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I am submitting herewith a dissertation written by John David Nichols entitled "Total Sequences in Topological Vector Spaces." I recommend that it be accepted in partial fulfillment of the requirements for the degree of Doctor of Philosophy, with a major in Mathematics.

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TOTAL SEQUENCES IN TOPOLOGICAL VECTOR SPACES

A Dissertation

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DEDICATION

This dissertation is dedicated to the author's wife, Edna.

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ABSTRACT

This work is concerned with the extension to more general topological vector spaces of two results concerning sequences of functions in $L^2[0,1]$.

The first result to be extended is due to J. Price and R. Zink and states that there exists a nonnegative, bounded, measurable function which converts a given sequence into a total sequence if and only if for each positive ϵ there exists a set S_ϵ whose measure exceeds $1 - \epsilon$ and such that the original sequence is total in $L^2(S_\epsilon)$. The second by A. Talalyan is that such a set exists if and only if the original sequence is total in measure.

In Chapter II we extend the first result by showing that in a Fréchet space or a complete p-normed space there is a scalar operator which converts a given sequence into a total sequence provided that there exists a nested, equicontinuous family of commuting projections which has the additional properties that the union of the ranges of these projections is dense and that each projected sequence is total in each projected space. Conversely, we show that such a family of projections exists in a space which possesses scalar operators provided that such an operator exists and is one-to-one.

In Chapter III we use the results of Chapter II to show that the Price-Zink result is valid not only in $L^2[0,1]$, but also in a larger class of spaces, called the function metric spaces. Included in these spaces are the classical L^p spaces, $0 < p < \infty$, the Orlicz spaces, the

normed Köthe spaces, and others. In addition, we show that the work of Talalyan can be extended to these spaces. Finally, we show that in certain function metric spaces a total sequence can have any finite subset and certain infinite subsets deleted without destroying totality.

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CHAPTER I

PRELIMINARIES

1.1. Introduction

It is assumed that the reader has a familiarity with basic concepts concerning topological vector spaces including resolutions of the identity and vector valued (here operator valued) integration. Standard references are Dunford and Schwartz [5], Köthe [10], Rudin [15], and Schaefer [18]. For completeness we have included some definitions and basic results in sections 1.3 and 1.4 and some deeper results in the appendix.

1.2. History of the Problem

In 1948 R. Boas and H. Pollard [3] wrote a short paper concerned with the possibility of finding a function m, called a multiplier, such that $\{m\phi_n\}_1^\infty$ is total (complete) in $L^2[a,b]$; i.e., the linear span of $\{m\phi_n\}_1^\infty$ is dense in $L^2[a,b]$. While Boas and Pollard were unable to find necessary and sufficient conditions for the existence of such a multiplier m, they did show that if $\{\phi_n\}_1^\infty$ is an orthonormal set which can be made total by the addition of a finite number of functions, then there does exist a multiplier m such that $\{m\phi_n\}_1^\infty$ is total. Thus if $\{\phi_n\}_1^\infty$ is a complete orthonormal set in $L^2[a,b]$, then for any k Boas and Pollard showed that there exists a multiplier m such that $\{m\phi_n\}_k^\infty$ is total. In addition, Boas and Pollard showed through an example that this necessary condition is not sufficient.

The search for necessary and sufficient conditions which guaranteed the existence of m such that $\{m\phi_n\}_1^\infty$ is total in $L^2[a,b]$ for an arbitrary sequence $\{\phi_n\}_1^\infty$ ended in 1965 when J. Price and R. Zink [14] showed the equivalence of the following conditions:

BP - there exists a bounded, measurable, nonnegative function m such that $\{m\phi_n\}_1^\infty$ is total in $L^2[0,1]$;

T - for each $\varepsilon > 0$, there exists a measurable set $S_\varepsilon \subseteq [0,1]$ whose measure exceeds $1 - \varepsilon$ and such that $\{\phi_n\}_1^\infty$ is total in $L^2(S_\varepsilon)$.

We are adhering to the notation of Price and Zink who labeled BP in honor of Boas and Pollard and T in honor of the Armenian mathematician A. Talalyan. Talalyan [20] showed in 1960 that condition T is equivalent to (again in the notation of Price and Zink):

M - $\{\phi_n\}_1^\infty$ is total in measure.

In addition to showing the equivalence of M and T, Talalyan also showed that if $\{\phi_n\}_1^\infty$ possesses property M in $L^2[0,1]$, then so does $\{\phi_n\}_k^\infty$ for any k; i.e., from a sequence which satisfies M, we may delete finitely many terms and still retain M. This result is very similar to the result of Boas and Pollard which states that from a total orthonormal sequence in $L^2[a,b]$, we may delete finitely many

terms and find a multiplier which restores totality. This material concerning deletions was culminated by Goffman and Waterman [7] who produced the same result as Talalyan in a much shorter and more elegant manner and also proved that from a sequence satisfying M certain infinite subsets may be deleted without destroying M.

With these results as background material, we observe that the Boas-Pollard question concerning the existence of a multiplier which converts a given sequence into a total sequence can naturally be raised in an arbitrary topological vector space X. To do this, we reconsider a sequence $\{\phi_n\}_1^\infty$ in $L^2[0,1]$ and a bounded, nonnegative, measurable multiplier m such that $\{m\phi_n\}_1^\infty$ is total. The map $f \mapsto mf$ defines a continuous, linear operator $A : L^2[0,1] \to L^2[0,1]$ since m is bounded. Moreover, since m is nonnegative, A is a positive operator on the Hilbert space $L^2[0,1]$. Being positive, A is self-adjoint and thus scalar (see sections 1.3 and 1.4). Hence the Boas-Pollard question can be restated for a sequence $\{\phi_n\}_1^\infty$ in an arbitrary topological vector space which possesses scalar operators by asking, "When does there exist a scalar operator A such that $\{A\phi_n\}_1^\infty$ is total?"

We propose to answer this question by restating condition T in suitable terms and showing that this restatement provides a necessary condition in certain spaces and a sufficient condition in other spaces for the existence of such an operator A. As a consequence of using an operator theoretic approach to the problem, the Price-Zink result can be shown to hold not only in $L^2[0,1]$, but also in $L^p[0,1]$ for $1 \leq p < \infty$ on the one hand, and $0 < p < \infty$ on the other.

While the $L^p[0,1]$ spaces form a natural class of spaces for examples for this work, they are not very far removed from $L^2[0,1]$ where the Price-Zink result is already known. We propose to create a much larger class of spaces for examples, which are called the function metric spaces. This class of spaces, which is a slight generalization of some work by Zaanen [23], includes not only the L^p spaces, but also the normed Köthe spaces, the Orlicz spaces, L^1_{loc}, and others. We propose to show that the work of Price and Zink and the work of Talalyan apply to these spaces. Finally, we will show that the work of Goffman and Waterman is also applicable to some of these spaces.

For more insight into the history of the problem, the reader is referred to a very informative article by J. Price [13]. This narrative in the American Mathematical Monthly discusses the above material and reviews related work concerning orthogonal sequences in $L^2[0,1]$.

1.3. Notation and Definitions

We wish now to establish certain notational conventions and to state a few definitions. The reader is referred to Appendices A.1 and A.2 where some of this material is covered in greater detail.

Unless otherwise stated, X is a complex, Hausdorff topological vector space and $\{\phi_n\}_1^\infty$ a sequence in X. The linear span of $\{\phi_n\}_1^\infty$ is denoted by Λ. The collection of continuous, linear operators on X is $L(X)$. We let Γ denote a linearly ordered set which possesses a cofinal sequence. Naturally, a sequence $\{\gamma_n\}_1^\infty$ is <u>cofinal</u> in Γ if

given $\gamma \in \Gamma$, there exists N such that $\gamma_n \geq \gamma$ for all $n \geq N$.

When X is a locally convex topological vector space, E is an equicontinuous resolution of the identity; i.e., E is a weakly countably additive map from a σ-algebra $\mathcal{F}$ on some set S into $L(X)$ such that

(1) $E(S) = I$;

(2) $E(\delta \cap \omega) = E(\delta)E(\omega)$;

(3) the image of E is an equicontinuous subset of $L(X)$.

From (2) we see that $E(\omega)$ is a projection for each $\omega \in \mathcal{F}$. The indicator function of ω is denoted by χ_ω. The collection of all bounded, $\mathcal{F}$-measurable, complex valued functions on S is denoted by M^∞.

We recall that a Fréchet space is a complete, locally convex, topological vector space whose topology is generated by a translation invariant metric, and that a p-normed space is a topological vector space whose topology is generated by a translation invariant metric which has the additional property that $d(ax,0) = |a|^p d(x,0)$. Since any normed space is a p-normed space, we see that p-normed spaces need not be complete. Moreover, since $L^p[0,1]$, $0 < p < 1$, is a p-normed space, we see p-normed spaces need not be locally convex.

When X is a locally convex topological vector space and E is a resolution of the identity, an operator $A \in L(X)$ is called scalar if $A = \int_S f(\lambda)E(d\lambda)$ for some $f \in M^\infty$ (see Appendix A.2). This operator A is actually the strong (pointwise) limit of a sequence of finite sums of scalar multiples of projections. When X is a p-normed space (which need not be locally convex), we say that $A \in L(X)$ is scalar if there exists an equicontinuous family $\mathcal{P}$ of commutative projections

such that A lies in the strong closure of the linear hull of P. This definition is motivated by the observation above and the material in Appendix A.3.

We need some basic material concerning spectra and adjoints. While these concepts can be defined in a more general setting, we need them only for the Banach spaces. Let X and Y be Banach spaces with duals X^* and Y^*. Let $T : X \to Y$ be continuous and linear. It is well-known that there exists a unique map $T^* : Y^* \to X^*$ which satisfies

$$\langle Tx, y^* \rangle = \langle x, T^* y^* \rangle$$

for all $x \in X$ and $y^* \in Y^*$. This map is called the adjoint of T. It is well known that if X is a Hilbert space, then $X = X^*$. Therefore, when X is a Hilbert space and $T \in L(X)$, we conclude that $T^* \in L(X)$ also. If, in this situation, $T = T^*$, we say T is self-adjoint. If $TT^* = T^*T$, we say T is normal. When X is a Banach space and $T \in L(X)$, the spectrum of T is $\sigma(T) = \{\lambda \in \mathbb{C} \mid T - \lambda I \text{ has no inverse in } L(X)\}$.

1.4. Some Known Results

This section contains some well-known results. They are included here for reference purposes. Some of this material appears in the appendix in greater detail. We begin with some propositions concerning scalar operators.

1.4.1. Proposition. Let X be a complex Hilbert space and let $T \in L(X)$. The following are equivalent:

(a) $\langle Tx, x \rangle \geq 0$ for all x in X;

(b) $T = T^*$ and $\sigma(T) \subseteq [0,\infty)$.

Proof. See Rudin [15, p. 313]. ///

If T satisfies either (a) or (b), then we say that T is positive. We write $T \geq 0$ when T is positive.

1.4.2. Proposition. Let $S, T \in L(X)$ where X is a complex Hilbert space and let $TS = ST$. If $T \geq 0$ and $S \geq 0$, then $T + S \geq 0$ and $ST \geq 0$.

Proof. See Rudin [15, p. 324]. ///

1.4.3. Proposition. Let X be a complex Banach space and $T \in L(X)$. Then $\sigma(T)$ is a nonempty compact set.

Proof. See Rudin [15, p. 235]. ///

1.4.4. Proposition. Let X be a complex Hilbert space and $A \in L(X)$ be normal. Then there exists a unique resolution of the identity E such that $A = \int_{\sigma(A)} \lambda E(d\lambda)$.

Proof. See Rudin [15, p. 308]. ///

1.4.5. Proposition. Let E be an equicontinuous resolution of the identity from $(S,\mathcal{F})$ into $L(X)$ where X is a real, locally convex, sequentially complete, topological vector space. Then there exists an algebraic homomorphism $f \longmapsto \int_S f(\lambda)E(d\lambda)$ from M^∞ into $L(X)$. (For notation, see section 1.3.)

Proof. See Walsh [21] or Appendix A.2. ///

1.4.6. Proposition. Let E be as in Proposition 1.4.5, but let X be a complex, barreled, locally convex, sequentially complete topological vector space. Then there exists an algebraic homomorphism $f \longmapsto \int_S f(\lambda)E(d\lambda)$ from M^∞ into $L(X)$.

Proof. See Ionescu Tulcea [9]. ///

1.4.7. Proposition. Let X be a locally convex topological vector space with dual X^*. Let $f \in M^\infty$ and let E be an equicontinuous resolution of the identity. Then

$$|\langle \int_S f(\lambda)E(d\lambda)x, x^* \rangle| \leq 4||f||_\infty \sup_{\delta \in F} |\langle E(\delta)x, x^* \rangle|$$

for any $x \in X$ and $x^* \in X^*$.

Proof. See Walsh [21]. ///

In section 2.4 of Chapter II we will need what is sometimes referred to as the Bipolar Theorem. In a locally convex space X with dual X^*, the polar of a circled set M is

$$M^0 = \{x^* \in X^* | \quad |\langle x, x^* \rangle| \leq 1 \text{ for all } x \in M\}.$$

The bipolar of M is

$$M^{00} = \{x \in X \mid |\langle x, x^* \rangle| \leq 1 \text{ for all } x^* \in M^0\}.$$

Clearly, $M \subset M^{00}$.

1.4.8. Proposition. Let M be convex and weakly closed. Let $0 \in M$. Then $M = M^{00}$.

Proof. See Köthe [10, p. 246]. ///

We will also need a monotone convergence theorem for ordered topological vector spaces (see Appendix A.2 for definitions).

1.4.9. Proposition. (The Monotone Convergence Theorem of Dini-Schaefer). Let X be a locally convex topological vector space and let $\leq$ be an order on X such that the associated cone is normal. If $\{x_n\}_1^\infty$ is a monotone sequence in X and if x_n converges to x weakly, then x_n converges to x.

Proof. See Schaefer [16]. ///

CHAPTER II

THE MAIN RESULTS

2.1. Introduction

As mentioned in section 1.2, Price and Zink [14] showed that there exists a bounded, nonnegative, measurable multiplier m such that $\{m\phi_n\}_1^\infty$ is total in $L^2[0,1]$ if and only if for each $\varepsilon > 0$ there exists a set S_ε whose measure exceeds $1 - \varepsilon$ and such that $\{\phi_n\}_1^\infty$ is total in $L^2(S_\varepsilon)$. This result answered a question posed by Boas and Pollard [3] concerning the existence of a multiplier m such that $\{m\phi_n\}_1^\infty$ becomes total.

