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THEORY AND APPLICATIONS OF THE DOPPLER EFFECT

A THESIS
PRESENTED FOR THE
MASTER OF SCIENCE DEGREE
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SHELLEY DEE WEBB

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Abstract

This paper describes the manner in which sound and electromagnetic waves are affected by the Doppler principle. The discovery of the Doppler effect is discussed, the mathematical basis for the Doppler phenomenon as it applies to the one-dimensional wave equation is examined, and several Doppler applications are mentioned. Applications include Doppler echocardiography, Doppler radar, and underwater echo-ranging.

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Introduction

The purpose of this paper is a bit different than that of most mathematical research papers. It seeks to give the reader a logical and lucid understanding of the Doppler effect and some of its applications. In researching this subject, the writer has been overwhelmed by the vast number of ways that the Doppler effect "comes into play." Dan Censor [5], an extensive researcher on the Doppler effect, points out, "The literature concerned with applications of the Doppler effect (has) mushroomed to such an extent that a comprehensive review paper in journal article form is practically impossible. Dipping into a data pool of engineering and scientific publications with the phrase 'Doppler effect' yields thousands of titles." For that reason, this writer limits herself to several specific acoustical and electromagnetic applications of the Doppler principle, and she concentrates upon providing a clear mathematical basis for Doppler phenomena.

The initial idea for the present topic arose from an interest in applied mathematics, particularly differential equations and partial differential equations. Because of this desire to work with ordinary and partial differential equations, the writer originally began to delve into the one-dimensional wave equation. Investigation of sound waves led to the Doppler effect.

Although the Doppler principle in its most basic sense is not very complicated, it can be rather deceiving. Hopefully, this paper will educate the reader concerning Doppler phenomena and will eliminate some of the inherent deception. This paper will look at the discovery of the Doppler effect, at the mathematics behind the Doppler frequency change, and at applications of the Doppler principle.

Chapter 1

History

The discovery of the Doppler effect dates back to Johann Christian Doppler. He was an Austrian physicist who discovered the principle that bears his name. His discovery came while he was a professor of elementary mathematics and practical geometry at the State Technical Academy in Prague, Czechoslovakia. In Doppler's enunciation of the Doppler principle, he related the observed frequency of a wave to the motion of the source (or the observer) relative to the medium in which the wave was emitted. On May 25, 1842, he stated this principle in his article "Ueber das farbige Licht der Doppelsterne und einigeranderer Gestirne des Himmels" [8]. In this work, an accurate mathematical formula was derived for relating the frequency of a wave to the motion of a source or of an observer along the line between them. Doppler stated that this result could be applied to both acoustics and optics, yet his argument for the application to optics and astronomy failed to be convincing to the scientific community [8]. Doppler incorrectly believed that all stars emit white light and that differences in color were observed on earth, because the motion of the stars affected the observed frequency of the light. This unsound assumption upon which he based his astronomical applications was criticized by many in the scientific community. Still, considering the fact that Doppler was probably the earliest important physicist in Austria in the nineteenth century and therefore worked primarily in isolation, his discovery was quite remarkable [1].

Doppler did relate the acoustical Doppler effect quite accurately. He explained that sound waves emitted from a source travelling towards an observer will reach the observer at a greater frequency than if the source is stationary; consequently, the observed frequency is increased and the pitch of the sound is raised. In like manner, sound waves emitted from a source moving away from the observer reach the observer more slowly, which causes a delay in the observed frequency and a lowering of pitch. Verification of Doppler's observation was first made in 1845 at Utrecht in Holland by Christoph Buys Ballot. Buys Ballot used a locomotive to transport a group of trumpeters in an open car to and fro past some musicians who could sense the pitch of the notes being played. The variation in pitch produced by the movement of the trumpeters was in agreement with Doppler's theory [1].

Although this writer concentrates upon the Doppler effect as it applies to sound and electromagnetic waves, it is beneficial to be aware that the Doppler principle affects light waves also. The examination of light spectra is of primary importance to the understanding of how Doppler phenomena affect light waves. A spectral line is one of a series of images formed when a beam of light is dispersed so that component waves are arranged in the order of their wavelengths. Armand Fizeau is the scientist who first conducted research on the Doppler effect as it applies to light [1]. Much is written and much research is still being done in areas such as optics and spectroscopy because of Fizeau's work with light waves.

Christian Doppler first stated the Doppler effect, but Armand Fizeau was the French physicist who accurately linked the principle to astronomy. Fizeau did much work with the speed of light, and this work led him to point out in 1848 that shifts in the spectral lines of stars could be observed and attributed to the Doppler effect and hence enabled their motion to be determined. He theorized that motion away from the earth caused a red shift in the spectral lines, whereas motion towards the earth produced a blue shift. Fizeau's discovery later became the foundation for the primary method of determining the distances of galaxies and other distant bodies [1].

Chapter 2

Theory

Mathematically speaking, what exactly is the Doppler effect? One must have a clear understanding of the Doppler principle in its most basic sense before one can truly see how it is utilized. For this reason, the following explanation is provided.

When the source of a periodic disturbance moves with respect to a medium, wave patterns are altered. The simplest case is that of a source moving in a straight line at a constant rate. French [6] describes the two situations that may occur. These situations depend upon whether the speed of the source is greater or less than the speed of the waves generated by it. Let S be the source generating a short pulse at regular time intervals, and let S_n represent the position of the source at time n. The speed of the source is u, and the speed of the wave is c. Logically, at time t after initiating one of the circular waves, the radius of the wavefront is ct while the source has travelled a distance ut. The situation in which $u < c$ is illustrated in Figure 2.1.

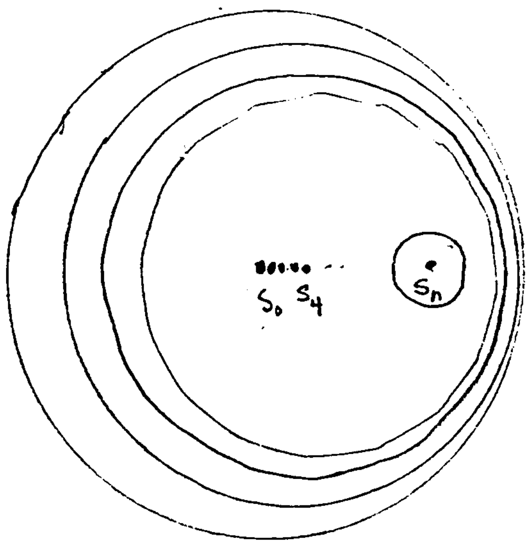

FIGURE 2.1 WAVE PATTERN WHEN $u < c$.

