To the Graduate Council:

I am submitting herewith a dissertation written by Tomasz Dubejko entitled "Branched Circle Packings, Discrete Complex Polynomials, and the Approximation of Analytic Functions." I have examined the final copy of this dissertation for form and content and recommend that it be accepted in partial fulfillment of the requirements for the degree of Doctor of Philosophy, with a major in Mathematics.

Kenneth Stephenson

Kenneth Stephenson, Major Professor

We have read this dissertation
and recommend its acceptance:

William R Wade

Accepted for the Council:

Associate Vice Chancellor
and Dean of The Graduate School

BRANCHED CIRCLE PACKINGS, DISCRETE COMPLEX POLYNOMIALS, AND THE APPROXIMATION OF ANALYTIC FUNCTIONS

A Dissertation

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Abstract

The aim of this dissertation was to investigate circle packings more general than those which had been studied in the past. In particular, new techniques were developed to explore these broader concepts. We found necessary and sufficient conditions for the existence of finite and infinite branched circle packings; we proved the uniqueness of hexagonal circle packings of finite valence; we also obtained a finite-valence generalization of the Ring Lemma. As a result, the theory of circle packings was extended to incorporate branched circle packings. Moreover, the extended theory was shown to be, in many aspects, a discrete parallel of the theory of analytic functions. Especially, via branched circle packings, we created objects termed *discrete Blaschke products* and *discrete complex polynomials* which are discrete analogues of their classical counterparts. Then we proved that any classical complex polynomial (resp. Blaschke product) can be approximated uniformly on compacta by discrete complex polynomials (resp. Blaschke products).

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Introduction

Over the last few years many researchers ([BSt1,2],[H],[RS],[Sc],[T1,2], etc – please see the bibliography list) have been working on questions related to circle packings, which are collections of circles in the plane with prescribed patterns of tangencies of a combinatorial nature. Most of the results that have been proved deal with univalent circle packings, i.e. "genuine" packings of circles (Figure 1).

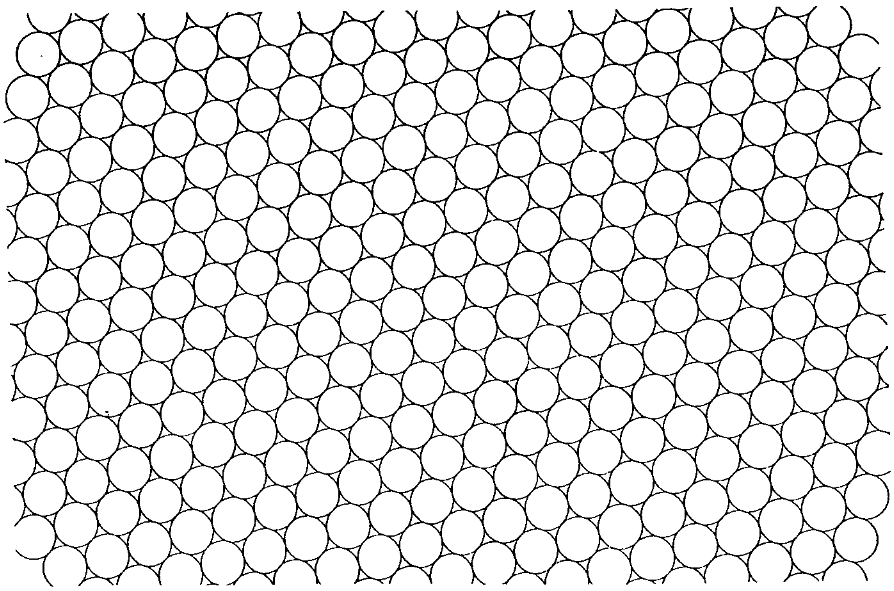

FIGURE 1. Example: regular hexagonal circle packing

The techniques that have been used in the proofs show profound influence of geometric function theory and complex analysis ([BSt2],[HSc],[RS],[Sc]), although ideas from graph theory ([HSc2],[St2]) and probability ([St2]) have also turned out to be very useful.

One may ask the following question: If geometric function theory and complex analysis have such an impact on the theory of circle packings, why have people been concentrating their efforts primarily on univalent circle packings rather than on non-univalent or even branched circle packings? On one hand the answer to this question is that circle packing theory is such a new theory that there was "no time" to investigate these more general packings. On the other hand there has not yet been enough machinery developed to study these broader concepts. Our attempt in this work is to establish such machinery and extend the theory so it will incorporate non-univalent and branched circle packings. It will be seen that in our process of creating such an expansion we are guided by the theory of analytic functions. In particular, via branched circle packings, we create objects like *discrete Blaschke products* and *discrete complex polynomials* which are discrete analogues of their classical counterparts.

Let us now describe briefly what the reader will find on the next several pages.

First of all, we need to stress that we will be assuming one's familiarity with the basic terms and results of the theory of circle packings. However, the first chapter contains a brief introduction to the subject. Moreover, we would like to point out that there is a vast list of references ([BSt2],[BDSt],[Sc]), with [DSt] being probably most comprehensive, anyway the best suited for the reader from our point of view.

In Chapter II we deal with finite branched circle packings which one may think of as finite circle packings (in euclidean metric) on surfaces associated with complex polynomials. We find necessary and sufficient conditions for the existence of such packings subject to given patterns of tangencies (in a form of finite triangulations of a disc) and given branch sets (Theorem II.2). These conditions are of combinatorial type and are encoded in a phrase *branch structure* (see Definition II.1). It is worth mentioning that our proof of existence of branched packings is an (practical) algorithm that allows one to construct them if the conditions are satisfied.

Among finite branched circle packings there is a special class of packings which we call Bl-type packings (Definiton II.6). These are branched versions of Andreev packings, where an Andreev packing is a univalent finite circle packing that "fills" the unit disc. Moreover, such packings induce an important group of simplicial maps – *discrete Blaschke products.*

In general, let us consider two circle packings with the same patterns of tangencies: one which is univalent and to which we will refer as the domain packing, the other which is not necessarily univalent and to which we will refer as the range packing. Then there is a naturally induced map by these two packings which is a simplicial map, whose domain is a triangulation in $\mathbf{C}$ obtained by joining the centers of tangent circles in the domain packing by segments, and which maps centers of the circles in the domain packing to centers of the corresponding circles in the range packing and extends affinely to the faces in its domain.

Now, a discrete Blaschke product is a simplicial map "from" an Andreev packing "to" a Bl-type packing. It is shown (Theorem III.4) that such maps are, modulo a homeomorphism, classical Blaschke products. As each discrete Blaschke product is uniquely determined by its values on a discrete set of points, which is the set of centers of all circles from the domain packing, the term *discrete* is quite valid. Moreover, as one of our main results, we prove that any classical Blaschke product can be approximated uniformly on compacta of the unit disc by discrete ones (Theorem V.8.3).

The generalization of techniques applied in case of finite branched circle packings, the notion of discrete Blaschke products, and some other methods are used in Chapter IV to show existence of <u>infinite</u> branched circle packings (Theorem IV.2). As in the finite case, we find necessary and sufficient conditions for the existence of infinite packings to be of combinatorial nature and equivalent to the branch structure condition.

The existence of infinite branched circle packings allows us to define discrete complex polynomials which are simplicial maps "from" univalent circle packings that "fill" the complex plane **C** "to" infinite branched circle packings that are "branched coverings" of **C**. In particular, each such map can be written as a composition of a self-homeomorphism of **C** followed by a classical complex polynomial. The main result about discrete complex polynomials is Theorem VII.2.1, which states that any classical complex polynomial can be approximated uniformly on compacta of **C** by discrete ones. The key observation in our proof of the approximation is Theorem VII.1.1, which we believe deserves special attention. It shows that a family of simplicial maps, induced by some infinite circle packings (hexagonal packings), having a uniform finite bound on their valences is a K-quasiregular family.

We finish this dissertation with a uniqueness result for infinite hexagonal branched circle packings (Theorem VIII.2) and the classification of infinite hexagonal circle packings whose underlying surfaces are of finite valence (Corollary VIII.3).

Chapter I

Preliminaries

We are going to introduce briefly some terminology and definitions related to circle packings. For a more complete picture we refer the reader to [BSt2] or [DSt].

1. Packings. Let $\mathbf{T}$ be a (locally finite) triangulation of a disc such that $\mathbf{T}$ is finite when the disc is closed, infinite without boundary when the disc is open, and infinite with boundary when the disc has partial boundary. We will be assuming that $\mathbf{T}$ has an orientation induced from an orientation of the disc. The sets of vertices, boundary vertices, and interior vertices of $\mathbf{T}$ will be denoted by $V(\mathbf{T})$, $bd\,\mathbf{T}$, and $int\,\mathbf{T}$, respectively.

Let P be a collection of circles in the plane. Then we say that P is a **circle packing** for $\mathbf{T}$ if:

- there exists 1-to-1 correspondence between the vertices $v \in V(\mathbf{T})$ and the circles $C_P(v)$ of P such that circles $C_P(u)$ and $C_P(w)$ are externally tangent whenever u and w are vertices sharing an edge in the complex $\mathbf{T}$,
- P is orientation preserving, i.e. if v_1, v_2, v_3 are the vertices of a face in $\mathbf{T}$ taken in the positive order (with respect to the orientation of $\mathbf{T}$) then $C_P(v_1)$, $C_P(v_2)$, $C_P(v_3)$ form a positively oriented triple of circles in the plane.

If P has all its elements contained in the unit disc $\Delta = \{|z| < 1\}$ then it can be viewed as a circle packing in the hyperbolic plane (i.e. in Δ with the hyperbolic metric), and in such case we consider P as a **hyperbolic** circle packing.

A **flower** in P associated with a vertex $v \in V(\mathbf{T})$ is a collection of the following circles from P: the circle $C_P(v)$, called the **center**, and circles that correspond to the neighboring vertices of v, called the **petals**. The number of petals is the **degree** of v.

If a circle packing has the property that all its circles have mutually disjoint interiors then it is called **univalent**. The reader should be aware of the fact that for any triangulation $\mathbf{T}$ of a disc there exists a univalent circle packing associated with $\mathbf{T}$ ([BSt2]). If $\mathbf{T}$ is finite then there is an important class of univalent circle packings determined by $\mathbf{T}$ called **Andreev packings**. These are packings contained in the unit disc such that their boundary circles are internally tangent to the unit circle. Among univalent circle packings associated with infinite triangulations there is one which will be of special interest for us. Its complex is an infinite constant 6-degree complex identified with an infinite regular hexagonal lattice in $\mathbf{C}$, and the packing itself is a **regular hexagonal circle packing** in $\mathbf{C}$ with all circles having the same radii.

2. Branched packings. If P is a circle packing for $\mathbf{T}$ then a function $r : V(\mathbf{T}) \mapsto (0,\infty)$ which assigns to each vertex in $V(\mathbf{T})$ the euclidean radius of the corresponding circle in P is called the **radius function** of P. The function r determines for each interior vertex of $\mathbf{T}$ its **angle sum** in the following way: Write $s_P(u)$ for the center of $C_P(u)$ and let $v \in int\,\mathbf{T}$, then the angle sum $\theta_r(v)$ at v is $\sum \alpha_r(v,\triangle)$, where the summation is over all faces $\triangle$ in the star of v in $\mathbf{T}$, and $\alpha_r(v,\triangle) = \alpha_r\big(v,\triangle(v,u_1,u_2)\big)$ is the angle at a vertex $s_P(v)$ in the euclidean triangle $\triangle s_P(v)s_P(u_1)s_P(u_2)$ with the vertices $s_P(v)$, $s_P(u_1)$, $s_P(u_2)$.

It follows almost immediately from the definition of a circle packing that all angle sums of P are multiples of 2π. A vertex in $\mathbf{T}$ with an angle sum determined by P equal to $2n\pi$, $n \geq 2$, is called a **branch point** (vertex) of P of **multiplicity**

n. If we want to stress that a circle packing has (does not have) a branch point then we refer to it as the **branched (unbranched) circle packing**. We observe that univalent circle packings are necessarily unbranched.

3. Mappings. A circle packing P for $\mathbf{T}$ naturally induces a **simplicial** function $\mathcal{S}_P : \mathbf{T} \mapsto \mathbf{C}$ which maps each vertex of $\mathbf{T}$ to the (euclidean) center of the corresponding circle in P and then is extended via barycentric coordinates to edges and faces of $\mathbf{T}$. The geometric complex $\mathcal{S}_P(\mathbf{T})$ is called the (euclidean) **carrier** of P and denoted $carr(P)$. If P is univalent then the induced simplicial map $\mathcal{S}_P$ is an embedding and the carrier of P is a subset of $\mathbf{C}$ equal to the union of images of all faces of $\mathbf{T}$ under $\mathcal{S}_P$.

Let Q be another circle packing for $\mathbf{T}$, and let r_P and r_Q denote the radius functions for packings P and Q respectively. Then the map $G_{Q,P} := \mathcal{S}_P \circ \mathcal{S}_Q^{-1}$: $carr(Q) \mapsto carr(P)$ is called the **simplicial map from Q to P**. The map $G^{\#}_{Q,P} : carr(Q) \mapsto (0, \infty)$ defined by $G^{\#}_{Q,P}(\mathcal{S}_Q(v)) = r_P(v)/r_Q(v)$ on the set of vertices of $\mathcal{S}_Q(\mathbf{T})$ and then extended affinely to faces of $\mathcal{S}_Q(\mathbf{T})$ is called the **ratio function from Q to P** (cf. [RS], [DSt]).

4. Radii and angle sums. One way of working with circle packings is by working with their radius functions. Such an approach allows for certain generalizations. In particular, for any function $\rho : V(\mathbf{T}) \mapsto (0, \infty)$ (an "abstract" radius function) we can define an "abstract" angle sum at any interior vertex of $\mathbf{T}$. To achieve this we do the following construction:

> Let $\triangle = \triangle(u_0, u_1, u_2)$ be the face in $\mathbf{T}$ with the vertices u_0, u_1, u_2 listed in the positive order. Let $C(u_0)$, $C(u_1)$, $C(u_2)$ be a positively oriented triple of mutually and externally tangent circles in the plane with radii $\rho(u_0)$, $\rho(u_1)$, $\rho(u_2)$ and centers $s(u_0)$, $s(u_1)$, $s(u_2)$ respec-

tively. Write $\alpha_\rho(u_j, \triangle(u_0, u_1, u_2))$, $j = 0, 1, 2$, to denote the angle at the vertex $s(u_j)$ in the triangle $\triangle s(u_0)s(u_1)s(u_2)$. For $v \in int\,\mathbf{T}$ we define $\theta_\rho(v) := \sum \alpha_\rho(v, \triangle)$, where the summation is over all faces $\triangle$ in the star of v in $\mathbf{T}$. Then $\theta_\rho(v)$ is called the ("abstract") angle sum at v determined by ρ.

It is almost immediate from the above construction that if ρ happens to be the radius function r_P of a circle packing P for $\mathbf{T}$ then $\theta_\rho(v) = \theta_{r_P}(v)$ for any $v \in int\,\mathbf{T}$. It is common, therefore, to refer to a collection $\{\rho(v)\}_{v \in V(\mathbf{T})}$ as a collection of **radii,** even if $\rho : V(\mathbf{T}) \mapsto (0, \infty)$ is not the radius function for any circle packing for $\mathbf{T}$. Necessary and sufficient conditions for this are easily established in terms of angle sums: $\rho : V(\mathbf{T}) \mapsto (0, \infty)$ is the radius function for some circle packing for $\mathbf{T}$ if and only if all values of $\theta_\rho : int\,\mathbf{T} \mapsto (0, \infty)$ are multiples of 2π. This depends on the fact that $\mathbf{T}$ is "simply connected" and was proven in [BSt1] for $\theta_\rho \equiv 2\pi$; the proof can easily be extended for values of θ_ρ which are integral multiples of 2π. In such case, that is when the values of θ_ρ are multiples of 2π, a circle packing whose radius function is ρ is called a **circle packing induced by** ρ. Such circle packing is determined uniquely by the location in the plane of a pair of its tangent circles associated with a given edge of $\mathbf{T}$, i.e. if u_1u_2 is an edge in $\mathbf{T}$, and $P = \{C(u)\}_{u \in V(\mathbf{T})}$ and $P' = \{C'(u)\}_{u \in V(\mathbf{T})}$ are circle packings induced by ρ such that $C(u_j) = C'(u_j)$ for $j = 1, 2$, then $P = P'$. In other words, any circle packing induced by ρ is unique up to rigid motions.

Whether or not ρ is a radius function, for $w \in bd\,\mathbf{T}$ we define the interior angle $\gamma_\rho(w)$ at w by $\gamma_\rho(w) := \sum \alpha_\rho(v, \triangle)$, where the summation is over all faces $\triangle$ in $\mathbf{T}$ with the vertex w. Then $\theta_\rho(\cdot)$ and $\gamma_\rho(\cdot)$ are related by the following euclidean version of the Gauss-Bonnet formula

(GB) $$\sum_{v\in int\mathbf{T}} (2\pi - \theta_\rho(v)) = 2\pi - \sum_{w\in bd\mathbf{T}} (\pi - \gamma_\rho(w)).$$

Finally, we would like to mention some monotonicity properties of $\theta_\rho(\cdot)$. Suppose ρ', $\rho : V(\mathbf{T}) \mapsto (0,\infty)$ are such that $\rho'(v_0) < \rho(v_0)$ for some $v_0 \in int\,\mathbf{T}$ and $\rho' \equiv \rho$ in $V(\mathbf{T}) \setminus \{v_0\}$. Let v_0, u_1, u_2 be the vertices of a face in $\mathbf{T}$. Then $\alpha_{\rho'}\big(v_0, \triangle(v_0,u_1,u_2)\big) > \alpha_\rho\big(v_0, \triangle(v_0,u_1,u_2)\big)$, and $\alpha_{\rho'}\big(u_j, \triangle(v_0,u_1,u_2)\big) < \alpha_\rho\big(u_j, \triangle(v_0,u_1,u_2)\big)$ for $j = 1,2$. In particular, $\theta_\rho(v_0) < \theta_{\rho'}(v_0)$ and $\theta_\rho(v) > \theta_{\rho'}(v)$ for $v \in int\,\mathbf{T} \setminus \{v_0\}$. Moreover, if maps $\rho_n : V(\mathbf{T}) \mapsto (0,\infty)$ are such that $\lim_{n\to\infty} \rho_n(v_0) = 0$ ($\lim_{n\to\infty} \rho_n(v_0) = \infty$) and $\rho_n \equiv \rho$ in $V(\mathbf{T}) \setminus \{v_0\}$ then $\lim_{n\to\infty} \alpha_{\rho_n}\big(v_0, \triangle(v_0,u_1,u_2)\big) = \pi$ ($\lim_{n\to\infty} \alpha_{\rho_n}\big(v_0, \triangle(v_0,u_1,u_2)\big) = 0$).

Since our prime interest will be in euclidean circle packings, we have introduced radii and angle sums from the euclidean point of view, but the reader should be aware that similar constructions and properties as above can be achieved in the hyperbolic setting as well ([BSt2]).

Chapter II

Finite Branched Circle Packings

Let $\mathbf{T}$ be a finite triangulation of a disc. Suppose that $\{v_i\}_{i=1}^{N}$ and $\{w_i\}_{i=1}^{M}$ are the sets of interior and boundary vertices of $\mathbf{T}$, respectively. Let $b_1, \ldots, b_m$ be designated interior vertices of $\mathbf{T}$. Suppose that $k_1, \ldots, k_m$ are positive integers.

We want to construct a branched circle packing satisfying the following requirement:

$$(*) \qquad \begin{aligned} \theta(b_j) &= (1 + k_j)2\pi \quad \text{for } j = 1, \ldots, m, \text{ and} \\ \theta(v) &= 2\pi \quad \text{for } v \in int\,\mathbf{T} \setminus \{b_1, \ldots, b_m\}, \end{aligned}$$

where $\theta(v)$ denotes the angle sum at the vertex v determined by the collection of radii of circles from the packing. If there exists a circle packing P that satisfies $(*)$ then b_j will be called a branch vertex of **multiplicity** $1 + k_j$ (**order** k_j) and

$$br(P) = \{\underbrace{b_1, b_1, \ldots, b_1}_{k_1 \text{ times}}, \ldots, \underbrace{b_m, b_m, \ldots, b_m}_{k_m \text{ times}}\} \ \left(\text{or } \{(b_1, k_1), (b_2, k_2), \ldots, (b_m, k_m)\}\right)$$

will be called the **branch set** for P. If $k_j = 1$ then b_j will be called a **simple** branch vertex. If all branch vertices of P are simple then we will write shortly $br(P) = \{b_1, b_2, \ldots, b_m\}$.

Before we settle the existence question of branched circle packings we need the following

Definition 1. *A set $\mathfrak{B} = \{u_1, u_2, \ldots, u_k\}$ of interior vertices of $\mathbf{T}$, perhaps with repetitions, is called a branch structure for $\mathbf{T}$ if every simple closed edge-path γ in $\mathbf{T}$ has at least $2\ell + 3$ edges, where ℓ is the number of points from $\mathfrak{B}$ enclosed by γ (counting multiplicities).*

We are now ready to state

Theorem 2. *There exists an euclidean branched circle packing P for $\mathbf{T}$ with the branch set $\mathfrak{B} = \{(b_1, k_1), (b_2, k_2), \ldots, (b_m, k_m)\}$ if and only if $\mathfrak{B}$ is a branch structure for $\mathbf{T}$.*

Proof. We will first prove the existence part of the theorem. This will be done by introducing an algorithm for construction of a branched circle packing, and then by showing that such an algorithm really works.

Suppose $\mathfrak{B}$ is a branch structure for $\mathbf{T}$. Let A be any unbranched circle packing for $\mathbf{T}$ (e.g. an Andreev packing). Write $r(v)$ for the (euclidean) radius of the circle in A associated with the interior vertex v, and write $R(w)$ for the (euclidean) radius of the circle in A associated with the boundary vertex w. Let $\Phi(v)$ be the angle sum at v in A, i.e. $\Phi(v)$ is the angle sum at v $(= 2\pi)$ determined by the collection of radii $\{r(v_i)\}_{i=1}^N \bigcup \{R(w_i)\}_{i=1}^M$.

We now proceed to describe an algorithm for construction of a branched circle packing determined by $\mathbf{T}$ and the branch set $\mathfrak{B}$. The algorithm starts with the radii of A, proceeds through a sequence of loops and their outputs, and ends with a collection of limit radii. In each loop we will successively adjust radii of all interior circles, and the results of adjustments will be called the output of the loop. The boundary radii $R(w_i)$ will remain unchanged throughout. The loops are going to be defined inductively as follows.

Loop(0): We set $r_0(v_i) := r(v_i)$ for $i = 1, \ldots, N$.

Output(0): The collection of radii $\{r_0(v_i)\}_{i=1}^N$.

Suppose that Loop(n) has been defined and produced the output $\{r_n(v_i)\}_{i=1}^N$. We are now going to describe Loop($n+1$).

Loop($n+1$): Let us start with the vertex v_1. Since $\mathfrak{B}$ is a branch structure for $\mathbf{T}$, v_1 has at least $1 + \theta(v_1)/\pi$ neighboring vertices. Thus, we can adjust, if necessary, $r_n(v_1)$ to a value $r_{n+1}(v_1)$ such that the angle sum at v_1 determined by $\{r_{n+1}(v_1), r_n(v_2), \ldots, r_n(v_N)\} \bigcup \{R(w_i)\}_{i=1}^{M}$ is equal to $\theta(v_1)$.

Next, we consider the vertex v_2. Since $\mathfrak{B}$ is a branch structure for $\mathbf{T}$, v_2 has at least $1 + \theta(v_2)/\pi$ neighboring vertices. Therefore, we can adjust, if necessary, $r_n(v_2)$ to a value $r_{n+1}(v_2)$ such that the angle sum at v_2 determined by $\{r_{n+1}(v_1), r_{n+1}(v_2), r_n(v_3), \ldots, r_n(v_N)\} \bigcup \{R(w_i)\}_{i=1}^{M}$ is equal to $\theta(v_2)$.

We continue doing these adjustments to and including v_N.

Output($n+1$): The collection of radii $\{r_{n+1}(v_i)\}_{i=1}^{N}$.

Finally, we complete the description of the algorithm by defining the *final output* as the collection of radii $\{\mathcal{R}(v_i)\}_{i=1}^{N}$, where $\mathcal{R}(v_i) = \liminf_{n\to\infty} r_n(v_i)$.

In what will follow we shall show that the collection of radii $\{\mathcal{R}(v_i)\}_{i=1}^{N} \bigcup \{R(w_i)\}_{i=1}^{M}$ determines the desired branched circle packing.

We need to make a few comments about the loops and their outputs. First, unless $\mathbf{T}$ has only one interior vertex, the output of Loop($n+1$) is never equal to the output of Loop(n). In the case when $\mathbf{T}$ has only one interior vertex, Output(n) = Output(1) for any $n > 1$.

Second, if $\Phi_n(v_i)$ denotes the angle sum at the vertex v_i determined by the collection of radii $\{r_n(v_1), \ldots, r_n(v_i), r_{n-1}(v_{i+1}), \ldots, r_{n-1}(v_N)\} \bigcup \{R(w_i)\}_{i=1}^{M}$ then $\Phi_n(v_i) = \theta(v_i)$.

Third, since $\Phi(v_i) \leq \theta(v_i)$ for $i = 1, \ldots, N$, the monotonicity features of angle sums imply the following two properties:

1) $\{r_n(v)\}_{n=1}^{\infty}$ is a non-increasing sequence of positive numbers for any $v \in$ *int* $\mathbf{T}$,

2) if $\theta_n(v)$ denotes the angle sum at v given by $\{r_n(v_i)\}_{i=1}^{N} \bigcup \{R(w_i)\}_{i=1}^{M}$ then $\theta_n(v) \leq \theta(v)$ for any n and $v \in$ *int* $\mathbf{T}$.

From property 1) we get that $\lim_{n\to\infty} r_n(v)$ exists for any vertex v, so $\mathcal{R}(v) = \lim_{n\to\infty} r_n(v)$, though a priori $\mathcal{R}(v)$ may be zero. We also observe

Remark 3. *If $\mathcal{R}(v) > 0$ then $\{\theta_n(v)\}_{n=1}^{\infty}$ converges to $\theta(v)$ (even if $\mathcal{R}(v_j) = 0$ for some (or all) neighboring vertices v_j of v).*

Proof. Let $u_1, u_2, \ldots, u_l$ be all neighboring vertices of v in $\mathbf{T}$ listed in the positive order as the boundary vertices of the star of v in $\mathbf{T}$. Recall that if $\rho : V(\mathbf{T}) \mapsto (0,\infty)$ then $\theta_\rho(v)$ denotes the angle sum at v determined by ρ. It follows easily from the definition of $\theta_\rho(\cdot)$ that $\theta_\rho(v)$ depends on the values of ρ at v and its neighbors $u_1, u_2, \ldots, u_l$. More precisely, $\theta_\rho(v) = \theta_\rho\big(\rho(u_1), \ldots, \rho(u_l); \rho(v)\big)$, where

$$\theta_\rho\big(\rho(u_1), \ldots, \rho(u_l); \rho(v)\big) =$$
$$\sum_{j=1 \bmod l}^{l} \arccos\left[\frac{\big(\rho(v)+\rho(u_j)\big)^2 + \big(\rho(v)+\rho(u_{j+1})\big)^2 - \big(\rho(u_j)+\rho(u_{j+1}\big)^2}{2\big(\rho(v)+\rho(u_j)\big)\big(\rho(v)+\rho(u_{j+1})\big)}\right].$$

From this formula we see that for any $\delta > 0$, $\theta_\rho(v)$ is continuous as a function of ρ for ρ in $\underbrace{[0,\infty)\times\cdots\times[0,\infty)}_{l \text{ times}}\times[\delta,\infty)$, and is uniformly continuous in any compact subset of the above product.

