of characteristics with a conservative finite volume algorithm resulting in a numerical scheme with increased accuracy through the use of enhanced shock capturing capability, and reduced levels of numerical dissipation. Characteristic boundary conditions are developed for subsonic and supersonic flow regimes, and for solid wall boundaries. The numerical scheme is further broadened by the use of an unstructured triangular grid system with generalized placement of internal boundary conditions.

The approach is to cast the Euler Equations in an uncoupled or approximately uncoupled form based on the classical eigenvalue study. Using the similarity transformation matrices from this study, a matrix operator is derived which splits a conservatively calculated residual into components which are propagated along specified paths. The matrix operator is derived for the quasi one-dimensional Euler Equations as well as the two-dimensional Euler Equations. Boundary conditions are derived based on a first order approximation of the compatibility equations for both the one and two-dimensional forms and shown to be consistent with the interior scheme.

The numerical algorithm is validated by various test cases with comparisons with theory and other numerical flow solvers. The algorithm is shown to give near exact solutions for the one-dimensional steady state ducts with area variation. Salient features of the two-dimensional algorithm include excellent shock capturing capability, use of general unstructured grids for complex geometries, as well as the ability to accurately model low and high speed flows with little or no artificial dissipation required for stability.

Table of Contents

1.0 INTRODUCTION	1
2.0 ONE-DIMENSIONAL CONCEPTS	8
2.1 THEORY	8
2.2 NUMERICAL ALGORITHM	14
2.3 BOUNDARY CONDITIONS	15
2.4 EXAMPLES	23
3.0 TWO-DIMENSIONAL EULER SOLUTIONS.....	26
3.1 THEORY	26
3.2 NUMERICAL ANALYSIS	34
3.3 BOUNDARY CONDITIONS	44
3.4 ORGANIZATION OF THE COMPUTATION	48
3.4.1 SUBSONIC DIVERGING CHANNEL	52
3.4.2 SUPERSONIC FLOW PAST A CYLINDER	53
3.4.3 TRANSONIC FLOW OVER A NACA-0012 AIRFOIL.....	54
3.4.4 FLOW OVER AN AIRFOIL WITH FLAP	55
3.4.5 FLOW OVER A FOUR FLAP AIRFOIL	56
4.0 CONCLUSIONS AND FUTURE WORK	58
BIBLIOGRAPHY	61
APPENDIX	64
VITA	104

Table of Figures

FIGURE		PAGE
1.	Mach Number comparison for divergent subsonic channel	65
2.	Static pressure comparison for divergent subsonic channel	66
3.	Total temperature comparison for subsonic divergent channel	67
4.	Total pressure comparison for subsonic divergent channel	68
5.	Mach Number comparison for converging-diverging nozzle	69
6.	Static pressure comparison for converging-diverging nozzle	70
7.	Total temperature comparison for converging-diverging nozzle	71
8.	Total pressure comparison for converging-diverging nozzle	72
9.	Example of a Nodal Connectivity Map (NCM)	73
10.	Orientation of $\alpha^{(j)}$ with corresponding vertex	74
11.	Unstructured grid used for divergent channel	75
12.	Structured grid for PARC code for divergent channel	76
13.	Contours of constant Mach Number for divergent channel	77
14.	Contours of constant static pressure for divergent channel	78
15.	Mach Number comparison for divergent channel	79
16.	Static pressure comparison for divergent channel	80
17.	Total pressure comparison for divergent channel	81
18.	Structured grid for PARC for flow around cylinder	82
19.	Unstructured grid for flow around cylinder	83
20.	Contours of constant static pressure using PARC for flow around cylinder	84

21.	Contours of constant static pressure for present study for flow around cylinder	85
22.	Sonic line for PARC for flow around cylinder	86
23.	Sonic line for present work for flow around cylinder.....	87
24.	Comparison plot of C_p for flow around cylinder.....	88
25.	Unstructured coarse grid for NACA-0012.....	89
26.	Unstructured fine grid for NACA-0012.....	90
27.	Contours of constant static pressure for coarse grid solution of NACA-0012.....	91
28.	Contours of constant static pressure for fine grid solution NACA-0012	92
29.	Comparison of pressure distribution for NACA-0012 coarse and fine grid solution	93
30.	Unstructured grid for Karman-Trefftz double airfoil	94
31.	Contours of constant static pressure for Karman-Trefftz double airfoil.....	95
32.	Contours of constant Mach Number for Karman-Trefftz double airfoil	96
33.	Velocity vector plot of main airfoil and flap for Karman-Trefftz double airfoil.....	97
34.	Comparison plot of C_p for Karman-Trefftz double airfoil	98
35.	Unstructured grid for four flap airfoil.....	99
36.	Contour plot of constant Mach Number for four flap airfoil	100
37.	Velocity vector plot around leading edge of four flap airfoil	101
38.	Velocity vector plot around trailing edge of main foil of four flap airfoil.....	102

39.	Velocity vector plot around trailing edge of third flap of four flap airfoil.....	103
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1.0 INTRODUCTION

With the introduction of the McCormack 1969 explicit Euler Solver [1], the numerical solution of the Euler Equations received much attention. Since then, considerable research effort has been expended increasing the accuracy and reliability of Euler simulations for practical applications. The methods of solution still in current use range from techniques based on non-conservative formulations, to finite difference and finite volume techniques, as well as the finite element method. Each of these schemes has strengths which make it appealing for specific applications and unacceptable for others. The current study takes advantage of many attributes of each of these schemes resulting in a unique solution procedure.

Researchers Morretti [2] and Verhoff [3] have extended the concept of steady state supersonic characteristics to space-time characteristics using finite difference concepts to integrate local Riemann style variables. Techniques such as these render exceptional resolution for shocks by applying the unsteady Rankine-Hugoniot jump relations after each non-conservative pass to grid intervals which exhibit the characteristics of a shock. This process, while viable in one dimension, creates tremendous bookkeeping problems in multidimensional space and is not common place in the aerospace industry.

Finite difference and finite volume techniques in conservative form are by far the most commonly used techniques for solving the Euler and Navier-Stokes Equations [4,5]. An integral or differential form of the equations is solved on a regular structured grid system taking advantage of the implied connectivity of its neighbors to define a differencing

stencil, for the differential form, or a surface volume for the integral form. These techniques have been extended to include overlapped [6] and embedded [7] systems of grids which communicate by some interpolation process at the grid boundaries. The use of structured grid systems have many advantages including the use of implicit solution schemes, high vectorization on vector architecture machines, and general ease of programming. Disadvantages of these schemes include difficulty in achieving acceptable local grid point distribution in areas of high gradients without overcompensating the grid density elsewhere and difficulty in modeling complex geometries. It is these disadvantages that the finite element technique compensates by the use of unstructured grids.

Finite element techniques originally were applied to the solution of structural design problems where their generality and applicability were shown superior to any previous method. As applied to field theory, the continuum is discretized into a set of simple polygons or elements over which the variation of properties is locally approximated by a given functional form¹. This functional form is defined in terms of unresolved constants determined by the application of boundary conditions and the solution of each of the other elements in the system. Typically the elements, defined by the nodes on the element vertices, are related to the other elements in the system by a hashing table that relates element number to node numbers for each cell. The technique has been applied

¹ commonly referred to as shape functions

successfully in the field of fluid mechanics as demonstrated by modeling the flow over a complete aircraft [8]. This flexibility of unstructured grids is further extended by node enrichment [9,10,11] and transport [12] in regions of large field gradients.

The general applicability of unstructured grids is also employed by the finite volume technique as demonstrated by Jameson in which the techniques used on structured quadrilateral grid systems were amended and applied on regular [13], and general unstructured triangular grids [14]. The current study is similar to the work of Jameson in that unstructured grids are used with a finite volume integration scheme to define a cell residual by summing fluxes about the boundary of each cell. One-third of the residual is distributed to each node in the cell resulting in what is analogous to a central difference scheme. To suppress instabilities, artificial dissipation is added based on second and fourth order differences. The upshot of the current study is to distribute the residual based on characteristic information so as to enhance stability of the scheme and thus minimize or eliminate the need for artificial dissipation. This distribution process will henceforth be referred to as the method of Characteristically Distributed Residual or (CDR) technique.

Published work abounds on schemes which use characteristic information in the solution of the Euler or Navier-Stokes equations. Coakley [15] extended the approximate factorization technique of Beam and Warming to include extensive use of an upwind diagonal form of the Jacobian matrices developed from the work by Pulliam and Chausse. Their work reduces a block tri-diagonal system of implicit equations to a system of uncoupled scalar equations through similarity transformations. Similarly, Steger

[16] makes use of characteristic theory for replacing the coefficient matrices used by his block tri-diagonal solver with simplified (sparse) coefficient matrices derived through characteristic theory. This effort shows that the Jacobian matrices that maintain similarity with the original Jacobian matrices can be used in place of the original without any "deleterious properties" while reducing the computational work. More extensive use of characteristic information can be found in the work of Roe [17] who uses a locally exact solution of the Riemann problem for the unsteady one-dimensional case or steady two-dimensional supersonic case. Of more curious nature is the work of Johnston and Pal [18] who integrate along one of the bicharacteristics of the system obtaining a new integral system of equations.

Powell and VanLeer [19] describe a conservative cell-vertex method in which a distribution function is derived based on stability analysis and an approximate decomposition of the Euler Equations. This decomposition procedure, stemming from the work of Hirsch [20], is based on the premise that terms of the compatibility relations which restrict the Euler equations from decoupling can be minimized or zeroed if the unit vector defining the coordinate transformation is selected specifically for each of the compatibility relations. This effort could be characterized as a CDR technique in that a conservatively calculated residual is calculated for a cell and distributed to the cell vertices or nodes using information derived from characteristic theory.

More recent work which parallels closely the current study is the work of Giles et. Al. [21]. This paper describes a new algorithm which is described as an "upwind control volume approach in which a standard flux residual is distributed with a downwind bias".

This technique uses unstructured triangular meshes and claims to handle distorted grids without loss of accuracy. Flows which can be characterized as irrotational and isentropic are treated by the addition of artificial dissipation based on entropy and vorticity smoothing. For cases which fall outside this regime, such as flows with a strong shock which generate vorticity downstream of the shock, the author resorts back to Jameson [13] type smoothing. The residual distribution function is based on a combination of geometrically and characteristically derived weights which bias the distribution of the residual to the downstream nodes. The current study varies from this methodology in that the distribution function is based on a classical formulation of the transformation Jacobians and its similarity matrices. Both of these methods are upwind schemes and are naturally diffusive, thereby reducing the dependence on externally introduced artificial dissipation.

An obvious difficulty in using characteristic information in any conservative scheme is that conservative forms of the multidimensional Euler equations can not be represented as an uncoupled set of total derivative equations. While alternate forms exist which approximately decouple the equations, such as in reference [3], these forms are non-conservative and therefore do not treat discontinuities. By writing the Euler equations in a non-conservative form [19], it is observed that the equations can be represented by the propagation of entropy along the streamlines, and by the propagation of "acoustic-like" variables along paths for which ∇p is zero. Therefore, the simple upwinding of conservative quantities based on the streamline eigenvalue, such as the Lax class schemes, will propagate entropy correctly, but will distort the propagation of the acoustic variables which are responsible for the placement of shocks. This distortion

accounts for the smearing of shocks for first order schemes, and oscillations for higher order schemes. The goal of this study is to find common ground by which characteristic information can be used to control the distribution of a conservatively calculated residual.

Following the classical eigenvalue and eigenvector analysis for hyperbolic systems, the current study decouples the Euler equations (approximately in multidimensions) into a system of scalar equations, each of which propagate information along characteristic paths. With this, a matrix operator is found which decomposes a conservatively calculated residual into individual components acting along the classical characteristic paths. Based on the sign and magnitude of the eigenvalue along each characteristic path, the decomposed residual is distributed to the vertices of the cell. These concepts are explored by first considering the quasi one-dimensional Euler equations in which the decoupling is complete. A numerical algorithm is constructed along with characteristically well-posed boundary conditions for supersonic and subsonic flows. The numerical scheme is then exercised with a small suite of test cases and compared to one-dimensional theory. Using the one-dimensional analysis as a model, the two-dimensional Euler equations are extended similarly, resulting in an approximate decomposition of the equations. The computational domain is discretized into triangular cells in which the vertices contain the conservative variables. An explicit finite volume integration is applied for each cell producing a conservatively calculated residual. An upwind distribution function is derived based on the similarity matrices of the equations. The boundary conditions developed for the one-dimensional algorithm are extended to two dimensions using a characteristic reference plane concept. The interior scheme along

with consistent boundary conditions are applied to a set of classical test cases and results are compared with exact solutions where available and against the solutions of an established flow solver. To demonstrate the generality of the technique, the algorithm is then applied to a series of test cases which include flow through a subsonic diverging channel, supersonic flow (Mach Number = 5.0) over a cylinder, followed by a series of test cases using single and multiple body airfoils in low subsonic to transonic flow regimes. For some of the simple cases, triangular grids are generated from structured quadrilateral grids. For the airfoil problems, true unstructured grids were used. Use of these unstructured grids allows a large variation of cell area over a small space, and in general, demonstrates the generality of the algorithm. The study is then summarized with recommendations for future research.

2.0 ONE-DIMENSIONAL CONCEPTS

A one-dimensional treatment of the Euler equations is presented as a predecessor to the multidimensional investigation as a means of simplifying the presentation of the underlying concepts. Working equations for the interior scheme are derived after a brief treatise of the classical eigenvalue study. The interior scheme is modified at the boundaries and combined with external constraints to provide characteristically based boundary conditions for supersonic and subsonic flow. Examples are given demonstrating the accuracy of the interior scheme and the boundary conditions for subsonic and mixed supersonic-subsonic flow regimes.

2.1 THEORY

The quasi one-dimensional Euler equations can be written in weak conservation form as

$$\frac{\partial U}{\partial t} + \frac{\partial F}{\partial x} = H \quad (2.1.1)$$

where

$$U = \begin{Bmatrix} \rho a \\ \rho u a \\ E a \end{Bmatrix}, F = \begin{Bmatrix} \rho u a \\ (\rho u^2 + p)a \\ u(E + p)a \end{Bmatrix}, H = \begin{Bmatrix} 0 \\ p \frac{\partial a}{\partial x} \\ 0 \end{Bmatrix} \quad (2.1.2)$$

with

$$p = (\gamma - 1) \left(E - \frac{1}{2} \rho u^2 \right) \quad (2.1.3)$$

Using Euler's Theorem for homogeneous functions of order one, Equation (2.1.1) written in non-conservation form is

$$\frac{\partial U}{\partial t} + A \frac{\partial U}{\partial x} = H \quad (2.1.4)$$

$$A = \frac{\partial F}{\partial U} = \begin{bmatrix} 0 & 1 & 0 \\ \frac{(\gamma-3)}{2}u^2 & (3-\gamma)u & \gamma-1 \\ (\gamma-1)u^3 - \gamma \frac{u}{\rho} & \gamma \frac{E}{\rho} - (\gamma-1)3 \frac{u^2}{2} & \gamma u \end{bmatrix} \quad (2.1.5)$$

It is convenient to express equation (2.1.4) in terms of primitive variables

$\tilde{U} = (\rho a, ua, pa)^T$ by application of the chain rule on the derivatives

$$\frac{\partial U}{\partial t} = \left(\frac{\partial U}{\partial \tilde{U}} \right) \left(\frac{\partial \tilde{U}}{\partial t} \right) = M \left(\frac{\partial \tilde{U}}{\partial t} \right) \quad (2.1.6)$$

$$\frac{\partial U}{\partial x} = \left(\frac{\partial U}{\partial \tilde{U}} \right) \left(\frac{\partial \tilde{U}}{\partial x} \right) = M \left(\frac{\partial \tilde{U}}{\partial x} \right) \quad (2.1.7)$$

followed by pre-multiplication of M^{-1} to give

$$\frac{\partial \tilde{U}}{\partial t} + \tilde{A} \frac{\partial \tilde{U}}{\partial x} = M^{-1}H \quad (2.1.8)$$

where

$$\tilde{A} = \begin{bmatrix} u & \rho & 0 \\ 0 & u & \frac{1}{\rho} \\ 0 & \gamma p & u \end{bmatrix} \quad (2.1.9)$$

$$M = \begin{bmatrix} 1 & 0 & 0 \\ u & \rho & 0 \\ \frac{u^2}{2} & \rho u & \frac{1}{\gamma-1} \end{bmatrix} \quad (2.1.10)$$

$$M^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ -\frac{u}{\rho} & \frac{1}{\rho} & 0 \\ \frac{\gamma-1}{2}u^2 & -(\gamma-1)u & \gamma-1 \end{bmatrix} \quad (2.1.11)$$

Using the corresponding right eigenvectors of $\tilde{A}$ as column vectors, we can form a matrix T which along with its inverse T^{-1} gives the similarity transformation which diagonalizes $\tilde{A}$, namely,

$$\tilde{A} = T \Lambda T^{-1} \quad (2.1.12)$$

where

$$T^{-1} = \begin{bmatrix} 0 & -\rho c & 1 \\ c^2 & 0 & -1 \\ 0 & \rho c & 1 \end{bmatrix} \quad (2.1.13)$$

$$T = \begin{bmatrix} \frac{1}{2c^2} & \frac{1}{c^2} & \frac{1}{2c^2} \\ -\frac{1}{2\rho c} & 0 & \frac{1}{2\rho c} \\ \frac{1}{2} & 0 & \frac{1}{2} \end{bmatrix} \quad (2.1.14)$$

$$\Lambda = \begin{bmatrix} u - c & 0 & 0 \\ 0 & u & 0 \\ 0 & 0 & u + c \end{bmatrix} \quad (2.1.15)$$

Pre-multiplying equation (2.1.8) by T^{-1} and using equation (2.1.12) gives

$$T^{-1} \frac{\partial \tilde{U}}{\partial t} + \Lambda T^{-1} \frac{\partial \tilde{U}}{\partial x} = T^{-1} M^{-1} H \quad (2.1.16)$$

which decouples the system of partial differential equations into a set of ordinary differential equations² acting along characteristic directions. Defining T_i^{-1} as the i^{th} row of T^{-1} and λ_i as the i^{th} diagonal of Λ , then

$$T_i^{-1} \frac{\partial \tilde{U}}{\partial t} + \lambda_i T_i^{-1} \frac{\partial \tilde{U}}{\partial x} = T_i^{-1} M^{-1} H \quad (2.1.17)$$

defines the i^{th} equation along the path defined by the derivatives

$$\frac{dx}{dt} = \lambda_{1,2,3} = u - c, u, u + c \quad (2.1.18)$$

Therefore, equation (2.1.17) can be written as

$$T_i^{-1} \frac{d\tilde{U}}{dt} = T_i^{-1} M^{-1} H \quad (2.1.19)$$

along the path defined in (2.1.18). The equations can be solved in this non-conservative form by integrating the compatibility equations along the characteristic paths by interpolating between time levels. This technique requires that special attention be given to

² commonly referred to as compatibility equations

shocks as the non-conservative formulation does not admit discontinuous solutions. A technique is sought which takes advantage of characteristic information while retaining a conservative formulation of the partial differential equations without the need for interpolation between time levels.

Substituting equations (2.1.6) and (2.1.7) into equation (2.1.16) gives

$$T^{-1}M^{-1}\frac{\partial U}{\partial t} + \Lambda T^{-1}M^{-1}\frac{\partial U}{\partial x} = T^{-1}M^{-1}H \quad (2.1.20)$$

For convenience, matrix N is defined as

$$N = MT \quad (2.1.21)$$

$$N^{-1} = T^{-1}M^{-1} = (MT)^{-1} \quad (2.1.22)$$

Substituting equation (2.1.22) into equation (2.1.20) gives

$$N^{-1}\frac{\partial U}{\partial t} + \Lambda N^{-1}\frac{\partial U}{\partial x} = N^{-1}H \quad (2.1.23)$$

Equation (2.1.23) represents a decoupled system of equations in terms of conservation variable U . Departure is made at this point from the classical characteristic analysis by introducing a pair of diagonal matrices S^+, S^- to separate the equations which propagate waves to the right from those that propagate waves to the left, respectively.

The matrices are related by

$$S = S^+ + S^- = I \quad (2.1.24)$$

The diagonal entries of S^+ , denoted S_i^+ , are defined by the sign of the i^{th} eigenvalue λ_i such that

$$S_i^+ = \begin{bmatrix} 1 & \text{if } \lambda_i > 1 \\ 0 & \text{if } \lambda_i \leq 1 \end{bmatrix} \quad (2.1.25)$$

Pre-multiplying equation (2.1.23) by NS gives

$$NSN^{-1} \frac{\partial U}{\partial t} + NS\Lambda N^{-1} \frac{\partial U}{\partial x} = NSN^{-1}H \quad (2.1.26)$$

Note the second term can be written as

$$NS\Lambda N^{-1} = NSN^{-1}N\Lambda N^{-1} = ZA \quad (2.1.27)$$

where

$$Z = NSN^{-1} \quad (2.1.28)$$

$$A = N\Lambda N^{-1} \quad (2.1.29)$$

Substituting equations (2.1.27) through (2.1.29) into equation (2.1.23) gives

$$Z \frac{\partial U}{\partial t} + ZA \frac{\partial U}{\partial x} = ZH \quad (2.1.30)$$

$$Z \left(\frac{\partial U}{\partial t} + A \frac{\partial U}{\partial x} = H \right) \quad (2.1.31)$$

Equation (2.1.31) is the original set of partial differential equations acted upon by a matrix operator Z . The significance of this operator is that it can be used to decompose a conservative residual into its upwind components. Section 2.2 illustrates how this splitting operator is used in a numerical algorithm for the solution of the Euler equations.

2.2 NUMERICAL ALGORITHM

Equation (2.1.1) is written as a conservative finite difference approximation as

$$\delta U_{i+\frac{1}{2}}^n = \delta t \left(H_{i+\frac{1}{2}}^n - \frac{F_{i+1}^n - F_i^n}{\Delta x} \right) \quad (2.2.1)$$

which represents a second order spatial accurate Euler integration. The momentum term in H is approximated at $i + \frac{1}{2}$ by

$$p \frac{\partial a}{\partial x} \Big|_{i+\frac{1}{2}}^n = (p_{i+1}^n + p_i^n) \frac{(a_{i+1} - a_i)}{2\Delta x} \quad (2.2.2)$$

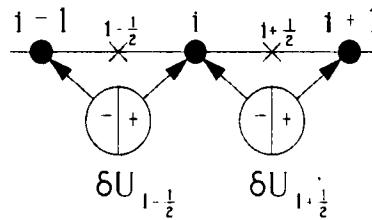
The residual in equation (2.2.1) is divided into two components using the matrix operator in equation (2.1.28) such that

$$Z \delta U_{i+\frac{1}{2}}^n = (Z^- + Z^+) \delta u_{i+\frac{1}{2}}^n = \delta U^- + \delta U^+ = \delta U_{i+\frac{1}{2}}^n \quad (2.2.3)$$

with

$$Z = NSN^{-1} = N(S^- + S^+)N^{-1} = NS^-N^{-1} + NS^+N^{-1} = Z^- + Z^+ = I \quad (2.2.4)$$

The components Z^+ and Z^- of the Z operator provide the distribution function for δU^n resulting in δU^+ being summed into the $i + 1$ node and δU^- being summed into the i node. Note that Z itself is an identity matrix.



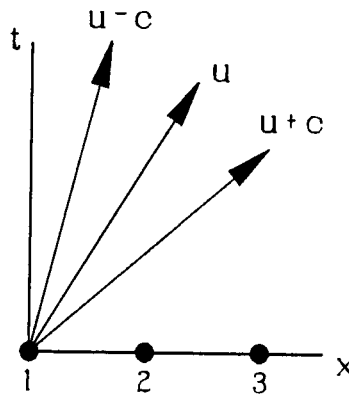
Equation (2.1.24) represents the statement of local conservation for each interval. Roe [22] averaging is used to evaluate the matrix components at $i + \frac{1}{2}$.

Although the integration stencil as defined in equation (2.2.1) is second order accurate as defined at the midnode, the distribution function reduces the scheme to first order when considering a single equation in which the sign of the eigenvalues are the same. It is unclear for subsonic cases which contain mixed eigenvalues if a single order of accuracy is definable when each of the scalar equations are upwinded. Numerical results as presented in Section 2.4 indicate that this distribution function produces excellent results in continuous and discontinuous regions of the flow.

2.3 BOUNDARY CONDITIONS

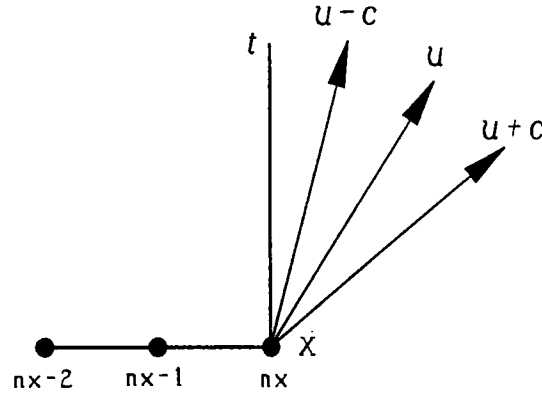
Boundary conditions determine the controlling influences which are passed to the interior points of the continuum. The present effort includes a study of well posed boundary conditions shown to be consistent with the interior scheme. The discussion of boundary conditions includes steady supersonic and subsonic inflow and outflow conditions along with characteristic wall boundary conditions. The boundary conditions are shown to be physically consistent with characteristic theory by taking advantage of residuals calculated from the interior scheme where appropriate.