One observes that the circular wavefronts lie one inside the next one in succession, and the distance between successive wavefronts is least along the direction of motion of the source. If T is the interval of time between one wavefront and its immediate successor, then the distance between successive wavefronts is $(c-u)T$ in the direction that S is moving, and $(c+u)T$ at a 180° angle from this direction. There is a systematic change of wavelength with time for the waves propagated from a moving source. This variation of wavelength is the Doppler effect. If $\lambda_0 = cT$ represents the distance between wavefronts in any direction when the source is stationary, one can say that when the source moves,

$$\lambda_{\mathrm{min}} = \lambda_0 \left(1 - \frac{u}{c}\right)$$

$$\lambda_{\mathrm{max}} = \lambda_0 \left(1 + \frac{u}{c}\right)$$

where λ_{min} is the minimum distance between wavefronts and λ_{max} is the maximum distance between wavefronts. Unfortunately, this observation only concerns two directions of observation. What can be done to determine the situation in other directions from the source? It is necessary to assume that the distance from the source to the observer is very large compared to one wavelength. Figure 2.2 illustrates a source moving from S_0 to S_n. Because the speed of the source is u, one concludes that $S_0\,S_n = x_n = u(nT)$. The angle $\angle S_0\,P\,S_n$ is very small due to the fact that P is far away by assumption. The wavefronts arriving at P from S_0 and S_n are therefore almost parallel.

Suppose that the wavefront W_0 from S_0 has just reached P. From this information one derives a time $t_p = \frac{r_0}{c}$ in which W_0 travels from S_0 to P_0. Now the wave W_n from S_n started out at $t = nT$; therefore, at time t_p it has been moving

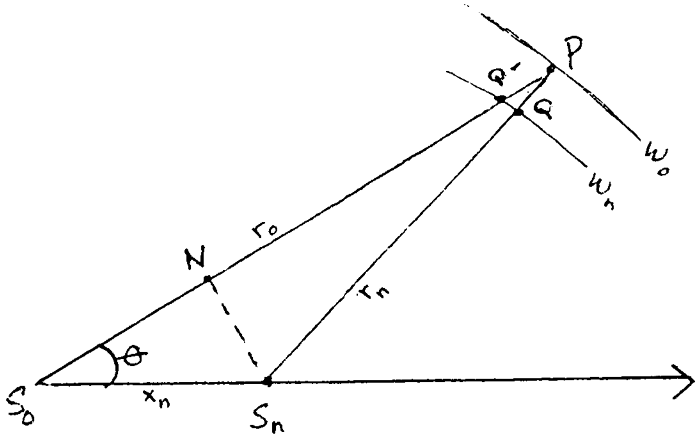

FIGURE 2.2 SOURCE MOVING FROM S_0 TO S_n.

for only a time $t_p - nT$ and one finds that

$$S_nQ = c(t_p - nT) = r_0 - c_nT$$

Now one can take either QP or $Q'P$ to be the distance between the wavefronts. Putting $S_nP = r_n$ and substituting leads to $QP = r_n - S_nQ = r_n - r_0 + cnT$. Dropping a perpendicular from S_n onto the line S_0P, one also has $NP \approx r_n$ (because of the smallness of the $\angle S_0PS_n$); therefore,

$$r_0 - r_n \approx S_0N = x_n \cos\Theta$$

$$\Longrightarrow r_0 - r_n \approx unT\cos\Theta$$

$$\Longrightarrow Q'P \approx QP \approx cnT - unT\cos\Theta$$

$$\Longrightarrow Q'P \approx n\lambda_0\left(1 - \frac{u\cos\Theta}{c}\right)$$

Since $Q'P$ and QP span n wavelengths of the disturbance in the direction Θ to the moving sources,

$$\lambda(\Theta) = \lambda_0\left(1 - \frac{u\cos\Theta}{c}\right)$$

where $\lambda(\Theta)$ is the distance from one wavefront to the next. This result says that the Doppler effect depends on the component of source velocity in the direction of the observer. The frequency, ν, at which successive wavefronts pass through the observation point P is the wave speed divided by the wavelength.

$$\nu(\Theta) = \frac{c}{\lambda(\Theta)} = \frac{c}{\lambda_0\left(1 - \frac{u\cos\Theta}{c}\right)}$$

$$\Longrightarrow \nu(\Theta) = \frac{\nu_0}{1 - \frac{u\cos\Theta}{c}} \tag{2.1}$$

where ν_0 is the initial frequency. Equation 2.1 is the best statement of the Doppler effect as it applies to accoustics [6].

What happens when the source velocity is greater than the wave velocity? French illustrates this situation with Figure 2.3.

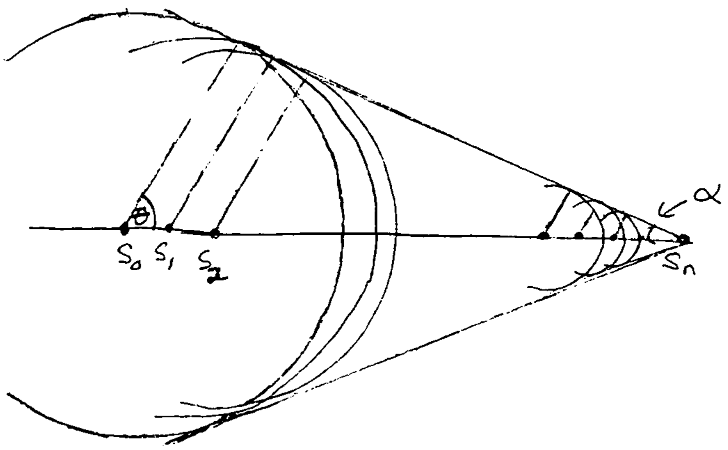

FIGURE 2.3 WAVE PATTERN WHEN $u > c$.