As was observed in Chapter I, the Boas-Pollard question can be raised concerning a given sequence $\{\phi_n\}_1^\infty$ of vectors in an arbitrary topological vector space X. In this setting we ask, "When does there exist a continuous, linear, scalar operator A such that $\{A\phi_n\}_1^\infty$ is total in X?" We will show that the necessary conditions in the Price-Zink result gives rise to a nested equicontinuous family of commuting projections which in addition has the properties that the union of their ranges is dense in X and that the projected sequence is total in each projected space. On the other hand, we will show that the sufficient condition of Price and Zink gives rise to a continuous, linear, scalar operator A such that $\{A\phi_n\}_1^\infty$ is total in X. Further, we will show that in either a Fréchet space or a p-normed space the existence of such a family of projections implies the existence of such an operator, and

in spaces where scalar operators exist we will show that such an operator which is also 1-1 implies the existence of such a family of projections.

2.2. Development of BP^* and T^*

In order to formulate and prove the main results, we first need to examine the Price-Zink result [14] more carefully. They showed the equivalence of the following:

BP - there exists a nonnegative, bounded, measurable multiplier m such that $\{m\phi_n\}_1^\infty$ is total in $L^2[0,1]$;

T - for every $\varepsilon > 0$, there exists a set $S_\varepsilon \subseteq [0,1]$ whose measure exceeds $1 - \varepsilon$ and such that $\{\phi_n\}_1^\infty$ is total in $L^2(S_\varepsilon)$.

Without loss of generality, the sets S_ε can be assumed to be nested since the work of Price and Zink shows that we may take $S_\varepsilon = m^{-1}\{x | x > \varepsilon\}$.

Examining condition BP more closely, we see that the map $f \longmapsto mf$ defines a continuous linear operator from $L^2[0,1]$ into $L^2[0,1]$ since m is bounded and measurable. Moreover, since m is nonnegative, it follows quickly that this operator is a positive operator on the Hilbert space $L^2[0,1]$. Hence the operator is scalar (normal). Thus in a topological vector space X which possesses scalar operators we restate BP as:

BP^* - there exists a scalar operator A such that $\{A\phi_n\}_1^\infty$ is total in X.

To extend condition T to a more general setting, we consider the space $L^2(S_\varepsilon)$ to be a subspace of $L^2[0,1]$ by identifying $f \in L^2(S_\varepsilon)$ with $f' \in L^2[0,1]$ where

$$f'(x) = \begin{cases} f(x) & x \in S_\varepsilon \\ 0 & x \notin S_\varepsilon \end{cases} .$$

Now for each $\varepsilon > 0$, $L^2(S_\varepsilon)$ can easily be seen to be a closed subspace of $L^2[0,1]$. Since $L^2[0,1]$ is a Hilbert space, there is an orthogonal projection P_ε of $L^2[0,1]$ onto $L^2(S_\varepsilon)$ for each $\varepsilon > 0$. The family of projections $\{P_\varepsilon\}_{\varepsilon>0}$ forms a nested equicontinuous family of commuting projections since each P_ε has operator norm 1 and since the sets S_ε, and hence the spaces $L^2(S_\varepsilon)$, are nested. Moreover, since $\bigcup_{\varepsilon>0} S_\varepsilon$ has full measure, $\bigcup_{\varepsilon>0} L^2(S_\varepsilon) = \bigcup_{\varepsilon>0} P_\varepsilon L^2[0,1]$ is dense in $L^2[0,1]$. Finally, under condition T, the sequence $\{\phi_n\}_1^\infty$ is total in $P_\varepsilon L^2[0,1]$ for each $\varepsilon > 0$. Hence each projected sequence is total in each projected space. Therefore in an arbitrary topological vector space X, we extend condition T to:

T^* - there exists a nested, equicontinuous family $\{P_\gamma\}_{\gamma\in\Gamma} = P$ of commuting projections such that

(1) $\bigcup_{\gamma\in\Gamma} P_\gamma X$ is dense in X;

(2) for each $\gamma \in \Gamma$, $\{P_\gamma\phi_n\}_1^\infty$ is total in $P_\gamma X$.

Whenever $\{\phi_n\}_1^\infty$ is a sequence in a topological vector space X for which there exists a nested, equicontinuous family P of commuting projections such that (1) and (2) in T^* hold, we say that *P satisfies T^* with respect to $\{\phi_n\}_1^\infty$*.

2.3. Statement of the Results

We will prove the following two main theorems.

2.3.1. Theorem. Let X be either a Fréchet space or a complete p-normed space. Then T^* implies BP^*.

2.3.2. Theorem. Let X be a locally convex topological vector space in which scalar operators exist. If the operator in BP^* is 1-1, then BP^* implies T^*.

The proofs of these theorems will be delayed until section 2.5.

2.4. Preparatory Lemmas

Before proving our main results, we state and prove several necessary lemmas.

2.4.1. Lemma. Let P be as in T^*. Then for $y \in X$, $\lim_\gamma P_\gamma y = y$.

Proof. Fix $y \in X$ and let U be a neighborhood of zero in X. Pick a circled neighborhood V of zero in X such that $V + V \subseteq U$. By equicontinuity of P, pick a circled neighborhood W of zero in X such that $P_\gamma W \subseteq V$ for every γ. Now since $\bigcup_{\gamma\in\Gamma} P_\gamma X$ is dense in X, there exists $x \in X$ such that $P_\gamma x - y \in W$. Hence $P_\gamma x - P_\gamma y \in V$. Therefore $P_\gamma y - y = P_\gamma y - P_\gamma x + P_\gamma x - y \in V + W \subseteq V + V \subseteq U$.

Thus for every neighborhood U of 0 there exists an index γ such that $P_\gamma y - y \in U$. To see that $\lim_\gamma P_\gamma y = y$, we again fix a neighborhood U of 0 and find circled neighborhoods V and W of 0 such that $W \subseteq V$, $V + V \subseteq U$, and $P_\gamma W \subseteq V$ for every γ. We pick γ such that $P_\gamma y - y \in W$. Now for $\beta \geq \gamma$, $P_\gamma y \in P_\beta X$. Hence $P_\beta P_\gamma y = P_\gamma y$. Therefore it follows that for $\beta \geq \gamma$, $P_\beta y - y = P_\beta y - P_\gamma y + P_\gamma y - y = P_\beta(y - P_\gamma y) + P_\gamma y - y$ $\in P_\beta W + W \subseteq V + V \subseteq U$. Hence $\lim P_\gamma y = y$. ///

2.4.2. Lemma. Let X be locally convex and let $\mathcal{P}$ be as above. Let $\{P_n\}_1^\infty \subseteq \mathcal{P}$ and $c_n \in [0,1]$. Let $A_n = P_n + c_n(I - P_n)$ and $\overline{A}_n = A_n \ldots A_1$. Then $\{\overline{A}_n\}_1^\infty$ is equicontinuous.

Proof. We first claim that $\overline{A}_n = (\prod_1^n c_i)I + (1 - c_1)P_1 + \sum_{i=2}^n (\prod_{j=1}^{i-1} c_j - \prod_{j=1}^{i} c_j)P_i$. For n = 1 this is clearly true since $\overline{A}_1 = A_1 = P_1 + c_1(I - P_1) = c_1 I + (1 - c_1)P_1$. Suppose the claim is true for n. Since $\mathcal{P}$ is nested, $j \leq k$ implies $P_j P_k = P_k P_j = P_j$. Hence it follows that

$$\overline{A}_{n+1} = A_{n+1}\overline{A}_n = A_{n+1}\left((\prod_1^n c_i)I + (1 - c_1)P_1 + \sum_{i=2}^n (\prod_{j=1}^{i-1} c_j - \prod_{j=1}^{i} c_j)P_i\right)$$

$$= (\prod_1^n c_i)P_{n+1} + (1 - c_1)P_{n+1}P_1 + \sum_{i=2}^n (\prod_{j=1}^{i-1} c_j - \prod_{j=1}^{i} c_j)P_{n+1}P_i$$

$$+ c_{n+1}(\prod_1^n c_i)I + c_{n+1}(1 - c_1)P_1$$

$$+ c_{n+1}\sum_{i=2}^n (\prod_{j=1}^{i-1} c_j - \prod_{j=1}^{i} c_j)P_i - c_{n+1}(\prod_1^n c_i)P_{n+1}$$

$$- c_{n+1}(1 - c_1)P_{n+1}P_1 - c_{n+1}\sum_{i=2}^{n}\left(\prod_{j=1}^{i-1} c_j - \prod_{j=1}^{i} c_j\right)P_{n+1}P_i$$

$$= \left(\prod_{1}^{n+1} c_i\right)I + (1 - c_1)P_1 + \left(\prod_{1}^{n} c_i\right)P_{n+1}$$

$$+ \sum_{i=2}^{n}\left(\prod_{j=1}^{i-1} c_j - \prod_{j=1}^{i} c_j\right)P_i - \left(\prod_{1}^{n+1} c_i\right)P_{n+1}$$

$$= \left(\prod_{1}^{n+1} c_i\right)I + (1 - c_1)P_1 + \sum_{i=2}^{n+1}\left(\prod_{j=1}^{i-1} c_j - \prod_{j=1}^{i} c_j\right)P_i.$$

Thus the claim holds for all n.

To see equicontinuity of $\{\overline{A}_n\}_1^\infty$, fix a convex, circled neighborhood U of 0. Pick a convex, circled neighborhood V of 0 such that $P_\gamma V \subseteq U$ for all $\gamma \in \Gamma$ and such that $V \subseteq U$. Then for any n

$$\overline{A}_n V \subseteq \left(\prod_{1}^{n} c_i\right)V + (1 - c_1)P_1 V + \sum_{i=2}^{n}\left(\prod_{j=1}^{i-1} c_j - \prod_{j=1}^{i} c_j\right)P_i V$$

$$\subseteq \left[\prod_{1}^{n} c_i + (1 - c_1) + \sum_{i=2}^{n}\left(\prod_{j=1}^{i-1} c_j - \prod_{j=1}^{i} c_j\right)\right]U$$

by equicontinuity of P and convexity of U. Thus $\overline{A}_n V \subseteq U$ for any n since the above constant which multiplies U is 1. Hence $\{\overline{A}_n\}_1^\infty$ is equicontinuous. ///

The above lemma is needed in the Frechet space setting. The next lemma is number theoretic in nature and will be used in Lemma 2.4.4.

2.4.3. Lemma. Let $\{c_n\}_1^\infty \subseteq [0,1]$ and let $c_n \geq c_{n+1}$. Let p be positive and let

$$K_n = \prod_1^n c_i^p + (1 - c_1)^p + \sum_{i=2}^n \left(\prod_{j=1}^{i-1} c_j - \prod_{j=1}^{i} c_j\right)^p.$$

Then there exists M such that $K_n \leq M$ for all n.

Proof. If $c_n = 1$ for all n, then $K_n = 1$ for all n. Hence M can be 1. Without loss of generality, we may assume $c_1 < 1$. Then $\sum_1^\infty c_1^{pi}$ is finite. Expanding the right side of the term above we see that

$$K_n \leq c_1^p + (1 - c_1)^p + \sum_{i=2}^n \left(\prod_{j=1}^{i-1} c_j\right)^p (1 - c_i)^p$$

$$\leq c_1^p + (1 - c_1)^p + \sum_{i=1}^{n-1} c_1^{pi}.$$

Hence if we let $M = c_1^p + (1 - c_1)^p + \sum_{i=1}^\infty c_1^{pi}$, we see that $K_n \leq M$ for all n. ///

Our next lemma will also be used in the p-normed setting. We recall that p-normed spaces are metrizable (see section 1.3).

2.4.4. Lemma. Let X be a p-normed space with metric d, $0 < p \leq 1$. Let $\{\overline{A}_n\}_1^\infty$ be as in Lemma 2.4.2 and let $\{c_n\}_1^\infty$ be as in Lemma 2.4.3. Then $\{\overline{A}_n\}_1^\infty$ is equicontinuous.

Proof. Let $\varepsilon > 0$ be given and let M be as in Lemma 2.4.3. Pick $\delta < \frac{\varepsilon}{M}$ such that $P_n B_\delta \subseteq B_{\varepsilon/M}$ for every n, where B_δ and $B_{\varepsilon/M}$ are the

balls centered at 0 with radii δ and ε/M respectively.

Now for $x \in B_\delta$,

$$d(\bar{A}_n x,0) \leq d((\prod_1^n c_i)x, 0) + d((1 - c_1)P_1 x,0)$$

$$+ \sum_{i=2}^{n} d((\prod_{j=1}^{i} c_j - \prod_{j=1}^{i-1} c_j)P_i x,0)$$

$$= (\prod_1^n c_i^p)d(x,0) + (1 - c_1)^p d(P_1 x,0)$$

$$+ \sum_{i=2}^{n} [\prod_{j=1}^{i} c_j - \prod_{j=1}^{i-1} c_j]^p \, d(P_i x,0)$$

$$\leq (\prod_{i=1}^{n} c_i)^p \delta + (1 - c_1)^p \frac{\varepsilon}{M} + \sum_{i=2}^{n} [\prod_{j=1}^{i} c_j - \prod_{j=1}^{i-1} c_j]^p \frac{\varepsilon}{M}$$

$$\leq K_n \frac{\varepsilon}{M} \leq \varepsilon.$$

Thus $\{\bar{A}_n\}_1^\infty$ is equicontinuous since $\bar{A}_n B_\delta \subset B_\varepsilon$ for all n. ///

2.4.5. Lemma. Suppose P satisfies T^* with respect to $\{\phi_n\}_1^\infty$. Then P satisfies T^* with respect to $\{A_\alpha \phi_n\}_1^\infty$, where $A_\alpha = P_\alpha + c_\alpha(I - P_\alpha)$ and $c_\alpha \neq 0$.

Proof. Since P is still a nested, equicontinuous family of commuting projections which has the property that $\bigcup_{\gamma \in \Gamma} P_\gamma X$ is dense in X, we need only show that $\{P_\gamma A_\alpha \phi_n\}_1^\infty$ is total in $P_\gamma X$ for all $\gamma \in \Gamma$.

