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I am submitting herewith a dissertation written by Edward Caston Nichols entitled "Maximum Principles for P-Functions in Elliptic Partial Differential Equations." I have examined the final copy of this dissertation for form and content and recommend that it be accepted in partial fulfillment of the requirements for the degree of Doctor of Philosophy, with a major in Mathematics.

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MAXIMUM PRINCIPLES FOR P-FUNCTIONS IN ELLIPTIC  
PARTIAL DIFFERENTIAL EQUATIONS

A Dissertation  
Presented for the  
Doctor of Philosophy  
Degree  
The University of Tennessee, Knoxville

Edward Caston Nichols  
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## DEDICATION

To my mother, Margaret Nichols, and the memory of my father,  
George Nichols.

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## ABSTRACT

A function is said to satisfy a maximum principle in a domain  $\Omega$  if the function either is constant or does not attain a maximum in  $\Omega$ . Moreover, if  $\Omega$  is a bounded domain, then a function which satisfies a maximum principle in  $\Omega$  and which is continuous on  $\bar{\Omega}$  must attain its maximum value on  $\partial\Omega$ . Maximum principles are developed for certain P-functions; i.e., functions defined on solutions of an elliptic partial differential equation. In turn, these maximum principles are used to obtain information about solutions of some second order semilinear elliptic equations as well as solutions of higher order elliptic equations. Maximum principles for P-functions associated with a system of second order elliptic equations are also considered.

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## CHAPTER I

### INTRODUCTION

One of the most useful tools in the study of elliptic partial differential equations is the maximum principle. In general, we say that a function satisfies a maximum principle in a domain  $\Omega$  if the function either is constant or does not attain a maximum in  $\Omega$ . Maximum principles allow us to obtain information about solutions of differential equations without explicitly finding the solution. Numerous applications of maximum principles for solutions of second order linear equations can be found in the excellent text of Protter and Weinberger [17]. We shall be interested in obtaining maximum principles for functions associated with solutions of some second order nonlinear equations as well as elliptic equations of higher order. We adopt the terminology in Sperb's text [23] and refer to these functions as P-functions.

Let  $u(x_1, x_2, \dots, x_N)$  be a real-valued function defined on a domain  $\Omega$  in Euclidean N-space  $\mathbb{R}^N$ . The Laplace operator or Laplacian  $\Delta$  is defined by

$$\Delta := \frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2} + \dots + \frac{\partial^2}{\partial x_N^2}.$$

The function  $u \in C^2(\Omega)$  is said to be subharmonic (superharmonic) in  $\Omega$  if  $\Delta u \geq 0$  ( $\leq 0$ ). If  $\Delta u = 0$  in  $\Omega$ , then  $u$  is said to be harmonic. It is well known, e.g. [17], that a subharmonic (superharmonic) function satisfies the following maximum (minimum) principle.

Theorem 1.1. Suppose  $\Delta u \geq 0$  ( $\leq 0$ ) in  $\Omega$ . If  $u$  attains a maximum (minimum) value at any point of  $\Omega$ , then  $u$  is constant. Consequently, a harmonic function cannot attain a maximum or a minimum value at any point of  $\Omega$  unless it is constant.

Now, if  $\Omega$  is a bounded domain with boundary  $\partial\Omega$  and if  $u$  is continuous on  $\bar{\Omega} := \Omega \cup \partial\Omega$ , then  $u$  attains a maximum and a minimum value on  $\bar{\Omega}$ . From Theorem 1.1 then, a subharmonic (superharmonic) function which is continuous on  $\bar{\Omega}$  attains its maximum (minimum) value on  $\partial\Omega$ . We can use Theorem 1.1 to prove the following result.

Theorem 1.2. Let  $\Omega$  be a bounded domain with smooth boundary  $\partial\Omega$ . If  $u \in C^3(\Omega) \cap C^1(\bar{\Omega})$  is harmonic in  $\Omega$ , then

$$P := |\text{grad } u|^2$$

takes its maximum on  $\partial\Omega$ .

We note that Theorem 1.2 follows directly from the subharmonicity of  $|\text{grad } u|^2$ .

A related result is valid for biharmonic functions. A function  $u \in C^4(\Omega)$  is said to be biharmonic in  $\Omega$  if  $\Delta^2 u = 0$ , where  $\Delta^2 u = \Delta(\Delta u)$ . Miranda [8] proved the following.

Theorem 1.3. Let  $\Omega$  be a bounded domain in the plane with smooth boundary  $\partial\Omega$ . If  $u \in C^4(\Omega) \cap C^2(\bar{\Omega})$  is biharmonic in  $\Omega$ , then

$$P := |\text{grad } u|^2 - u\Delta u$$

takes its maximum on  $\partial\Omega$ .

Again, the result follows directly from the subharmonicity of  $|\text{grad } u|^2 - u\Delta u$ .

An important example of a biharmonic function is the torsion function  $\psi$ , which is a solution of

$$\begin{aligned}\Delta\psi + 2 &= 0 \quad \text{in } \Omega, \\ \psi &= 0 \quad \text{on } \partial\Omega,\end{aligned}$$

where  $\Omega$  is a bounded domain in the plane. If we apply Theorem 1.3 to the torsion function  $\psi$ , we have that

$$P_1 := |\text{grad } \psi|^2 + 2\psi$$

takes its maximum on  $\partial\Omega$ . Moreover, since  $\psi$  is superharmonic in  $\Omega$ , from Theorem 1.1 we conclude that  $\psi > 0$  in  $\Omega$ . Consequently,  $|\text{grad } \psi|^2$  takes its maximum on  $\partial\Omega$ . Motivated by Miranda's result, Payne [10] showed that for the torsion function  $\psi$ , the associated function

$$P_2 := |\text{grad } \psi|^2 + 4\psi$$

takes its maximum where  $|\text{grad } \psi| = 0$  provided  $\Omega$  is a convex domain. In particular,  $P_2$  takes its maximum in  $\Omega$  where  $\psi$  is a maximum. Payne used this information to obtain bounds for the maximum shearing stress, defined (up to a constant factor) as

$$\tau := \max_{\partial\Omega} |\text{grad } \psi|,$$

and the torsional rigidity given by

$$T := \int_{\Omega} |\text{grad } \psi|^2 dx .$$

Several authors have extended this technique of Payne to functions defined on solutions of second order linear and nonlinear elliptic equations under Dirichlet or Robin type boundary conditions, e.g., Payne and Stakgold [13], [14]; Payne, Sperb, and Stakgold [12]; and Schaefer and Sperb [19], [20]. Also, Sperb [22] and Protter [15] have adapted this technique for more general uniformly elliptic operators. These authors give various applications in the case where the P-function attains a maximum value at a critical point of the solution to the differential equation. Alternatively, we shall consider P-functions which take their maximum on  $\partial\Omega$ .

Instead of the Laplacian, we consider more general elliptic differential operators. In the following, we denote the partial derivative  $\frac{\partial u}{\partial x_i}$  by  $u_{,i}$  and adopt the summation convention for repeated indices in a single term; e.g., for  $u(x_1, x_2, \dots, x_N)$  and  $v(x_1, x_2, \dots, x_N)$  we have

$$\text{grad } u \cdot \text{grad } v := u_{,i} v_{,i}$$

and

$$\Delta u := u_{,ii} .$$

Using this notation, we define for  $u \in C^2(\Omega)$  the elliptic operator

$L$  by

$$(1.1) \quad Lu := a_{ij}(x)u_{,ij} + b_i(x)u_{,i} ,$$

where without loss of generality we assume  $a_{ij}(x) = a_{ji}(x)$ . The operator  $L$  is said to be uniformly elliptic if there exists a positive constant  $\mu$  such that

$$a_{ij}(x)\xi_i\xi_j \geq \mu|\xi|^2, \quad x \in \Omega \subset \mathbb{R}^N ,$$

for any  $N$ -vector  $\xi$ . A proof of the following maximum principle, due to E. Hopf, can be found in Protter and Weinberger [17].

Theorem 1.4. Suppose  $u \in C^2(\Omega)$  satisfies the differential inequality

$$Lu := a_{ij}(x)u_{,ij} + b_i(x)u_{,i} \geq 0$$

in a domain  $\Omega$ , where  $L$  is uniformly elliptic. Suppose the coefficients  $a_{ij}$  and  $b_i$  are uniformly bounded. If  $u$  attains a maximum value at any point of  $\Omega$ , then  $u$  is constant.

As with the Laplacian, a corresponding minimum principle holds for solutions of  $Lu \leq 0$ .

Protter and Weinberger [16] used the Hopf maximum principle to give a simple extension of Theorem 1.2 to second order elliptic equations with variable coefficients. They showed that if  $u$  is a solution of the linear elliptic equation

$$Lu + c(x)u := a_{ij}(x)u_{,ij} + b_i(x)u_{,i} + c(x)u = 0 ,$$

where  $L$  is uniformly elliptic and  $c(x) \leq 0$ , then for  $\gamma > 0$  sufficiently large the function

$$P := |\text{grad } u|^2 + \gamma u^2,$$

and more generally, the function

$$P := [\alpha(x)]^2 |\text{grad } u|^2 + \gamma u^2,$$

does not attain its maximum in  $\Omega$  unless it is constant. In Chapter II we present a direct analogue for the function

$$(1.2) \quad P := [\alpha(x)]^2 |\text{grad } u|^2 + \gamma \int_0^u f(t) dt,$$

where  $u$  is now a solution of the related semilinear equation

$$Lu + c(x)f(u) := a_{ij}(x)u_{,ij} + b_i(x)u_{,i} + c(x)f(u) = 0,$$

for  $c(x) \leq 0$  and  $f'(u)$  bounded below by a positive constant. Moreover, we show this latter condition can be relaxed by making a suitable adjustment in the function  $P$  defined in (1.2). Also, we develop a maximum principle for a function associated with solutions of the corresponding inhomogeneous equation

$$Lu + c(x)f(u) := a_{ij}(x)u_{,ij} + b_i(x)u_{,i} + c(x)f(u) = q(x),$$

where  $c(x) \leq 0$ .

More recently, for the equation

$$(1.3) \quad Lu + f(u) := a_{ij}(x)u_{,ij} + b_i(x)u_{,i} + f(u) = 0$$

Lu and Yiang [6] established conditions under which the function

$$(1.4) \quad P := |\text{grad } u|^2 g(u) + \gamma u^2 - 2 \int_0^u f(t)g(t) dt$$

takes its maximum on  $\partial\Omega$ , where  $\gamma$  is a sufficiently large positive constant and  $g(u)$  is a sufficiently smooth positive function. In Chapter II we shall develop a corresponding maximum principle for the function

$$P := [\alpha(x)]^2 |\text{grad } u|^2 g(u) + \gamma u^2 - 2 \int_0^u f(t)g(t) dt$$

defined on solutions of the uniformly elliptic equation

$$Lu + c(x)f(u) := a_{ij}(x)u_{,ij} + b_i(x)u_{,i} + c(x)f(u) = 0$$

with  $c(x) \geq 0$ . Moreover, we show that if  $u$  is a solution of (1.3), then we can drop the integral term in (1.4) and still obtain a maximum principle for the resulting function.

There is an extensive literature available on maximum principles for second order elliptic equations and their applications to boundary value problems. However, less attention has been given to maximum principles for higher order elliptic equations. This can be partly attributed to the fact that Theorem 1.1 does not immediately

generalize to equations of the form  $\Delta^P u = 0$ , where  $\Delta^P u = \Delta(\Delta^{P-1} u)$ . For example,  $u(x,y) = 1 - x^2 - y^2$  on the unit disk  $\Omega = \{(x,y): x^2 + y^2 < 1\}$  satisfies  $\Delta^2 u = 0$  in  $\Omega$  and although  $u$  does take its minimum value on  $\partial\Omega = \{(x,y): x^2 + y^2 = 1\}$ ,  $u$  attains an interior maximum at the origin. Nevertheless, a number of results have been proven. For example, Schaefer [18] extended the maximum principle of Miranda for the biharmonic equation, Theorem 1.3, to the fourth order semilinear elliptic equation

$$\Delta^2 u + p(x,y)f(u) = 0 ,$$

where  $p(x,y) > 0$  and  $f$  satisfies the inequality

$$sf(s) \geq 0 .$$

Also, Payne [11] established various maximum principles for certain P-functions associated with solutions of the equation

$$\Delta^2 u = f(u)$$

in  $N \geq 2$  dimensions based on specific assumptions on the form of the function  $f$ ; e.g.,  $f(u) > 0$  or  $f'(u) < 0$ . In general, these P-functions depend on the dimension  $N$ .

D.R. Dunninger [2] obtained a maximum principle for the P-function

$$P := (\Delta u)^2 + cu^2 ,$$

where  $c$  is a positive constant and  $u$  is a solution of the fourth order elliptic equation

$$\Delta^2 u + cu = 0$$

in a bounded domain  $\Omega$  in  $\mathbb{R}^N$ . Moreover, if  $u$  satisfies the boundary condition  $\Delta u = 0$  on  $\partial\Omega$ , Dunninger was able to deduce a maximum principle for  $|u|$ . Chow and Dunninger [1] then developed a corresponding P-function for the equation

$$\Delta^2 u + c(x)u = 0 ,$$

where  $c(x) > 0$ , from which a pointwise estimate for  $|u|$  resulted. Recently, Wang and Wang [25] established maximum principles for some functions determined by solutions of the nonlinear fourth order elliptic equation

$$(1.5) \quad \Delta(a(x)g(u)\Delta u) + b(x)f(u) = 0 .$$

Lu and Wang [5] also considered the equation (1.5) although most of the results are given for  $g(u) \equiv 1$ . In Chapter III, we improve the result in [25] which greatly simplifies some of the theorems in [5]. Also, in Chapter III we establish a maximum principle for several P-functions associated with the system of second order elliptic equations

$$Lu + F_1(x, u, v)h_1(u)g_1(v) = 0 ,$$

(1.6)

$$Lv + F_2(x, u, v)h_2(u)g_2(v) = 0 ,$$

where  $L$  is the uniformly elliptic operator (1.1). Moreover, we show that the system (1.6) includes the system considered by Lu and Wang in [5].

Finally, in Chapter IV we consider the metaharmonic equation

$$\Delta^n u + a_{n-1} \Delta^{n-1} u + \dots + a_0 u = 0 ,$$

(1.7)

where  $a_0, \dots, a_{n-1}$  are real constants and  $\Delta^m := \Delta(\Delta^{m-1})$ ,  $m \geq 2$ . Under appropriate conditions on the coefficients  $a_0, \dots, a_{n-1}$  in (1.7), Chow and Dunninger [1] found that certain solutions to (1.7) in a bounded domain  $\Omega$  satisfy a pointwise estimate of the form

$$|u(x)| \leq k |u(x_0)| ,$$

(1.8)

where  $k$  is a positive constant and  $x_0 \in \partial\Omega$ . Dunninger [2] had earlier obtained this estimate with  $k = 1$  for the specific case  $n = 2$ . Also, Schaefer and Walter [21] extended the pointwise estimate (1.8) of Chow and Dunninger to a wider class of metaharmonic equations without explicit knowledge of the constant  $k$ . In Chapter IV, we give an estimate similar to (1.8) for the cases  $n = 2$  and  $n = 3$  which is sharper than the corresponding estimate of Chow and Dunninger. We conjecture that sharper estimates can be obtained for  $n \geq 4$  also.

## CHAPTER II

### SECOND ORDER EQUATIONS

We consider the uniformly elliptic equation

$$(2.1) \quad Lu + c(x)f(u) := a_{ij}(x)u_{,ij} + b_i(x)u_{,i} + c(x)f(u) = 0$$

in a bounded domain  $\Omega$  in  $\mathbb{R}^N$ ,  $N \geq 2$ , with coefficients  $a_{ij}, b_i, c \in C^1(\overline{\Omega})$  and where  $f \in C^1(\mathbb{R})$ . By uniformly elliptic, we mean the symmetric matrix  $(a_{ij})$  is positive definite and satisfies the uniform ellipticity condition

$$a_{ij}(x)\xi_i\xi_j \geq \eta|\xi|^2, \quad x \in \Omega, \quad \xi \in \mathbb{R}^N$$

for some positive constant  $\eta$ .

In [16], Protter and Weinberger showed that if  $u \in C^3(\Omega)$  is a solution of (2.1) with  $c(x) \leq 0$  and  $f(u) = u$ , then for  $\alpha(x) \in C^2(\overline{\Omega})$  there exists a positive constant  $\gamma$  such that the function

$$P(x) = [\alpha(x)]^2 |\text{grad } u|^2 + \gamma u^2$$

cannot attain its maximum in  $\Omega$  unless it is constant.

More generally, we have the following, where for convenience we set

$$(2.2) \quad F(u) = \int_0^u f(t) dt .$$

Theorem 2.1. Let  $u \in C^3(\Omega)$  be a solution of (2.1) where  $c(x) \leq 0$ . If  $f'(u) \geq K > 0$ ,  $K$  constant, then for  $\alpha(x) \in C^2(\bar{\Omega})$  there exists a positive constant  $\gamma$  such that the function

$$(2.3) \quad P(x) = [\alpha(x)]^2 |\text{grad } u|^2 + \gamma F(u)$$

cannot attain its maximum in  $\Omega$  unless it is constant.