Suppose the source is at S_0 when $t = 0$, and it is at S_n when $t = nT$; consequently, $S_0S_n = nuT$, and the wavefront from S_0 has a radius of ncT. At this instant, the waves emitted from S_n are just beginning to be generated. Tangent lines from S_n to the circular wavefront emanating from S_0 are also tangent to all the other intermediate circles. The angle α made by the wavefronts with the line of motion of the source is defined by the relation

$$\sin\alpha = \frac{S_0P}{S_0S_n} = \frac{c}{u}.$$

The Mach number is defined to be the ratio $\frac{u}{c}$, and the angle α is the Mach angle (which only exists if the Mach number is greater than 1). One may use this information to mathematically analyze the situation. To consider the times of arrival of successive circular wavefronts at a point P far away from the moving source, again one refers to Figure 2.2. Suppose that a wave begins from S_0 at $t = 0$, and that a wave starts out from S_n at $t = nT$. The arrival times of these waves at point P are then described by

$$\begin{aligned} t_0 &= \frac{r_0}{u} \\ t_n &= nT + \frac{r_n}{c} \\ \text{thus} \qquad t_n - t_0 &= nT - \frac{r_0 - r_n}{c} \end{aligned}$$

Again using $r_0 - r_n \approx x_n \cos\Theta = nuT\cos\Theta$ one has

$$\begin{aligned} t_n - t_0 &\approx nT - \frac{nu\,T\,\cos\Theta}{c} \\ t_n - t_0 &\approx nT\left(1 - \frac{u\cos\Theta}{c}\right) \end{aligned}$$

When $u > c$, there is a value of Θ for which all the wavefronts arrive at P at the same time. If one calls this angle Θ_0, one has

$$\cos\Theta_0 = \frac{c}{u}$$

Note that Θ_0 is the complement of the Mach angle and gives the direction, prependicular to the straight wavefront itself, along which this concentrated region of the circular wavefront travels [6].

Before investigating the relationship of the Doppler effect to the one-dimensional wave equation, it is helpful to look at a brief analysis of a sound wave relative to a source and observer. Lonngren [12] says that if a stationary sound source emits a sound wave of frequency ν and an observer is moving away from the source with constant velocity u_0 relative to the air, which is stationary, the apparent sound velocity relative to the observer is $c - u_0$, where c is the sound velocity. Since the wave source is stationary, it emits sound waves in all directions with the same wavelength λ. Now $\lambda = \frac{c}{\nu}$. The observer should always observe the same wavelength. Whether the observer is moving or not moving should not make a difference in the wavelength. On the other hand, a moving observer should detect a frequency change, since the apparent sound velocity is $c-u_0$. The new frequency ν' is determined by

$$\lambda = \frac{c}{\nu} = \frac{c - u_0}{\nu'}$$
$$\Longrightarrow \nu' = \nu\left(\frac{c - u_0}{c}\right)$$

Similarly, if one has a moving sound source and a stationary observer, the new Doppler-shifted frequency is calculated using

$$\lambda = \frac{c - u_s}{\nu} = \frac{c}{\nu'}$$
$$\Longrightarrow \nu' = \nu\left(\frac{c}{c - u_s}\right)$$

where c is the sound wave velocity and u_s is the velocity of the source. If both the source and the observer are moving, the apparent frequency of the sound wave is given by

$$\nu' = \nu\left(\frac{c - u_0}{c - u_s}\right)$$

To illustrate this situation, suppose as in [12], a fire truck is approaching a solid wall with velocity u_s. If it is sounding a siren of frequency ν, what frequency of reflected sound waves do the firemen hear? To answer this query, one can regard the wall as an observer who hears the Doppler-shifted frequency ν' of the siren, where ν' is defined by

$$\nu' = \nu\left(\frac{c}{c - u_s}\right)$$

When the sound waves are reflected, the wall serves as a new source with frequency ν'. The fireman are the observers moving at velocity u_s toward the new, stationary, sound source; consequently, the sign of u_s is reversed, and the observed frequency is determined by

$$\begin{aligned} \nu'' &= \frac{c+u_s}{c}\nu' \\ \Longrightarrow \nu'' &= \frac{c+u_s}{c}\left(\frac{c}{c-u_s}\right)\nu \\ \Longrightarrow \nu'' &= \frac{c+u_s}{c-u_s}\nu \ . \end{aligned}$$

Hopefully, the reader can conceptualize the Doppler effect now. The previous discussion explains the Doppler phenomenon in terms of individual wavefronts. Examining wavefronts at equally spaced intervals is a useful method for understanding the Doppler effect, yet real life is not so discrete. Time is continuous; consequently, a more accurate mathematical representation of the wave itself is needed for a complete analysis of the Doppler effect. This fact leads us to the one-dimensional wave equation. Using this equation, the reader will see that the expected frequency change occurs when considering a wave propagated along the line between a moving source or observer.

Now we would like to look at the wave equation to investigate the behavior of waves. Each different situation should be handled separately. Gill [7] considers the case in which the transmitter T moves along a straight line passing through a fixed receiver R. Here the wave motion is described by the one-dimensional wave equation, $f_{tt} - c^2 f_{xx} = 0$, where c is the velocity of the wave and $f(x,t)$ represents the displacement of the wave from its initial position. The condition is imposed that at a point moving with velocity u, the disturbance is a periodic function of t, say $g(t)$ of period P. Using a sufficiently smooth function, $f_1(x-ct)$, as a solution of the wave equation and using the periodicity condition, we know that at a moving point defined by $x = x_0 + ut$, the disturbance is a certain periodic function $g(t)$. In other words,

$$f_1[x_0 - (c-u)t] = g(t)$$

is a function which has a period P and, therefore,

$$f_1[x - (c-u)t + (x_0 - x)]$$

has period P; consequently,

$$f_1[x - (c-u)t] = h(t) \tag{2.2}$$

with $h(t+P) = h(t)$. Let us consider

$$f(x,t) = f_1(x - ct) = q(t) \tag{2.3}$$

We wish to show that $q(t)$ has period

$$P' = \left(\frac{c-u}{c}\right) P. \tag{2.4}$$

If we let $t = \left(\frac{c-u}{c}\right)\tau$, then

$$\begin{aligned} q(t+P') &= q\left[\left(\frac{c-u}{c}\right)\tau + \left(\frac{c-u}{c}\right)P\right] \\ &= q\left[\left(\frac{c-u}{c}\right)(\tau+P)\right] . \end{aligned} \tag{2.5}$$

Substituting equation 2.5 into equation 2.3, we have

$$\begin{aligned} q(t+P') &= f_1\left\{x - c\left[\left(\frac{c-u}{c}\right)(\tau+P)\right]\right\} \\ &= f_1\left(x - (c-u)(\tau+P)\right) \\ &= f_1\left(x - (c-u)\tau\right) \end{aligned}$$

because $h(t)$ is periodic with period P. Since $ct = (c-u)\tau$, we may say that

$$q(t+P') = f_1(x-ct) = q(t)$$

and we have shown that $f_1(x-ct)$ has period P'. Substituting $P' = \frac{1}{\nu'}$ and $P = \frac{1}{\nu}$ into equation 2.4 yields

$$\nu' = \nu\left(\frac{c}{c-u}\right),$$

which is the normal Doppler expression for a moving source and stationary observer. This analysis shows how an incident wave is Doppler-shifted [7].