Supersonic inflow boundary conditions are the simplest to treat with no additional information needed to complete the description. For supersonic flow, all three characteristics $\lambda_1 = u - c$, $\lambda_2 = u$, $\lambda_3 = u + c$ are positive-valued or incoming indicating that all information propagates into the interior from the boundary node.



Requiring that all properties be supplied from outside the domain is physically consistent with characteristic theory.

The supersonic outflow boundary condition is equally trivial by virtue of the correct propagation of information from the interior. For this case, all three characteristics are positive-valued or outgoing indicating that all information propagates out of the interior.



The interior scheme contributes a residual from the last interval which is complete with no additional information necessary to complete its description at the outflow node. Thus having all information supplied from the interior is physically consistent with characteristic theory.

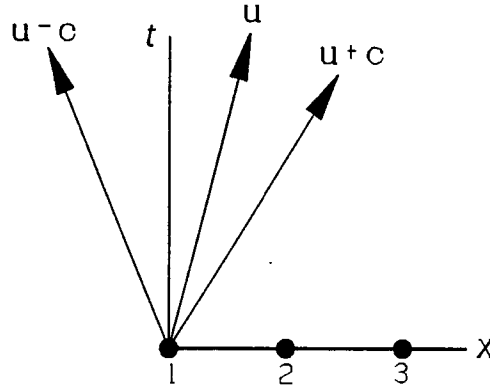
For subsonic boundary conditions where the signs of the characteristics are mixed at the boundary node, special treatment is necessary to complete the information supplied from the interior with information provided externally. This requires that the residual at the boundary node be segregated into right and left moving components. This splitting is by the linearization of equation (2.1.16) resulting in

$$T^{-1} \delta \bar{U} = \delta R \quad (2.3.1)$$

with δR calculated from the known state at the boundary node at time level n , and $\delta \bar{U}$ being the component of the residual contributed from the interior scheme. The rows of T^{-1} represent a system of equations each of which correspond to a specific characteristic direction. By eliminating those rows of T^{-1} which correspond to incoming characteristics and replacing them with additional equations relating to some desired constraint, the

system of equations defined in equation (2.3.1) can be solved for an unknown $\delta\bar{U}$, provided that the resulting set of equations is non-singular. The row elimination is generated by pre-multiplying T^{-1} by S^+ or S^- corresponding to the component of the residual contributed by the interior. Details of this procedure pertaining to subsonic inflow and outflow boundary conditions are described individually.

For subsonic inflow conditions, two characteristics $\lambda_2 = u$, $\lambda_3 = u + c$ are positive-valued (incoming) corresponding to the second and third rows, respectively, of T^{-1} , with one characteristic $\lambda_1 = u - c$ negative-valued (outgoing), corresponding to row one.



To extract the existing portion of T^{-1} representing the propagation of information from the interior to the boundary node, equation (2.3.1) is multiplied by S^- to give

$$S^- T^{-1} \delta\bar{U} = S^- \delta R \quad (2.3.2)$$

explicitly given by

$$\begin{bmatrix} 0 & -\rho c & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{Bmatrix} \delta\rho \\ \delta u \\ \delta p \end{Bmatrix} = \begin{Bmatrix} R_1 \\ 0 \\ 0 \end{Bmatrix} \quad (2.3.3)$$

the diagonal components of $S^- = (1, 0, 0)$. Since two equations are discarded, two additional constraints are provided by requiring that total pressure and total temperature be specified at the inflow plane. Convenient forms of these classic equations can be found from [24] in the form

$$p = p_T \left[1 - \frac{(\gamma-1)q^2}{2c_T^2} \right]^{\frac{\gamma}{\gamma-1}} \quad (2.3.4)$$

$$\rho = \rho_T \left[1 - \frac{(\gamma-1)q^2}{2c_T^2} \right]^{\frac{1}{\gamma-1}} \quad (2.3.5)$$

$$T = T_T \left[1 - \frac{(\gamma-1)q^2}{2c_T^2} \right] \quad (2.3.6)$$

The utility in this particular form is that each static property is only a function of q , assuming total properties are constant. Equations (2.3.4) through (2.3.6) are an integrated form of the pressure-energy or Euler's [25] energy equation, which when approximated to first order is

$$\delta p + \rho q \delta q = 0 \quad (2.3.7)$$

with q equivalent to u in one dimension. The first equation from the system of equations (2.3.3)

$$\delta p - \rho c \delta u = R_1 \quad (2.3.8)$$

can be subtracted from equation (2.3.7) to give an expression for δu in the form

$$\delta u = \frac{-R_1}{\rho(u + c)} \quad (2.3.9)$$

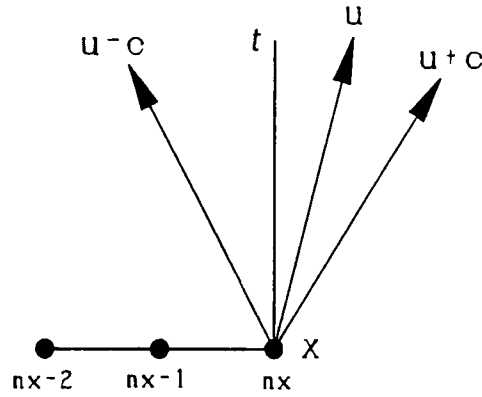
The step is completed by updating u to the new time level,

$$u^{n+1} = u^n + \delta u \quad (2.3.10)$$

calculating p from equation (2.3.4), ρ from equation (2.3.5), and calculating other properties from these and the equation of state.

Numerical considerations aside, this boundary condition, at the node it is applied, simulates an upstream isentropic reservoir capable of maintaining constant total properties. If a steady state solution is required, this boundary condition is appropriate with no further explanation. If temporal accuracy is required, one of two possible conditions must be met for this boundary condition to be appropriate: (1) the boundary node must be far enough upstream such that it effectively represents the reservoir, or alternately, (2) the process taking place upstream must be known in order to correctly calculate δR_1 and δR_2 for equation (2.3.1), since this information is known to propagate to the boundary node from the exterior. If the latter is true, the right hand vector can be correctly calculated and the system of equations solved for $\delta \vec{U}$.

For subsonic outflow boundary conditions, two characteristics $\lambda_2 = u$, $\lambda_3 = u + c$ are positive-valued (outgoing) corresponding to the second and third rows, respectively, of T^{-1} with one characteristic $\lambda_1 = u - c$, negative-valued (incoming) corresponding to row one.



To extract the existing portion of T^{-1} which represents the propagation of information from the interior to the boundary node, equation (3.3.1) is pre-multiplied by S^{-} to give

$$S^{+}T^{-1}\delta\bar{U} = S^{+}\delta R \quad (2.3.11)$$

explicitly given by

$$\begin{bmatrix} 0 & 0 & 0 \\ c^2 & 0 & -1 \\ 0 & \rho c & 1 \end{bmatrix} \begin{Bmatrix} \delta p \\ \delta u \\ \delta p \end{Bmatrix} = \begin{Bmatrix} 0 \\ R_2 \\ R_3 \end{Bmatrix} \quad (2.3.12)$$

the diagonal components of $S^{+} = (0, 1, 1)$. For the equation discarded, one additional equation is provided by requiring that the static pressure p be specified at the exit plane or

$$\delta p = p_{exit} - p^n = \delta p^* \quad (2.3.13)$$

Substituting equation (2.3.13) into equation (2.3.12) gives

$$\begin{bmatrix} 0 & 0 & 1 \\ c^2 & 0 & -1 \\ 0 & \rho c & 1 \end{bmatrix} \begin{Bmatrix} \delta p \\ \delta u \\ \delta p \end{Bmatrix} = \begin{Bmatrix} \delta p^* \\ R_2 \\ R_3 \end{Bmatrix} \quad (2.3.14)$$

Each equation in the system of equations (2.3.14) can be solved explicitly for δp and δu giving

$$\frac{R_2 + \delta p^*}{c^2} = \delta p \quad (2.3.15)$$

$$\frac{R_3 - \delta p^*}{\rho} = \delta u \quad (2.3.16)$$

The static pressure, velocity, and density are then updated followed by the calculation of all other properties from these.

Similar treatments are given to other boundary conditions such as the specification of constant mass flow. Given a constant mass flow rate $\dot{m}$, the product ρu is known at the boundary node and can be linearized to first order as

$$u \delta \rho + \rho \delta u = (\delta \rho u)^* = \frac{\dot{m}}{a} - (\rho u)^n \quad (2.3.17)$$

Equation (2.3.17) completes the system of equations (2.3.14) if used at the outflow boundary. However, if used at the inflow boundary, an additional equation would be necessary to complete the system of equations.

The combination of constraints at a boundary as applied to equation (2.3.14) are determined consistent with characteristic theory given the resulting set of linear equations have the same number of equations as unknowns and is non-singular.

2.4 EXAMPLES

To evaluate the results of the examples, a brief description of the nondimensionalization procedure is given. Static pressure and density are defined as a ratio of the dimensional reference properties p_r and ρ_r given in equations (2.4.1) and (2.4.2).

$$\bar{p} = \frac{p}{p_r} \quad (2.4.1)$$

$$\bar{\rho} = \frac{\rho}{\rho_r} \quad (2.4.2)$$

The remainder of the properties derived from the previous are

$$\bar{c}^2 = \gamma \frac{\bar{p}}{\bar{\rho}} = \frac{c^2}{c_r^2} = \gamma \bar{T}, \quad c_r^2 = \frac{p_r}{\rho_r} \quad (2.4.3)$$

$$\bar{T} = \frac{\bar{p}}{\bar{\rho}} = \frac{T}{T_r}, \quad T_r = \frac{p_r}{R \rho_r} \quad (2.4.4)$$

$$\bar{E} = \frac{1}{2} \bar{\rho} \bar{u}^2 + \frac{\bar{p}}{\gamma - 1} = \frac{E}{E_r}, \quad E_r = p_r \quad (2.4.5)$$

$$\bar{u} = \frac{u}{u_r}, \quad u_r = c_r \quad (2.4.6)$$

In addition, the reference state is chosen to be the upstream stagnation state such that, for example, $\bar{p} = p/p_r = p/p_T$.

The first example is a simple subsonic diverging nozzle with exit to entrance area ratio of 2.0. The exit static pressure is held fixed at 0.95 with upstream total pressure and density held at 1.0. The initial condition is uniform flow with Mach Number equal to 0.2. Plots of Mach Number, static pressure, total temperature, and total pressure are shown in Figures 1-4³ along with the exact solution calculated using simple isentropic relations. The test case indicates the boundary conditions correctly model the specified exit static pressure, specified entrance total pressure and density. The boundary conditions and the interior flow match the exact solution to at least plotting accuracy.

Test case two is a transonic converging/diverging nozzle with exit to throat area ratio of 4.57. The exit static pressure is held fixed at 0.55 with upstream total pressure and density held at 1.0. The initial condition is impulsive flow with Mach Number equal to 0.2. The area variation along the nozzle is given by

$$a(x) = x^2 + \frac{1}{2}x + \frac{1}{2} \quad (2.4.7)$$

with $-1 \leq x \leq 1$, giving an exit to throat area ratio of approximately 4.57:1. These conditions generate a subsonic flow upstream of the sonic throat which expands through the diverging portion of the nozzle after which the flow undergoes a shock and compresses up to a subsonic exit. Plots of Mach Number, static pressure, total temperature and

³ All figures are in the Appendix

pressure are shown in Figures 5-8 and compared to exact theory calculated using isentropic theory and the normal shock jump tables. This test case demonstrates that the scheme is stable and accurate for subsonic and mixed supersonic and subsonic flows with shocks.

3.0 TWO-DIMENSIONAL EULER SOLUTIONS

The concepts explored in Section 2 for the quasi one-dimensional Euler equations is extended into two dimensions. The theory is developed along with boundary conditions for subsonic and supersonic flows consistent with the interior scheme. A numerical algorithm is presented which illustrates the effectiveness of the interior scheme with a series of test cases involving low and high speed flows. The algorithm is developed for triangular meshes which may be structured arbitrarily. The results of the test cases are then compared to known solutions or another well known flow solver.

3.1 THEORY

The development of the working relationships for the two-dimensional Euler Equations parallels closely the one-dimensional development in Section 2.1. The two-dimensional Euler Equations written in conservative vector form is

$$\frac{\partial U}{\partial t} + \frac{\partial F}{\partial x} + \frac{\partial G}{\partial y} = 0 \quad (3.1.1)$$

where U , F , and G are defined as

$$U = \begin{Bmatrix} \rho \\ \rho u \\ \rho v \\ E \end{Bmatrix}, F = \begin{Bmatrix} \rho u \\ \rho u^2 + p \\ \rho u v \\ u(E + p) \end{Bmatrix}, G = \begin{Bmatrix} \rho v \\ \rho u v \\ \rho v^2 + p \\ v(E + p) \end{Bmatrix} \quad (3.1.2)$$

with

$$p = (\gamma - 1) \left(E - \frac{1}{2} \rho (u^2 + v^2) \right) \quad (3.1.3)$$

Equation (3.1.3) written in non-conservative form is

$$\frac{\partial U}{\partial t} + A \frac{\partial U}{\partial x} + B \frac{\partial U}{\partial y} = 0 \quad (3.1.4)$$

$$A = \frac{\partial F}{\partial U} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ \frac{\gamma-3}{2}u^2 + \frac{\gamma^2-1}{2} & (3-\gamma)u & (1-\gamma)v & \gamma-1 \\ -uv & v & u & 0 \\ -\frac{\gamma E u}{\rho} + (\gamma-1)u(u^2+v^2) & \frac{\gamma E}{\rho} - \frac{(\gamma-1)}{2}(3u^2+v^2) & -(\gamma-1)uv & \gamma u \end{bmatrix} \quad (3.1.5)$$

$$B = \frac{\partial G}{\partial U} = \begin{bmatrix} 0 & 0 & 1 & 0 \\ -uv & v & u & 0 \\ \frac{\gamma-3}{2}v^2 + \frac{(\gamma-1)}{2}u^2 & -(\gamma-1)u & (3-\gamma)v & \gamma-1 \\ -\frac{\gamma E u}{\rho} + (\gamma-1)v(u^2+v^2) & -(\gamma-1)uv & \frac{\gamma E}{\rho} - \frac{(\gamma-1)}{2}(3u^2+v^2) & \gamma v \end{bmatrix} \quad (3.1.6)$$

The dependent variable U will be replaced in equation (3.1.4) by a primitive variable formulation given by $\tilde{U} = (\rho, u, v, p)^T$ where

$$\frac{\partial U}{\partial t} = \left(\frac{\partial U}{\partial \tilde{U}} \right) \left(\frac{\partial \tilde{U}}{\partial t} \right) = M \left(\frac{\partial \tilde{U}}{\partial t} \right) \quad (3.1.7)$$

$$\frac{\partial U}{\partial x} = \left(\frac{\partial U}{\partial \tilde{U}} \right) \left(\frac{\partial \tilde{U}}{\partial x} \right) = M \left(\frac{\partial \tilde{U}}{\partial x} \right) \quad (3.1.8)$$

$$\frac{\partial U}{\partial y} = \left(\frac{\partial U}{\partial \tilde{U}} \right) \left(\frac{\partial \tilde{U}}{\partial y} \right) = M \left(\frac{\partial \tilde{U}}{\partial y} \right) \quad (3.1.9)$$

After substituting equations (3.1.7) through (3.1.9) into equation (3.1.4) gives

$$M \frac{\partial \tilde{U}}{\partial t} + AM \frac{\partial \tilde{U}}{\partial x} + BM \frac{\partial \tilde{U}}{\partial y} = 0 \quad (3.1.9)$$

$$M = \frac{\partial U}{\partial \tilde{U}} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ u & \rho & 0 & 0 \\ v & 0 & \rho & 0 \\ \frac{u^2 + v^2}{2} & \rho u & \rho v & \frac{1}{\gamma - 1} \end{bmatrix} \quad (3.1.10)$$

$$M^{-1} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ -\frac{u}{\rho} & \frac{1}{\rho} & 0 & 0 \\ -\frac{v}{\rho} & 0 & \frac{1}{\rho} & 0 \\ \frac{(\gamma - 1)}{2}(u^2 + v^2) & -(\gamma - 1)u & -(\gamma - 1)v & \gamma - 1 \end{bmatrix} \quad (3.1.11)$$

Pre-multiplying equation (3.1.9) by M^{-1} gives the non-conservative form using primitive variables $\tilde{U}$ given by

$$\frac{\partial \tilde{U}}{\partial t} + \tilde{A} \frac{\partial \tilde{U}}{\partial x} + \tilde{B} \frac{\partial \tilde{U}}{\partial y} = 0 \quad (3.1.12)$$

with the primitive Jacobians $\tilde{A}, \tilde{B}$ defined as

$$\tilde{A} = M^{-1} A M = \begin{bmatrix} u & \rho & 0 & 0 \\ 0 & u & 0 & \frac{1}{\rho} \\ 0 & 0 & u & 0 \\ 0 & \rho c^2 & 0 & u \end{bmatrix} \quad (3.1.13)$$

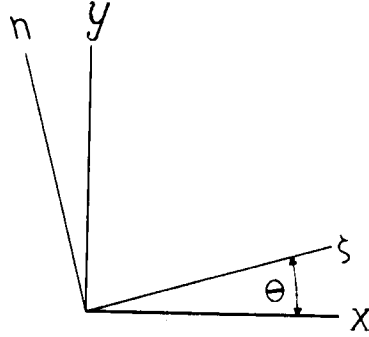
$$\tilde{B} = M^{-1} B M = \begin{bmatrix} v & 0 & \rho & 0 \\ 0 & v & 0 & 0 \\ 0 & 0 & v & \frac{1}{\rho} \\ 0 & 0 & \rho c^2 & v \end{bmatrix} \quad (3.1.14)$$

To control the propagation direction of the wave solutions in equation (3.1.12), an orthogonal coordinate transformation is introduced

$$\zeta = k_1 x + k_2 y \quad (3.1.15)$$

$$\eta = -k_2 x + k_1 y$$

with constants $k_1 = \cos \theta, k_2 = \sin \theta$.



Solving equation (3.1.15) for x and y and differentiating gives

$$\begin{bmatrix} k_1 & -k_2 \\ k_2 & k_1 \end{bmatrix} \begin{Bmatrix} \frac{\partial}{\partial \zeta} \\ \frac{\partial}{\partial \eta} \end{Bmatrix} = \begin{Bmatrix} \frac{\partial}{\partial x} \\ \frac{\partial}{\partial y} \end{Bmatrix} \quad (3.1.16)$$

Substituting equation (3.1.16) in equation (3.1.12) gives

$$\frac{\partial \bar{U}}{\partial t} + \bar{P} \frac{\partial \bar{U}}{\partial \zeta} + \bar{Q} \frac{\partial \bar{U}}{\partial \eta} = 0 \quad (3.1.17)$$

with the transformed primitive Jacobians $\bar{P}, \bar{Q}$ given by

$$\bar{P} = k_1 \bar{A} + k_2 \bar{B} \quad (3.1.18)$$

$$\bar{Q} = k_2 \bar{A} - k_1 \bar{B}$$

Note that $\bar{P}$ and P are related through equations (3.1.13) such that

$$\bar{P} = k_1 M^{-1} A M + k_2 M^{-1} B M \quad (3.1.19)$$

$$M \bar{P} M^{-1} = k_1 A + k_2 B = P \quad (3.1.20)$$

Similarly for $\tilde{Q}$,

$$M\tilde{Q}M^{-1} = k_2A - k_1B = Q \quad (3.1.21)$$

As a consequence of equation (4.1.19), $\tilde{P}$ and P have the same eigenvalues, namely,

$$\lambda_1 = \lambda_2 = k_1u + k_2v, \lambda_3 = \lambda_1 + c, \lambda_4 = \lambda_1 - c \quad (3.1.22)$$

Using the corresponding right eigenvectors of $\tilde{P}$ as column vectors, a matrix T is formed

$$T = \begin{bmatrix} 1 & 0 & \frac{\rho}{2c} & \frac{\rho}{2c} \\ 0 & k_2 & \frac{k_1}{2} & -\frac{k_1}{2} \\ 0 & -k_1 & \frac{k_2}{2} & -\frac{k_2}{2} \\ 0 & 0 & \frac{\rho c}{2} & \frac{\rho c}{2} \end{bmatrix} \quad (3.1.23)$$

which along with its inverse T^{-1}

$$T^{-1} = \begin{bmatrix} 1 & 0 & 0 & -\frac{1}{c^2} \\ 0 & k_2 & -k_1 & 0 \\ 0 & k_1 & k_2 & \frac{1}{\rho c} \\ 0 & -k_1 & -k_2 & \frac{1}{\rho c} \end{bmatrix} \quad (3.1.24)$$

gives the similarity transformation which diagonalizes $\tilde{P}$

$$T^{-1}\tilde{P}T = \Lambda = \begin{bmatrix} \lambda_1 & 0 & 0 & 0 \\ 0 & \lambda_2 & 0 & 0 \\ 0 & 0 & \lambda_3 & 0 \\ 0 & 0 & 0 & \lambda_4 \end{bmatrix} \quad (3.1.25)$$

It is important to note that any one linear combination of the Jacobians $\tilde{P}$, $\tilde{Q}$ can be diagonalized, but not both simultaneously. Pre-multiplying equation (4.1.17) by T^{-1} and recognizing from equation (3.1.25) that $T^{-1}\tilde{P} = \Lambda T^{-1}$ allows equation (4.1.17) to be rewritten as

$$T^{-1}\frac{\partial \tilde{U}}{\partial t} + \Lambda T^{-1}\frac{\partial \tilde{U}}{\partial \zeta} + T^{-1}\tilde{Q}\frac{\partial \tilde{U}}{\partial \eta} = 0 \quad (3.1.26)$$

Equation (3.1.26), unlike its one-dimensional analog in equation (2.1.16), does not decouple into a set of ordinary differential equations acting along characteristic paths. Instead, information propagates into time along a characteristic surface defined by coupled partial derivatives along what is referred to as the characteristic conoid. The inability to decouple is due to the similarity matrices T and T^{-1} to diagonalize both primitive Jacobians simultaneously.

Equation (3.1.26) is recast in terms of the conservation variable U using equations (3.1.7) through (3.1.9) as

$$N^{-1}\frac{\partial U}{\partial t} + \Lambda N^{-1}\frac{\partial U}{\partial \zeta} + T^{-1}\tilde{Q}M^{-1}\frac{\partial U}{\partial \eta} = 0 \quad (3.1.27)$$

Substituting for $\tilde{Q}$ from equation (3.1.21) into the third term of equation (3.1.27) and pre-multiplying by NS gives

$$NSN^{-1}\frac{\partial U}{\partial t} + NS\Lambda N^{-1}\frac{\partial U}{\partial \zeta} + NSN^{-1}Q\frac{\partial U}{\partial \eta} = 0 \quad (3.1.28)$$

with S a diagonal matrix to be defined. Note that the coefficient from the second term can be written as $NS\Lambda N^{-1} = NSN^{-1}N\Lambda N^{-1}$ which when substituted into equation (3.1.27) gives

$$NSN^{-1}\frac{\partial U}{\partial t} + NSN^{-1}N\Lambda N^{-1}\frac{\partial U}{\partial \zeta} + NSN^{-1}Q\frac{\partial U}{\partial \eta} = 0 \quad (3.1.29)$$

Substituting equation (3.1.20) into equation (3.1.25) gives

$$N\Lambda N^{-1} = P \quad (3.1.30)$$

which when substituted into the second term of equation (3.1.29) gives

$$Z\left(\frac{\partial U}{\partial t} + P\frac{\partial U}{\partial \zeta} + Q\frac{\partial U}{\partial \eta}\right) = 0 \quad (3.1.31)$$

Z defined by

$$Z = NSN^{-1} \quad (3.1.32)$$

The original partial differential equation, being acted upon by a matrix operator Z , is reproduced despite the fact that the second Jacobian is not diagonalized.

3.2 NUMERICAL ANALYSIS

The fluid continuum is discretized as a finite number of triangular sub-domains with fluid properties stored at each vertex. The connectivity between the sub-domains is

maintained in a table indexed by cell number referencing the three node numbers in each cell. This table is referred to as a Nodal Connectivity Map (NCM) with an example given in Figure 9.

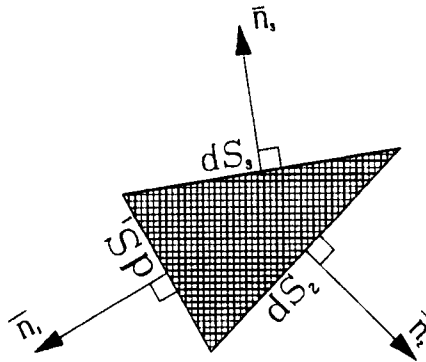
The numerical analysis begins by integrating equation (3.1.1) over each triangular sub-domain

$$\int \frac{\partial U}{\partial t} dV + \int \left(\frac{\partial F}{\partial x} + \frac{\partial G}{\partial y} \right) dV = 0 \quad (3.2.1)$$

Applying the Mean Value Theorem to the first term and Gauss's Theorem to the second term, equation (3.2.1) is rewritten in the form of a temporal residual as

$$\delta U_{cell} = -\frac{\delta t}{V} \int \vec{F} \cdot \vec{n} dS \quad (3.2.2)$$

where $\vec{F} = (F, G)$, $\vec{n}$ is a surface normal on the bounded surface S and V is the area of the triangle.