Proof. By a straightforward computation, we have

$$\begin{aligned} P_{,i} &= 2\alpha^2 u_{,k} u_{,ik} + 2\alpha\alpha_{,i} u_{,k} u_{,k} + \gamma f(u) u_{,i} , \\ P_{,ij} &= 2\alpha^2 u_{,ik} u_{,jk} + 2\alpha^2 u_{,k} u_{,ijk} + 4\alpha\alpha_{,j} u_{,k} u_{,ik} \\ &\quad + 4\alpha\alpha_{,i} u_{,k} u_{,jk} + 2\alpha\alpha_{,ij} u_{,k} u_{,k} + 2\alpha_{,i} \alpha_{,j} u_{,k} u_{,k} \\ &\quad + \gamma f' u_{,i} u_{,j} + \gamma f u_{,ij} , \end{aligned}$$

where  $f'$  denotes differentiation with respect to  $u$ . Hence,

$$\begin{aligned} (2.4) \quad LP &:= a_{ij} P_{,ij} + b_i P_{,i} \\ &= 2\alpha^2 a_{ij} u_{,ik} u_{,jk} + 2\alpha^2 (a_{ij} u_{,ijk} + b_i u_{,ik}) u_{,k} \\ &\quad + 8\alpha\alpha_{,i} a_{ij} u_{,k} u_{,jk} + 2\alpha (a_{ij} \alpha_{,ij} + b_i \alpha_{,i}) u_{,k} u_{,k} \\ &\quad + 2\alpha_{,i} \alpha_{,j} a_{ij} u_{,k} u_{,k} + \gamma f' a_{ij} u_{,i} u_{,j} + \gamma f (a_{ij} u_{,ij} + b_i u_{,i}) . \end{aligned}$$

By differentiating (2.1) with respect to  $x_k$ , we get

$$(2.5) \quad a_{ij}^u{}_{,ijk} + b_i^u{}_{,ik} = -a_{ij,k}^u{}_{,ij} - b_{i,k}^u{}_{,i} - cf^u{}_{,k} - fc_{,k}.$$

We substitute (2.5) into (2.4) and use the differential equation (2.1) to obtain

$$(2.6) \quad \begin{aligned} LP = & 2\alpha^2 a_{ij}^u{}_{,ik}{}^u{}_{,jk} - 2\alpha^2 a_{ij,k}^u{}_{,k}{}^u{}_{,ij} + 8\alpha\alpha_{,i} a_{ij}^u{}_{,k}{}^u{}_{,jk} \\ & - 2\alpha^2 b_{i,k}^u{}_{,i}{}^u{}_{,k} - 2\alpha^2 cf^u{}_{,k}{}^u{}_{,k} - 2\alpha^2 fc_{,k}{}^u{}_{,k} \\ & + 2\alpha\alpha_{,ij} a_{ij}^u{}_{,k}{}^u{}_{,k} + 2\alpha\alpha_{,i} b_i^u{}_{,k}{}^u{}_{,k} + 2\alpha_{,i}\alpha_{,j} a_{ij}^u{}_{,k}{}^u{}_{,k} \\ & + \gamma f^u{}_{,i} a_{ij}^u{}_{,i}{}^u{}_{,j} - \gamma cf^2. \end{aligned}$$

In order to remove the term  $-2\alpha^2 fc_{,k}{}^u{}_{,k}$  from the right side of (2.6), we define the elliptic operator  $\hat{L}$  by

$$\hat{L}P := LP + 2\gamma^{-1}\alpha^2 c_{,i}^p{}_{,i},$$

where

$$\begin{aligned} 2\gamma^{-1}\alpha^2 c_{,i}^p{}_{,i} = & 4\gamma^{-1}\alpha^4 c_{,i}^u{}_{,k}{}^u{}_{,ik} + 4\gamma^{-1}\alpha^3 \alpha_{,i} c_{,i}^u{}_{,k}{}^u{}_{,k} \\ & + 2\alpha^2 fc_{,i}^u{}_{,i}. \end{aligned}$$

Hence from (2.6) we have

$$\begin{aligned}
(2.7) \quad \hat{L}P = & 2\alpha^2 a_{ij}^u, ik^u, jk - 2\alpha^2 a_{jk, i}^u, i^u, jk + 8\alpha\alpha, i a_{ij}^u, k^u, jk \\
& + 4\gamma^{-1}\alpha^4 c_{,j}^u, k^u, jk - 2\alpha^2 b_{i, k}^u, i^u, k - 2\alpha^2 c f^u, k^u, k \\
& + 4\gamma^{-1}\alpha^3 \alpha, i c_{,i}^u, k^u, k + 2\alpha\alpha, ij a_{ij}^u, k^u, k + 2\alpha\alpha, i b_{i, k}^u, k^u, k \\
& + 2\alpha, i \alpha, j a_{ij}^u, k^u, k + \gamma f^u a_{ij}^u, i^u, j - \gamma c f^2.
\end{aligned}$$

For  $x \in \Omega$ , if  $\alpha(x) = 0$  (2.7) reduces to

$$\hat{L}P = \gamma f^u a_{ij}^u, i^u, j - \gamma c f^2 + 2a_{ij} \alpha, i \alpha, j |\text{grad } u|^2$$

which is clearly nonnegative. If  $\alpha(x) \neq 0$ , the first four terms on the right side of (2.7) can be combined by using an inequality from [16]. Let  $(A_{ij})$  be the inverse of the positive definite matrix  $(a_{ij})$ . Since

$$a_{ij}(u, ik + \frac{1}{2} A_{ip} s_{pk})(u, jk + \frac{1}{2} A_{jq} s_{qk}) \geq 0$$

for any  $N \times N$  matrix  $(s_{ij})$ , we have

$$(2.8) \quad s_{jk}^u, jk \geq -a_{ij}^u, ik^u, jk - \frac{1}{4} A_{pq} s_{pk} s_{qk}.$$

By choosing

$$s_{jk} = -a_{jk, i}^u, i + 4\alpha^{-1} \alpha, i a_{ij}^u, k + 2\gamma^{-1} \alpha^2 c_{,j}^u, k$$

we get from (2.8)

$$(2.9) \quad 2\alpha^2 a_{ij}^u,_{ik}^u,_{jk} + 2\alpha^2 (-a_{jk,i}^u,_{i} + 4\alpha^{-1} a_{\alpha,i} a_{ij}^u,_{k} + 2\gamma^{-1} \alpha^2 c_{,j}^u,_{k})^u,_{jk} \geq -\frac{1}{2} \alpha^2 A_{pq} s_{pk} s_{qk}.$$

Using (2.9) we have

$$\begin{aligned} \hat{L}P \geq & -\frac{1}{2} \alpha^2 A_{pq} (-a_{pk,m}^u,_{m} + 4\alpha^{-1} a_{\alpha,m} a_{mp}^u,_{k} + 2\gamma^{-1} \alpha^2 c_{,p}^u,_{k}) \\ & \times (-a_{qk,n}^u,_{n} + 4\alpha^{-1} a_{\alpha,n} a_{nq}^u,_{k} + 2\gamma^{-1} \alpha^2 c_{,q}^u,_{k}) \\ & + [\gamma f' a_{ij} + (2\alpha a_{,pq} + 2\alpha_{,p} a_{,q}) a_{pq} \delta_{ij} + 2\alpha a_{,p} b_p \delta_{ij} \\ & - 2\alpha^2 b_{i,j} - 2\alpha^2 c f' \delta_{ij} + 4\gamma^{-1} \alpha^3 a_{,p} c_{,p} \delta_{ij}] u_{,i}^u,_{j} - \gamma c f^2 \end{aligned}$$

which after simplification results in

$$\begin{aligned} (2.10) \quad \hat{L}P = & \{\gamma f' a_{ij} + [2(\alpha a_{,pq} - 3\alpha_{,p} a_{,q}) a_{pq} + 2\alpha a_{,p} b_p - 2\alpha^2 c f' \\ & - 4\gamma^{-1} \alpha^3 a_{,p} c_{,p} - 2\gamma^{-2} \alpha^6 A_{pq} c_{,p} c_{,q}] \delta_{ij} - \frac{1}{2} \alpha^2 A_{pq} a_{pk,i} a_{qk,j} \\ & - 2\alpha^2 b_{i,j} + 2\alpha(2\alpha_{,q} + \gamma^{-1} \alpha^3 A_{pq} c_{,p}) a_{qi,j}\} u_{,i}^u,_{j} - \gamma c f^2 \end{aligned}$$

where  $\delta_{ij}$  denotes the Kronecker delta. Now, for  $\gamma$  sufficiently large the right side of (2.10) is nonnegative and the theorem follows directly from the maximum principle. Moreover, the size of  $\gamma$  depends only on  $\alpha$ ,  $f$ , and the coefficients in (2.1). ■

The preceding theorem can be used to obtain interior estimates for  $|\text{grad } u|$ . For example, suppose that  $f$  is an odd function which satisfies  $f'(u) \geq K > 0$ . If  $|u| \leq m$  and  $|\text{grad } u| \leq M$  on

$\partial\Omega$  , then by Theorem 2.1,

$$(2.11) \quad [\alpha(x)]^2 |\text{grad } u|^2 \leq \alpha_1^2 M^2 + \gamma \int_0^m f(t) dt$$

for  $x \in \Omega$  , where  $\alpha_1 = \max_{\partial\Omega} |\alpha|$ . Moreover, if  $\alpha(x)$  vanishes on  $\partial\Omega$  , then for  $x_0 \in \Omega$  , (2.11) becomes

$$[\alpha(x_0)]^2 |(\text{grad } u)(x_0)|^2 \leq \gamma \int_0^m f(t) dt .$$

Finally, by normalizing  $\alpha$  such that  $\alpha(x_0) = 1$  we obtain

$$|(\text{grad } u)(x_0)| \leq \sqrt{\gamma} \left( \int_0^m f(t) dt \right)^{1/2} .$$

In Theorem 2.1, the positivity condition on  $f'$  in  $\Omega$  is somewhat restrictive. We can formulate a weaker hypothesis if we consider the function

$$(2.12) \quad P(x) = |\text{grad } u|^2 + \gamma F(\phi(u))$$

instead of (2.3), where we take  $\alpha(x) = 1$  for simplicity and where  $\phi(u)$  is a function to be determined. For convenience, we let  $f = f(u) = f(\phi(u))$  and use a prime to indicate differentiation with respect to argument. We then have the following.

Theorem 2.2. Let  $u \in C^3(\Omega)$  be a solution of (2.1) with  $c(x) \leq 0$  and  $f'$  bounded below. Suppose there exists a function  $\phi(u) \in C^2(\mathbb{R})$  such that for some positive constant  $\beta$  the inequality

$$(2.13) \quad f'(\phi')^2 + f\phi'' \geq \beta$$

holds. Assume further that  $|f\phi'| > 0$ ,  $f\phi' \geq 0$ , and that  $|f(f\phi')^{-1}|$  is bounded. Then for  $\gamma > 0$  sufficiently large, the function  $P(x)$  given by (2.12) cannot attain its maximum in  $\Omega$  unless it is constant.

Proof. We first note that nontrivial  $f$  and  $\phi$  which satisfy the conditions in Theorem 2.2 for a solution which is bounded below are  $f(u) = \arctan u^2$  and  $\phi(u) = e^u$ . Now, proceeding as before,

$$P_{,i} = 2u_{,k}^u u_{,ik} + \gamma f\phi' u_{,i} ,$$

$$P_{,ij} = 2u_{,ik}^u u_{,jk} + 2u_{,k}^u u_{,ijk} + \gamma[f'(\phi')^2 + f\phi''] u_{,i} u_{,j} \\ + \gamma f\phi' u_{,ij} .$$

Then,

$$(2.14) \quad LP := a_{ij}P_{,ij} + b_i P_{,i} = 2a_{ij}u_{,ik}^u u_{,jk} + 2(a_{ij}u_{,ijk} + b_i u_{,ik}^u) u_{,k} \\ + \gamma[f'(\phi')^2 + f\phi''] a_{ij} u_{,i} u_{,j} + \gamma f\phi' (a_{ij} u_{,ij} + b_i u_{,i}) .$$

We again use (2.5) and the differential equation (2.1) in (2.14) to obtain

$$(2.15) \quad LP = 2a_{ij}u_{,ik}^u u_{,jk} - 2a_{ij,k}^u u_{,k}^u u_{,ij} - 2b_{i,k}^u u_{,i}^u u_{,k} - 2cf' u_{,k}^u u_{,k} \\ - 2fc_{,k}^u u_{,k} + \gamma[f'(\phi')^2 + f\phi''] a_{ij} u_{,i} u_{,j} - \gamma cf f\phi' .$$

In order to eliminate the undesirable term  $-2fc_{,k}^u{}_{,k}$  from the right side of (2.15), we consider the elliptic operator  $\hat{L}$  defined by

$$\hat{L}P := LP + 2f(\gamma f\phi')^{-1}c_{,i}^P{}_{,i}.$$

Since

$$2f(\gamma f\phi')^{-1}c_{,i}^P{}_{,i} = 4f(\gamma f\phi')^{-1}c_{,i}^u{}_{,k}{}^u{}_{,ik} + 2fc_{,i}^u{}_{,i},$$

we have from (2.15)

$$\begin{aligned} (2.16) \quad \hat{L}P &= 2a_{ij}^u{}_{,ik}{}^u{}_{,jk} - 2a_{jk,i}^u{}_{,i}{}^u{}_{,jk} + 4f(\gamma f\phi')^{-1}c_{,j}^u{}_{,k}{}^u{}_{,jk} \\ &\quad - 2b_{i,k}^u{}_{,i}{}^u{}_{,k} - 2cf'^u{}_{,k}{}^u{}_{,k} + \gamma[f'(\phi')^2 + f\phi'']a_{ij}^u{}_{,i}{}^u{}_{,j} \\ &\quad - \gamma cff\phi'. \end{aligned}$$

We combine the first three terms on the right side of (2.16) by using the inequality (2.8) where

$$s_{jk} = -a_{jk,i}^u{}_{,i} + 2f(\gamma f\phi')^{-1}c_{,j}^u{}_{,k}.$$

Thus,

$$\begin{aligned} \hat{L}P &\geq -\frac{1}{2}A_{pq}[-a_{pk,i}^u{}_{,i} + 2f(\gamma f\phi')^{-1}c_{,p}^u{}_{,k}] \\ &\quad \times [-a_{qk,j}^u{}_{,j} + 2f(\gamma f\phi')^{-1}c_{,q}^u{}_{,k}] + \gamma[f'(\phi')^2 + f\phi'']a_{ij}^u{}_{,i}{}^u{}_{,j} \end{aligned}$$

$$- 2b_{i,j}u_{,i}u_{,j} - 2cf'u_{,k}u_{,k} - \gamma cff\phi'$$

which after simplification results in

$$(2.17) \quad \hat{L}P \geq \{\gamma[f'(\phi')^2 + f\phi'']a_{ij} - 2cf'\delta_{ij} - 2b_{i,j} - \frac{1}{2}(A_{pq}a_{pk,i}a_{qk,j} \\ - 4f(\gamma f\phi')^{-1}A_{pq}c_{,p}a_{qi,j} + 4f^2(\gamma f\phi')^{-2}A_{pq}c_{,p}c_{,q}\delta_{ij})\}u_{,i}u_{,j} \\ - \gamma cff\phi' .$$

Because of (2.13), the right side of (2.17) can be made nonnegative by choosing  $\gamma$  large enough. Hence, the theorem follows from the maximum principle. ■

We now present an analogue of Theorem 2.1 for an inhomogeneous uniformly elliptic equation, where for simplicity we let  $\alpha(x) \equiv 1$  in (2.3). For  $a_{ij}, b_i, c \in C^1(\overline{\Omega})$  and  $f \in C^1(\mathbb{R})$ , let  $u$  be a solution of the equation

$$(2.18) \quad Lu + c(x)f(u) := a_{ij}(x)u_{,ij} + b_i(x)u_{,i} + c(x)f(u) = q(x) ,$$

where  $q(x) \in C^1(\overline{\Omega})$ . In addition, let  $\psi(x)$  be any nonnegative function which satisfies in  $\Omega$  the differential inequality

$$(2.19) \quad a_{ij}\psi_{,ij} + b_i\psi_{,i} \leq -1 .$$

Then, we have the following theorem.

Theorem 2.3. Let  $u \in C^3(\Omega)$  be a solution of (2.18) with  $c(x) \leq c_0 < 0$ ,  $c_0$  constant, and suppose  $\psi \in C^2(\Omega) \cap C^1(\bar{\Omega})$  is a nonnegative solution of the differential inequality (2.19). If  $f'(u) \geq K_1 > 0$  and  $F(u) \geq K_2$ ,  $K_1$  and  $K_2$  constants, where  $F(u)$  is given by (2.2), then there exists a positive constant  $\gamma$  such that the function

$$(2.20) \quad P(x) = |\text{grad } u|^2 + \gamma F(u) - [F(u) + \beta] \psi$$

with

$$\beta = -K_2 + \max_{\bar{\Omega}} [|\text{grad } q|^2 + \frac{3}{2} \gamma^2 q^2]$$

cannot attain its maximum in  $\Omega$  unless it is constant.