Let $\lambda := \max\{r(v_j), R(w_j) : v_j, w_j \in \{u_1, \ldots, u_l\}\}$. We observe that each argument of the angle function $\theta_n(v)$ is at least as big as the corresponding argument of the angle function $\Phi_{n+1}(v)$ but not bigger than the corresponding argument of $\Phi_n(v)$. Thus, if $\mathcal{R}(v) > 0$ then $\theta_\rho(v)$ is uniformly continuous for ρ in $[0,\lambda]^l \times [\mathcal{R}(v)/2, r(v)]$ and, by property 1), one has

$$\theta(v) \equiv \Phi_n(v) = \lim_{n\to\infty} \Phi_n(v) = \lim_{n\to\infty} \theta_n(v);$$

that is, the limit of $\{\theta_n(v)\}_{n=1}^{\infty}$ exists and is equal to $\theta(v)$. $\square$

Since $\mathcal{R}(v)$ may be zero, we have to consider two cases.

CASE 1. $\mathcal{R}(v) > 0$ *for all* $v \in int\,\mathbf{T}$.

It follows from Remark 3 that $\{\theta_n(v_i)\}_{i=1}^{\infty}$ converges to $\theta(v_i)$ for any i, and, as the $\theta(v_i)$ are multiples of 2π, the collection of radii $\{\mathcal{R}(v_i)\}_{i=1}^{N} \bigcup \{R(w_i)\}_{i=1}^{M}$ defines the sought after branched circle packing.

CASE 2. *There exists* i_0, $1 \le i_0 \le N$, *such that* $\mathcal{R}(v_{i_0}) = 0$.

We will show that this case is void.

We extend the definition of $r_n(\cdot)$ and $\mathcal{R}(\cdot)$ to the boundary vertices of $\mathbf{T}$ by $r_n(w_i) := R(w_i)$ and $\mathcal{R}(w_i) := \lim_{n\to\infty} r_n(w_i) = R(w_i)$. Now let us call a vertex u a **solid dot** if $\mathcal{R}(u) > 0$ and an **open dot** if $\mathcal{R}(u) = 0$. Let OD be the collection of all open dots and SD be the collection of all solid dots. Notice that all boundary vertices w_i are solid dots. Hence $SD \neq \emptyset$. Also, by hypothesis, $OD \neq \emptyset$ (because $v_{i_0} \in OD$).

We now need to introduce some technical notation which otherwise is quite natural. If X is a subcomplex of $\mathbf{T}$ (not necessarily a 2-complex) then $V(X)$ will denote set of **vertices** of X. We will call $v \in V(X)$ an **interior vertex** of X if v is an interior vertex of $\mathbf{T}$ and all neighboring vertices of v in $\mathbf{T}$ belong to $V(X)$. If $v \in V(X)$ is not interior then it will be called a **boundary vertex** of X. The sets of interior and boundary vertices of X will be denoted by $intX$ and bdX respectively. If γ is an edge-path in $\mathbf{T}$ and $X \subset V(\mathbf{T})$ then we will say that γ is **in** X if $V(\gamma) \subset X$.

If γ is a closed oriented edge-path in $\mathbf{T}$, possibly with self-intersections, then the set $\{u_1, \ldots, u_l\}$ of **consecutive vertices** of γ will be denoted by $V_c(\gamma)$ and u_{j+1} will be called the **successor** of u_j (j mod l, of course). We observe that $V(V_c(\gamma)) = V(\gamma) \subseteq V_c(\gamma)$ with equality if and only if γ is a simple closed path (i.e. $V_c(\gamma)$ has exactly the same elements as $V(\gamma)$ but they may appear with repetitions in $V_c(\gamma)$ if γ has self-intersections). Finally, $\mathbf{u_{j-1}u_ju_{j+1}}$ will denote an oriented path given by the part of γ between u_{j-1} and u_{j+1} that contains u_j.

A subset $X \subset V(\mathbf{T})$ is called **connected** if any two vertices in X can be joined by an edge-path in X; it is **simply connected** if there is no closed edge-path in X which encloses an element from $V(\mathbf{T}) \setminus X$. If $X \subset V(\mathbf{T})$ then $hull(X)$ will denote the **combinatorial hull** of X, i.e. a union of the set X and the collection of all vertices in $V(\mathbf{T})$ such that each vertex (from the collection) is enclosed by some edge-path in X.

The next few definitions are only for subsets of $int\mathbf{T}$. If $X \subset int\mathbf{T}$ then the **outer boundary** $outX$ of X is the set of all vertices in $\mathbf{T}$ which are not in $hull(X)$ but which have a neighboring vertex in $hull(X)$. A subset $X \subset int\mathbf{T}$ is said to be **bounded** by an edge-path γ (or γ bounds X) if $hull(X)$ is connected and γ is the boundary curve of the union $U(X)$ of open stars associated with the vertices of $hull(X)$; in such case, γ is closed and carries an orientation induced by the orientation of the set $U(X)$ (with respect to which $U(X)$ stays to the left of γ). We notice that any subset $X \subset int\mathbf{T}$ with connected $hull(X)$ is bounded by some edge-path that in fact encloses X and, in general, such a path may have vertices or even edges of self-tangency (but not transverse). If X is bounded by a path γ then we will write $\partial_e X$ for γ and $V_c(X)$ for $V_c(\partial_e X)$, and we observe that $V(\partial_e X) = outX$ (Figure 2). If a subset $X \subset int\mathbf{T}$ is simply connected then $\big(bd(V(\mathbf{T}) \setminus X) \setminus bd\,\mathbf{T}\big) \cup \big(bd\,\mathbf{T} \cap outX\big) = outX$.

Let C_{OD} and C_{SD} denote components of OD and SD respectively. We say that an edge-path γ in $\mathbf{T}$ is a path of open (solid) dots if $V(\gamma) \subset OD$ $(V(\gamma) \subset SD)$. We also remark that C_{OD} (C_{SD}) is simply connected if and only if there is no closed edge-path in C_{OD} (C_{SD}) that encloses at least one point from OD (SD).

We will now state a lemma; only the first four parts, (i)–(iv), will be needed

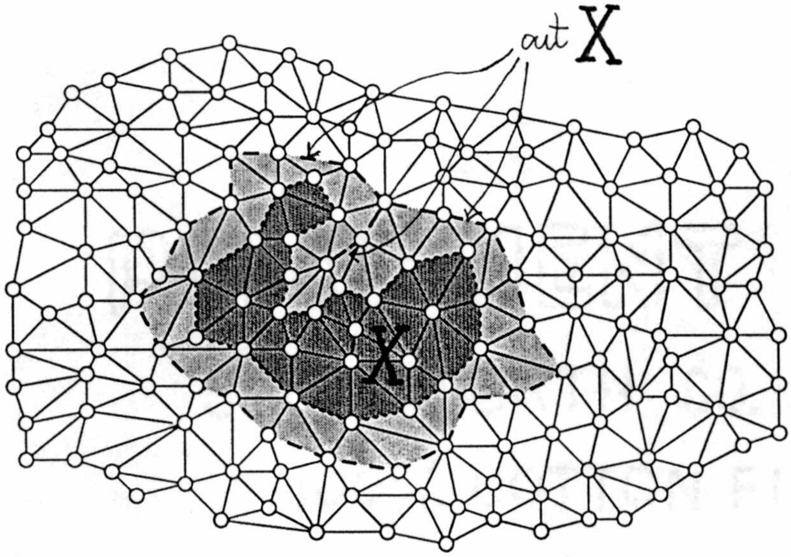

FIGURE 2. A simply connected set of vertices X and its outer boundary $out\, X$

to prove Theorem 2, the other parts, (v)–(vii), are stated for completeness and will be used later.

Lemma 4. *Under the assumption that* $\{(b_1, k_1), \ldots, (b_m, k_m)\}$ *is a branch structure for the finite triangulation* $\mathbf{T}$ *and that sets* OD *and* SD *are defined for the function* $\mathcal{R}$ *as above, the following hold:*

(i) *There is no simply connected component* C_{OD} *of* OD*;*

(ii) *There is no simply connected component* C_{SD} *of* SD *except for* $V(\mathbf{T})$*;*

(iii) *There is no subset* $S \subset int\mathbf{T}$ *that contains at least one solid dot and is bounded by a path of open dots;*

(iv) *There is no subset* $S \subset int\mathbf{T}$ *that contains at least one open dot and is bounded by a path of solid dots;*

(v) *There is no subset* $S \subset SD \cap int\mathbf{T}$ *so that* S *is simply connected and* $outS$ *consists only of open dots except one solid dot;*

(vi) *There is no connected subset S of $SD \cap int\mathbf{T}$ with $outS$ having only one solid dot;*

(vii) *There is no connected subset S in $int\mathbf{T}$ so that S contains at least one solid dot and $outS$ has only one solid dot.*

Proof. (i). Suppose C_{OD} is a simply connected component of OD. Since $bd\mathbf{T} \subset SD$, $C_{OD} \subset int\mathbf{T}$ and $\partial_e C_{OD}$ is in SD (Figure 3).

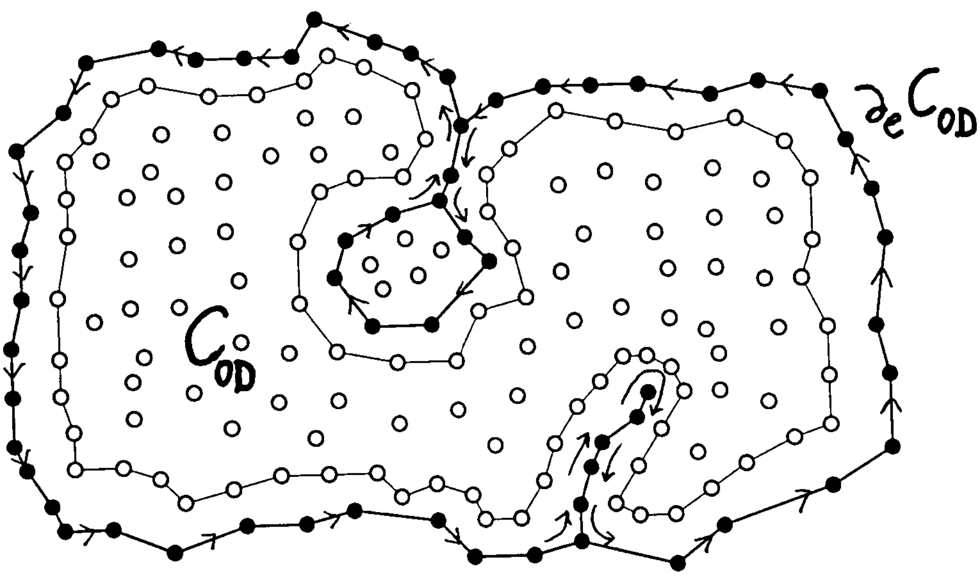

FIGURE 3. A set C_{OD} and a path $\partial_e C_{OD}$ with its orientation indicated by arrows

For three consecutive vertices u_{j-1}, u_j, u_{j+1} on $\partial_e C_{OD}$ let S_j denote the part of the star of u_j that stays to the left of the path $u_{j-1}u_ju_{j+1}$ (Figure 4).

Let $r_n : V(\mathbf{T}) \mapsto (0, \infty)$ be defined as above, and let $\theta_n(\cdot)$, $\alpha_n(\cdot,\cdot)$ denote $\theta_{r_n}(\cdot)$, $\alpha_{r_n}(\cdot,\cdot)$, respectively (see the notation for (GB)). Let $\gamma_n(u_j) := \sum_{\triangle \in S_j} \alpha_n(u_j, \triangle)$, i.e. $\gamma_n(u_j)$ is an interior angle with respect to $u_{j-1}u_ju_{j+1} \subset \partial_e C_{OD}$ at $u_j \in V_c(C_{OD})$ determined by $r_n(\cdot)$. Since each triangle $\triangle \in S_j$ has a

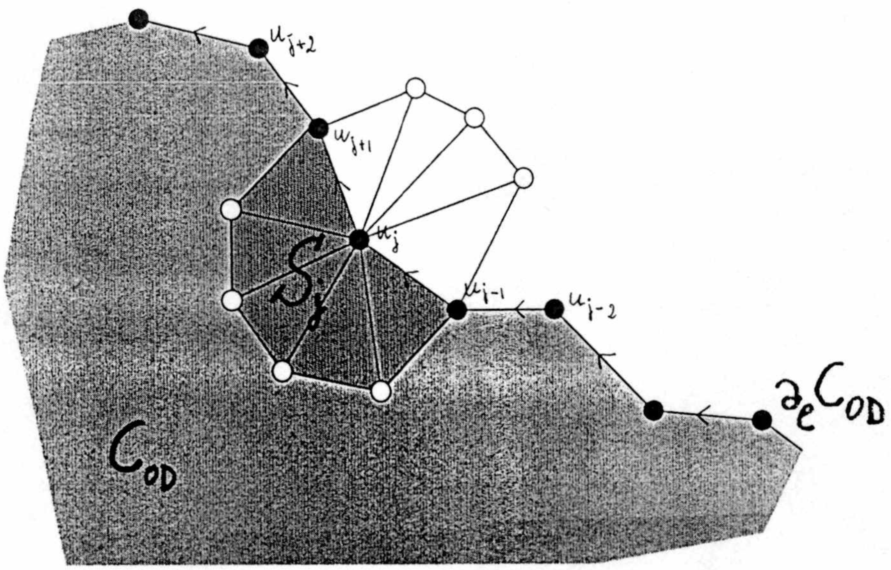

FIGURE 4. The set S_j for C_{OD}

vertex in C_{OD}, we have that $\lim_{n\to\infty}\gamma_n(u_j)=0$. Also, by 2), $\theta_n(v_i)\le\theta(v_i)$ for any $v_i\in C_{OD}$. By (GB)

$$(**)\qquad \sum_{v_i\in C_{OD}}(2\pi-\theta_n(v_i)) = 2\pi - \sum_{u_j\in V_c(C_{OD})}(\pi-\gamma_n(u_j)).$$

Let $b_{i_1},\ldots,b_{i_L}$ be all b_i's that are in C_{OD}. The hypothesis (i.e. $\{(b_1,k_1),\ldots,(b_m,k_m)\}$ is a branch structure for $\mathbf{T}$) implies that the number of u_j's in $V_c(C_{OD})$ is at least $2(\sum_{j=1}^{L}k_{i_j}+1)+1$. Thus

$$(**)_1\quad \lim_{n\to\infty}\left[2\pi-\sum_{u_j\in V_c(C_{OD})}(\pi-\gamma_n(u_j))\right] = 2\pi-\sum_{u_j\in V_c(C_{OD})}\pi \le$$
$$2\pi-\left[2(\sum_{j=1}^{L}k_{i_j}+1)+1\right]\pi = -\pi-2(\sum_{j=1}^{L}k_{i_j})\pi.$$

On the other hand

$$(**)_2\ \sum_{v_i\in C_{OD}}(2\pi-\theta_n(v_i)) \ge \sum_{v_i\in C_{OD}}(2\pi-\theta(v_i)) = \sum_{j=1}^{L}(2\pi-\theta(b_{i_j})) = -\sum_{j=1}^{L}2k_{i_j}\pi$$

for any n.

Comparing $(**)_1$ and $(**)_2$ one obtains a contradiction.

(ii). Suppose there exists $C_{SD} \neq V(\mathbf{T})$ which is simply connected. Then $bd\mathbf{T} \cap C_{SD} = \emptyset$, for otherwise the edge-path of boundary edges of $\mathbf{T}$ is a closed path in C_{SD} that encloses some $v_{i_0} \in OD$, implying that C_{SD} not simply connected. Hence $C_{SD} \subset int\mathbf{T}$ and $\partial_e C_{SD}$ is in OD.

For three consecutive (open) dots u_{j-1}, u_j, u_{j+1} on $\partial_e C_{SD}$ let S_j denote the part of the star of u_j that stays to the left of the path $u_{j-1}u_ju_{j+1}$ (Figure 5).

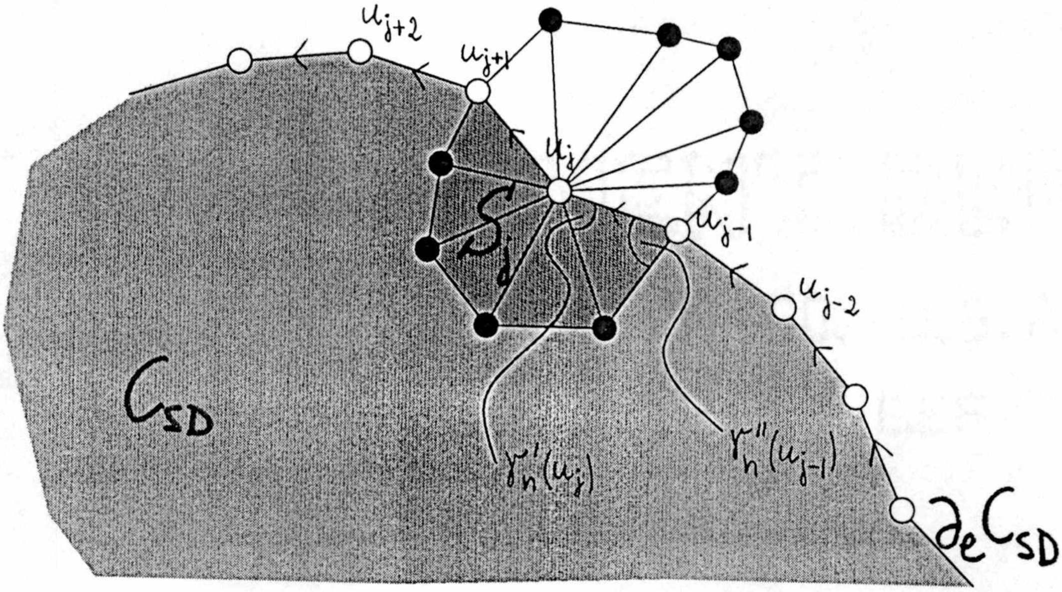

FIGURE 5. The set S_j for C_{SD}

Let $\gamma_n(u_j) := \sum_{\triangle \in S_j} \alpha_n(u_j, \triangle)$ (notation as in the proof of Part (i)), i.e. $\gamma_n(u_j)$ is an interior angle with respect to $u_{j-1}u_ju_{j+1} \subset \partial_e C_{SD}$ at $u_j \in V_c(C_{SD})$ determined by $r_n(\cdot)$. Since each triangle in S_j has a vertex in C_{SD}, for each $u_j \in \partial_e C_{SD}$ there exists $v^j \in C_{SD}$ such that v^j, u_{j-1}, u_j are the vertices of a triangle in S_j. Let $\gamma_n'(u_j) := \alpha_n(u_j, \triangle(v^j, u_j, u_{j-1}))$ and $\gamma_n''(u_{j-1}) := \alpha_n(u_{j-1}, \triangle(v^j, u_j, u_{j-1}))$, i.e. $\gamma_n'(u_j)$ ($\gamma_n''(u_{j-1})$) is the part of $\gamma_n(u_j)$ "restricted to $\triangle(v^j, u_j, u_{j-1})$" (the part of $\gamma_n(u_{j-1})$ "restricted to $\triangle(v^j, u_j, u_{j-1})$"). Since

$\mathcal{R}(u_{j-1}) = \mathcal{R}(u_j) = 0$ and $\mathcal{R}(v^j) > 0$, $\lim_{n\to\infty}(\gamma_n''(u_{j-1}) + \gamma_n'(u_j)) = \pi$. From this remark it follows that

$$(***) \qquad \lim_{n\to\infty}\left[\sum_{u_j \in V_c(C_{SD})} (\pi - \gamma_n(u_j))\right] = A, \text{ where } A \in (-\infty, 0].$$

Now, if $v_i \in C_{SD}$ then $\lim_{n\to\infty}\theta_n(v_i) = \theta(v_i)$ by Remark 3.

Since $\theta(v_i) \geq 2\pi$, we have

$$(***)_1 \qquad \lim_{n\to\infty}\left[\sum_{v_i \in C_{SD}} (2\pi - \theta_n(v_i))\right] = -\sum_{j=1}^{L} 2k_{i_j}\pi,$$

where $b_{i_1}, \ldots, b_{i_L}$ are all b_i's that are in C_{SD}. By (GB)

$$0 \geq -\sum_{j=1}^{L} 2k_{i_j}\pi = \lim_{n\to\infty}\left[\sum_{v_i \in C_{SD}} (2\pi - \theta_n(v_i))\right] =$$

$$\lim_{n\to\infty}\left[2\pi - \sum_{u_j \in V_c(C_{SD})} (\pi - \gamma_n(u_j))\right] = 2\pi - A \geq 2\pi,$$

and one gets a contradiction.

(iii). Suppose there is such a subset S. Let S_{SD} be a component in $hull(S)$ of solid dots. We notice that S_{SD} is one of the C_{SD}'s because $outS = out(hull(S)) \subset OD$. By Part (ii), S_{SD} is not simply connected. Hence there is a path in S_{SD} that encloses at least one open dot in $hull(S)$. Let S_{OD} be the component in $hull(S)$ of open dots that contains the above open dot. Since S_{OD} is enclosed by a path in S_{SD}, S_{OD} is one of the C_{OD}'s, and, by Part (i), is not simply connected. We now apply to S_{OD} the same type of argument as we did above to S_{SD}, and by continuing this process we will eventually end up with a simply connected component of one type of dots. The last yields a contradiction to Part (i) or (ii).

(iv). This part follows from the same arguments that were used in the proof of Part (iii).

(v). Suppose that S exists. Let us denote the only solid vertex of $outS$ by u'. Then u' can not separate $outS$, i.e. $outS \setminus \{u'\}$ must be connected. For if u' separates $outS$ then there is a closed path of solid dots (including u') that bounds a region in $int\mathrm{T}$ which contains one of the components of $outS \setminus \{u'\}$. But by Part (iv) that is impossible.

Hence u' does not separate $outS$ and therefore there exists unique $u_{j_0} \in V_c(S)$ such that $u' = u_{j_0}$, i.e. u_{j_0} is the only solid dot in $V_c(S)$. Moreover, if S_{j_0} is the part of the star of u_{j_0} that is in S, then S_{j_0} consists of solid dots only (Figure 6).

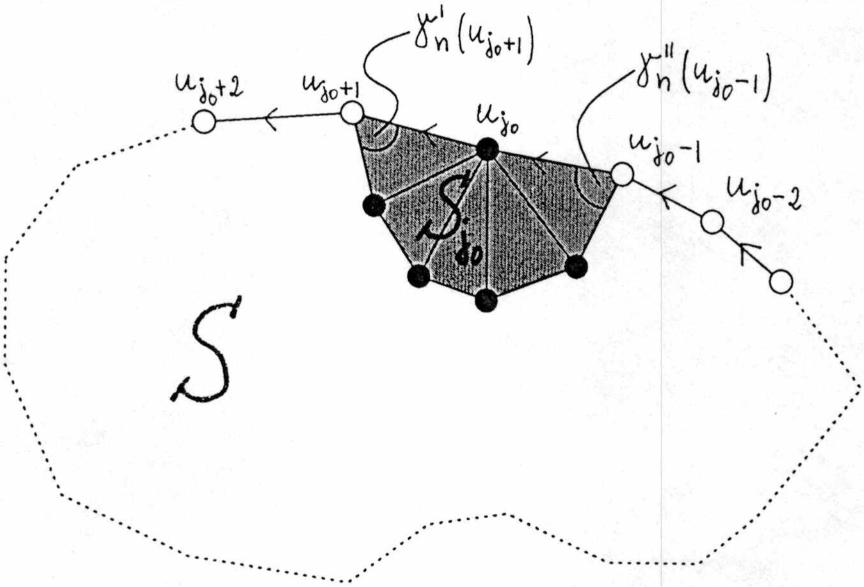

FIGURE 6. The sets S and S_{j_0}

Using the notation as in the proof of Part (ii), we easily obtain from the definition of solid and open dots that $\lim_{n\to\infty} \gamma_n''(u_{j_0-1}) = \pi = \lim_{n\to\infty} \gamma_n'(u_{j_0+1})$. Applying the arguments from the proof of Part (ii), the above observation, and the fact that u_{j_0} is the only solid dot in $V_c(S)$, we conclude that

$$(1) \qquad \lim_{n\to\infty} \left[\sum_{u_j \in V_c(S)\setminus\{u_{j_0}\}} (\pi - \gamma_n(u_j)) \right] = C, \text{ where } C \in (-\infty, 0].$$

Since $\gamma_n(u_{j_0}) \geq 0$, from (1) we have

$$(2) \qquad \sum_{u_j \in V_c(S)} (\pi - \gamma_n(u_j)) \leq 3/2\pi \text{ for big } n.$$

As in the proof of Part (ii)

$$(3) \qquad \lim_{n\to\infty} \left[\sum_{v_i \in S} (2\pi - \theta_n(v_i)) \right] = -\sum_{j=1}^{L} 2k_{i_j}\pi,$$

where $b_{i_1}, \dots, b_{i_L}$ are all b_i's that are in S. Using (2), (3) and (GB)

$$0 \geq \sum_{v_i \in S} (2\pi - \theta_n(v_i)) = 2\pi - \sum_{u_j \in V_c(S)} (\pi - \gamma_n(u_j)) \geq \pi/2 \text{ for sufficiently large } n,$$

but this is a contradiction.

(vi). Suppose that S exists. Then, by Part (v), S can not be simply connected, and we have only two possibilities. The first one is that there exists a closed edge-path in $S \subset SD$ that encloses an element from OD, but this contradicts Part (iv). Hence we are left with the second possibility which is: $hull(S) \subset SD$. Since $hull(S)$ is simply connected and $outS = out(hull(S))$, we get a contradiction to Part (v).

(vii). Suppose that S exists. Let us take the component C of solid dots in $S \cup outS$ that contains the only boundary solid dot $u' \in outS$. If $C = \{u'\}$ then there is a region which is bounded by a path of only open dots in $out\,(S \cup (outS \setminus \{u'\})) \subset S \cup outS$ and which contains at least one solid dot from S. But this contradicts Part (iii).

If $C \neq \{u'\}$ then Part (vi) applies to $C \setminus \{u'\}$, and we again get a contradiction. $\square$

Let us now finish the proof of Theorem 2.

Part (iv) of Lemma 4 says that our assumption that $\mathcal{R}(v_{i_0}) = 0$ is false. Hence we are in Case 1 and one implication in our theorem is proved, i.e. the existence.

The other implication (i.e. the necessity) follows easily from (GB), and this completes the proof of Theorem 2. □

Before we go on, we observe that using (GB) and the arguments as in the proof of Lemma 4 (even for $\{b_1, \ldots, b_m\} = \emptyset$) one easily verifies

Remark 5. *Let $\{P_n\}_{n=1}^{\infty}$ be a sequence of branched circle packings associated with* $\mathbf{T}$ *and having identical branch sets. Write $r_n : V(\mathbf{T}) \mapsto (0, \infty)$ for the radius function of P_n. Suppose that $\mathcal{R}(u) := \lim_{n\to\infty} r_n(u) \in [0, \infty]$ exists for any $u \in V(\mathbf{T})$. We say that a vertex u is a* **solid dot** *if $\mathcal{R}(u) \in (0, \infty]$, otherwise u is called an* **open dot**. *Let SD and OD denote the sets of solid and open dots respectively. Using the terminology established in the proof of Theorem 2, the conclusions (iii)–(vii) of Lemma 4 follow. Moreover, if $bd\mathbf{T} \subset SD$ then the conclusions (i) and (ii) of Lemma 4 hold as well.*

We will now state a theorem that addresses the boundary value problem for branched circle packings in $\mathbf{R}^2$ and in the hyperbolic plane; i.e., having been given geometry and prescribed radii for boundary vertices, can one construct a packing in the given geometry with boundary circles having those prescribed radii?