We now go on to discuss the Doppler principle as it applies to a *reflection* of an incident wave from a moving boundary. Again, the one-dimensional wave equation becomes a useful tool. Changing notation slightly from the previous section, the one-dimensional wave equation can be written

$$u_{tt} - c^2 u_{xx} = 0$$

where c is the velocity of the wave. This partial differential equation describes situations that occur in acoustics, small vibrations of a bar, waves on a string, etc. Censor [4] uses the D'Alembert solution of the wave equation, which is given by

$$u(x,t) = f\left(t - \frac{x}{c}\right) + g\left(t + \frac{x}{c}\right)$$

where f and g are sufficiently smooth functions describing waves traveling at speed c. A right travelling wave is described by $f\left(t - \frac{x}{c}\right)$, while a left travelling wave is described by $g\left(t + \frac{x}{c}\right)$. The solution $u(x,t)$ is therefore a superposition of an incident wave and its reflection. The D'Alembert solution leads one directly to an expression for the Doppler effect [4].

Censor [4] considers as a model for wave propagation a semi-infinite string extending from $-\infty$ to the boundary $x = x(t)$. The boundary is moving; therefore, its position is dependent upon time. For simplicity, one assumes density and tension are uniform and that the string is not a lossy media. The D'Alembert solution

$$u(x,t) = f\left(t - \frac{x}{c}\right) + g\left(t + \frac{x}{c}\right)$$

describes the displacement of the string. For this situation the boundary condition $u(x(t),t) = 0$ exists, because the wave disappears at $x = x(t)$ for all t; consequently, at the boundary, one discovers that

$$g\left[t + \frac{x(t)}{c}\right] = -f\left[t - \frac{x(t)}{c}\right]. \tag{2.6}$$

Consider the case in which the boundary moves with constant velocity $\hat{u}$. In other words

$$x(t) = \hat{u}t \tag{2.7}$$

Substituting equation 2.7 into equation 2.6 yields

$$g\left[t + \frac{\hat{u}t}{c}\right] = -f\left[t - \frac{\hat{u}t}{c}\right] \tag{2.8}$$

Letting $\zeta = t\left(1 + \frac{\hat{u}}{c}\right)$, equation 2.8 becomes

$$g[\zeta] = -f\left[\zeta\frac{\left(1 - \frac{\hat{u}}{c}\right)}{\left(1 + \frac{\hat{u}}{c}\right)}\right] \tag{2.9}$$

Here is the expression for the scattered wave $g(\zeta)$, no matter what ζ represents. Using equation 2.9, one determines

$$g\left(t + \frac{x}{c}\right) = -f\left\{\left(t + \frac{x}{c}\right)\left[\frac{1 - \frac{\hat{u}}{c}}{1 + \frac{\hat{u}}{c}}\right]\right\}$$

which is the desired expression [4]. The Doppler effect for the reflected wave is derived, the result being analogous to the fire truck example.

The problem just discussed is time dependent. One would like to derive the same result using a more general representation for the solution to the wave equation. Censor presents an argument using such an expression. He lets the incident wave be $e^{-w_i(t-\frac{x}{c})}$ where w_i is the frequency of the incident wave. Again, $x(t) = \hat{u}t$ at the boundary. Censor then represents the reflected wave, which is unknown, in a general manner as

$$g(w_r, x, t) = \int_{-\infty}^{\infty} e^{[-iw_r(t+\frac{x}{c})]} F(w_r) dw_r \tag{2.10}$$

where w_r, the frequency of the reflected wave, and $F(w_r)$ are determined by the boundary conditions. As before, the wave vanishes at the boundary; hence, $e^{[-iw_i t(1-\frac{\hat{u}}{c})]} = -\int_{-\infty}^{\infty} e^{-iw_r t(1+\frac{\hat{u}}{c})} F(w_r) dw_r$. One recognizes this equation as the Fourier transform such that its inverse is

$$\begin{aligned} F(w_r) &= -\frac{1}{2\pi} \int_{-\infty}^{\infty} e^{\left\{it(1+\frac{\hat{u}}{c})\left[w_r - w_i\left(\frac{1-\frac{\hat{u}}{c}}{1+\frac{\hat{u}}{c}}\right)\right]\right\}} \left(1 + \frac{u}{c}\right) dt \\ &= -\delta\left[w_r - w_i\left(\frac{1-\frac{\hat{u}}{c}}{1+\frac{\hat{u}}{c}}\right)\right] \end{aligned}$$

Substituting this expression into equation 2.10, the δ-function chooses the frequency $w_r = w_i\left[\frac{1-\frac{\hat{u}}{c}}{1+\frac{\hat{u}}{c}}\right]$, which is the expected Doppler frequency change. As in the fire truck example, one expects the frequency of the reflected wave to be double-Doppler shifted [4].

Chapter 3

Applications

As mentioned earlier in this paper, the Doppler effect is applicable in many situations. For example, the Doppler principle can be applied to devices such as burglar alarms. An ultra high frequency electromagnetic field or a supersonic sound field is formed in the region to be guarded. Any moving object in the area reflects a Doppler shifted wave which beats with the original signal and is therefore detected. Police radar detectors work along the same principle. The officer's radar gun could be pointed toward any moving object, even a bird, to determine its speed. When the radar gun sends out a signal, a Doppler-shifted wave is reflected back from the moving object and then the speed of the object is calculated. This method of determining the velocity of vehicles is extremely accurate, and consequently very difficult to refute in court. These two examples of the usefulness of Doppler's principle are not even " the tip of the iceberg." Due to the proliferation of information on Doppler, this writer shall concentrate upon medical and navigational applications.

A. Medical uses.

The Doppler effect is used extensively in echocardiography, the determination of blood velocity, and gynecology. The type of signal initially sent is called an ultrasonic wave. By definition, ultrasound is beyond the audible range of sound and consequently has a frequency greater than 20,000 hertz (Hz). One hertz is equal to one cycle per second. A high frequency indicates a small wavelength. Medical applications require very small wavelengths ranging from .1 to 1.5 mm because the wavelength determines the resolution of the imaging system. If two structures are closer than one wavelength apart, they cannot be identified as separate entities in an ultrasound image. The primary use of the ultrasound is in the detection and measurement of the flow of blood. Red blood cells become a main source of ultrasonic scattering. Woodcock says that the higher the ultrasonic frequency used to investigate whole blood, the stronger is the scattering [16]. An ultrasonic beam is echoed from blood cells and altered in frequency. This frequency shift is detected and applied to a loudspeaker to produce an audio signal. The continuous-wave, nondirectional Doppler technique has been very useful in the detection of fetal life, the determination of arterial pulsations, and the evaluation of varicose veins.