Proof. From (2.20) we find

$$P_{,i} = 2u_{,k}u_{,ik} + \gamma f(u)u_{,i} - [F(u) + \beta]\gamma_{,i} - f(u)\psi_{u,i},$$

$$\begin{aligned} P_{,ij} = & 2u_{,ik}u_{,jk} + 2u_{,k}u_{,ijk} + \gamma f_{,i}u_{,j} + \gamma f'u_{,i}u_{,j} \\ & - [F(u) + \beta]\psi_{,ij} - f\psi_{,i}u_{,j} - fu_{,i}\psi_{,j} - f\psi_{u,i}u_{,j} - f'\psi_{u,i}u_{,j} \end{aligned}$$

and thus

$$\begin{aligned} (2.21) \quad LP := & a_{ij}P_{,ij} + b_iP_{,i} = 2a_{ij}u_{,ik}u_{,jk} + 2(a_{ij}u_{,ijk} + b_iu_{,ik})u_{,k} \\ & + \gamma f(a_{ij}u_{,ij} + b_iu_{,i}) + \gamma f'a_{ij}u_{,i}u_{,j} - f'\psi a_{ij}u_{,i}u_{,j} \\ & - 2fa_{ij}u_{,i}\psi_{,j} - f\psi(a_{ij}u_{,ij} + b_iu_{,i}) \end{aligned}$$

$$- [F(u) + \beta](a_{ij}\psi_{,ij} + b_{i\psi,i}) .$$

Differentiating (2.18) with respect to  $x_k$ , we obtain

$$(2.22) \quad a_{ij}^u{}_{,ijk} + b_{i,u}{}_{,ik} = -a_{ij,k}^u{}_{,ij} - b_{i,k}^u{}_{,i} - cf'_{u,k} - fc_{,k} + q_{,k} .$$

We substitute (2.22) into (2.21) and use the differential equation (2.18) to get

$$(2.23) \quad \begin{aligned} LP = & 2a_{ij}^u{}_{,ik}^u{}_{,jk} - 2a_{ij,k}^u{}_{,k}^u{}_{,ij} - 2b_{i,k}^u{}_{,k}^u{}_{,i} - 2cf'_{u,k}^u{}_{,k} \\ & - 2fc_{,k}^u{}_{,k} + 2q_{,k}^u{}_{,k} + (\gamma - \psi)qf - (\gamma - \psi)cf^2 - 2fa_{ij}^u{}_{,i\psi,j} \\ & + f'(\gamma - \psi)a_{ij}^u{}_{,i}^u{}_{,j} - [F(u) + \beta](a_{ij}\psi_{,ij} + b_{i\psi,i}) . \end{aligned}$$

We employ the elementary inequalities

$$-2fc_{,k}^u{}_{,k} \geq -\frac{1}{6}f^2 - 6(c_{,k}^u{}_{,k})^2 ,$$

$$2q_{,k}^u{}_{,k} \geq -|\text{grad } q|^2 - u_{,k}^u{}_{,k} ,$$

$$(\gamma - \psi)qf \geq -\frac{1}{6}f^2 - \frac{3}{2}(\gamma - \psi)^2 q^2 ,$$

$$-2fa_{ij}^u{}_{,i\psi,j} \geq -\frac{1}{6}f^2 - 6(a_{ij}^u{}_{,i\psi,j})^2 ,$$

and inequality (2.8) with  $s_{jk} = -a_{jk,i}^u{}_{,i}$ . Also, we use (2.19) to obtain from (2.23)

$$\begin{aligned}
(2.24) \quad LP \geq & [f'(\gamma - \psi)a_{ij} - \frac{1}{2}A_{pq}a_{pk,i}a_{qk,j} - 2b_{i,j} - (2cf' + 1)\delta_{ij} \\
& - 6c_{,i}c_{,j} - 6a_{im}a_{jn}\psi_{,m}\psi_{,n}]u_{,i}u_{,j} - [(\gamma - \psi)c + \frac{1}{2}]f^2 \\
& + F(u) + \beta - |\text{grad } q|^2 - \frac{3}{2}(\gamma - \psi)^2q^2.
\end{aligned}$$

Since  $c(x) \leq c_0 < 0$ , we initially take  $\gamma = \gamma_0 > \max_{\bar{\Omega}} \psi(x)$  such that  $(\gamma_0 - \psi)c + \frac{1}{2} \leq 0$  in  $\Omega$ . Finally, since  $f' \geq K_1 > 0$  and by the choice of  $\beta$ , the right side of (2.24) is nonnegative for  $\gamma \geq \gamma_0$  sufficiently large. The theorem follows directly from the maximum principle. ■

We note that when  $f(u) = u$  the restriction  $c(x) \leq c_0 < 0$  can be relaxed; i.e., Theorem 2.3 is valid for  $c(x) \leq 0$ . This follows easily since for  $f(u) = u$ , we have  $F(u) = \frac{1}{2}u^2$  so that (2.24) reduces to

$$\begin{aligned}
LP \geq & [(\gamma - \psi)a_{ij} - \frac{1}{2}A_{pq}a_{pk,i}a_{qk,j} - 2b_{i,j} - (2c + 1)\delta_{ij} \\
& - 6c_{,i}c_{,j} - 6a_{im}a_{jn}\psi_{,m}\psi_{,n}]u_{,i}u_{,j} - (\gamma - \psi)cu^2 + \beta \\
& - |\text{grad } q|^2 - \frac{3}{2}(\gamma - \psi)^2q^2.
\end{aligned}$$

We now investigate a class of P-functions defined on solutions of (2.1) where  $c(x) \geq 0$ . In particular, we consider the function

$$(2.25) \quad P(x) = [\alpha(x)]^2 |\text{grad } u|^2 g(u) + \gamma u^2 - 2H(u),$$

where  $\gamma > 0$  is constant,  $\alpha \in C^2(\bar{\Omega})$ ,  $g \in C^2(\mathbb{R})$  with  $g(u) > 0$ ,

and where

$$(2.26) \quad H(u) = \int_0^u h(t) dt, \quad h(t) = f(t)g(t).$$

We also assume  $f$  and  $g$  and their first derivatives are bounded.

We differentiate the function  $P$  in (2.25) and obtain

$$\begin{aligned} P_{,i} &= 2\alpha^2 g u_{,k}^u{}_{,ik} + \alpha^2 g' u_{,i}^u{}_{,k}^u{}_{,k} + 2\alpha\alpha_{,i} g u_{,k}^u{}_{,k} \\ &\quad + 2\gamma u u_{,i} - 2h(u)u_{,i}, \\ P_{,ij} &= 2\alpha^2 g u_{,ik}^u{}_{,jk} + 2\alpha^2 g u_{,k}^u{}_{,ijk} + 2\alpha^2 g' u_{,k}^u{}_{,ik}^u{}_{,j} \\ &\quad + 4\alpha\alpha_{,j} g u_{,k}^u{}_{,ik} + 2\alpha^2 g' u_{,i}^u{}_{,k}^u{}_{,jk} + \alpha^2 g' u_{,ij}^u{}_{,k}^u{}_{,k} \\ &\quad + \alpha^2 g'' u_{,i}^u{}_{,j}^u{}_{,k}^u{}_{,k} + 2\alpha\alpha_{,j} g' u_{,i}^u{}_{,k}^u{}_{,k} + 4\alpha\alpha_{,i} g u_{,k}^u{}_{,jk} \\ &\quad + 2\alpha\alpha_{,i} g' u_{,j}^u{}_{,k}^u{}_{,k} + 2\alpha\alpha_{,ij} g u_{,k}^u{}_{,k} + 2\alpha_{,i}\alpha_{,j} g u_{,k}^u{}_{,k} \\ &\quad + 2\gamma u u_{,ij} + 2\gamma u_{,i}^u{}_{,j} - 2h u_{,ij} - 2h' u_{,i}^u{}_{,j}. \end{aligned}$$

Then, we have

$$\begin{aligned} (2.27) \quad LP &:= a_{ij} P_{,ij} + b_i P_{,i} = 2\alpha^2 g a_{ij} u_{,ik}^u{}_{,jk} \\ &\quad + 2\alpha^2 g (a_{ij} u_{,ijk} + b_i u_{,ik}) u_{,k} + 4\alpha^2 g' a_{ij} u_{,k}^u{}_{,ik}^u{}_{,j} \\ &\quad + 8\alpha\alpha_{,j} g a_{ij} u_{,k}^u{}_{,ik} + \alpha^2 g' (a_{ij} u_{,ij} + b_i u_{,i}) u_{,k}^u{}_{,k} \\ &\quad + \alpha^2 g'' a_{ij} u_{,i}^u{}_{,j}^u{}_{,k}^u{}_{,k} + 4\alpha\alpha_{,j} g' a_{ij} u_{,i}^u{}_{,k}^u{}_{,k} \\ &\quad + 2\alpha g (a_{ij} \alpha_{,ij} + b_i \alpha_{,i}) u_{,k}^u{}_{,k} + 2g a_{ij} \alpha_{,i} \alpha_{,j} u_{,k}^u{}_{,k} \end{aligned}$$

$$\begin{aligned}
& + 2\gamma u(a_{ij}^u,_{ij} + b_i^u,_{i}) + 2\gamma a_{ij}^u,_{i} u,_{j} - 2h'a_{ij}^u,_{i} u,_{j} \\
& - 2h(a_{ij}^u,_{ij} + b_i^u,_{i}) .
\end{aligned}$$

To eliminate the undesirable third and seventh terms from the right side of (2.27), we consider the elliptic operator  $\hat{L}$  defined by

$$\hat{L}P := LP - 2g'g^{-1}a_{ij}^u,_{j} P,_{i} ,$$

where

$$\begin{aligned}
2g'g^{-1}a_{ij}^u,_{j} P,_{i} &= 4\alpha^2 g'a_{ij}^u,_{k} u,_{ik} u,_{j} + 2\alpha^2 (g')^2 g^{-1} a_{ij}^u,_{i} u,_{j} u,_{k} u,_{k} \\
&+ 4\alpha_{,i} g'a_{ij}^u,_{j} u,_{k} u,_{k} + 4\gamma g^{-1} u g'a_{ij}^u,_{i} u,_{j} - 4fg'a_{ij}^u,_{i} u,_{j} .
\end{aligned}$$

Moreover, if  $u \in C^3(\Omega)$ , we use (2.5), (2.1), and (2.27) to get

$$\begin{aligned}
(2.28) \quad \hat{L}P &= 2\alpha^2 g a_{ij}^u,_{ik} u,_{jk} - 2\alpha^2 g a_{ij}^u,_{k} u,_{k} u,_{ij} + 8\alpha_{,j} g a_{ij}^u,_{k} u,_{ik} \\
&- 2\alpha^2 g b_i^u,_{k} u,_{k} u,_{i} - 2\alpha^2 c f' g u,_{k} u,_{k} - 2\alpha^2 f g c,_{k} u,_{k} \\
&- \alpha^2 c f g' u,_{k} u,_{k} + \alpha^2 [g'' - 2(g')^2 g^{-1}] a_{ij}^u,_{i} u,_{j} u,_{k} u,_{k} \\
&+ 2\alpha g (a_{ij}^{\alpha},_{ij} + b_i^{\alpha},_{i}) u,_{k} u,_{k} + 2g a_{ij}^{\alpha},_{i} u,_{j} u,_{k} u,_{k} - 2\gamma c u f \\
&+ 2\gamma a_{ij}^u,_{i} u,_{j} + 2(fg' - f'g) a_{ij}^u,_{i} u,_{j} + 2cf^2 g \\
&- 4\gamma g^{-1} u g'a_{ij}^u,_{i} u,_{j} .
\end{aligned}$$

If  $\alpha(x) = 0$  at some  $x \in \Omega$ , (2.28) reduces to

$$(2.29) \quad \hat{L}P = 2ga_{ij}^{\alpha, i, j, u, k, u, k} - 2\gamma cuf + 2(\gamma + fg' - f'g)a_{ij}^{u, i, u, j} \\ + 2cf^2g - 4\gamma g^{-1}ug'a_{ij}^{u, i, u, j}.$$

Since  $f$  and  $g$  and their first derivatives are bounded, the right side of (2.29) is nonnegative for  $\gamma > 0$  sufficiently large if, in addition to the sign conditions  $c(x) \geq 0$  and  $g(u) > 0$ , we require  $uf(u) \leq 0$  and  $ug'(u) \leq 0$ .

We now assume  $\alpha(x) \neq 0$ . Here, we use inequality (2.8) with

$$s_{jk} = -a_{jk, i, i}^u + 4\alpha^{-1}\alpha_{, i}a_{ij}^{u, k}$$

to combine the first three terms on the right side of (2.28). Also, we combine the sixth and fourteenth terms in (2.28) by

$$(2.30) \quad 2cf^2g - 2\alpha^2fgc_{, k}^{u, k} = cf^2g + c^{-1}g[c^2f^2 - 2\alpha^2cfc_{, k}^{u, k} \\ + \alpha^4(c_{, k}^{u, k})^2] - \alpha^4c^{-1}g(c_{, k}^{u, k})^2 \\ \geq cf^2g + c^{-1}g(cf - \alpha^2c_{, k}^{u, k})^2 \\ - \alpha^4c^{-1}g|\text{grad } c|^2|\text{grad } u|^2,$$

where we assume  $c(x)$  is strictly positive. Thus, from (2.28) we obtain

$$\hat{L}P \geq -\frac{1}{2}\alpha^2gA_{pq}(-a_{pk, \ell}^{u, \ell} + 4\alpha^{-1}\alpha_{, \ell}a_{\ell p}^{u, k}) \\ \times (-a_{qk, m}^{u, m} + 4\alpha^{-1}\alpha_{, m}a_{mq}^{u, k}) - 2\alpha^2gb_{i, k}^{u, k}u_{, i}$$

$$\begin{aligned}
& - 2\alpha^2 c f' g u_{,k} u_{,k} - \alpha^2 c f g' u_{,k} u_{,k} + \alpha^2 [g'' - 2(g')^2 g^{-1}] \\
& \times a_{ij} u_{,i} u_{,j} u_{,k} u_{,k} + 2\alpha g (a_{ij} \alpha_{,ij} + b_{i\alpha, i}) u_{,k} u_{,k} + 2g a_{ij} \alpha_{,i} \alpha_{,j} u_{,k} u_{,k} \\
& - 2\gamma c u f + 2(\gamma + f g' - f' g) a_{ij} u_{,i} u_{,j} - 4\gamma g^{-1} u g' a_{ij} u_{,i} u_{,j} + c f^2 g \\
& + c^{-1} g (c f - \alpha^2 c_{,k} u_{,k})^2 - \alpha^4 c^{-1} g |\text{grad } c|^2 |\text{grad } u|^2
\end{aligned}$$

which after simplification becomes

$$\begin{aligned}
(2.31) \quad \hat{L}P \geq & \{2(\gamma + f g' - f' g) a_{ij} + [2g a_{pq} (\alpha \alpha_{,pq} - 3\alpha_{,p} \alpha_{,q}) + 2\alpha g b_{p\alpha, p} \\
& - 2\alpha^2 c f' g - \alpha^2 c f g' - \alpha^4 c^{-1} g |\text{grad } c|^2] \delta_{ij} - \frac{1}{2} \alpha^2 g A_{pq} a_{pk, i} a_{qk, j} \\
& - 2\alpha^2 g b_{i, j} + 4\alpha g a_{qi, j} \alpha_{,q} \} u_{,i} u_{,j} \\
& - \alpha^2 g^{-2} (g^{-1})'' a_{ij} u_{,i} u_{,j} |\text{grad } u|^2 - 2\gamma c u f \\
& - 4\gamma g^{-1} u g' a_{ij} u_{,i} u_{,j} + c f^2 g + c^{-1} g (c f - \alpha^2 c_{,k} u_{,k})^2 .
\end{aligned}$$

The right side of (2.31) is nonnegative for  $\gamma > 0$  sufficiently large in the case  $\alpha(x) \neq 0$  if we impose the conditions

- (a)  $|\text{grad } c|^2 c^{-1}$  bounded
- (b)  $(g^{-1})'' \leq 0$
- (c)  $u f(u) \leq 0$
- (d)  $u g'(u) \leq 0$

in addition to the sign conditions and smoothness assumptions already given. Thus, we have  $\hat{L}P \geq 0$  in  $\Omega$ . By the maximum principle we have the following theorem.