Theorem 6. *Let* $\mathbf{T}$ *be a finite triangulation of a disc. Suppose that $\mathfrak{B} = \{(b_1, k_1), \ldots, (b_m, k_m)\}$ is a branch structure for* $\mathbf{T}$. *Then, in either euclidean or hyperbolic geometry, there exists a branched circle packing determined by* $\mathbf{T}$ *and $\mathfrak{B}$ with any prescribed boundary radii. Moreover, such a packing is unique up to isometries in the corresponding geometry.*

Proof. Euclidean Case. Let $\widetilde{P}$ be an unbranched circle packing for $\mathbf{T}$ that has prescribed boundary radii. Such a packing exists by [CR]. We now replace $\mathcal{A}$ in the proof of Theorem 2 by $\widetilde{P}$, and we follow the method of construction of a branched circle packing in that proof. We observe that this construction does not change the boundary radii of $\widetilde{P}$. Hence we obtain the desired branched circle packing with prescribed boundary radii.

Regarding uniqueness, suppose P_1 and P_2 are two branched circle packings determined by $\mathfrak{B}$ and $\mathbf{T}$ having the same boundary radii. Let $\mathcal{R}_i(\cdot) : V(\mathbf{T}) \mapsto (0, \infty)$ be the radius function for P_i, and define $\frac{\mathcal{R}_1}{\mathcal{R}_2}(\cdot) : V(\mathbf{T}) \mapsto (0, \infty)$ by $\frac{\mathcal{R}_1}{\mathcal{R}_2}(u) = \mathcal{R}_1(u)/\mathcal{R}_2(u)$. Since P_1 and P_2 have the same angle sums at corresponding vertices, one easily verifies that $\frac{\mathcal{R}_1}{\mathcal{R}_2}(\cdot)$ satisfies the maximum and minimum principle. This implies that $\mathcal{R}_1(\cdot) \equiv \mathcal{R}_2(\cdot)$, as $\frac{\mathcal{R}_1}{\mathcal{R}_2}(\cdot)$ restricted to $bd\mathbf{T}$ is equal to 1.

Hyperbolic Case. Theorem 2 gives the existence of a branched circle packing determined by $\mathfrak{B}$ and $\mathbf{T}$ in the hyperbolic plane (one just takes any euclidean branched circle packing determined by $\mathfrak{B}$ and $\mathbf{T}$, scales it down so it will be contained in the unit disc). Using this information together with some straightforward generalizations of the techniques used in [BSt1] for the unbranched setting, like the Perron method, one easily obtains the conclusion of the theorem in the hyperbolic setting. □

We finish this chapter with

Definition 7. *Let $\mathbf{T}$ be a finite triangulation of a disc. Suppose $\mathfrak{B}$ is a branch structure for $\mathbf{T}$. Let P be a branched circle packing in the unit disc determined by $\mathbf{T}$ and $\mathfrak{B}$ with all boundary circles being horocycles. Then P will be called a Bl-type packing determined by $\mathbf{T}$ and $\mathfrak{B}$.*

CHAPTER III

EXTENSIONS OF SIMPLICIAL MAPS

Suppose that $\mathfrak{B} = \{(b_1, k_1), \ldots, (b_m, k_m)\}$ is a branch structure for a finite triangulation $\mathbf{T}$ of a disc. Let A be an (orientation preserving) Andreev packing for $\mathbf{T}$, and let B be a (orientation preserving) Bl-type packing determined by $\mathbf{T}$ and $\mathfrak{B}$. Write f for the simplicial map from A to B. Our goal in this chapter is to show that one can think about f as a discrete Blaschke product.

As in the previous chapter, let $\{w_i\}$ denote the set of boundary vertices of $\mathbf{T}$. Assume that the w_i's are indexed so that they induce counterclockwise orientation of the boundary of $\mathbf{T}$. For any two points a and b in the unit circle ∂D, $\wr a, b \wr$ will represent the oriented arc of ∂D from a to b, where the orientation of ∂D is counterclockwise. Also, for any two points $a, b \in \mathbf{C}$, we will write $[a, b]$ for the segment in $\mathbf{C}$ with the ends a and b. For a vertex $u \in V(\mathbf{T})$, $C_A(u)$ and $C_B(u)$ will denote the circles that correspond to u in A and B, respectively; $\widehat{u}$ will denote the center of $C_A(u)$, so that $f(\widehat{u})$ is the center of $C_B(u)$.

Suppose that a, b, and c are the points of tangency of $C_B(w_2)$ with ∂D, $C_B(w_3)$, and $C_B(w_1)$, respectively. Let all circles in A and B have counterclockwise orientation. Since B is orientation preserving one has

Lemma 1. *The ordered triple $< a, b, c >$ has the order that agrees with the orientation of $C_B(w_2)$, i.e. starting at a and going along the circle $C_B(w_2)$ according to its orientation first we encounter b then c.*

Proof. Let $w_3, u_1, \ldots, u_l, w_1$ be a listing of neighboring vertices of w_2 in counterclockwise order. Notice that segments

$$[f(\widehat{w}_3), f(\widehat{u}_1)],\ [f(\widehat{u}_1), f(\widehat{u}_2)], \ldots,\ [f(\widehat{u}_{l-1}), f(\widehat{u}_l)],\ [f(\widehat{u}_l), f(\widehat{w}_1)]$$

are in D but exterior to $C_B(w_2)$.

For $x, y \in D$ let $L(x,y)$ be the half line $\{t(y-x)+x : t \geq 0\}$. Let $w_3'' = \partial D \cap L(f(\widehat{w}_2), f(\widehat{w}_3))$ then, since B is orientation preserving, $f(\widehat{u}_1)$ is in the region bounded by $[a, f(\widehat{w}_2)]$, $[f(\widehat{w}_2), w_3'']$, and $\wr w_3'', a \wr \subset \partial D$ (Figure 7).

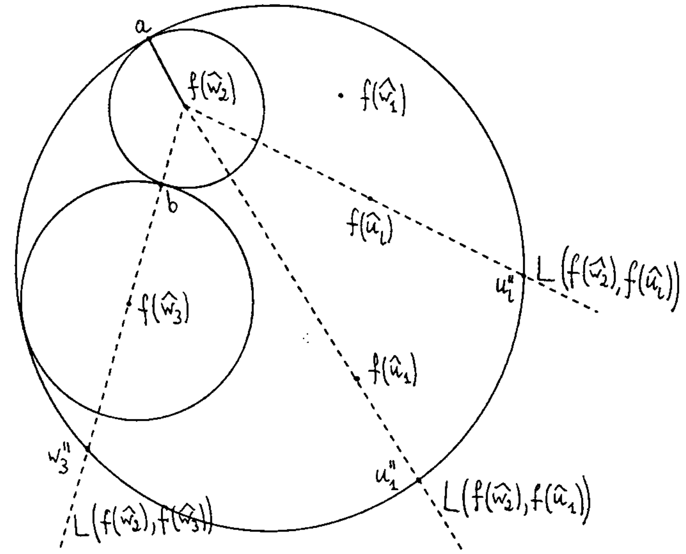

FIGURE 7. Points $f(\widehat{u}_j)$

Now, since B is orientation preserving, $f(\widehat{u}_2)$ lies in the region bounded by $[a, f(\widehat{w}_2)]$, $[f(\widehat{w}_2), u_1'']$, and $\wr u_1'', a\wr$, where $u_1'' = \partial D \cap L(f(\widehat{w}_2), f(\widehat{u}_1))$. Next, as above, $f(\widehat{u}_3)$ is in the region bounded by $[a, f(\widehat{w}_2)]$, $[f(\widehat{w}_2), u_2'']$, and $\wr u_2'', a\wr$, where $u_2'' = \partial D \cap L(f(\widehat{w}_2), f(\widehat{u}_2))$. Continuing this process one obtains that $f(\widehat{w}_1)$ is in the region bounded by $[a, f(\widehat{w}_2)]$, $[f(\widehat{w}_2), u_l'']$, and $\wr u_l'', a\wr$, where $u_l'' = \partial D \cap L(f(\widehat{w}_2), f(\widehat{u}_l))$. But $\{c\} = C_B(w_2) \cap [f(\widehat{w}_2), f(\widehat{w}_1)]$, hence $< a, b, c >$ has an order that agrees with the orientation of $C_B(w_2)$. $\square$

Remark 2. *Let $\overline{w}_i' = \partial D \cap C_B(w_i)$. Then, using Lemma 1, one easily verifies that $< \overline{w}_i', \overline{w}_{i+1}', \overline{w}_{i+2}' >$ has the order that agrees with the orientation of ∂D.*

We are now going to build an extension of f to a map from $\overline{D}$ onto $\overline{D}$ as follows:

Let $\overline{w}_i = \partial D \cap C_A(w_i)$. In each arc $\wr \overline{w}_i, \overline{w}_{i+1} \wr \subset \partial D$ pick a point x_i such that $[\widehat{w}_i, x_i] \cap C_A(w_{i+1}) = \emptyset = [\widehat{w}_{i+1}, x_i] \cap C_A(w_i)$. Similarly, in any arc $\wr \overline{w}'_i, \overline{w}'_{i+1} \wr \subset \partial D$ pick a point y_i such that $[f(\widehat{w}_i), y_i] \cap C_B(w_{i+1}) = \emptyset = [f(\widehat{w}_{i+1}), x_i] \cap C_B(w_i)$. Then we extend our map f in the following way:

1) Each closed triangle $\Delta x_i \widehat{w}_i \widehat{w}_{i+1}$ is mapped simplicially onto $\Delta y_i f(\widehat{w}_i) f(\widehat{w}_{i+1})$ by mapping $x_i, \widehat{w}_i, \widehat{w}_{i+1}$ onto $y_i, f(\widehat{w}_i), f(\widehat{w}_{i+1})$ respectively. Hence the extended map agrees with f on each segment $[\widehat{w}_i, \widehat{w}_{i+1}]$, so is well defined.

2) Next, each closed triangular region $\Delta(x_i x_{i+1} \widehat{w}_{i+1})$ (bounded by $[x_i, \widehat{w}_{i+1}]$, $[x_{i+1}, \widehat{w}_{i+1}]$ and $\wr x_i, x_{i+1} \wr \subset \partial D$) is mapped homeomorphically onto the closed triangular region $\Delta(y_i y_{i+1} f(\widehat{w}_{i+1}))$ so that:

- $x_i, x_{i+1}, \widehat{w}_{i+1}$ are mapped onto $y_i, y_{i+1}, f(\widehat{w}_{i+1})$ respectively,
- an arc $\wr x_i, x_{i+1} \wr$ is mapped onto $\wr y_i, y_{i+1} \wr$, and the homeomorphism agrees on segments $[x_i, \widehat{w}_{i+1}], [x_{i+1}, \widehat{w}_{i+1}]$ with the extension done in step 1).

This completes the construction of an extension of f.

Let us call the extension defined above $\bar{f}$. We observe that the "star" of $\widehat{w}_i$ (a union of: all faces in $carr(A)$ having $\widehat{w}_i$ as a vertex, two triangles $\Delta \widehat{w}_{i-1} \widehat{w}_i x_i$ and $\Delta \widehat{w}_{i+1} \widehat{w}_i x_i$, and a triangular region $\Delta(x_i x_{i+1} \widehat{w}_i)$) is mapped by $\bar{f}$ onto the "star" of $f(\widehat{w}_i)$ (a union of: all faces in $carr(B)$ having $f(\widehat{w}_i)$ as a vertex, two triangles $\Delta f(\widehat{w}_{i-1}) f(\widehat{w}_i) x_i$ and $\Delta f(\widehat{w}_{i+1}) f(\widehat{w}_i) x_i$, and a triangular region $\Delta(x_i x_{i+1} f(\widehat{w}_i))$) which contains $C_B(w_i)$. Also, from the definition of $\bar{f}$ and Lemma 1 it follows that $\bar{f}|_{\overline{D} \setminus carr(A)}$ is a local homeomorphism. Since $\bar{f}|_{carr(A) \setminus \{\widehat{b}_1, \ldots, \widehat{b}_m\}} = f|_{carr(A) \setminus \{\widehat{b}_1, \ldots, \widehat{b}_m\}}$ is also a local homeomorphism and the map $f|_{carr(A)\setminus\partial\left(carr(A)\right)}$ is an open map, we get that $\bar{f}|_{\overline{D}\setminus\{\widehat{b}_1,\ldots,\widehat{b}_m\}}$ is a local homeomorphism and $\bar{f}|_D$ is an open map. Using the definition of f and the fact that $\bar{f}|_{\overline{D}\setminus\{\widehat{b}_1,\ldots,\widehat{b}_m\}}$ is a local homeomorphism one easily verifies that f is light, i.e. every component of $\bar{f}^{-1}(\xi)$ is a single point for any $\xi \in \bar{f}(D)$. Thus, we have the following

Lemma 3. *$\bar{f} : \overline{D} \mapsto \overline{D}$ is a local homeomorphism in $\overline{D} \setminus \{\widehat{b}_1, \ldots, \widehat{b}_m\}$, $\bar{f}|_D$ is open and light, and $\bar{f}(\partial D) \subseteq \partial D$.*

Since, by Lemma 3, $\bar{f}$ restricted to D is continuous, open and light, Stoilow's Theorem (see [LV]) implies that $\bar{f}|_D = \varphi \circ h$, where h is a homeomorphism of D and φ is a holomorphic function in $h(D)$. A priori $h(D)$ may or may not be equal to $\mathbf{C}$. If $h(D) = \mathbf{C}$ then φ is an entire analytic function that is bounded (because $\bar{f} : \overline{D} \mapsto \overline{D}$). Hence φ is constant. But $\bar{f}$ is not constant — contradiction. Therefore $h(D) \neq \mathbf{C}$. Since $h(D)$ is simply connected, by the Riemann Mapping Theorem there exists a conformal map $g : h(D) \mapsto D$. Thus $\bar{f}|_D = \varphi_1 \circ h_1$, where $h_1 : D \mapsto D$ is a homeomorphism and $\varphi_1 : D \mapsto D$ is analytic.

Let $\{z_n\}_{n=1}^{\infty}$ be a sequence of points from D that converges to ∂D. Then $\{h_1^{-1}(z_n)\}$ converges to ∂D. Let $h_1^{-1}(z_n) = z'_n$. Since $\bar{f}$ is continuous in $\overline{D}$ and $\bar{f}(\partial D) \subseteq \partial D$, $\{\bar{f}(z'_n)\}$ converges to ∂D. Hence points $\varphi_1(z_n) = \bar{f}(z'_n)$ converge to ∂D. Therefore φ_1 is a proper analytic map from D into D, so φ_1 is a finite Blaschke product (please refer to the characterization of Blaschke products in Chapter V).

All that we can summarize in

Theorem 4. *Let $\bar{f}$ be an extended map from A to B described as above. Then $\bar{f}|_D = \varphi \circ h$, where h is a homeomorphism of D and φ is a finite Blaschke product having branch points $h(\widehat{b}_1), \ldots, h(\widehat{b}_m)$ with orders $k_1, \ldots, k_m$, respectively.*

Since φ is a local homeomorphism near ∂D and, by construction, $\bar{f}$ is also a local homeomorphism near ∂D we have

Fact 5. *h can be extended to the homeomorphism $\bar{h} : \overline{D} \mapsto \overline{D}$ such that $\bar{f} = \varphi \circ \bar{h}$ in $\overline{D}$.*

Theorem 4 justifies us to state the following

Definition 6. *Suppose that $\mathfrak{B} = \{(b_1, k_1), \ldots, (b_m, k_m)\}$ is a branch structure for a finite triangulation $\mathbf{T}$ of a disc. Let A be an Andreev circle packing for $\mathbf{T}$, and let B be a Bl-type circle packing determined by $\mathbf{T}$ and $\mathfrak{B}$. If f is the simplicial map from A to B then f will be called the **discrete Blaschke product** from A to B. Moreover, a map f will be called a discrete Blaschke product **induced** by B if f is the discrete Blaschke product from A to B for some A.*

We finish this chapter with a lemma which says that overlapping petals of a flower in a Bl-type packing must trap a branch value of any discrete Blaschke product induced by the packing.

Lemma 7. *Let $\mathbf{T}$ be a finite triangulation of a disk. Suppose B is a Bl-type circle packing determined by $\mathbf{T}$ and the branch set $\mathfrak{B} = \{(b_1, k_1), \ldots, (b_m, k_m)\}$. Write f for a discrete Blaschke product induced by B. Let v_0 be an interior vertex of $\mathbf{T}$. Suppose there are non-adjacent vertices v_1 and v_2, both neighboring v_0, such that $C(v_1) \cap C(v_2) \neq \emptyset$, where $C(v_j)$ is the circle in B associated with v_j, $j = 1, 2$. Then $f(\widehat{\mathfrak{B}}) \cap \Delta f(\widehat{v}_0) f(\widehat{v}_1) f(\widehat{v}_2) \neq \emptyset$, where $\widehat{v}$ denotes the vertex in the domain of f that corresponds to $v \in V(\mathbf{T})$, and $\widehat{\mathfrak{B}} = \{(\widehat{b}_1, k_1), \ldots, (\widehat{b}_m, k_m)\}$.*

Proof. Let $\bar{f} : D \to D$ be an extension of f described earlier. Choose $a \in C(v_1) \cap C(v_2) \cap [f(\widehat{v}_1), f(\widehat{v}_2)]$. Let a_1 and a_2 be the preimages of a under $\bar{f}$ in the star of $\widehat{v}_1$ and in the star of $\widehat{v}_2$ respectively. Suppose that γ is the curve built out of "segments" γ_1, $[\widehat{v}_1, \widehat{v}_0]$, $[\widehat{v}_0, \widehat{v}_2]$ and γ_2, where γ_1 is the component of $\bar{f}^{-1}\big([a, f(\widehat{v}_1)]\big)$ which contains a_1, and γ_2 is the component of $\bar{f}^{-1}\big([f(\widehat{v}_2), a]\big)$ which contains a_2. Then γ is a simple (**not closed**) curve such that $\bar{f}(\gamma) = \partial(\triangle f(\widehat{v}_0)f(\widehat{v}_1)f(\widehat{v}_2))$. By Theorem 4, $\bar{f} : D \setminus \bar{f}^{-1}(f(\widehat{\mathfrak{B}})) \to D \setminus f(\widehat{\mathfrak{B}})$ is a covering map. In particular, if $\mathcal{D} \subset D \setminus f(\widehat{\mathfrak{B}})$ is a closed topological disk then $\bar{f}^{-1}(\partial\mathcal{D})$ is a disjoint union of simple closed curves $\{\beta_1, \ldots, \beta_l\}$ such that $\bar{f}|\beta_j$ is a homeomorphism onto $\partial\mathcal{D}$ for $j = 1, \ldots, l$. Now, if $f(\widehat{\mathfrak{B}}) \cap \triangle f(\widehat{v}_0)f(\widehat{v}_1)f(\widehat{v}_2) = \emptyset$ then $\triangle f(\widehat{v}_0)f(\widehat{v}_1)f(\widehat{v}_2) \subset D \setminus f(\widehat{\mathfrak{B}})$ is a closed topological disk and γ is forced to be a simple closed curve, as its image under $\bar{f}$ is $\partial(\triangle f(\widehat{v}_0)f(\widehat{v}_1)f(\widehat{v}_2))$ – contradiction. $\square$

Chapter IV

Infinite Branched Circle Packings

Suppose that $\mathbf{T}$ is an infinite triangulation of a disc. Let $b_1, \ldots, b_m$ be interior vertices of $\mathbf{T}$, and let $k_1, \ldots, k_m$ be positive integers. Our main objective in this chapter is to give necessary and sufficient conditions for the existence of an infinite branched circle packing for $\mathbf{T}$ with the branch set $\mathfrak{B} = \{(b_1, k_1), \ldots, (b_m, k_m)\}$. We need to allow for the possibility of infinite branch sets, though our results concern finite ones.

Definition 1. *Let $\mathbf{T}$ be an infinite triangulation, with or without boundary, of a disc. A set $\mathfrak{B} = \{u_1, u_2, u_3, \ldots\}$, possibly finite, of interior vertices of $\mathbf{T}$, perhaps with finite repetitions, is called a branch structure for $\mathbf{T}$ if every simple closed edge-path γ in $\mathbf{T}$ has at least $2\ell + 3$ edges, where ℓ is the number of points from $\mathfrak{B}$ enclosed by γ (counting multiplicities).*

As in Chapter II, if $\mathfrak{B} = \{\underbrace{b_1, b_1, \ldots, b_1}_{k_1 \text{ times}}, \ldots, \underbrace{b_m, b_m, \ldots, b_m}_{k_m \text{ times}}, \ldots\}$ then we will write shortly $\{(b_1, k_1), (b_2, k_2), \ldots, (b_m, k_m), \ldots\}$ for $\mathfrak{B}$, and call b_j a branch point (vertex) of multiplicity $k_j + 1$ (order k_j).

We can now state

Theorem 2. *Let $\mathbf{T}$ be an infinite triangulation, with or without boundary, of a disc. There exists an infinite branched circle packing determined by $\mathbf{T}$ and $\mathfrak{B} = \{(b_1, k_1), \ldots, (b_m, k_m)\}$ if and only if $\mathfrak{B}$ is a branch structure for $\mathbf{T}$.*

Before we start the proof we need to give a general description of things to come. Since the necessity part of the theorem is easy to show, the real work will be in showing the existence part. We will construct the required infinite packing as the "Bl-limit" of a sequence of circle packings associated with finite subcomplexes of the infinite complex. To be more precise, we require the following definition

Definition 3. *Let* $\mathbf{T}$ *be an infinite triangulation of a disc. Let* $\{\mathbf{T}(n)\}_{n=1}^{\infty}$ *be a sequence of finite, simply connected 2-subcomplexes of* $\mathbf{T}$ *such that* $\mathbf{T}(n) \subset \mathbf{T}(n+1)$ *and* $\bigcup_{n=1}^{\infty} \mathbf{T}(n) = \mathbf{T}$. *Suppose that* $P(n)$ *is a circle packing for* $\mathbf{T}(n)$. *Write* $r_n : V(\mathbf{T}(n)) \mapsto (0, \infty)$ *for the radius function of* $B(n)$. *Let* $\{v_n\}_{n=1}^{\infty}$ *be the set of vertices of* $\mathbf{T}$ *with* $v_0 \in \mathbf{T}(1)$. *Suppose that* $R(v_j) := \lim_{n\to\infty}\left(\frac{r_n(v_j)}{r_n(v_0)}\right)$ *exists for any* j. *If* $R(\cdot)$ *is the radius function for a circle packing* P *of* $\mathbf{T}$, *then we say that the sequence* $\{P(n)\}_{n=1}^{\infty}$ *converges to* P *normalized at* v_0, *which we shortly write as* $P = \varinjlim(P(n); \mathbf{T}(n), v_0)$. *If all the packings* $P(n)$ *are Bl-type then we say that* P *is the Bl-limit of the sequence* $\{P(n)\}_{n=1}^{\infty}$ *of packings for* $\{\mathbf{T}(n)\}_{n=1}^{\infty}$ *normalized at* v_0, *and we write* $P = \text{Bl-}\varinjlim(P(n); \mathbf{T}(n), v_0)$.

In order to obtain our infinite packing as a Bl-limit one must first come up with an appropriate sequence of subcomplexes and Bl-type packings associated with them. To do this we will rely on a diagonalization argument which is akin to the method of "normal families" in the classical setting. Let us outline the method in general terms before beginning our proof.

Given an infinite triangulation $\mathbf{T}$ with a finite branch structure $\mathfrak{B} = \{(b_1, k_1), \ldots, (b_m, k_m)\}$, we fix a sequence $\{\mathbf{T}(n)\}_{n=1}^{\infty}$ of finite, simply connected 2-subcomplexes of $\mathbf{T}$ such that $\mathbf{T}(n) \subset \mathbf{T}(n+1)$, $\{b_1, \ldots, b_m\} \subset \mathbf{T}(1)$, and $\bigcup_{n=1}^{\infty} \mathbf{T}(n) = \mathbf{T}$. Suppose that for each n we have a circle packing $P(n)$ for $\mathbf{T}(n)$ with $br(P(n)) = \mathfrak{B}$. Write $R_n(\cdot)$ for the radius function of $P(n)$. Suppose further that the sequence $\{\mathbf{P}(n)\}$ has the following property: Given any subsequence $\{l_k\}$ of indices and any vertex $v \in V(\mathbf{T})$, there exists a further subsequence $\{l'_k\}$ so that $\lim_{k\to\infty} R_{l'_k}(v)$ exists and lies in $(0, \infty)$. Then, by a standard diagonalization argument, there exists a subsequence $\{n_k\}$ of indices so that

$$R(v) = \lim_{k\to\infty} R_{n_k}(v) \in (0,\infty)$$

exists for every $v \in V(\mathbf{T})$. Since angle sums depend continuously on radii, one can verify that the limit function $R(\cdot) : V(\mathbf{T}) \mapsto (0,\infty)$ is the radius function of a circle packing P for $\mathbf{T}$ with branch set $\mathfrak{B}$.

Proof of Theorem 2. An infinite branched circle packing will be constructed by applying the diagonalization method to a sequence of (scaled) Bl-type packings whose complexes will exhaust $\mathbf{T}$ and will have the same branch set $\mathfrak{B}$. The difficulty in completing the proof will be to show that the limit radius function obtained in the diagonalization process has values in $(0,\infty)$ at each vertex of $\mathbf{T}$. This very last goal will be obtained by generalization of the ideas from the proof of Theorem II.2 and by applying properties of Blaschke products. More precisely, we will show that if the limit radius function has value in $\{0,\infty\}$ at some vertex then we can define non-empty sets OD and SD of open and solid dots, respectively, in the similar way as in Chapter II. Then the whole effort will be to contradict "non-emptyness" of one of these sets. For example, if one of the components of OD or SD is finite then we will reach a contradiction in exactly the same way as in the proof of Theorem II.2. Unfortunately, since $\mathbf{T}$ is infinite, all components of OD and SD may be infinite and we will be forced to use some other techniques to show the contradiction. We will construct an edge-path, which will be a part of the outer boundary of a component of OD, with the property that, no matter how far along we are in our sequence of (scaled) Bl-type packings, we can always find a point in that path which has more elements in its preimage under a (scaled) discrete Blaschke product determined by one of the Bl-type packings then the degree of such a product. The last will constitute an obvious contradiction and will complete the proof of the existence part of the theorem. We now proceed to the actual proof.

<u>*NECESSITY*</u>. If $\mathfrak{B}$ is not a branch structure for $\mathbf{T}$ then $\mathfrak{B}$ is not a branch structure for some subcomplex $\mathbf{K} \subset \mathbf{T}$ which is finite and is a triangulation of a disc. Now Theorem II.2 implies that there is no branched circle packing determined by $\mathbf{K}$ and $\mathfrak{B}$. Thus there is no branched circle packing determined by $\mathbf{T}$ and $\mathfrak{B}$.

<u>*EXISTENCE*</u>. We have to consider two cases depending on whether the boundary of $\mathbf{T}$ is empty or not.

Case 1. $\mathbf{T}$ *has no boundary.*

The proof of this case will be given in several parts.

<u>*Part 1*</u>. In this part we will construct a sequence of scaled Bl-type circle packings to which we will apply the diagonalization argument. Moreover, if the limiting radius function obtained from the diagonalization method turns out to have values in $(0, \infty)$ then it will determine the sought after infinite branched circle packing.

Let $\left\{\mathbf{T}(n)\right\}_{n=1}^{\infty}$ be a sequence of finite, simply connected 2-subcomplexes of $\mathbf{T}$ such that $\mathbf{T}(n) \subset \mathbf{T}(n+1)$, $\{b_1, \ldots, b_m\} \subset int\,\mathbf{T}(1)$, and $\bigcup_{n=1}^{\infty} \mathbf{T}(n) = \mathbf{T}$. Write $\{v_n\}_{n=0}^{\infty}$ for the set of vertices of $\mathbf{T}$ with $v_0 \in int\,\mathbf{T}(1)$. Suppose that $B(n)$ is a Bl-type circle packing determined by $\mathbf{T}(n)$ and $\mathfrak{B}$. Write $r_n(\cdot) : V(\mathbf{T}(n)) \mapsto (0, \infty)$ for the radius function of $B(n)$ defined on the set $V(\mathbf{T}(n))$ of vertices of $\mathbf{T}(n)$. Let us normalize circles in the packings $B(n)$ associated with v_0 by scaling each $B(n)$ by a factor $1/r_n(v_0)$ to obtain a new packing $\frac{1}{r_n(v_0)}B(n)$ and its radius function $R_n(\cdot) := \frac{r_n(\cdot)}{r_n(v_0)} : V(\mathbf{T}(n)) \mapsto (0, \infty)$.