Equation (3.2.2) applied to each of the three piece-wise segments of the surface with equation (3.2.2) is written as

$$\delta U_{cell} = -\frac{\delta t}{V} \sum_{i=1}^3 \vec{F}(U) \cdot \vec{n} dS \quad (3.2.3)$$

where ΔU_{cell} is defined at the geometric centroid of the cell. To evaluate $\vec{F}(U)$ in equation (3.2.3) along each of the three surfaces, a linear distribution of the conservative variables U is assumed, namely,

$$U_e = \frac{(U_i + U_k)}{2} + O(\Delta x^2, \Delta y^2) \quad (3.2.4)$$

U_i and U_k are nodal values of U along edge e . Having a cell residual δU_{cell} calculated, conservation requires that the total cell residual δU_{cell} be distributed to the three nodes such that

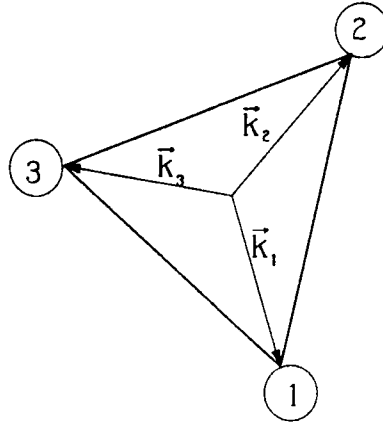
$$\delta U_{cell} = \delta U_1 + \delta U_2 + \delta U_3 \quad (3.2.5)$$

Each node in the global system potentially receives a residual contribution from each cell for which that node is common. The total residual for each node is therefore the cumulative sum of all contributions from all cells that have that node in common. The remaining task is to find the function that will generate the nodal distribution within each cell as described in equation (3.2.5).

To construct a distribution function analogous to the one-dimensional model, the splitting operator Z (equation (3.1.32)), which approximately decomposes the Euler equations into individual waves, is applied to the discrete form of the conservative residual (equation (3.2.2)) and appears in the form

$$Z(\vec{k}_1)\delta U_{cell} + Z(\vec{k}_2)\delta U_{cell} + Z(\vec{k}_3)\delta U_{cell} = \delta U_{cell} \quad (3.2.6)$$

The unit vector $\vec{k}_i$ in equation (3.2.6) substitutes for the general unit vector $\vec{k}$ defining the Z matrix and is defined by the three vectors originating at the geometric centroid of each cell extending to each of the three vertices.



Equation (3.2.6) requires that

$$Z(\vec{k}_1) + Z(\vec{k}_2) + Z(\vec{k}_3) = I \quad (3.2.7)$$

The Z is defined in terms of a similarity transformation matrix, its inverse, and a yet undefined diagonal matrix S . Therefore the only degrees of freedom available to satisfy the system of equations in (3.2.7) is in the definition of the diagonal matrix S . Carrying out the matrix multiplies in equation (3.2.7) and equating to the identity matrix provides a

system of eight linear equations which must be simultaneously satisfied. Defining $S_i^{(j)}$ as the i^{th} diagonal of the j^{th} node, the following equations are given as the general solution to Equations (3.2.7).

$$\sum_{j=1}^3 S_1^{(j)} = 1 \quad (3.2.8)$$

$$\sum_{j=1}^3 k_x^{(j)} (S_3^{(j)} - S_4^{(j)}) = 0 \quad (3.2.9)$$

$$\sum_{j=1}^3 K_y^{(j)} (S_3^{(j)} - S_4^{(j)}) = 0 \quad (3.2.10)$$

$$\sum_{j=1}^3 S_3^{(j)} + S_4^{(j)} = 2 \quad (3.2.11)$$

$$\sum_{j=1}^3 k_x^{(j)} k_y^{(j)} \left(\frac{S_3^{(j)} + S_4^{(j)}}{2} - S_2^{(j)} \right) = 0 \quad (3.2.12)$$

$$\sum_{j=1}^3 S_2^{(j)} k_y^{(j)2} + k_x^{(j)2} \frac{(S_3^{(j)} + S_4^{(j)})}{2} = 1 \quad (3.2.13)$$

$$\sum_{j=1}^3 S_2^{(j)} k_x^{(j)2} + k_y^{(j)2} \frac{(S_3^{(j)} + S_4^{(j)})}{2} = 1 \quad (3.2.14)$$

Combining equations (3.2.12) through (3.2.14) gives

$$\sum_{j=1}^3 S_2^{(j)} = 1 \quad (3.2.15)$$

Equation (3.2.12) gives the constraint

$$\frac{S_3^{(j)} + S_4^{(j)}}{2} - S_2^{(j)} = 0, j = 1, 2, 3 \quad (3.2.16)$$

Combining equations (3.2.9), (3.2.10) and (3.2.16) leaves

$$S_3^{(j)} = S_2^{(j)} + B' \sin \alpha^{(j)}, j = 1, 2, 3 \quad (3.2.17)$$

$$S_4^{(j)} = S_2^{(j)} - B' \sin \alpha^{(j)}, j = 1, 2, 3 \quad (3.2.18)$$

where

$$\sin \alpha_1 = k_x^{(2)} k_y^{(3)} - k_x^{(3)} k_y^{(2)} \quad (3.2.19)$$

$$\sin \alpha_2 = k_x^{(3)} k_y^{(1)} - k_y^{(3)} k_x^{(1)} \quad (3.2.20)$$

$$\sin \alpha_3 = k_x^{(1)} k_y^{(2)} - k_y^{(1)} k_x^{(2)} \quad (3.2.21)$$

with B' an arbitrary constant. Therefore, $\alpha^{(j)}$ is the angle opposite node j (See Figure 10).

Equations (3.2.8), (3.2.15), (3.2.17), and (3.2.18) represent the general solution to Equation (3.2.7). Unfortunately, the general solution only provides constraints that must be observed to satisfy equation (3.2.7) and does not uniquely identify the diagonal components of the S matrix. To identify these components, the one-dimensional equivalent of this scheme (Section 2.1) is examined for analogies that can be used in the

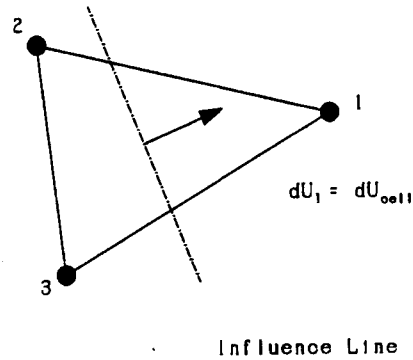
multidimensional case. Of immediate note is that in one dimension there are only two ways information can propagate in the spatial sense. This suggests that the components of the respective S^+ or S^- matrices have only positive-valued entries. The implication is that the residual can only contribute to upwind nodes which follows the domain of influence principle. This requires that the components of S be positive-valued. Note also from the one-dimensional study that the components of the S matrices are not coupled, for example, $S_2^{(+)}$ is not a function of $S_1^{(+)}$ such as the case for the multidimensional splitting operator as presented. A plausible explanation for this is that the one-dimensional equations fully decouple whereas the multidimensional equations only approximately decouple.

The requirement that the components of S be positive-valued presents a dilemma when forced to comply with equation (3.2.7). Following the one dimensional model, the i^{th} diagonal element of matrix S is related by some function of the i^{th} eigenvalue of the matrix $\tilde{P}$ or zero if the corresponding eigenvalue is negative. Examination of equations (3.2.8), (3.2.15), (3.2.17), and (3.2.18) would imply B' would have to be zero. This condition reduces the splitting operator to a scalar operation where the residual distributed to each node is simply a scalar multiple of the residual for the cell. This distribution is feasible for supersonic flow, but does not work for subsonic flow under the requirement that the diagonal entries of S be positive valued. Therefore the distribution function, while being the obvious extension of the one-dimensional function, is not an acceptable general upwind distribution function in this form.

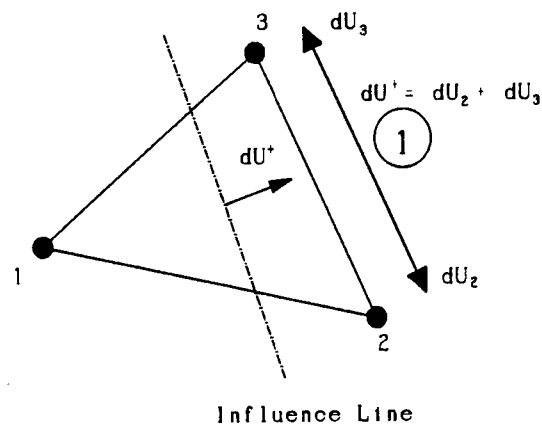
The inability of the fore-mentioned two-dimensional distribution function to provide a physical distribution of the cell residual is attributed to the failure of the transformation matrices to simultaneously diagonalize both Jacobians as is possible for the one-dimensional case. This distribution function is documented in this study as the most natural extension of the one dimensional distribution function. However, it has been found that it does not provide proper upwinding features for general two-dimensional flow problems, but alternate distribution functions are found which apply the same basic principles based on the similarity matrices and how been found to work well in two-dimensional problems.

To define an upwind distribution function, a cell residual is calculated using the finite volume concept as previously defined in Equation (3.2.3). The cell residual is split in the direction of the streamline, resulting in $\delta U^- = 0$ if the flow is supersonic, a δU^+ and a δU^- if the flow is subsonic. For each regime, there are two possible cases to consider based on the orientation of the cell nodes with respect to an imaginary line constructed through the cell centroid normal to the stream-wise direction. This line subdivides the cell continuum into an area which is influenced by right moving waves or left moving waves. The distribution proceeds based on the position of the nodes with respect to this line. The diagonal components of S are determined by first calculating the eigenvalues in the direction of the split and then applying Equation (2.1.25).

For supersonic flow, if the cell is oriented such that one node is downstream of the influence line, the downstream node gets all of the residual for the cell.

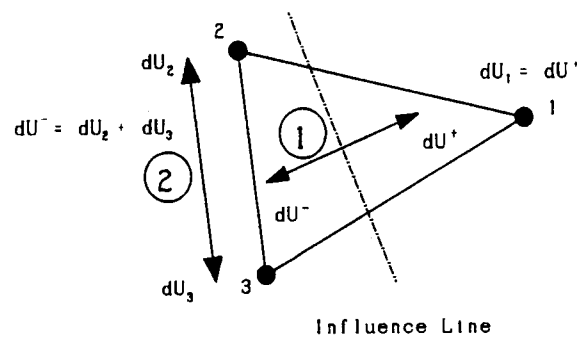


If two nodes are downstream of the influence line, then the cell residual is shared between the two downstream nodes, being split in the direction of the vector between the nodes using the splitting operator in Equation (3.1.32).

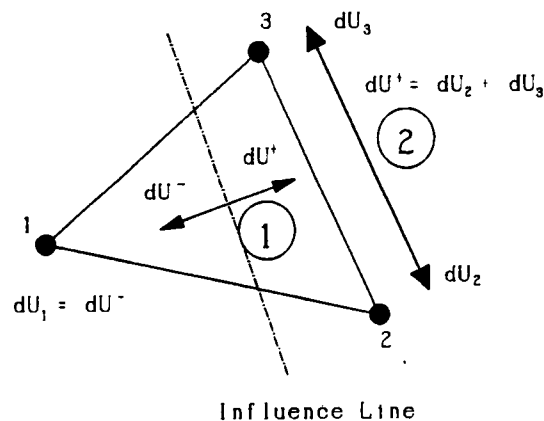


For both cases since δU^- is zero, the downstream nodes receive no contribution of the cell residual. Therefore for supersonic flow, zero or one application of the splitting operator is required.

For the subsonic case, splitting the cell residual in the direction of the influence line will result in a non-zero δU^+ and δU^- . For the case where one node is downstream of the cell centroid, the downstream node receives δU^+ . The δU^- is shared between the two upstream nodes being split in the direction of the vector between the nodes.



For the case where two nodes are downstream of the influence line, the upstream node receives δU^+ with the downstream nodes sharing δU^- being split in the direction of the vector between the two nodes.



Thus subsonic flow requires two applications of the splitting operation.

For flows containing shocks, a small amount of fourth order artificial dissipation is necessary to damp large transients and non-linear convective instabilities. The artificial dissipation model used is derived from Mavriplis [26] with a simplification made to reduce floating point operations. A Laplacian stencil is approximated by

$$dq_i = \sum_{k=1}^n (U_k - U_i) \quad (3.2.22)$$

where the k index represents the neighbors of node i . A biharmonic operator is then formed by repeating the above step

$$\delta U'_i = \varepsilon_2 \delta t \sum_{k=1}^n \frac{dq_k - dq_i}{V} \omega_e \quad (3.2.23)$$

where ε_2 is an empirical constant and V is the average area of the cells either side of the line segment formed by the i and j node pair. The scaling factor ω_e in Equation (3.2.23) is given by

$$\omega_e = |\bar{u}_e \cdot \bar{\eta}| + c_e |\bar{\eta}| \quad (3.2.24)$$

with local velocity and sound speed averaged between the ik node pair. The dissipation residual $\delta U'_i$ is then subtracted from the flux residual at each node after the distribution function is applied. The range of coefficient ε_2 used in the two-dimensional examples ranged from 0.0 for some flows with no shocks, to 3.0/256.0 for flows with strong shocks or other complexities.

3.3 BOUNDARY CONDITIONS

Development of boundary conditions is based on a concept where eigenvalues are restricted to one direction with derivatives in the other direction being represented as source terms. The restriction is due to the inability to fully diagonalize the Euler Equations. In all cases, the ζ direction is chosen to be aligned with an outward normal to the boundary at the boundary nodes. Success of this technique depends on correct information transported to the boundary nodes from the interior and vice versa. As in the one-dimensional case, the emphasis is to develop boundary conditions primarily for steady flows, though much of the analysis is applicable to unsteady flows as well.

For supersonic outflow normal to the boundary, all eigenvalues are outgoing with information propagating to the boundary node from the interior, therefore, no additional information need be applied to the boundary. Similarly for supersonic inflow, all eigenvalues are negative-valued with all information propagating into the interior from the boundary, therefore, all information must be specified at the boundary node. For supersonic cases, the interior scheme coupled with boundary conditions are complete and are consistent with the characteristic theory. Subsonic boundary nodes, however, contain eigenvalues with mixed signs and therefore have pieces of information which must be supplied from the interior as well as from some external constraints.

For subsonic outflow, the eigenvalues $\lambda_1 = \lambda_2 = k_1 u + k_2 v$, $\lambda_3 = \lambda_1 + c$ are greater than zero with $\lambda_4 = \lambda_1 - c$ less than zero. The negative eigenvalue represents information

which enters from the boundary and therefore must be supplied externally, classically, the specification of static pressure, P_{exit} . As in the one-dimensional case, a first order approximate form of the compatibility equations is used in the analysis with

$$T^{-1}\delta\tilde{U} = \delta R \quad (3.3.1)$$

The inappropriate information is removed from the equation by pre-multiplying equation (3.3.1) by S^+ where $S_i^+ = (1, 1, 1, 0)$, effectively removing the last equation from the system. Replacing the lost equation with $\delta p = \delta p^* = p_{exit} - p^n$ results in the non-singular system of linear equations given by

$$\begin{bmatrix} 1 & 0 & 0 & -\frac{1}{c^2} \\ 0 & k_2 & -k_1 & 0 \\ 0 & k_1 & k_2 & \frac{1}{\rho c} \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{Bmatrix} \delta p \\ \delta u \\ \delta v \\ \delta p \end{Bmatrix} = \begin{Bmatrix} \delta R_1 \\ \delta R_2 \\ \delta R_3 \\ \delta R_4 \end{Bmatrix} \quad (3.3.2)$$

Solution of this system gives

$$\delta p = \delta p^* \quad (3.3.3)$$

$$\delta p = \delta R_1 + \frac{\delta p^*}{c^2} \quad (3.3.4)$$

$$\delta u = \delta R_2 k_2 + k_1 \left(\delta R_3 - \frac{\delta p}{\rho c} \right) \quad (3.3.5)$$

$$\delta v = k_2 \left(\delta R_3 - \frac{\delta p}{\rho c} \right) - k_1 \delta R_2 \quad (3.3.6)$$

The primitive variables ρ, u, v, p are updated with total internal energy calculated from

$$E = \frac{1}{2} \rho (u^2 + v^2) + \frac{p}{\gamma - 1} \quad (3.3.7)$$

Subsonic inflow also has mixed eigenvalues with $\lambda_{1,2}$, and λ_3 less than zero with λ_4 greater than zero. Thus, there are three pieces of information incoming and one outgoing requiring that three addition equations be provided. Typically, these constraints are the specification of two total conditions and flow angle. To eliminate the unwanted equations, equation (3.3.1) is multiplied by S^+ , with $S_i^+ = (0, 0, 0, 1)$ leaving only

$$-k_1 \delta u - k_2 \delta v + \frac{\delta p}{\rho c} = \delta R_4 \quad (3.3.8)$$

To solve the system, as for the one-dimensional case the steady form of the energy equation is used in the form

$$\delta p + \rho u \delta u + \rho v \delta v = 0 \quad (3.3.9)$$

along with the specification of the flow angle of the form

$$\vec{t} \cdot \delta \vec{q} = 0 \quad (3.3.10)$$

where $\vec{t}$ is a unit vector tangent to the surface, $\delta \vec{q}$ with components $(\delta u, \delta v)$. This linear system is written in matrix notation as

$$\begin{bmatrix} 1 & \rho u & \rho v \\ 1 & -k_1 \rho c & -k_2 \rho c \\ 0 & t_1 & t_2 \end{bmatrix} \begin{Bmatrix} \delta p \\ \delta u \\ \delta v \end{Bmatrix} = \begin{Bmatrix} 0 \\ \rho c \delta R_4 \\ 0 \end{Bmatrix} \quad (3.3.11)$$

Solving equation (3.3.11) for δu and δv gives

$$\delta u = \frac{-c t_2 \delta R_4}{(u + k_1 c) t_2 - (v + k_2 c) t_1} \quad (3.3.12)$$

$$\delta v = \frac{c t_1 \delta R_4}{(u + k_1 c) t_2 - (v + k_2 c) t_1} \quad (3.3.13)$$

After updating u and v in time, static pressure and temperature are calculated using

$$p = p_t \left[1 - \frac{\gamma - 1}{2} \frac{(u^2 + v^2)}{c_t^2} \right]^{\frac{\gamma}{\gamma - 1}} \quad (3.3.14)$$

$$T = T_t \left[1 - \frac{\gamma - 1}{2} \frac{(u^2 + v^2)}{c_t^2} \right] \quad (3.3.15)$$

For characteristically based slip wall boundary conditions, the process is similar to that for the subsonic outflow. By taking ζ normal to the wall, λ_1, λ_2 are zero by requiring that flow normal to the wall be zero, or equivalently,

$$\vec{\eta} \cdot \vec{u} = 0 \quad (3.3.16)$$

This implies from the compatibility equations that the first two equations propagate information normal to the ζ, t plane. The third equation of the compatibility equation is

kept since its corresponding eigenvalue is always outgoing. The fourth equation is be discarded since its corresponding eigenvalue is incoming representing information propagating from outside the boundary. This process can be automated by multiplying equation (3.3.1) by $S^+, S_i^+ = (1, 1, 1, 0)$. Replacing the discarded equation by the differential form of equation (3.3.16) gives

$$\begin{bmatrix} 1 & 0 & 0 & -\frac{1}{c^2} \\ 0 & k_2 & -k_1 & 0 \\ 0 & k_1 & k_2 & \frac{1}{\rho c} \\ 0 & \eta_x & \eta_y & 0 \end{bmatrix} \begin{Bmatrix} \delta\rho \\ \delta u \\ \delta v \\ \delta p \end{Bmatrix} = \begin{Bmatrix} \delta R_1 \\ \delta R_2 \\ \delta R_3 \\ 0 \end{Bmatrix} \quad (3.3.17)$$

Solving the linear system of equation (3.3.17) gives

$$\delta u = \eta_y \delta R_2 \quad (3.3.18)$$

$$\delta v = -\eta_x \delta R_2 \quad (3.3.19)$$

$$\delta p = \rho c (\delta R_3 - k_1 \delta u - k_2 \delta v) \quad (3.3.20)$$

$$\delta\rho = \delta R_1 + \frac{\delta p}{c^2} \quad (3.3.21)$$

3.4 ORGANIZATION OF THE COMPUTATION

The following examples are designed to demonstrate the theoretical concepts of the

interior scheme and boundary conditions. As with any numerical scheme, however, there are mechanical processes which occur outside the theory of the scheme which makes the computations possible. The section begins with a brief explanation of the tools created to generate the examples for this effort. Afterward, a detailed description of each of the test cases are given, preceded by a statement defining the goals each test case is directed to address. Definition of the geometry and initial conditions for each example are given in some detail. Comparisons are made between the current study, known exact solutions where possible, and numerical computations produced by a well established flow solver.

The use of unstructured grids for fluid dynamics is a relatively new practice in the aerospace industry with few tools available to the researcher. The flow solver is designed internally to store arbitrarily configured unstructured grids, however, for early experimentation of the flow solver for simple geometries, this generality may not be needed. Instead, grids were generated using existing structured grid programs [26] and then processed into a node file containing node coordinates indexed by node number, and a cell file consisting of three node numbers indexed by cell number. This process similar to that of Jameson [13], results in a regular grid composed of triangulated cells with node numbers on the boundaries either contiguously numbered, or containing a constant skip factor, reducing the work of specifying boundary conditions. Boundary conditions are generated and edited by an external program which requests a boundary condition type, followed by the node numbers involved in the boundary condition, which can be individually specified, or specified alternately by a start/stop/increment indexing scheme useful for grids originating from structured grids. The flow solver reads the description of the

boundary condition, allocates memory for the storage of the boundary condition data, and checks the continuity of the nodes to make sure each subsequent node in the boundary condition specification shares a cell face with the previous node. To observe the node numbers needed in specifying boundary conditions and to visualize the results of the flow solver, a CAD style program is written which optionally displays node and cell numbers, internal and external cell faces, and solid and line contours of the most common static and total properties. A future version will contain a preprocessing boundary condition module for visual specification of boundary conditions, replacing the earlier simplistic program. Because of the generality of the unstructured grid technique with the seemingly random distribution of node numbers along boundary faces, the automation of boundary condition specification is necessary for large scale practical use of the method. The use of graphics workstations or personal computers is indicated for this function.

There are many techniques available for generating true unstructured grids in the literature. The technique used in the present study for airfoils is to generate an "O" mesh about each airfoil using regular quadrilaterals. Any vertices which fall into the volume of an airfoil are removed and the remaining vertices connected using Delaunay triangulations[26]. A Laplacian operator is applied over the mesh to further remove any skewness in the grid. This process allows a large local gradient in grid density to accommodate large changes in grid density over a small distance. This effect is easily seen in the airfoil test cases presented. Points are heavily clustered near the foil surfaces, in particular, at the leading and trailing edges accounting for several hundred points around the surfaces.

The farfield boundary, however, only contains 128 points around the entire circumference which is over a hundred chord lengths away. This stretching requires a large variation in cell area as the surface is approached from the farfield.

The structure of the flow solver, written in the portable ANSI C computer language, is quite simple by design, but allows a great deal of flexibility in the application of the interior scheme and boundary conditions. From the command line, the program optionally accepts the names of the node file, cell file, boundary condition file, and restart file, each of which have default names if not specified. The program first reads through the node file to ascertain the number of nodes, dynamically allocates the memory needed to store the nodes, then loads the node data into memory. The program then performs the similar function for the cell data after which a table is built (called a Cell Connectivity Map) defining neighboring cells which share common faces with the cells, or alternately, read off of disk from a previous run since this task can be time consuming the first time performed. If an initial condition or restart file exists, the data is read from that file for each node, the memory having been allocated at the time the nodes were defined. If the initial condition file does not exist, a routine is called which creates user defined impulsive start values at each node. With data in memory, the program starts its computation cycle for a maximum number of time steps, during which the program generates data on convergence, mass flow rates (input as a pseudo boundary condition), and C_p . On termination, the program generates a new restart file for plotting of data and as initial conditions for the next start of the program.

3.4.1 SUBSONIC DIVERGING CHANNEL

The first example is a simple source flow through a diverging channel with exit to entrance area ratio of 2:1. This example is chosen because it has a simple exact solution and exhibits large defects in entrance and exit static conditions if the computations has even small losses in total conditions. The grid for this test case (Figure 11) is formed by taking a 31 by 47 structured grid (Figure 12), removing every other grid point in the theta direction, and constructing an regular triangular mesh consisting of 775 nodes. The sides of the channel extend to meet at the origin with entrance radius of 8.0 and exit radius of 16.0 representing a 0.3 radian slice. Initial conditions are impulsive flow at Mach Number of 0.2 with flow angle at each node defined by the position vector from each node to the origin. The boundary conditions are specified upstream total pressure, total temperature, and flow angle, slip wall boundary conditions on the sides of the channel, with specified static pressure specified at the exit. Results of this test case are shown in Figures 13 and 14, with contour plots of Mach number and static pressure. Comparison with the exact solution, and the industry standard PARC code [5] are shown in Figures 15, 16, and 17 with plots of static pressure, Mach Number, and total pressure, respectively, along the bottom wall. The PARC code is run to convergence with second order dissipation set to zero and fourth order dissipation set to default values. No artificial dissipation is necessary for the present method with results reproducing the exact solution to plottable accuracy. Note that the present computations exhibits no losses in total quantities and thus produces an accurate solution. The PARC code indicates a significant

error in static properties as well as an oscillatory nature in total temperature and pressure at the boundaries. This demonstrates not only the viability of the scheme, but also the accuracy for this class of problems.