Theorem 2.4. Let  $u \in C^3(\Omega)$  be a solution of (2.1), where  $c(x) > 0$  and the function  $f$  is such that  $uf(u) \leq 0$ . In addition, assume  $|\nabla c|^2 c^{-1}$  and  $f, f'$  are bounded in  $\Omega$ . Let  $\alpha(x) \in C^2(\overline{\Omega})$  and suppose  $g(u) \in C^2(\mathbb{R})$  is a positive function such that  $(g^{-1})'' \leq 0$  and  $ug'(u) \leq 0$  where  $g, g'$  are bounded in  $\Omega$ . Then, for  $\gamma > 0$  sufficiently large, the function

$$P(x) = [\alpha(x)]^2 |\text{grad } u|^2 g(u) + \gamma u^2 - 2H(u)$$

cannot attain its maximum in  $\Omega$  unless it is constant.

Again, we can use Theorem 2.4 to obtain interior estimates for  $|\text{grad } u|$ . For simplicity, assume  $f$  is an odd function which satisfies the hypotheses of Theorem 2.4. If we set  $g(u) \equiv 1$  in (2.25), then the function

$$P(x) = [\alpha(x)]^2 |\text{grad } u|^2 + \gamma u^2 - 2F(u)$$

takes its maximum on  $\partial\Omega$ . If  $|u| \leq m$  and  $|\text{grad } u| \leq M$  on  $\partial\Omega$ , then

$$[\alpha(x)]^2 |\text{grad } u|^2 \leq \alpha_1^2 M^2 + \gamma m^2 + 2 \int_0^m |f(t)| dt$$

for  $x \in \Omega$ , where  $\alpha_1 = \max_{\partial\Omega} |\alpha(x)|$ . Here we have used the fact that  $uf(u) \leq 0$  implies  $-F(u) \geq 0$ . Further, if  $\alpha(x) = 0$  on  $\partial\Omega$ , we have

$$[\alpha(x_0)]^2 |(\text{grad } u)(x_0)|^2 \leq \gamma m^2 + 2 \int_0^m |f(t)| dt$$

for  $x_0 \in \Omega$ . Thus,

$$|(\text{grad } u)(x_0)| \leq |\alpha(x_0)|^{-1} (\gamma m^2 + 2 \int_0^m |f(t)| dt)^{1/2}.$$

Now, suppose  $u$  is a solution of (2.1) where  $c(x) \geq c_0 > 0$ ; i.e.,  $c(x)$  is bounded below by a positive constant. Here, we can divide equation (2.1) by  $c(x)$  and consider the resulting uniformly elliptic equation

$$(2.32) \quad Lu + f(u) := a_{ij}(x)u_{,ij} + b_i(x)u_{,i} + f(u) = 0,$$

where  $a_{ij} = a_{ij}c^{-1}$ ,  $b_i = b_ic^{-1}$ . In [6], Lu and Yiang prove the analogue of Theorem 2.4 with  $\alpha(x) \equiv 1$  in (2.25) and  $u$  a solution of (2.32). However, in this case we can show that the simpler function

$$(2.33) \quad P(x) = |\text{grad } u|^2 g(u) + \gamma u^2$$

satisfies the maximum principle in  $\Omega$ . We prove the following theorem.

Theorem 2.5. Let  $u \in C^3(\Omega)$  be a solution of (2.32), where  $f, f'$  are bounded in  $\Omega$  and  $uf(u) \leq 0$ . Suppose  $g(u) \in C^2(\mathbb{R})$  is a positive function such that  $g$  and  $g'$  are bounded in  $\Omega$ . If  $(g^{-1})'' \leq 0$  and  $ug'(u) \leq 0$ , then for  $\gamma > 0$  sufficiently large, the function

$$P(x) = |\text{grad } u|^2 g(u) + \gamma u^2$$

cannot attain its maximum in  $\Omega$  unless it is constant.

Proof. As in the proof of Theorem 2.4, from (2.33) we compute

$$\begin{aligned}
 (2.34) \quad LP &:= a_{ij}^P{}_{,ij} + b_i^P{}_{,i} = 2ga_{ij}^u{}_{,ik}u{}_{,jk} + 2g(a_{ij}^u{}_{,ijk} + b_i^u{}_{,ik})u{}_{,k} \\
 &\quad + 4g'a_{ij}^u{}_{,k}u{}_{,ik}u{}_{,j} + g'(a_{ij}^u{}_{,ij} + b_i^u{}_{,i})u{}_{,k}u{}_{,k} \\
 &\quad + g''a_{ij}^u{}_{,i}u{}_{,j}u{}_{,k}u{}_{,k} + 2\gamma u(a_{ij}^u{}_{,ij} + b_i^u{}_{,i}) + 2\gamma a_{ij}^u{}_{,i}u{}_{,j} .
 \end{aligned}$$

We differentiate (2.32) with respect to  $x_k$  to obtain

$$(2.35) \quad a_{ij}^u{}_{,ijk} + b_i^u{}_{,ik} = -a_{ij,k}^u{}_{,ij} - b_{i,k}^u{}_{,i} - f'u{}_{,k} .$$

Thus, we use (2.35) and the differential equation (2.32) to write

(2.34) as

$$\begin{aligned}
 (2.36) \quad LP &= 2ga_{ij}^u{}_{,ik}u{}_{,jk} - 2ga_{ij,k}^u{}_{,k}u{}_{,ij} - 2gb_{i,k}^u{}_{,k}u{}_{,i} - 2f'gu{}_{,k}u{}_{,k} \\
 &\quad + 4g'a_{ij}^u{}_{,k}u{}_{,ik}u{}_{,j} - fg'u{}_{,k}u{}_{,k} + g''a_{ij}^u{}_{,i}u{}_{,j}u{}_{,k}u{}_{,k} - 2\gamma uf \\
 &\quad + 2\gamma a_{ij}^u{}_{,i}u{}_{,j} .
 \end{aligned}$$

Now, proceeding as in the proof of Theorem 2.4, consider the elliptic operator  $\hat{L}$  defined by

$$\hat{L}P := LP - 2g'g^{-1}a_{ij}^u{}_{,j}P{}_{,i} .$$

Then, from (2.36) we have

$$\begin{aligned}
(2.37) \quad \hat{L}P = & 2ga_{ij}^u,_{ik}^u,_{jk} - 2ga_{jk,i}^u,_{i}^u,_{jk} - 2gb_{i,k}^u,_{k}^u,_{i} \\
& - 2f'gu,_{k}^u,_{k} - fg'u,_{k}^u,_{k} + [g'' - 2(g')^2g^{-1}]a_{ij}^u,_{i}^u,_{j}^u,_{k}^u,_{k} \\
& - 2\gamma uf + 2\gamma a_{ij}^u,_{i}^u,_{j} - 4\gamma g^{-1}ug'a_{ij}^u,_{i}^u,_{j}
\end{aligned}$$

since

$$P_{,i} = 2gu,_{ik}^u,_{k} + g'u,_{i}^u,_{k}^u,_{k} + 2\gamma uu,_{i}.$$

We next combine the first two terms on the right side of (2.37) by using inequality (2.8) for the coefficient matrix  $(a_{ij})$  and its inverse  $(A_{ij})$  with the choice  $s_{jk} = -a_{jk,i}^u,_{i}$ . This gives the differential inequality

$$\begin{aligned}
(2.38) \quad \hat{L}P \geq & \{2\gamma a_{ij} - 2gb_{i,j} - (2f'g + fg')\delta_{ij} \\
& - \frac{1}{2}gA_{pq}a_{pk,i}a_{qk,j}\}u,_{i}^u,_{j} - g^{-2}(g^{-1})''a_{ij}^u,_{i}^u,_{j}|\text{grad } u|^2 \\
& - 2\gamma uf - 4\gamma g^{-1}ug'a_{ij}^u,_{i}^u,_{j}.
\end{aligned}$$

The theorem follows from the maximum principle since the right side of (2.38) is nonnegative for  $\gamma > 0$  sufficiently large. ■

We note also that Lu and Yiang [6] claim that if  $f$  is an odd function such that  $uf(u) \leq 0$  with  $f'$  bounded above, then a solution  $u$  of (2.32) satisfies

$$(2.39) \quad \sup_{\Omega} |u| = \sup_{\partial\Omega} |u|.$$

In fact, the requirement that  $f$  be odd is superfluous as is the boundedness assumption on  $f'$ . Indeed, if  $u$  is a solution of (2.32) where  $uf(u) \leq 0$  we have

$$\begin{aligned} L(u^2) &= 2a_{ij}u_{,i}u_{,j} + 2u(a_{ij}u_{,ij} + b_i u_{,i}) \\ &= 2a_{ij}u_{,i}u_{,j} - 2uf(u) \geq 0. \end{aligned}$$

Therefore,  $u^2$  satisfies the maximum principle. From this, we obtain the conclusion (2.39), but we have not assumed that  $f$  is an odd function or that  $f'$  is bounded above. Of course, (2.39) also applies to solutions of (2.1) where  $c(x) \geq 0$  and  $uf(u) \leq 0$ .

We note that in [15] Protter obtained a slightly different maximum principle. Following the Payne technique introduced in [10], Protter showed that if  $u \in C^3(\Omega)$  is a solution of (2.32) and if  $f'(u) \geq K > 0$ ,  $K$  constant, then for  $\gamma > 0$  sufficiently large the function

$$G(x) = |\text{grad } u|^2 + \gamma F(u)$$

cannot attain its maximum at a point in  $\Omega$  where  $|\text{grad } u| \neq 0$  unless  $G$  is constant. We observe that  $G$  coincides with the function (2.3) of Theorem 2.1 where  $\alpha(x) \equiv 1$ . However, the conclusion here allows  $G$  to possibly take a maximum at a critical point of  $u$ ; i.e., where  $|\text{grad } u| = 0$ . We avoid this by using the function (2.33) and imposing different requirements on  $f$ .

We further note that Hile and Protter [4] have extended the work of Protter mentioned above to the equation

$$a_{ij}(x)u_{,ij} + b_i(x)u_{,i} + g(x,u) = 0$$

and to weakly coupled nonlinear elliptic systems of the form

$$a_{ij}(x)v_{k,ij} + b_i(x)v_{k,i} + h_k(x,v_0,\dots,v_m) = 0 \quad , k = 0,\dots,m .$$

Again, the P-functions used can attain a maximum in  $\Omega$  where  $|\text{grad } u| = 0$  ,  $|\text{grad } v_0| = 0$  in the case of elliptic systems.

Finally, in his text [23], Sperb has examined the equation (2.32) using differential geometry. Although similar in consequence to those of Protter [15], the principles presented there depend heavily on differential geometric quantities and the geometry of the domain. One may consult [23] for numerous applications of principles of this type.

## CHAPTER III

### FOURTH ORDER EQUATIONS AND ELLIPTIC SYSTEMS

We recall that Miranda [8] obtained a maximum principle for the function

$$P := |\text{grad } u|^2 - u\Delta u ,$$

where  $u$  is a solution of the biharmonic equation

$$\Delta^2 u = 0 \quad \text{in } \Omega \subset \mathbb{R}^2 ,$$

although we see the solution  $u$  need not satisfy a maximum principle. For example,  $u(x,y) = 1 - x^2 - y^2$  satisfies  $\Delta^2 u = 0$  in  $\Omega = \{(x,y): x^2 + y^2 < 1\}$  but takes its maximum at the origin rather than on  $\partial\Omega = \{(x,y): x^2 + y^2 = 1\}$ . However, in [2], Dunninger was able to obtain a maximum principle for solutions of the fourth order equation

$$(3.1) \quad \Delta^2 u + cu = 0 ,$$

where  $c$  is a positive constant. In particular, if  $u \in C^4(\Omega) \cap C^2(\overline{\Omega})$  is a nonconstant solution of (3.1) in a bounded domain  $\Omega$  in  $\mathbb{R}^N$ , Dunninger showed that the function

$$P(x) = c[u(x)]^2 + [\Delta u(x)]^2$$

cannot attain its maximum in  $\Omega$ . Consequently, if  $u$  is a nonconstant

solution of (3.1) such that  $\Delta u = 0$  on  $\partial\Omega$ , then  $|u|$  cannot attain its maximum in  $\Omega$ ; i.e., there exists a point  $x_0 \in \partial\Omega$  such that

$$|u(x)| < |u(x_0)|$$

for all  $x \in \Omega$ . Chow and Dunninger [1] obtained a corresponding pointwise estimate of the form

$$(3.2) \quad |u(x)| \leq k|u(x_0)|,$$

where  $k$  is a constant and  $x_0 \in \partial\Omega$ , for certain solutions of the equation

$$\Delta^2 u + c(x)u = 0,$$

where  $c(x) \in C^2(\Omega) \cap C^0(\overline{\Omega})$  is a positive function such that either  $c$  or  $1/c$  is superharmonic in  $\Omega$ . We note that if  $\Delta c \leq 0$ , then  $k = 1$  in (3.2) whereas if  $\Delta(\frac{1}{c}) \leq 0$ ,  $k$  is given by

$$k = [c(x_0)]^{1/2} [\min_{\overline{\Omega}} c(x)]^{-1/2}.$$

More recently, Lu and Wang [5] considered the equation

$$(3.3) \quad \Delta(a(x)g(u)\Delta u) + b(x)f(u) = 0.$$

If  $c(x) = a(x)b(x)$  is a positive function on  $\overline{\Omega}$  such that  $\Delta c \leq 0$  and if  $fg, (fg)' > 0$  on  $\mathbb{R}$ , they showed that a nonconstant solution

$u \in C^4(\Omega) \cap C^2(\overline{\Omega})$  of (3.3) for which  $\Delta u = 0$  on  $\partial\Omega$  cannot attain its maximum in  $\Omega$ . We note that under the conditions specified there are no constant solutions of (3.3). Moreover, if  $u \in C^4(\Omega) \cap C^2(\overline{\Omega})$  is a solution of (3.3) such that  $\Delta u = 0$  on  $\partial\Omega$ , then we can obtain the same conclusion as Lu and Wang without the assumptions that  $c$  is superharmonic and  $fg$  is strictly increasing. Indeed, suppose  $c(x) = a(x)b(x) > 0$  in  $\Omega$  and  $fg > 0$  on  $\mathbb{R}$ . We assume without loss of generality that  $a(x), b(x) > 0$  in  $\Omega$  and  $f(u), g(u) > 0$  on  $\mathbb{R}$ . If we set

$$v(x) = a(x)g(u)\Delta u,$$

then from (3.3)

$$\Delta v = -b(x)f(u) < 0 \quad \text{in } \Omega.$$

Also, since  $\Delta u = 0$  on  $\partial\Omega$ ,  $v(x) = 0$  on  $\partial\Omega$ . By the minimum principle for superharmonic functions,  $v > 0$  in  $\Omega$  and, therefore,  $\Delta u > 0$  in  $\Omega$ . By the maximum principle for subharmonic functions,  $u$  cannot attain its maximum in  $\Omega$ .

We now extend the result of Chow and Dunninger without the assumption of Lu and Wang that  $fg$  is positive. We have the following theorem.

Theorem 3.1. Suppose  $u \in C^4(\Omega) \cap C^2(\overline{\Omega})$  is a solution of the equation

$$\Delta(a(x)g(u)\Delta u) + b(x)f(u) = 0$$

in  $\Omega$ , where  $f(u(x))$  is not identically zero in  $\Omega$ . Define the function

$$(3.4) \quad P(x) = \frac{1}{2}[c(x)]^{-1}[v(x)]^2 + \int_0^{u(x)} h(t) dt$$

where

$$(3.5) \quad \begin{aligned} v(x) &= a(x)g(u(x))\Delta u(x), \\ c(x) &= a(x)b(x), \\ h(t) &= f(t)g(t). \end{aligned}$$

If  $h \in C^1(\mathbb{R})$ ,  $h'(t) \geq 0$  and if  $c(x) \in C^2(\Omega) \cap C^0(\bar{\Omega})$ ,  $c(x) > 0$  on  $\bar{\Omega}$  with  $\Delta c \leq 0$  in  $\Omega$ , then there exists a point  $x_0 \in \partial\Omega$  such that

$$P(x) < P(x_0), \quad x \in \Omega.$$

Proof. We first note that since  $f(u) \not\equiv 0$  and  $b(x) \not\equiv 0$  in  $\Omega$ ,  $\Delta v \not\equiv 0$  in  $\Omega$ . Moreover,  $v$  is not constant. Taking derivatives in (3.4), we find

$$\begin{aligned} P_{,i} &= (1/c)v v_{,i} - \frac{1}{2}v^2 c^{-2} c_{,i} + h(u)u_{,i}, \\ P_{,ii} &= (1/c)(v v_{,ii} + v_{,i} v_{,i}) - 2c^{-2} v v_{,i} c_{,i} - \frac{1}{2}c^{-2} v^2 c_{,ii} \\ &\quad + v^2 c^{-3} c_{,i} c_{,i} + h(u)u_{,ii} + h'(u)u_{,i} u_{,i}. \end{aligned}$$

Thus, rearranging terms we have

$$(3.6) \quad \Delta P = (1/c)v\Delta v + h(u)\Delta u + h'(u)u_{,i}u_{,i} - \frac{1}{2}c^{-2}v^2\Delta c \\ + (1/c)v_{,i}v_{,i} - 2c^{-2}vv_{,i}c_{,i} + c^{-3}v^2c_{,i}c_{,i} .$$

From (3.3) and (3.5),

$$(1/c)v\Delta v = [a(x)b(x)]^{-1}[a(x)g(u)\Delta u][-b(x)f(u)] = -h(u)\Delta u$$

so that the first two terms on the right side of (3.6) cancel. Also, an easy computation gives

$$(v/c)_{,i} = (1/c)v_{,i} - c^{-2}vc_{,i} , \\ |\text{grad}(v/c)|^2 = c^{-2}v_{,i}v_{,i} - 2c^{-3}vv_{,i}c_{,i} + c^{-4}v^2c_{,i}c_{,i} .$$

Therefore, (3.6) becomes

$$(3.7) \quad \Delta P = h'(u)|\text{grad } u|^2 + c|\text{grad}(v/c)|^2 - \frac{1}{2}c^{-2}v^2\Delta c$$

and since  $h'(u) \geq 0$ ,  $c(x) > 0$  with  $\Delta c \leq 0$ , the right side of (3.7) is nonnegative; i.e.,  $P$  is subharmonic.