We are now going to apply the diagonalization process to the sequence of packings $\left\{\frac{1}{r_n(v_0)}B(n)\right\}_{n=1}^{\infty}$, or equivalently to the sequence of corresponding radius functions $\{R_n(\cdot)\}_{n=1}^{\infty}$. Suppose that the diagonalization argument can be applied to the sequence $\{R_n(\cdot)\}_{n=1}^{\infty}$. Then, by the earlier discussion, we get the radius function of an infinite branched circle packing for $\mathbf{T}$ with branch set $\mathfrak{B}$, and the theorem is proved. In the remaining parts of the proof of the theorem we will show that we can actually apply the diagonalization argument to $\{R_n(\cdot)\}_{n=1}^{\infty}$.

Part 2. Suppose that we can not apply the diagonalization argument to $\{R_n(\cdot)\}_{n=1}^{\infty}$. It means that there exist a subsequence $\{n_l\}$ of indices and a vertex in $\mathbf{T}$, say v_t, so that $\lim_{l\to\infty} R_{n_l}(v_j) \in (0,\infty)$, $j = 0,\ldots,t-1$, and $\lim_{l\to\infty} R_{n_l}(v_t) \in \{0,\infty\}$. Notice that $t \geq 1$ because $R_n(v_0) \equiv 1$. Moreover, without loss of generality we may assume that $\lim_{l\to\infty} R_{n_l}(v_t) = \infty$. (Otherwise, i.e. if $\lim_{l\to\infty} R_{n_l}(v_t) = 0$, we would substitute the sequence $\left\{\frac{1}{R_{n_l}(v_t)} R_{n_l}(\cdot)\right\}_{n=1}^{\infty}$ for the sequence $\{R_n(\cdot)\}_{n=1}^{\infty}$ and then, with the role previously played by v_0 now played by v_t, we would proceed as in Part 1, which would eventually bring us to Part 2, with the role of v_t now played possibly by v_{t-1}.)

Let

$$\mathcal{R}_n^j(\cdot) := \frac{R_{n_l}(\cdot)}{R_{n_l}(v_t)} \text{ for } j = 0, 1, \ldots, t.$$

For $j > t$ we define $\{\mathcal{R}_n^j(\cdot)\}_{n=1}^{\infty}$ by induction as follows:

- if it is possible we choose from $\{\mathcal{R}_n^j(\cdot)\}_{n=1}^{\infty}$ a subsequence $\{\mathcal{R}_n^{j+1}(\cdot)\}_{n=1}^{\infty}$ such that $\lim_{n\to\infty} \mathcal{R}_n^{j+1}(v_{j+1}) \in (0,\infty)$,
- if the above is not possible we choose a subsequence $\{\mathcal{R}_n^{j+1}(\cdot)\}_{n=1}^{\infty}$ such that $\lim_{n\to\infty} \mathcal{R}_n^{j+1}(v_{j+1}) = \infty$,
- if the last is impossible we choose a subsequence $\{\mathcal{R}_n^{j+1}(\cdot)\}_{n=1}^{\infty}$ such that $\lim_{n\to\infty} \mathcal{R}_n^{j+1}(v_{j+1}) = 0$.

Finally we define the sequence $\{\mathcal{R}_n(\cdot)\}_{n=1}^{\infty}$ by $\mathcal{R}_n(\cdot) := \mathcal{R}_n^n(\cdot)$. Notice that for any k

$$\mathcal{R}(v_k) := \lim_{n\to\infty} \mathcal{R}_n(v_k) = \lim_{n\to\infty} \mathcal{R}_n^l(v_k) \in [0,\infty] \quad \text{for any } l \geq k.$$

Also, by the construction, $\mathcal{R}(v_i) = 0$ for $0 \leq i < t$, and $\mathcal{R}(v_t) = 1 \equiv \mathcal{R}_n(v_t)$ for any $n \geq t$. Finally, we want to make the following

Remark 4. *$\mathcal{R}_n(\cdot)$ is a scaled version of a radius function $r_{n'}(\cdot)$ of $B(n')$ for some $n' \geq n$.*

Part 3. In this part we will classify vertices of $\mathbf{T}$ into two groups with respect to the values of the function $\mathcal{R}(\cdot)$ constructed in Part 2.

The set $V(\mathbf{T})$ is a disjoint union of the sets SD (solid dots) and OD (open dots) which are defined as follows:

— $v_k \in SD$ if $\mathcal{R}(v_k) \in (0,\infty]$,

— $v_k \in OD$ if $\mathcal{R}(v_k) = 0$.

We observe that both sets are not empty because $v_0 \in OD$ and $v_t \in SD$.

We now remind the reader that terms such as component, simply connected region, outer boundary etc. for subsets in $V(\mathbf{T})$ were defined in Chapter II, and they will be used below.

Let C_{SD} and C_{OD} denote components of SD and OD, respectively. Since $\mathfrak{B}$ is a branch structure for $\mathbf{T}$, by applying Remark II.5 to finite subsets $\mathbf{T}(n)$ of $\mathbf{T}$, one gets the following properties:

(i) C_{SD} and C_{OD} can not be finite,

(ii) There is no finite region S in $\mathbf{T}$ that contains at least one solid dot and whose outer boundary $outS$ has only one solid dot,

(iii) If S is the star in $\mathbf{T}$ of a solid dot, and C_1 and C_2 are components in $bd\,S$ of open dots, then C_1 and C_2 are not contained in the same component of OD.

Let C_{OD} be the component of OD that contains v_0. We recall that, by the definition, the outer boundary $outC_{OD}$ of C_{OD} is contained in SD. Unlike in the finite case, $outC_{OD}$ may have several components.

We may assume that there exist a component Σ of $outC_{OD}$ and a vertex $v_{k_0} \in \Sigma$ such that $\mathcal{R}_n(v_{k_0}) \equiv 1$ for any n, otherwise we can redefine SD and OD in the following way:

> Let v_{k_0} be a vertex in $outC_{OD}$. Let $\widehat{\mathcal{R}}_n^0(\cdot) := \frac{\mathcal{R}_n(\cdot)}{\mathcal{R}_n(v_{k_0})}$. Then, as for the $\mathcal{R}_n^j(\cdot)$, we construct inductively sequences $\{\widehat{\mathcal{R}}_n^j(\cdot)\}_{n=1}^{\infty}$, $j = 0, 1, \ldots$. Next we define the sequence $\{\mathcal{R}'_n(\cdot)\}_{n=1}^{\infty}$ by $\mathcal{R}'_n(\cdot) := \widehat{\mathcal{R}}_n^n(\cdot)$ and the limit radius function $\mathcal{R}'(\cdot)$ by $\mathcal{R}'(v_k) := \lim_{n\to\infty} \mathcal{R}'_n(v_k)$. We observe that $\mathcal{R}'_n(\cdot)$ is a scaled version of $r_{n'}(\cdot)$ for some $n' \geq n$, hence Remark 4 will hold for $\mathcal{R}_n(\cdot)$ replaced by $\mathcal{R}'_n(\cdot)$. Then we redefine SD and OD by saying that $v_k \in OD$ if $\mathcal{R}'(v_k) = 0$, otherwise $v_k \in SD$. This new definition of the sets SD and OD implies that both are non-empty ($v_0 \in OD$ and $v_{k_0} \in SD$), and the "new" set OD contains the "old" one. In particular, if C_{OD} is the ("new") component of OD that contains v_0 then $v_{k_0} \in outC_{OD}$ and $\mathcal{R}'_n(v_{k_0}) \equiv 1$ for any n.

Choose any point from Σ and call it α. Let $\beta \in bd(star(\alpha)) \cap C_{OD}$, where $star(\alpha)$ is the star of α. Let us start walking from β along the boundary of $star(\alpha)$ with respect to the counterclockwise orientation (i.e. the positive orientation of $bd(star(\alpha))$ induced from the orientation of $\mathbf{T}$). Suppose that γ is the first solid dot encountered in such a walk along the boundary of $star(\alpha)$. Obviously $\gamma \in \Sigma$, and we will call this point the *previous* point on Σ prior to α. Notice that, by (iii), γ does not depend on the choice of β. Since we now know what we mean by the next (or previous) point on Σ after given point, we have established an orientation on Σ.

We are now going to change notation slightly. Let us call vertex v_{k_0} by a_0, the vertex next after it in Σ (with respect to the orientation that we have established) by a_1, the next vertex after the last one by a_2 and so on. In similar way we denote vertices on Σ when we "walk in opposite direction" by $a_{-1}, a_{-2}, a_{-3}, \ldots$. This defines a doubly-infinite sequence with no repeats; i.e., if one starts walking from a vertex $a_j \in \Sigma$ along Σ (with respect to the orientation that we have established) one will never return to a vertex that has already been visited (Figure 8).

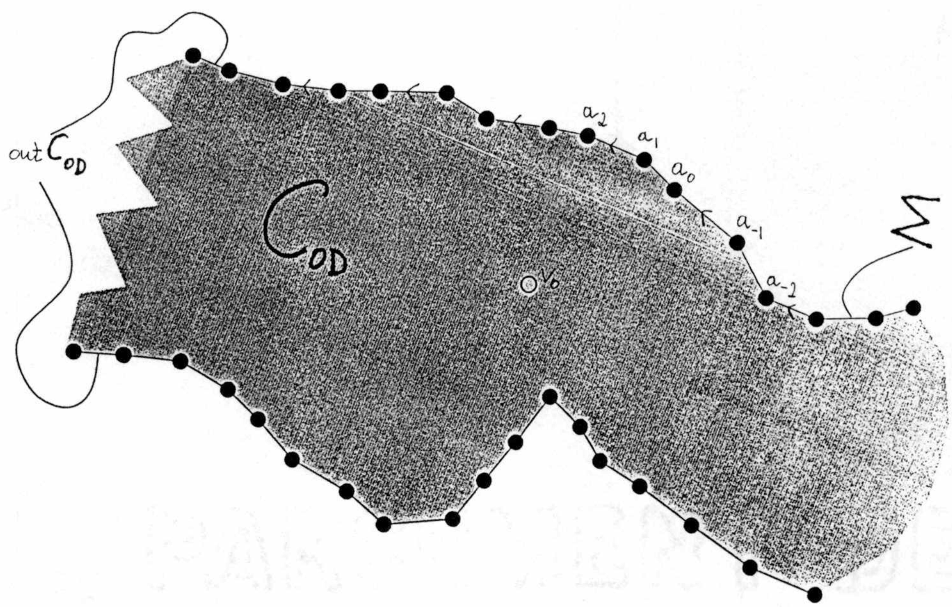

FIGURE 8. C_{OD} and Σ

Part 4. Let us consider a part of Σ, say an edge path $[a_k, a_{k+1}, \ldots, a_{k+l}]$. Then the set of all vertices of C_{OD} that have at least one neighbor in $\{a_k, \ldots, a_{k+l}\}$ will be denoted by $\Omega([a_k, a_{k+l}])$. The property (ii) and the established orientation on Σ imply that $\Omega([a_k, a_{k+l}])$ is "on the left side" of the path $[a_k, a_{k+1}, \ldots, a_{k+l}]$. Also it is easy to see that $\Omega([a_k, a_{k+l}])$ is a connected set.

Part 5. In this part we will be dealing with the following setup:

Suppose that $\ell \geq 3$. Let $\{r_n(0), r_n(1), \ldots, r_n(\ell)\}_{n=1}^{\infty}$ be a sequence of $(\ell + 1)$-

tuples of positive numbers such that $\lim_{n\to\infty} r_n(j) \in (0,\infty]$ for $j = 0, 1, \ell$, and $\lim_{n\to\infty} r_n(j) = 0$ for $j = 2, 3, \ldots, \ell - 1$. Let $C_n(j)$ and $s_n(j)$ denote a circle of radius $r_n(j)$ and its center respectively. Moreover, suppose that the $C_n(j)$ have the following properties:

- $C_n(j)$ is tangent and exterior to $C_n(0)$, $j = 1, \ldots, \ell$;
- $C_n(j)$ is tangent and exterior to $C_n(j+1)$, $j = 1, \ldots, \ell - 1$;
- an orientation in $\partial(\Delta s_n(0)s_n(j)s_n(j+1))$ induced by the ordered triple $< s_n(0), s_n(j), s_n(j+1) >$ and the counterclockwise orientation on $\partial(\Delta s_n(0)s_n(j)s_n(j+1))$ are compatible for $j = 1, \ldots, \ell - 1$.

This finishes the setup (Figure 9).

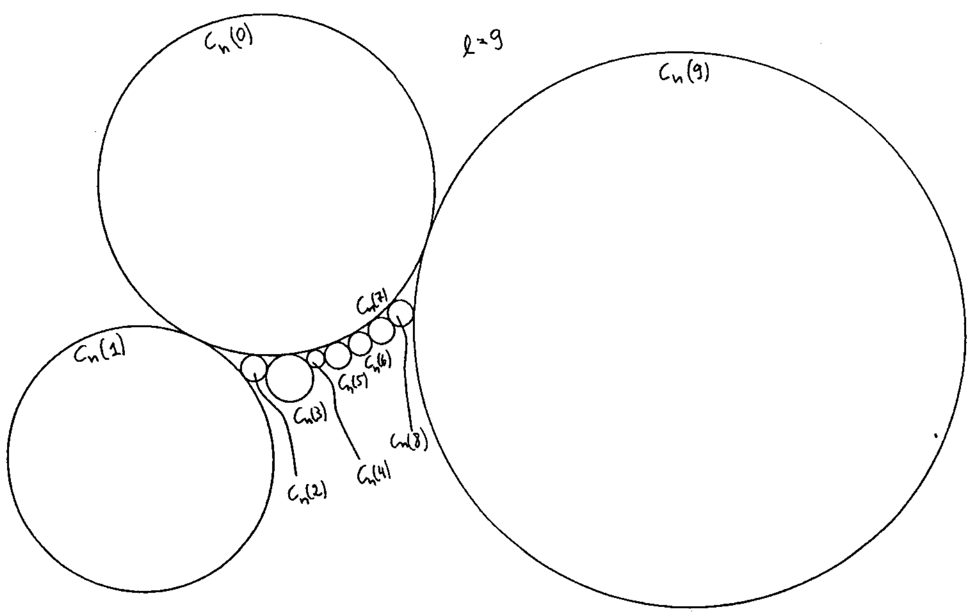

FIGURE 9. The setup

Let us now normalize things slightly. We will assume that the point of tangency of $C_n(0)$ and $C_n(1)$ is always the origin in $\mathbf{R}^2$, and that $s_n(0)$ and $s_n(1)$ belong to the positive and negative part of the y-axis, respectively.

Let $r(j) = lim_{n\to\infty} r_n(j)$. Suppose that $\angle_n(j)$ denotes the angle at the vertex

$s_n(0)$ in the triangle $\Delta s_n(0)s_n(j)s_n(j+1)$. As $r(j)=0$ for $j=2,\ldots,\ell-1$, we have the following elementary

Lemma 5.

(i) $\lim_{n\to\infty}\angle_n(j)=0$ *for* $j=1,\ldots,i-1$. *In particular,* $\lim_{n\to\infty}\angle_n=0$, *where* $\angle_n:=\sum_{j=1}^{\ell-1}\angle_n(j)$.

(ii) *If* $r(1)<\infty$ *or* $r(0)<\infty$ *then* $C_n(2)$ *converges to a point which is the origin.*

(iii) *Suppose that* $r(0)=\infty=r(1)$. *Let* L_n *denote the line that goes through the point* $s_n(2)$ *and is parallel to the* y*-axis. Let* p_n *and* q_n *be the points in* $\mathbf{R}^2$ *such that* $\{p_n,q_n\}=L_n\cap C_n(2)$ *and* $\Pi_y(p_n)>\Pi_y(q_n)$, *where* Π_y *is the projection onto the* y*-axis. Then, for any* $\epsilon\in(0,1/4)$, *there exists* N *such that for any* $n>N$

$$\left[\Pi_x(p_n)-\frac{\epsilon}{2},\Pi_x(p_n)+\frac{\epsilon}{2}\right]\times\left[\Pi_y(p_n)+\frac{\epsilon}{2},\Pi_y(p_n)+\frac{1}{4}+\frac{\epsilon}{2}\right]\subset intC_n(0)$$

and

$$\left[\Pi_x(q_n)-\frac{\epsilon}{2},\Pi_x(q_n)+\frac{\epsilon}{2}\right]\times\left[\Pi_y(q_n)-\frac{1}{4}-\frac{\epsilon}{2},\Pi_y(q_n)-\frac{\epsilon}{2}\right]\subset intC_n(1),$$

where Π_x *is the projection onto the* x*-axis, and* $intC_n(j)$ *is the interior of the circle* $C_n(j)$.

<u>*Part 6*</u>. Before we state the goal of this part we need to introduce some notation.

Recall that in Part 2 we have defined the sequence $\{\mathcal{R}_n(\cdot)\}_{n=1}^{\infty}$ of radius functions. Due to Remark 4, $\mathcal{R}_n(\cdot)=c_nr_{n'}(\cdot)$ for some $c_n\in(0,\infty)$ and $n'\geq n$, where $r_{n'}(\cdot)$ is the radius function of the branched circle packing $B(n')$. Hence $\mathcal{R}_n(\cdot)$ defines a branched circle packing which is just a scaled version of the packing

$B(n')$; later we will refer to the scaled version as $\mathcal{B}(n)$. Let f_n be a discrete Blaschke product induced by $B(n)$ and let $\bar{f}_n : \overline{D} \mapsto \overline{D}$ be its extension defined as in Chapter III. Then $\bar{f}_n = \varphi_n \circ h_n$, where $h_n : \overline{D} \mapsto \overline{D}$ is a homeomorphism and $\varphi_n : \overline{D} \mapsto \overline{D}$ is a Blaschke product of degree $M = 1 + \sum_{i=1}^{m} k_i$ because each of the $B(n)$'s is a Bl-type packing with the branch set $\mathfrak{B}$. Let $F_n := c_n f_n$ then F_n is at most an M-to-1 mapping.

We are ready now to state our goal in this part. We will show, with the help of the edge-path $[\dots, a_{-1}, a_0, a_1, \dots]$, that for n sufficiently large we can find a point which will have more elements in its preimage under F_n then the valence of F_n. This will give a contradiction and the proof of Theorem 2 will be completed.

Let $S(a_i, a_{i+1})$ denote the union of closed stars in $\mathbf{T}$ associated with vertices a_i and a_{i+1}. Let N be a positive integer such that there exists a sequence $\{k_j\}_{j=1}^{M+1}$ with the following properties:

- the sets $S(a_{k_j}, a_{k_j+1})$ are pairwise disjoint for $j = 1, \dots, M+1$,
- $k_1 = 0$ and $k_{M+1} = N - 1$.

From now on we will be interested in the path $[a_0, ..., a_N]$ only.

If L is a line and ab is the line that contains the interval $[a, b]$, $a, b \in \mathbf{R}^2$, then $AD_L(ab)$ is defined as follows:

> take the bigger of two angles which are obtained from the intersection of L and ab if they intersect, or the angle π if the lines do not intersect, then subtract $\pi/2$ from it and the result is $AD_L(ab)$.

As an immediate consequence of Lemma 5 one has

Fact 6. *Let a_k^n denote the vertex in the domain of f_n (which is the carrier of some Andreev packing associated with $\mathbf{T}(n)$) that corresponds to the vertex*

$a_k \in V(\mathbf{T}(n)) \subset V(\mathbf{T})$. *Then*

$$\forall \epsilon > 0\ \exists N_\epsilon\ \forall n > N_\epsilon\ \sum_{k=0}^{N-1} AD_{l_n}\left(F_n(a_k^n)F_n(a_{k+1}^n)\right) < \epsilon,$$

where l_n *is a line perpendicular to the line* $F_n(a_0^n)F_n(a_1^n)$; *in other words, if* n *is sufficiently large then all segments* $[F_n(a_k^n), F_n(a_{k+1}^n)]$, $k = 0, 1, \ldots, N-1$, *are nearly parallel to* $[F_n(a_0^n), F_n(a_1^n)]$.

Let us now normalize things once again. Let $\mathcal{C}_n(a_k)$ denote a circle in $\mathcal{B}(n)$ associated with the vertex a_k. We will be assuming without loss of generality that the tangency point of $\mathcal{C}_n(a_0)$ and $\mathcal{C}_n(a_1)$ is always at the origin (in $\mathbf{R}^2$), and that $F_n(a_1^n)$ and $F_n(a_0^n)$ belong to the positive and the negative part of y-axis respectively. Let $\Delta(n)$ denote the smallest disc that contains all circles in $\mathcal{B}(n)$ associated with vertices $\Omega([a_0, a_N])$. Write $\rho(n)$ for the radius of $\Delta(n)$. Since $\Omega([a_0, a_N])$ is connected, finite and contained in OD, $\lim_{n\to\infty} \rho(n) = 0$. Recall that $\mathcal{R}_n(a_0) \equiv 1$, and observe that any two vertices a_k and a_{k+1}, $0 \le k \le N-1$, have a common neighboring vertex among vertices in $\Omega([a_0, a_N])$. This together with Part (ii) of Lemma 5 implies that $\Delta(n)$ converges to a point which is the origin. Moreover, if $\mathcal{R}(a_k) < \infty$ or $\mathcal{R}(a_{k+1}) < \infty$, $0 \le k \le N-1$, then the tangency point of $\mathcal{C}_n(a_k)$ and $\mathcal{C}_n(a_{k+1})$ will converge to the origin and, by Fact 6, there exists a sequence $\{\delta_n(k)\}_{n=0}^{\infty}$ such that

$$\delta_n(k) > 0,\ \lim_{n\to\infty} \delta_n(k) = 0 \quad \text{and}$$

$$\Big(\{0\} \times \left[\delta_n(k), \mathcal{R}(a_k, a_{k+1})\right]\Big) \bigcup \Big(\{0\} \times \left[-\mathcal{R}(a_k, a_{k+1}), -\delta_n(k)\right]\Big) \subset$$
$$int\mathcal{C}_n(a_k) \cup int\mathcal{C}_n(a_{k+1}) \quad \text{for sufficiently large } n,$$

where $\mathcal{R}(a_k, a_{k+1}) = \min\{\mathcal{R}(a_k), \mathcal{R}(a_{k+1})\}$. Similarly, if $\mathcal{R}(a_k) = \infty = \mathcal{R}(a_{k+1})$, $0 \le k \le N-1$, then Part (iii) of Lemma 5, Fact 6 and the convergence of $\Delta(n)$

to the origin imply that there exists a sequence $\{\delta_n(k)\}_{n=0}^{\infty}$ such that

$$\delta_n(k) > 0, \ \lim_{n\to\infty} \delta_n(k) = 0 \quad \text{and}$$

$$\left(\{0\} \times [\delta_n(k), \frac{1}{4}]\right) \bigcup \left(\{0\} \times [-\frac{1}{4}, -\delta_n(k)]\right) \subset$$
$$int\mathcal{C}_n(a_k) \cup int\mathcal{C}_n(a_{k+1}) \quad \text{for sufficiently large } n.$$

Hence, there exists a point $p \in \{0\} \times [-1, 0)$ such that

$$p \in \bigcap_{j=1}^{M+1} \left(int\mathcal{C}_n(a_{k_j}) \cup int\mathcal{C}_n(a_{k_j+1})\right) \quad \text{for any sufficiently large } n.$$

Let $S(a_{k_j}, a_{k_j+1})^n$ denote the set in the domain of f_n that corresponds to the set $S(a_{k_j}, a_{k_j+1}) \subset \mathbf{T}(n) \subset \mathbf{T}$, i.e. $S(a_{k_j}, a_{k_j+1})^n$ is a union of closed stars in the domain of f_n associated with the vertices $a^n_{k_j}$ and $a^n_{k_j+1}$. Since $int\mathcal{C}_n(a_{k_j}) \cup int\mathcal{C}_n(a_{k_j+1}) \subset F_n\left(S(a_{k_j}, a_{k_j+1})^n\right)$ and the sets $S(a_{k_j}, a_{k_j+1})$ are pairwise disjoint (for $j = 1, \ldots, M+1$), one gets that $F_n^{-1}(p)$ has at least $M+1$ elements. As each F_n is at most M-to-1, the last conclusion is a contradiction. In other words, we contradicted our assumption of Part 2 that the diagonalization argument is not applicable to the sequence $\{R_n(\cdot)\}$ (which was converted in Part 3 to the equivalent assumption that both sets OD and SD are not empty). This concludes the proof of Case 1.

Case 2. T ***has boundary.***

We will show this case by discussing briefly how the proof of Case 1 can be modified for $\mathbf{T}$ with boundary.

Let the objects $\mathbf{T}(n)$, $B(n)$, v_n, and $r_n(\cdot)$ be defined as in Part 1 of Case 1. Write $C_n(v)$ for the circle in $B(n)$ associated with $v \in V(\mathbf{T}(n))$. We will follow the steps of the proof of Case 1, however we will work with the packings $B(n)$

not their scaled versions to show that the diagonalization argument applies to the sequence $\{B(n)\}$.

First we are going to define a collection of sequences of radius functions. Let $\{\mathcal{R}^0_n(\cdot)\}_{n=1}^{\infty}$ be the sequence $\{r_n(\cdot)\}_{n=1}^{\infty}$. For $j \geq 0$ we define $\{\mathcal{R}^{j+1}_n(\cdot)\}_{n=1}^{\infty}$ by induction as follows:

- if it is possible we choose from $\{\mathcal{R}^j_n(\cdot)\}_{n=1}^{\infty}$ a subsequence $\{\mathcal{R}^{j+1}_n(\cdot)\}_{n=1}^{\infty}$ such that $\lim_{n\to\infty} \mathcal{R}^{j+1}_n(v_{j+1}) \in (0, \infty)$,
- if the above is not possible we choose from $\{\mathcal{R}^j_n(\cdot)\}_{n=1}^{\infty}$ a subsequence $\{\mathcal{R}^{j+1}_n(\cdot)\}_{n=1}^{\infty}$ such that $\lim_{n\to\infty} \mathcal{R}^{j+1}_n(v_{j+1}) = 0$.

We observe that each $\mathcal{R}^j_n(\cdot)$ is the radius function of one of the packings $B(n)$, and therefore the values of $\mathcal{R}^j_n(\cdot)$ are always in $(0, 1]$. Hence we do not need to be concerned with $\lim_{n\to\infty} \mathcal{R}^j_n(v_j)$ being ∞ in the above definition.

Finally we define a sequence $\{\mathcal{R}_n(\cdot)\}_{n=1}^{\infty}$ by $\mathcal{R}_n(\cdot) := \mathcal{R}^n_n(\cdot)$. Notice that for any j

$$\mathcal{R}_n(v_j) := \lim_{n\to\infty} \mathcal{R}_n(v_j) = \lim_{n\to\infty} \mathcal{R}^l_n(v_j) \in [0, 1] \quad \text{for any } l \geq j.$$

Let $SD := \mathcal{R}^{-1}((0, 1]) \subset V(\mathbf{T})$ and $OD := \mathcal{R}^{-1}(\{0\}) \subset V(\mathbf{T})$. We will now show that SD is not empty by proving that any boundary vertex of $\mathbf{T}$ belongs to SD. Let v_* be a boundary vertex of $\mathbf{T}$ with $v_{**} \in int\,\mathbf{T}$ as a neighbor. If we view the packings $B(n)$ as hyperbolic packings then the $C_n(v_*)$ are horocycles, and the hyperbolic distance of the hyperbolic center of $C_n(v_{**})$ to the origin and the hyperbolic radius of $C_n(v_{**})$ are both decreasing as $n \to \infty$ (see [BSt1]). This implies that $r_n(v_*)$ is bounded away from zero as $n \to \infty$. Thus $\mathcal{R}(v_*) > 0$, and $v_* \in SD$. An easy argument shows that any boundary vertex belongs to SD.