In 1950 the first succesfull attempt to measure blood velocity in a specific blood vessel from the surface of the body was reported. An instrument called a Doppler-shift flow-velocity meter was used. This device sent ultrasound waves to the blood cells, which in turn scattered the incident radiation; a second transducer then received the scattered waves. The signals emitted and received were mixed, the result of which was the Doppler-shift frequency. Why did the mixture produce the Doppler-shift frequency? The answer lay in the fact that the scatterers, the red blood cells, were moving.

In the Doppler-shift flow-velocity meter, the Doppler shift is founded upon two changes in apparent frequency between the emitting and receiving transducers. Woodcock [16] explains that the first shift occurs when the ultrasound arrives at the scattering cell, and the second shift occurs as the ultrasound exits the scatterer. If the blood cells are moving toward the emitter, the initial change in apparent frequency is given by

$$f_1 = f_0 \left(\frac{c - v \cos \Theta}{c} \right)$$

where f_0 is the emitter frequency, c is the speed of ultrasound in the blood, Θ is the angle of inclination of the emitted wave to the direction of blood flow, and v

is the velocity of the blood cells. If one assumes that the emitted and scattered waves are both at angle Θ to the blood flow direction, the second alteration in frequency is given by

$$f_2 = f_1 \left[\frac{c}{c + v\cos\Theta}\right]$$

The Doppler shift Δf can therefore be represented by

$$\begin{aligned}\Delta f &= f_0 - f_2 \\ &= f_0 - \left[\frac{f_0(c - v\cos\Theta}{c + v\cos\Theta})\right]\end{aligned}$$

Then since $c \gg v$, one may deduce the common expression for the Doppler-shift in blood flow [16]:

$$\Delta f = \frac{2fv\cos\Theta}{c}$$

There are three methods of measuring blood velocity with ultrasonic beams. In order to examine how the Doppler shift plays an important role in the calculation of blood velocity with each method, one must first know the angle of inclination of the ultrasonic beam to the direction of blood flow. Of course in theory, no Doppler-shift frequency is observed when the ultrasonic beam is at right angles to the blood flow direction. On the other hand, there is usually a tiny signal present in practice.

The first method of measuring blood flow requires the probe to be situated such that one gets the minimum signal, which is logically assumed to be at a right angle to the direction of blood flow. From this point, Θ can be discovered by rotating through a known angle from the original position. This method only involves one probe.

The second method of measuring blood flow involves a device consisting of two Doppler-shift flow probes oriented at right angles to each other. If Θ and ϕ are the angles of incidence of the ultrasonic beams from these probes, the Doppler-shifts are

$$\Delta f_1 = \frac{2fv(\cos\Theta)}{c}$$

and

$$\Delta f_2 = \frac{2fv(\cos\phi)}{c}.$$

Since $\Theta = (90° - \phi)$, it is valid to say

$$\Delta f_1 = \Delta f_2 \tan\phi.$$

One can find ϕ and Θ by measuring the two Doppler shifts Δf_1 and Δf_2.

The last method involves three transducers and is the result of work done by Hansen [9] in 1976. Two of the transducers are used as transmitters while the third transducer is a receiver. The ultrasonic fields from the transmitters produce an interference field. By oscillating these two transmitters at different frequencies, the fringe pattern can be made to move. The scattered signal is amplitude-modulated when blood moves within the interference field; the modulated frequency is dependent upon whether the blood is moving with or against the fringe movement. To a first approximation, this is independent of the angle of inclination of the beam to the blood flow direction. Figure 3.1 illustrates method 3. The Doppler-shift frequency is calculated as $\Delta f = \frac{2fv\cos\phi\sin\frac{1}{2}\Theta}{c}$ [16],[9].

One of the major Doppler calculations made to interpret abnormal signals is the modified Bernoulli equation. The Bernoulli equation for fluid flow past an obstruction relates pressures before and after the obstruction, P_1 and P_2 respectively, with other properties of the fluid, such as the density ρ, the fluid velocities in front of and behind the obstruction (V_1 and V_2), and the quantitites particular to the fluid and obstruction. These other quantities involve flow acceleration and viscous friction, $R(V)$, which retards the flow. The equation is

$$P_1 - P_2 = \frac{1}{2}\rho\left(V_2^2 - V_1^2\right) + \rho\int_1^2 \frac{dV}{dt}ds + R(V)$$

(Pressure change) = (Corrective acceleration) + (Flow acceleration) + (Friction)
Labovitz [11] considers this equation and points out that if we maximize velocity, flow acceleration is zero.

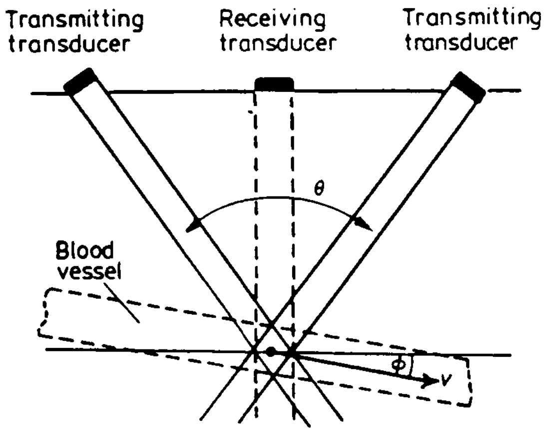

FIGURE 3.1 HANSEN'S METHOD OF MEASURING BLOOD FLOW VELOCITY.

In stenotic valves, valves subject to constriction, the diameter of the orifice is wide in relation to the length; therefore, friction forces are very small until the diameter of the valve orifice is less than 2mm, which is a condition not seen in life. For these reasons, friction forces and flow acceleration can be ignored, thus simplifying the equation. Expressing the density of blood in appropriate units, the equation becomes

$$P_1 - P_2 = 4(V_2^2 - V_1^2)$$

When V_1 is less than 1m/sec, it can even be ignored, leaving the equation:

$$P_1 - P_2 = 4V_{\max}^2$$

Using this equation, one can determine the mean pressure gradient by measuring blood velocity at equally spaced points through the use of Doppler methods [11].