We now show  $P$  is nonconstant. Indeed, if  $P$  is constant, then  $\Delta P \equiv 0$ . Necessarily, each term on the right side of (3.7) must be identically zero. Now, since  $c(x) > 0$ , the second term of (3.7) vanishes if and only if  $v/c$  is constant; i.e.,  $v(x) = kc(x)$

for some nonzero constant  $k$  . However, the third term of (3.7) becomes

$$-\frac{1}{2}c^{-2}(kc)^2\Delta c = -\frac{1}{2}k^2\Delta c ,$$

which vanishes if and only if  $\Delta c \equiv 0$  . In this case,  $\Delta v \equiv 0$  which contradicts the fact that  $\Delta v \not\equiv 0$  in  $\Omega$  . Since  $P$  is subharmonic and nonconstant,  $P$  cannot attain its maximum in  $\Omega$  . The theorem now follows from the continuity of  $P$  . ■

We remark that Theorem 3.1 is a direct consequence of a weak maximum principle for the function (3.4) given by Wang and Wang in [25]. As an immediate corollary of the preceding theorem, we can state a maximum principle for solutions of (3.3) when  $h(t)$  is an odd function or  $h(t) \geq 0$  .

Corollary 3.2. Suppose the hypotheses of Theorem 3.1 are satisfied and let  $u$  be a solution of (3.3) such that  $\Delta u = 0$  on  $\partial\Omega$  .

(a) If  $h(t)$  is odd, then there exists a point  $x_0 \in \partial\Omega$  such that for  $x \in \Omega$

$$|u(x)| < |u(x_0)| .$$

(b) If  $h(t) \geq 0$  , then there exists a point  $x_0 \in \partial\Omega$  such that for  $x \in \Omega$

$$u(x) < u(x_0) .$$

Proof. From Theorem 3.1, there exists  $x_0 \in \partial\Omega$  such that for  $x \in \Omega$

$$(3.8) \quad \frac{1}{2} [c(x)]^{-1} [v(x)]^2 + \int_0^{u(x)} h(t) dt < \frac{1}{2} [c(x_0)]^{-1} [v(x_0)]^2 + \int_0^{u(x_0)} h(t) dt .$$

Since  $\Delta u = 0$  on  $\partial\Omega$ ,  $v(x) = 0$  on  $\partial\Omega$  and since  $c(x) > 0$  on  $\overline{\Omega}$ , from (3.8) we get

$$(3.9) \quad \int_0^{u(x)} h(t) dt < \int_0^{u(x_0)} h(t) dt$$

for  $x \in \Omega$ . Now,  $h(t)$  is nondecreasing. Therefore, if  $h(t) \geq 0$ , from (3.9)

$$u(x) < u(x_0) .$$

If  $h(t)$  is odd and nondecreasing, then  $th(t) \geq 0$  so that

$$\int_0^s h(t) dt = \int_0^{-s} h(t) dt \geq 0$$

for each real number  $s$ . From (3.9), we conclude

$$|u(x)| < |u(x_0)| . \quad \blacksquare$$

We remark that (a) of Corollary 3.2 can be applied to a nontrivial solution of the equation

$$(3.10) \quad \Delta(a(x)\Delta u) + b(x)u^p = 0 ,$$

where  $p \geq 1$  is odd and  $c(x) = a(x)b(x)$  satisfies the hypotheses of Theorem 3.1, for which  $\Delta u = 0$  on  $\partial\Omega$ . Therefore, there exists a point  $x_0 \in \partial\Omega$  such that for  $x \in \Omega$

$$|u(x)| < |u(x_0)| .$$

Lu and Wang [5] deduced this same conclusion for the equation (3.10) but under much more stringent conditions on the coefficients  $a(x)$  and  $b(x)$ .

We now consider the system of elliptic equations

$$(3.11) \quad \begin{aligned} Lu + f_1(x, u, v)g(v) &= 0 , \\ Lv + f_2(x, u, v)h(u) &= 0 \end{aligned}$$

in a bounded domain  $\Omega$  in Euclidean  $N$ -space  $\mathbb{R}^N$  with boundary  $\partial\Omega$ , where  $L$  is the uniformly elliptic operator defined by

$$(3.12) \quad L := a_{ij}(x) \frac{\partial^2}{\partial x_i \partial x_j} + b_i(x) \frac{\partial}{\partial x_i}$$

such that

$$a_{ij} = a_{ji} , \quad a_{ij}(x)\xi_i\xi_j \geq \eta|\xi|^2 , \quad x \in \Omega , \quad \xi \in \mathbb{R}^N$$

for some positive constant  $\eta$ . If  $u, v \in C^2(\Omega)$  is a solution pair

of (3.11), then Lu and Wang [5] showed the following.

- (i) If  $g \in C^2(\Omega)$ ,  $h \in C^1(\Omega)$  and  $gh(f_1 + g'f_2) \leq 0$ ,  $h' \geq 0$ ,  $(g^2)'' \geq 0$ , then the function

$$\Phi_1(x) = 2H(u) + [g(v)]^2,$$

where  $H'(u) = h(u)$ , cannot attain its maximum in  $\Omega$  unless it is constant.

- (ii) If  $g, h \in C^1(\Omega)$  and  $gh(f_1 + f_2) \leq 0$ ,  $h' \geq 0$ ,  $g' \geq 0$ , then the function

$$\Phi_2(x) = H(u) + G(v),$$

where  $H'(u) = h(u)$ ,  $G'(v) = g(v)$ , cannot attain its maximum in  $\Omega$  unless it is constant.

- (iii) If  $g, h \in C^2(\Omega)$  and  $gh(h'f_1 + g'f_2) \leq 0$ ,  $(h^2)'' \geq 0$ ,  $(g^2)'' \geq 0$ , then the function

$$\Phi_3(x) = [h(u)]^2 + [g(v)]^2$$

cannot attain its maximum in  $\Omega$  unless it is constant.

Similarly, we can show that for a solution pair  $u, v \in C^2(\Omega)$  of (3.11) if  $g \in C^1(\Omega)$ ,  $h \in C^2(\Omega)$  and  $gh(h'f_1 + f_2) \leq 0$ ,  $g' \geq 0$ ,  $(h^2)'' \geq 0$ , then the function

$$(3.13) \quad \Phi_4(x) = [h(u)]^2 + 2G(v),$$

where  $G'(v) = g(v)$ , cannot attain its maximum in  $\Omega$  unless it is constant. This follows as an easy consequence of the next theorem.

Suppose we have the system of elliptic equations

$$(3.14) \quad \begin{aligned} Lu + F_1(x, u, v)h_1(u)g_1(v) &= 0, \\ Lv + F_2(x, u, v)h_2(u)g_2(v) &= 0 \end{aligned}$$

in a bounded domain  $\Omega$  in Euclidean  $N$ -space  $\mathbb{R}^N$  with boundary  $\partial\Omega$ , where  $L$  is again the uniformly elliptic operator (3.12). We note that the system (3.14) includes the system (3.11), where we make the identification  $h_1(u) = g_2(v) \equiv 1$ ,  $g_1 = g$ ,  $h_2 = h$ , and  $F_i = f_i$ ,  $i = 1, 2$ . We have the following theorem.

Theorem 3.3. Let  $u, v \in C^2(\Omega)$  be a solution pair of (3.14).

- (a) If  $g_1g_2 \in C^2(\Omega)$ ,  $h_1h_2 \in C^1(\Omega)$  and  $g_1h_2[h_1^2F_1 + g_2^2(g_1g_2)'F_2] \leq 0$ ,  $(h_1h_2)' \geq 0$ ,  $(g_1^2g_2^2)'' \geq 0$ , then the function

$$(3.15) \quad P_1(x) = 2H(u) + [g_1(v)g_2(v)]^2,$$

where  $H'(u) = h_1(u)h_2(u)$ , cannot take its maximum in  $\Omega$  unless it is constant.

- (b) If  $g_1g_2, h_1h_2 \in C^1(\Omega)$  and  $g_1h_2[h_1^2F_1 + g_2^2F_2] \leq 0$ ,  $(h_1h_2)' \geq 0$ ,  $(g_1g_2)' \geq 0$ , then the function

$$(3.16) \quad P_2(x) = 2H(u) + 2G(v),$$

where  $H'(u) = h_1(u)h_2(u)$  ,  $G'(v) = g_1(v)g_2(v)$  , cannot attain its maximum in  $\Omega$  unless it is constant.

- (c) If  $g_1g_2, h_1h_2 \in C^2(\Omega)$  and  $g_1h_2[h_1^2(h_1h_2)'F_1 + g_2^2(g_1g_2)'F_2] \leq 0$  ,  $(h_1^2h_2^2)'' \geq 0$  ,  $(g_1^2g_2^2)'' \geq 0$  , then the function

$$(3.17) \quad P_3(x) = [h_1(u)h_2(u)]^2 + [g_1(v)g_2(v)]^2$$

cannot attain its maximum in  $\Omega$  unless it is constant.

- (d) If  $g_1g_2 \in C^1(\Omega)$  ,  $h_1h_2 \in C^2(\Omega)$  and  $g_1h_2[h_1^2(h_1h_2)'F_1 + g_2^2F_2] \leq 0$  ,  $(g_1g_2)' \geq 0$  ,  $(h_1^2h_2^2)'' \geq 0$  , then the function

$$(3.18) \quad P_4(x) = [h_1(u)h_2(u)]^2 + 2G(v) ,$$

where  $G'(v) = g_1(v)g_2(v)$  , cannot attain its maximum in  $\Omega$  unless it is constant.

Proof. We prove (b) and (c) noting that (a) and (d) are proved in a similar manner.

- (b) From (3.16),

$$P_{2,i} = 2h_1(u)h_2(u)u_{,i} + 2g_1(v)g_2(v)v_{,i} ,$$

$$P_{2,ij} = 2h_1(u)h_2(u)u_{,ij} + 2(h_1h_2)'u_{,i}u_{,j} + 2g_1(v)g_2(v)v_{,ij} + 2(g_1g_2)'v_{,i}v_{,j} .$$

Hence,

$$(3.19) \quad LP_2 := a_{ij}P_{2,ij} + b_iP_{2,i} = 2h_1h_2Lu + 2(h_1h_2)'a_{ij}u_{,i}u_{,j} \\ + 2g_1g_2Lv + 2(g_1g_2)'a_{ij}v_{,i}v_{,j} .$$

We use (3.14) to write (3.19) as

$$(3.20) \quad LP_2 = -2g_1h_2[h_1^2F_1 + g_2^2F_2] + 2(h_1h_2)'a_{ij}u_{,i}u_{,j} \\ + 2(g_1g_2)'a_{ij}v_{,i}v_{,j} .$$

Since  $L$  is uniformly elliptic and  $(h_1h_2)' \geq 0$ ,  $(g_1g_2)' \geq 0$ ,  $g_1h_2[h_1^2F_1 + g_2^2F_2] \leq 0$ , the right side of (3.20) is nonnegative.

Thus,  $P_2$  cannot attain its maximum in  $\Omega$  unless it is constant. ■

(c) From (3.17),

$$P_{3,i} = 2[h_1(u)h_2(u)](h_1h_2)'u_{,i} + 2[g_1(v)g_2(v)](g_1g_2)'v_{,i} , \\ P_{3,ij} = 2[h_1(u)h_2(u)](h_1h_2)'u_{,ij} + (h_1^2h_2^2)''u_{,i}u_{,j} \\ + 2[g_1(v)g_2(v)](g_1g_2)'v_{,ij} + (g_1^2g_2^2)''v_{,i}v_{,j} .$$

Hence

$$(3.21) \quad LP_3 := a_{ij}P_{3,ij} + b_iP_{3,i} = 2h_1h_2(h_1h_2)'Lu + (h_1^2h_2^2)''a_{ij}u_{,i}u_{,j} \\ + 2g_1g_2(g_1g_2)'Lv + (g_1^2g_2^2)''a_{ij}v_{,i}v_{,j} .$$

Again, using (3.14), (3.21) becomes

$$LP_3 = -2g_1h_2[h_1^2(h_1h_2)']F_1 + g_2^2(g_1g_2)'F_2] + (h_1^2h_2)''a_{ij}u_{,i}u_{,j} \\ + (g_1^2g_2)''a_{ij}v_{,i}v_{,j} .$$

From the uniform ellipticity of  $L$  and since  $(h_1^2h_2)'' \geq 0$ ,  $(g_1^2g_2)'' \geq 0$ , and  $g_1h_2[h_1^2(h_1h_2)']F_1 + g_2^2(g_1g_2)'F_2 \leq 0$ , we have  $LP_3 \geq 0$ . The result follows from the maximum principle. ■

As an illustration of the preceding theorem, consider the system

$$Lu - ue^{u^2-v^2} = 0 , \\ (3.22) \\ Lv - ve^{v^2-u^2} = 0 .$$

If we set  $F_1(x,u,v) = F_2(x,u,v) \equiv -1$ ,  $h_1(u) = ue^{u^2}$ ,  $h_2(u) = e^{-u^2}$ ,  $g_1(v) = e^{-v^2}$ , and  $g_2(v) = ve^{v^2}$ , then the hypotheses in each of the parts (a)-(d) of Theorem 3.3 are satisfied. In this case, we have that the functions  $P_n$ ,  $n = 1,2,3,4$ , all reduce to

$$P(x) = [u(x)]^2 + [v(x)]^2$$

which by Theorem 3.3 cannot attain a maximum in  $\Omega$  unless it is constant. Moreover, if  $u, v \in C^2(\Omega) \cap C^0(\overline{\Omega})$ , then there exists a point  $x_0 \in \partial\Omega$  such that for  $x \in \overline{\Omega}$

$$|u(x)|, |v(x)| \leq ([u(x_0)]^2 + [v(x_0)]^2)^{1/2} .$$

We observe that for the system (3.22), only case (iii) of Lu and Wang

with  $h(u) = g(v) \equiv \pm 1$  is applicable. However, in this case

$$\Phi_3(x) = [h(u)]^2 + [g(v)]^2 \equiv 2$$

which does not lead to pointwise estimates for  $u$  or  $v$ .

Motivated by the above example, we can formulate the following.

Corollary 3.4. Let  $u, v \in C^2(\Omega) \cap C^0(\bar{\Omega})$  be a solution pair of

(3.14) such that  $h_1(u)h_2(u) = u$ ,  $g_1(v)g_2(v) = v$ , and

$$g_1h_2(h_1^2F_1 + g_2^2F_2) \leq 0.$$

- (1) If  $u = 0$  on  $\partial\Omega$ , then there exists a point  $x_0 \in \partial\Omega$  such that for  $x \in \bar{\Omega}$

$$|u(x)|, |v(x)| \leq |v(x_0)|.$$

- (2) If  $v = 0$  on  $\partial\Omega$ , then there exists a point  $x_0 \in \partial\Omega$  such that for  $x \in \bar{\Omega}$

$$|u(x)|, |v(x)| \leq |u(x_0)|.$$

- (3) If  $u = v = 0$  on  $\partial\Omega$ , then  $u = v \equiv 0$  in  $\bar{\Omega}$ .

Proof. Since the hypotheses in each of the parts (a)-(d) of Theorem 3.3 are satisfied and  $u, v \in C^0(\bar{\Omega})$ , there exists a point  $x_0 \in \partial\Omega$  such that for  $x \in \bar{\Omega}$

$$(3.23) \quad |u(x)|, |v(x)| \leq ([u(x_0)]^2 + [v(x_0)]^2)^{1/2}.$$

Under the appropriate assumption that  $u = 0$  on  $\partial\Omega$  or  $v = 0$  on

$\partial\Omega$  we have the desired result. Also, if  $u = v = 0$  on  $\partial\Omega$ , then from (3.23),  $u(x) = v(x) = 0$  for every  $x \in \overline{\Omega}$ . ■

Finally, we consider the equation

$$(3.24) \quad L^2 v + f(x, Lv, v) = 0 ,$$

where  $L$  is defined by (3.12). We can express (3.24) equivalently as the system

$$(3.25) \quad \begin{aligned} Lu + f(x, u, v) &= 0 , \\ Lv - u &= 0 . \end{aligned}$$

Moreover, the system (3.25) has the form of (3.14) where

$F_1(x, u, v) = f(x, u, v)$  ,  $F_2(x, u, v) \equiv -1$  ,  $h_1(u) = g_1(v) = g_2(v) \equiv 1$  , and  $h_2(u) = u$  . We immediately obtain the following corollaries of Theorem 3.3.