Having shown that $bd\,\mathbf{T} \subset SD$, we may proceed precisely as in the proof of Case 1; namely, using discrete Blaschke products associated with the packings

$B(n)$, one shows that of the two sets SD and OD, one has to be empty. Since $SD \supset bd\,\mathbf{T} \neq \emptyset$, we get that $OD = \emptyset$. The last implies that $\mathcal{R}\big(V(\mathbf{T})\big) \subset (0,1]$ and the proof of Case 2 is completed.

This concludes the proof of Theorem 2. $\square$

From the proof of Case 2 of Theorem 1 we have the following

Remark 7. *Let* $\mathbf{T}$ *be an infinite triangulation of a disc which has boundary vertices. Suppose that* $\mathfrak{B}$ *is a branch structure for* $\mathbf{T}$*. Then there exists an infinite branched circle packing* B *for* $\mathbf{T}$ *with* $br(B) = \mathfrak{B}$ *so that* B *is contained in the unit disc.*

We finish this chapter with

Remark 8. *The infinite branched packing constructed in Case 1 in the proof of Theorem 1 is a Bl-limit packing. The branched packing constructed in Case 2 is a scaled version of a Bl-limit packing.*

Chapter V

Blaschke Products

In this chapter we will show (Theorem 8.3) that:

Any (classical) finite Blaschke product can be approximated uniformly on compacta of $D = \{z : |z| < 1\}$ by discrete Blaschke products.

The chapter is divided into several sections with the last one containing a proof of the above assertion.

Section 1. We will recall now some facts about finite Blaschke products. Let $\overline{D}$ and ∂D denote the closure and the boundary of the unit disc D respectively. Suppose that $x_1, \ldots, x_m$ are points in D, and that $k_1, \ldots, k_m$ are non-negative integers. A holomorphic map $F : \overline{D} \mapsto \overline{D}$ given by $F(z) = c\prod_{i=1}^{i=m}(\frac{z-x_i}{1-\overline{x}_i z})^{k_i}$, $|c| = 1$, is called a finite Blaschke product with branch points $x_1, \ldots, x_m$ of orders $k_1, \ldots, k_m$, respectively (or equivalently, with branch set $\{(b_1, k_1), \ldots, (b_m, k_m)\}$). Finite Blascke products have also another characterization: if $F : D \mapsto D$ is a holomorphic, proper mapping then F is a finite Blaschke product.

Let $a, b \in \partial D$ and $a \neq b$. Suppose that γ is a simple curve in D with the end points a and b. Let $\mathcal{D}$ be a component of $D \setminus \gamma$. We will be assuming that $\{x_1, \ldots, x_m\} \cap \mathcal{D} = \emptyset$. Then we have the following two facts.

Fact 1.1. *If $F|_{\mathcal{D}}$ is a homeomorphism onto $F(\mathcal{D})$ then there exists i, $1 \leq i \leq m$, such that $F(x_i) \notin F(\mathcal{D})$.*

Fact 1.2. *Let $\gamma' = \gamma \setminus \{a, b\}$. If $F|_{\mathcal{D}}$ is not a homeomorphism onto $F(\mathcal{D})$ then $F(\gamma')$ has at least one self-intersection. Moreover, the length of the path $F(\gamma)$ is at least $\inf_{1 \leq i \leq m}\{dist(F(x_i), \partial D)\}$.*

Section 2. Suppose that $\mathbf{T}$ is a triangulation of a disc and $r : V(\mathbf{T}) \mapsto (0, \infty)$. Then $r(\cdot)$ is said to be **subharmonic** at $v \in V(\mathbf{T})$ if $r(v) < \frac{1}{n}\big(r(v_1) + \cdots + r(v_n)\big)$, where $v_1, \ldots, v_n$ are the neighboring vertices of v. $\mathbf{T}$ is called **hexagonal** if each interior vertex has exactly 6 neighboring vertices and each boundary vertex has at most 6 neighbors. We have the following

Lemma 2.1. *Let $r(\cdot)$ be the radius function of a circle packing P of a hexagonal complex $\mathbf{T}$. Then $r(\cdot)$ is subharmonic at interior vertices of $\mathbf{T}$.*

Proof. It was shown in [BFP] (see also [R1]) that $r(\cdot)$ is subharmonic at any interior vertex with the angle sum 2π and with non-overlapping petals of the flower associated with the vertex. However, one can easily verify that the condition that petals are disjoint is irrelevant as long as the angle sum is 2π. Thus, the radius function $r(\cdot)$ of P is subharmonic at any interior vertex which is not a branch point.

Suppose that b is a branch vertex and $v_1, \ldots, v_6$ are its neighboring vertices. Recall that the angle sum at b is 4π. The monotonicity properties of the angle sums imply that if one keeps $r(v_1), \ldots, r(v_6)$ fixed and continuously increases $r(b)$ to ∞ then the angle sum at b continuously decreases to 0. Thus, there is a value $r_0(b)$, $r_0(b) > r(b)$, such that the angle sum at b given by radii $r_0(b), r(v_1), \ldots, r(v_6)$ is equal to 2π. But then $r(b) < r_0(b) \leq \frac{1}{6}\big(r(v_1) + \cdots + r(v_6)\big)$, i.e. the radius function $r(\cdot)$ of B is subharmonic at b. $\square$

Section 3. The property that the radius function for a branched circle packing is subharmonic at interior vertices is very special. Since we will need such a property later, from now on we are going to work in this chapter only with branched circle packings associated with hexagonal complexes, namely, simply connected 2-subcomplexes of the infinite regular hexagonal 2-complex. Therefore we need to introduce some notation.

Let $\mathbf{H}$ denote an infinite regular hexagonal 2-complex of mesh 2 with vertices at points $2k + l(1 + \sqrt{3}i)$, $k, l \in \mathbf{Z}$. Write $\mathbf{H}(n)$ for a subcomplex of $\mathbf{H}$ that consists of all faces of $\mathbf{H}$ whose closures are contained in $\{z : |z| < n - 1\}$. Let $H(n) := \frac{1}{n}\mathbf{H}(n)$ and let $H'(n)$ be a subcomplex of $H(n)$ obtained by removing boundary faces from $H(n)$ (where a face is a boundary face if it has a vertex in $bd\, H(n)$). We will refer to $\mathbf{H}$, $\mathbf{H}(n)$, $H(n)$, and $H'(n)$ sometimes as simplicial complexes (triangulations) and sometimes as subsets of $\mathbf{C}$, but the reader should not have any problems with this duality of notation, as it will be clear from the context to which one we refer.

Let $P_{\mathbf{H}}$ be a regular hexagonal circle packing whose carrier is $\mathbf{H}$. Write $P_{\mathbf{H}(n)}$ for the part of $P_{\mathbf{H}}$ which corresponds to $\mathbf{H}(n) \subset \mathbf{H}$, and $P_{H(n)}$ for $\frac{1}{n}P_{\mathbf{H}(n)}$. We remark that $P_{H(n)}$ is contained in the unit disc.

Let us fix n. Suppose that B is a Bl-type packing for $H(n)$. Let g be the simplicial map from $P_{H(n)}$ to B and $g^{\#}$ be the corresponding ratio map (i.e. $g = \mathcal{S}_B \circ \mathcal{S}_{P_{H(n)}}^{-1}$ and $g^{\#} = (\mathcal{S}_B \circ \mathcal{S}_{P_{H(n)}}^{-1})^{\#}$). Using a method analogous to that in Chapter III, one constructs an extension of g to a map $\bar{g} : \overline{D} \mapsto \overline{D}$ such that: 1) $\bar{g} = \varphi \circ h$, where h is a self-homeomorphism of $\overline{D}$ and φ is a Blaschke product with the branch set $\{h(b_1), \ldots, h(b_m)\}$, where $\{b_1, \ldots, b_m\}$ is the branch set for B, 2) the image under $\bar{g}$ of the "star" (see Chapter III) of a boundary vertex of $H(n)$ contains the circle of B associated with this vertex. If it is clear from the context which $H(n)$ is complex for B then we will refer to g and $\bar{g}$ as the simplicial map induced by B and an extended simplicial map associated with B, respectively. Finally, we will say that B fixes a point $p \in H(n)$ and maps $q \in H(n)$ to $\mathbf{R}_+$ if the induced map g fixes p and $g(q) \in \mathbf{R}_+$.

Let $A(n)$ be an Andreev circle packing associated with $H(n)$ which fixes the origin and maps $1/2$ to $\mathbf{R}_+$. Then, by [St] or [RS], the euclidean radii of boundary circles of $A(n)$ converge uniformly to zero as $n \to \infty$. Moreover, due to the normalization of $A(n)$, the simplicial maps induced by the $A(n)$ converge uniformly on compacta of D to the identity function. In particular, for fixed $r < 1$, $D(r) = \{z : |z| < r\}$ is mapped under such simplicial functions into $D(\bar{r})$ for sufficiently large n, where $\bar{r} = \frac{1+r}{2}$.

Although our goal in this chapter is to proof the approximation of classical Blaschke products by discrete ones, due to technicalities, most of our work will be in showing that extended simplicial maps approximate classical Blaschke products. Let us be more specific. For a given finite classical Blaschke product F we will construct an appropriate sequence $\{B_F(n)\}$ of Bl-type packings associated with the sequence of complexes $H(n)$. Suppose that F is normalized so that $F(0) = 0$ and $F(1/2) \in \mathbf{R}_+$, then the packings $B_F(n)$ will fix the origin and map $1/2$ to $\mathbf{R}_+$. Let f_n be an extension of the simplicial map from $P_{H(n)}$ to $B_F(n)$ (Figure 10). Let $A(n)$ be as above. Write $\mathbb{F}_n$ for the discrete Blaschke product from $A(n)$ to $B_F(n)$ and G_n for the simplicial map from $P_{H(n)}$ to $A(n)$. We want to show that $\mathbb{F}_n \mapsto F$ uniformly on compacta of D. In order to prove this, we will first show that $f_n \mapsto F$ uniformly on compacta of D. Then, since $G_n \mapsto id_D$ uniformly on compacta of D, we may conclude that $\mathbb{F}_n = f_n \circ G_n^{-1} \mapsto F$ uniformly on compacta of D.

Section 4. Let F be a finite Blaschke product with $\{(x_1, k_1), \ldots, (x_m, k_m)\}$ as its branch set. Furthermore, suppose that F is normalized so that $F(0) = 0$ and $F(1/2) \in \mathbf{R}_+$. We remark that such a normalization is merely a convenience, not a restriction for what we are going to do. We need to introduce the following

Definition 4.1. *If $\{B(n)\}_{n=N}^{\infty}$ is a sequence of Bl-type packings such that*

(1) *$B(n)$ is determined by $H(n)$ and a branch set $\{v_{i,j}(n)\}_{j=1,i=1}^{k_i,m}$, fixes the origin, and maps $1/2$ to $\mathbf{R}_+$,*

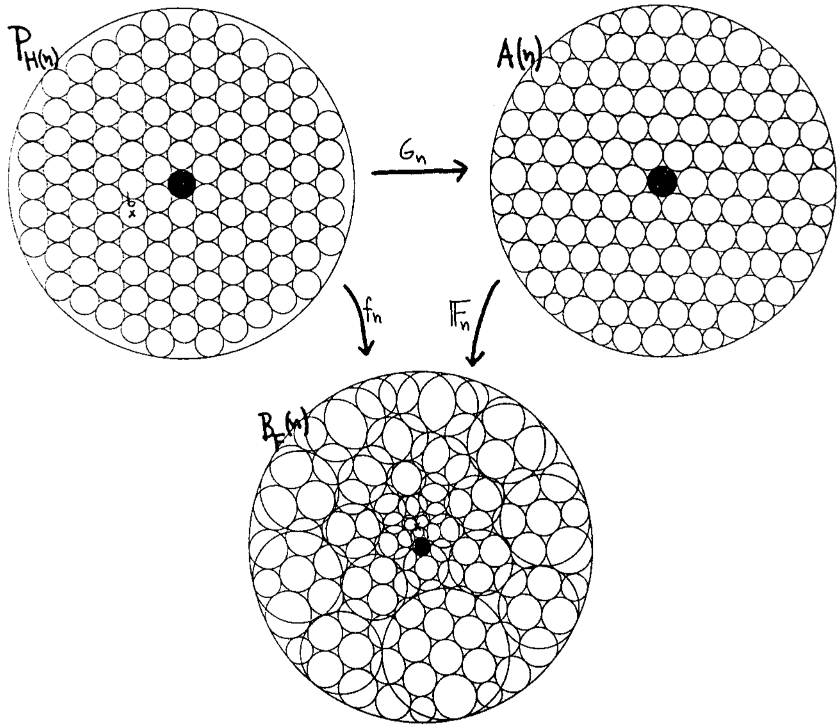

FIGURE 10. Packings and simplicial maps induced by them; $br\big(B_F(n)\big) = \big\{(b, 1)\big\}$

(2) *there exists* $L \in \mathbf{N}$ *such that the combinatorial distance of* $v_{i,j}(n)$, *for any* n *and* $j = 1, \ldots, k_i$, *to the star in* $H(n)$ *that contains* x_i *is at most* L

then $\{B(n)\}$ *will be called an approximating sequence of Bl-type packings for* F.

The reader should be aware of

Remark 4.2. *Because of the nature of* $\mathbf{H}$, *and Theorems II.2 and II.6, there always exists an approximating sequence of Bl-type packings for a given* F. *Moreover, the similar definition to Definition 4.1 can be given for* F *without the normalization by the use of an appropriate Möbius transformation which preserves the unit disc.*

Let us fix an approximating sequence for F and denote it by $\{B_F(n)\}$. Our aim in the reminder of this chapter is to show that the introduced terminology is justified, namely, that discrete Blaschke products induced by the packings $B_F(n)$ converge uniformly to F.

Section 5. Let $v''(n)$ and $\tilde{v}''(n)$ denote the centers of the circle associated with vertex $v(n) \in H(n)$ in $A(n)$ in euclidean and hyperbolic geometry respectively. Write $v'(n)$ and $\tilde{v}'(n)$ for the centers in $B_F(n)$ in euclidean and hyperbolic geometry, respectively. Let $v_0(n)$ denote the vertex in $H(n)$ that corresponds to the vertex at the origin in $\mathbf{H}(n)$.

Proposition 5.1. *There exists $\bar{r} \in (0,1)$ such that $v'_{i,j}(n) \in D(\bar{r})$ for $i = 1,\ldots,m$, $j = 1,\ldots,k_i$ and any n.*

Proof. Since $\{x_1,\ldots x_m\} \subset D(r)$ for some $r \in (0,1)$, it follows from the choice of the $v_{i,j}(n)$ and Section 3 that there is $M \in (0,\infty)$ such that $\rho\big(0,\tilde{v}''_{i,j}(n)\big) \leq M$ for $i = 1,2,\ldots,m$, $j = 1,2,\ldots,k_i$ and any n, where ρ is the hyperbolic metric in D. By Thm[4] of [BSt1], $\rho(0,\tilde{v}'_{i,j}(n)) = \rho(\tilde{v}'_0(n),\tilde{v}'_{i,j}(n)) \leq \rho(\tilde{v}''_0(n),\tilde{v}''_{i,j}(n)) = \rho(0,\tilde{v}''_{i,j}(n))$, and the assertion of our proposition is proved. □

Section 6. In this section we will prove

Lemma 6.1. *The (euclidean) radii of circles in $B_F(n)$ go uniformly to 0 as $n \to \infty$.*

Proof. The following proof is a generalization of the Rodin-Sullivan length-area argument (see [RS]). Let f_n be an <u>extended</u> simplicial map associated with $B_F(n)$, then $f_n(v_0(n)) = f_n(0) = v'_0(n) = 0$ and $f_n(1/2) \in \mathbf{R}_+$. Moreover, by Theorem III.4, f_n can be written as a composition of a self-homeomorphism of D and a Blaschke product with the values of branch points at $f_n(v_{i,j}(n))$'s.

a). Recall that $f_n(v_{i,j}(n)) = v'_{i,j}(n) \in D(\bar{r})$ for any i,j and n. Let us fix n. Let S_j be a chain of n_j circles in $B_F(n)$ of radii r_{ji} that starts and ends at the boundary of D. Then, by the Cauchy-Schwarz inequality,

$$\left(\sum_{i=1}^{n_j} r_{ji}\right)^2 \leq n_j \sum_{i=1}^{n_j} r_{ji}^2.$$

Let $l_j = 2\sum_{i=1}^{n_j} r_{ji}$. Simple computations show

$$(*) \qquad \sum_j l_j^2 \frac{1}{n_j} \leq 4 \sum_{i,j} r_{ji}^2.$$

Now, let us consider a boundary vertex v_* of $H(n)$ and the collection of edge paths s_j in $H(n)$ that are as in Figure 11 (i.e. s_j starts and ends at ∂D, it omits the $v_{i,j}(n)$ and it cuts out a "wedge" of D, denoted by D_j, that does not contain the $v_{i,j}(n)$ and has v_* in "the middle" of $\partial H(n) \cap D_j$).

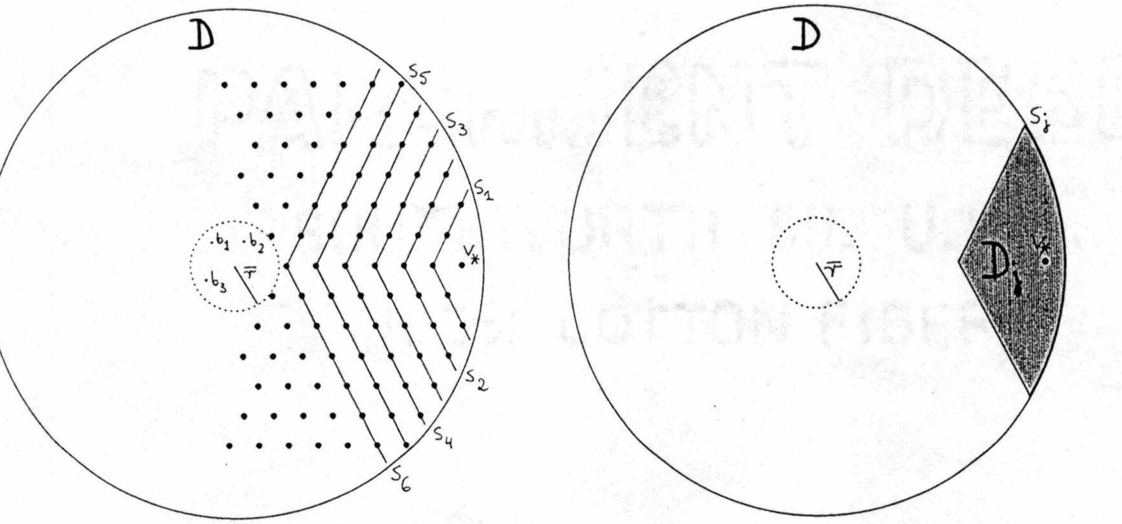

FIGURE 11. Paths s_j and a wedge D_j

Let $\mathcal{S}_j$ be a chain of circles in $B_F(n)$ associated with vertices of s_j. We will show the following inequality

$$(**) \qquad l_j \geq (1 - \bar{r}) r_n(v_*),$$

where $r_n(\cdot)$ is the radius function of $B_F(n)$. By Part 2, we have two possibilities for fixed j:

(1) $f_n|_{D_j}$ is a homeomorphism,

(2) the path $f_n(s_j)$ has self-intersection.

In case (1) the circle in $B_F(n)$ associated with v_* is contained in $f_n(D_j)$ and, by Fact 1.1, $D \setminus f_n(D_j)$ contains at least one value $f_n(v_{i,j}(n))$. Thus

$$l(f_n(s_j)) \geq \min\left\{\min_{i,j}\left\{dist\big(f_n(v_{i,j}(n)), \partial D\big)\right\}, r_n(v_*)\right\} \geq \min\{1-\bar{r}, r_n(v_*)\},$$

where $l(f_n(s_j))$ denotes the length of the path $f_n(s_j)$. Since $l_j = l(f_n(s_j))$, one has $(**)$ in case (1).

In case (2), by Fact 1.2 we get

$$l(f_n(s_j)) \geq \min_{i,j}\left\{dist\big(f_n(v_{i,j}(n)), \partial D\big)\right\} \geq 1-\bar{r}.$$

Thus $(**)$ holds in case (2) as well.

Combining inequalities $(*)$ and $(**)$ one obtains

$$(***) \qquad r_n^2(v_*) \leq \frac{4}{(1-\bar{r})^2}\left(\sum_{i,j} r_{ji}^2\right)\left(\sum_j \frac{1}{n_j}\right)^{-1}.$$

b). We will now divide the set of double indices ji into four pairwise disjoint groups J_1, J_2, J_3, J_4. First let us mark the vertices in the paths s_j as in Figure 12 (i.e. if j is odd then every second vertex in s_j is marked by 1 and every other vertex in s_j is marked by 2, if j is even then every second vertex in s_j is marked by 3 and every other vertex in s_j is marked by 4).

Then we say that a double index ji is in J_k if r_{ji} is associated with the vertex that was marked by k, $k = 1,2,3,4$. Notice that the stars in $H(n)$ associated with distinct vertices in J_k are disjoint. Moreover, if $C_n(v)$ is the circle in $B_F(n)$

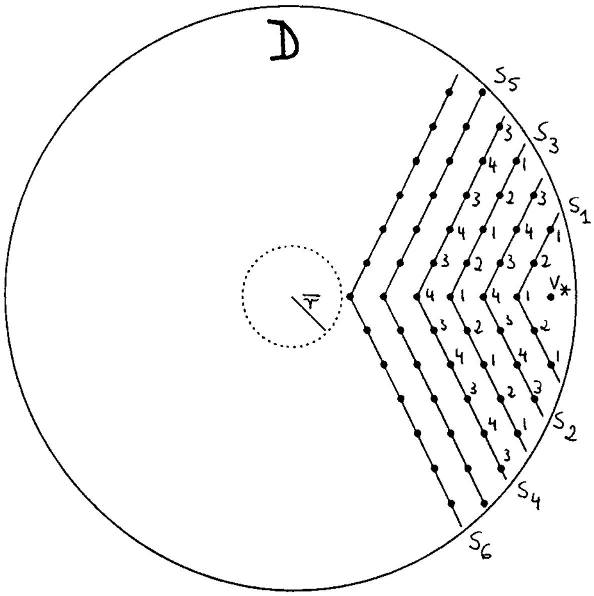

FIGURE 12. Marking of vertices

associated with the vertex v in $H(n)$ then $C_n(v)$ is contained in $f_n(star(v))$, where as before $star(v)$ denotes the star in $H(n)$ associated with the vertex v. We also observe that the star in D is mapped homeomorphically by f_n, except for the stars that correspond to the vertices $v_{i,j}(n)$. Thus, since f_n is M-to-1, we have

$$\sum_{ji\in J_k} \pi r_{ji}^2 \leq M\pi \quad \text{for } k = 1, 2, 3, 4,$$

where $M = 1 + \sum_{i=1}^{m} k_i$. Hence

$$\sum_{j,i} \pi r_{ji}^2 \leq 4M\pi.$$

The last inequality and $(***)$ show that

$$r_n(v_*) \leq \frac{16}{(1-\bar{r})^2} M \left(\sum_j \frac{1}{n_j} \right)^{-1}.$$

Now, as $n \to \infty$ the series $\sum_j \frac{1}{n_j}$ diverges because $n_j = O(j)$, and the last inequality implies that the radii of circles in $B_F(n)$ associated with boundary vertices of $H(n)$ go to 0 uniformly in n. By Lemma 1.1, the radius function of $B_F(n)$ is subharmonic, which gives that the radii of **all** circles in $B_F(n)$ go uniformly to 0 as $n \to \infty$. $\square$

Section 7. In this section we will be assuming that <u>all</u> branch points of F are simple. In particular, since $k_i = 1$ for $i = 1, \ldots, m$, we will write $v_i(n)$ for $v_{i,1}(n)$.

Let f_n be an <u>extended</u> simplicial map associated with $B_F(n)$. We recall that $f_n = \varphi_n \circ h_n$, where $h_n : \overline{D} \mapsto \overline{D}$ is a homeomorphism and φ_n is a Blaschke product with branch points $h_n(v_i(n))$. Since $f_n(0) = 0$ and $f_n(1/2) \in \mathbf{R}_+$, we will assume that $h_n(0) = 0 = \varphi_n(0)$ and $h_n(1/2) \in \mathbf{R}_+$. (If necessary we use the appropriate Möbius transformation and its inverse composed with h_n and φ_n, respectively, to obtain a new factorization of f_n.) Moreover, we observe that all the products φ_n are of the same order $m+1$.

In order to keep our notation as clear as possible, we will write $f_n(v_i)$ and $h_n(v_i)$ for $f_n(v_i(n))$ and $h_n(v_i(n))$, respectively, if there is no reason for confusion.

Let us now recall some facts from the theory of quasiconformal and quasiregular functions that will be stated in two parts **a′)** and **a″)**. All of them can be found in [LV] or [V].

a′). If $\mathcal{C} \subset \mathcal{A} \subset \mathbf{R}^2$ are such that $\mathcal{A}$ is open and $\mathcal{C}$ is compact then $(\mathcal{A}, \mathcal{C})$ will be called a **condenser** and *cap*$(\mathcal{A}, \mathcal{C})$ will be the **conformal capacity** of $(\mathcal{A}, \mathcal{C})$. All the $(\mathcal{A}, \mathcal{C})$ condensers we consider will be **bounded**, which simply means that $\mathcal{A}$ is a bounded set. If $E = (\mathcal{A}, \mathcal{C})$ is a bounded condenser in $\mathbf{R}^2$ then

$$cap\, E = M(\Delta(\mathcal{C}, \partial\mathcal{A}; \mathcal{A})),$$

where $M(\Delta(\mathcal{C}, \partial\mathcal{A}; \mathcal{A}))$ is the modulus of the curve family $\Delta(\mathcal{C}, \partial\mathcal{A}; \mathcal{A})$ of paths in $\mathcal{A}$ that connect $\mathcal{C}$ with $\partial\mathcal{A}$. Moreover, if $\mathcal{C}' \subset \mathcal{C}$, $\mathcal{C}'$ is compact, then $E' := (\mathcal{A}, \mathcal{C}')$

is a bounded condenser in $\mathbf{R}^2$ and

$$M(\Gamma) \geq M(\Gamma'),$$

where $\Gamma = \Delta(\mathcal{C}, \partial\mathcal{A}; \mathcal{A})$ and $\Gamma' = \Delta(\mathcal{C}', \partial\mathcal{A}; \mathcal{A})$. In particular $cap\, E \geq cap\, E'$.