3.4.2 SUPERSONIC FLOW PAST A CYLINDER

The second example is Mach 5.0 freestream flow over a circular cylinder. This example is chosen because the shock detachment distance is sensitive to losses in total pressure across the shock. The strength of the shock is great enough that many CFD codes require large amounts of second order dissipation near the shock to maintain stability. Since the present work requires no second order dissipation for stability and only small amounts of fourth order dissipation, this test case demonstrates the increased accuracy for mixed regime flows. Since no exact solution is known for this problem, the present work is compared with the PARC code with the detachment distance compared to experimental data in reference [28]. The grid size for the quadrilateral grid (Figure 18) used with the PARC code is 91x91 resulting in 8281 nodes. To generate a triangular grid of approximately the same number of nodes, a structured quadrilateral grid is generated of dimension 161x101 resulting in 8231 nodes for the triangular grid (Figure 19). Figures 20 and 21 are contour plots of static pressure for the PARC code and present work, respectively. The plots indicate that while the PARC code generates a sharper shock, there is odd-even decoupling in the upstream supersonic region. The present computation requires fourth order dissipation ($\epsilon_4=8/256$) to maintain stability because of the strong shock in front of the cylinder followed by stagnation and rapid expansion.

However, the shock is non-oscillatory and the computed solution is smoother than that obtained by the PARC code. Figures 22 and 23 are contour plots of the sonic line for the PARC code and present work, respectively. A comparison plot of Mach Number along the center-line is given in Figure 24 between the PARC code and present work. These results show the shock smearing of the present effort and the oscillations of the PARC code approaching the shock, substantiating the results of the contour plots. Note however, that both codes correctly positioned the Mach 1.0 location as given in reference [28].

3.4.3 TRANSONIC FLOW OVER A NACA-0012 AIRFOIL

The third example is a NACA-0012 airfoil in a uniform flow of Mach = 0.8 at 1.25 degrees angle of attack. The grid is a general unstructured system generated with the use of Delaunay triangulation [26] and containing 2048 cells and 1088 nodes (Figure 25). Example 4 is also a NACA-0012 airfoil with the same run conditions as the previous example, but with a finer grid of 4088 nodes and 2010 cells (Figure 26). These plots indicate a higher concentration of nodes on the fine grid (228 points) as compared to the coarse grid (84 points). Figures 27 and 28 are contours of constant static pressure for the coarse and fine grid solutions, respectively. The shock on the top and bottom surface is sharper for the fine grid solution. Figure 29a is a comparison plot of Pressure Coefficient (C_p) for the coarse grid solution and Reference 26. Figure 29b is a comparison plot of Pressure Coefficient for the fine grid solution and Reference 26. Note that the coarse solution nominally predicts the correct levels of C_p , but does not capture the shock loca-

tion as well as the fine grid solution due to the lack of surface grid resolution. However, the coarse grid solution of the present scheme predicted nearly as good as Mavriplis [26] who uses the fine mesh in his test case. In the transonic case, the present scheme requires minimal fourth order dissipation ($\epsilon_4=.8/256$) and no second order artificial dissipation. This test case demonstrates the better solution quality obtainable by the present scheme for transonic flows with unstructured grids.

3.4.4 FLOW OVER AN AIRFOIL WITH FLAP

The fifth example is a Karman-Trefftz dual airfoil with the main airfoil at 0° incidence and the flap at 10° incidence. The geometry of the airfoils originates from a conformal mapping and has an incompressible exact potential flow solution [27]. To compare these results with the present scheme, the freestream Mach Number is set to 0.125 which should result in a compressibility error of less than one percent, according to the Prandtl-Glauert rule. The unstructured grid used for this example is shown in Figure 30. This grid is constructed by taking two separate "O" grids for the airfoil and the flap, merging the two sets of points, and applying Delaunay triangulation. This produces a truly unstructured grid not only at the interface between the two grids but also near the trailing edge of the main airfoil and the leading edge of the flap. The grid has a large variation in the size of the triangles. The ratio of the smallest to largest areas of triangles is about 1 to 10^9 . Thus this example provides an excellent test case for the present unstructured grid flow solver.

Contours of constant static pressure and Mach Number are shown in Figures 31 and 32, respectively. These contours indicate a smooth solution with the leading edge stagnation point of the main foil well below the centerline, and a large suction along the upper leading edge. A close-up vector plot of the trailing edge of the main airfoil and leading edge of the flap are shown in Figure 33 and indicate the tangency condition at the surfaces as well as the acceleration of the flow between the two airfoils. A comparison plot of C_p with the exact solution is shown in Figure 34 and indicates good overall agreement. This test case demonstrates the accuracy of the solution obtained by the current scheme for a complex grid.

3.4.5 FLOW OVER A FOUR FLAP AIRFOIL

This test case is flow over an airfoil with a leading edge flap and two trailing edge flaps in a landing configuration at a Mach Number of 0.125 with the incidence angle at zero degrees. The geometry for the following test case (Figure 35) does not represent a real aircraft. Its original authors' intent was to experiment with different grid generation concepts. Its inclusion in this study is to demonstrate the flexibility of the flow code in solving problems with complex grids and boundary conditions. This example represents a more severe test than the previous for an unstructured grid algorithm with an area ratio of about 1 to 10^{10} between the smallest and largest triangles. This grid resolution is not practical with a structured grid system and illustrates the capability of unstructured grids to have locally large grid point distributions.

Figure 36 is a contour plot of constant Mach Number for the four flap airfoil. These contours indicate a smooth solution with large gradients on the upper and lower leading edge of the main airfoil. Various close-up velocity vector plots are shown in Figures 37-39 with the vectors colored according to Mach Number. These plots illustrate how the combination of the leading and trailing edge flaps move the stagnation point down the leading edge of the main foil effectively increasing the angle of attack and thus the lift.

4.0 CONCLUSIONS AND FUTURE WORK

The method of Characteristically Distributed Residual (CDR) as applied to the quasi one-dimensional Euler equations results in a numerical scheme that reproduces near exact steady state solutions. The high accuracy of the scheme is attributed to the exact decoupling of the Euler equations, predicating a matrix operator which splits a conservatively calculated temporal residual into upwind and downwind spatial components. Appropriate application of the classical compatibility equations coupled with residuals contributed from the interior scheme is shown to provide accurate, consistent, and well-posed boundary conditions for steady state in the supersonic and subsonic flow regimes. Examples are provided that exercise examples from the subsonic and supersonic regimes with excellent comparison to theory.

Simple extension of the quasi one-dimensional algorithm to two dimensions is not trivial. The inability of the transformation Jacobians to simultaneously diagonalize results in inability of the analogous distribution function to provide a physically meaningful upwind scheme. As a result, an alternate distribution function is presented which is intuitively upwind based on the Domain of Influence Principle. While exhibiting many of the same properties as the one-dimensional scheme, the two-dimensional algorithm requires that fourth order artificial dissipation be added for stability. Weak shocks are captured well, but strong shocks tend to smear over several cells. This is credited to the inaccurate treatment of the waves not propagated in the direction of the streamline which

control the position of the shock. However, static properties at the shock do not exhibit oscillations and smears the shock over the equivalent number of cells as the structured flow solver.

The use of unstructured grids is shown to be an excellent method to use in the solution of the Euler equations. The modeling of complex geometries as well as the ease in specification of boundary conditions are salient features of this philosophy. However, for the method to compete with implicit flow solvers for computational efficiency, some form of acceleration is needed such that the time step is not restricted to the smallest cell in the field. There are many methods available for this. The simplest is to use local time stepping where each cell uses its own maximum allowable time step. However, for this technique to work well, residual smoothing is required due to the artificially large variation of residual possible in the field due to large area variation. This process distributes locally large variations in temporal residual to its neighboring nodes, thus aiding stability and over-coming the time step constraint. Enthalpy dampening often accompanies this technique which makes the steady state assumption that total enthalpy is locally constant. This technique tends to damp the temporal residual, and like residual smoothing, introduces no error in the steady state. Another method to accelerate time convergence is the use of implicit solution algorithms. With this technique, the residuals are simultaneously solved for in the flow field, thus requiring the solution of a large linear system of equations. An improvement to this is the concept of a "locally implicit" scheme [29,30] which implicitly solves for the solution using only immediate neighbors for the implicit solution. This technique has the appeal that only a small number of equations are

required to be solved while retaining unconditional stability. However, there is no obvious technique for extending the current solution algorithm such that it has an implicit form. The discontinuous nature of the algorithm would suggest that a multi-gridding technique would be appropriate, and indeed, has already been explored in the literature [26].

As was true in going from the one-dimensional case to the two-dimensional case, extending to three dimensions from two-dimensions is not trivial. Consider an influence plane passing through the centroid of a tetrahedral cell and normal to the flow direction. As in the two-dimensional case, nodes downstream of this influence plane would receive the contribution of δU^+ and the nodes upstream of this plane would receive δU^- , if it existed. If two nodes are upstream with two nodes downstream, then each corresponding upwind and downwind residual would be split between each respective node pair. Similarly, if one node is downstream of the influence plane and the flow is supersonic, then that node receives the entire residual. However, if three nodes are to one side of the influence plane, there is no obvious way to treat the distribution between the three nodes. There are many approaches which might solve this problem, such as weighting the nodes based on orthogonal distance to the influence plane, or as a series of one or more two-dimensional distributions, and many others. The author hopes that the work presented in this study motivates others to continue research on this topic.