Corollary 3.5. Suppose  $sf(x, s, t) \leq 0$  for  $x \in \Omega$  and  $s, t \in \mathbb{R}$  .

If  $f(x, 0, t) \neq 0$  for  $x \in \Omega$  and  $t \in \mathbb{R}$  , then there is no solution  $v \in C^4(\Omega) \cap C^2(\overline{\Omega})$  of the equation (3.24) for which  $Lv = 0$  on  $\partial\Omega$  .

Moreover, if  $f(x, 0, t) = 0$  for  $x \in \Omega$  and  $t \in \mathbb{R}$  , then  $v \equiv 0$  is the only solution of (3.24) such that  $v = Lv = 0$  on  $\partial\Omega$  .

Proof. We express (3.24) as the system (3.25) and note that the hypotheses (a) in Theorem 3.3 are satisfied since  $uf(x, u, v) \leq 0$  by assumption and  $h_1 h_2 = u$  ,  $g_1 g_2 \equiv 1$  . Therefore, from (3.15), the function

$$P_1(x) = [u(x)]^2 + 1$$

takes its maximum on  $\partial\Omega$ . Consequently, there exists  $x_0 \in \partial\Omega$  such that for  $x \in \overline{\Omega}$

$$|u(x)| \leq |u(x_0)|.$$

However, from (3.25),  $u = Lv$  so that if  $Lv = 0$  on  $\partial\Omega$ , we have  $Lv \equiv 0$  in  $\overline{\Omega}$ . Hence, if  $f(x,0,t) \neq 0$  we reach the contradiction  $0 = L(Lv) = -f(x,Lv,t) = -f(x,0,t) \neq 0$ . If  $f(x,0,t) = 0$  for  $x \in \Omega$ ,  $t \in \mathbb{R}$  and  $v = 0$  on  $\partial\Omega$ , then we have  $Lv = 0$  in  $\Omega$ ,  $v = 0$  on  $\partial\Omega$ . By the maximum principle,  $v \equiv 0$  in  $\overline{\Omega}$ . ■

Corollary 3.6. Suppose  $s[f(x,s,t) - 1] \leq 0$  for  $x \in \Omega$  and  $s, t \in \mathbb{R}$ . If  $v \in C^4(\Omega) \cap C^2(\overline{\Omega})$  is a solution of (3.24) such that  $Lv = 0$  on  $\partial\Omega$ , then there exists a point  $x_0 \in \partial\Omega$  such that for  $x \in \overline{\Omega}$

$$v(x) \leq v(x_0).$$

Proof. Again, using (3.25), the hypotheses (d) in Theorem 3.3 are satisfied. Thus, from (3.18), the function

$$P_4(x) = [u(x)]^2 + 2v(x)$$

takes its maximum at a point  $x_0 \in \partial\Omega$ . Since  $u = Lv = 0$  on  $\partial\Omega$ , we have

$$2v(x) \leq P_4(x) \leq P_4(x_0) = 2v(x_0)$$

for  $x \in \overline{\Omega}$  from which the result follows. ■

We note that  $f(x,s,t) = -s^3$  satisfies the condition on  $f$  in Corollary 3.5 since  $sf(x,s,t) = -s^4 \leq 0$ . Likewise,  $f(x,s,t) = e^{-s}$  satisfies the condition on  $f$  in Corollary 3.6 since  $s[f(x,s,t) - 1] = s(e^{-s} - 1) \leq 0$ .

## CHAPTER IV

### METAHARMONIC EQUATIONS

It is well-known (Gilbarg and Trudinger [3], Corollary 3.2, p. 33) that if  $u \in C^2(\Omega) \cap C^0(\overline{\Omega})$  is a solution of

$$(4.1) \quad \Delta u + a_0 u = 0$$

in a bounded domain  $\Omega \subset \mathbb{R}^N$ , where  $a_0 \leq 0$  is constant, then

$$\sup_{\Omega} |u| = \sup_{\partial\Omega} |u|.$$

Moreover, if  $u$  is a nonconstant solution of (4.1), then  $|u|$  cannot attain its maximum in  $\Omega$ . To see this, consider the function

$$P(x) = [u(x)]^2.$$

Then,

$$\Delta P = 2u\Delta u + 2|\text{grad } u|^2 = -2a_0 u^2 + 2|\text{grad } u|^2 \geq 0$$

and since  $u$  is nonconstant, we have by the maximum principle for subharmonic functions that  $P$  cannot attain its maximum in  $\Omega$ .

Since  $P$  is continuous on  $\overline{\Omega}$ , there exists a point  $x_0 \in \partial\Omega$  such that

$$[u(x)]^2 < [u(x_0)]^2$$

for all  $x \in \Omega$  and the result follows by taking square roots.

In [2], Dunninger extended this result to fourth-order equations of the form

$$(4.2) \quad \Delta^2 u + a_1 \Delta u + a_0 u = 0 ,$$

where the constants  $a_0, a_1$  satisfy  $a_1 \leq 0$  and  $a_0 > 0$ . In particular, if  $u \in C^4(\Omega) \cap C^2(\overline{\Omega})$  is a nonconstant solution of (4.2) such that  $\Delta u = 0$  on  $\partial\Omega$ , then Dunninger showed that  $|u|$  cannot attain its maximum in  $\Omega$ . As before, the result follows as a direct consequence of the subharmonicity of the function

$$(4.3) \quad P(x) = [\Delta u(x)]^2 + a_0 [u(x)]^2 .$$

In general, consider a differential equation of the form

$$(4.4) \quad \Delta^n u + a_{n-1} \Delta^{n-1} u + \dots + a_1 \Delta u + a_0 u = 0$$

where  $a_0, \dots, a_{n-1}$  are real constants. A function  $u \in C^{2n}(\Omega) \cap C^{2(n-1)}(\overline{\Omega})$  which satisfies (4.4) in a bounded domain  $\Omega \subset \mathbb{R}^N$  is called an  $n$ -metaharmonic function. Moreover, if we set  $u_1 = u$ ,  $u_2 = \Delta u$ ,  $\dots$ ,  $u_n = \Delta^{n-1} u$  then

$$U = \begin{bmatrix} u_1 \\ \vdots \\ u_n \end{bmatrix}$$

satisfies the vector equation

$$(4.5) \quad \Delta U = -AU ,$$

where  $A$  is the  $n \times n$  matrix

$$A = \begin{bmatrix} 0 & -1 & 0 & \dots & 0 \\ 0 & 0 & -1 & \dots & 0 \\ \vdots & & & & \\ 0 & 0 & 0 & \dots & -1 \\ a_0 & a_1 & a_2 & \dots & a_{n-1} \end{bmatrix} ,$$

Now, define the characteristic equation

$$(4.6) \quad \xi^n + a_{n-1}\xi^{n-1} + \dots + a_1\xi + a_0 = 0$$

associated with (4.4) and suppose the roots  $\xi_i$  of (4.6) satisfy both of the conditions

- (a)  $\operatorname{Re} \xi_i \geq 0$  ,
- (4.7)
- (b) If  $\operatorname{Re} \xi_i = 0$  , then  $\xi_i$  is a simple root.

In this case, Chow and Dunninger [1] showed that if  $u$  is a solution of (4.4) such that

$$(4.8) \quad \Delta u = \Delta^2 u = \dots = \Delta^{n-1} u = 0 \quad \text{on } \partial\Omega ,$$

then there exists a positive constant  $k$  and  $x_0 \in \partial\Omega$  such that for all  $x \in \overline{\Omega}$

$$|u(x)| \leq k|u(x_0)| .$$

More specifically, since the eigenvalues  $\mu_i$  of  $A$  are given by  $\mu_i = -\xi_i$ , we have from a well-known result in Liapunov stability theory (Massera [7], Theorem 4\*, p. 183) that there exists a symmetric positive definite matrix  $B$  and a symmetric positive semidefinite matrix  $C$  satisfying

$$(4.9) \quad A^t B + BA = -C .$$

Consequently, the function

$$(4.10) \quad V = U^t B U$$

is subharmonic since from (4.5) and (4.9)

$$\begin{aligned} (4.11) \quad \Delta V &= (\Delta U)^t B U + U^t B (\Delta U) + 2 \sum_{i=1}^N (U_{x_i})^t B (U_{x_i}) \\ &= -U^t (A^t B + BA) U + 2 \sum_{i=1}^N (U_{x_i})^t B (U_{x_i}) \\ &= U^t C U + 2 \sum_{i=1}^N (U_{x_i})^t B (U_{x_i}) \geq 0 \end{aligned}$$

and thus  $V$  attains its maximum at some  $x_0 \in \partial\Omega$ . From the boundary

conditions (4.8) and since  $B$  is symmetric positive definite, it follows that for  $x \in \overline{\Omega}$

$$\lambda_1(B)[u(x)]^2 \leq \lambda_1(B)u^t u \leq V(x) \leq V(x_0) = b_{11}[u(x_0)]^2,$$

where  $\lambda_1(B)$  is the minimum eigenvalue of  $B$ . This yields the estimate

$$(4.12) \quad |u(x)| \leq (b_{11})^{1/2}[\lambda_1(B)]^{-1/2}|u(x_0)|, \quad x \in \overline{\Omega}$$

given by Chow and Dunninger. Of course, since  $B$  is symmetric positive definite,  $b_{11}$  and  $\lambda_1(B)$  are positive. Finally, we remark that if  $u$  is a nonconstant solution of (4.4), then  $V$  cannot attain its maximum in  $\Omega$ . Thus, for  $x \in \Omega$ , we obtain strict inequality in (4.12).

The question as to whether the estimate in (4.12) can be improved arises naturally. Moreover, since the preceding analysis applies for the particular choice  $n = 2$  in equation (4.4), a comparison of the bound (4.12) can be made with Dunninger's original work for the equation (4.2). In the remainder of this chapter we examine the matrix  $B$  in the function (4.10) for  $n = 2$  and  $n = 3$ . Using a variation of the Chow and Dunninger proof we give an improvement of their estimate in these two cases. Moreover, when  $n = 2$  we can obtain the function (4.3) of Dunninger by an appropriate choice of the matrix  $B$  in (4.10).

Suppose  $u \in C^4(\Omega) \cap C^2(\overline{\Omega})$  is a nonconstant solution of (4.2), where  $a_0$  and  $a_1$  are real constants, in a bounded domain  $\Omega \subset \mathbb{R}^N$

such that  $\Delta u = 0$  on  $\partial\Omega$  ; i.e.,

$$(4.13) \quad \Delta^2 u + a_1 \Delta u + a_0 u = 0 \quad \text{in } \Omega ,$$

$$\Delta u = 0 \quad \text{on } \partial\Omega .$$

Moreover, assume that the roots  $\xi_1, \xi_2$  of the characteristic equation

$$(4.14) \quad \xi^2 + a_1 \xi + a_0 = 0$$

satisfy (4.7). Consider the function

$$(4.15) \quad V = [u \quad \Delta u] B \begin{bmatrix} u \\ \Delta u \end{bmatrix} ,$$

where  $B$  is a symmetric positive definite matrix which satisfies (4.9). Without loss of generality, we assume

$$(4.16) \quad B = \begin{bmatrix} b_{11} & -b_{12} \\ -b_{12} & 1 \end{bmatrix} ,$$

where  $b_{11} > 0$  ,  $b_{11} - b_{12}^2 > 0$  (Noble and Daniel [9], Theorem 11.4, p. 427). For convenience, we set

$$(4.17) \quad b_{11} = b_{12}^2 + \alpha ,$$

where  $\alpha = \det B > 0$  . From the form (4.16) and (4.17), (4.15) becomes

$$V(x) = (b_{12}^2 + \alpha)u^2 - 2b_{12}u\Delta u + (\Delta u)^2$$

and completing the square we have

$$(4.18) \quad V(x) = (\Delta u - b_{12}u)^2 + \alpha u^2 .$$

Now, from the calculation (4.11)  $V$  is subharmonic and nonconstant.

Hence, there exists a point  $x_0 \in \partial\Omega$  such that

$$(4.19) \quad V(x) < V(x_0) , \quad x \in \Omega .$$

Using the boundary condition in (4.13), it follows from (4.18) and (4.19) that for  $x \in \Omega$

$$\alpha[u(x)]^2 \leq [\Delta u(x) - b_{12}u(x)]^2 + \alpha[u(x)]^2 < (b_{12}^2 + \alpha)[u(x_0)]^2 ,$$

or since  $\alpha > 0$  ,

$$|u(x)| < (b_{12}^2 + \alpha)^{1/2} \alpha^{-1/2} |u(x_0)| = (b_{11})^{1/2} (\det B)^{-1/2} |u(x_0)| .$$

We thus have the following theorem.

Theorem 4.1. Suppose the roots  $\xi_1, \xi_2$  of (4.14) satisfy both of the conditions

$$(4.20) \quad (a) \quad \operatorname{Re} \xi_i \geq 0 ,$$

(b) If  $\operatorname{Re} \xi_i = 0$  , then  $\xi_i$  is a simple root.

If  $u \in C^4(\Omega) \cap C^2(\overline{\Omega})$  is a nonconstant solution of

$$(4.21) \quad \Delta^2 u + a_1 \Delta u + a_0 u = 0$$

which satisfies on  $\partial\Omega$

$$\Delta u = 0 ,$$

then there exists a symmetric positive definite matrix  $B$  and  $x_0 \in \partial\Omega$  such that

$$(4.22) \quad |u(x)| < (b_{11})^{1/2} (\det B)^{-1/2} |u(x_0)| , \quad x \in \Omega .$$

We remark that the bound (4.22) is an improvement over that of Chow and Dunninger in (4.12). Indeed, consider the characteristic polynomial for  $B$

$$(4.23) \quad \begin{aligned} p(\lambda) &:= \det(\lambda I - B) = \lambda^2 - (b_{11} + 1)\lambda + (b_{11} - b_{12}^2) \\ &= \lambda^2 - (b_{12}^2 + \alpha + 1)\lambda + \alpha , \end{aligned}$$

where we have used (4.17). Then,  $p(0) = \alpha > 0$  and  $p(\alpha) = -b_{12}^2 \alpha \leq 0$  . It suffices to show that  $0 < \lambda_1 \leq \det B$  . We have the following two cases.

Case 1. If  $b_{12} \neq 0$ , then  $p(\alpha) < 0$ . Hence,  
 $0 < \lambda_1(B) < \alpha = \det B$ .

Case 2. If  $b_{12} = 0$ , then  $p(\alpha) = 0$  so that  $\alpha$  is an eigenvalue of  $B$ . Consequently, by the definition of  $\lambda_1(B)$  we have  $0 < \lambda_1(B) \leq \alpha$ . More specifically, if  $b_{12} = 0$  the characteristic polynomial (4.23) becomes

$$p(\lambda) = \lambda^2 - (\alpha + 1)\lambda + \alpha = (\lambda - \alpha)(\lambda - 1).$$

Thus, since  $\alpha = \det B$  is an eigenvalue of  $B$ ,  $0 < \lambda_1(B) < \det B$  iff  $\det B > 1$  and  $0 < \lambda_1(B) = \det B$  iff  $0 < \det B \leq 1$ .

We now give an explicit method for constructing the symmetric positive definite matrix  $B$  in Theorem 4.1. First, however, we note that since the roots  $\xi_1$  and  $\xi_2$  of (4.14) are given by

$$\xi_1 = \frac{1}{2}[-a_1 - (a_1^2 - 4a_0)^{1/2}], \quad \xi_2 = \frac{1}{2}[-a_1 + (a_1^2 - 4a_0)^{1/2}],$$

it follows that  $\xi_1$  and  $\xi_2$  satisfy (4.20) iff one of the following holds:

- (1)  $a_0 > 0$  and  $a_1 \leq 0$ ;
- (2)  $a_0 = 0$  and  $a_1 < 0$ .

Therefore, for convenience, we express equation (4.21) in the form

$$(4.24) \quad \Delta^2 u - q\Delta u + pu = 0,$$

where  $p, q \geq 0$  and  $p + q \neq 0$ .

As a vector equation (4.24) becomes

$$\Delta U = -AU ,$$

where

$$U = \begin{bmatrix} u \\ \Delta u \end{bmatrix} , \quad A = \begin{bmatrix} 0 & -1 \\ p & -q \end{bmatrix} .$$

Now, from (4.16) and (4.17), the desired matrix  $B$  has the form

$$B = \begin{bmatrix} b_{12}^2 + \alpha & -b_{12} \\ -b_{12} & 1 \end{bmatrix}$$

for  $\alpha > 0$  and satisfies

$$A^t B + BA = -C ,$$

where  $C$  is symmetric positive semidefinite. An easy calculation gives

$$C = \begin{bmatrix} 2b_{12}p & b_{12}(b_{12} - q) - p + \alpha \\ b_{12}(b_{12} - q) - p + \alpha & 2(q - b_{12}) \end{bmatrix} .$$

Now,  $C$  is positive semidefinite iff (1)  $2b_{12}p, 2(q - b_{12}) \geq 0$  and (2)  $\det C \geq 0$  (Strang [24], Theorem 6E, p. 244). We consider the following two cases.