Right now, the Doppler evaluation of a stenotic aortic valve is one of the most important clinical applications of Doppler technology. Stenosis is a condition in which some bodily passageway such as the aortic valve is narrowed or constricted. Continuous-wave and high-pulse repetition frequency (high-PRF) Doppler analyses have been shown to yield accurate quantification of transvalvular gradients. This quantification is possible because the velocity of blood flow increases in proportion to the degree of stenosis. The peak transvalvular gradient can be determined using the modified Bernoulli equation:

$$\text{Gradient } = 4V_{\mathbf{max}}^2$$

where $V_{\mathbf{max}}$ is the maximal transvalvular aortic velocity. In patients with aortic stenosis, there is usually an increase in velocity at the valve level [11].

Another very useful aspect of Doppler techniques is the detection of malfunctioning prosthetic valves. Prior to the advent of Doppler techniques, the non-invasive evaluation of prosthetic valves was limited. X-ray or echocardiographic visualization could seldom identify problems. Most of the malfunctioning prostheses required cardiac catheterization for confirmation, something which is painful and expensive [11].

How is the Doppler principle useful in obstetrics? Doppler instruments are useful in early detection of pregnancy, diagnosis of fetal death in utero, detection of a remote fetal heart, intermittent observation of the rate and rhythm of the fetal pulse, placental localization, diagnosis of multiple pregnancy, the diagnosis of vaginal bleeding during pregnancy, and fetal monitoring in the third trimester and during labor.

How do obstetric Doppler devices work? They funtion in much the same way as Doppler devices used to detect blood velocity. Doppler machinery for obstetrics consists of an abdominal wall transducer containing both ultrasonic transmitting and receiving crystals. The detected frequency changes are recorded in a central unit which can translate the signals into any of several output formats. Formats include audible sounds, waves on an oscilloscope screen, or a tracing on a continuously moving paper chart. In addition to being quite useful, Doppler ultrasound is considered to be safe. Sanders and James [14] indicate that "Doppler is to traditional ultrasound what fluoroscopy is to x-ray." In other words, it is a relatively continuous, low-intensity energy beam. Doppler ultrasound employs intensities several orders of magnitude lower than therapeutic ultrasound. Women who experience difficult and protracted labors may discover that they and their fetuses are continuously exposed to Doppler ultrasound for many hours. These women have little to fear from Doppler ultrasound [14].

B. Radar Applications.

The Doppler effect has extensive applications to radar systems. There are many kinds of radar sets that have been developed for various applications, and each application emphasizes a certain type of target. For instance, a marine navigational radar system is designed to indicate land masses with as much accuracy as possible. Although an aircraft navigation radar system targets the land below, an aircraft-warning radar should show other aircraft and moving formations clearly, not land masses. These examples should suffice to show that response to some targets is desired while response to others is not wanted.

E.J. Barlow [3] points out that radar systems are built which use the Doppler effect to distinguish between fixed and moving targets. An ordinary pulse radar system concentrates upon fixed objects unless MTI, moving-target indication, features are employed. On the other hand, cw (continuous-wave) radar systems tend to deemphasize stationary objects and concentrate upon moving entities; therefore, such a system might be useful in detecting the presence of a moving object among a large number of fixed objects. For example, the system could be used as a watchman to detect a man walking in the vicinity of trees or buildings. For this particular application, an ordinary pulse system would be practically useless because a man would be unnoticed amongst the stationary objects. Another application for a simple cw Doppler system would be the precise measurement of the velocity of bullets and other projectiles, and the investigation of their trajectories.

There are drawbacks to Doppler radar systems, but first one must understand the fundamental principles of Doppler radar. Figure 3.2 illustrates the simple radar system. As explained by Barlow [3], "the transmitter output is an unmodulated carrier which is fed into the directive transmitting antenna. The receiver consists of a directive antenna followed by a detector and an audio amplifier." The antennas are placed side by side, and they rotate together. This system causes a leakage signal to spill over from the transmitting antenna into the receiver; the leakage signal is always present. Any objects in the path of the transmitter beam reflect energy back to the receiver also. As shown in Figure 3.3, there are primarily three types of signals to enter the receiver. There are signals reflected from fixed targets, signals that bounce off of moving targets, and the leakage signal. Because of the Doppler effect, the moving target will reflect a signal of a different frequency.

The phase difference between the leakage signal and the signal from the moving target depends upon the distance to the target and on the wavelength of the

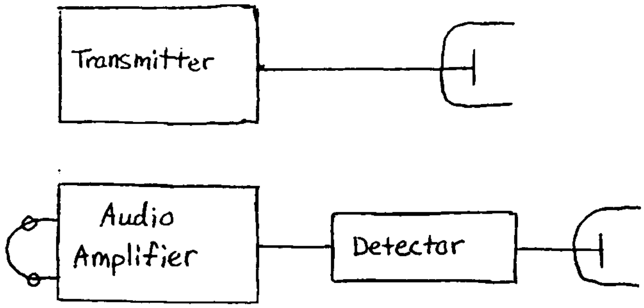

FIGURE 3.2 A SIMPLE RADAR SYSTEM.

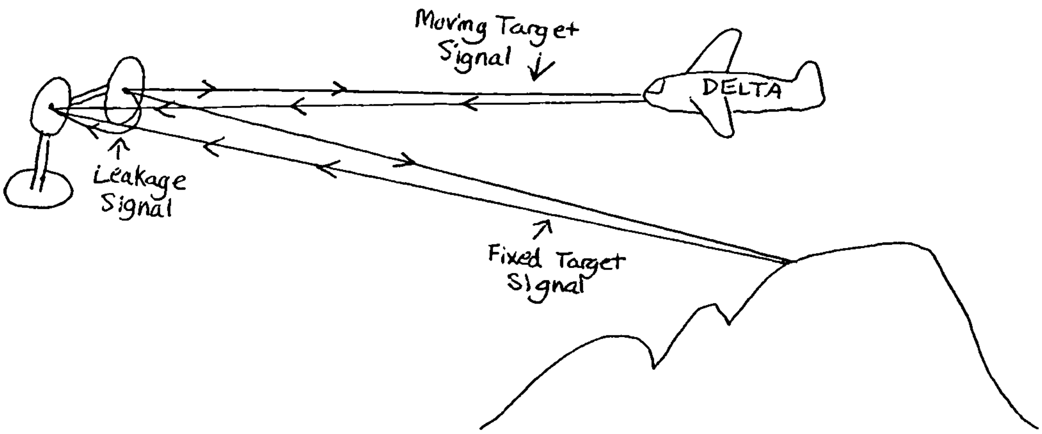

FIGURE 3.3 ILLUSTRATION OF SIGNALS ASSOCIATED WITH TARGET DOPPLER RADAR.