If $\mathcal{D} = (\mathcal{C}_0, \mathcal{C}_1)$ is a condenser and $\mathcal{C}_0 \setminus \mathcal{C}_1$ is a doubly connected domain then $\mathcal{D}$ is called a **ring**. The **modulus** of $\mathcal{D}$ is

$$mod\, \mathcal{D} = \left(\frac{M(\Delta(\mathcal{C}_1, \partial\mathcal{C}_0; \mathcal{C}_0 \setminus \mathcal{C}_1))}{2\pi} \right)^{-1}.$$

Moreover, $mod\, \mathcal{D} = t$ if and only if $\mathcal{D}$ can be mapped conformally onto the annulus $\{z : 1 < |z| < e^t\}$. Also, the capacity $cap\, \mathcal{D}$ ($\equiv cap(\mathcal{C}_0, \mathcal{C}_1)$) of the ring $\mathcal{D}$ is equal to $M(\Delta(\mathcal{C}_1, \partial\mathcal{C}_0; \mathcal{C}_0 \setminus \mathcal{C}_1))$, so

$$mod\, \mathcal{D} = \frac{2\pi}{cap\, \mathcal{D}}.$$

If $\mu(r) := mod\big((D, [0, r])\big)$ then $\mu(r)$ is continuous in $(0,1)$, monotonically decreasing, and $\lim_{r\to 1} \mu(r) = 0$, $\lim_{r \to 0} \mu(r) = \infty$. In addition one has

Modulus Theorem. *If the ring $\mathcal{D}$ separates the points 0 and r from the circle $\{z : |z| = 1\}$ then $mod\, \mathcal{D} \leq \mu(r)$.*

a$''$). Let $f : G \mapsto \mathbf{R}^2$ be a continuous function. We say that $(\mathcal{A}, \mathcal{C})$ is a **normal condenser** for f if $(\mathcal{A}, \mathcal{C})$ is a condenser such that $\overline{\mathcal{A}} \subset G$ and $f(\partial\mathcal{A}) = \partial f(\mathcal{A})$. If f is non-constant and quasiregular then

$$cap(\mathcal{A}, \mathcal{C}) \leq K_0(f) N(f, \mathcal{A}) cap\big(f(\mathcal{A}, \mathcal{C})\big),$$

for all normal condensers $(\mathcal{A}, \mathcal{C})$ for f in G, where $K_0(f)$ denotes outer dilatation of f, $N(f, \mathcal{A})$ is the maximal number of preimagies in $\mathcal{A}$ of any point from $f(\mathcal{A})$, and $f(\mathcal{A},\mathcal{C}) = \big(f(\mathcal{A}), f(\mathcal{C})\big)$. In particular, if f is a Blaschke product of order $m+1$ then

$$cap(D,\mathcal{C}) \leq (m+1)cap(D, f(\mathcal{C})),$$

for any compact subset $\mathcal{C}$ of D. Moreover, if $\mathcal{C}'$ is simply connected, $\overline{\mathcal{C}'} \subset D$, and $\mathcal{C}$ is a component of $f^{-1}(\mathcal{C}')$, then

$$mod\,\mathcal{D}_2 \leq (m+1)mod\,\mathcal{D}_1,$$

where $\mathcal{D}_1 := (D,\mathcal{C})$ and $\mathcal{D}_2 := (D,\mathcal{C}')$.

After all these preliminaries, we return now to our functions h_n and φ_n.

a). Recall that, by Section 5, the images $f_n(v_i)$ of branch points $h_n(v_i)$ of φ_n are in $B(0,\bar{r})$, where $B(z_0,\varrho) = \{z : |z-z_0| < \varrho\}$. We will now use K-quasiregular arguments for K=1 to show that the sets $\varphi_n^{-1}(f_n(v_i))$ are contained in a compact subset of D. Since each φ_n is a Blaschke product of order $m+1$, by part **a**$''$) we get that

$$-\ln\bar{r} = mod\big(D, B(0,\bar{r})\big) \leq (m+1)mod\big(D, \varphi_n^{-1}(B(0,\bar{r}))\big).$$

The Modulus Theorem implies that there exists $\delta > 0$ such that $\varphi_n^{-1}(B(0,\bar{r})) \subset B(0,1-\delta)$ for any n. Thus we have

$$\exists\delta > 0 \ni \varphi_n^{-1}(f_n(v_i)) \subset B(0,1-\delta) \text{ for any } n \text{ and } i = 1,2,\ldots,m.$$

In particular, all the points $h_n(v_i)$ are contained in $B(0,1-\delta)$ for any n.

b). Suppose that $\epsilon > 0$. Let $d_i^\epsilon(n) := \sup\{diam\,C_i^\epsilon(n)\}$, where sup is taken over all components $C_i^\epsilon(n)$ of $\varphi_n^{-1}\big(B(f_n(v_i),\epsilon)\big)$ (Figure 13). Let $d^\epsilon(n) := \sup_i\{d_i^\epsilon(n)\}$. Once again, by quasiregular arguments and the fact that images of branch points of functions φ_n are in $B(0,\bar{r})$ we obtain that $d^\epsilon(n) \to 0$ as $\epsilon \to 0$ uniformly in

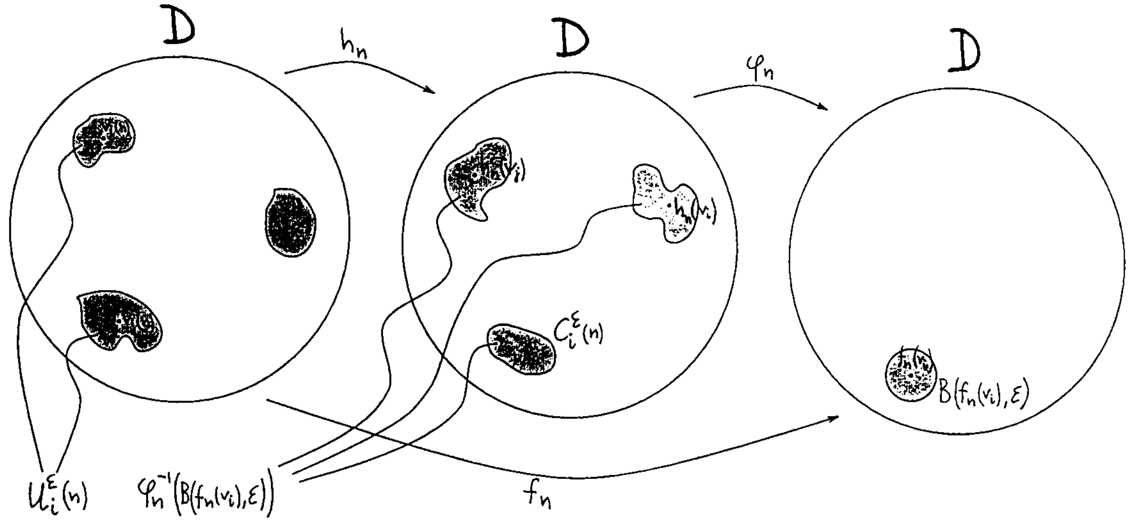

FIGURE 13. Mappings and sets

n (i.e. for any $\sigma > 0$ there exists $\epsilon(\sigma)$ such that $d^\epsilon(n) < \sigma$ for any n and any ϵ, $\epsilon < \epsilon(\sigma)$).

c). We will now prove a lemma about quasiconformality of the functions h_n.

Lemma 7.1. *Let $\mathcal{U}_i^\epsilon(n)$ be the union of all components of $f_n^{-1}(B(f_n(v_i), \epsilon))$ that have non-empty intersection with the set $\{v_j(n) : j = 1, 2, \ldots, m\}$. Let $H_\epsilon(n) := H'(n) \setminus \bigcup_i \overline{\mathcal{U}_i^\epsilon(n)}$. Then for any $\epsilon > 0$ there exists N_ϵ such that h_n is K-quasiconformal in $H_\epsilon(n)$ for any $n > N_\epsilon$, where K does not depend on n or ϵ.*

Proof. Let $\epsilon > 0$. Let $S_k(v, n)$ denote the simplicial complex of faces of k generations in $H(n)$ around vertex $v \in H'(n)$. By Lemma 6.1, there exists N_ϵ such that for any $n > N_\epsilon$

$$f_n(S_3(v, n)) \bigcap \{f_n(v_i)\}_{i=1}^m = \emptyset,$$

for any vertex $v \in H'(n) \setminus \bigcup_i \overline{f_n^{-1}(B(f_n(v_i), \epsilon))}$. Thus, according to Lemma III.7, for any vertex $v \in H'(n) \setminus \bigcup_i \overline{f_n^{-1}(B(f_n(v_i), \epsilon))}$ no flower in $B_F(n)$ associated with a vertex of $S_1(v,n)$ has overlapping petals. Hence, the Ring Lemma in [RS] (applied to $f_n|_{S_3(v,n)}$) implies the existence of a constant K (associated with the complex **H**) such that h_n restricted to $H'(n) \setminus \bigcup_i \overline{f_n^{-1}(B(f_n(v_i),\epsilon))}$ is K-quasiconformal for any n, $n > N_\epsilon$.

Let $\mathcal{V}_i^\epsilon(n)$ denote a component of $f_n^{-1}(B(f_n(v_i),\epsilon)) \bigcap H_\epsilon(n)$. To complete the proof of the lemma we only need to show now that there exists N_ϵ such that $f_n|_{\mathcal{V}_i^\epsilon(n)}$ is K-quasiconformal for $n > N_\epsilon$ and $i = 1, \ldots, m$.

A property of Blaschke products tells us that $\varphi_n|_{h_n(\mathcal{V}_i^\epsilon(n))}$ is a homeomorphism, thus $f_n|_{\mathcal{V}_i^\epsilon(n)}$ is a homeomorphism. Lemma III.7 and Lemma 6.1 imply that there exists N_ϵ such that the following hold:

(1) If $n > N_\epsilon$ and $p \in H'(n)$ such that $f_n(p) \in D \setminus \bigcup_i B(f_n(v_i), \epsilon/2)$ then no flower in $B_F(n)$ associated with any vertex of $S_{10}(v_p, n)$ has overlapping petals, where v_p is a vertex in $H'(n)$ closest to the point p,

(2) If $n > N_\epsilon$ and $p \in H'(n)$ such that $f_n(p) \in B(f_n(v_i), \epsilon/2)$ for some i then $f_n(S_4(v_p, n)) \subset B(f_n(v_i), \epsilon)$, where v_p is defined as in (1).

Let w be a vertex in $\mathcal{V}_i^\epsilon(n)$, then:

1) If $n > N_\epsilon$ and w is such that $S_3(w,n) \subset \mathcal{V}_i^\epsilon(n)$ then there are no overlapping circles in $B_F(n)$ associated with vertices of $S_2(w,n)$ because $f_n|_{S_3(w,n)}$ is a homeomorphism,

2) If $n > N_\epsilon$ and w is such that $f_n(w) \in D \setminus \bigcup_i B(f_n(v_i), \epsilon/2)$ then, by (1), no flower in $B_F(n)$ associated with any vertex of $S_3(w,n)$ has overlapping petals,

3) If $n > N_\epsilon$ and there exists i' such that $f_n(w) \in B(f_n(v_{i'})), \epsilon/2)$ then $w \in \mathcal{C}$ for some component $\mathcal{C}$ of $f_n^{-1}(B(f_n(v_{i'})), \epsilon))$. By (2), $S_3(w,n) \subset \mathcal{C}$. Moreover, $\mathcal{C}$ does not contain any of $v_i(n)$'s, otherwise, according to the

definition of $H_\epsilon(n)$, $C \cap H_\epsilon(n) = \emptyset$ and we get a contradiction with the fact that $w \in C \cap H_\epsilon(n)$. In particular, $f_n|_C$ is a homeomorphism and so is $f_n|_{S_3(w,n)}$. Thus there are no overlapping circles in $B_F(n)$ associated with vertices of $S_2(w,n)$.

Let p be a point in $\mathcal{V}_i^\epsilon(n)$ but not a vertex. Let w_p be a vertex in $H'(n)$ closest to the point p, then:

4) The cases 1)–3), as above, follow for w replaced by w_p.

From the cases 1)–4), the fact that $p \in star(w_p)$, and the Ring Lemma in [RS], we obtain that $h_n|_{\mathcal{V}_i^\epsilon(n)}$ is K-quasiconformal for $n > N_\epsilon$. $\square$

d). In this part we will establish properties of the families $\{\varphi_n\}$ and $\{h_n\}$.

Let us start with the family $\{\varphi_n\}$. We observe that $\{\varphi_n\}$ is a normal family, and each φ_n is a Blaschke product of order $m+1$. Thus, there exists a subsequence $\{\varphi_{n_k}\}$ that converges uniformly on compact subsets of D to a function φ. From part a) and Hurwitz's Theorem it follows that φ is a Blaschke product of order $m+1$ with branch points $b_1, b_2, \ldots, b_l \in B(0,\bar{r})$. We will now prove

Lemma 7.2. *If $\{\varphi_{n_k}\}$ and $b_1, \ldots, b_l$ are defined as above then $l = m$.*

Proof. Since φ_n's have m simple branch points, it follows that $1 \le l \le m$. Thus, if $m = 1$ there is nothing to prove, so let us assume that $m > 1$. From the definition of φ we get that $\{b_1, \ldots, b_l\}$ is the limit set of the set $\{h_{n_k}(v_1), \ldots, h_{n_k}(v_m)\}_{k=0}^{\infty}$ and, without loss of generality, we may assume that there exist $1 \le t_1 < \cdots < t_l \le m$ such that $\lim_{k\to\infty} h_{n_k}(v_i) = b_j$ for $i = t_{j-1}+1, \ldots, t_j$. In order to prove that $l = m$ one has to show that $\lim_{k\to\infty} h_{n_k}(v_i) \neq \lim_{k\to\infty} h_{n_k}(v_{i'})$ when $i \neq i'$.

Let $C_i^\epsilon(n)$ be as in part b). Let $\tilde{C}_i^\epsilon(n)$ denote $C_i^\epsilon(n)$ that has non-empty intersection with the set $\{h_n(v_j)\}_{j=1}^m$. Write $\tilde{d}_i^\epsilon(n) := \sup\left\{diam\, h_n^{-1}\left(\tilde{C}_i^\epsilon(n)\right)\right\}$, where the sup is taken over all $\tilde{C}_i^\epsilon(n)$'s.

Proposition 7.3. *If $\tilde{d}_i^\epsilon(n)$ is defined as above then*

$$(\Omega) \qquad \forall\sigma > 0\ \exists\epsilon_\sigma\ \forall\epsilon < \epsilon_\sigma\ \exists N_\epsilon\ \forall n > N_\epsilon\ \forall i\ \tilde{d}_i^\epsilon(n) < \sigma.$$

Proof. Since, by Lemma 6.1, $f_n(\partial H'(n))$ approaches ∂D uniformly as $n \to \infty$ (i.e. $f_n(\partial H'(n))$ is in a small neighborhood of ∂D for large n), the Schwarz Lemma applied to φ_n shows that $\partial\big(h_n(H'(n))\big) = h_n(\partial H'(n))$ approaches ∂D uniformly as $n \to \infty$. Thus, we may assume without loss of generality that $B(0, 1-\delta/2) \subset h_n(H'(n))$ for any n, where δ is as in part a) (i.e. δ is such that $h_n(v_i)$ is in $B(0, 1-\delta)$ for any i and n). The last and part b) show that there exists ϵ_0 such that $\tilde{C}_i^\epsilon(n) \subset B(0, 1-\delta/2)$ for any i, n and $\epsilon < \epsilon_0$.

Let $C^\epsilon(n) := \bigcup \tilde{C}_i^\epsilon(n)$ and $\widetilde{E}^\epsilon(n) := \Big(h_n(H'(n)), \overline{C^\epsilon(n)}\Big)$, where $\overline{C^\epsilon(n)}$ denotes the closure of $C^\epsilon(n)$. Then $\widetilde{E}^\epsilon(n)$ is a bounded condenser and, by quasiregular arguments, part b), and above remarks, $cap\,\widetilde{E}^\epsilon(n) \to 0$ as $\epsilon \to 0$ uniformly in n.

Let $\mathcal{U}^\epsilon(n) := \bigcup_i \mathcal{U}_i^\epsilon(n)$, using the notation of Lemma 7.1, and note that $h_n\big(\overline{\mathcal{U}^\epsilon(n)}\big) = \overline{C^\epsilon(n)}$. Writing $E^\epsilon(n)$ for the bounded condenser $\Big(H'(n), \overline{\mathcal{U}^\epsilon(n)}\Big)$, we have by part a'),

$$(*) \qquad cap\, E^\epsilon(n) \le \mathrm{K} cap\, \widetilde{E}^\epsilon(n) \text{ for } n > N_\epsilon,$$

where N_ϵ is as in Lemma 7.1. Thus

$$(**) \qquad \forall\sigma > 0\ \exists\epsilon_\sigma\ \forall\epsilon < \epsilon_\sigma\ \exists N_\epsilon\ \forall n > N_\epsilon\ \ cap\, E^\epsilon(n) < \sigma.$$

Since $\widetilde{E}^\epsilon(n)$ is a bounded condenser for any i, n, and $\epsilon < \epsilon_0$, so is the ring $\Big(h_n(H'(n)), \overline{\tilde{C}_i^\epsilon(n)}\Big)$ and, by part a'),

$$(***) \qquad cap\left(H'(n), h_n^{-1}\left(\overline{\tilde{C}_i^\epsilon(n)}\right)\right) \le cap\, E^\epsilon(n) \text{ for any } i,\ n, \text{ and } \epsilon < \epsilon_0.$$

The Modulus Theorem, $(**)$, and $(***)$ imply the assertion (Ω). $\square$

Proposition 7.4. $\lim_{k\to\infty} h_{n_k}(v_i) = b_p$ *if and only if* $\lim_{k\to\infty} h_{n_k}(x_i) = b_p$, *where* x_i *is one of the branch points of* F.

Proof. The definition of $v_i(n)$ and the fact that the mesh of $H(n)$ goes to 0 as $n \to \infty$ imply that $v_i(n) \to x_i$ as $n \to \infty$ for $1 \leq i \leq m$. Now Proposition 7.3 gives

$$(*) \qquad \forall \sigma > 0 \; \exists \epsilon_\sigma \; \exists N_\sigma \; \forall n > N_\sigma \; \bigcup_i \mathcal{U}_i^{\epsilon_\sigma}(n) \subset \bigcup_{i=1}^m B(x_i, \sigma),$$

where $\mathcal{U}_i^\epsilon(n)$ is defined as in Lemma 7.1. Let

$$\chi = \min\left\{ \min_{i \neq j} |x_i - x_j|, \min_{1 \leq i \leq m} dist(x_i, \partial D) \right\}.$$

Suppose that $\sigma < \chi/100$. Let ϵ_σ and N_σ be as in $(*)$. We may again assume without loss of generality that $\bigcup_i B(x_i, \chi/2) \subset H'(n)$ for any n. Let

$$\mathcal{D}_i(\sigma) := \left(B(x_i, \chi/2), \overline{B(x_i, \sigma)} \right).$$

Then $\mathcal{D}_i(\sigma) \subset H'(n) \setminus \bigcup_{i=1}^m B(x_i, \sigma)$ for $n > N_\sigma$, and by $(*)$ one has

$$\mathcal{D}_i(\sigma) \subset H_{\epsilon_\sigma}(n).$$

Lemma 7.1 implies that for every σ, $\sigma < \chi/100$, the homeomorphism h_n is K-quasiconformal in $\mathcal{D}_i(\sigma)$ for sufficiently large n, so we can assume, by making N_σ larger if necessary, that h_n is K-quasiconformal in $\mathcal{D}_i(\sigma)$ for $n > N_\sigma$. In particular, $mod\, \mathcal{D}_i(\sigma) \leq \mathrm{K}\, mod\, h_n(\mathcal{D}_i(\sigma))$ for $n > N_\sigma$, and there exists K_σ such that

$$(**) \qquad mod\, \mathcal{D}_i(\sigma) \leq \mathrm{K}\, mod\, h_{n_k}(\mathcal{D}_i(\sigma)) \text{ for } k > K_\sigma.$$

Since the rings $\mathcal{D}_i(\sigma)$ are such that $\lim_{\sigma\to 0} mod\, \mathcal{D}_i(\sigma) = \infty$, the Modulus Theorem and $(**)$ give

$$\forall \tau > 0 \; \exists \sigma_\tau \; \forall \sigma < \sigma_\tau \; \exists K_\sigma \; diam\, h_{n_k}(B(x_i, \sigma)) < \tau \text{ for } k > K_\sigma.$$

As $\lim_{n\to\infty} v_i(n) = x_i$, the last shows that $\lim_{k\to\infty} |h_{n_k}(v_i) - h_{n_k}(x_i)| = 0$. This finishes the proof of our proposition. $\square$

Proposition 7.5. $\lim_{k\to\infty} h_{n_k}(x_i) \neq \lim_{k\to\infty} h_{n_k}(x_{i'})$ *for* $i \neq i'$.

Proof. Suppose that the assertion of the proposition is not true. Then, without loss of generality, we can assume that $\lim_{k\to\infty} h_{n_k}(x_1) = \lim_{k\to\infty} h_{n_k}(x_2)$ and, by Propostion 7.4, $\lim_{k\to\infty} h_{n_k}(x_1) = b_1$. Part b) and the definitions of $\tilde{\mathcal{C}}_i^\epsilon(n)$ and $\mathcal{U}_i^\epsilon(n)$ imply

$$(*) \qquad \forall \sigma > 0\ \exists \epsilon_\sigma, K_\sigma\ \forall k > K_\sigma\ \bigcup_i h_{n_k}(\mathcal{U}_i^{\epsilon_\sigma}(n_k)) \subset \bigcup_{i=1}^{l} B(b_i, \sigma).$$

Let

$$\tilde{\chi} = \min\left\{ \min_{i\neq j} |b_i - b_j|,\ \min_{1\leq i\leq l} dist(b_i, \partial D) \right\}.$$

Since φ_{n_k} fixes 0 and $\varphi_{n_k}(h_{n_k}(\partial H'(n_k))) = f_{n_k}(\partial H'(n_k))$ approaches ∂D as $k \to \infty$, the Schwarz Lemma applied to φ_{n_k} shows that $\partial\big(h_{n_k}(H'(n_k))\big)$ approaches ∂D as $k \to \infty$. Therefore, we can assume without loss of generality that $B(b_1, \tilde{\chi}/2) \subset h_{n_k}(H'(n_k))$ for any $k > K_\sigma$ and $\sigma < \tilde{\chi}/100$, where K_σ is defined as in $(*)$. Moreover, for any $\sigma < \tilde{\chi}/100$ and $k > K_\sigma$ one has

$$\widetilde{\mathcal{D}}_1(\sigma) := \Big(B(b_1, \tilde{\chi}/2), \overline{B(b_1,\sigma)}\Big) \subset h_{n_k}(H'(n_k)) \setminus \bigcup_{i=1}^{l} B(b_i, \sigma) \subset h_{n_k}(H_{\epsilon_\sigma}(n_k)).$$

Now Lemma 7.1 implies that for any σ, $\sigma < \tilde{\chi}/100$, the homeomorphism $h_{n_k}^{-1}$ is K-quasiconformal in $H_{\epsilon_\sigma}(n_k) \supset h_{n_k}^{-1}\left(\widetilde{\mathcal{D}}_1(\sigma)\right)$ for sufficiently large k. Therefore, by making K_σ large enough if necessary, we can assume that $h_{n_k}^{-1}$ is K-quasiconformal in $h_{n_k}^{-1}\left(\widetilde{\mathcal{D}}_1(\sigma)\right)$ for $k > K_\sigma$. Hence

$$(**) \qquad mod\, \widetilde{\mathcal{D}}_1(\sigma) \leq \mathrm{K} mod\, h_{n_k}^{-1}(\widetilde{\mathcal{D}}_1(\sigma)) \text{ for } k > K_\sigma \text{ and } \sigma < \tilde{\chi}/100.$$

Since $\lim_{k\to\infty} h_{n_k}(x_1) = b_1 = \lim_{k\to\infty} h_{n_k}(x_2)$, for any σ there exists k_σ such that $h_{n_k}(v_1), h_{n_k}(v_2) \in B(b_1, \sigma)$ for $k > k_\sigma$, and without loss of generality we may assume that $K_\sigma \geq k_\sigma$. In particular, $v_1(n_k), v_2(n_k) \in h_{n_k}^{-1}(B(b_1,\sigma))$ for $k > K_\sigma$. Now recall that $\lim_{k\to\infty} v_1(n_k) = x_1$ and $\lim_{k\to\infty} v_2(n_k) = x_2$, hence $diam\, h_{n_k}^{-1}(B(b_1,\sigma)) > \chi/2$ for sufficiently large k, say $k > K_\sigma$, where χ is defined as in the proof of Proposition 7.4. The Modulus Theorem shows that for any $\sigma < \tilde{\chi}/100$

$$(***) \qquad mod\, h_{n_k}^{-1}(\widetilde{\mathcal{D}}_1(\sigma)) \leq const(\tilde{\chi}) \text{ for } k > K_\sigma \;,$$

where $const(\tilde{\chi})$ depends only on $\tilde{\chi}$. Combining $(**)$ and $(***)$, and letting $\sigma \to 0$, we obtain a contradiction. This finishes the proof of the proposition. □

The above proposition completes the proof of Lemma 7.2. □

We can summarize the results about subsequences $\{\varphi_{n_k}\}$ and $\{h_{n_k}\}$ which we have obtained so far in the following

Lemma 7.6. *The limit function φ of the sequence $\{\varphi_{n_k}\}$ is a Blaschke product with simple branch points at $b_1, \ldots, b_m$, where $b_i = \lim_{k\to\infty} h_{n_k}(x_i)$ for $i = 1, 2, \ldots, m$.*

We now turn our attention to quasiconformal properties of the family $\{h_{n_k}\}$.

Let $D_n := \{z : |z| < 1 - 1/n\} \setminus \bigcup_{i=1}^{m} B(x_i, 1/n)$ and $\widetilde{D}_n := \{z : |z| < 1 - 1/n\} \setminus \bigcup_{i=1}^{m} B(b_i, 1/n)$. Since $\partial H'(n)$ tends to ∂D, $(*)$ in the proof of Proposition 7.4 shows that for any $l \in \mathrm{N}$ there exist ϵ_l and N_l such that $D_l \subset H_{\epsilon_l}(n)$ for $n > N_l$. In particular, from Lemma 7.1 it follows

$$(\bigstar) \qquad \forall l \in \mathrm{N}\ \exists K_l\ \forall k > K_l\ h_{n_k}|_{D_l} \text{ is K-quasiconformal.}$$

Similarly, $(*)$ in the proof of Proposition 7.5 and the fact that $\partial\big(h_{n_k}(H'(n_k))\big)$ tends to ∂D show that for any $l \in \mathrm{N}$ there exist ϵ_l and $\widetilde{K}'_l$ such that $\widetilde{D}_l \subset h_{n_k}(H_{\epsilon_l}(n_k))$ for $k > \widetilde{K}'_l$. Thus

$$\widetilde{(\bigstar)} \qquad \forall l \in \mathrm{N}\ \exists \widetilde{K}_l\ \forall k > \widetilde{K}_l\ h_{n_k}^{-1}|_{\widetilde{D}_l} \text{ is K-quasiconformal.}$$

The way the functions h_{n_k} are defined, ($\bigstar$), $\widetilde{(\bigstar)}$, and standard quasiconformal arguments show

Lemma 7.7. *The family $\{h_{n_k}\}$ of uniformly bounded functions has the following properties:*

(1) *$\{h_{n_k}\}$ is a normal family in $D \setminus \{x_1, x_2, \ldots, x_m\}$,*

(2) *any convergent subsequence of $\{h_{n_k}\}$ tends uniformly on compacta of $D \setminus \{x_1, x_2, \ldots, x_m\}$ to some K-quasiconformal function h,*

(3) *the limit function h is the restriction of a K-quasiconformal homeomorphism $\tilde{h}$, $\tilde{h} : D \mapsto D$, with $\tilde{h}(x_i) = b_i$ for $i = 1, 2, \ldots, m$, and*

(4) *due to the normalization condition on the functions h_{n_k}, the function $\tilde{h}$ fixes 0 and $\tilde{h}(1/2) \in \mathbf{R}_+$.*

We are now going to prove

Lemma 7.8. *Let $\{h_{n'_k}\}$ be a convergent subsequence of $\{h_{n_k}\}$. Let h be the limit function of $\{h_{n'_k}\}$ and $\tilde{h}$ be the function described as in Lemma 7.7 (3). Then $\tilde{h} = id_D$.*

Proof. In order to prove our statement we will show that h is 1-quasiconformal, i.e. conformal. Then also $\tilde{h}$ is 1-quasiconformal, and the Riemann Mapping Theorem implies that $\tilde{h} = id_D$.

Let $z_0 \in D \setminus \{x_1, \ldots, x_m\}$. Choose $\delta > 0$ so that $\overline{B(z_0, \delta)} \subset D \setminus \{x_1, \ldots, x_m\}$. Then $h_{n'_k}|_{\overline{B(z_0,\delta)}}$ converges uniformly to $h|_{\overline{B(z_0,\delta)}}$. Since $h(z_0) \in D \setminus \{b_1, \ldots, b_m\}$, $h(z_0)$ is not a branch point of φ, and there are ϵ and k_0 such that $\varphi_{n_k}|_{B(h(z_0),\epsilon)}$ are 1-to-1 for $k > k_0$. The equicontinuity of the normal family $\{h_{n'_k}\}$ implies the existence of σ, $\sigma \le \delta$, and k_1, $k_1 \ge k_0$, such that $h_{n'_k}(B(z_0, \sigma)) \subset B(h(z_0), \epsilon)$ for $k > k_1$. Thus $f_{n_k}|_{B(z_0,\sigma)}$ is 1-to-1 for $k > k_1$.