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APPENDIX

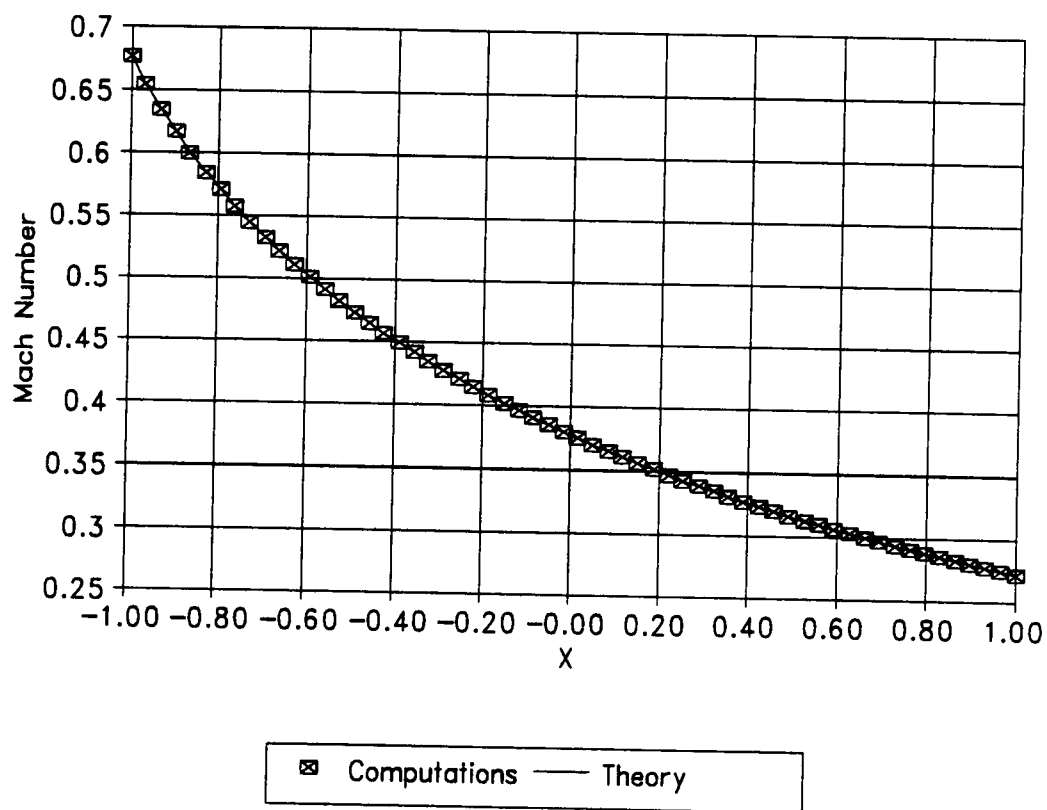


Figure 1. Mach Number comparison for divergent subsonic channel

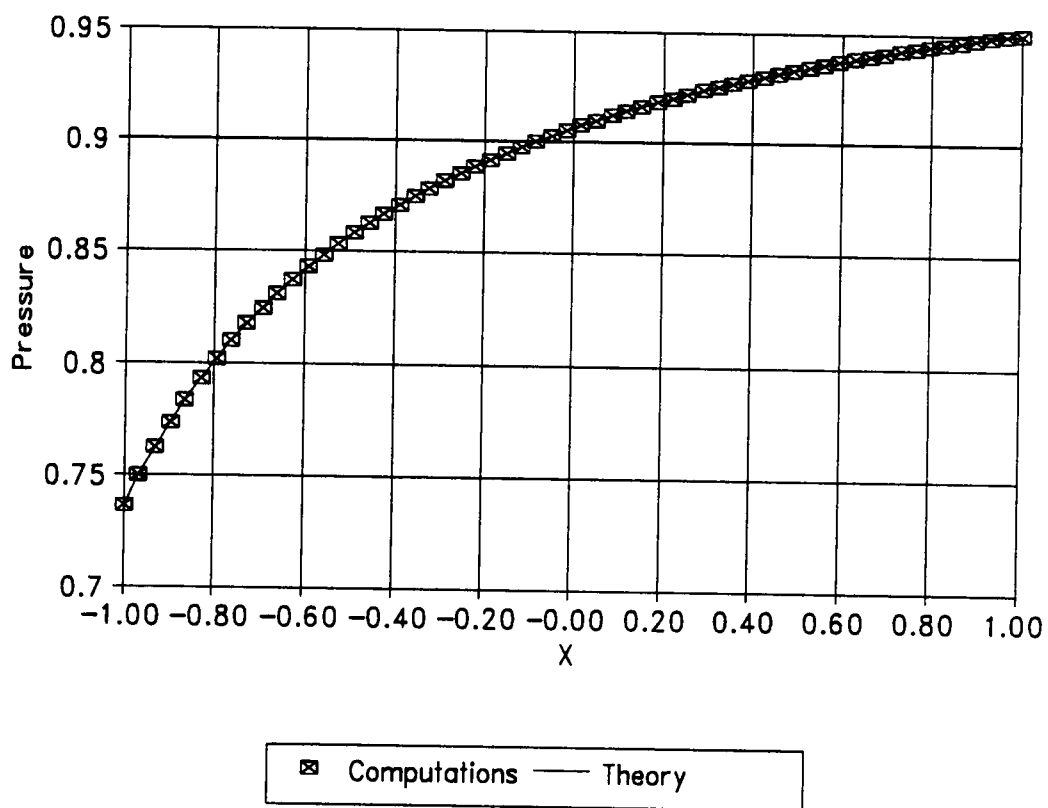


Figure 2. Static pressure comparison for divergent subsonic channel

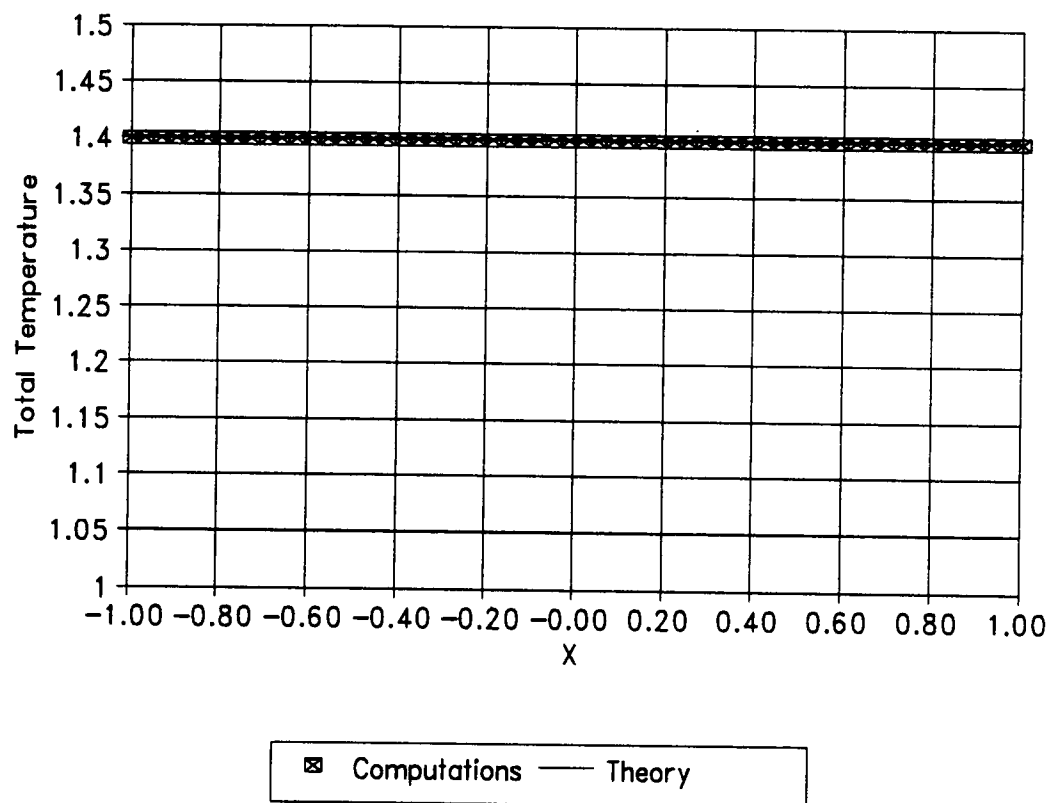


Figure 3. Total temperature comparison for subsonic divergent channel

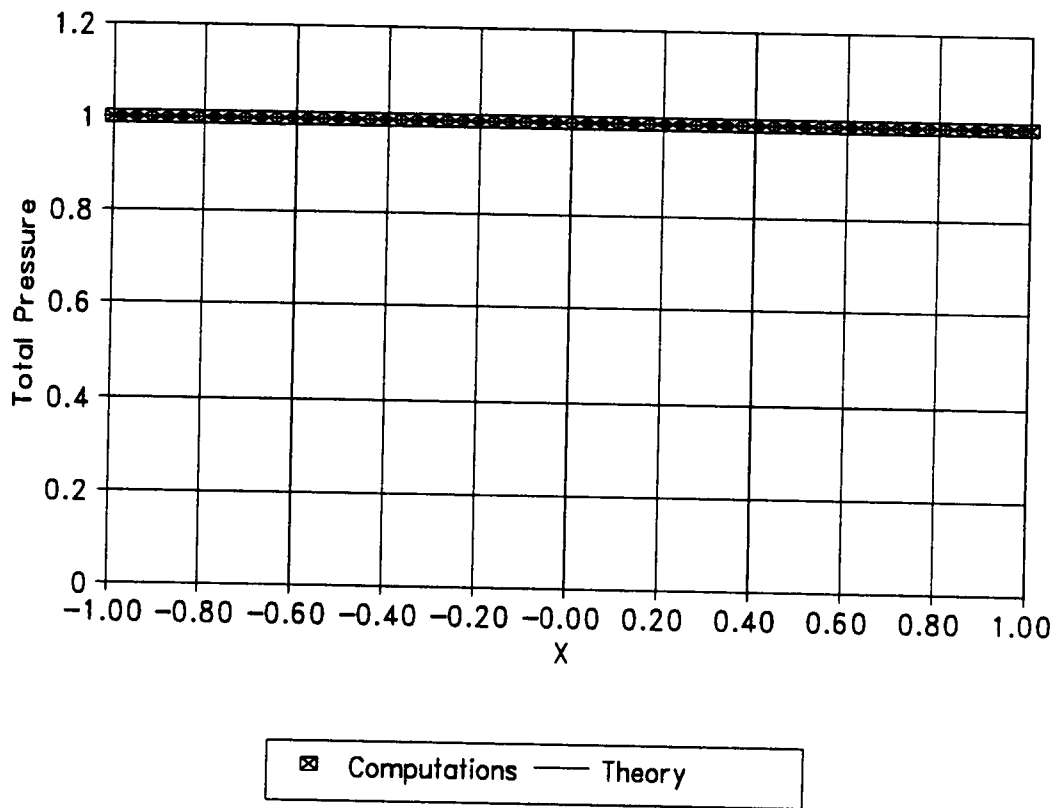


Figure 4. Total pressure comparison for subsonic divergent channel

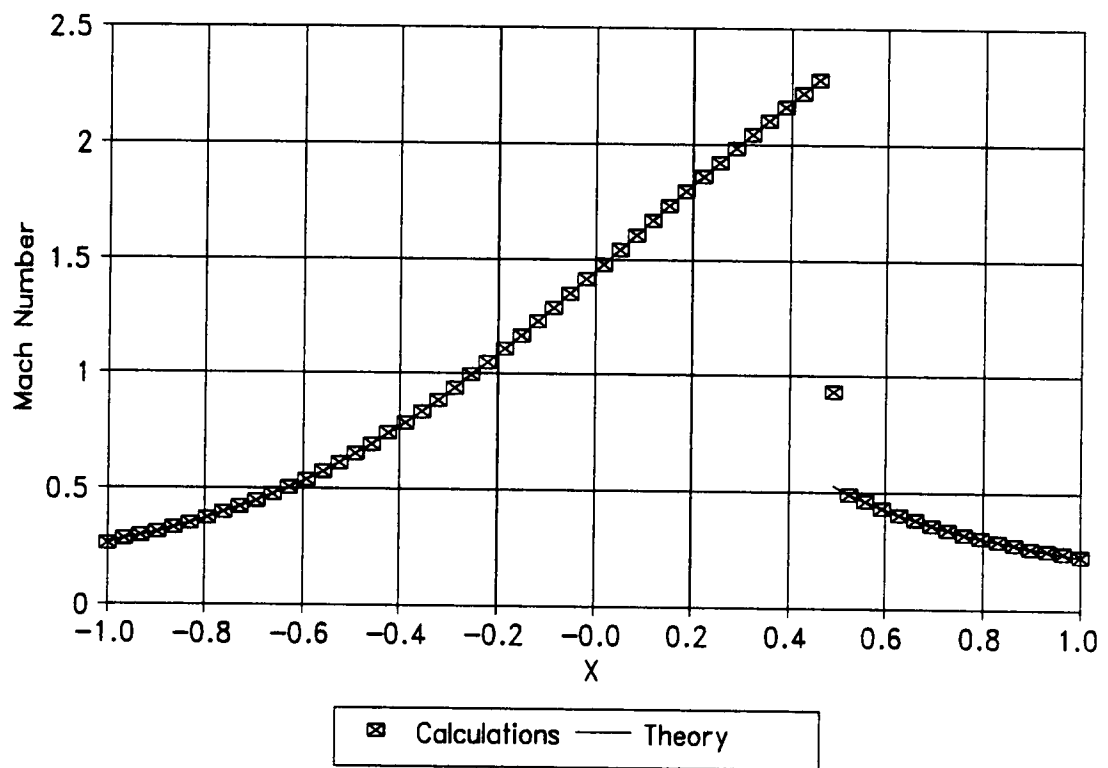


Figure 5. Mach Number comparison for converging-diverging nozzle

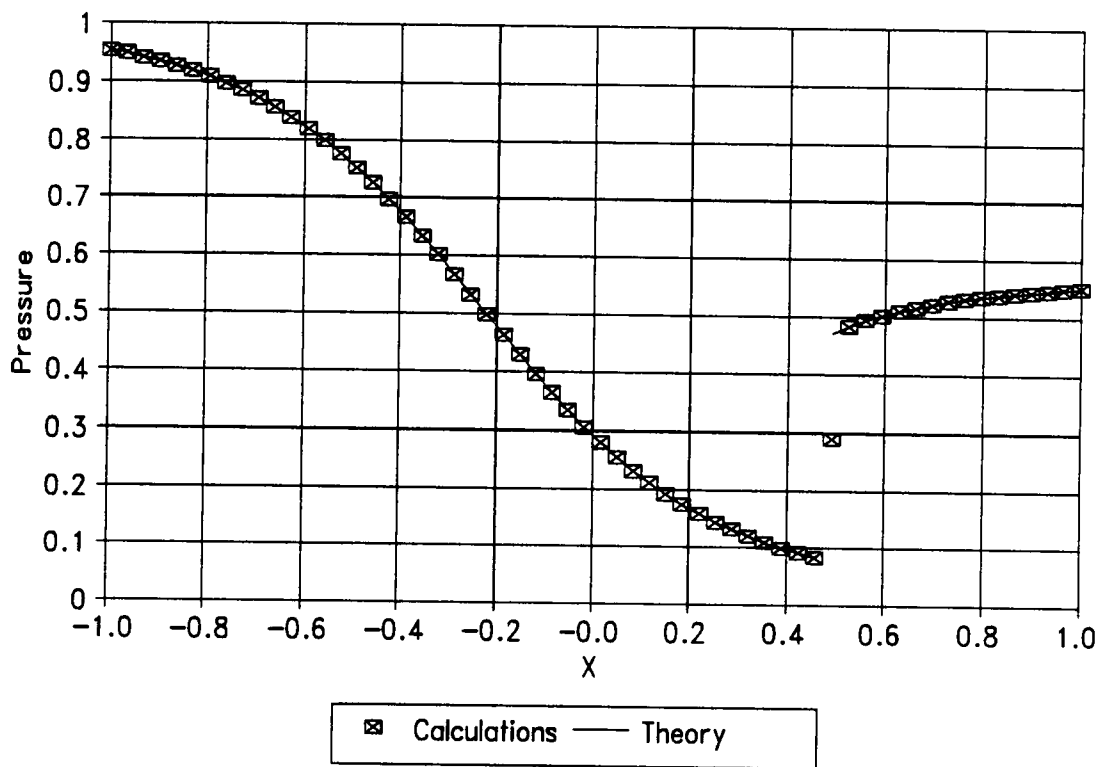


Figure 6. Static pressure comparison for converging-diverging nozzle

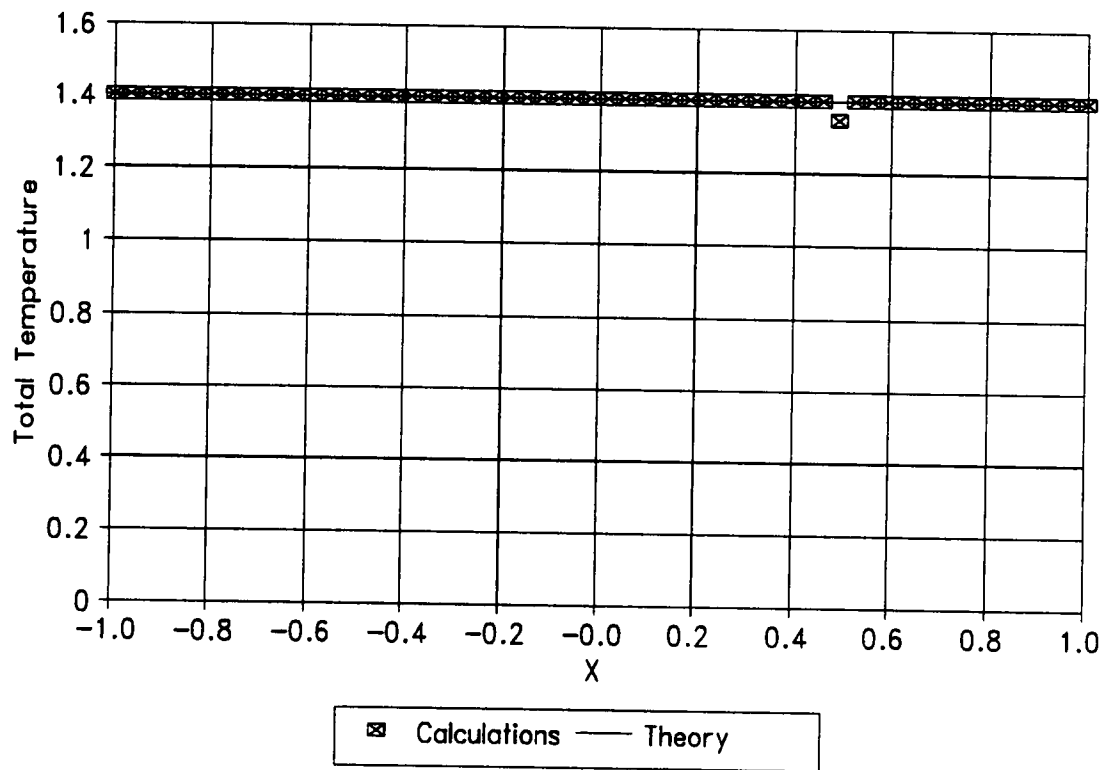


Figure 7. Total temperature comparison for converging-diverging nozzle

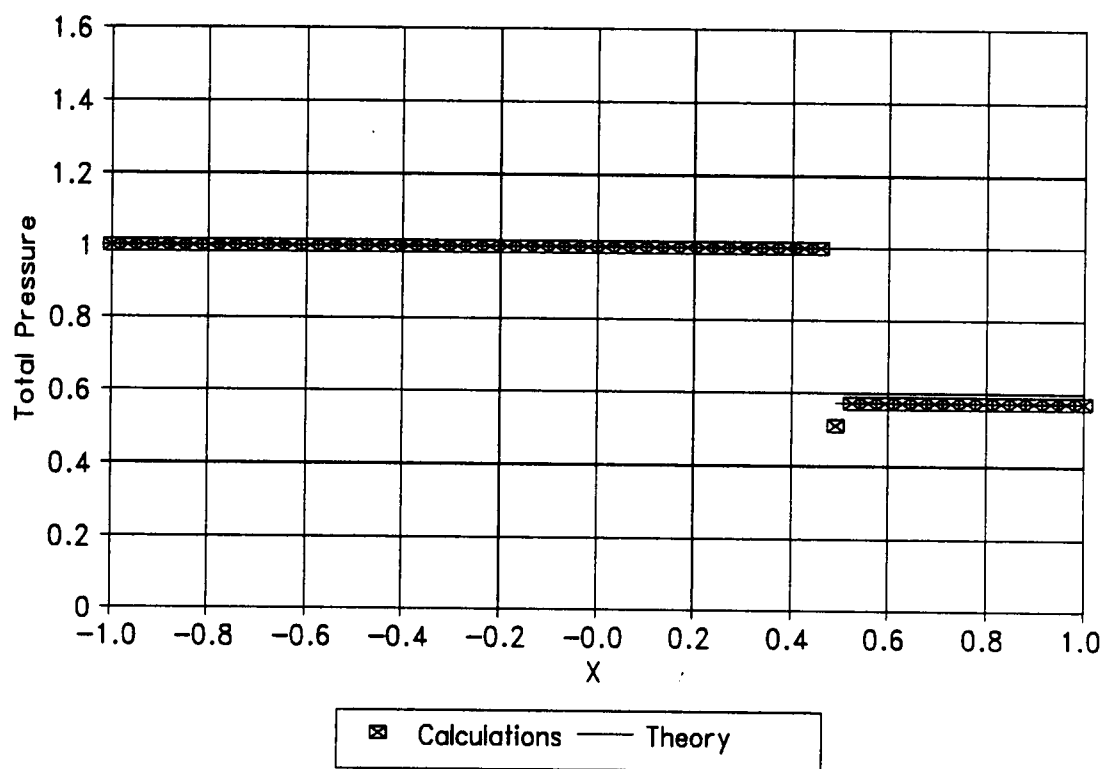
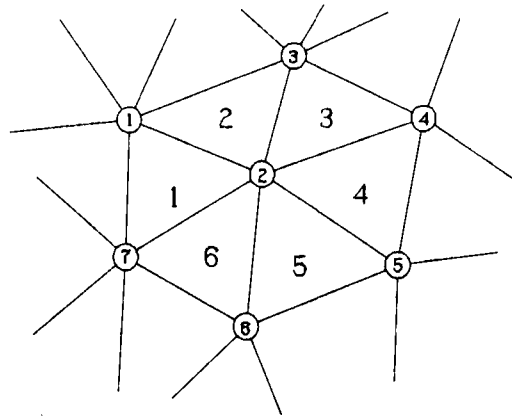


Figure 8. Total pressure comparison for converging-diverging nozzle



cell node
s

1	7	2	1
2	2	3	1
3	2	4	3
4	2	5	4
5	6	2	7

Figure 9. Example of a Nodal Connectivity Map (NCM)

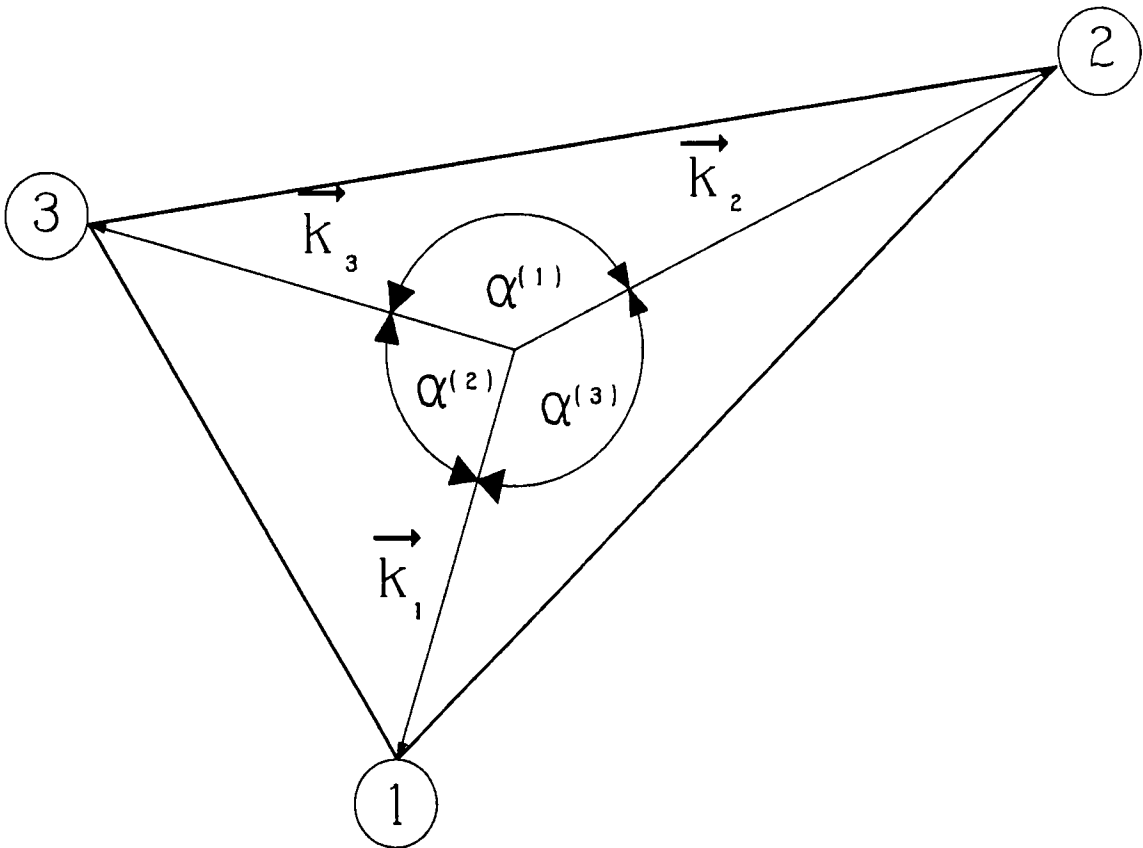


Figure 10. Orientation of $\alpha^{(j)}$ with corresponding vertex

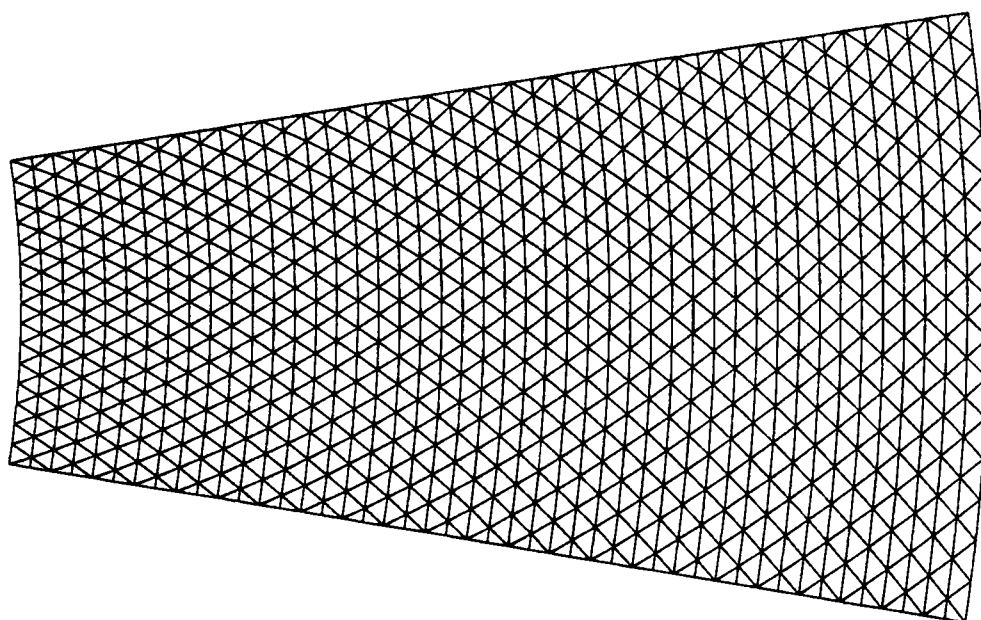


Figure 11. Unstructured grid used for divergent channel

GEOMETRY

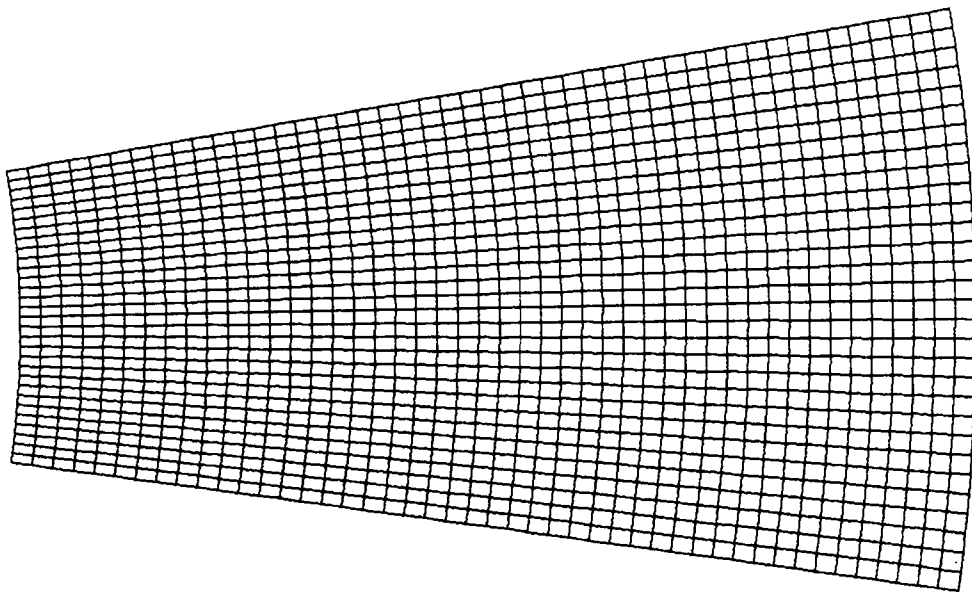
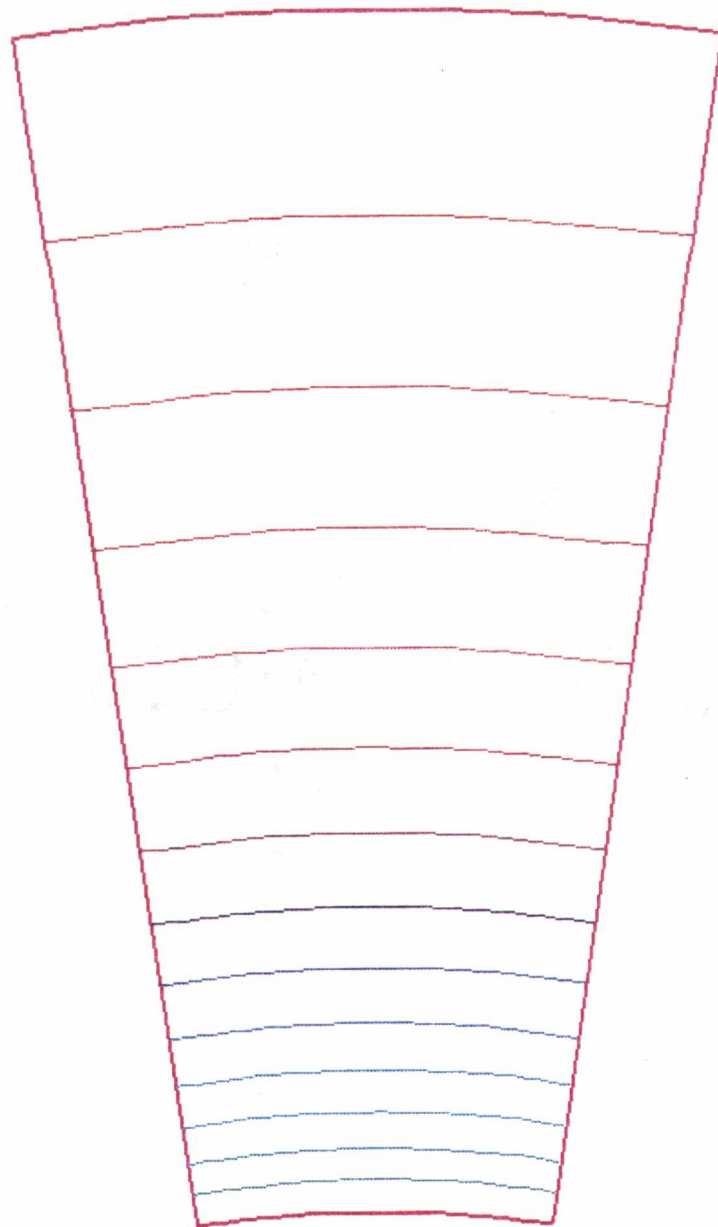


Figure 12. Structured grid for PARC code for divergent channel



Mach	Number
1	0.270823
2	0.299445
3	0.328067
4	0.356688
5	0.385310
6	0.413931
7	0.442553
8	0.471175
9	0.499796
10	0.528418
11	0.557040
12	0.585661
13	0.614283
14	0.642904
15	0.671526