To see this, we first claim that for each $\gamma \in \Gamma$ the operator A_α maps $P_\gamma X$ onto $P_\gamma X$. Since P is a family of commuting operators,

it follows that $A_\alpha P_\gamma = P_\gamma A_\alpha$. Hence A_α maps $P_\gamma X$ into $P_\gamma X$ for each γ. To see that A_α maps $P_\gamma X$ onto $P_\gamma X$, fix $x \in P_\gamma x$. If $\gamma \leq \alpha$, then $P_\alpha x = x$. Thus $A_\alpha x = P_\alpha x + c_\alpha(I - P_\alpha)x = x + c_\alpha x - c_\alpha x = x$. If, on the other hand, $\gamma > \alpha$, then for $z = P_\alpha x + c_\alpha^{-1}(I - P_\alpha)x$, we see that

$$A_\alpha z = A_\alpha P_\alpha x + c_\alpha^{-1} A_\alpha x - c_\alpha^{-1} A_\alpha P_\alpha x$$

$$= P_\alpha x + c_\alpha P_\alpha x - c_\alpha P_\alpha x + c_\alpha^{-1} P_\alpha x + x - P_\alpha x - c_\alpha^{-1} P_\alpha x - P_\alpha x + P_\alpha x$$

$$= x.$$

Hence z maps onto x. But since $\gamma > \alpha$, $P_\alpha x \in P_\gamma X$. Hence $z \in P_\gamma X$. Thus A_α does indeed map $P_\gamma X$ onto $P_\gamma X$.

To finish the proof, we need only show that $\{P_\gamma A_\alpha \phi_n\}_1^\infty$ is total in $P_\gamma X$ for each $\gamma \in \Gamma$. To see this, fix $\gamma \in \Gamma$, $x \in P_\gamma X$, and a neighborhood U of 0. By continuity of A_α, pick a neighborhood W of 0 such that $A_\alpha W \subseteq U$. Pick also $y \in P_\gamma X$ such that $A_\alpha y = x$. Since P satisfies T^* with respect to $\{\phi_n\}_1^\infty$, there exists $g \in \Lambda$ such that $y - P_\gamma g \in W$. Hence $A_\alpha y - A_\alpha P_\gamma g = x - P_\gamma A_\alpha g \in U$. Thus $\{P_\gamma A_\alpha \phi_n\}_1^\infty$ is total in $P_\gamma X$. ///

The previous lemmas are needed for Theorem 2.3.1. We need the following lemmas for Theorem 2.3.2. We begin by recalling that a scalar operator A on a locally convex space X is an operator which can be written in the form $A = \int_S f(\lambda)E(d\lambda)$ where f is a bounded, measurable, complex valued function on S and $E : F \to L(X)$ is a resolution of the identity (see section 1.3 and Appendix A.2).

2.4.6. Lemma. Let X be a space which possesses scalar operators. Let $A = \int_S f(\lambda)E(d\lambda)$ be a scalar operator. Then the null space of A is $E(f^{-1}\{0\})X$.

Proof. Let $\omega_0 = f^{-1}\{0\}$. Now $f(\lambda)\chi_{\omega_0}(\lambda) = 0$ for all $\lambda \in S$. Hence $\int_S f(\lambda)\chi_{\omega_0}(\lambda)E(d\lambda) = 0$. Thus $\int_S f(\lambda)E(d\lambda) \int_S \chi_{\omega_0}(\lambda)E(d\lambda) = 0$, since the map $f \longmapsto \int_S f(\lambda)E(d\lambda)$ is an algebraic homomorphism from F into $L(X)$ (see Appendix A.2). But $\int_S \chi_{\omega_0}(\lambda)E(d\lambda) = E(\omega_0)$. Hence it follows that

$$AE(\omega_0) = 0. \tag{2.1}$$

Now if $x \in E(\omega_0)X$, then $E(\omega_0)x = x$. Hence by (2.1), $Ax = AE(\omega_0)x = 0$. Therefore $E(\omega_0)X$ lies in the null space of A.

Conversely, let $\omega_n = \{\lambda \in S \mid |f(\lambda)| \geq \frac{1}{n}\}$. Define f_n by

$$f_n(\lambda) = \begin{cases} \frac{1}{f(\lambda)} & \lambda \in \omega_n \\ 0 & \lambda \notin \omega_n \end{cases}.$$

Now $f_n(\lambda)f(\lambda) = \chi_{\omega_n}(\lambda)$. Hence

$$\int_S f_n(\lambda)E(d\lambda) \int_S f(\lambda)E(d\lambda) = \int_S \chi_{\omega_n}(\lambda)E(d\lambda).$$

Thus $\int_S f_n(\lambda)E(d\lambda)Ax = E(\omega_n)x$. Now if $Ax = 0$, then $E(\omega_n)x = 0$. Since $E(\cdot)x$ is a vector valued measure (see Appendix A.2) and since $S = \omega_0 \dot\cup (\dot\bigcup_1^\infty \omega_n)$ ($\dot\cup$ denotes a disjoint union), it now follows that

$$x = E(S)x = E(\omega_0)x + E(\bigcup_1^\infty \omega_n)x$$

$$= E(\omega_0)x + \lim_{k\to\infty} E(\bigcup_1^k \omega_n)x = E(\omega_0)x + 0.$$

Thus if $Ax = 0$, then $x = E(\omega_0)x$. Hence the null space of A lies in $E(\omega_0)X$. ///

2.4.7. Lemma. (Walsh [21]). Let E be an equicontinuous resolution of the identity (see section 1.3) and let X be a locally convex topological vector space possessing scalar operators. Then $\{\int_S f(\lambda)E(d\lambda)$ where $f \in M^\infty$ and $||f||_\infty \leq 1\}$ is a family of equicontinuous operators in $L(X)$.

Proof. Let W be a closed, convex, circled neighborhood of 0. Pick a closed, convex, circled neighborhood V of 0 such that $E(\omega) \subset \frac{1}{4} W$ for all $\omega \in F$. Now if $x \in V$ and $y^* \in W^0$, the polar of W, and if $||f||_\infty \leq 1$, then by Proposition 1.4.7,

$$|\langle \int_S f(\lambda)E(d\lambda)x, y^* \rangle| \leq 4||f||_\infty \sup_{\omega\in F} |\langle E(\omega)x, y^* \rangle|$$

$$\leq 1.$$

Thus $\int_S f(\lambda)E(d\lambda)x \in W^{00}$. Hence $\left(\int_S f(\lambda)E(d\lambda)\right) V \subseteq W^{00} \subset W$ (recall Proposition 1.4.8) for all $f \in M^\infty$ which satisfy $||f||_\infty \leq 1$. Thus this class of operators is indeed equicontinuous. ///

2.4.8. Lemma. (Walsh [21]). Let E be an equicontinuous resolution of the identity and X be a locally convex topological vector space in which scalar operators exist. Let $\{V_\alpha\}$ be a neighborhood base of 0 which consists of circled, convex sets. Then there exists another neighborhood base $\mathcal{U} = \{U_\alpha\}$ such that

(1) $\mathcal{U}$ generates the same topology as $\{V_\alpha\}$;

(2) $|f(\lambda)| \leq |g(\lambda)|$ for all λ implies $[\int_S f(\lambda)E(d\lambda)]U_\alpha \subseteq [\int_S g(\lambda)E(d\lambda)]U_\alpha$ for each α;

(3) U_α is convex and circled for each α.

Proof. For each α define

$$U_\alpha = \bigcap_{\{f\in M^\infty \mid ||f||_\infty \leq 1\}} [\int_S f(\lambda)E(d\lambda)]^{-1}V_\alpha.$$

Since V_α is convex and circled and $\int_S f(\lambda)E(d\lambda)$ is linear, it follows that U_α is circled and convex. So (3) holds.

Now since $||\chi_S||_\infty = 1$ and since $[\int_S \chi_S(\lambda)E(d\lambda)]^{-1} V_\alpha = I^{-1}V_\alpha = V_\alpha$, we see that $U_\alpha \subseteq V_\alpha$ for each α. Hence the topology generated by $\mathcal{U}$ is finer than the original topology. However, since $\{\int_S f(\lambda)E(d\lambda) \mid f \in M^\infty \text{ and } ||f||_\infty \leq 1\}$ is equicontinuous by Lemma 2.4.7, it follows easily that each U_α contains an open set (see Schaefer [18, p. 83]). Thus the original topology is finer than the topology generated by $\mathcal{U}$. Hence, the topologies agree, and (1) also holds.

To see that (2) holds, define

$$h(\lambda) = \begin{cases} 0 & g(\lambda) = 0 \\ \dfrac{f(\lambda)}{g(\lambda)} & g(\lambda) \neq 0. \end{cases}$$

Then $||h||_\infty \leq 1$. Hence $[\int_S h(\lambda)E(d\lambda)]U_\alpha \subset U_\alpha$ for each α by definition of U_α. Multiplying by $\int_S g(\lambda)E(d\lambda)$ and recalling that $f \longmapsto \int_S f(\lambda)E(d\lambda)$ is an algebraic homomorphism from M^∞ into $L(X)$, we see that $[\int_S g(\lambda)h(\lambda)E(d\lambda)]U_\alpha \subset [\int_S g(\lambda)E(d\lambda)]U_\alpha$. But $g(\lambda)h(\lambda) = f(\lambda)$ for all λ; hence (2) follows readily. ///

2.4.9. Lemma. Let E, X, and $\mathcal{U}$ be as above. Let $A = \int_S f(\lambda)E(d\lambda)$ be scalar. For each $\varepsilon > 0$, let $T_\varepsilon = \{x \in X \mid \varepsilon x \in U$ whenever $Ax \in U$ and $U \in \mathcal{U}\}$, and let $\omega_\varepsilon = \{\lambda \in S \mid |f(\lambda)| \geq \varepsilon\}$. Then $E(\omega_\varepsilon)X \subseteq T_\varepsilon$.

Proof. Fix $x \in E(\omega_\varepsilon)X$. Then $E(\omega_\varepsilon)x = x$ since $E(\omega_\varepsilon)$ is a projection (see section 1.3). Suppose that $Ax \in U$ for some $U \in \mathcal{U}$. Then

$$\begin{aligned} Ax &= AE(\omega_\varepsilon)x \\ &= [\textstyle\int_S f(\lambda)E(d\lambda) \int_S \chi_{\omega_\varepsilon}(\lambda)E(d\lambda)]x \\ &= [\textstyle\int_S f(\lambda)\chi_{\omega_\varepsilon}(\lambda)E(d\lambda)]x \in U. \end{aligned}$$

But $|f(\lambda)\chi_{\omega_\varepsilon}(\lambda)| \geq \varepsilon\chi_{\omega_\varepsilon}(\lambda)$ for all λ. Hence by Lemma 2.4.8, $[\int_S \varepsilon\, \chi_{\omega_\varepsilon}(\lambda)E(d\lambda)]x \in U$. Thus $\varepsilon E(\omega_\varepsilon)x \in U$. But since $E(\omega_\varepsilon)x = x$, it now follows that $x \in T_\varepsilon$. Hence $E(\omega_\varepsilon)X \subseteq T_\varepsilon$. ///

2.5. Proof of the Main Results

Before proceeding with the proof of Theorem 2.3.1, it is worth noting that if either BP^* or T^* holds, then X is separable. This is clearly true if BP^* holds. On the other hand, since Γ possesses a cofinal sequence, and since $\bigcup_{\gamma\in\Gamma} P_\gamma X$ is dense, and since each $P_\gamma X$ is separable by property (2) of T^*, we conclude that T^* also implies separability.

Proof of Theorem 2.3.1. Let $\{y_n\}_1^\infty$ be a dense subset of X in which each element appears infinitely often. We claim it suffices to construct $A \in L(X)$, $\{g_n\}_1^\infty \subseteq \Lambda$, and $\delta_n \downarrow 0$ such that $d(y_n, Ag_n) \leq \delta_n$, where d is the metric on X. (See section 1.3 for notational conventions.) For if we can construct such objects, then given $\varepsilon > 0$ and $x \in X$, pick $y_k \in \Lambda$ such that $d(y_k, x) < \varepsilon/2$. Now since $\delta_n \downarrow 0$, there exists k' such that $\delta_{k'} < \varepsilon/2$. Moreover, since y_k appears infinitely often in $\{y_n\}_1^\infty$, it follows that there exists $k'' \geq k'$ such that $y_{k''} = y_k$. By our hypothesized construction, there exists $g_{k''} \in \Lambda$ such that $d(y_{k''}, Ag_{k''}) \leq \delta_{k''}$. Hence

$$d(x, Ag_{k''}) \leq d(x, y_{k''}) + d(y_{k''}, Ag_{k''})$$

$$\leq \varepsilon/2 + \delta_{k''} \leq \varepsilon.$$

Thus $\{A\phi_n\}_1^\infty$ is indeed total in X.

To construct $A \in L(X)$, $\{g_n\}_1^\infty \subset \Lambda$, and $\delta_n \downarrow 0$, pick $\{\varepsilon_n\}_1^\infty$, a positive sequence such that $\sum_1^\infty \varepsilon_n$ is finite. Since by Lemma 2.4.1

$\lim_{\alpha} P_\alpha y_1 = y_1$, there is an index N_1 so large that $d(y_1, P_{N_1} y_1) \leq \varepsilon_1/6$. But since $\{P_{N_1}\phi_n\}_1^\infty$ is total in $P_{N_1}X$, there exists $g_1 \in \Lambda$ such that $d(P_{N_1}y_1, P_{N_1}g_1) \leq \frac{\varepsilon_1}{6}$. Let $A_1 = P_{N_1} + c_1(I - P_{N_1})$, where $c_1 \leq 1$ is so small that $d(c_1 g_1, c_1 P_{N_1} g_1) \leq \frac{\varepsilon_1}{6}$. By continuity of $d : X \times X \to \mathbb{R}$ such a c_1 exists. Thus we see that

$$d(y_1, A_1 g_1) = d(y_1, P_{N_1} g_1 + c_1 g_1 - c_1 P_{N_1} g_1)$$

$$\leq d(y_1, P_{N_1} y_1) + d(P_{N_1} y_1, P_{N_1} g_1) + d(c_1 P_{N_1} g_1, c_1 g_1)$$

$$\leq \frac{\varepsilon_1}{6} + \frac{\varepsilon_1}{6} + \frac{\varepsilon_1}{6}.$$

Hence

$$d(y_1, A_1 g_1) \leq \frac{\varepsilon_1}{2} \tag{2.2}$$

By Lemma 2.4.1, $\lim_{\alpha} P_\alpha(y_1 - A_1 g_1) = y_1 - A_1 g_1$ and $\lim_{\alpha} P_\alpha y_2 = y_2$. Hence there exists an index $N_2 > N_1$ so large that

$$d(P_{N_2} y_1 - P_{N_2} A_1 g_1, y_1 - A_1 g_1) \leq \frac{\varepsilon_1}{6} \tag{2.3}$$

and

$$d(P_{N_2} y_1, y_1) \leq \frac{\varepsilon_2}{6} \tag{2.4}$$

and

$$d(P_{N_2} y_2, y_2) \leq \frac{\varepsilon_2}{6}. \tag{2.5}$$

Moreover, by Lemma 2.4.5, P satisfies T^* with respect to $\{A_1\phi_n\}_1^\infty$. Hence there exists $g_2 \in \Lambda$ such that

$$d(P_{N_2} y_2, P_{N_2} A_1 g_2) \leq \frac{\varepsilon_2}{6}. \tag{2.6}$$

Pick now $c_2 \in (0, c_1]$ such that

$$d(c_2 A_1 g_1, c_2 P_{N_2} A_1 g_1) \leq \frac{\varepsilon_2}{6} \tag{2.7}$$

and

$$d(c_2 A_1 g_2, c_2 P_{N_2} A_1 g_2) \leq \frac{\varepsilon_2}{6}. \tag{2.8}$$

Let $A_2 = P_{N_2} + c_2(I - P_{N_2})$.