Case 1. If  $p = 0$ , then  $\det C = -[b_{12}(b_{12} - q) + \alpha]^2 \leq 0$ . Hence,  $\det C = 0$  iff  $\alpha = b_{12}(q - b_{12})$ . Therefore, since  $\alpha > 0$ ,  $C$  is positive semidefinite iff  $0 < b_{12} < q$  and  $\alpha = b_{12}(q - b_{12})$ .

Case 2. For  $p > 0$  we have that  $C$  is positive semidefinite iff  $0 \leq b_{12} \leq q$  and  $\det C \geq 0$ . Moreover, since

$$\det C = 4pb_{12}(q - b_{12}) - [b_{12}(b_{12} - q) - p + \alpha]^2$$

it follows that  $\det C \geq 0$  iff

$$([b_{12}(q - b_{12})]^{1/2} - p^{1/2})^2 \leq \alpha \leq ([b_{12}(q - b_{12})]^{1/2} + p^{1/2})^2.$$

Of course, we must have  $\alpha > 0$ . Thus, for the equation (4.24) the entries of  $B$  are chosen as follows:

(1) If  $p = 0$ , then for  $0 < b_{12} < q$  choose  $\alpha = b_{12}(q - b_{12})$ .  
(4.25)

(2) If  $p > 0$ , then for  $0 \leq b_{12} \leq q$  choose  $\alpha \neq 0$  such that

$$([b_{12}(q - b_{12})]^{1/2} - p^{1/2})^2 \leq \alpha \leq ([b_{12}(q - b_{12})]^{1/2} + p^{1/2})^2.$$

From (4.17), the estimate (4.22) becomes

$$(4.26) \quad |u(x)| \leq (b_{12}^2 + \alpha)^{1/2} \alpha^{-1/2} |u(x_0)|$$

so that by choosing  $b_{12} = 0$  we have

$$|u(x)| < |u(x_0)| ,$$

which coincides with the result of Dunninger [2]. Of course, from the discussion above, the choice  $b_{12} = 0$  is possible only when  $p > 0$  in equation (4.24). In this case, from (4.25) when  $b_{12} = 0$  we have  $\alpha = p$  and the function (4.18) reduces to Dunninger's function (4.3).

Moreover, in terms of the coefficients of the differential equation (4.24), if  $b_{12} = 0$ , then from (4.23) we have

$$p(\lambda) = (\lambda - 1)(\lambda - p) ,$$

since  $\alpha = p$  in this case, and using (4.17) we can write (4.12) as

$$(4.27) \quad |u(x)| < p^{1/2} [\min(1, p)]^{-1/2} |u(x_0)| .$$

Therefore, if  $b_{12} = 0$ , it follows that (4.26) and (4.27) yield the same estimate provided  $0 < p \leq 1$ . Otherwise, (4.26) is the better estimate since for  $p > 1$ ,  $\min(1, p) = 1$ .

We now proceed to derive an improved estimate for the equation

$$(4.28) \quad \Delta^3 u + a_2 \Delta^2 u + a_1 \Delta u + a_0 u = 0 ,$$

where  $a_0$ ,  $a_1$ , and  $a_2$  are real constants. To this end, let  $u \in C^6(\Omega) \cap C^4(\overline{\Omega})$  be a nonconstant solution of (4.28) in a bounded domain  $\Omega \subset \mathbb{R}^N$  such that

$$(4.29) \quad \Delta u = \Delta^2 u = 0 \quad \text{on } \partial\Omega .$$

Again, assume that the roots  $\xi_1, \xi_2, \xi_3$  of the characteristic equation

$$(4.30) \quad f(\xi) := \xi^3 + a_2 \xi^2 + a_1 \xi + a_0 = 0$$

satisfy (4.7). Consider the function

$$(4.31) \quad V = [u \quad \Delta u \quad \Delta^2 u] B \begin{bmatrix} u \\ \Delta u \\ \Delta^2 u \end{bmatrix}$$

where  $B$  is a symmetric positive definite matrix which satisfies

(4.9). Without loss of generality, we assume

$$(4.32) \quad B = \begin{bmatrix} b_{11} & -b_{12} & b_{13} \\ -b_{12} & b_{22} & -b_{23} \\ b_{13} & -b_{23} & 1 \end{bmatrix} ,$$

where from [9]

- (i)  $b_{11}, b_{22} > 0$  ;
- (ii)  $b_{11} - b_{13}^2 > 0, b_{22} - b_{23}^2 > 0$  ;
- (iii)  $b_{11}b_{22} > b_{12}^2$  ;
- (iv)  $\det B > 0$  .

For convenience, we set

$$(4.33) \quad b_{11} = b_{13}^2 + \alpha, \quad b_{22} = b_{23}^2 + \beta$$

where  $\alpha, \beta > 0$  are given by

$$\alpha = \begin{vmatrix} b_{11} & b_{13} \\ b_{13} & 1 \end{vmatrix}, \quad \beta = \begin{vmatrix} b_{22} & -b_{23} \\ -b_{23} & 1 \end{vmatrix}.$$

By the form (4.32) and (4.33), (4.31) becomes

$$\begin{aligned} V(x) = & (b_{13}^2 + \alpha)u^2 - 2b_{12}u\Delta u + 2b_{13}u\Delta^2 u + (b_{23}^2 + \beta)(\Delta u)^2 \\ & - 2b_{23}\Delta u\Delta^2 u + (\Delta^2 u)^2 \end{aligned}$$

and completing the square, we have

$$\begin{aligned} (4.34) \quad V(x) = & (\Delta^2 u - b_{23}\Delta u + b_{13}u)^2 + \beta(\Delta u - [b_{12} - b_{13}b_{23}]^{-1}u)^2 \\ & + (\alpha - z)u^2, \end{aligned}$$

where

$$(4.35) \quad z = (b_{12} - b_{13}b_{23})^2 \beta^{-1}.$$

Since  $V$  is subharmonic and nonconstant, there exists  $x_0 \in \partial\Omega$  such that

$$(4.36) \quad V(x) < V(x_0), \quad x \in \Omega.$$

From the boundary condition (4.29), it follows from (4.34) and (4.35) that for  $x \in \Omega$

$$(4.37) \quad (\alpha - z)[u(x)]^2 < (b_{13}^2 + \alpha)[u(x_0)]^2 .$$

Now, using (4.33) we have

$$\det B = \beta(\alpha - z)$$

and since  $\beta > 0$  and  $\det B > 0$ , necessarily  $\alpha - z > 0$ . Hence, we divide (4.37) by  $\alpha - z$  to obtain the estimate

$$\begin{aligned} |u(x)| &< (b_{13}^2 + \alpha)^{1/2}(\alpha - z)^{-1/2} |u(x_0)| \\ &= b_{11}^{1/2}[(\det B)(\det B_{23})^{-1}]^{-1/2} |u(x_0)| , \end{aligned}$$

where

$$B_{23} = \begin{bmatrix} b_{22} & -b_{23} \\ -b_{23} & 1 \end{bmatrix} .$$

We have the following theorem.

Theorem 4.2. Suppose the roots  $\xi_1, \xi_2, \xi_3$  of (4.30) satisfy both of the conditions

$$(4.38) \quad (a) \quad \operatorname{Re} \xi_i \geq 0$$

$$(b) \quad \text{If } \operatorname{Re} \xi_i = 0, \text{ then } \xi_i \text{ is a simple root.}$$

If  $u \in C^6(\Omega) \cap C^4(\overline{\Omega})$  is a nonconstant solution of

$$\Delta^3 u + a_2 \Delta^2 u + a_1 \Delta u + a_0 u = 0$$

which satisfies on  $\partial\Omega$

$$\Delta u = \Delta^2 u = 0 ,$$

then there exists a symmetric positive definite matrix  $B$  and  $x_0 \in \partial\Omega$  such that

$$(4.39) \quad |u(x)| < b_{11}^{1/2} [(\det B)(\det B_{23})^{-1}]^{-1/2} |u(x_0)| , \quad x \in \Omega ,$$

where  $B_{23}$  is the submatrix formed by deleting the first row and column of  $B$ .

We now show that the bound (4.39) is again an improvement over (4.12). Indeed, consider the characteristic polynomial for  $B$

$$\begin{aligned} p(\lambda) &:= \det(\lambda I - B) = \lambda^3 - (b_{11} + b_{22} + 1)\lambda^2 \\ &\quad + (b_{11}b_{22} - b_{12}^2 + b_{11} - b_{13}^2 + b_{22} - b_{23}^2)\lambda \\ &\quad - [b_{11}(b_{22} - b_{23}^2) - b_{13}^2 b_{22} - (b_{12}^2 - 2b_{12}b_{13}b_{23})] . \end{aligned}$$

We complete the square in the constant term and use (4.33) and (4.35) to write

$$(4.40) \quad p(\lambda) = \lambda^3 - (b_{13}^2 + b_{23}^2 + \alpha + \beta + 1)\lambda^2 \\ + ([b_{13}^2 + \alpha][b_{23}^2 + \beta] - b_{12}^2 + \alpha + \beta)\lambda - \beta(\alpha - z) .$$

Then,  $p(0) = -\beta(\alpha - z) < 0$  . Again, it suffices to show that  $0 < \lambda_1(B) \leq (\det B)(\det B_{23})^{-1}$  , where

$$(\det B)(\det B_{23})^{-1} = \alpha - z .$$

After a rearrangement of terms and simplification, we find

$$(4.41) \quad p(\alpha - z) = (\alpha - z)[(b_{13}^2 + z)(b_{23}^2 + \beta) - b_{12}^2 + z - (\alpha - z)(b_{13}^2 + z)] .$$

Now, from (4.35),  $z \geq 0$  and hence  $b_{13}^2 + z \geq 0$  . However, we shall later show that  $b_{13}^2 + z \neq 0$  . Therefore, assume  $b_{13}^2 + z > 0$  . In this case, we express (4.41) as

$$p(\alpha - z) = (\alpha - z)(b_{13}^2 + z)[b_{23}^2 + \beta + 1 - R - (\alpha - z)] ,$$

where

$$(4.42) \quad R = (b_{13}^2 + b_{12}^2)(b_{13}^2 + z)^{-1} .$$

We consider the following three cases.

Case 1. If  $\alpha - z < b_{23}^2 + \beta + 1 - R$  , then  $p(\alpha - z) > 0$  .

Hence,  $0 < \lambda_1(B) < \alpha - z$  .

Case 2. If  $\alpha - z = b_{23}^2 + \beta + 1 - R$ , then  $p(\alpha - z) = 0$  so that  $\alpha - z$  is an eigenvalue of  $B$ , and hence  $0 < \lambda_1(B) \leq \alpha - z$ . Moreover, the characteristic polynomial (4.40) becomes

$$p(\lambda) = (\lambda - [\alpha - z])r(\lambda),$$

where

$$r(\lambda) = \lambda^2 - (b_{13}^2 + b_{23}^2 + \beta + z + 1)\lambda + \beta.$$

Now,  $r(0) = \beta > 0$  and

$$r(\alpha - z) = -[R(b_{23}^2 + \beta + 1 - R) - \beta] - (b_{13}^2 + z)(\alpha - z).$$

A straightforward algebraic manipulation using the definition of  $R$  given by (4.42) and the definition of  $z$  in (4.35) shows that

$$(4.43) \quad R(b_{23}^2 + \beta + 1 - R) - \beta \geq 0.$$

Hence,  $r(\alpha - z) \leq -(b_{13}^2 + z)(\alpha - z) < 0$  and hence  $\lambda_1(B)$  must be the smaller root of  $r(\lambda)$ . Therefore,  $0 < \lambda_1(B) < \alpha - z$ .

Case 3. If  $\alpha - z > b_{23}^2 + \beta + 1 - R$ , then  $p(\alpha - z) < 0$ . However, from (4.43) we have  $b_{23}^2 + \beta + 1 - R > 0$  and if we evaluate the characteristic polynomial (4.40) at  $b_{23}^2 + \beta + 1 - R$ , we obtain after simplification

$$p(b_{23}^2 + \beta + 1 - R) = [(\alpha - z) - (b_{23}^2 + \beta + 1 - R)] \\ \times [R(b_{23}^2 + \beta + 1 - R) - \beta] .$$

Since  $\alpha - z > b_{23}^2 + \beta + 1 - R$  by hypothesis and by (4.43), we have  $p(b_{23}^2 + \beta + 1 - R) \geq 0$ . Therefore,  $0 < \lambda_1(B) \leq b_{23}^2 + \beta + 1 - R < \alpha - z$ .

We now show that  $b_{13}^2 + z \neq 0$ . From (4.35) we have  $b_{13}^2 + z = 0$  if and only if  $b_{12} = b_{13} = 0$ . We claim that there does not exist a symmetric positive definite matrix  $B$  with  $b_{12} = b_{13} = 0$  which satisfies (4.9). Indeed, with

$$B = \begin{bmatrix} b_{11} & 0 & 0 \\ 0 & b_{22} & -b_{23} \\ 0 & -b_{23} & 1 \end{bmatrix}, \quad A = \begin{bmatrix} 0 & -1 & 0 \\ 0 & 0 & -1 \\ a_0 & a_1 & a_2 \end{bmatrix}$$

we find

$$A^t B + B A = - \begin{bmatrix} 0 & b_{11} + a_0 b_{23} & -a_0 \\ b_{11} + a_0 b_{23} & 2a_1 b_{23} & (a_2 b_{23} + b_{22}) - a_1 \\ -a_0 & (a_2 b_{23} + b_{22}) - a_1 & -2(b_{23} + a_2) \end{bmatrix} := -C ,$$

where  $C$  is symmetric positive semidefinite. However, if the symmetric matrix  $C$  is positive semidefinite, then all principal submatrices have nonnegative determinants (Strang [24], Theorem 6.3, p. 244). Necessarily,  $a_0 = 0$  which in turn gives  $b_{11} = 0$ .

Obviously, this contradicts the fact that  $B$  is symmetric positive definite. Thus,  $b_{13}^2 + z \neq 0$  and therefore the estimate (4.39)

is sharper than the estimate (4.12) for any suitable choice of the matrix  $B$  since for  $b_{13}^2 + z \neq 0$  we have  $0 < \lambda_1(B) < (\det B)(\det B_{23})^{-1}$ .

We claim that the roots  $\xi_1, \xi_2, \xi_3$  of the characteristic equation (4.30) satisfy (4.38) iff one of the following holds:

- (1)  $a_0 = 0$ ,  $a_1 > 0$ , and  $a_2 \leq 0$ ;
- (2)  $a_0 < 0$ ,  $a_1 > 0$ ,  $a_2 < 0$ , and  $|a_1 a_2| \geq |a_0|$ .

Indeed, if  $a_0 = 0$ , then the roots of (4.30) are  $\xi_1 = 0$  and

$$\xi_2 = \frac{1}{2}[-a_2 - (a_2^2 - 4a_1)^{1/2}], \quad \xi_3 = \frac{1}{2}[-a_2 + (a_2^2 - 4a_1)^{1/2}].$$

Now,  $a_1 \neq 0$  since  $\xi = 0$  must be a simple root. Moreover, if  $a_1 < 0$ , then  $\operatorname{Re} \xi_2 = \xi_2 < 0$ . Thus,  $a_1 > 0$  and consequently  $a_2 \leq 0$  for otherwise  $\operatorname{Re} \xi_3 < 0$ .

If  $a_0 \neq 0$ , then there are two cases in which the roots of the characteristic equation (4.30) satisfy (4.38). Either  $f(\xi) = 0$  has three positive real roots or it has one positive real root and two complex conjugate roots with nonnegative real parts. If  $f(\xi) = 0$  has exactly three positive real roots, then there must be exactly three variations of sign in (4.30); i.e.,  $a_2 < 0$ ,  $a_1 > 0$ ,  $a_0 < 0$ . If  $f(\xi) = 0$  has exactly one positive real root  $\xi^*$ , then the characteristic equation (4.30) can be written as

$$f(\xi) := (\xi - \xi^*)(\xi^2 + [a_2 + \xi^*]\xi + [a_1 + \xi^*(a_2 + \xi^*)]) = 0$$

and thus has roots  $\xi_1 = \xi^*$  and

$$\xi_2 = \frac{1}{2}[-(a_2 + \xi^*) + \{(a_2 + \xi^*)^2 - 4[a_1 + \xi^*(a_2 + \xi^*)]\}^{1/2}] ,$$

$$\xi_3 = \frac{1}{2}[-(a_2 + \xi^*) - \{(a_2 + \xi^*)^2 - 4[a_1 + \xi^*(a_2 + \xi^*)]\}^{1/2}] ,$$

where by hypothesis

$$(4.44) \quad (a_2 + \xi^*)^2 - 4[a_1 + \xi^*(a_2 + \xi^*)] < 0 .$$

It follows that  $\xi_2$  and  $\xi_3$  have nonnegative real parts iff

$a_2 + \xi^* \leq 0$  . Hence,  $a_2 \leq -\xi^* < 0$  . Consequently, from (4.44) we obtain

$$a_1 > \frac{1}{4}(a_2 + \xi^*)(a_2 - 3\xi^*) \geq 0 .$$

Finally, since  $a_2 < 0$  and  $a_1 > 0$  , we must have  $a_0 < 0$  for otherwise the characteristic equation (4.30) has two variations in sign and hence we reach the contradiction that  $f(\xi) = 0$  has at least one negative real root. Thus, if  $a_0 \neq 0$  , we must have the sign conditions  $a_0 < 0$  ,  $a_1 > 0$  ,  $a_2 < 0$  . Consequently, for convenience we rewrite (4.30) in this case as

$$(4.45) \quad f(\xi) := \xi^3 - r\xi^2 + q\xi - p = 0 ,$$

where  $p, q, r > 0$  . Let us now show that the roots of (4.45) satisfy (4.38) iff  $rq \geq p$  . We consider the following two cases.