Now we are ready to make a crucial step in showing that h is 1-qusiconformal. As $k \to \infty$, we have an ever increasing number of generations of $H(n'_k)$'s around z_0 in $B(z_0, \sigma)$ and associated with them univalent (i.e. non-overlapping) hexagonal circle packings that are parts of the corresponding packings $B_F(n'_k)$. From the Hexagonal Packing Lemma in [RS] we conclude that the quasiconformal distortion of the $f_{n'_k}$ at z_0 goes to 0 as $k \to \infty$. (We remark here that in [RS] circle packings are, by definition, univalent packings.) Hence $f := \varphi \circ h = \lim_{k\to\infty}(\varphi_{n'_k} \circ h_{n'_k})$ has the quasiconformal distortion at z_0 equal to 0. As z_0 was arbitrary, this shows that f is 1-quasiregular and h is 1-quasiconformal. □

As an immediate consequence of the above lemma we have

Corollary 7.9. *The sequence $\{h_{n_k}\}$ converges uniformly on compacta of D to id_D.*

From Corollary 7.9 and Lemma 7.6 it follows that $x_i = b_i$. Since φ and F are finite Blaschke products with identical branch sets and satisfy $\varphi(0) = F(0)$ and $\varphi(1/2), F(1/2) > 0$, we conclude that $\varphi \equiv F$. Hence $\{f_{n_k}\}$ converges uniformly on compacta of D to F. Thus, by standard arguments, the full sequence $\{f_n\}$ converges uniformly on compacta of D to F.

Recall that f_n is an extension $\overline{F}_n$ of the simplicial map F_n from $P_{H(n)}$ to $B_F(n)$. Therefore, we have shown that these maps F_n converge uniformly on compacta of D to F.

Section 8. We must discuss how the restriction in Section 7 that F has only simple branch points can be removed. The only reason why we did not prove all the results in Section 7 in the general case was that we would be overwhelmed by notation, due to double indices for example, that would shadow the essence of the arguments.

In general, suppose that F is a finite Blaschke product with a branch set $\{(x_1, k_1), \ldots, (x_m, k_m)\}$ (not necessarily simple branch points) such that $F(0) = 0$ and $F(1/2) \in \mathbf{R}_+$. Let $\{B_F(n)\}$ be an approximating sequence of Bl-type packings for F with branch sets $\{v_{i,j}(n)\}_{j=1,i=1}^{k_i,m}$ (see Section 4). Let f_n be an extended simplicial map associated with $B_F(n)$. Then Lemma 6.1 and Proposition 5.1 imply that the results in Section 7 a), b) and c) hold with the various quantities $v_i(n)$, $C_i^\epsilon(n)$, $d_i^\epsilon(n)$, etc, of Section 7, replaced by their doubly-indexed counterparts. As in Section 7 d), $\{\varphi_n\}$ has a convergent subsequence $\{\varphi_{n_k}\}$ with a limit function φ, and, since all the $v_{i,j}(n)$ are in $B(0, \bar{r})$, φ is a Blaschke product with branch points $b_1, \ldots, b_l \in B(0, \bar{r})$. We also remark that the degree of φ is the same as degree of each φ_{n_k}. Using the same quasiregular techniques as in the proofs of Propositions 7.3 – 7.5 one shows

Lemma 8.1. *The limit function φ of the subsequence $\{\varphi_{n_k}\}$ is a Blaschke product with the branch set $\{(b_1, k_1), \ldots, (b_m, k_m)\}$, where $b_i = \lim_{k\to\infty} h_{n_k}(x_i)$.*

Finally, since all quasiconformal properties of the family $\{h_{n_k}\}$ discussed in Section 7 carry over to our current setting, $\{f_{n_k}\}$ converges uniformly on compacta of D to F. Moreover, since f_n is an extension $\overline{F}_n$ of the simplicial map F_n from $P_{H(n)}$ to $B_F(n)$, Theorem 1 of [DSt] (see Comments after the proof of the theorem) implies that the functions $F_{n_k}^\#$ converge uniformly on compacta of D to $|F'|$. Thus we have

Theorem 8.2. *Given a finite Blaschke product F, let $\{B_F(n)\}$ be an approximating sequence of Bl-type packings for F, and write F_n for the simplicial map from $P_{H(n)}$ to $B_F(n)$. Then functions F_n and $F_n^\#$ converge uniformly on compacta of D to $F(z)$ and $|F'(z)|$ respectively.*

An almost immediate consequence of the above result is

Theorem 8.3. *Let F be a finite classical Blaschke product. Then there exists a sequence $\{\mathbb{F}_n\}$ of discrete Blaschke products such that functions $\mathbb{F}_n$ and $\mathbb{F}_n^{\#}$ converge uniformly on compacta of D to F and $|F'|$ respectively.*

Proof. Since a discrete (classical) Blaschke product followed by a map $\mathcal{M}(z) = c\frac{z-a}{1-z\bar{a}}$, $|a| < 1, |c| = 1$, is also a discrete (classical) Blaschke product, we may assume without loss of generality that $F(0) = 0$ and $F(1/2) \in \mathbf{R}_+$. Suppose that $A(n)$ is an Andreev circle packing for $H(n)$ that fixes the origin and maps $1/2$ to $\mathbf{R}_+$. Let $B_F(n)$ be an approximating sequence of Bl-type packings for F. Write $\mathbb{F}_n$ for the discrete Blaschke product from $A(n)$ to $B_F(n)$. Let F_n and G_n be the simplicial maps from $P_{H(n)}$ to $B_F(n)$ and from $P_{H(n)}$ to $A(n)$ respectively. We observe that $\mathbb{F}_n = F_n \circ G_n^{-1}$. By [RS], functions G_n converge uniformly on compacta of D to the identity map of D while, by Theorem 8.2, maps F_n converge uniformly on compacta of D to F. Hence the functions $\mathbb{F}_n$ converge uniformly on compacta of D to F.

The approximation of $|F'|$ by maps $\mathbb{F}_n^{\#}$ is a consequence of Theorem 1 of [DSt]. $\square$

Before we finish this part let us make three remarks.

Remark 8.4. Let $\alpha_n : \mathbf{H}(n) \mapsto \mathbf{H}$ be the natural inclusion and let $\beta_n : H(n) \mapsto \mathbf{H}$ be the map $\alpha_n \circ t_n$, where $t_n : H(n) \mapsto \mathbf{H}(n)$ is a scaling by multiplication by n. Recall that $v_0(n) = 0$ for any n, so $\beta_n(v_0(n)) = 0$ for any n. Suppose that vertices $v_i(n)$, $i = 1, \ldots, m$, have the same combinatorial location in $H(n)$ with respect to $v_0(n)$ for any n, i.e. $\beta_n(v_i(n)) = \beta_{n'}(v_i(n'))$ for any $n, n' \in \mathbf{N}$ and any $i = 1, 2, \ldots, m$ (in other words, for a fixed i, the vertices $\beta_n(v_i(n))$ have fixed location in $\mathbf{H}$ as n changes).

Let $B(n)$ be the Bl-type packing for $H(n)$ with $br(B(n)) = \{v_1(n), \ldots, v_m(n)\}$

fixing 0 and mapping 1/2 to $\mathbf{R}_+$. Then $B(n)$ is an approximating sequence of Bl-type packings for $F(z) = z^{m+1}$. In particular, if F_n is the simplicial map from $P_{H(n)}$ to $B(n)$ then the functions F_n converge uniformly on compacta of D to the function z^{m+1}.

Remark 8.5. *Let $F(z) = z^m$ and let $\{B_F(n)\}$ be an approximating sequence of Bl-type packings for F. Suppose that $0 < t_0 < t_1 < 1$. Then for any $\delta > 0$ there exists N_δ such that*

$$\frac{mt_0^{m-1} - \delta}{mt_1^{m-1} + \delta} \le \frac{r_n(w)}{r_n(v)} \le \frac{mt_1^{m-1} + \delta}{mt_0^{m-1} - \delta} \text{ for } n > N_\delta,$$

where v, w are vertices of $H(n)$ that are in $\{z : \ t_0 \le |z| \le t_1\}$, and $r_n(\cdot)$ is the radius function of $B_F(n)$.

Proof. Since $F_n^\#(v) = r_n(v)/(1/n)$ for any vertex v in $H(n)$ and $|F'(z)| = mt^{m-1}$ for $|z| = t$, the assertion of our remark follows from Theorem 8.2. $\square$

Remark 8.6. *Suppose that Ω is a bounded simply connected domain in $\mathbf{C}$. Let $F : \Omega \mapsto D$ be a proper analytic map with a branch set $\{(x_1, k_1), \dots, (x_m, k_m)\}$ (i.e. $F = G \circ \omega$, where $\omega : \Omega \mapsto D$ is a Riemann mapping and $G : D \mapsto D$ is a finite Blaschke product with a branch set $\{(\omega(x_1), k_1), \dots, (\omega(x_m), k_m)\}$). Write $H_\Omega(n)$ for a subcomplex of $\frac{1}{n}\mathbf{H}$ which is a union of all closed faces of $\frac{1}{n}\mathbf{H}$ contained in Ω. Let $P_{H(n)}(\Omega)$ be the part of the circle packing $\frac{1}{n}P_{\mathbf{H}}$ which corresponds to the subcomplex $H_\Omega(n)$ of $\frac{1}{n}\mathbf{H}$. We define points $v_{i,j}(n)$ and Bl-type packings $B_F(n)$ as we did in Section 4 but this time for the complexes $H_\Omega(n)$ instead of the $H(n)$. If F_n is the simplicial map from $H_\Omega(n)$ to $B_F(n)$ then functions F_n and $F_n^\#$ converge uniformly on compacta of Ω to F and $|F'|$ respectively.*

Chapter VI

Infinite Hexagonal

Branched Circle Packings

In this chapter we return to infinite branched circle packings. Let $\mathbf{H}$ and $\mathbf{H}(n)$ be as in Chapter V. Suppose that B is a branched circle packing for $\mathbf{H}$ (i.e. an infinite hexagonal branched circle packing) which is obtained as the Bl-limit Bl-$\varinjlim(B(n_k); \mathbf{H}(n_k), 0)$ (see Chapters IV and V for notation). Let $f : \mathbf{C} \mapsto \mathbf{C}$ be the simplicial map from $P_{\mathbf{H}}$ to B. Our purpose in this chapter is to prove that f serves as a discrete analogue of a classical complex polynomial.

We begin with preliminary results on the radius function $R(\cdot) : V(\mathbf{H}) \mapsto \mathbf{R}_+$ of the packing B.

Section 1. The goal of this section is to show the following

Theorem 1.1. *The function $R(\cdot)$ is subharmonic with its infimum obtained in $\{b_1, b_2, \ldots, b_m\} = br(B)$.*

Proof. Let $r_{n_k} : V(\mathbf{H}(n_k)) \mapsto \mathbf{R}_+$ be the radius function of $B(n_k)$. Let $R_{n_k} : V(\mathbf{H}(n_k)) \mapsto \mathbf{R}_+$ be defined by $R_{n_k}(\cdot) = \frac{r_{n_k}(\cdot)}{r_{n_k}(0)}$. Recall that $R(\cdot)$ is the limit of $\{R_{n_k}(\cdot)\}$. We may assume without loss of generality that $n_k = n$, i.e. that B is the Bl-limit of a sequence $\{B(n)\}$ of Bl-type packings for $\{\mathbf{H}(n)\}$ normalized at 0, and that $R(\cdot)$ is the limit of radius functions $R_n(\cdot) = \frac{r_n(\cdot)}{r_n(0)}$. Let $B'(n)$ be a Bl-type packing for $\mathbf{H}(n)$, with the branch set $\{b_1, \ldots, b_m\}$, such that it fixes the origin and maps $1/2$ to $\mathbf{R}_+$ (that is, the simplicial map from $\frac{1}{n}P_{\mathbf{H}(n)}$ to $B'(n)$ fixes 0 and maps $1/2$ to $\mathbf{R}_+$, where $\frac{1}{n}P_{\mathbf{H}(n)}$ is defined as in Section V.4). We will now prove that B is also the Bl-limit of a more rigid sequence of packings, i.e. the sequence $\{B'(n)\}$, in which all the packings fix 0 and map $1/2$ to $\mathbf{R}_+$, and therefore are uniquely determined due to Theorem II.6.

Lemma 1.2. *$\{B'(n)\}$ is a sequence of Bl-type packings for $\{\mathbf{H}(n)\}$ that converges to B normalized at 0, i.e. $B = Bl\text{-}\varinjlim\big(B'(n); \mathbf{H}(n), 0\big)$.*

Proof. By Theorem II.6, $B(n) = \psi_n(B'(n))$ for some $\psi_n(z) = e^{i\lambda_n}\frac{p_n - z}{1 - z\bar{p}_n}$, where $\lambda_n \in [0, 2\pi)$, $p_n \in \mathbf{C}$, and $|p_n| < 1$. Let $r'_n : V(\mathbf{H}(n)) \mapsto \mathbf{R}_+$ be the radius function of $B'(n)$. Write $s_n(v)$ and $s'_n(v)$ for the centers of circles associated with the vertex $v \in V(\mathbf{H}(n))$ in $\{B(n)\}$ and $\{B'(n)\}$ respectively. Let us fix our attention on some vertex $a \in V(\mathbf{H})$. Suppose n is big enough so that $a \in V(\mathbf{H}(n))$. From Lemma V.6.1 we have that $d_a(n) := dist(s'_0(n), s'_a(n))$ and $r'_n(a)$ go to 0 as $n \to \infty$. We will show that

$$(*) \qquad \lim_{n\to\infty} \frac{r'_n(a)}{r'_n(0)} = R(a).$$

For simplicity of computation we may assume without loss of generality that $\lambda_n = 0$ and $s'_a(n) \in [0, 1)$. Then

$$r_n(a) = \frac{r'_n(a)(1 - |p_n|^2)}{|1 - d_a(n)(p_n + \bar{p}_n) + |p_n|^2(d_a(n)^2 - r'_n(a)^2)|}, \text{ and}$$

$$r_n(0) = \frac{r'_n(0)(1 - |p_n|^2)}{1 - |p_n|^2 r'_n(0)^2}.$$

Hence

$$\frac{r_n(a)}{r_n(0)} = \frac{r'_n(a)}{r'_n(0)} \cdot \frac{1 - |p_n|^2 r'_n(0)^2}{|1 - d_a(n)(p_n + \bar{p}_n) + |p_n|^2(d_a(n)^2 - r'_n(a)^2)|}, \text{ and}$$

$$\lim_{n\to\infty} \frac{r'_n(a)}{r'_n(0)} = \lim_{n\to\infty} \frac{r_n(a)}{r_n(0)} = R(a).$$

Thus we have shown $(*)$ and proved the lemma. □

Since our goal is to show that $R(\cdot)$ has its minimum at one of the vertices b_i, we will be assuming that $\{B(n)\}$ is in fact $\{B'(n)\}$.

Fix an index k such that $V(\mathbf{H}(k))$ contains the vertices b_i. We will show that for any $v \in V(\mathbf{H}(k))$ there is N such that $\min_i\{R_n(b_i)\} \le R_n(v)$ for $n \ge N$.

Suppose this is not true. Then there exist $v_0 \in V(\mathbf{H}(k)) \setminus \{b_1, b_2, \ldots, b_m\}$ and a subsequence $\{n_l\}$ such that $R_{n_l}(v_0) < R_{n_l}(b_i)$ for $i = 1, 2, \ldots, m$ and any l.

In the subsequent discussion, for $p < n$, $\mathbf{H}(n,p)$ will denote a subcomplex of $\mathbf{H}$ whose boundary is a simple closed edge path that lies in $\mathbf{H}(n) \setminus \mathbf{H}(p)$. It follows from Lemma V.2.1 and [BFP] that $R(\cdot)$, $R_n(\cdot)$, $1/R(\cdot)$, and $1/R_n(\cdot)$ are subharmonic in $V(\mathbf{H})$, $int\mathbf{H}(n)$, $V(\mathbf{H}) \setminus \{b_1, b_2, \ldots, b_m\}$, and $int\mathbf{H}(n) \setminus \{b_1, b_2, \ldots, b_m\}$, respectively. Since $1/R_n(\cdot)$ is subharmonic in $int\mathbf{H}(n) \setminus \{b_1, \ldots, b_m\}$, the maximum of $1/R_n(\cdot)|_{V(\mathbf{H}(n,p))}$ is obtained in $bd\mathbf{H}(n,p) \bigcup \{b_1, b_2, \ldots, b_m\}$ for $p = k, k+1, \ldots, n-1$. Our hypothesis about v_0 says that the maximum of $1/R_{n_l}(\cdot)|_{\mathbf{H}(n_l,p)}$ is obtained in $bd\mathbf{H}(n_l,p)$ for any p, $k \le p \le n_l - 1$; i.e., the minimum of $R_{n_l}(\cdot)|_{\mathbf{H}(n_l,p)}$ is obtained in $bd\mathbf{H}(n_l,p)$. Since $R_{n_l}(\cdot)$ is subharmonic in $int\mathbf{H}(n_l)$, $R_{n_l}(\cdot)|_{\mathbf{H}(n_l,p)}$ attains its maximum and minimum in $bd\mathbf{H}(n_l,p)$ for any p, $k \le p \le n_l - 1$. Let $v_1(p)$ and $v_2(p)$ be the points in $bd\mathbf{H}(n_l,p)$ such that $R_{n_l}(\cdot)|_{\mathbf{H}(n_l,p)}$ attains its maximum and minimum, respectively. Then

$$(*) \qquad \frac{R_{n_l}(v_2(p))}{R_{n_l}(v_1(p))} \le \frac{R_{n_l}(v)}{R_{n_l}(w)} \le \frac{R_{n_l}(v_1(p))}{R_{n_l}(v_2(p))} \text{ for all } v, w \in V(\mathbf{H}(n_l,p)).$$

Let $t_0, t_1 \in (1/4, 3/4)$ and $t_0 < t_1$. Suppose that $t_1 - t_0$ and $\delta > 0$ are small, then

$$(**) \qquad \frac{(m+1)t_0^m - \delta}{(m+1)t_1^m + \delta} \quad \text{and} \quad \frac{(m+1)t_1^m + \delta}{(m+1)t_0^m - \delta} \quad \text{are close to 1.}$$

It is evident that there exists l_0 such that for any l, $l > l_0$, we have

1) $\frac{1}{n_l}\mathbf{H}(k) \subset B(0, t_0/2)$ and
2) there exist p_l, $k \le p_l \le n_l - 1$, and a subcomplex $\mathbf{H}(n_l, p_l)$ such that $bd\mathbf{H}(n_l,p_l) \subset \{z : \; t_0 \le |z| \le t_1\}$.

Let us recall now that $R_n(\cdot) = r_n(\cdot)/r_n(0)$. From Remarks V.8.4 and V.8.5, there exists l_δ, $l_\delta \geq l_0$, such that if $l \geq l_\delta$ then

$$\frac{(m+1)t_0^m - \delta}{(m+1)t_1^m + \delta} \leq \frac{R_{n_l}(v)}{R_{n_l}(w)} \leq \frac{(m+1)t_1^m + \delta}{(m+1)t_0^m - \delta} \quad \text{for all } v, w \in bd\mathbf{H}(n_l, p_l),$$

where $\mathbf{H}(n_l, p_l)$ is as in 2) above. By applying the last double inequality to $v := v_1(p_l)$ and $w := v_2(p_l)$, and using $(*)$ and $(**)$, one obtains that $\frac{R_{n_l}(v)}{R_{n_l}(w)} \approx 1$ for any $v, w \in V(\mathbf{H}(n_l, p_l))$ and $l > l_\delta$. In particular, $\frac{R_{n_l}(v)}{R_{n_l}(w)} \approx 1$ for any $v, w \in \mathbf{H}(k)$ and $l > l_\delta$. Thus $\frac{R_{n_l}(b_i)}{R_{n_l}(w)} \approx 1$ for $l > l_\delta$, where $w \in V(star(b_i))$. The last approximation implies that the angle sum at vertex b_i is approximately 2π rather than 4π, as required of a simple branch point, and we end up with a contradiction.

Therefore, for any vertex v in $\mathbf{H}(k)$ there exists N such that $\min_i\{R_n(b_i)\} \leq R_n(v)$ for $n \geq N$. Hence $\min_i\{R(b_i)\} \leq R(v)$ for any vertex v in $\mathbf{H}(k)$ and, as k was arbitrary, this finishes the proof. □

Section 2. In this section we prove that the function $f : \mathbf{C} \mapsto \mathbf{C}$, which is the simplicial map from $P_{\mathbf{H}}$ to B, is a polynomial-like map. Although this conclusion can be reached as a corollary to a theorem from the next chapter, which states that f is in fact quasiregular, we present here alternative arguments that prove our assertion.

Using a translation and rotation we may assume without loss of generality that $f(0) = 0$ and $f(2) \in \mathbf{R}_+$. By Lemma 1.2 we can assume that the packings $B(n_k)$, which give B as the Bl-limit, are such that the center of the circle in $B(n_k)$ associated with $0 \in V(\mathbf{H}(n_k))$ is the origin and the center of the circle in $B(n_k)$ associated with $2 \in V(\mathbf{H}(n_k))$ is on $\mathbf{R}_+$. Recall that $H(n) = \frac{1}{n}\mathbf{H}(n)$, and let $\tilde{f}_k : H(n_k) \mapsto D$ be the simplicial map from $P_{H(n_k)}$ to $B(n_k)$ ($P_{H(n)}$ defined as in Section V.3). Let $f_k : \mathbf{H}(n_k) \mapsto \mathbf{C}$ be defined by $f_k(z) := \tilde{f}_k(\frac{1}{n_k}z)/r_{n_k}(0)$, where $r_{n_k}(\cdot)$ is the radius function of $B(n_k)$. Since all functions $\tilde{f}_k$ are at most

$m+1$-to-1, all functions f_k are at most $m+1$-to-1. Moreover, as centers of circles of the packings $\frac{1}{n_k}B(n_k)$ converge to centers of the corresponding circles of B, $f_k \to f$ uniformly on compacta of $\mathbf{C}$. Thus, since the maps f_k and f are open and discrete, we get

Proposition 2.1. *The function f is at most $m+1$-to-1.*

We have the following

Lemma 2.2. *The function $f : \mathbf{C} \mapsto \mathbf{C}$ is a proper map with $f(\mathbf{C}) = \mathbf{C}$.*

Proof. Using properties of $R(\cdot)$ we will first investigate distances between certain points in the carrier of B.

From Theorem 1.2 we have that $R(v) \geq \min_{1\leq i\leq m}\{R(b_i)\}$ for any $v \in V(\mathbf{H})$. For our purposes, we may assume without loss of generality that infimum of $R(\cdot)$ is 1 and is obtained at b_1. Let $\triangle v_1v_2v_3$ denote a face in $\mathbf{H}$ with vertices v_1, v_2, v_3. Then $f(\triangle v_1v_2v_3) = \triangle f(v_1)f(v_2)f(v_3)$ is a triangle with heights of length at least 1. Let $\triangle v_1'v_2'v_3$ be the triangle with vertices v_1', v_2', v_3, where $v_i' \in [v_3, v_i]$ is chosen so that $|v_i'v_3| = 10/6$ for $i = 1, 2$, where $|ab|$ denotes the length of the segment $[a, b]$. Then f maps $\triangle v_1'v_2'v_3$ onto $\triangle f(v_1')f(v_2')f(v_3)$ in an affine way. In particular, the segments $f(v_1')f(v_2')$ and $f(v_1)f(v_2)$ are parallel and $|f(v_1')f(v_3)|/|f(v_1)f(v_3)| = 5/6 = |f(v_2')f(v_3)|/|f(v_2)f(v_3)|$. Let h be the height of $\triangle f(v_1)f(v_2)f(v_3)$ from vertex $f(v_3)$, and h' be the height of the trapezoid $\square f(v_1')f(v_1)f(v_2)f(v_2')$ from vertex $f(v_1')$ to the side $f(v_1)f(v_2)$. Then $(|h| - |h'|)/|h| = |f(v_1')f(v_3)|/|f(v_1)f(v_3)| = 5/6$, and $|h'| = |h|/6 \geq 1/6$. Therefore the distance of any point from $\triangle f(v_1')f(v_2')f(v_3)$ to the side $f(v_1)f(v_2)$ is at least $1/6$.

Let $st(v_1, \ldots, v_6; v_0)$ denote the star of v_0 in $\mathbf{H}$ with $v_1, \ldots, v_6$ as its boundary vertices. Let $st(v_1', \ldots, v_6'; v_0) := \triangle v_6'v_1'v_0 \cup \bigcup_{i=1}^{5} \triangle v_i'v_{i+1}'v_0$, where points v_i' are defined as above. Suppose $v_0 \notin \{b_1, b_2, \dots, b_m\}$, then f maps $st(v_1, \dots, v_6; v_0)$ homeomorphically onto $f\big(st(v_1, \dots, v_6; v_0)\big)$. Since $|h'| \geq 1/6$, it follows that the distance from the boundary of $f\big(st(v_1, \dots, v_6; v_0)\big)$ to any point in the set $f\big(st(v_1', \dots, v_6'; v_0)\big)$ is at least 1/6. Finally, we observe that for any point in $\mathbf{C}$ there is a vertex $v_0 \in V(\mathbf{H})$ whose star $st(v_1', \dots, v_6'; v_0)$ contains that point.

Using what we have just established, we will now show that if $\{p_n\}$ is a sequence of points in $\mathbf{C}$ that converges to ∞ as $n \to \infty$, then $\lim_{n\to\infty} f(p_n) = \infty$.

Suppose that this is not true. Then there exist a constant $M > 0$ and a subsequence $\{p_{n_k}\}$ such that $|f(p_{n_k})| \leq M$ as $k \to \infty$. Since $\{f(p_{n_k})\}$ is a bounded sequence we can assume that it is convergent. Let $\tilde{p}$ be the limit of $\{f(p_{n_k})\}$. Then $|\tilde{p}| \leq M$. Now, for every p_{n_k} there exists a vertex $v_0(n_k)$ in $V(\mathbf{H})$ such that $p_{n_k} \in S'_{n_k} := st(v_1'(n_k), \dots, v_6'(n_k); v_0(n_k)) \subset S_{n_k} := st(v_1(n_k), \dots, v_6(n_k); v_0(n_k))$. Since $p_{n_k} \to \infty$, we can assume without loss of generality that the sets S_{n_k} are pairwise disjoint and have empty intersection with the set $\{b_1, b_2, \dots, b_m\}$. Since $f(p_{n_k}) \to \tilde{p}$ and the distance of $f(p_{n_k})$ to the boundary of $f(S_{n_k})$ is at least 1/6, there exists k_0 such that $\bigcap_{k\geq k_0} f(S_{n_k}) \neq \emptyset$. The last gives a contradiction to Proposition 2.1. Thus we have proved that $\lim_{n\to\infty} f(p_n) = \infty$.

To finish the proof of the lemma it is sufficient to show that $\mathbb{S}^2 \setminus f(\mathbf{C}) = \{\infty\}$, where $\mathbb{S}^2$ is the Riemann sphere.