Now by (2.2) and (2.3), it follows that

$$d(P_{N_2} y_1, P_{N_2} A_1 g_1) = d(P_{N_2} y_1 - P_{N_2} A_1 g_1, 0)$$

$$\leq d(P_{N_2} y_1 - P_{N_2} A_1 g_1, y_1 - A_1 g_1) + d(y_1 - A_1 g_1, 0)$$

$$\leq \frac{\varepsilon_1}{6} + \frac{\varepsilon_1}{2} \leq \frac{3\varepsilon_1}{4}.$$

Hence we see that

$$d(P_{N_2}y_1,\ P_{N_2}A_1g_1) \leq \frac{3\varepsilon_1}{4} \tag{2.9}$$

Moreover, by (2.5), (2.6), and (2.7),

$$d(y_2,\ A_2A_1g_2) \leq d(y_2,\ P_{N_2}y_2) + d(P_{N_2}y_2,\ P_{N_2}A_1g_2)$$

$$+ d(c_2A_1g_2,\ c_2P_{N_2}A_1g_2)$$

$$\leq \frac{\varepsilon_2}{6} + \frac{\varepsilon_2}{6} + \frac{\varepsilon_2}{6} = \frac{\varepsilon_2}{2}\ .$$

Thus we see that

$$d(y_2,\ A_2A_1g_2) \leq \frac{\varepsilon_2}{2}. \tag{2.10}$$

Finally, the operator A_2 does not move A_1g_1 very far from y_1. Indeed by (2.4), (2.9), and (2.7)

$$d(y_1,\ A_2A_1g_1) \leq d(y_1,\ P_{N_2}y_1) + d(P_{N_2}y_1,\ P_{N_2}A_1g_1)$$

$$+ d(c_2A_1g_1,\ c_2P_{N_2}A_1g_1)$$

$$\leq \frac{\varepsilon_2}{6} + \frac{3\varepsilon_1}{4} + \frac{\varepsilon_2}{6} \leq \frac{3}{4}(\varepsilon_1 + \varepsilon_2).$$

Hence it follows that

$$d(y_1, A_2A_1g_1) \leq \frac{3}{4}(\varepsilon_1 + \varepsilon_2). \tag{2.11}$$

Proceeding inductively, we may construct $\{A_j\}_1^\infty \subseteq L(X)$ and $\{g_j\}_1^\infty \subseteq \Lambda$ such that

$$d(y_j, \overline{A}_n g_j) \leq (1 - 1/2n) \sum_{k=j}^{\infty} \varepsilon_k,$$

where

$$\overline{A}_n = A_n \ldots A_1 \text{ and } 1 \leq j \leq n.$$

Now for $x \in \bigcup_{\gamma\in\Gamma} P_\gamma X$, we claim that $\lim_{n\to\infty} \overline{A}_n x$ exists. This is so since $x \in P_\alpha X$ for some α and since $A_n : P_\gamma X \to P_\gamma X$ for each n and each γ, as was seen in the proof of Lemma 2.4.5. Now since Γ possesses a cofinal sequence (see section 1.3), without loss of generality, we may assume $\{N_n\}_1^\infty$ is cofinal. Hence there exists N_k such that $N_k \geq \alpha$. Therefore $x \in P_{N_k} X$. Hence $\overline{A}_k x \in P_{N_k} X$. However, $\overline{A}_{k+1} x = A_{k+1}\overline{A}_k x$ $= P_{N_k}\overline{A}_k x + c_n(I - P_{N_k})\overline{A}_k x = \overline{A}_k x + c_n\overline{A}_k x - c_n\overline{A}_k x = \overline{A}_k x$. Hence x is moved by only finitely many $\overline{A}_n$. Thus $\lim_{n\to\infty} \overline{A}_n x$ exists on $\bigcup_{\gamma\in\Gamma} P_\gamma X$. We let A denote the extension of this limit to all of X. Since $\{\overline{A}_n\}_1^\infty$ is equicontinuous by either Lemma 2.4.2 or by Lemma 2.4.4, it follows that $A \in L(X)$.

We now claim that this is the operator which we are seeking. Since $d(y_k, \overline{A}_n g_k) \leq \sum_{j=k}^{\infty} \varepsilon_j$, $1 \leq k \leq n$, and since $\overline{A}_n g_k \to Ag_k$, it follows that $d(y_k, Ag_k) \leq \sum_{j=k}^{\infty} \varepsilon_j$. Letting $\delta_k = \sum_{j=k}^{\infty} \varepsilon_j$, we have

constructed $A \in L(X)$, $\{g_n\}_1^\infty \subseteq \Lambda$, and $\{\delta_n\}_1^\infty$ as desired.

To finish the proof, we need only note that each operator A_n is scalar since it is a sum of bounded scalar multiples of projections. Thus each $\overline{A}_n$ is also scalar. Hence by the work in Appendix A.3, we conclude that A is scalar. ///

Before proceeding to the proof of the next theorem, it is worth noting that in a Hilbert space the operator A which was constructed above is positive and 1-1. To see this, we first observe that each A_n is positive since

$$\langle A_n x,x \rangle = \langle P_n x,x \rangle + \langle c_n Ix,x \rangle - \langle c_n P_n x,x \rangle$$

$$= (1 - c_n) \langle P_n x,x \rangle + c_n \langle x,x \rangle \geq 0.$$

Thus since A_n and A_m commute for all m and n, it follows from Proposition 1.4.2 that each $\overline{A}_n$ is positive. Now by continuity of the inner product,

$$\langle Ax,x \rangle = \langle \lim_{n\to\infty} \overline{A}_n x,x \rangle$$

$$= \lim_{n\to\infty} \langle \overline{A}_n x,x \rangle \geq 0.$$

Hence A is positive.

To see that A is 1-1, we recall that by Proposition 1.4.1, A is not only positive, but also self adjoint. Now if $Ax = 0$ for

some $x \in X$, then for each ϕ_n we see that $0 = \langle Ax, \phi_n \rangle = \langle x, A^* \phi_n \rangle$ $= \langle x, A\phi_n \rangle$. But since $\{A\phi_n\}_1^\infty$ is total, it now follows that $x = 0$; hence A is 1-1.

We now prove Theorem 2.3.2. We assume that X is a locally convex topological vector space which possesses scalar operators. Two sufficient conditions for this are that x is either

(1) a complex, barreled, sequentially complete, topological vector space (see Ionescu Tulcea [9]),

or

(2) a real, sequentially complete, topological vector space (see Walsh [21] or Appendix A.2).

We recall that $A \in L(X)$ is called scalar if $A = \int_S f(\lambda)E(d\lambda)$ for some equicontinuous resolution of the identity E and some $f \in M^\infty$ (see section 1.3 and Appendix A.2).

Proof of Theorem 2.3.2. Since A is scalar, there is a equicontinuous resolution of the identity E and bounded measurable function f such that $A = \int_S f(\lambda)E(d\lambda)$. Let $\omega_n = \{\lambda \in S \mid |f(\lambda)| \geq \frac{1}{n}\}$, for $n \geq 1$, and let $\omega_0 = f^{-1}\{0\}$. Then $S = \bigcup_{n=0}^{\infty} \omega_n$. Moreover, by Lemma 2.4.6, $E(\omega_0)X = \ker A$. Since A is 1-1 by hypothesis, it follows that $E(\omega_0)X = \{0\}$.

Let $P_n = E(\omega_n)$. Since E is an equicontinuous resolution of the identity, it follows easily that $P = \{P_n\}_1^\infty$ is a nested equicontinuous family of commuting projections. We claim that P satisfies (1) and (2) of condition T^*.

To see that (1) of T^* holds, fix $x \in X$. Then $x = Ix = E(S)x = E(\omega_0)x + E(\bigcup_1^\infty \omega_n)x = 0 + \lim_{k\to\infty} E(\bigcup_1^k \omega_n)x = \lim_{k\to\infty} P_k x$ since E is a resolution of the identity and $\{\omega_n\}_1^\infty$ is nested. Hence $\bigcup_1^\infty P_n X$ is dense in X. Therefore, (1) of condition T^* holds.

To see that (2) of condition T^* holds, fix k, $x \in P_k$, and $U \in \mathcal{U}$, where $\mathcal{U}$ is as in Lemma 2.4.8. Let

$$T_\alpha = \{x : \alpha x \in W \text{ whenever } Ax \in W, \text{ and } W \in \mathcal{U}\}.$$

Then by Lemma 2.4.9, $P_k X \subseteq T_{\frac{1}{k}}$. By continuity of P_k, fix $V \in \mathcal{U}$ such that $P_k V \subseteq \frac{1}{k}U$. By totality of $\{A\phi_n\}_1^\infty$, pick $g \in \Lambda$ such that $Ax - Ag \in V$. Then $P_k A(x-g) = AP_k(x-g) \in \frac{1}{k}U$. But since $P_k X \subseteq T_{\frac{1}{k}}$, it follows that $P_k A(x-g) \in T_{\frac{1}{k}}$. By definition of $T_{\frac{1}{k}}$, it follows that $AP_k(x-g) \in \frac{1}{k}U$ implies that $\frac{1}{k} P_k(x-g) \in \frac{1}{k}U$. Hence $P_k x - P_k g = x - P_k g \in U$. Therefore $\{P_k\phi_n\}_1^\infty$ is total in $P_k X$. Hence (2) of condition T^* holds. Thus the proof is finished. ///

A careful examination of the proof of Theorem 2.3.1 shows that the technique of the proof involves a countable neighborhood base at 0. Hence any generalization of Theorem 2.3.1 beyond metric spaces will require new techniques. The topological vector space dealt with in Theorem 2.3.2 is much more general than the space dealt with in Theorem 2.3.1 since the space in Theorem 2.3.2 is any locally convex space in which scalar operators exist. However, the operator used in Theorem 2.3.2 is somewhat restricted since it is 1-1 and scalar.

Possible generalizations of this theorem would evidentially involve removing these conditions from the operator. One direction which is possible is to consider spectral operators. A continuous linear operator N is quasi-nilpotent if its spectrum equals $\{0\}$. We say T is spectral if $T = S + N$ where S is scalar and N is quasi-nilpotent. The theory of spectral operators on a Banach space is covered extensively in Dunford and Schwartz [5].

The following example is included to illustrate that T^* may hold for a given sequence in a Hilbert space, but that an operator which converts this sequence into a total sequence need not be scalar nor 1-1. Consider the Hilbert space ℓ^2 and the canonical basis $\{\phi_n\}_1^\infty$. Thus

$$\phi_n(k) = \begin{cases} 1 & n = k \\ 0 & n \neq k \end{cases} .$$

Let P_n be the orthogonal projection onto the space generated by $\{\phi_k\}_1^n$. Clearly, $\{P_n\}_1^\infty$ is a nested equicontinuous family of commuting projections and T^* holds. Let A be the left shift operator; i.e.,

$$(Ax)(k) = x(k + 1).$$

Then $A\phi_1 = 0$ and $A\phi_n = \phi_{n-1}$ for $n \geq 2$, consequently, $\{A\phi_n\}_1^\infty$ is total. However, A is neither 1-1 nor scalar. To see that A is not scalar, we first observe that the adjoint of the left shift operator is the right shift operator. However, the left and right shift operators do not commute. Hence A is not normal. Thus A is not

scalar (see Dunford and Schwartz [5]). Fortunately, the operator which would be constructed by Theorem 2.3.1 would be the identity since $\{\phi_n\}_1$ is already total.

CHAPTER III

FUNCTION METRIC SPACES

3.1. Introduction

In this chapter we will show that results of Chapter II can be used to answer the Boas-Pollard question not only in $L^2[0,1]$, as was done by Price and Zink, but also in a much broader class of spaces, the L_d spaces (defined later). Included in these spaces are the $L^p[0,1]$ spaces for all values of p. In addition, we will show that the work of Talalyan [20] and of Goffman and Waterman [7] is also applicable to a wider class of spaces.

This material on function metric spaces is covered in slightly less generality by Zaanen [23], who discusses normed Köthe spaces. We extend his concept of a function norm to a function metric. Therefore included in this discussion will be the normed Köthe spaces which include the Orlicz spaces which include the classical L^p spaces.

3.2. Some Definitions and Basic Properties

Throughout this chapter (S,F,μ) will be a measure space, M the family of extended complex valued, measurable functions on S where we identify functions which are equal almost everywhere, and M^+ the family of $f \in M$ such that $f(\lambda) \geq 0$ for almost all λ in S.

3.2.1. Definition. A function semimetric is a map $d : M^+ \times M^+ \to [0,\infty]$ (infinity may be attained) having the properties:

(1) $u, v, u + w, v + w \in M^+$ implies that $d(u,v) = d(u + w, v + w)$;

(2) $d(0,0) = 0$;

(3) $0 \leq u \leq v$ implies $d(u,0) \leq d(v,0)$;

(4) $u, v \in M^+$ implies that $d(u + v, 0) \leq d(u,0) + d(v,0)$;

(5) $d(u,0) = d(0,u)$;

(6) for some $u \in M^+$ we have $0 < d(u,0) < \infty$.