Case 1.  $rq = p$  . Here,

$$f(\xi) = (\xi - \kappa)(\xi^2 + q)$$

and  $f(\xi) = 0$  has roots  $\xi_1 = \kappa$  and  $\xi_2, \xi_3 = \pm(q^{1/2})i$ . Hence, (4.38) is satisfied.

Case 2.  $\kappa q \neq p$ . Here,

$$f(\xi) = \xi^2(\xi - \kappa) + (q\xi - p).$$

If  $\kappa q < p$ ,  $f(\xi) < 0$  for  $\xi \leq \kappa$  and  $f(\xi) > 0$  for  $\xi \geq p/q$ .

Therefore, all real roots of (4.45) lie in the interval  $(\kappa, p/q)$ .

If  $\kappa q > p$ ,  $f(\xi) < 0$  for  $\xi \leq p/q$  and  $f(\xi) > 0$  for  $\xi \geq \kappa$ .

Therefore, all real roots of (4.45) lie in the interval  $(p/q, \kappa)$ .

Now, (4.45) has at least one real root  $\xi^* > 0$  and hence we write

$$f(\xi) = (\xi - \xi^*)(\xi^2 + [\xi^* - \kappa]\xi + [q + \xi^*(\xi^* - \kappa)])$$

so that  $f(\xi) = 0$  has roots  $\xi_1 = \xi^*$  and

$$\xi_2 = \frac{1}{2}[(\kappa - \xi^*) + \{(\kappa - \xi^*)(\kappa + 3\xi^*) - 4q\}^{1/2}],$$

$$\xi_3 = \frac{1}{2}[(\kappa - \xi^*) - \{(\kappa - \xi^*)(\kappa + 3\xi^*) - 4q\}^{1/2}].$$

If  $\kappa q < p$ , then  $0 < \kappa < \xi^* < p/q$  so that  $\kappa - \xi^* < 0$  and hence

$(\kappa - \xi^*)(\kappa + 3\xi^*) - 4q < 0$ ; i.e.,  $\xi_2$  and  $\xi_3$  are complex with

$\text{Re } \xi_2 = \text{Re } \xi_3 = \frac{1}{2}(\kappa - \xi^*) < 0$ . Therefore, (4.38) does not hold for

$\kappa q < p$ . If, however,  $\kappa q > p$ , then  $0 < p/q < \xi^* < \kappa$  so that

$\kappa - \xi^* > 0$ . Hence, if  $\xi_2$  and  $\xi_3$  are complex, we have

$\operatorname{Re} \xi_2 = \operatorname{Re} \xi_3 = \frac{1}{2}(\kappa - \xi^*) > 0$ . Also, if  $\xi_2$  and  $\xi_3$  are real, then we have already shown  $0 < p/q < \xi_2$ ,  $\xi_3 < \kappa$ . Thus, (4.38) is satisfied.

We next construct the matrix  $B$  of Theorem 4.2 under each of the conditions (1) and (2), where in either case we can consider an equation of the form

$$(4.46) \quad \Delta^3 u - \kappa \Delta^2 u + q \Delta u - pu = 0,$$

where  $p \geq 0$ ,  $q > 0$ ,  $\kappa \geq 0$  such that  $\kappa q \geq p$ . Equivalently, as a vector equation, (4.46) becomes

$$\Delta U = -AU,$$

where

$$U = \begin{bmatrix} u \\ \Delta u \\ \Delta^2 u \end{bmatrix}, \quad A = \begin{bmatrix} 0 & -1 & 0 \\ 0 & 0 & -1 \\ -p & q & -\kappa \end{bmatrix}$$

From (4.32) and (4.33), the symmetric positive definite matrix  $B$  of Theorem 4.2 has the form

$$(4.47) \quad B = \begin{bmatrix} b_{13}^2 + \alpha & -b_{12} & b_{13} \\ -b_{12} & b_{23}^2 + \beta & -b_{23} \\ b_{13} & -b_{23} & 1 \end{bmatrix}$$

where  $\alpha, \beta > 0$ ,  $(b_{13}^2 + \alpha)(b_{23}^2 + \beta) > b_{12}^2$  and  $\det B > 0$ , and  $B$

is such that

$$A^t B + BA = -C$$

for a symmetric positive semidefinite matrix  $C$ . Again, an easy calculation gives

$$(4.48) \quad C = \begin{bmatrix} 2pb_{13} & b_{13}^2 + \alpha - (qb_{13} + pb_{23}) & p + \kappa b_{13} - b_{12} \\ b_{13}^2 + \alpha - (qb_{13} + pb_{23}) & 2(qb_{23} - b_{12}) & b_{23}^2 + \beta + b_{13} - (q + \kappa b_{23}) \\ p + \kappa b_{13} - b_{12} & b_{23}^2 + \beta + b_{13} - (q + \kappa b_{23}) & 2(\kappa - b_{23}) \end{bmatrix}$$

It is easy to see that a special class of symmetric positive definite matrices of the form (4.47) are those for which

$$b_{12} = b_{13}b_{23}.$$

For simplicity, we suppose that the matrix  $B$  of Theorem 4.2 has this form. In this case, the matrix  $C$  in (4.48) takes the form

$$C = \begin{bmatrix} 2pb_{13} & \alpha - b_{13}(q - b_{13}) - pb_{23} & p + b_{13}(\kappa - b_{23}) \\ \alpha - b_{13}(q - b_{13}) - pb_{23} & 2b_{23}(q - b_{13}) & \beta - b_{23}(\kappa - b_{23}) - (q - b_{13}) \\ p + b_{13}(\kappa - b_{23}) & \beta - b_{23}(\kappa - b_{23}) - (q - b_{13}) & 2(\kappa - b_{23}) \end{bmatrix}$$

Again,  $C$  is positive semidefinite iff

- (1)  $2pb_{13}, 2b_{23}(q - b_{13}), 2(\kappa - b_{23}) \geq 0$  ;
- (2)  $4pb_{13}(\kappa - b_{23}) - [p + b_{13}(\kappa - b_{23})]^2 \geq 0$  ;

- (3)  $4pb_{13}b_{23}(q - b_{13}) - [\alpha - b_{13}(q - b_{13}) - pb_{23}]^2 \geq 0$  ;  
 (4)  $4b_{23}(q - b_{13})(\kappa - b_{23}) - [\beta - b_{23}(\kappa - b_{23}) - (q - b_{13})]^2 \geq 0$  ;  
 (5)  $\det C \geq 0$  ;

or equivalently, after algebraic simplification,

- (1')  $pb_{13} , b_{23}(q - b_{13}) , \kappa - b_{23} \geq 0$  ;  
 (2')  $[p - b_{13}(\kappa - b_{23})]^2 \leq 0$  ;  
 (3')  $\alpha^2 - 2\alpha[b_{13}(q - b_{13}) + pb_{23}] + [b_{13}(q - b_{13}) - pb_{23}]^2 \leq 0$  ;  
 (4')  $\beta^2 - 2\beta[b_{23}(\kappa - b_{23}) + (q - b_{13})] + [b_{23}(\kappa - b_{23}) - (q - b_{13})]^2 \leq 0$  ;  
 (5')  $b_{23}(q - b_{13})Q^2 + [\beta - b_{23}(\kappa - b_{23}) - (q - b_{13})] \times Q(b_{23}Q + b_{13}\beta - \alpha) + (\kappa - b_{23})(b_{23}Q + b_{13}\beta - \alpha)^2 \leq 0$  ,  
 where  $Q = p - b_{13}(\kappa - b_{23})$  .

Immediately, from (2') ,

$$(4.49) \quad p = b_{13}(\kappa - b_{23}) ; \text{ i.e., } Q = 0 ,$$

and from (3') and (4') we get the sign restrictions

$$(4.50) \quad \begin{aligned} b_{13}(q - b_{13}) + pb_{23} &> 0 , \\ b_{23}(\kappa - b_{23}) + (q - b_{13}) &> 0 . \end{aligned}$$

Moreover, from (3) and (4) we obtain the inequalities

$$(4.51) \quad |\alpha - [b_{13}(q - b_{13}) + pb_{23}]| \leq 2[pb_{13}b_{23}(q - b_{13})]^{1/2},$$

$$|\beta - [b_{23}(\kappa - b_{23}) + (q - b_{13})]| \leq 2[b_{23}(q - b_{13})(\kappa - b_{23})]^{1/2}.$$

Now, since  $Q = 0$ , (5') reduces to

$$(4.52) \quad (\kappa - b_{23})(b_{13}\beta - \alpha)^2 \leq 0.$$

We first suppose  $p = 0$ . By (4.49),  $b_{13} = 0$  or  $b_{23} = \kappa$ . However, if  $p = b_{13} = 0$ , the first sign condition of (4.50) does not hold. Therefore,  $b_{23} = \kappa$ . From (4.51), we have

$$\alpha = b_{13}(q - b_{13}),$$

$$\beta = q - b_{13}$$

and, consequently,  $0 < b_{13} < q$  since  $\alpha, \beta > 0$ . Thus, for  $p = 0$  the matrix  $B$  has the form

$$B = \begin{bmatrix} qb_{13} & -\kappa b_{13} & b_{13} \\ -\kappa b_{13} & \kappa^2 + q - b_{13} & -\kappa \\ b_{13} & -\kappa & 1 \end{bmatrix},$$

where  $0 < b_{13} < q$ .

Now, assume  $p > 0$  and suppose  $\kappa q > p$ . From (1') and (4.49),

$$b_{13} > 0, \quad \kappa - b_{23} > 0.$$

Consequently,

$$b_{23} \geq 0, \quad q - b_{13} \geq 0,$$

where  $b_{23} + (q - b_{13}) \neq 0$ , since  $b_{23}(q - b_{13}) \geq 0$  and the sign restrictions (4.50) must hold. In particular, from (4.49), we must have

$$0 < b_{13} = p(\kappa - b_{23})^{-1} \leq q$$

with strict inequality if  $b_{23} = 0$ . Finally, from (4.52), we must have  $b_{13}\beta = \alpha$ . We note that by multiplying the second inequality of (4.51) by  $b_{13} > 0$ , we get

$$\begin{aligned} & |b_{13}\beta - [b_{13}b_{23}(\kappa - b_{23}) + b_{13}(q - b_{13})]| \\ & \leq 2[b_{13}^2b_{23}(q - b_{13})(\kappa - b_{23})]^{1/2}, \end{aligned}$$

which is precisely the first inequality of (4.51) since  $b_{13}\beta = \alpha$  and  $p = b_{13}(\kappa - b_{23})$ . Thus, for  $p > 0$  and  $\kappa q > p$  the matrix  $B$  has the form

$$B = \begin{bmatrix} b_{13}(b_{13} + \beta) & -b_{13}b_{23} & b_{13} \\ -b_{13}b_{23} & b_{23}^2 + \beta & -b_{23} \\ b_{13} & -b_{23} & 1 \end{bmatrix}$$

where  $0 \leq b_{23} < \kappa$ ,  $b_{13} = p(\kappa - b_{23})^{-1}$ , and

$$|\beta - [b_{23}(\kappa - b_{23}) + (q - b_{13})]| \leq 2[b_{23}(q - b_{13})(\kappa - b_{23})]^{1/2}.$$

For  $p > 0$  and  $\kappa q = p$ , we must look outside the class of symmetric positive definite matrices for which  $b_{12} = b_{13}b_{23}$ . If  $\kappa q = p$ , the matrix  $C$  in (4.48) takes the form

$$C = \begin{bmatrix} 2\kappa qb_{13} & \alpha - b_{13}(q - b_{13}) - \kappa qb_{23} & \kappa(q + b_{13}) - b_{12} \\ \alpha - b_{13}(q - b_{13}) - \kappa qb_{23} & 2(qb_{23} - b_{12}) & \beta - b_{23}(\kappa - b_{23}) - (q - b_{13}) \\ \kappa(q + b_{13}) - b_{12} & \beta - b_{23}(\kappa - b_{23}) - (q - b_{13}) & 2(\kappa - b_{23}) \end{bmatrix}$$

Now,  $C$  is positive semidefinite iff

- (1'')  $2\kappa qb_{13}, 2(qb_{23} - b_{12}), 2(\kappa - b_{23}) \geq 0$  ;
- (2'')  $4\kappa qb_{13}(\kappa - b_{23}) - [\kappa(q + b_{13}) - b_{12}]^2 \geq 0$  ;
- (3'')  $4\kappa qb_{13}(qb_{23} - b_{12}) - [\alpha - b_{13}(q - b_{13}) - \kappa qb_{23}]^2 \geq 0$  ;
- (4'')  $4(\kappa - b_{23})(qb_{23} - b_{12}) - [\beta - b_{23}(\kappa - b_{23}) - (q - b_{13})]^2 \geq 0$  ;
- (5'')  $\det C \geq 0$  .

Since  $\kappa, q, p > 0$ , we have  $b_{13} \geq 0$ ,  $qb_{23} - b_{12} \geq 0$ , and  $\kappa - b_{23} \geq 0$  from (1''). Also, by combining terms and completing the square, we obtain from (2'')

$$0 \leq -4\kappa b_{13}(qb_{23} - b_{12}) - [\kappa(q - b_{13}) - b_{12}]^2.$$

Necessarily,

$$b_{13}(qb_{23} - b_{12}) = 0 \quad \text{and} \quad b_{12} = \kappa(q - b_{13}).$$

If  $qb_{23} > b_{12}$ , then  $b_{13} = 0$  and hence  $b_{12} = \kappa q$ . This, however, leads to the contradiction

$$\kappa q = b_{12} < qb_{23} \leq \kappa q .$$

Therefore,

$$(4.53) \quad qb_{23} = b_{12} = \kappa(q - b_{13}) .$$

Immediately, from (3'') and (4'') we have

$$(4.54) \quad \alpha = b_{13}(q - b_{13}) + \kappa qb_{23} ,$$

$$\beta = b_{23}(\kappa - b_{23}) + (q - b_{13}) .$$

Furthermore, from (4.53) and (4.54), it follows that  $\det C = 0$ .

Since  $\alpha, \beta > 0$ , we use (4.54) to deduce that  $0 \leq b_{13} < q$  and  $0 < b_{23} = \kappa q^{-1}(q - b_{13}) \leq \kappa$ . From (4.53), we can express (4.54) in terms of  $b_{13}$  as

$$\alpha = (q - b_{13})(\kappa^2 + b_{13}) ,$$

$$\beta = (q - b_{13})(1 + \kappa^2 q^{-2} b_{13}) .$$

Hence, for  $p > 0$  and  $\kappa q = p$  the matrix  $B$  has the form

$$B = \begin{bmatrix} (q - b_{13})\kappa^2 + qb_{13} & -\kappa(q - b_{13}) & b_{13} \\ -\kappa(q - b_{13}) & (q - b_{13})(1 + \kappa^2 q^{-1}) & -\kappa q^{-1}(q - b_{13}) \\ b_{13} & -\kappa q^{-1}(q - b_{13}) & 1 \end{bmatrix}$$

and  $B$  is positive definite iff  $0 < b_{13} < q$ .

It appears that the procedure used in this chapter to obtain bounds can be extended to metaharmonic functions of higher order. Hopefully, this extension can be done for the general equation (4.4) without having to consider each value of  $n$  separately as we have done here for  $n = 2$  and  $n = 3$ . Furthermore, if the roots  $\xi_i$  of the characteristic equation (4.6) satisfy (4.7), it appears that an entirely similar procedure can be used to obtain bounds for  $\Delta^i u$ ,  $i = 1, 2, \dots, n-1$ , where  $u$  is a nonconstant solution of (4.4) such that

$$u = \Delta u = \dots = \Delta^{i-1} u = \Delta^{i+1} u = \dots = \Delta^{n-1} u = 0 \quad \text{on } \partial\Omega,$$

which could possibly improve those obtained by Chow and Dunninger [1].

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## VITA

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