Figure 13. Contours of constant Mach Number for divergent channel

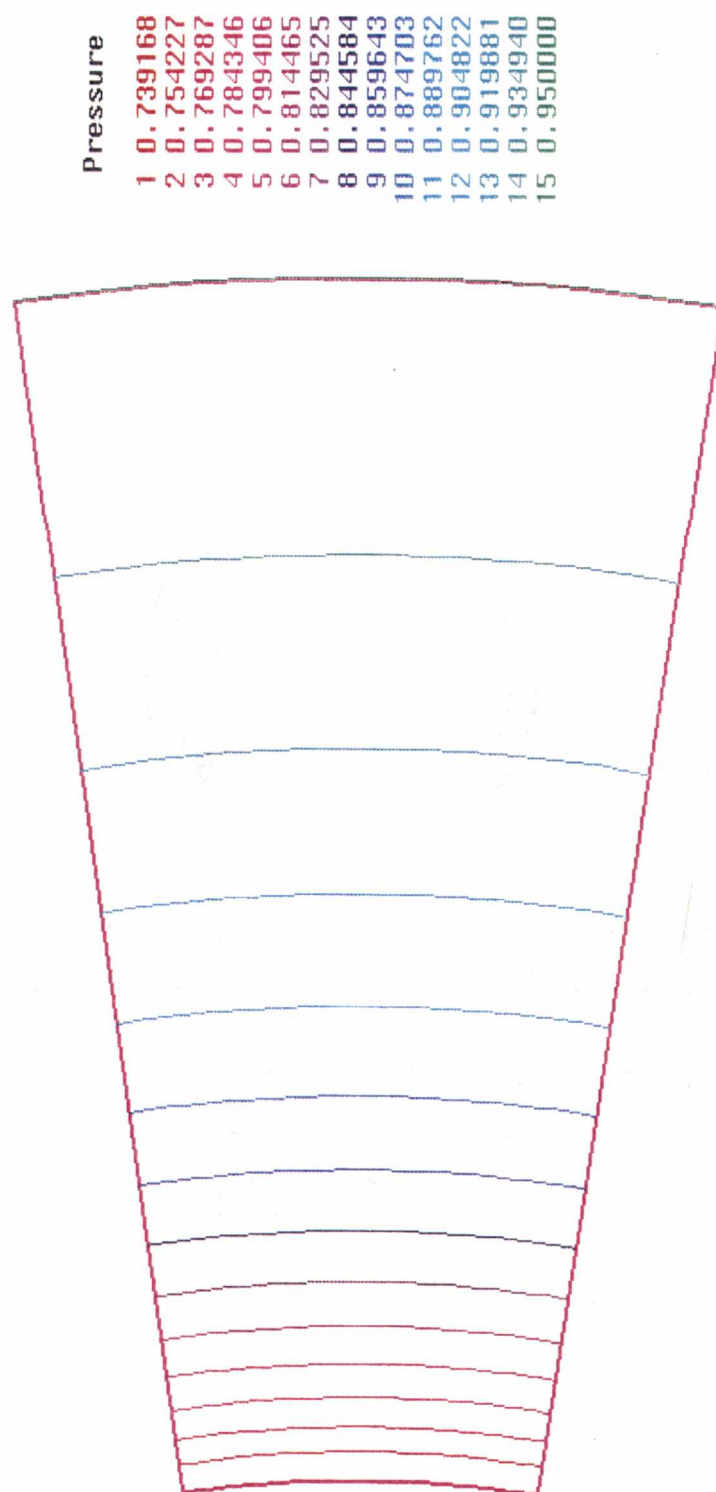


Figure 14. Contours of constant static pressure for divergent channel

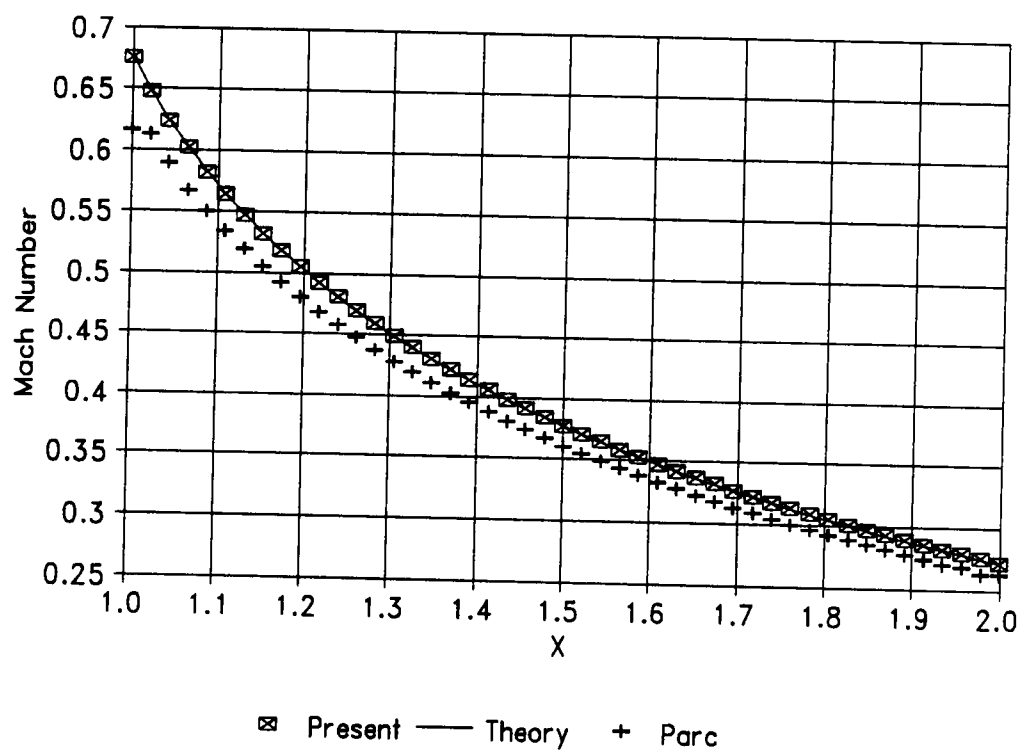


Figure 15. Mach Number comparison for divergent channel

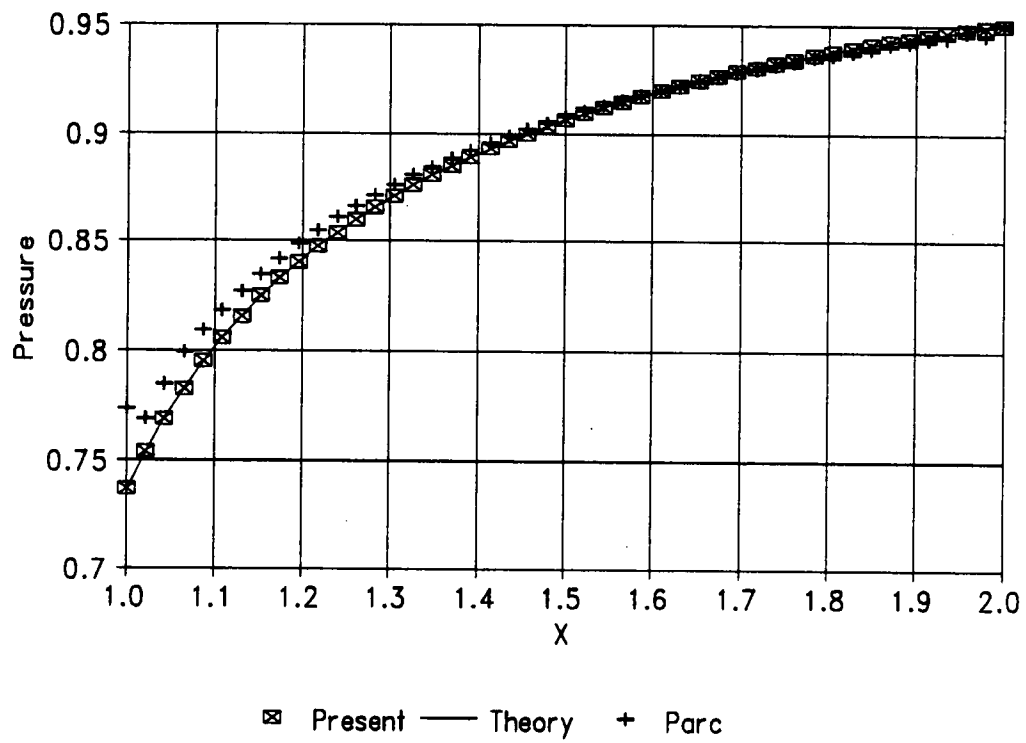


Figure 16. Static pressure comparison for divergent channel

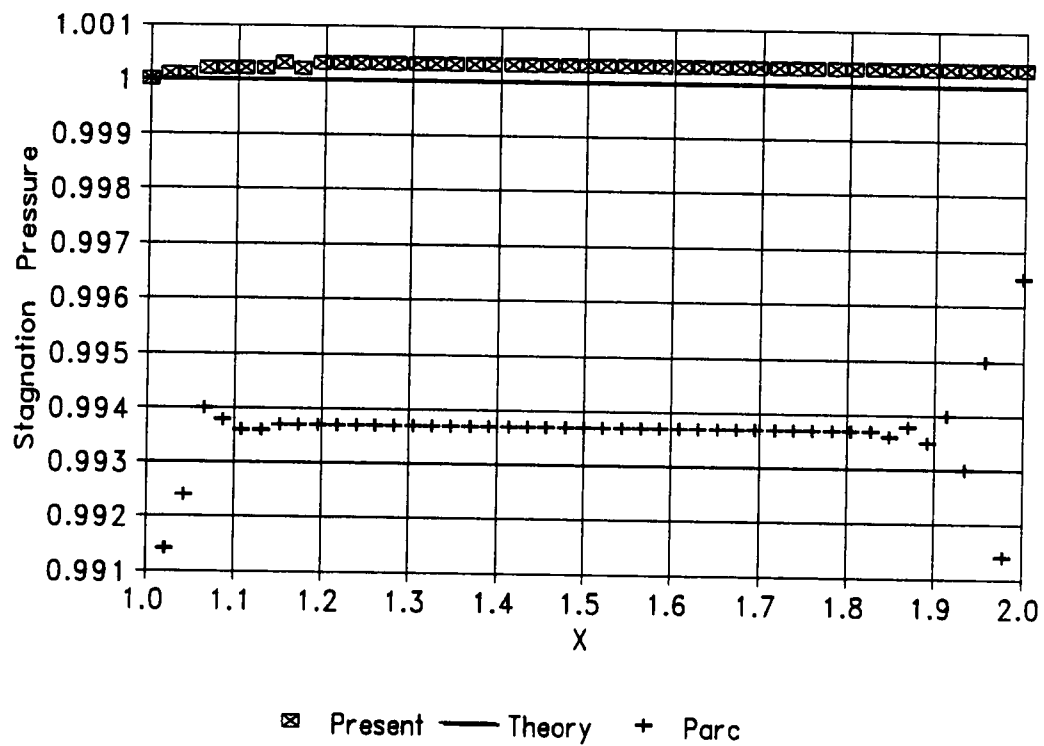


Figure 17. Total pressure comparison for divergent channel

GEOMETRY

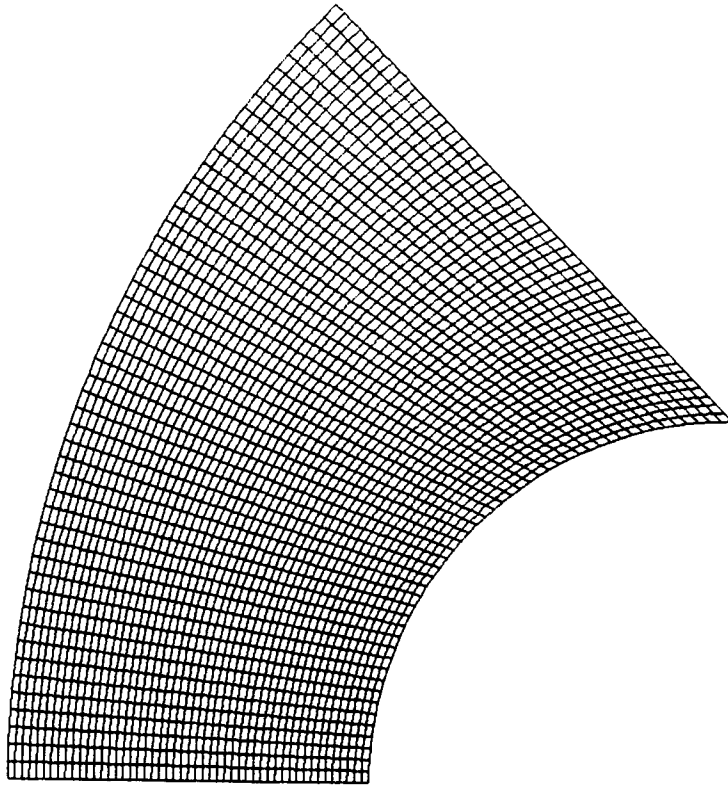


Figure 18. Structured grid for PARC for flow around cylinder

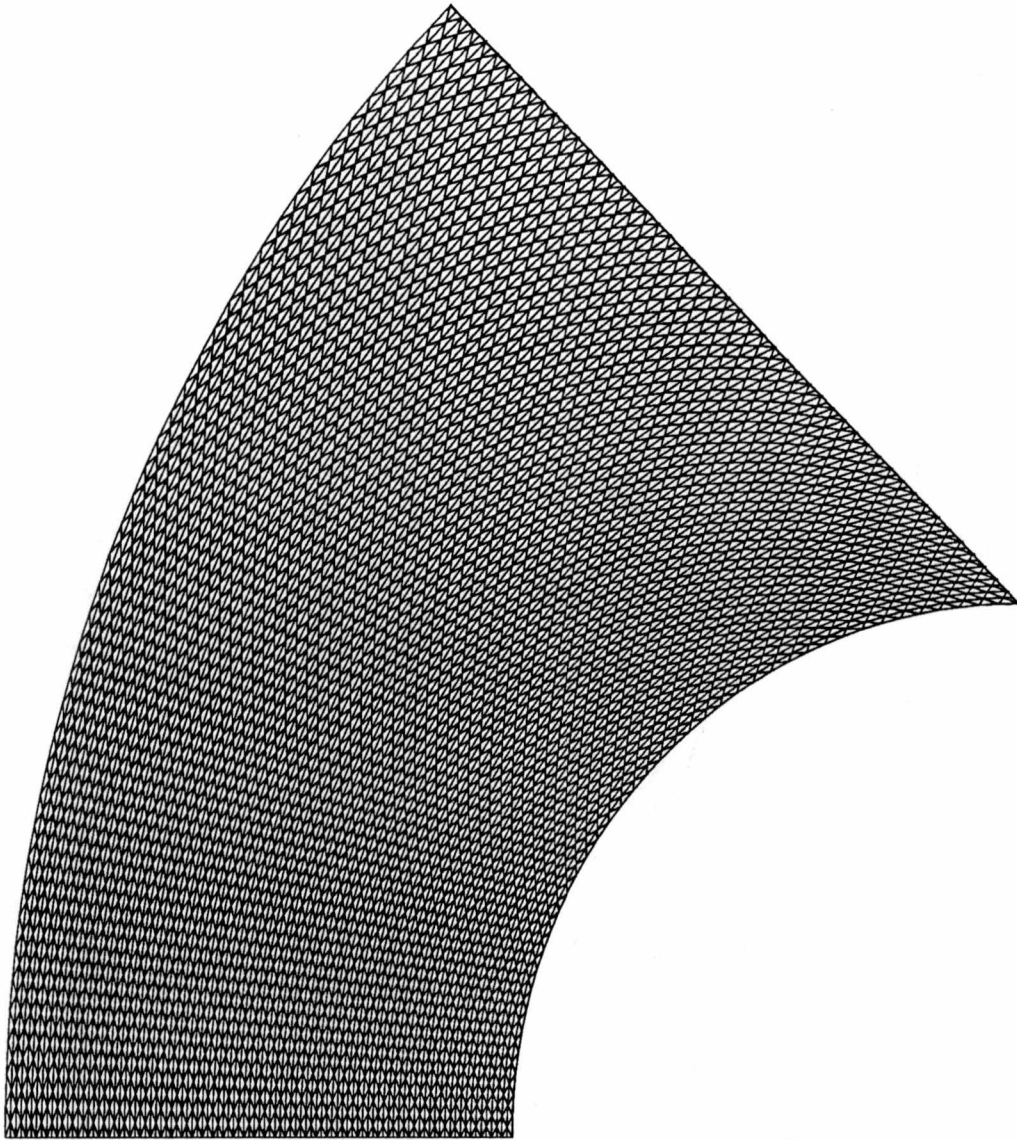


Figure 19. Unstructured grid for flow around cylinder

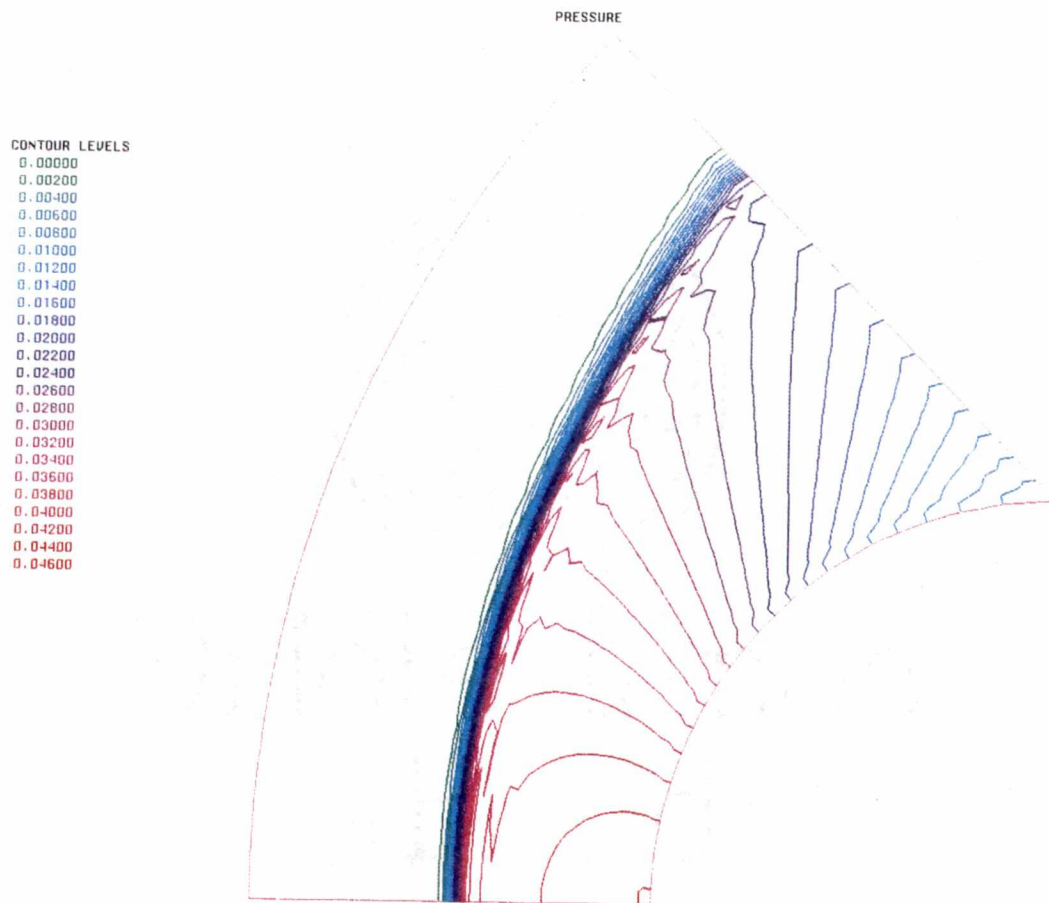
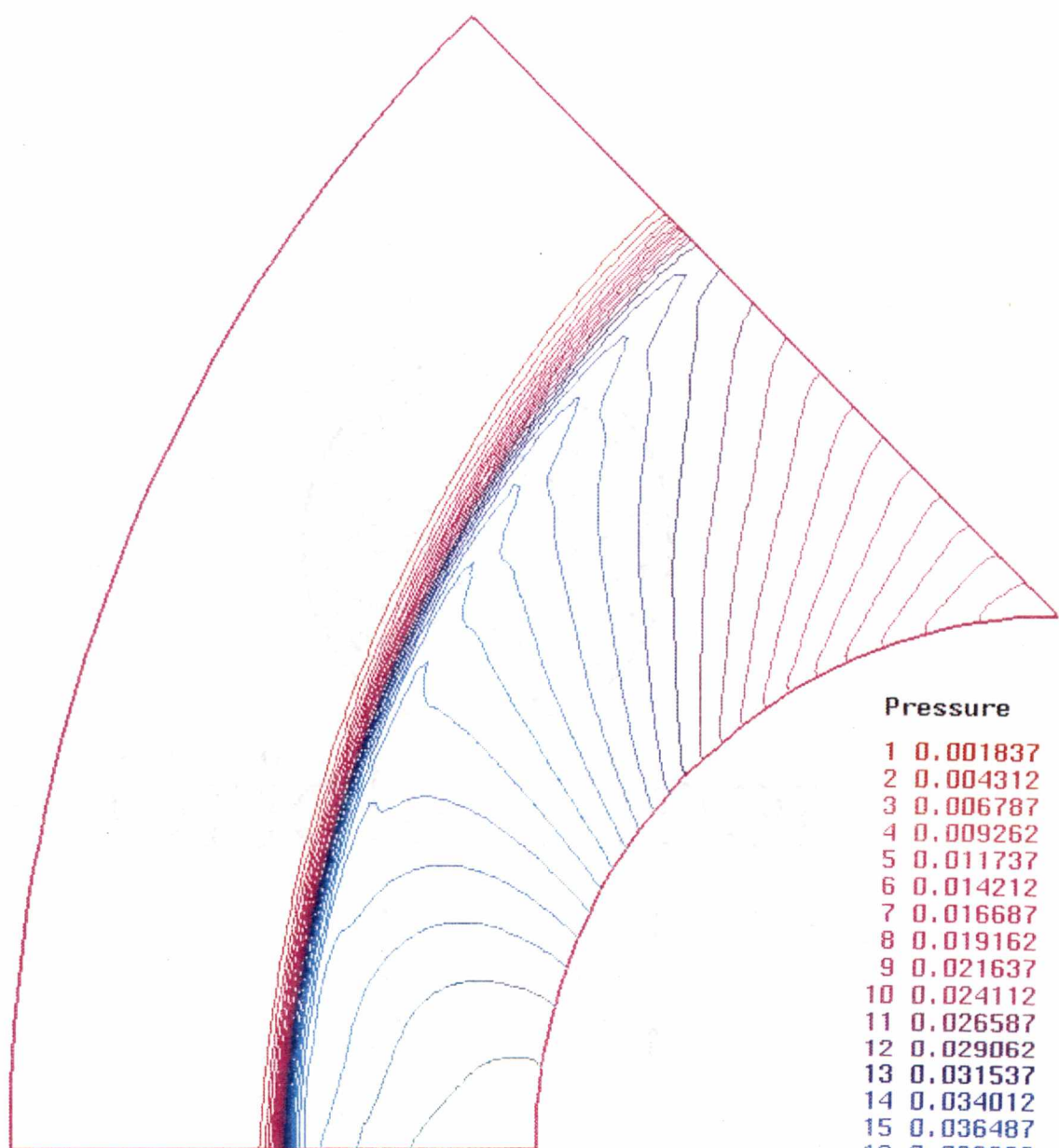


Figure 20. Contours of constant static pressure using PARC for flow around cylinder



Pressure

1	0.001837
2	0.004312
3	0.006787
4	0.009262
5	0.011737
6	0.014212
7	0.016687
8	0.019162
9	0.021637
10	0.024112
11	0.026587
12	0.029062
13	0.031537
14	0.034012
15	0.036487
16	0.038962
17	0.041437
18	0.043912
19	0.046387
20	0.048862
21	0.051337
22	0.053812
23	0.056287
24	0.058762
25	0.061237

Figure 21. Contours of constant static pressure for present study for flow around cylinder