We say d is a function metric if in addition

(7) $d(u,0) = 0$ implies $u = 0$ almost everywhere.

We say d is proper if in addition

(8) $d(u,0) < \infty$ implies $u < \infty$ almost everywhere.

We say d possesses a homogeneity function if there exists a function $h : [0,\infty) \to [0,\infty)$ satisfying

(9) $d(\alpha u,0) = h(\alpha)d(u,0)$ for all $\alpha \in \mathbb{C}$ and $u \in M^+$.

If d is a function semimetric which possesses a homogeneity function h, then h must have a special form. Precisely, we have the following proposition.

3.2.2. Proposition. Let d be a function semimetric which possesses a homogeneity function h, then $h(\lambda) = \lambda^p$ for some $p > 0$ or $h(\lambda) = \chi_{\{0\}^c}(\lambda)$.

Proof. We first fix u such that $0 < d(u,0) < \infty$. Now since $0 = d(0,0) = d(0u,0) = h(0)d(u,0)$, it follows that $h(0) = 0$. Moreover, $d(u,0) = d(1u,0) = h(1)d(u,0)$. Hence $h(1) = 1$. Also $h(\alpha\beta)d(u,0) = d(\alpha\beta u,0) = h(\alpha)d(\beta u,0) = h(\alpha)h(\beta)d(u,0)$. Hence

$h(\alpha\beta) = h(\alpha)h(\beta)$.

With these basic properties, the proposition now follows easily. This is so since $h(2) = 2^p$ for some p. But $h(2^2) = h(2\cdot 2) = h(2)h(2) = (2^2)^p$. Inductively, if $h(2^k) = (2^k)^p$, then $h(2^{k+1}) = h(2^k 2) = h(2^k)h(2) = (2^k)^p 2^p = (2^{k+1})^p$. Hence $h(2^n) = (2^n)^p$ for all $n \in \mathbb{N}$. Moreover, $1 = h(2^n 2^{-n}) = h(2^n)h(2^{-n})$ $= (2^n)^p h(2^{-n})$. Hence $h(2^{-n}) = (2^{-n})^p$ for all $n \in \mathbb{N}$. Now $h(2^{1/n})^n = h(2) = 2^p$. Thus $h(2^{1/n}) = (2^{1/n})^p$ for all $n \in \mathbb{N}$. Moreover, $h(2^{m/n}) = h(2^{1/n})^m = (2^{m/n})^p$. Thus for all rational q we have $h(2^q) = (2^q)^p$. To finish the proof we merely need to note that h is monotonically increasing since $\alpha \leq \beta$ implies $\alpha u \leq \beta u$ which implies $d(\alpha u,0) \leq d(\beta u,0)$ which implies $h(\alpha)d(u,0) \leq h(\beta)d(u,0)$. Hence $\alpha \leq \beta$ implies $h(\alpha) \leq h(\beta)$. Thus since $h(\lambda) = \lambda^p$ on a dense subset of $(0,\infty)$ and since h is monotone, it follows that $h(\lambda) = \lambda^p$ for all $\lambda \in (0,\infty)$. Now since h is monotonically increasing, $p \geq 0$. If $p = 0$, then $h(\lambda) = \chi_{\{0\}^c}(\lambda)$. Otherwise $h(\lambda) = \lambda^p$. ///

The function semimetric d is defined only on $M^+ \times M^+$. We extend d to $M_0 \times M_0$, where M_0 is the collection of finite almost everywhere functions in M, by letting $d(f,g) = d(f - g,0) = d(|f-g|,0)$. Since f and g are finite almost everywhere, $f - g$ makes sense. We extend d to $M \times \{0\}$ by defining $d(f,0) = d(|f|,0)$.

3.2.3. Definition. Let d be a function metric. Then $L_d = \{f \in M \mid d(f,0) < \infty\}$.

The set L_d is a vector space provided d is proper. In general L_d need not be a vector space. To see this, consider (S, F, μ) to be $([0,1], \mathcal{B}, m)$ where $\mathcal{B}$ is the Borel σ-algebra on $[0,1]$ and m is Lebesgue measure. Let d be given by

$$d(f,g) = \begin{cases} 0 & f = g \\ 1 & f \neq g \end{cases} .$$

It is easy to see that d is a function metric and that $L_d = M$. Naturally, d is not proper and L_d is not a vector space since M contains extended valued functions. Even if we restrict ourselves to $L_d \cap M_0$ we do not obtain a topological vector space, although we do obtain a vector space since M_0 is the class of finite almost everywhere functions. To see that $L_d \cap M_0$ is not a topological vector space, we merely observe that for $f \neq 0$, $\frac{1}{n}f$ does not converge to zero in L_d.

To remedy this unfortunate situation, we will restrict ourselves to those proper metrics d such that L_d is a topological vector space. These metrics are characterized by Proposition 3.2.4.

3.2.4. Proposition. Let d be a proper function metric. Then L_d is a topological vector space if and only if the map $(\alpha,f) \longmapsto \alpha f$ is continuous.

Proof. If L_d is a topological vector space, then $(\alpha,f) \longmapsto \alpha f$ is continuous by default. Conversely, suppose $(\alpha,f) \longmapsto \alpha f$ is continuous. Then scalar multiplication is continuous. Continuity of addition also holds since $d(f,g) \leq d(f,0) + d(g,0)$. Finally, since d is a

metric on L_d, L_d is Hausdorff. Consequently, L_d is a topological vector space. ///

When d is a function metric which possesses a homogeneity function h, we can give a necessary and sufficient condition on h which will guarantee that L_d is a topological vector space. This is done in our next proposition.

3.2.5. Proposition. Let d be a function metric with a homogeneity function h. Then L_d is a topological vector space if and only if $h(\alpha) = \alpha^p$ for some $p > 0$. Moreover, if h is of this form, then d is also proper.

Proof. By Proposition 3.2.2, $h(\alpha) = \alpha^p$ or $h(\alpha) = \chi_{\{0\}}c(\alpha)$. If $h(\alpha) = \chi_{\{0\}}c(\alpha)$, then for $f \in L_d\backslash\{0\}$ we see that $d(\frac{1}{n}f,0) = h(\frac{1}{n})d(f,0) = d(f,0) > 0$. Hence $\frac{1}{n}f$ does converge to zero in L_d. Therefore L_d is not a topological vector space. Conversely, suppose that $h(\alpha) = \alpha^p$. Let $u \in M^+$ be such that $d(u,0) < \infty$. Let $A = \{\lambda | u(\lambda) = \infty\}$. Then for each $n \in \mathbb{N}$, $n\chi_A \leq u$. Hence for all n, $d(n\chi_A,0) \leq d(u,0) < \infty$. Thus for all n, $n^p d(\chi_A,0) < \infty$. Hence $d(\chi_A,0) = 0$. Hence $\chi_A = 0$ almost everywhere. Therefore A has measure zero, and u is finite almost everywhere. Thus d is indeed proper.

To see that L_d is a topological vector space we need only show that the map $(\alpha,f) \longmapsto \alpha f$ is continuous. But since $d(\alpha f,\beta g) \leq d(\alpha f,\alpha g) + d(\alpha g,\beta g) = |\alpha|^p\, d(f,g) + |\alpha-\beta|^p d(g,0)$, continuity of scalar multiplication holds. ///

Even though we are restricting ourselves to those proper metrics d such that L_d is a topological vector space, there is no guarantee that L_d is complete or that many of the standard theorems from analysis hold; e.g., Fatou's Lemma, the Monotone Convergence Theorem, Lebesgue's Dominated Convergence Theorem, etc. In order to guarantee the validity of results of this type, we now list various conditions which d might satisfy. For the remainder of this chapter, d will be a proper function metric such that L_d is a topological vector space, unless otherwise stated. The functional convergence below is pointwise almost everywhere.

The properties which d might satisfy are:

ST - d is <u>saturated</u> if given $F \in \mathcal{F}$ such that $\mu(F) > 0$, there exists $E \in \mathcal{F}$ with $E \subseteq F$ and such that $\mu(E) > 0$ and $\chi_E \in L_d$;

RF - d has the <u>Riesz-Fischer property</u> if $\{u_n\}_1^\infty \subseteq L_d$ and $u_n \geq 0$ for all n imply that $d(\sum_1^\infty u_n, 0) \leq \sum_1^\infty d(u_n,0)$;

WF - d is <u>weakly Fatou</u> if $\{u_n\}_1^\infty \subseteq M^+$ and $u_n \uparrow u$ and $\lim_{n\to\infty} d(u_n,0) < \infty$ imply that $d(u,0) < \infty$;

F - d is <u>Fatou</u> if $\{u_n\}_1^\infty \subseteq M^+$ and $u_n \uparrow u$ imply that $d(u_n,0) \uparrow d(u,0)$;

AC - d is <u>absolutely continuous</u> if $\{f_n\}_1^\infty \subseteq L_d$ and $|f_n| \downarrow 0$ imply that $d(f_n,0) \downarrow 0$;

C - μ is <u>continuous with respect to d</u> if given $\varepsilon > 0$, there exists $\delta > 0$ such that $d(\chi_A,0) < \delta$ implies that $\mu(A) < \varepsilon$.

It is clear that F implies WF and that F is a restatement of the Monotone Convergence Theorem for this setting. Fatou's Lemma also follows from F. We now prove Fatou's Lemma and examine other properties of d.

<u>3.2.6. Proposition</u>. (Fatou's Lemma). Let $\{u_n\}_1^\infty \subseteq M^+$ and let F hold. Then $d(\varliminf_{n\to\infty} u_n,0) \leq \varliminf_{n\to\infty} d(u_n,0)$.

<u>Proof</u>. First, since $u_n \in M^+$ for all n, $\varliminf_{n\to\infty} u_n$ exists. For each positive integer k, let $g_k = \inf \{u_{n+k}\}_{n=0}^\infty$; then $\{g_k\}_1^\infty \subseteq M^+$, $g_k \uparrow g$ for some $g \in M^+$, $g_k \leq u_k$ for all k, and $\varliminf_{k\to\infty} u_k = \varliminf_{k\to\infty} g_k$. Hence

$$d(\varliminf_{k\to\infty} u_k,0) = d(\varliminf_{k\to\infty} g_k,0)$$

$$= \varliminf_{k\to\infty} d(g_k,0), \text{ by F,}$$

$$\leq \varliminf_{k\to\infty} d(u_k,0). \quad ///$$

<u>3.2.7. Proposition</u>. Let AC and WF hold. Then F holds.

<u>Proof</u>. Let $\{u_n\}_1^\infty \subseteq M^+$ and $u_n \uparrow u$. Now either $d(u,0) = \infty$ or $d(u,0) < \infty$. If $d(u,0) = \infty$, then by WF, $\varliminf_{n\to\infty} d(u_n,0) = \infty$. Hence F holds. If $d(u,0) < \infty$, then since $u - u_n \downarrow 0$, it follows from AC that $d(u,u_n) \downarrow 0$. But $d(u,0) \leq d(u,u_n) + d(u_n,0)$. Hence

$$d(u,0) \leq \varliminf_{n\to\infty} d(u_n,0).$$

However, since $u_n \leq u$, we have $d(u_n,0) \leq d(u,0)$. Thus

$$\varlimsup_{n\to\infty} d(u_n,0) \leq d(u,0).$$

Thus $\lim_{n\to\infty} d(u_n,0) = d(u,0)$. Hence F holds. ///

3.2.8. Proposition. Let F hold. Then RF holds.

Proof. Let $\{u_n\}_1^\infty \subset M^+$. If $\sum_1^\infty d(u_n,0) = \infty$, then $d(\sum_1^\infty u_n,0) \leq \sum_1^\infty d(u_n,0)$ holds trivially.

Suppose on the other hand that $\sum_1^\infty d(u_n,0) < \infty$. Define s_k by $s_k = \sum_1^k u_n$. Then $s_k \in M^+$ and $s_k \uparrow \sum_1^\infty u_n$. Now by the triangle inequality, $d(s_k,0) \leq \sum_1^k d(u_n,0)$. Hence

$$\varliminf_{n\to\infty} d(s_n,0) \leq \sum_1^\infty d(u_n,0).$$

However, by Proposition 3.2.6,

$$d(\sum_1^\infty u_n,0) = d(\varliminf_{n\to\infty} s_n,0)$$

$$\leq \varliminf_{n\to\infty} d(s_n,0)$$

$$\leq \sum_1^\infty d(u_n,0).$$

Thus RF holds. ///

3.2.9 Proposition. Let RF hold. Then L_d is complete.

Proof. Let $\{f_n\}_1^\infty \subseteq L_d$ be Cauchy in the d metric. Then there is a subsequence $\{f_{n_k}\}_{k=1}^\infty$ such that $\sum_{k=1}^\infty d(f_{n_{k+1}}, f_{n_k}) < \infty$. Let $g = \sum_{k=1}^\infty |f_{n_{k+1}} - f_{n_k}|$. Then by RF, $d(g,0) \leq \sum_{k=1}^\infty d(|f_{n_{k+1}} - f_{n_k}|, 0)$ $= \sum_{k=1}^\infty d(f_{n_{k+1}}, f_{n_k}) < \infty$. Hence $g \in L_d$. Since d is proper by assumption, g is finite almost everywhere. Thus if we define f by $f = f_{n_1} + \sum_{k=1}^\infty (f_{n_{k+1}} - f_{n_k})$, it follows that f exists almost everywhere. Now by RF, $d(f,0) \leq d(f_{n_1},0) + \sum_{k=1}^\infty d(f_{n_{k+1}}, f_{n_k}) < \infty$. Hence $f \in L_d$. We now claim $f_{n_k} \to f$ in L_d. For any m, $f - f_{n_m} = \sum_{k=m}^\infty (f_{n_{k+1}} - f_{n_k})$. Hence by RF again, $d(f, f_{n_m}) \leq \sum_{k=m}^\infty d(f_{n_{k+1}}, f_{n_k})$. Thus $f_{n_k} \to f$ in L_d. But since $\{f_n\}_1^\infty$ is Cauchy and since a subsequence converges to an element of L_d, it follows that $\{f_n\}_1^\infty$ itself converges to the same element of L_d. Hence L_d is complete. ///

3.2.10. Proposition. (Dominated Convergence Theorem). Let AC hold and let $\{f_n\}_1^\infty \subset M$ and $f_n \to f$. Suppose there exists $g \in L_d$ such that $|f_n| \leq g$ for all n. Then $d(f, f_n) \to 0$.