system. The wavelength is given by

$$\lambda = \frac{4\pi r}{\Delta\phi}$$

where $\Delta\phi$ = phase difference, r = target range, and λ = system wavelength. As the target range changes, $\Delta\phi$ will be altered. Barlow [3] points out that a "change in the phase difference $\Delta\phi$ is equivalent to a frequency difference of the amount

$$\Delta f = \frac{1}{2\pi}\frac{d(\Delta\phi)}{dt}$$

which is equal to

$$\Delta f = \frac{4\pi}{\lambda} \cdot \frac{1}{2\pi} \cdot \frac{dr}{dt}$$

or simply

$$\Delta f = \frac{2v}{\lambda}, \text{ the Doppler frequency,}$$

where v = the radial velocity of the target." In the system described here, the leakage signal and moving-target signal beat simultaneously in the detector and produce a signal at the difference of these frequencies, in other words, at the Doppler frequency. The audio amplifier increases the intensity of the Doppler signal, which is then fed into earphones. Now the fixed-target signal is of the same frequency as the leakage signal; therefore, it goes unnoticed by the operator. Adding a band-rejection filter to the audio amplifier makes it possible to reject targets of a certain radial velocity. One could even select one radial velocity and eliminate all others. As seen in previous examples, the use of Doppler to discriminate among velocities is quite important and is widely utilized.

Unfortunately, the system described by figure 3.2 has a few drawbacks. Amidst those negative aspects is the fact that the presence of the leakage signal represents a limitation. The leakage signal makes it difficult to distinguish between approaching and receding targets. A target approaching with velocity v and one receding with this same velocity both produce the same Doppler frequency, $\Delta f = \frac{2v}{\lambda}$. This problem may be remedied if the system can be arranged so that a signal at some frequency f is produced for fixed targets. Approaching targets then have a frequency $f + \frac{2v}{c}$, and receding targets have a frequency $f - \frac{2v}{c}$. The leakage signal then must be filtered out if moving targets are to be detected. Sometimes the leakage signal is much stronger than the signals from the desired moving targets; hence, the leakage signal is difficult to filter out of the system.

The significance of the leakage signal and its noise modulation depends upon the absolute amount of leakage power, which is usually dependent upon transmitter power. For a small system of limited range, the problem may not be serious at all. The watchman application would fall in this category, because the range on a person probably wouldn't exceed one-half mile. On the other hand, a system designed to detect an airplane two hundred miles away, would utilize transmitter power of great magnitude; therefore, the leakage signal would be very large and very difficult to filter. The problem can be eliminated in this situation by separating the transmitter and receiver by an adequate distance. This solution is feasible in some applications and not feasible in others. Such a system could be permanently installed for ballistics studies at a firing range, but a system which must be transportable may cause physical separation of transmitter and receiver to be inconvenient.

The Doppler effect is an integral part of many navigation systems. The term navigation is used when a vehicle's position is determined from the vehicle itself; therfore, the term is used most of the time with manned vehicles such as aircraft, ships, or submarines. Gill [7] indicates that there are basically three types of navigation. The first, the reference grid type, is the kind upon which most practical systems of navigation by ships and aircraft are based. With this system, two coordinates are fixed by observing quantities which may vary over the surface of the earth in such a way as to uniquely determine a position if both are measured. Since the altitude of a ship is fixed, and since the altitude of an airplane usually varies by only a few miles, only two coordinates are necessary. The traditional system of navigation, which is based upon observations of bodies such as stars in the sky, is a system of the reference grid type. In the second type of navigation, the velocity of the manned vehicle is measured as a function of time and then integrated to determine position. Although this second type of navigation is used extensively at sea and in the air, errors tend to be cumulative; consequently, it is common to find it used in combination with the reference grid method. The third type of navigation takes the acceleration of the vehicle and then integrates twice to determine displacement from the original position. This method, referred to as inertial guidance, requires no contact with the outside world and is thus not vulnerable to being misled or sabotaged by an outside force. Inertial guidance systems are obviously of great interest for military purposes [7]. One would logically assume that navigation systems employing the Doppler effect would all be of the second type, since Doppler measures velocity directly. In reality, there are also Doppler navigation systems using the reference grid method. These systems use artificial satellites carrying radio transmitters whose radiation penetrates clouds, thus allowing an all weather navigation system.

All satellite navigation systems basically share common problems. One of these problems concerns the orbits of the satellites. The orbits must be computed and the information made accessible to the users of the system while the information is still valid. Another concern is that enough satellites are situated in place such that twenty-four hour world coverage is relayed without a navigator having to wait too long for a set of observations. Accurate directional aerials on vessels are requred in navigational systems in which satellites' angular positions are observed; Doppler systems are therefore quite successful.

One of the first widely utilized Doppler navigational systems, "Transit", developed in the United States was a reference grid type system. Gill [7] describes it as using four satellites in circular orbits at altitudes of about 500 miles. This height gave a good compromise between the negative results of an orbit being too high or too low. For example, if an orbit had too high of an altitude, the period would have been long, the Doppler shift small, and the perturbations due to the moon and sun would have complicated calculations. Alternatively, if the altitude of the orbit had been too low, perturbations due to atmospheric drag and the unsymmetrical gravitational field of the earth would have had a noticeable effect upon the calculations. Nonetheless, four satellites were used so that world coverage at all times could be accomplished [7]. Four Doppler stations provided and still can provide sufficient data to track a vehicle; consequently, a vessel carrying a receiver and frequency standard could be navigated if it received signals from four properly located transmitters functioning at known frequencies.

The second type of Doppler navigation mentioned has been utilized in aircraft navigation. As mentioned by J.D. Harmer and W.S. O'Hare [10], "Very precise measurement of ground velocity for navigational purposes using self-contained active continuous-wave radar has become a well established art." The ground is used as a target for accurately directed slant radar beams. Basically, an aircraft transmits a narrow beam of short radio waves to the ground below at a specific angle from the craft's direction of flight, as illustrated in Figure 3.4.