Suppose there is $a \in \partial f(\mathbf{C})$ such that $a \neq \infty$. Hence there exists a sequence $\{p_n\}$ of points in $\mathbf{C}$ such that $\{f(p_n)\}$ converges to a as $n \to \infty$. Suppose that $\{p_n\}$ has a bounded subsequence $\{p_{n_k}\}$. We can assume that $\{p_{n_k}\}$ is convergent. Let p be the limit of $\{p_{n_k}\}$, then $|p| < \infty$. Since f is continuous, $\lim_{k\to\infty} f(p_{n_k}) = f(p)$. Moreover, by the choice of $\{p_n\}$, $f(p) = a$. The fact that f is open says that a can not belong to $\partial f(\mathbf{C})$. Therefore $\{p_n\}$ does not have a bounded subsequence. The last implies that $\{p_n\}$ converges to ∞. Hence

$\{f(p_n)\}$ converges to ∞, which contradicts the assumption that $a \neq \infty$. $\square$

Now we are ready to prove

Theorem 2.5. *Let $f : \mathbf{C} \mapsto \mathbf{C}$ be the simplicial map from $P_{\mathbf{H}}$ to B, where $B = Bl\text{-}\varinjlim(B(n_k); \mathbf{H}(n_k), 0)$ is an infinite hexagonal branched circle packing with the branch set $\{b_1, \ldots, b_m\}$. Then f has a decomposition $f = \varphi \circ h$, where $h : \mathbf{C} \mapsto \mathbf{C}$ is a homeomorphism and $\varphi : \mathbf{C} \mapsto \mathbf{C}$ is a polynomial with simple branch points $h(b_1), h(b_2), \ldots, h(b_m)$. Moreover, if $f(0) = 0$ and $f(1) \in \mathbf{R}_+$ then φ and h may be chosen so that $\varphi(0) = 0 = h(0)$, $h(1) \in \mathbf{R}_+$ and $\varphi(h(1)) \in \mathbf{R}_+$.*

Proof. From the construction of f one has that f is continuous, open and discrete. Hence, by Stoilow's Theorem, $f = \varphi \circ h$, where $h : \mathbf{C} \mapsto h(\mathbf{C})$ is a homeomorphism and φ is analytic in $h(\mathbf{C})$. Moreover, if $f(0) = 0$ and $f(1) \in \mathbf{R}_+$ then we can obviously require that φ and h satisfy conditions: $\varphi(0) = 0 = h(0)$ and $h(1) \in \mathbf{R}_+$, $\varphi(h(1)) \in \mathbf{R}_+$.

Now suppose that $h(\mathbf{C}) \neq \mathbf{C}$. By the Riemann Mapping Theorem, there exists conformal map $\psi : D \mapsto h(\mathbf{C})$. Let $\tilde{\varphi} := \varphi \circ \psi$ and $\tilde{h} := \psi^{-1} \circ h$, then $f = \tilde{\varphi} \circ \tilde{h}$, where $\tilde{\varphi} : D \mapsto \mathbf{C}$ is analytic and $\tilde{h} : \mathbf{C} \mapsto D$ is a homeomorphism. By Proposition 2.1 and Lemma 2.2, $\tilde{\varphi}$ is proper, at most $m + 1$-to-1 map and maps D onto $\mathbf{C}$. But quasiregular arguments show that such map $\tilde{\varphi}$ does not exist. Hence $h(\mathbf{C}) = \mathbf{C}$, and φ is a proper entire function with branch points only at the $h(b_i)$. Thus φ is a polynomial with branch points $h(b_i)$. Moreover, since $f = \varphi \circ h$ and the b_i are simple branch points of f, it follows that the $h(b_i)$ are simple branch points of φ. $\square$

The above theorem justifies us to state the following

Definition 2.6. *A map $f : \mathbf{C} \mapsto \mathbf{C}$ will be called a discrete complex polynomial* *if the following are satisfied:*

(i) *there exist an infinite triangulation* $\mathbf{T}$ *of* $\mathbf{C}$ *and a univalent circle packing* P *for* $\mathbf{T}$ *whose carrier is* $\mathbf{C}$,

(ii) *there exists an infinite branched circle packing* B *for* $\mathbf{T}$ *such that* f *is the simplicial map from* P *to* B,

(iii) f *has a decomposition* $f = \varphi \circ h$, *where* h *is a self-homeomorphism of* $\mathbf{C}$ *and* φ *is a complex polynomial.*

We will say that f *is a discrete complex polynomial determined by* $\mathbf{T}$ *and the branch set* $\mathfrak{B} = \{(b_1, k_1), \ldots, (b_m, k_m)\}$ *if (i)-(iii) hold with* B *in (ii) being a branched circle packing for* $\mathbf{T}$ *with the branch set* $\mathfrak{B}$.

Chapter VII

Discrete Complex Polynomials: Distortion and Approximation Aspects

Our main goal in this chapter is to show (Theorem 2.1) that:

Any complex polynomial can be approximated uniformly on compacta of $\mathbf{C}$ *by discrete complex polynomials.*

Before a proof of the above assertion is given, we will establish some important results related to families of simplicial maps induced by circle packings.

Section 1. In this section we will show that certain families of maps are K-quasiregular.

Let $\mathbf{H}$ be the regular hexagonal complex defined in Chapter V. Any circle packing for $\mathbf{H}$ will be called an **infinite hexagonal circle packing**. We will say that a map $f : \mathbf{C} \mapsto \mathbf{C}$ is the simplicial map **induced** by an infinite hexagonal circle packing B if f is the simplicial map from $P_{\mathbf{H}}$ to B.

Let $\mathcal{F}_m$ be a family of simplicial maps f induced by infinite hexagonal circle packings (unbranched or branched, even with infinitely many branch points) such that f has valence at most m.

Theorem 1.1. *$\mathcal{F}_m$ is a family of K-quasiregular mappings, where K depends only on m.*

Proof. Trivially, $\mathcal{F}_m$ is not empty because $id \in \mathcal{F}_m$. Second, all maps in $\mathcal{F}_m$ are open and discrete, which follows from the construction of these maps. Moreover, if $f \in \mathcal{F}_m$ has branch points then they are simple branch points.

For $f \in \mathcal{F}_m$ let B_f denote the infinite hexagonal circle packing that induces f. Let $R_f(\cdot) : V(\mathbf{H}) \mapsto \mathbf{R}_+$ be the radius function of B_f. To prove the theorem we will show that there exists $K \geq 1$ such that

$$(\bigstar) \quad \frac{R_f(u)}{R_f(w)} \leq K \text{ for any } f \in \mathcal{F}_m \text{ and any neighboring vertices } u, w \in V(\mathbf{H}).$$

Suppose $(\bigstar)$ is not true. Then there are a sequence $\{f^n\}$ of functions from $\mathcal{F}_m$ and a sequence of vertices $\{u_n, w_n\}$ in $V(\mathbf{H})$ such that u_n and w_n are neighboring vertices, and $r_n(u_n)/r_n(w_n) \geq n$, where $r_n(\cdot) := R_{f^n}(\cdot)$. By applying a simplicial self-isometry of $\mathbf{H}$, we may assume without loss of generality that $w_n = 0 \in V(\mathbf{H})$ and $u_n = 2 \in V(\mathbf{H})$ for any n.

Let $v_0, v_1, v_2, \ldots$ denote vertices in $\mathbf{H}$ with $v_0 = 0$ and $v_1 = 2$. Notice that

$$(\star) \qquad r_n(v_1)/r_n(v_0) \geq n \quad \text{for any } n.$$

Let $\mathbf{H}(n)$ be defined as in Chapter V. We observe that $\mathbf{H}(n) \subset \mathbf{H}(n+1)$ and $\mathbf{H} = \bigcup_{n=1}^{\infty} \mathbf{H}(n)$. We will now follow the steps of the proof of Theorem IV.2 for $\mathbf{T} := \mathbf{H}$ with $\mathbf{T}(n) := \mathbf{H}(n)$. We start with the sequence $\{r_n(\cdot)\}$ and, because of $(\star)$, we can construct, using the same method as in Chapter IV (except that now our collections of radii are infinite), the sequence $\{\mathcal{R}_n(\cdot)\}$ with the following properties:

1) $\mathcal{R}_n(\cdot)$ is a scaled version of $r_{n'}(\cdot)$ for some $n' \geq n$, and
2) $\mathcal{R}(v_k) := \lim_{n\to\infty} \mathcal{R}_n(v_k) \in [0, \infty]$ for any $k \in \mathrm{N}$, and $\mathcal{R}(v_0) = 0$, $\mathcal{R}_n(v_1) \equiv 1$.

Since the family $\mathcal{F}_m$ is scale-invariant, we observe that the simplicial map f_n induced by a circle packing with the radius function $\mathcal{R}_n(\cdot)$ belongs to $\mathcal{F}_m$.

By choosing a subsequence of $\{f_n\}$ if necessary, one may assume one of two cases obtains:

Case 1.

(1.1) There are $v_{i_1}, v_{i_2}, \ldots$ (possibly a finite sequence) such that v_{i_j} is a branch point of B_{f_n} for any sufficiently large n.

(1.2) If $v_k \in V(\mathbf{H}) \setminus \{v_{i_j}\}_{j=1}^{\infty}$ then v_k fails to be a branch point of B_{f_n} for any sufficiently large n.

(1.3) The set $\{v_{i_j}\}_j \subset V(\mathbf{H})$ is a branch structure for $\mathbf{H}$.

Case 2.

(2.1) Any vertex $v_k \in V(\mathbf{H})$ fails to be a branch point of B_{f_n} for any sufficiently large n.

Notice that the condition (1.3) in Case 1 follows from the condition (1.1) and the definition of a branch structure for a finite set of branch points. In particular, for any $p \in \mathbf{N}$ there exists n_p such that

$$(*) \qquad \forall v_k \in V(\mathbf{H}(p)) \quad v_k \in \{v_{i_j}\}_{j=1}^{\infty} \iff \forall n > n_p \quad v_k \in br(B_{f_n}),$$

where in Case 2 we understand that $\{v_{i_j}\}_{j=1}^{\infty} = \emptyset$.

Let the sets SD and OD be defined as in Chapter IV (but for $\mathcal{R}(\cdot)$ defined as above), i.e.

- $v_k \in SD$ if $\mathcal{R}(v_k) \in (0, \infty]$,
- $v_k \in OD$ if $\mathcal{R}(v_k) = 0$.

Then $SD \neq \emptyset \neq OD$ because $v_0 \in OD$ and $v_1 \in SD$. Moreover, using $(*)$ above, Lemma II.4 and Remark II.5, and proceeding the same way as we did in Chapter IV, we obtain the properties (i)-(iii) stated in Part 3 of the proof of Theorem IV.2. Next we define C_{OD} and points a_i exactly the same way as in Chapter IV. Then, using arguments from Part 5 of the proof of Theorem IV.2, one finds a path $[a_0, \ldots, a_N]$ (notation as in Chapter IV) and then shows that for any sufficiently large n there is a point whose preimage under the function f_n defined as above (in Chapter IV it was the function F_n) has more then m elements. The last is a contradiction to the assumption that all the functions f_n are in $\mathcal{F}_m$, i.e. are at most m-to-1 maps. $\square$

We can now draw a few conclusions but first we need the following remark which is an easy consequence of Stoilow's Theorem.

Remark 1.2. *Let $f : \mathbf{C} \mapsto \mathbf{C}$ be a K-quasiregular mapping of finite valence. Then $f = \varphi \circ h$, where $h : \mathbf{C} \mapsto \mathbf{C}$ is K-quasiconformal and φ is a polynomial. In particular, f has finite branch set. If $\{(b_1, k_1), \ldots, (b_m, k_m)\}$ is the branch set of f then $\{(h(b_1), k_1), \ldots, (h(b_m), k_m)\}$ is the branch set of φ.*

Proposition VI.2.1, Theorem 1.1, and the above remark give an alternative proof of Theorem VI.2.5. However, from Theorem 1.1 and Remark 1.2 we obtain this more general

Corollary 1.3. *Let f be a simplicial map induced by an infinite, possibly branched, hexagonal circle packing B. If f has finite valence then it has at most finitely many branch points, say $b_1, \ldots, b_m$, and a decomposition $f = \varphi \circ h$, where $h : \mathbf{C} \mapsto \mathbf{C}$ is K-quasiconformal, K depending only on the valence, and φ is a polynomial with simple branch points $h(b_1), \ldots, h(b_m)$.*

Before we state the next result, we need the following

Definition 1.4. *Let P be a circle packing for a complex $\mathbf{K}$, finite or infinite. We say that P has valence m if for any univalent circle packing Q for $\mathbf{K}$ the simplicial map from Q to P has valence m.*

Applying the same methods of argument as in the proofs of Theorem IV.2 and Theorem 1.1 above, one easily gets

Corollary 1.5. *Let $\mathbf{L}_0 \subset \mathbf{L}_1 \subset \mathbf{L}_2 \ldots$ be a nested sequence of subcomplexes of $\mathbf{H}$, finite or infinite, such that $\bigcup_n \mathbf{L}_n = \mathbf{H}$. Let v_0 be a vertex in $\mathbf{L}_0$. Suppose that $\{B_n\}$ is a sequence of (branched) circle packings for the complexes $\mathbf{L}_n$. Write $r_n(\cdot)$ for the radius function of B_n. If all packings B_n have a uniform finite bound on* *their valences then there exists a constant K such that $1/K \le r_n(w)/r_n(v_0) \le K$ for any n and any neighboring vertex w of v_0.*

As a consequence of the last corollary we have

Corollary 1.6 (Ring Lemma in the Large). *Suppose $m > 0$ is an integer. Let $\mathbb{B}_m$ be the family of circle packings P such that*

(1) *if $\mathbf{L}_P$ is the complex of P then $\mathbf{L}_P$ is a subcomplex of $\mathbf{H}$,*

(2) *P has valence at most m.*

Then there exist an integer $N = N(m)$ and a positive constant $K = K(m)$ such that

$$P \in \mathbb{B}_m,\ \mathbf{H}(v, N) \subset \mathbf{L}_P \Longrightarrow \frac{1}{K} \le \frac{r_P(w)}{r_P(v)} \le K \text{for any neighboring vertex } w \text{ of } v,$$

where $r_P(\cdot)$ is the radius function of P, and $\mathbf{H}(v, N)$ denotes the subcomplex of $\mathbf{H}$ consisting of n generations of $\mathbf{H}$ around the vertex v.

Section 2. In this section we are going to show the following

Theorem 2.1. *Let F be a classical complex polynomial. There exists a sequence $\{f_n\}$ of discrete complex polynomials such that functions f_n and $f_n^{\#}$ converge uniformly on compacta of $\mathbf{C}$ to F and $|F'|$ respectively.*

Proof. The proof will be given in several parts; the maps f_n will be discrete complex polynomials associated with infinite hexagonal circle packings.

a). Suppose that $x_1, \ldots, x_m$ are the branch points of F of orders $k_1, \ldots, k_m$ respectively. We will be assuming that F is normalized so that $F(0) = 0$. Let $r > 0$ be such that $x_1, x_2, \ldots, x_m \in G := F^{-1}\big(B(0, r)\big)$. Let $p \in \{z \in \partial G : \ |z| = dist(0, \partial G)\}$. By multiplying F by $e^{i\lambda}$, $\lambda \in [0, 2\pi)$, we may assume that $p' := F(p) \in \mathbf{R}_+$.

b). Let $\mathbf{H}_n = \frac{1}{n}\mathbf{H}$. Write $P_{\mathbf{H}_n}$ for the regular hexagonal circle packing whose carrier is $\mathbf{H}_n$, i.e. $P_{\mathbf{H}_n} = \frac{1}{n}P_{\mathbf{H}}$.

We now define, for any sufficient large n, B_n to be an infinite hexagonal branched circle packing and $f_n : \mathbf{H}_n = \mathbf{C} \mapsto \mathbf{C}$ to be the simplicial map from $P_{\mathbf{H}_n}$ to B_n so the following properties are satisfied:

1) B_n has a branch set

$$br(B_n) = \{b_{1_1}(n), b_{1_2}(n), \ldots, b_{1_{k_1}}(n), \ldots, b_{m_1}(n), b_{m_2}(n), \ldots, b_{m_{k_m}}(n)\}$$

of simple branch points that are located in $\mathbf{H}_n$ in the following way:

For any $i \in \{1, 2, \ldots, m\}$, vertices $b_{i_1}(n), \ldots, b_{i_{k_i}}(n)$ are in a fixed combinatorial distance from x_i and this distance does not depend on n or i (i.e. there is a number ℓ (independent of i or n) such that $b_{i_1}(n), \ldots, b_{i_{k_i}}(n)$ are always within ℓ generations around the closest vertex in $\mathbf{H}_n$ to x_i, and this is for any i and n),

2) f_n has finite valence,

3) f_n is normalized (or rather B_n is normalized) so that $f_n(0) = 0$, $f_n(p) \in \mathbf{R}_+$, $f_n(G) \subseteq D(r)$ and $\partial f_n(G) \cap \partial D(r) \neq \emptyset$.

Let us notice that the existence of such maps f_n (or packings B_n) is given by Theorem IV.2 and Theorem VI.2.5, and that condition 1) above is analogous to condition (2) in Definition V.4.1.

It follows from Corollary 1.3 that there exists $\mathrm{K} > 0$ so that each f_n has a decomposition $f_n = \varphi_n \circ h_n$, where $h_n : \mathbf{C} \mapsto \mathbf{C}$ is K-quasiconformal, φ_n is a complex polynomial, and $h_n(0) = 0 = \varphi_n(0)$, $h_n(p) = p$. In particular, $\{h_n\}$ is a sequence of entire K-quasiconformal mappings that fix $0, p$ and ∞, therefore $\{h_n\}$ is a normal family.

Let $\{h_{n_k}\}$ be a subsequence of $\{h_n\}$ that converges uniformly on compacta of

$\mathbf{C}$ to some limit function h. Then h is an entire K-quasiconformal map that fixes 0, p, and ∞. In particular, there exist δ_1 and δ_2, $\delta_2 > \delta_1 > 0$, such that

$$(*) \qquad h_{n_k}(\partial G) \subset B(0,\delta_2) \setminus B(0,\delta_1) \text{ for all } k.$$

From the construction of the f_n, Corollary 1.3, and the fact that the φ_n are polynomials fixing 0, one has

$$(**) \qquad \varphi_n(z) = a_1(n)z + a_2(n)z^2 + \cdots + a_M(n)z^M,$$

where the $a_i(n)$ are complex numbers and $M = 1 + \sum_{i=1}^m k_i$. Moreover,

$$\varphi_{n_k}(B(0,\delta_1)) \subseteq D(r) \text{ for any } k$$

because $f_n(G) \subseteq D(r)$ and $B(0,\delta_1) \subseteq h_{n_k}(G)$ by $(*)$. Thus, by $(**)$, there exists $\Lambda > 0$ such that

$$(***) \qquad |a_i(n_k)| \leq \Lambda \text{ for any } k \text{ and } i = 1, \ldots, m.$$

From $(**)$ and $(***)$ it follows that $\{\varphi_{n_k}\}$ is a normal family. That means that there exists a subsequence $\{\varphi_{n'_k}\}$ that converges uniformly on compacta of $\mathbf{C}$ to some analytic function φ such that $\varphi(0) = 0$. Moreover, φ is not a constant function because of $(*)$ and the condition that $\partial f_n(G) \cap \partial D(r) \neq \emptyset$ for any n. The last observation and $(**)$ imply that φ is a polynomial with $\varphi(0) = 0$ and $\varphi(p) \in \mathbf{R}_+$. Finally, the uniform convergence of $\{h_{n_k}\}$ on compacta of $\mathbf{C}$ and the condition 1) show that φ is a polynomial with a branch set $br(\varphi) = \{(h(x_1), k_1), \ldots, (h(x_m), k_m)\}$.

c). Arguing as we did in Section V.8 (or Section V.7), we easily show that h is 1-quasiconformal in $\mathbf{C} \setminus \{x_1, x_2, \ldots, x_m\}$. Thus h is conformal in $\mathbf{C}$. Since $h(0) = 0$ and $h(p) = p$, we get that $h = id_{\mathbf{C}}$. Therefore φ is the polynomial with the branch set $br(\varphi) = \{(x_1, k_1), \ldots, (x_m, k_m)\}$, and $\varphi(0) = 0$, $\varphi(p) \in \mathbf{R}_+$. Hence $\varphi = F$, and by standard arguments we get that not only $\{f_{n'_k}\}$ but the full sequence $\{f_n\}$ converges uniformly on compacta of $\mathbf{C}$ to F.

Finally, since the f_n converge uniformly on compacta of $\mathbf{C}$ to F, by Theorem 1 of [DSt] we get that functions $f_n^{\#}$ converge uniformly on compacta to $|F'|$. This completes the proof. $\square$

Remark 2.2. *We observe that one can use any sequence of discrete polynomials* $f_n : \mathbf{H}_n = \mathbf{C} \mapsto \mathbf{C}$ *in an approximation of* F *in Theorem 2.1 as long as the conditions 1) and 3) from the above proof are satisfied, which translate, respectively, to the following conditions: i) the branch sets of the* f_n *have to converge to the branch set of* F *counting multiplicities, and ii) the* f_n *are normalized appropriately (which is easily achieved by applying similarities of* $\mathbf{C}$*).*

We also want to point out that, since the functions used in the proof of Theorem 2.1 were <u>*explicit*</u> *discrete complex polynomials, we have a practical way of approximating complex polynomials by discrete ones.*

The following result is an immediate consequence of Theorem 2.1 and Runge's Theorem

Corollary 2.3. *Let* $\Omega \subset \mathbf{C}$ *be simply connected. Any analytic function on* Ω *can be approximated uniformly on compacta of* Ω *by the restrictions of discrete complex polynomials.*

Chapter VIII

Uniqueness of Finite-Valence Infinite Hexagonal Branched Circle Packings

Our goal in this chapter is to prove the following

Theorem 1. *Any two infinite hexagonal branched circle packings of finite valence with identical branch sets are the same up to similarities in* $\mathbf{C}$.

Proof. Let B_f and B_g be finite-valence infinite hexagonal branched circle packings associated with simplicial maps f and g defined on $\mathbf{H}$ and assume that they have identical branch sets. Let $\tilde{f}_n(z) := f(s_n(z))$ and $\tilde{g}_n(z) := g(s_n(z))$, where $s_n(z) = nz$. Then $\tilde{f}_n$ and $\tilde{g}_n$ are simplicial maps from $P_{\mathbf{H}_n}$ to B_f and B_g respectively, where $P_{\mathbf{H}_n}$ is as in the previous chapter. From Corollary VII.1.3 and the proof of Theorem VII.2.1 (see also Remark VII.2.2) it follows that there are $c_n \in \mathbf{C}$ such that functions $f_n(z) := c_n\tilde{f}_n(z)$ converge uniformly on compacta of $\mathbf{C}$ to the function z^m, where $m-1$ is the number of branch points of f (and also g). Similarly, there are $d_n \in \mathbf{C}$ such that functions $g_n(z) := d_n\tilde{g}_n(z)$ converge uniformly on compacta of $\mathbf{C}$ to the function z^m. The definition of f_n shows that f_n is the simplicial map from $P_{\mathbf{H}_n}$ to $B_{f_n} := c_nB_f$. Similarly, g_n is the simplicial map from $P_{\mathbf{H}_n}$ to $B_{g_n} := d_nB_g$. Let $R_f, R_g : V(\mathbf{H}) \mapsto (0,\infty)$ and $R_{f_n}, R_{g_n} : V(\mathbf{H}_n) \mapsto (0,\infty)$ be the radius functions of $B_f, B_g, B_{f_n}, B_{g_n}$ respectively. Then, by Theorem VII.2.1, $f_n^{\#}(\cdot) = R_{f_n}(\cdot)/(1/n)$ and $g_n^{\#}(\cdot) = R_{g_n}(\cdot)/(1/n)$ converge uniformly on compacta of $\mathbf{C}$ to $|(z^m)'|$. In particular, for any $\epsilon > 0$ there exist $\delta > 0$ and N such that for $n > N$,

$$(*) \qquad \frac{1-\epsilon}{1+\epsilon} \leq \frac{R_{f_n}(v)}{R_{g_n}(v)} \leq \frac{1+\epsilon}{1-\epsilon} \text{ for any } v \in V(\mathbf{H}_n) \cap (D \setminus B(0, 1-\delta)).$$

Let n be fixed for the moment, and let C be any subcomplex of $\mathbf{H}_n$ bounded by a simple closed edge curve in $\mathbf{H}_n$. Then

$$(**)' \qquad \max_{w\in V(\mathcal{C})} \frac{R_{f_n}(w)}{R_{g_n}(w)} = \max_{w\in bd\,\mathcal{C}} \frac{R_{f_n}(w)}{R_{g_n}(w)}$$

because if the maximum were obtained in $int\,\mathcal{C}$ then (since B_{f_n} and B_{g_n} have the same angle sums at corresponding vertices) the ratio $\frac{R_{f_n}(\cdot)}{R_{g_n}(\cdot)}$ would have to stay constant in $V(\mathcal{C})$ otherwise one of packings B_{f_n} or B_{g_n} would not have the proper angle sum at one of the vertices of $\mathcal{C}$. Similarly

$$(**)'' \qquad \max_{w\in V(\mathcal{C})} \frac{R_{g_n}(w)}{R_{f_n}(w)} = \max_{w\in bd\,\mathcal{C}} \frac{R_{g_n}(w)}{R_{f_n}(w)}.$$

For $n > N$, let $\mathcal{C}_n$ be a subcomplex of $\mathbf{H}_n$ such that $0 \in \mathcal{C}_n$, $bd\mathcal{C}_n \subset D\backslash B(0, 1-\delta)$ and the boundary of $\mathcal{C}_n$ is a simple closed edge curve in $\mathbf{H}_n$. Then $(*)$, $(**)'$ and $(**)''$ give

$$(***)' \qquad \frac{|c_n|R_f(0)}{|d_n|R_g(0)} = \frac{R_{f_n}(0)}{R_{g_n}(0)} \le \max_{w\in bd\,\mathcal{C}_n} \frac{R_{f_n}(w)}{R_{g_n}(w)} \le \frac{1+\epsilon}{1-\epsilon}$$

and

$$(***)'' \qquad \frac{|d_n|R_g(0)}{|c_n|R_f(0)} = \frac{R_{g_n}(0)}{R_{f_n}(0)} \le \max_{w\in bd\,\mathcal{C}_n} \frac{R_{g_n}(w)}{R_{f_n}(w)} \le \frac{1+\epsilon}{1-\epsilon}.$$

Hence $\lim_{n\to\infty} |c_n|/|d_n| = R_g(0)/R_f(0)$.

Let $v \in V(\mathbf{H})$. If n is sufficiently large then $s_n^{-1}(v) \in int\,\mathcal{C}_n$ and, as in $(***)'$ and $(***)''$, we have

$$\frac{1-\epsilon}{1+\epsilon} \le \frac{|c_n|R_f(v)}{|d_n|R_g(v)} = \frac{R_{f_n}(s_n^{-1}(v))}{R_{g_n}(s_n^{-1}(v))} \le \frac{1+\epsilon}{1-\epsilon}.$$

Since $\lim_{n\to\infty} |c_n|/|d_n| = R_g(0)/R_f(0)$, by letting $\epsilon \to 0$, one gets

$$\frac{R_g(v)}{R_f(v)} = \lim_{n\to\infty} \frac{|c_n|}{|d_n|} = \frac{R_g(0)}{R_f(0)}.$$

As v is an arbitrary vertex of $\mathbf{H}$, the last shows that $R_f(\cdot)$ is a scaled version of $R_g(\cdot)$. $\square$

As a result of Theorem 1, Corollary VII.1.3, and the uniqueness of infinite univalent hexagonal circle packings (see [RS]), we get the following discrete analog of a classical theorem

Corollary 2. *The only infinite hexagonal circle packings of finite valence are the range packings of discrete complex polynomials. Moreover, any such packing is uniquely determined, up to similarities in* $\mathbf{C}$*, by its branch set.*

Remark 3. *The assumption about finiteness of valence in Corollary 3 is essential as the examples of infinite hexagonal spiral circle packings of [BDSt] show.*

We finish with an example which should be easily verified by the reader.

Example 4. Let B_1 denote an infinite hexagonal branched circle packing of finite valence and with only one branch point. Let B be an infinite univalent circle packing associated with complex $\mathcal{K}$, where $\mathcal{K}$ has all vertices of degree 6 except for one of degree 3. Then B and the projected image of B_1 are identical up to similarity. In particular, there is only one, up to similarity, infinite, unbranched circle packing of finite valence associated with complex $\mathcal{K}$.

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