Proof. Since $|f_n| \leq g$ and $g \in L_d$, it follows that $f_n \in L_d$ for all n and that $f \in L_d$. Let $u_n = \sup_{k>0} |f_{n+k} - f|$. Then $u_n \downarrow 0$ and $u_n \leq 2g$ for all n. Hence $u_n \in L_d$ for all n. Thus by AC, $d(u_n,0) \downarrow 0$. But $|f - f_n| \leq u_n$. Hence $d(f, f_n) \leq d(u_n,0)$. Thus $d(f,f_n) \to 0$. ///

3.2.11. Proposition. Let AC hold and let $f \in L_d$. Then given $\varepsilon > 0$, there exists $\delta > 0$ such that $\mu(E) < \delta$ implies $d(f\chi_E,0) < \varepsilon$.

Proof. Suppose the theorem is false. Then there exists $\varepsilon_0 > 0$ such that for each positive integer n there is a set E_n with $\mu(E_n) < 2^{-n}$, but such that $d(f\chi_{E_n},0) \geq \varepsilon_0$. Let $F_n = \bigcup_{k \geq n} E_k$. Then $\mu(F_n) \leq \sum_n^\infty \mu(E_k) \leq \sum_n^\infty 2^{-k}$. Hence $\mu(F_n) \downarrow 0$. Since $\{F_n\}_1^\infty$ are nested, it follows that $\chi_{F_n} \downarrow 0$ pointwise. Thus $|f\chi_{F_n}| \downarrow 0$. Hence by AC, $d(f\chi_{F_n},0) \downarrow 0$. But $E_n \subset F_n$. Hence $|f\chi_{E_n}| \leq |f\chi_{F_n}|$. Thus $d(f\chi_{E_n},0) \leq d(f\chi_{F_n},0)$. Therefore $d(f\chi_{E_n},0) \to 0$, a contradiction. ///

Proposition 3.2.11 states that, in those L_d spaces where d satisfies AC, if a set has small measure, then restriction of a function to that small set results in a function which is near the origin. This seems natural, but need not be true in all L_d spaces as the examples in Articles 3.2.12 and 3.3.7 show. If we consider the classical L^p spaces, $0 < p < \infty$, this relationship is referred to as continuity of the integral. (In these spaces $d(f,0) = \int_S |f(\lambda)|^p d\mu$. Because of the Lebesgue Dominated Convergence Theorem, AC holds in the classical L^p spaces, $0 < p < \infty$.) When $f \equiv 1$, the conclusion of Proposition 3.2.11 is transparent since

$$\mu(E) = \int_E d\mu = \int_S |\chi_E|^p d\mu = d(1\chi_E,0).$$

From the propositions listed above, we see that if AC and WF hold, then F holds. Hence RF holds and L_d must be complete. Moreover,

if AC and WF hold, then many of the standard theorems from analysis have valid restatements in the L_d spaces.

3.2.12. Examples. We conclude this section with a series of examples which will help to acquaint the reader with certain L_d spaces and their properties. Many of these examples come directly from Zaanen [23], where the reader is directed for details.

(1) As noted above, the classical L^p spaces, $0 < p < \infty$ form L_d spaces under the metric $d(f,0)=\int_S |f|^p d\mu$. By the Monotone Convergence Theorem, F holds, and by the Dominated Convergence Theorem, AC holds. Hence WF and RF hold. Therefore L_d is complete. Clearly, d possesses a homogeneity function given by $h(\alpha) = \alpha^p$. Therefore by Proposition 3.2.5, we conclude that d is proper and that L_d is a topological vector space. Moreover, if μ is σ-finite, then ST holds. (Of course, ST does not imply σ-finite as the counting measure μ on $\mathbb{R}$ shows, where $d(f,0) = \int_{\mathbb{R}} |f| d\mu$. Then ST holds, but μ is not σ-finite). As was noted above, C holds and is referred to as continuity of the integral.

(2) If d is a function metric which possesses a homogeneity function of the form $h(\alpha) = \alpha^p$, then d is proper and L_d is a topological vector space, as was seen in Proposition 3.2.5. When $0 < p \leq 1$, such spaces are referred to as p-normed spaces (see Köthe [10, p. 160]). These spaces can have duals which are trivial or which separate points, as $L^p[0,1]$ and ℓ^p, $0 < p < 1$, illustrate (see Rudin [15]). When $p = 1$,

such spaces are referred to as normed Köthe spaces (see Zaanen [23]). The class of normed Köthe spaces includes the Orlicz spaces. The reader is referred to Example 3.3.7 (3).

(3) Let d be the trivial metric. As noted earlier, L_d is not a vector space. However, many of the definitions do apply. We see quickly that ST holds, RF holds, F fails, AC fails, and C fails.

(4) Let μ be counting measure on $\mathbb{N}$, the natural numbers. Define d on $M^+ \times \{0\}$ by

$$d(u,0) = \sum_{1}^{\infty} 2^{-n}u(n) + \overline{\lim_{n\to\infty}}\ u(n).$$

Extend d in the natural fashion. Then ST holds, RF fails, WF fails, AC fails, and C fails. To see that WF fails, consider u_n given by

$$u_n(k) = \begin{cases} k & k \leq n \\ 0 & \text{otherwise.} \end{cases}$$

Clearly, $u_n \uparrow u$, where $u(k) = k$, and $d(u_n,0) = \sum_{1}^{n} k2^{-k}$, but $d(u,0) = \infty$. A similar example shows RF fails and that L_d is not complete. For more details, see Zaanen [23].

(5) For the same spaces as in the previous example, define d by

$$d(u,0) = \sup u(n) + 2\ \overline{\lim_{n\to\infty}}\ u(n).$$

Extend d in the natural manner. Then WF holds, but F fails. To see that WF holds, observe that if $u_k \geq 0$ and $u_k \uparrow u$, then $\sup_n u_k(n) \uparrow \sup_n u(n)$. Hence $d(u,0) \leq 3 \sup_n u(n)$ $= 3 \lim_{k\to\infty} \sup_n u_k(n) \leq 3 \lim_{k\to\infty} d(u_k,0)$. To see that F fails, consider u_n given by

$$u_n(k) = \begin{cases} 1 & k \leq n \\ 0 & \text{otherwise.} \end{cases}$$

This example illustrates that while F clearly implies WF, it is not valid that WF implies F.

(6) For the same space as above, let

$$d(u,0) = \begin{cases} \sup u(n) & \text{if } u(n) \to 0 \\ \infty & \text{otherwise.} \end{cases}$$

Then RF holds, but WF fails. The sequence in Example 3.2.12 (5) shows that WF fails. To see that RF holds, we observe that d is a norm and L_d is isometrically isomorphic to the classical Banach space c_0. But in c_0, $||\sum_1^\infty u_k|| \leq \sum_1^\infty ||u_k||$ holds. So RF holds. Hence while F implies RF, we see that RF does not imply F (since WF fails).

(7) Let $\mu(S) = \infty$ and let S possess no atoms. Define d on $M \times \{0\}$ by $d(f,0) = \sup_{\mu(E)=1} \int_E |f| d\mu$. Zaanen [23] shows that F holds. However, AC fails. (If s = $\mathbb{R}$, then

$\chi_{[n,\infty)} \in L_d$, $d(\chi_{[n,\infty)},0) = 1$, and $\chi_{[n,\infty)} \not\to 0$. Hence AC fails.) Thus while WF and AC imply F, we see that F does not imply AC.

Before proceeding to the next example, it is worth noting in passing that Zaanen [23, p. 215] showed that this L_d space contains all of the classical L^p spaces, $0 < p \leq \infty$, but that if f is positive, integrable, and unbounded, then $1 + f$ lies in L_d but does not lie in any of the classical L^p spaces, $0 < p \leq \infty$.

(8) Let $S = \mathbb{R}$, $\mathcal{F}$ be the Borel sets, and μ be Lebesgue measure. Let

$$d_n(f,g) = \int_{-n}^{n} |f-g| d\mu.$$

Define d by

$$d(f,g) = \begin{cases} \sum_{1}^{\infty} \dfrac{d_n(f,g)}{1 + d_n(f,g)} 2^{-n} & \text{otherwise} \\ \infty & \text{if } d_n(f,g) = \infty \text{ for some } n. \end{cases}$$

Then $L_d = L^1_{loc}$. By the Monotone Convergence Theorem, F holds. Hence WF and RF hold, and L_d is complete. The metric d is clearly proper, and L_d is a topological vector space (see Rudin [15]). Trivially, ST holds since $1 \in L_d$. However, C fails since $\mu([n,n+1]) = 1$ for all n, but $d(\chi_{[n,n+1]},0) \to 0$ as $n \to \infty$.

3.3. Relationships Between BP, M, and T in Function Metric Spaces

We will now explore the relationships between BP, M, and T in the L_d spaces. We again assume that d is a proper function metric and that L_d is a topological vector space which is complete. We recall that section 3.2 showed that sufficient conditions for completeness are that either

(1) WF and AC hold,

or

(2) F holds.

Naturally, (1) implies (2), but not conversely (see Example 3.2.12(7)). As was shown in section 3.2, many of the standard theorems from analysis have valid restatements in the L_d spaces.

In section 1.2, it was noted that Talalyan [20] and Price and Zink [14] showed that M, BP, and T are equivalent in $L^2[0,1]$. We wish to examine the relationships between these conditions in the L_d spaces. We will restrict ourselves to the conditions on d mentioned above and to those L_d spaces where $\mu(S) < \infty$. Naturally, we must restate M, BP, and T in terms of these L_d spaces. While the restatements should be clear, they are included here for completeness:

M - $\{\phi_n\}_1^\infty$ is total in L_d in measure;

T - for each $\varepsilon > 0$, there exists a set $S_\varepsilon \subset S$ such that $\mu(S_\varepsilon) \geq \mu(S) - \varepsilon$ and such that $\{\phi_n\}_1^\infty$ is total in $L_d(S_\varepsilon)$;

BP - there exists a nonnegative, bounded, measurable function m such that $\{m\phi_n\}_1^\infty$ is total in L_d.

We begin by examining under what conditions does BP imply T. We need two preparatory lemmas.

3.3.1. Lemma. Let ST hold and let $\{\phi_n\}_1^\infty$ satisfy BP. Let m be a multiplier as guaranteed by BP. Then $\mu\{x|m(x) = 0\} = 0$.

Proof. Let $F = \{x|m(x) = 0\}$. Suppose $\mu(F) > 0$. Then by ST, there exists $E \subseteq F$ such that $\mu(E) > 0$ and such that $\chi_E \in L_d$. Let $\delta = d(\chi_E, 0)$. Since $\mu(E) > 0$, it follows that $\delta > 0$. Now for any $g \in \Lambda$ (see section 1.3), we see that $mg = 0$ on E. Hence $|mg - \chi_E| \geq \chi_E$. Thus $d(mg, \chi_E) \geq d(\chi_E, 0) = \delta > 0$. Hence χ_E can not be approximated by any mg where $g \in \Lambda$, a contradiction to the totality of $\{m\phi_n\}_1^\infty$. Therefore

$$\mu\{x|m(x) = 0\} = 0. \quad ///$$

3.3.2. Lemma. Let $E : F \to L(L_d)$ be given by $E(\omega)f(\lambda) = \chi_\omega(\lambda)f(\lambda)$. Then E is an equicontinuous resolution of the identity (see section 1.3) and if L_d possesses scalar operators, then $[\int_S m(\xi)E(d\xi)]f(\lambda) = m(\lambda)f(\lambda)$, where $m \in M^\infty$.

Proof. Observe first that $E(\omega \cap \delta)f(\lambda) = \chi_{\omega \cap \delta}(\lambda)f(\lambda)$ $= \chi_\omega(\lambda)\chi_\delta(\lambda)f(\lambda) = E(\omega)E(\delta)f(\lambda)$. Hence $E(\omega \cap \delta) = E(\omega)E(\delta)$. Observe also that $E(S)f(\lambda) = \chi_S(\lambda)f(\lambda) = f(\lambda)$. Hence $E(S) = I$. Observe

also that $E(\bigcup_1^\infty \omega_i)f(\lambda) = \chi_{\bigcup \omega_i}(\lambda)f(\lambda) = \sum_1^\infty \chi_{\omega_i}(\lambda)f(\lambda) = \sum_1^\infty E(\omega_i)f(\lambda)$. Thus E is strongly countably additive and hence weakly countably additive. To see that E is equicontinuous, we merely observe that $d(\chi_\omega f,0) \leq d(f,0)$ for all ω. Thus E is indeed an equicontinuous resolution of the identity.

Suppose now that L_d possesses scalar operators. Then for $m \in M^\infty$, $\int_S m(\xi)E(d\xi)$ exists and is the strong (pointwise) limit of finite linear combinations of operators from the range of E (see Appendix A.2). In particular, if $\{g_n\}_1^\infty$ is a sequence of simple functions such that $g_n(\lambda) \to m(\lambda)$ for all λ in S, then the corresponding integrals converge strongly to $\int_S m(\xi)E(d\xi)$. But for a fixed $\lambda \in S$, we may select g_n such that $g_n(\lambda) = m(\lambda)$. Hence for any $f \in L_d$,

$$
\begin{aligned}
[\textstyle\int_S m(\xi)E(d\xi)]f(\lambda) &= [\lim_{n\to\infty} \textstyle\int_S g_n(\xi)E(d\xi)]f(\lambda) \\
&= [\lim_{n\to\infty} \sum_1^{k_n} \alpha_{n,i}\, E(\omega_{n,i})]f(\lambda) \\
&= \lim_{n\to\infty} [\sum_1^{k_n} \alpha_{n,i}\, E(\omega_{n,i})f(\lambda)] \\
&= \lim_{n\to\infty} [\sum_1^{k_n} \alpha_{n,i}\, \chi_{\omega_{n,i}}(\lambda)f(\lambda)] \\
&= \lim_{n\to\infty} \alpha_{n,i_0}\, f(\lambda),
\end{aligned}
$$

where α_{n,i_0} is the value of g_n at λ. But $g_n(\lambda) = m(\lambda)$. Hence

$$[\int_S m(\xi)E(d\xi)]f(\lambda) = m(\lambda)f(\lambda). \quad ///$$

3.3.3. Theorem. Let L_d possess scalar operators. Then BP implies T.

Proof. Since BP holds, there exists a bounded, nonnegative measurable function m such that $\{m\phi_n\}_1^\infty$ is total in L_d. Define $A : L_d \to L_d$ by $Af(\lambda) = m(\lambda)f(\lambda)$. Trivially, A is linear. By Lemma 3.3.1, A is 1-1. We claim that A is also continuous. To see this, let $f_n \to 0$ in L_d. Since m is bounded, there exists K such that $|m(\lambda)| \leq K$ for all $\lambda \in S$. Hence $|f_n(\lambda)m(\lambda)| \leq K|f_n(\lambda)|$ for all n. Therefore $d(Af_n,0) = d(mf_n,0) \leq d(Kf_n,0)$. Thus since $Kf_n \to 0$ in L_d, $Af_n \to 0$ in L_d. Hence A is continuous.