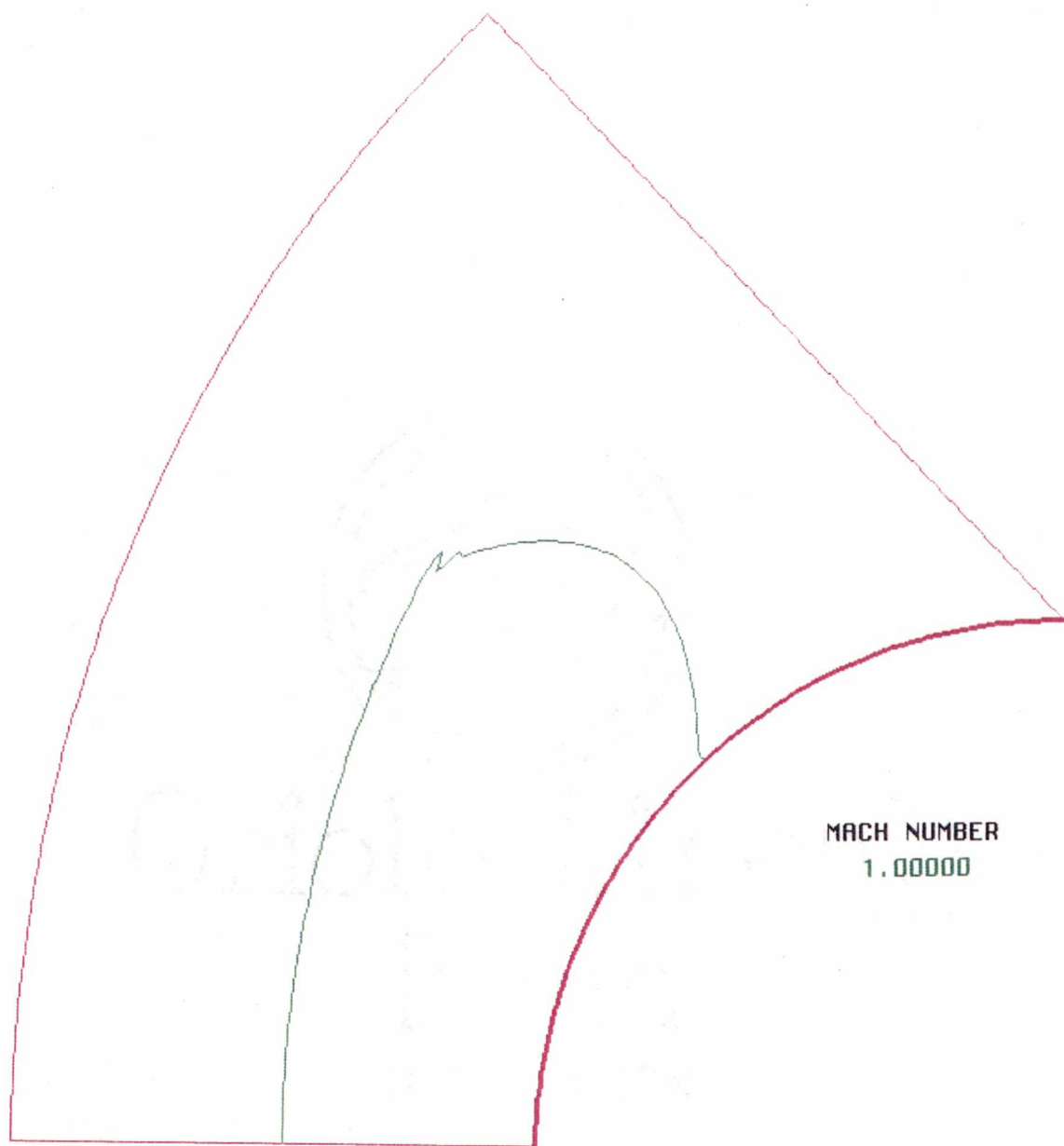


Figure 22. Sonic line for PARC for flow around cylinder

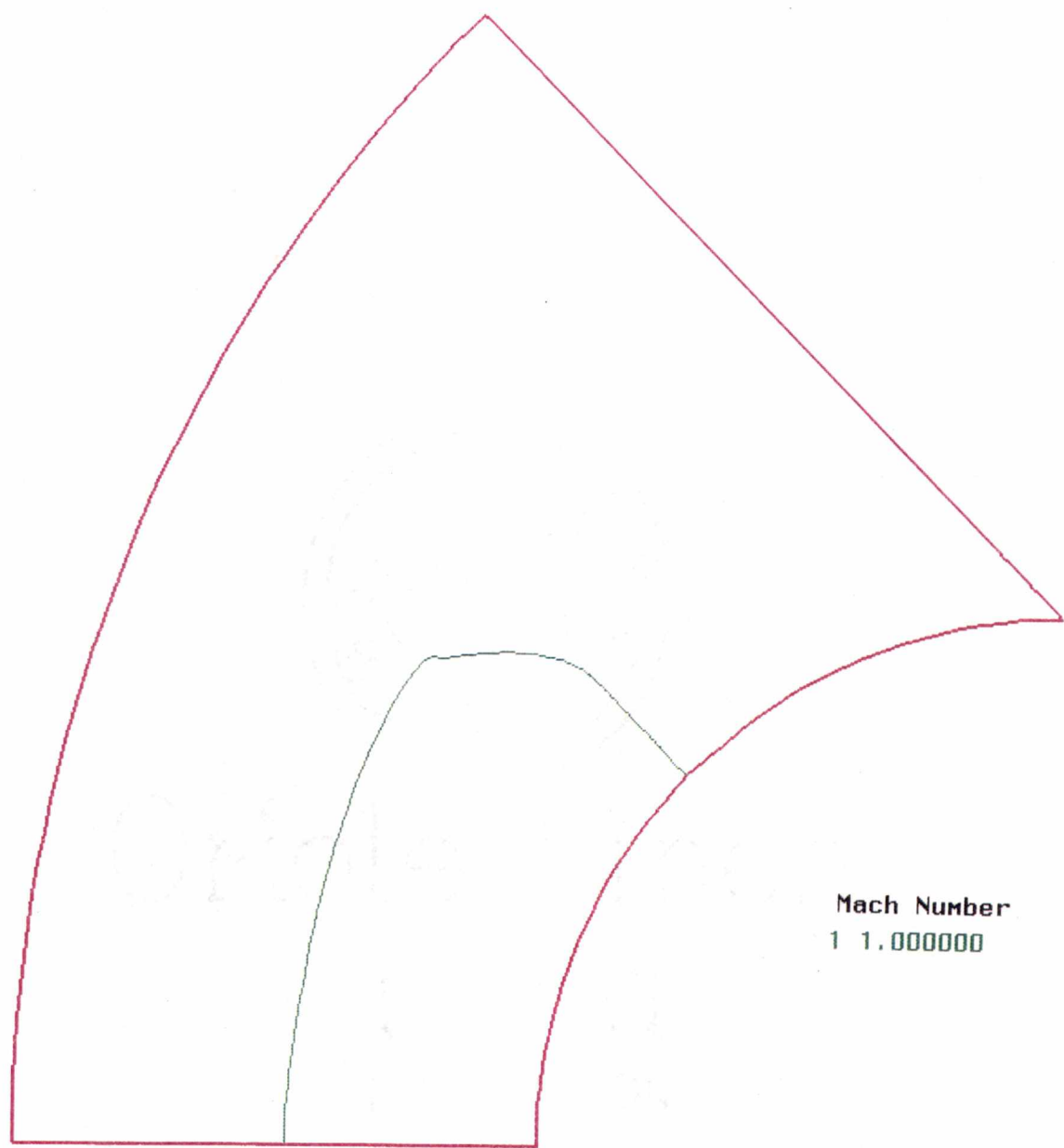


Figure 23. Sonic line for present work for flow around cylinder

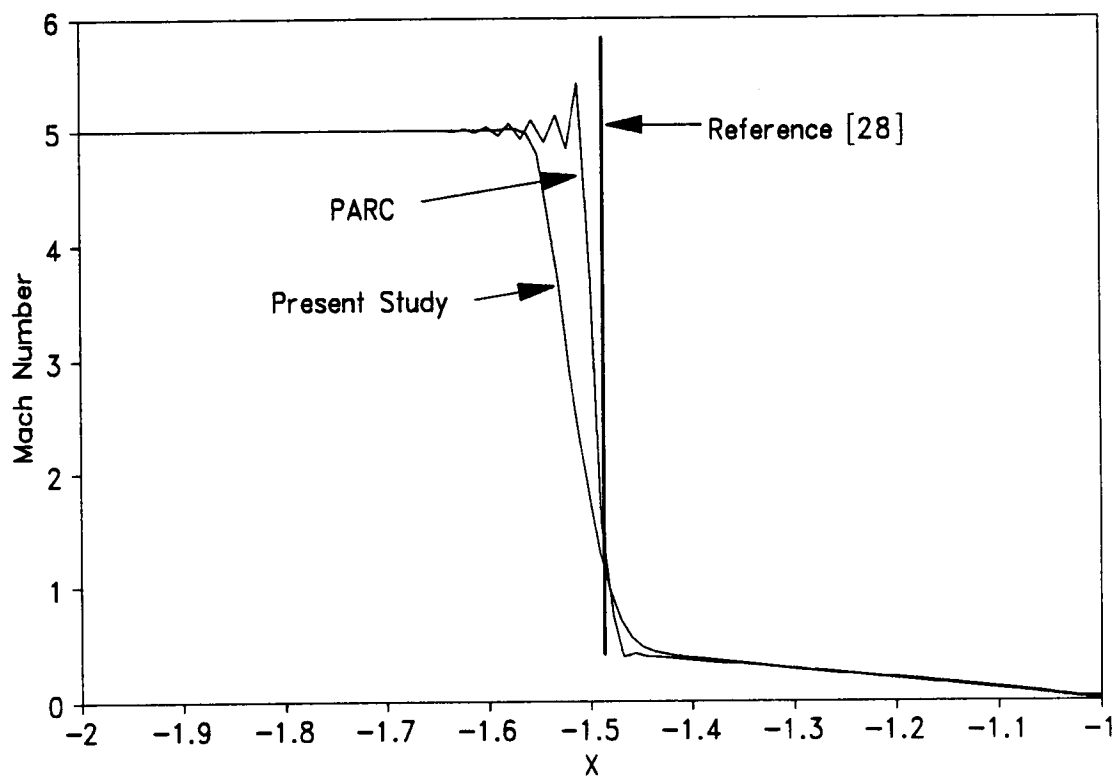


Figure 24. Comparison plot of C_p for flow around cylinder

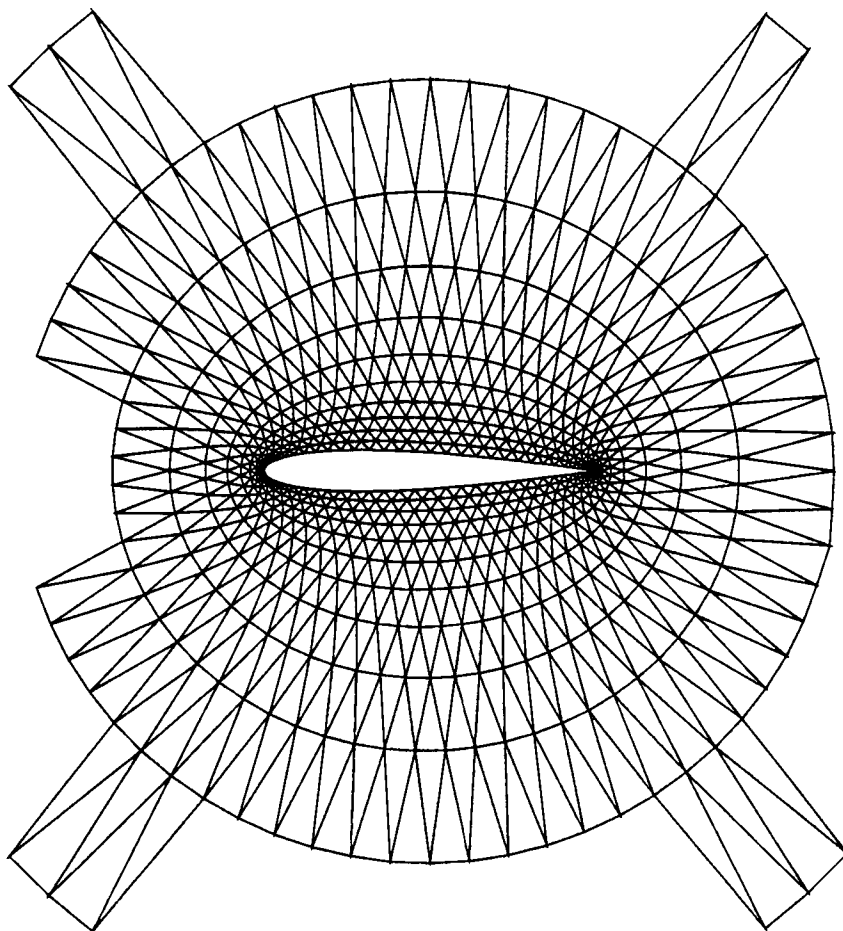


Figure 25. Unstructured course grid for NACA-0012

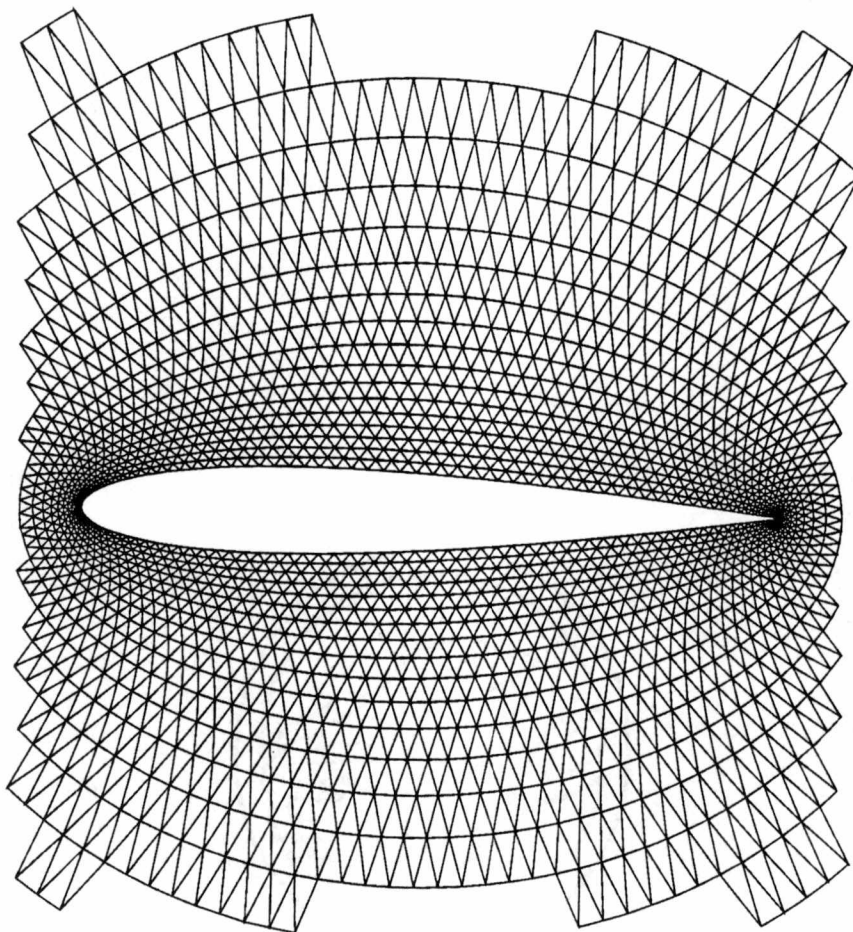


Figure 26. Unstructured fine grid for NACA-0012

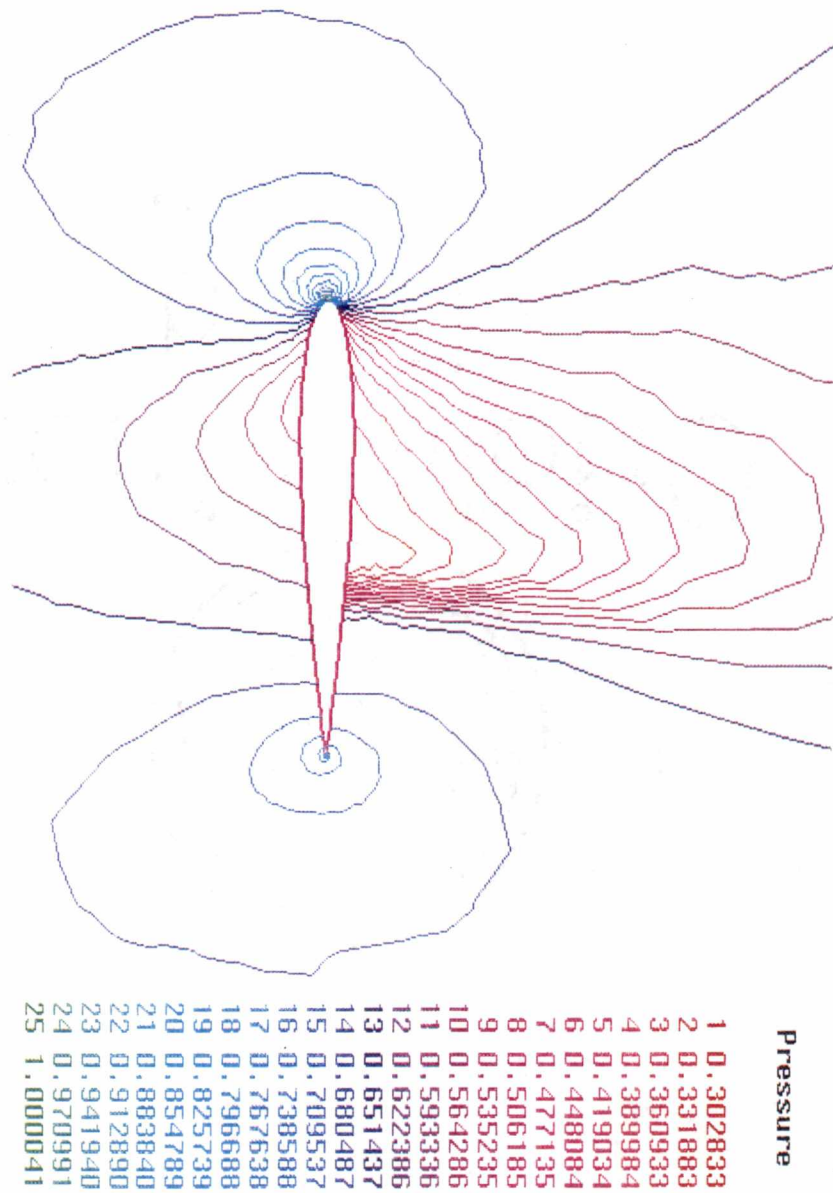


Figure 27. Contours of constant static pressure for coarse grid solution of NACA-0012

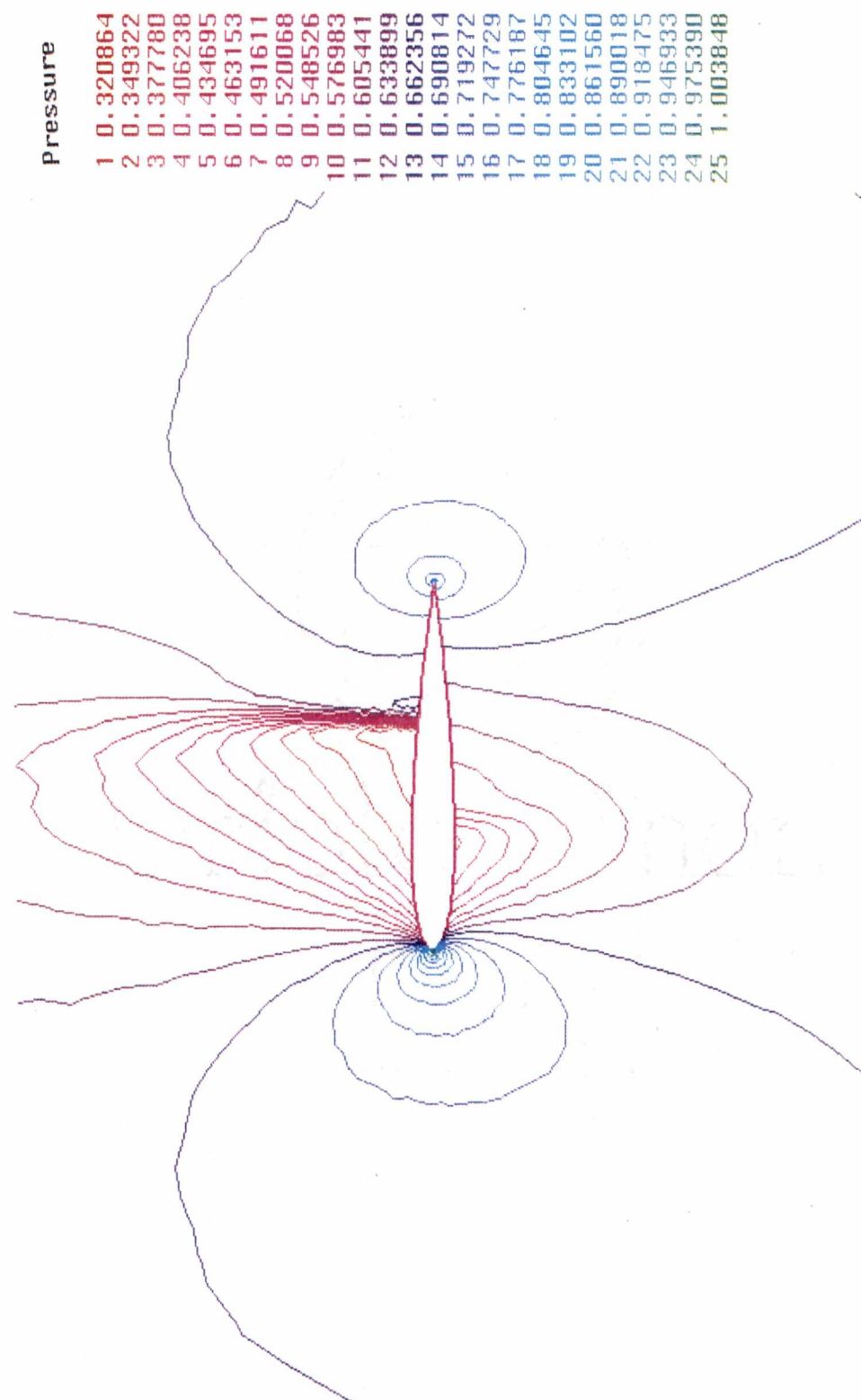


Figure 28. Contours of constant static pressure for fine grid solution NACA-0012

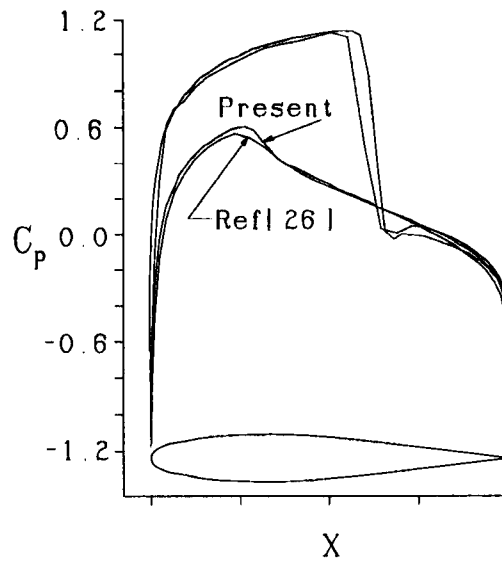


Figure 29a. Course grid solution

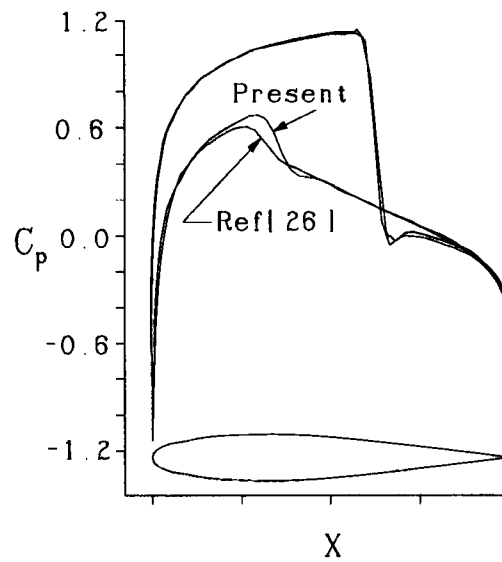


Figure 29b. Fine grid solution

Figure 29. Comparison of pressure distribution for NACA-0012 fine and course grid solution

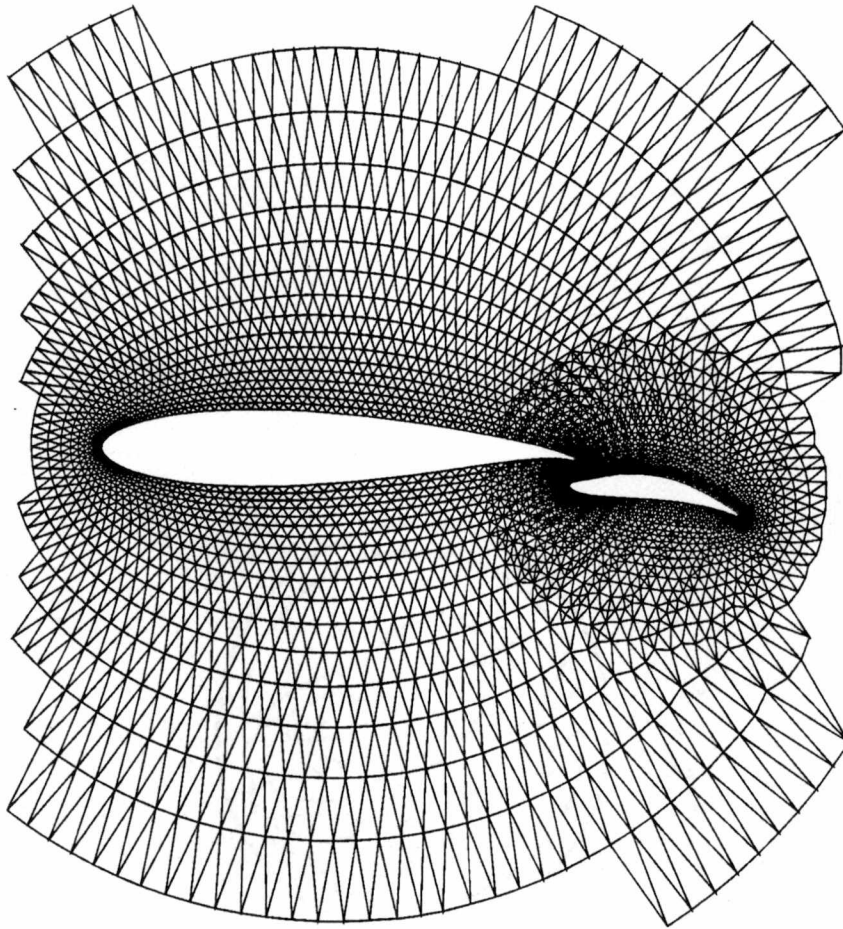


Figure 30. Unstructured grid for Karman-Trefftz double airfoil

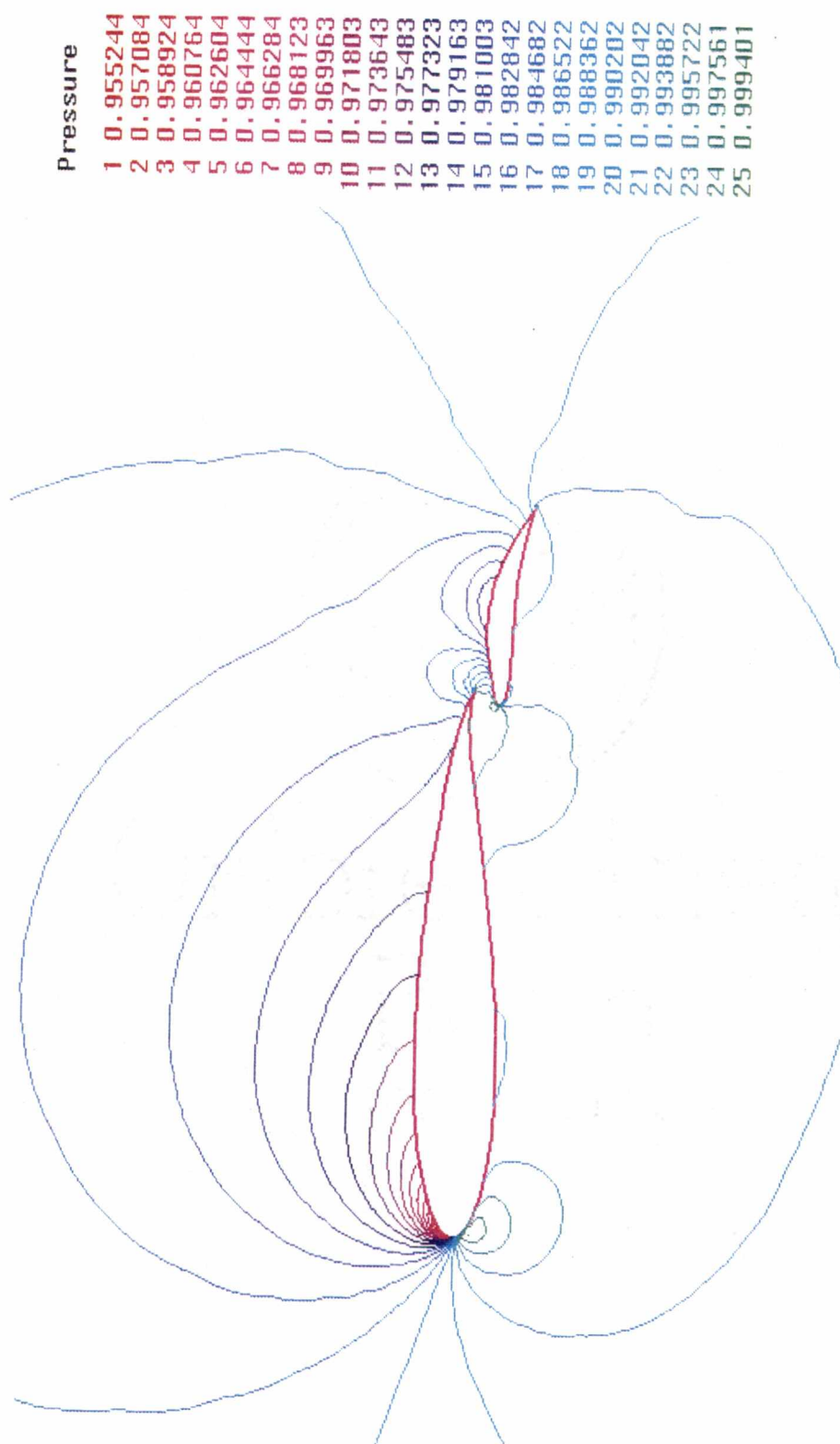


Figure 31. Contours of constant static pressure for Karman-Trefftz double airfoil

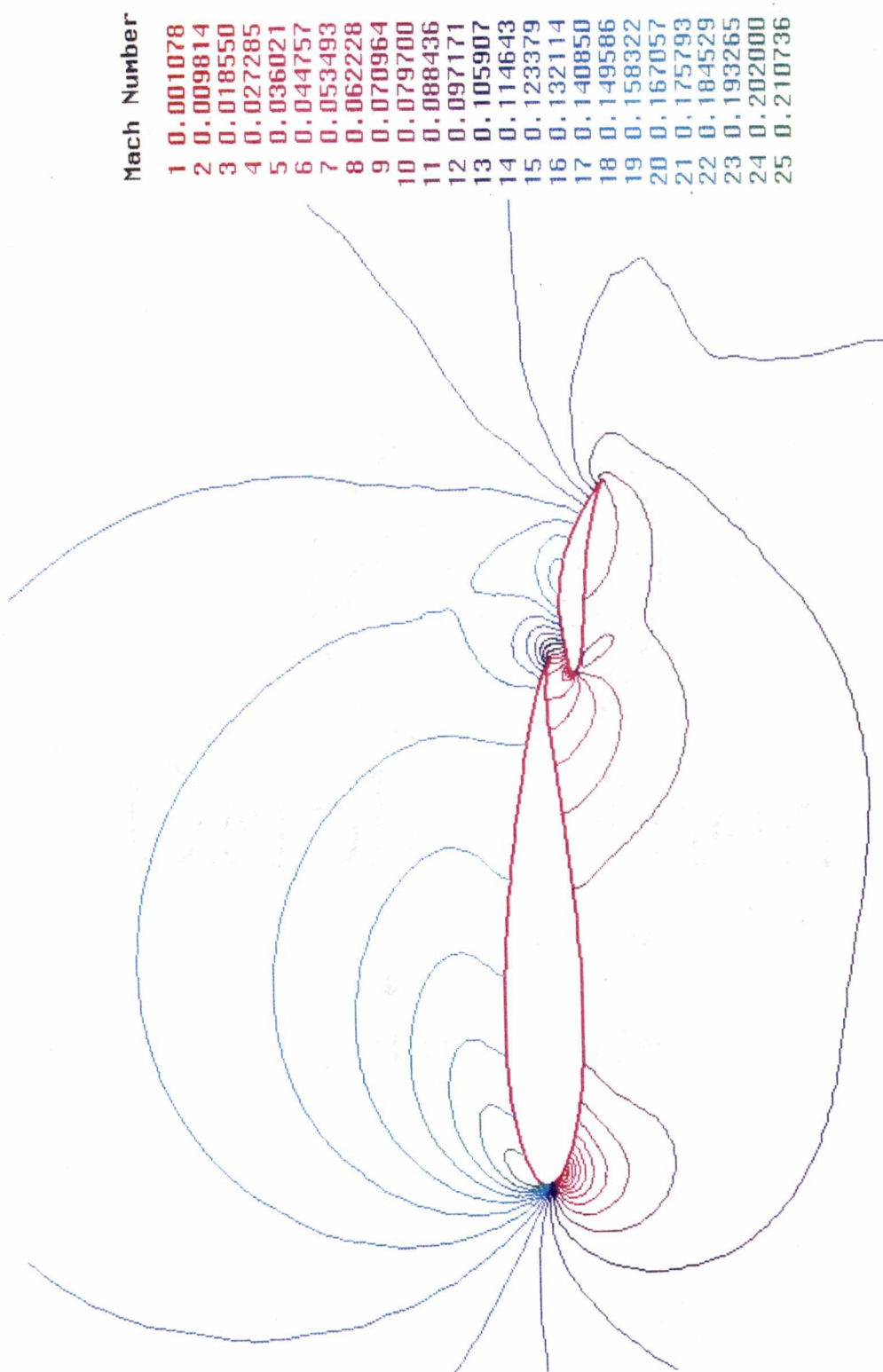


Figure 32. Contours of constant Mach Number for Karman-Trefftz double airfoil

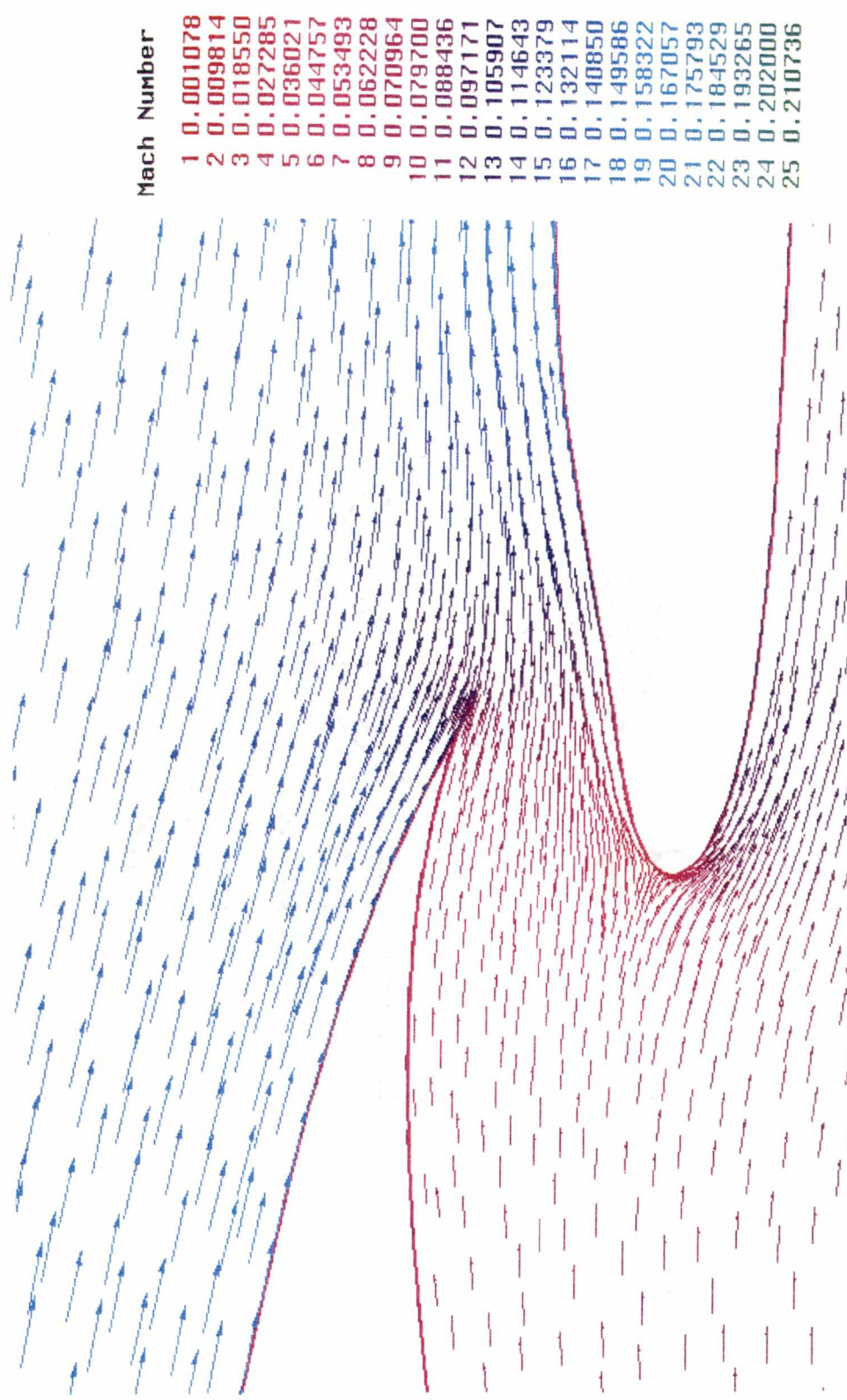


Figure 33. Velocity vector plot of main foil and flap for Karman-Trefftz double airfoil