These radio waves are reflected back from the irregularities in the ground below. The return signal is mixed with some of the transmitted signal. The mixture of two waves superimposes the waves and causes a variation of amplitude in the mixed signal. This growth and decay of amplitude is registered as 'beats' of strong reinforcement. The velocity of the plane relative to the ground can be determined from the beat frequency. Integrating this velocity with respect to time, one may deduce the distance traveled. The reason that the beam is directed downwards at

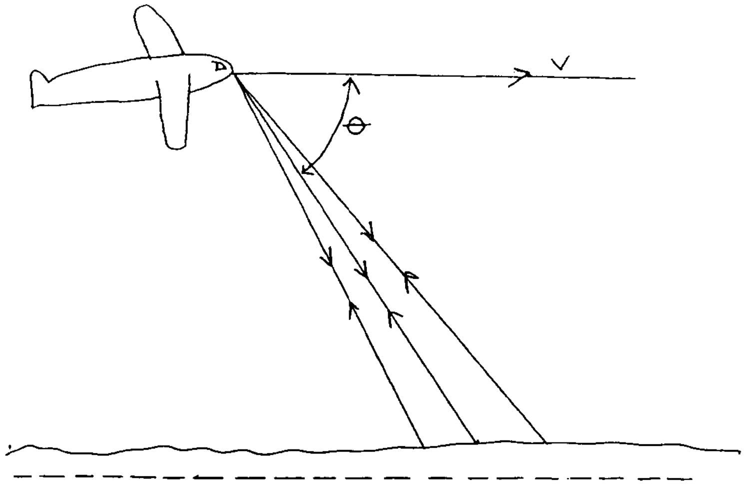

FIGURE 3.4 ILLUSTRATION OF HOW AIRCRAFT NAVIGATE USING SLANT RADAR BEAMS.

an angle Θ is to obtain a reflected signal; there is normally no reflecting surface directly ahead or behind an aircraft in flight. The Doppler effect is important to the determination of the aircraft's velocity in much the same way as in the determination of blood velocity. The Doppler shift Δf is again given by

$$\Delta f = \frac{2fv\cos\Theta}{c}$$

where f is the frequency of the emitted radio wave, c is the velocity of the wave, v is the speed of the aircraft and Θ is the angle of incidence of the wave with respect to the heading of the airplane. Because

$$v = \frac{c\Delta f}{2f}\sec\Theta,$$

a systematic error in Θ would be cumulative and would affect the calculation of distance travelled. It is therefore necessary to know $\cos\Theta$ with as much accuracy as possible. T.P. Gill explains why Θ is usually chosen to be between 60° and 70°.

If Θ is chosen close to 90°, the reflected signal would be strong, but a small error in Θ would represent a large error in v. If on the other hand, Θ is chosen near 0°, the intensity of the reflected signal may be too small to measure its frequency. Choosing Θ to be in the neighborhood of 60° to 70° reflects a good compromise [7].

C. Sonar and echo ranging.

Another place where Doppler phenomena can be observed is under the sea. Of all the forms of radiation known, sound travels through the sea best. Because underwater sound is propagated with relative ease, people have applied it to a variety of purposes in their use and exploration of the ocean. These uses of underwater sound form the engineering science of sonar.

How does a sonar system function? Robert Urick explains that in a sonar system, sound is purposely generated by one of the system components, the projector. The sound waves generated by the projector travel through the water to some target, and then are returned as sonar echoes to a device called a hydrophone. The hydrophone converts sound into electricity, which is then amplified and processed onto a display device so that the output can be read [15]. This process of using sound to discover information, such as size or speed, about a target is called echo-ranging. As one might expect, the Doppler effect is an important consideration in the processing of echo-ranging data. Before examining the target echo itself, it is desirable to understand the concept of reverberation. When a piano note is played in a large, empty room, the sound echoes and re-echoes from the walls, ceiling and floor for a considerable amount of time. A very similar phenomenon occurs, because of surface and ocean floor irregularities, when an echo-ranging pulse of sound is emitted into the sea. This phenomenon has been called "reverberation." Reverberation is the result of a large number of weak echoes. Surface waves, plankton, fish and even air bubbles act as scatterers of sound [13].

In echo ranging, the operator does not hear the outgoing ping, because the sonar equipment is on "send," and the receiver is blocked. For this reason, it is not possible for him to compare the frequency of the returning echo with that of the outgoing ping. Fortunately, he can compare the frequency of the echo with that of the reverberation heard immediately after the ping is emitted. The difference between the reverberation and echo frequency depends on the target's absolute motion through the water and its direction relative to the sound beam.

Suppose a ship is moving with velocity V, and its sound beam is directed straight ahead. The reverberation frequency is

$$f_R = f_0 + \frac{2F_0 V}{c}$$

where c is the sound wave speed, F_0 is the true frequency of the sound, and f_0 is the audio frequency of the echo for a zero range rate. It should be noted that the quantity $\pm\frac{2F_0 V}{c}$ is the absolute Doppler shift. If a submarine is approaching the echo-ranging ship with a speed V', the relative speed v is $v = V + V'$ and the audio frequency of the echo is

$$f_E = f_0 + \frac{2F_0 V}{c} + \frac{2F_0 V'}{c}.$$

From this information, one can easily see that the audio frequency of the echo exceeds that of the reverberation by

$$\Delta f = \frac{2F_0 V'}{c}.$$

This difference is called the "target Doppler." One may note that this expression is independent of the speed of the sonar vessel, a fact that enhances the usefulness of target Doppler.

Target Doppler is proportional to the speed of the target; therefore, it gives information concerning the motion of the target. If the echo is a higher frequency than the reverberation, this is referred to as up-Doppler, and the range between the target and the sonar vessel is closing. If the echo frequency is lower than that of the reverberation, the range is increasing [13]. One can readily see that the Doppler effect has contributed to the investigation of underwater phenomena.

Conclusion

Hopefully, the reader now has an understanding of the basic Doppler principle and some of its application. Several applications of the Doppler effect have been investigated. One has seen that Doppler is widely used in diagnostic procedures concerning blood velocity and cardiology. Ultrasound and Doppler often go "hand in hand." One has also become familiar with Doppler radar, its use as a detection system, and its uses as a navigational system. Lastly, the reader has been presented with how a sonar system utilizes the Doppler effect to examine the ocean through echo-ranging. Since the Doppler effect is basically a frequency shift associated with moving objects, one can only imagine the number of applications it has to the lives of everyone. Objects in motion are everywhere to be seen. This writer simply hopes that the concept of the Doppler effect has been made clear to the reader. Hopefully, the reader will at least note the frequency change in sound when a train approaches and passes him; maybe the reader will even think, "That is the Doppler effect!"

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Vita

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