The operator A constructed above is a 1-1 scalar operator by Lemma 3.3.2. Moreover, the resolution of the identity for A is given by $E(\omega)f(\lambda) = \chi_\omega(\lambda)f(\lambda)$. We let $P_n = E(\{\lambda|m(\lambda) \geq 1/n\})$. Then $P = \{P_n\}$ is a nested equicontinuous family of commuting projections. As was seen in the proof of Theorem 2.3.2, $\bigcup_1^\infty P_nL_d$ is dense in L_d and $\{\phi_n\}_1^\infty$ is total in P_nL_d.

Now by Lemma 3.3.1, $\bigcup_1^\infty \{\lambda|m(\lambda) \geq 1/n\}$ has full measure. Thus given $\varepsilon > 0$, there exists an integer N_ε such that $\mu(\{\lambda|m(\lambda) \geq \frac{1}{N_\varepsilon}\}) \geq \mu(S) - \varepsilon$. Hence T holds in $L_d(S_\varepsilon)$ with $S_\varepsilon = \{\lambda|m(\lambda) \geq \frac{1}{N_\varepsilon}\}$. ///

We now show that under suitable hypotheses on L_d that the reverse implication holds; namely, T implies BP.

3.3.4. Theorem. Let L_d be either locally convex or p-normed and let AC hold. Then T implies BP.

Proof. We recall that since L_d is assumed to be a complete topological vector space, the hypotheses of Theorem 2.3.1 hold. We will now show that if T is satisfied then T^* is also satisfied.

By hypothesis, for each $\varepsilon > 0$, there exists a set S_ε such that $\mu(S_\varepsilon) \geq \mu(S) - \varepsilon$ and such that $\{\phi_n\}_1$ is total in $L_d(S_\varepsilon)$. Define $P_\varepsilon : L_d \to L_d$ by

$$P_\varepsilon f(\lambda) = \chi_{S_\varepsilon}(\lambda) f(\lambda).$$

Since $\{S_\varepsilon\}_{\varepsilon>0}$ is a nested family of sets (see section 2.2), it follows that $\{P_\varepsilon\}_{\varepsilon>0}$ is a nested family of projections. These projections clearly commute since $P_\varepsilon P_\delta = P_\delta P_\varepsilon = P_{\min\{\varepsilon,\delta\}}$. Moreover, since $|f| \geq |\chi_{S_\varepsilon} f| = |P_\varepsilon f|$ for all $f \in L_d$ and $\varepsilon > 0$, it follows that $d(P_\varepsilon f, 0) \leq d(f,0)$. Therefore $\{P_\varepsilon\}_{\varepsilon>0}$ is also equicontinuous.

To apply Theorem 2.3.1, we need only to show that $\bigcup_{\varepsilon>0} P_\varepsilon L_d$ is dense in L_d since condition (2) of T^* holds by hypothesis. To see that this union is dense, fix $f \in L_d$ and $\delta > 0$. By Proposition 3.2.11, there exists $\eta > 0$ such that $\mu(A) < \eta$ implies $d(f\chi_A, 0) < \delta$. Now for $\varepsilon < \eta$, we observe that $\mu(S \backslash S_\varepsilon) < \eta$. Hence $d(f, P_\varepsilon f) = d(f - P_\varepsilon f, 0) = d(f\chi_{S\backslash S_\varepsilon}, 0) < \delta$. Therefore $\bigcup_{\varepsilon>0} P_\varepsilon L_d$ is indeed dense in L_d.

Thus T^* holds; consequently, we may apply Theorem 2.3.1 and conclude that there is an operator $A \in L(L_d)$ such that $\{A\phi_n\}_1^\infty$ is total.

Examining the proof of Theorem 2.3.1, we see that A is the strong limit of operators $\overline{A}_n$, each of which is a product of operators of the form

$$A_n = P_{\varepsilon_n} + c_n(I - P_{\varepsilon_n})$$

for some suitable choice of c_n and P_{ε_n}. Now

$$A_n f(\lambda) = P_{\varepsilon_n} f(\lambda) + c_n f(\lambda) - c_n P_{\varepsilon_n} f(\lambda)$$

$$= \chi_n(\lambda) f(\lambda) + c_n f(\lambda) - c_n \chi_n(\lambda) f(\lambda),$$

where χ_n is the indicator function on S_{ε_n}. Hence A_n is a multiplier m_n given by

$$m_n(\lambda) = (1 - c_n)\chi_n(\lambda) + c_n.$$

Since $0 < c_n \leq 1$ by construction, we see that $0 < m_n(\lambda) \leq 1$ for all λ. Therefore since $\overline{A}_n$ is the composition of such A_k, $1 \leq k \leq n$, we conclude that $\overline{A}_n$ is a multiplier $\overline{m}_n$ which is the product of m_k, $1 \leq k \leq n$. Moreover, since $0 < m_n(\lambda) \leq 1$ for all λ, it follows that $0 < \overline{m}_n(\lambda) \leq 1$ for all λ. Therefore $\overline{m}_n$ converges to some bounded, nonnegative, measurable function m. But for any $f \in L_d$, $\overline{m}_n f = \overline{A}_n f$ and $\overline{A}_n f \to Af$. Hence A is a multiplier and $Af = mf$. ///

We now wish to see when T implies M and conversely.

3.3.5. Theorem. Let C hold and let d possess a homogeneity function h. Then T implies M.

Proof. Fix $f \in L_d$. We first claim that for each n there exists a function $g_n \in \Lambda$ and a measurable set E_n such that

(1) $|g_n(\lambda) - f(\lambda)| \leq 2^{-n}$ for all λ in E_n,

(2) $\mu(S \backslash E_n) \leq 2^{-n}$.

To see this, fix n and pick S_n such that $\mu(S_n) \leq 2^{-n-1}$ and that $\{\phi_k\}_1^\infty$ is total in $L_d(S_n)$, as guaranteed by T. Since $\{\phi_k\}_1^\infty$ is total in $L_d(S_n)$, pick $\{g_{n,k}\}_{k=1}^\infty \subseteq \Lambda$ such that $g_{n,k} \to f$ in $L_d(S_n)$ as $k \to \infty$. Let $A_k = \{\lambda \mid |g_{n,k}(\lambda) - f(\lambda)| \geq 2^{-n}\}$. We claim that $\mu(A_k) \leq 2^{-n}$ for large k.

Since $\mu(A_k) = \mu(A_k \cap S_n) + \mu(A_k \cap S_n^c)$ and since $\mu(A_k \cap S_n^c) \leq \mu(S \backslash S_n) \leq 2^{-n-1}$, it suffices to show that $\mu(A_k \cap S_n) \leq 2^{-n-1}$. But since $|g_{n,k}\chi_{S_n \cap A_k} - f\chi_{S_n \cap A_k}| \leq |g_{n,k}\chi_{S_n} - f\chi_{S_n}|$, it follows from the definition of A_k that

$$d(g_{n,k}\chi_{S_n}, f\chi_{S_n}) \geq d(g_{n,k}\chi_{S_n \cap A_k} - f\chi_{S_n \cap A_k}, 0)$$

$$\geq d(2^{-n}\chi_{S_n \cap A_k}, 0)$$

$$\geq h(2^{-n}) d(\chi_{S_n \cap A_k}, 0).$$

Since $g_{n,k} \to f$ in $L_d(S_n)$, we conclude that $\chi_{S_n \cap A_k} \to 0$ in L_d and consequently that $\mu(S_n \cap A_k) < 2^{-n-1}$ for large k since C holds. Thus

$\mu(A_k) \leq 2^{-n}$ for large k as was claimed.

Fix k(n) suitably large such that $\mu(A_{k(n)}) \leq 2^{-n}$. Let $E_n = S \backslash A_{k(n)}$. Then $\mu(S \backslash E_n) \leq 2^{-n}$ as required by (1). Moreover, if $\lambda \in E_n$, then $|g_{n,k(n)}(\lambda) - f(\lambda)| \leq 2^{-n}$. Hence (2) holds if we let $g_n = g_{n,k(n)}$.

We now claim that from (1) and (2) we can conclude that for any $\delta > 0$ there exists a measurable set E_δ and an integer n_δ such that

(3) $\mu(S \backslash E_\delta) \leq \delta$

(4) $|g_n(\lambda) - f(\lambda)| \leq \delta$ for all λ in E_δ and all $n \geq n_\delta$.

To see this, fix $\delta > 0$ and pick n_δ such that $\sum_{n_\delta}^{\infty} 2^{-k} \leq \delta$. Let $E_\delta = \bigcap_{n_\delta}^{\infty} E_n$. Now $\mu(E_\delta^c) = \mu(\bigcup_{n_\delta}^{\infty} E_k^c) \leq \sum_{n_\delta}^{\infty} 2^{-k} \leq \delta$. Hence (3) holds. Moreover, for $n \geq n_\delta$ and $\lambda \in E_\delta$, $|g_n(\lambda) - f(\lambda)| \leq 2^{-n} \leq \delta$. Hence (4) also holds.

We now claim that from (3) and (4) we can conclude that $g_n \to f$ in measure. For let $\delta > 0$ be given. Let $B_n = \{\lambda \mid |g_n(\lambda) - f(\lambda)| > \delta\}$. For E_δ and n_δ as above,

$$\mu(B_n) = \mu(E_\delta \cap B_n) + \mu(E_\delta^c \cap B_n)$$

$$\leq \mu(E_\delta \cap B_n) + \delta.$$

But for $n \geq n_\delta$, $E_\delta \cap B_n = \emptyset$. For if $\lambda \in E_\delta \cap B_n$, then $|g_n(\lambda) - f(\lambda)| > \delta$ and $|g_n(\lambda) - f(\lambda)| \leq 2^{-n} \leq \delta$, a contradiction. Hence we see that for large n, $\mu(B_n) \leq \delta$. Thus $g_n \to f$ in measure. ///

3.3.6. Theorem. Let L_d be separable and let $1 \in L_d$. Then M implies T.

Proof. Let $\varepsilon > 0$ be given. Let $\{f_n\}_1^\infty$ be dense in L_d. Since $\{\phi_k\}_1^\infty$ is total in measure, for each n there exists a sequence $\{F_{n,k}\}_{k=1}^\infty \subseteq \Lambda$ such that $F_{n,k} \to f_n$ in measure as $k \to \infty$. Hence by Egoroff's Theorem, there exists a set S_n such that $\mu(S \backslash S_n) \leq \varepsilon/2^n$ and such that $F_{n,k} \to f_n$ uniformly on S_n. (We pass to a subsequence if necessary.) Let $S_\varepsilon = \bigcap_1^\infty S_n$. Then $\mu(S \backslash S_\varepsilon) \leq \varepsilon$ and $F_{n,k} \to f_n$ uniformly on S_ε as $k \to \infty$ for each n.

To complete the proof, we need only show that $\{\phi_n\}_1^\infty$ is total in $L_d(S_\varepsilon)$. For let $f \in L_d$ and $\delta > 0$ be given. Pick f_n such that $d(f,f_n) < \delta/2$. Now since $1 \in L_d$, it follows that $\frac{1}{k} \chi_{S_\varepsilon} \in L_d$. Moreover, since $\frac{1}{k} \chi_{S_\varepsilon} \to 0$ in L_d, there exists N such that $k \geq N$ implies $d(\frac{1}{k} \chi_{S_\varepsilon}, 0) < \delta/2$. Finally, since $F_{n,k} \to f_n$ uniformly on S_ε, we can select K such that $\ell \geq K$ implies $|F_{n,\ell}\chi_{S_\varepsilon} - f_n\chi_{S_\varepsilon}| \leq \frac{1}{N}$. Hence for $\ell \geq K$, $d(F_{n,\ell}\chi_{S_\varepsilon}, f\chi_{S_\varepsilon}) \leq d(F_{n,\ell}\chi_{S_\varepsilon}, f_n\chi_{S_\varepsilon}) + d(f_n\chi_{S_\varepsilon}, f\chi_{S_\varepsilon}) \leq d(\frac{1}{N} \chi_{S_\varepsilon}, 0)$ $+ d(f_n,f) \leq \delta/2 + \delta/2 = \delta$. Therefore $\{\phi_n\}_1^\infty$ is total in $L_d(S_\varepsilon)$. ///

3.3.7. Examples. We now present some examples to illustrate a few L_d spaces in which the previous results of this section hold. We recall that Price-Zink [14] and Talalyan [20] showed that the main results of this section are true in $L^2[0,1]$.

(1) Since $L^p[0,1]$, $0 < p < \infty$ is either a Fréchet space or a p-normed space, since ST, AC, WF, and C hold, since $L^p[0,1]$ is separable, and since the metric possesses a

homogeneity function, we quickly see that all of the results of this section hold except possibly Theorem 3.3.3, which holds when $1 \leq p < \infty$. Hence results of Price and Zink [14] and of Talalyan [20] can be extended to $L^p[0,1]$ for a large range of p.

(2) Let $S = [0,1]$, F be the Borel sets, and μ be Lebesgue measure. Let $d(f,0) = \sup \{\frac{1}{h} \int_0^h |f| d\mu\}$. Then d is in actuality a norm which possesses the Fatou property (see Zaanen [23]). Thus L_d is a Banach space. In addition, ST and C hold. However, AC fails since $f_n = \chi_{[0,\frac{1}{n}]} \downarrow 0$, but $d(f_n,0) = 1$. Evidently, all of the results of this section except possibly Theorem 3.3.4 hold for this space.