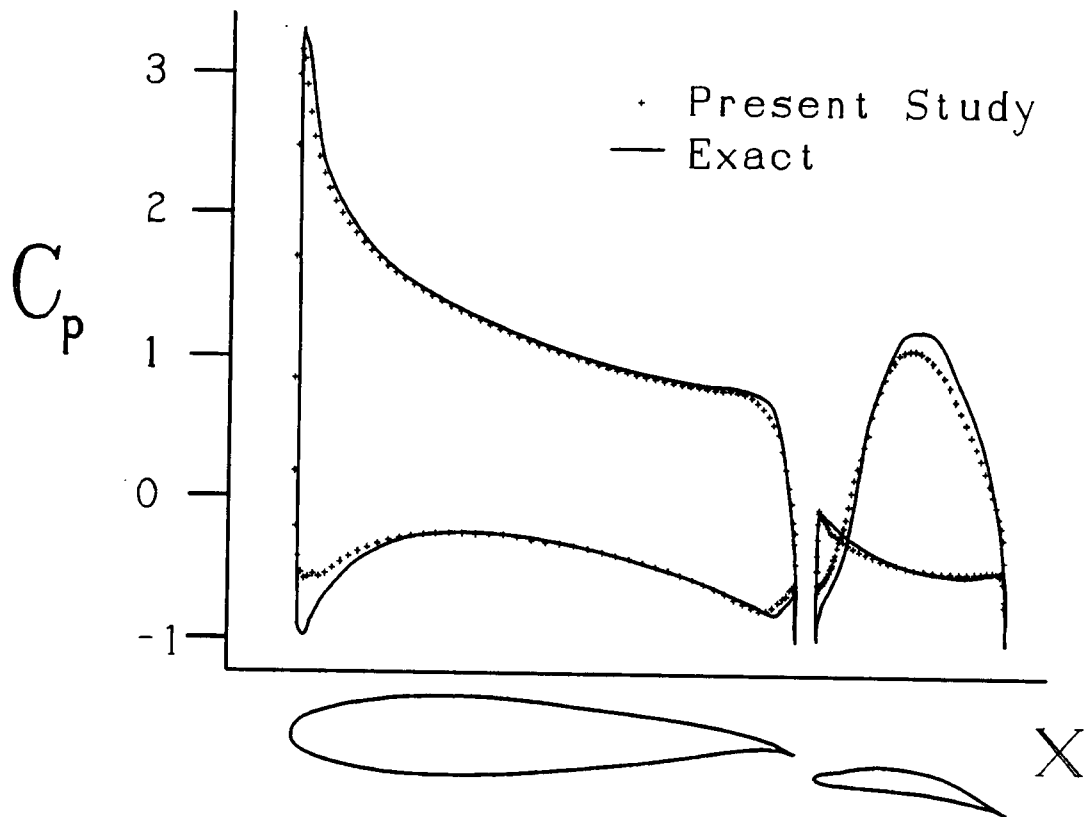


Figure 34. Comparison plot of C_p for Karman-Trefftz double airfoil

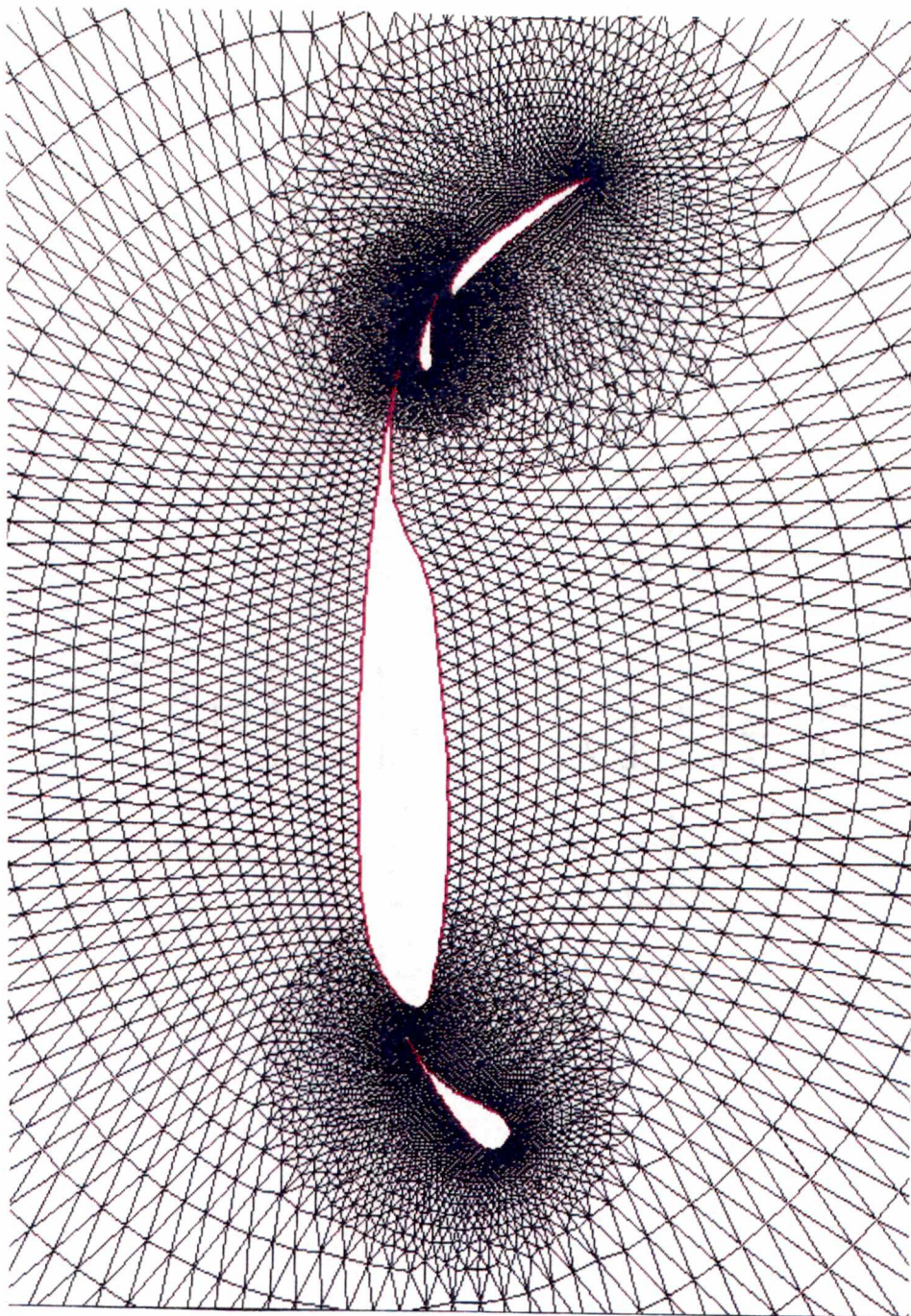


Figure 35. Unstructured grid for four flap airfoil

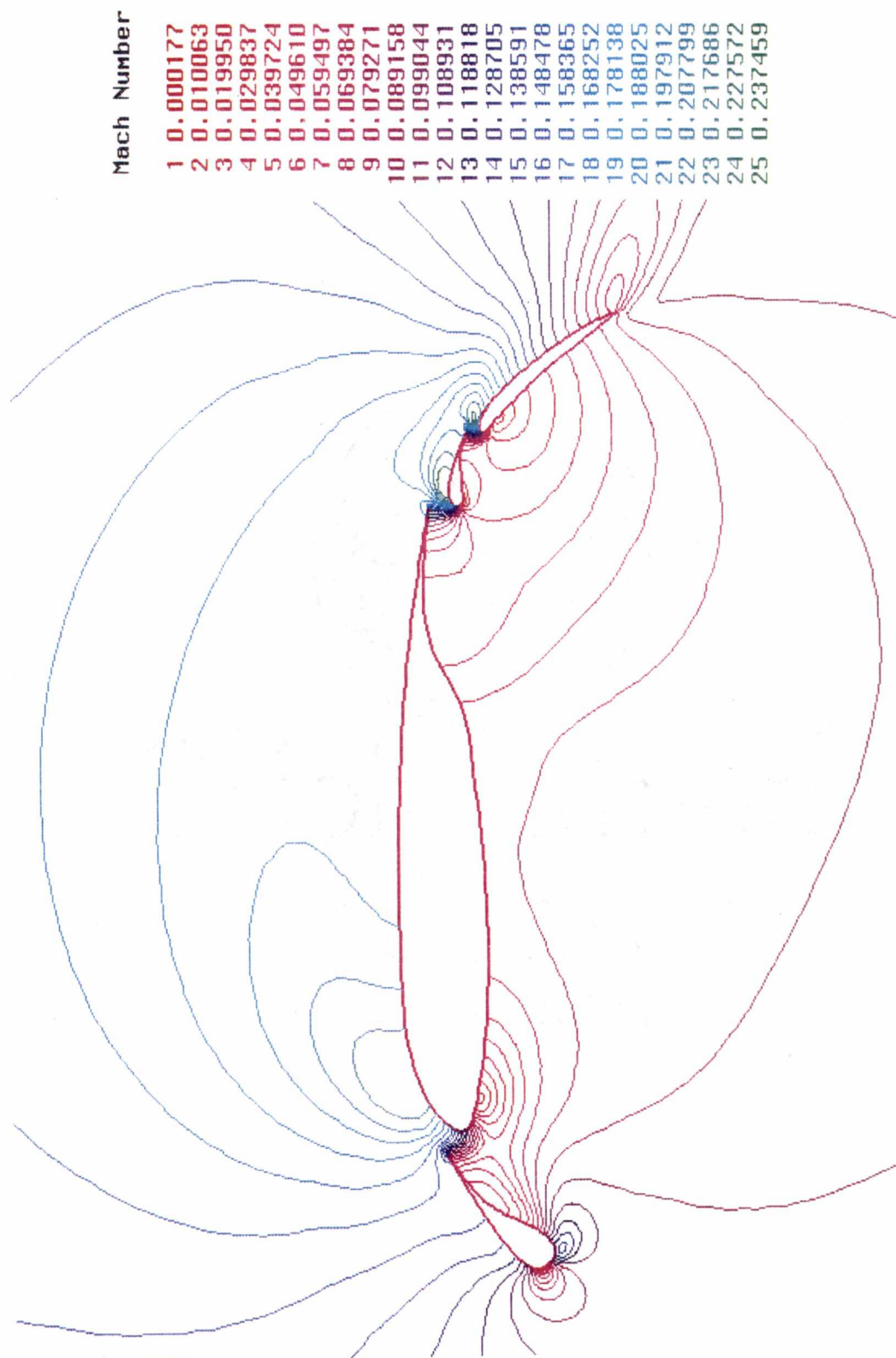


Figure 36. Contour plot of constant Mach number for four flap airfoil

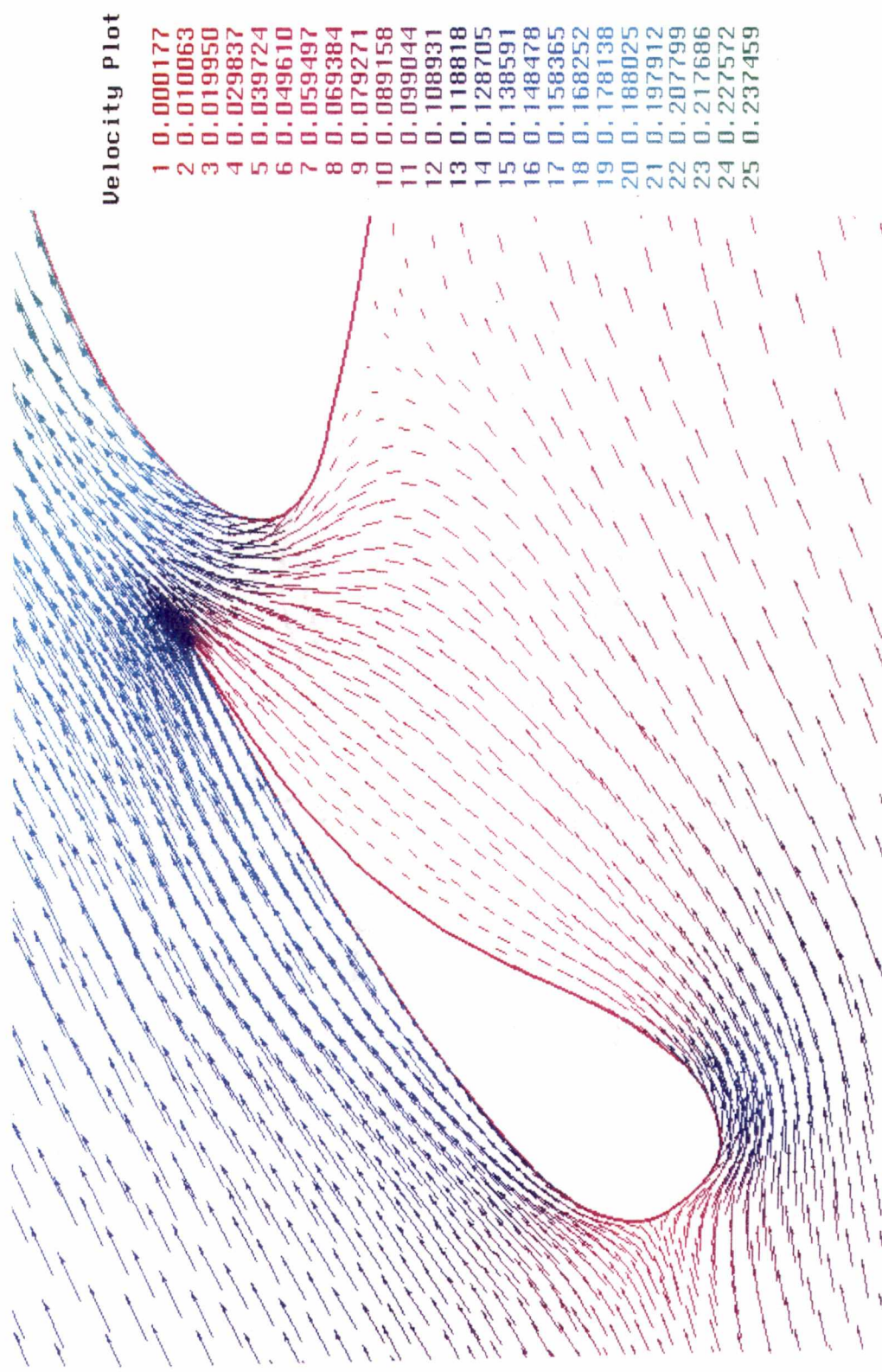


Figure 37. Velocity vector plot around leading edge of four flap airfoil

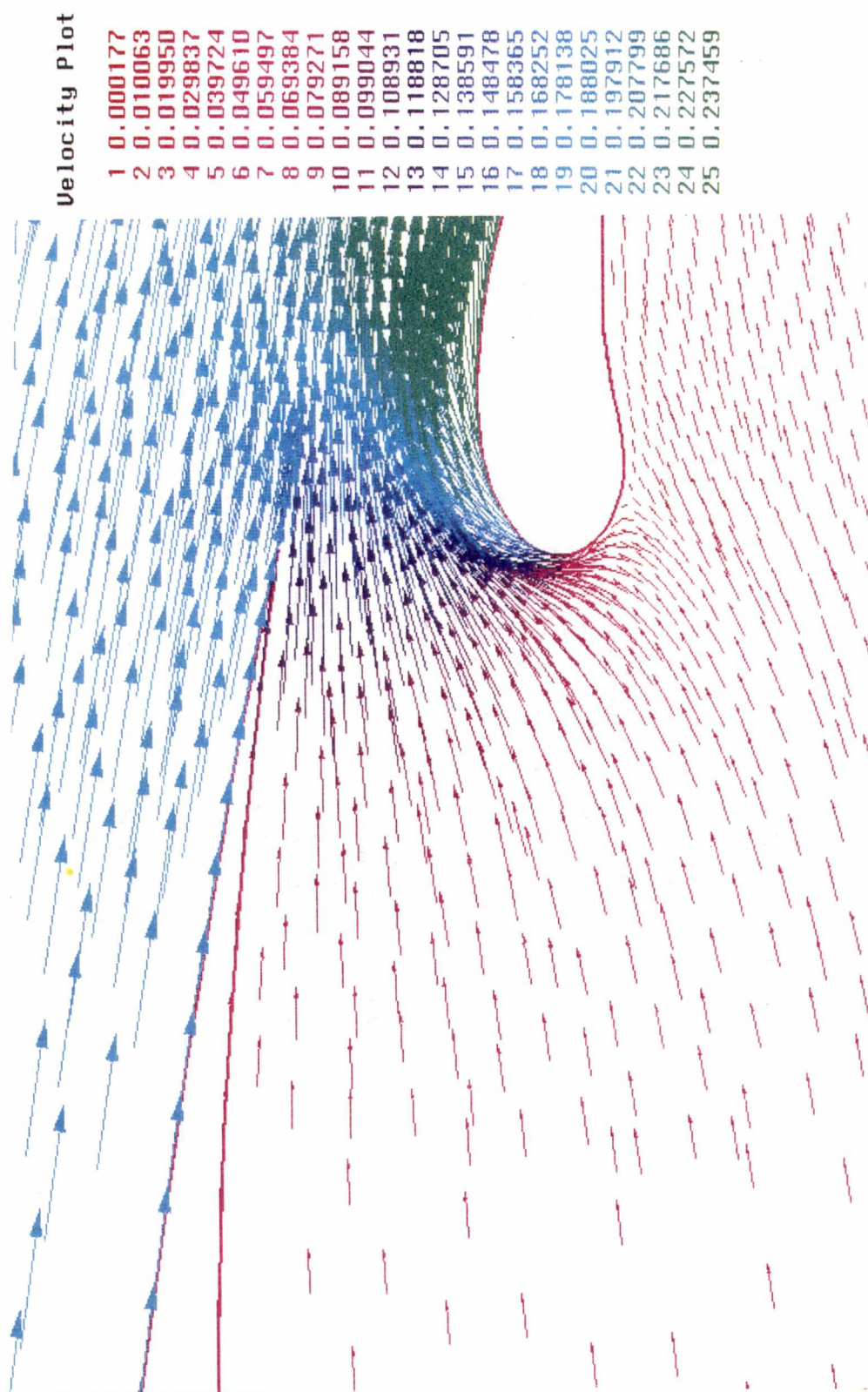


Figure 38. Velocity vector plot around trailing edge of main foil of four flap airfoil

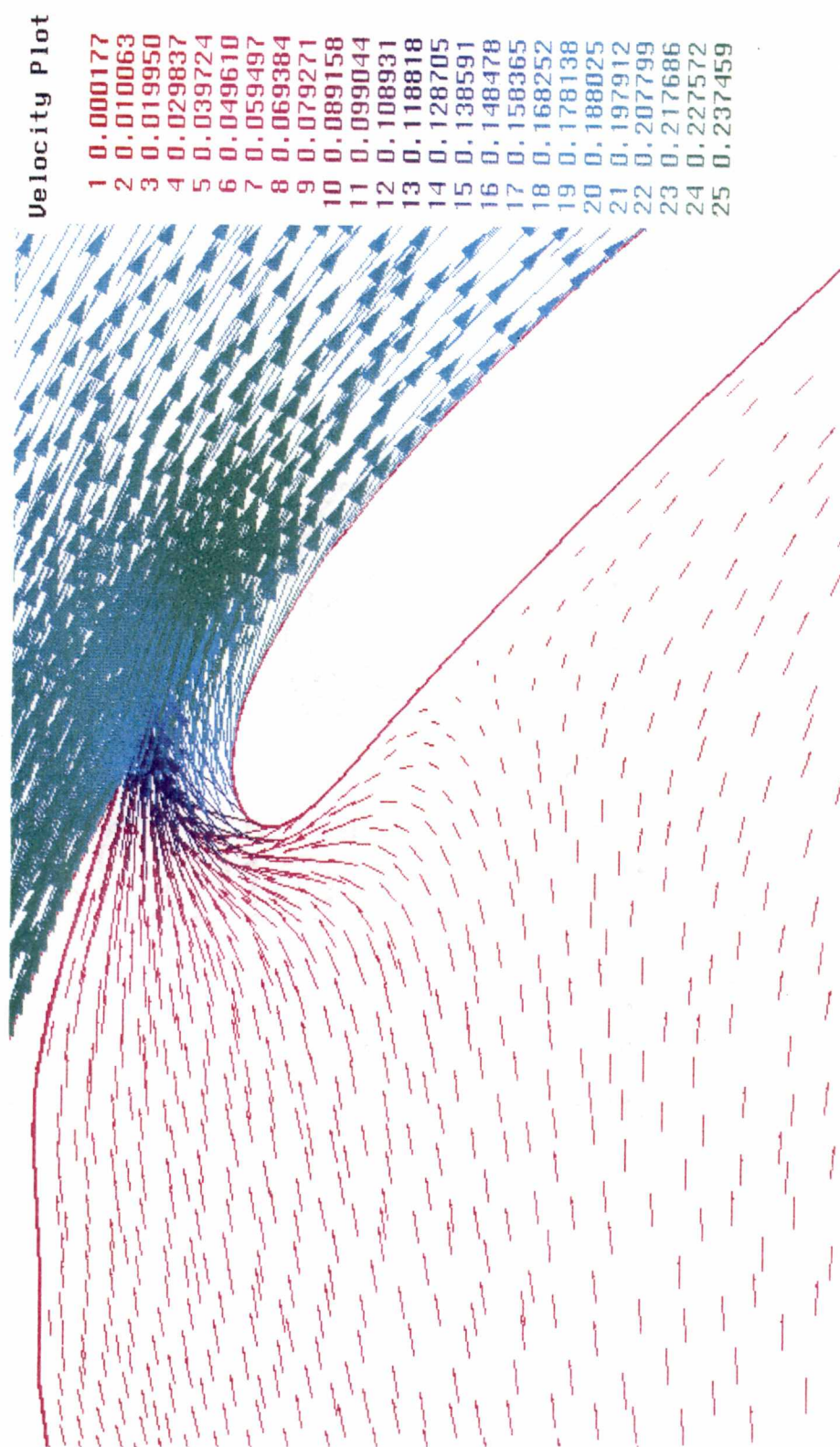


Figure 39. Velocity vector plot around trailing edge of third flap of four flap airfoil

VITA

Michael Lee Stokes was born in Columbia, Tennessee on February 24, 1956. He was graduated from Columbia Central High School in May 1974. The following September he entered Columbia State Community College, and in June 1976 he graduated with an Associate of Science degree in Freshman Engineering, Cum Laude. In September 1976, he entered Tennessee Technological University and in May 1979, received a Bachelor of Science degree in Mechanical Engineering. He remained at Tennessee Technological University where he received his Master of Science Degree in Mechanical Engineering in December 1981. He was employed by Sverdrup Technology, Inc. in April 1981 in Tullahoma, Tennessee.

Mr. Stokes was married on August 1, 1981 to Patricia Ann Youree of Dickson, Tennessee. They have two daughters, Lauren Elizabeth and Kathryn Anne Stokes.