(3) The Orlicz spaces also provide examples. Many of the results inserted here are stated without proof. For details see Zaanen [22] or Luxemburg and Zaanen [11].

Let $\phi : [0,\infty) \to [0,\infty)$ be a nondecreasing, left continuous function such that $\phi(0) = 0$ and such that ϕ is not identically zero. Let $\psi : [0,\infty) \to [0,\infty]$ be the left continuous "inverse" of ϕ defined by the conventions

(a) $\psi(0) = 0$;

(b) if ϕ has a jump discontinuity at a, then $\psi(x) = a$ for x in $(\phi(a),\phi(a^+))$;

(c) if $\lim_{x\to\infty} \phi(x) = \ell < \infty$, then $\psi(x) = \infty$ for x in (ℓ,∞);

(d) if $\phi(x) = c$ for all x in $(a,b]$ and if ϕ has a jump discontinuity at a then $\psi(c) = a$.

For example, let ϕ be 0, 4, and 8 on [0,1], (1,2], and $(2,\infty)$, respectively. Then ψ is 0, 1, 2, and ∞ on $\{0\}$, (0,4], (4,8] and $(8,\infty)$, respectively. We now define

$$\Phi(x) = \int_0^x \phi(t)dt$$

$$\Psi(x) = \int_0^x \psi(t)dt.$$

The functions Ψ and Φ are called <u>complementary Young's functions</u>. It can be shown that $\Phi : [0,\infty) \to [0,\infty)$ is a Young's function if and only if $\Phi(0) = 0$, Φ is convex, and Φ is not identically zero. We also define P_Φ to be the class of functions f with $M_\Phi(f) = \int_S \Phi(|f|)d\mu < \infty$, where (S,F,μ) is a σ-finite measure space containing no atoms. We define P_Ψ and M_Ψ similarly. For $f \in M$, we define

$$d_\Phi(f,0) = \sup\{\int_S |fg| d\mu \;\Big|\; M_\Psi(g) \leq 1\}.$$

We let L_Φ be the corresponding L_d space. The metric d is clearly a norm.

For L_Φ as defined above, Zaanen [22] and Luxemburg and Zaanen [11] showed that ST and F hold and that AC holds if and only if $L_\Phi = P_\Phi$ and that C holds in a finite measure space. Another sufficient condition for AC to hold is that there exist $K > 0$ and $u_0 \in M^+$, $u_0 > 0$ such that $\Phi(2u) \leq K\Phi(u)$ for all $u \geq u_0$. Moreover, if such K and u_0 exist, then $L_\Phi[0,1]$ is separable. Hence the results of this chapter apply to a rather large class of Orlicz spaces.

Before concluding this class of examples, a few particular cases are worth mentioning. If we let μ be Lebesgue

measure on [0,1] and $\Phi(x) = e^x - 1$, then for $f(x) = \log x^{-1/2}$ we see that $f \in P_\Phi$, but $kf \notin P_\Phi$ for $k \geq 2$. Hence P_Φ need not be a vector space. If we again let μ be Lebesgue measure on [0,1] and

$$\phi(x) = \begin{cases} 0 & 0 \leq x < 1 \\ x \log x & 1 \leq x \end{cases},$$

then the corresponding L_Φ space has all of the properties mentioned in the previous paragraph. This space arises quite naturally in Fourier analysis and is denoted by $L \log^+ L$. It is well-known that $L^p[0,1] \subset L \log^+ L \subset L'[0,1]$ for all $p > 1$.

(4) Let μ be a weighted counting measure on IN given by $\mu\{n\} = 2^{-n}$. Then μ is a finite measure. If we let d be as in Example 3.2.12 (6), then ST holds, L_d possesses scalar operators (since it is a Banach space), AC holds, and C holds. Hence all of the results of this section except possibly Theorem 3.3.6 hold in this space.

3.4. Deletions

This material is a generalization of some related work by Goffman and Waterman [7]. They showed that if $\{\phi_n\}_1^\infty$ is total in measure in M_0, the collection of finite almost everywhere Borel measurable complex valued functions on [0,1], then so is $\{\phi_n\}_k^\infty$

for any k. Moreover, by making judicious choices, they showed that infinitely many terms may be deleted from $\{\phi_n\}_1^\infty$ without destroying totality in measure. We wish to show that these results are valid in certain of the L_d spaces by observing that similar results hold in a metrizable topological vector space whose dual is trivial and then by restricting this result to the L_d setting. Since a topological vector space is metrizable if and only if it is First Axiom (see Rudin [15, p. 18]), we may examine any convergence by looking at sequences rather than nets. We begin with a lemma.

3.4.1. Lemma. Let X be a metrizable topological vector space and let $\{\phi_n\}_1^\infty$ be total in X. Let Y be the closure of the linear span of $\{\phi_n\}_2^\infty$ and Z be the linear span of $\{\phi_1\}$. Then either Y = X or $Y \oplus Z = X$ (see Appendix A.1).

Proof. If $\phi_1 \in Y$, then by hypothesis $\{\phi_n\}_1^\infty \subseteq Y$. But since $\{\phi_n\}_1^\infty$ is total in X, it follows that $X \subseteq Y$. Hence X = Y.

If, however, $\phi_1 \notin Y$, then $Y \cap Z = \{0\}$. Since Y and Z are both closed, we need only show that X = Y + Z in order to show that $X = Y \oplus Z$. To see this, fix $x \in X$. Since $\{\phi_n\}_1^\infty$ is total, there exists scalars α_n and there exists $y_n \in Y$ such that $\alpha_n\phi_1 + y_n \to x$. We claim that $\{\alpha_n\}_1^\infty$ is bounded. For suppose not; then there would exist a subsequence $\{\alpha_{n_k}\}_{k=1}^\infty$ such that $|\alpha_{n_k}| \to \infty$ as $k \to \infty$. Since $\alpha_{n_k}\phi_1 + y_{n_k} \to x$, it follows that $\phi_1 + y_{n_k}/\alpha_{n_k} \to 0$. Hence $y_{n_k}/\alpha_{n_k} \to -\phi_1$. But this is a contradiction since $\phi_1 \notin Y$ and Y is a closed subspace of X. Hence $\{\alpha_n\}_1^\infty$ is bounded.

Now since $\{\alpha_n\}_1^\infty$ is bounded, there exists a subsequence $\alpha_{n_k}\}_1^\infty$ and a scalar α such that $\alpha_{n_k} \to \alpha$. Since $\alpha_{n_k}\phi_1 + y_{n_k} \to x$ and $\alpha_{n_k}\phi_1 \to \alpha\phi_1$, it follows that $y_{n_k} \to x - \alpha\phi_1$. But since Y is closed, we see that $x - \alpha\phi_1 \in Y$. Since $x = (x - \alpha\phi_1) + \alpha\phi_1$, we see that $X = Y + Z$. Therefore $X = Y \oplus Z$. ///

Boas and Pollard [3] and Talalyan [20] proved theorems which dealt with finite deletions from total sequences (see section 1.2). We now prove a more general result of the same type. Following this result we will prove a result concerning the deletion of infinitely many elements from a total sequence.

3.4.2. Proposition. Let X be a metrizable topological vector space whose dual is trivial. Let $\{\phi_n\}_1^\infty$ be total in X. Then $\{\phi_n\}_k^\infty$ is total in X for any k.

Proof. It suffices to show this for k = 2. Let Y and Z be as in Lemma 3.4.1. Then either $X = Y$ or $X = Y \oplus Z$. If $X = Y$, we are finished. If $X = Y \oplus Z$, where Z is nontrivial, then define $f : X \to \mathbb{C}$ by $f(y + \alpha\phi_1) = \alpha$. The map f is clearly linear and continuous. Hence $f \in X^*$, the dual of X. But $X^* = \{0\}$ and f is not identically zero since Z is nontrivial, a contradiction. Hence $X = Y$, and consequently $\{\phi_n\}_2^\infty$ is total in X. ///

3.4.3. Proposition. Let X be a metrizable topological vector space. Let $\{\phi_n\}_1^\infty$ be such that $\{\phi_n\}_k^\infty$ is total in X for any k. Then by making judicious choices we may delete infinitely many elements from $\{\phi_n\}_1^\infty$ and still retain totality.

Proof. Let $\varepsilon_n \downarrow 0$. Select any n_1 and delete ϕ_{n_1} from $\{\phi_n\}_1^\infty$. Since $\{\phi_n\}_{n_1+1}^\infty$ is total by hypothesis, there is a finite linear combination of these terms which approximates ϕ_{n_1} to within ε_1. Let k_1 be the largest index used in this approximation. Select any $n_2 > k_1$ and delete ϕ_{n_2}. Again $\{\phi_n\}_{n_2+1}^\infty$ is total by hypothesis. Thus each of ϕ_{n_2} and ϕ_{n_1} may be approximated to within ε_2 by finite linear combinations of $\{\phi_n\}_{n_2+1}^\infty$. Let k_2 be the largest index used in either approximation. Select any $n_3 > k_2$ and delete ϕ_{n_3}. Continuing in this fashion, we may delete infinitely many elements from $\{\phi_n\}_1^\infty$ and still retain totality since the deleted elements may be approximated by the remaining terms. ///

3.4.4. Corollary. Let X be a metrizable topological vector space whose dual is trivial. If $\{\phi_n\}_1^\infty$ is total, then we may delete any finite number of terms, or certain infinite number of terms, and still retain totality.

Proof. This follows quickly from Propositions 3.4.2 and 3.4.3.

3.4.5. Corollary. Let $\{\phi_n\}_1^\infty$ be a total sequence in $L^p[0,1]$, $0 < p < 1$. Then from $\{\phi_n\}_1^\infty$ we may delete any finite subset, or certain infinite subsets, and still retain totality.

Proof. Since $L^p[0,1]$ is a metrizable topological vector space whose dual is trivial (see Rudin [15, p. 35]), this follows quickly from Corollary 3.4.4. ///

3.4.6. Corollary. For $L^p[0,1]$, $0 < p < \infty$, the sequence $\{x^n\}_{n=0}^{\infty}$ has the property that any finite subset, or certain infinite subsets, may be deleted and the remaining sequence is still total.

Proof. Since the polynomials are dense in $L^p[0,1]$, $0 < p < \infty$, we see that $\{x^n\}_{n=0}^{\infty}$ is total. For $0 < p < 1$, the result follows from Corollary 3.4.5. For $1 \leq p \leq \infty$, the result will follow provided $\{x^n\}_k^{\infty}$ is total for every k. To see this, we recall that a simple application of the Hahn-Banach Theorem shows that in a locally convex space X a sequence $\{\phi_n\}_1^{\infty}$ is total if and only if $\langle \phi_n, y^* \rangle = 0$ for all n implies $y^* = 0$, where $y^* \in X^*$, the dual of X. Fix $f \in L^q[0,1]$, where $\frac{1}{p} + \frac{1}{q} = 1$, such that $\langle x^n, f \rangle = 0$ for all $n \geq k$. Then $\int_0^1 x^n \overline{f(x)}\, dx = 0$ for all $n \geq k$. Hence $\int_0^1 x^n \overline{x^k f(x)}\, dx = 0$ for all $n \geq 0$. Since $\{x^n\}_0^{\infty}$ is total, it follows that $x^k f(x) = 0$ for almost all x. Hence $f = 0$ almost everywhere. Consequently, $\{x^n\}_k^{\infty}$ is total for every k and the result follows. ///

The propositions above also yield information concerning universal series. Let X be a topological vector space and $\{\phi_n\}_1^{\infty}$ be a sequence in X. We say $\sum_1^{\infty} \phi_n$ is universal if the collection of partial sums of $\sum_1^{\infty} \phi_n$ is dense in X. While it is easy to give necessary and sufficient conditions on X so that a universal series will exist, it is of greater difficulty to determine whether or not a given sequence gives rise to a universal series. We now prove two propositions which illustrate these facts.

3.4.7. Proposition. A metrizable topological vector space possesses a universal series if and only if it is separable.

Proof. Let X be a metrizable topological vector space. Since a topological vector space is metrizable if and only if it is First Axiom (see Rudin [15, page 18]), it suffices to consider sequences rather than nets. If $\sum_1^\infty \phi_n$ is universal, then $\{\sum_1^k \phi_n\}_{k=1}^\infty$ is dense. Hence X is separable. Conversely, suppose X is separable. Let $\{\phi_n\}_1^\infty$ be a dense subset of X. Let $\psi_1 = \phi_1$ and $\psi_n = \phi_n - \phi_{n-1}$ for $n \geq 2$. Then $\sum_1^\infty \psi_n$ is universal since $\sum_1^k \psi_n = \phi_k$. ///

3.4.8. Proposition. Let X be a metrizable topological vector space. Let $\{\phi_n\}_1^\infty$ be a sequence in X such that $\{\phi_n\}_k^\infty$ is total for every k. Then there exists a sequence $\{\alpha_n\}_1^\infty$ of scalars such that $\sum_1^\infty \alpha_n\phi_n$ is universal.

Proof. Let d be the metric on X. Then d is translation invariant. Since $\{\phi_n\}_1^\infty$ is total, X is separable. Let $\{x_n\}_1^\infty$ be a dense subset of X. Reorder $\{x_n\}_1^\infty$ so that each element appears infinitely often. Let $\varepsilon_n \downarrow 0$.

Now since $\{\phi_n\}_1^\infty$ is total, there exists scalars $\{\alpha_n\}_1^{k_1}$ and a sum $S_1 = \sum_1^{k_1} \alpha_n\phi_n$ such that $d(x_1,S_1) < \varepsilon_1$. But by hypothesis, $\{\phi_n\}_{k_1+1}^\infty$ is total; consequently, there exists scalars $\{\alpha_n\}_{k_1+1}^{k_2}$ and a sum $\sum_{k_1+1}^{k_2} \alpha_n\phi_n$ such that $d(x_2 - S_1, \sum_{k_1+1}^{k_2} \alpha_n\phi_n) < \varepsilon_2$. Hence $d(x_2,S_2) < \varepsilon_2$, where $S_2 = \sum_1^{k_2} \alpha_n\phi_n$. Since $\{\phi_n\}_{k_2+1}^\infty$ is again total by hypothesis, we

may continue in this fashion and create $\sum_1^\infty \alpha_n\phi_n$ such that any x_k may be approximated to within ε_k by a partial sum S_k. Since each x_k appears infinitely often in $\{x_k\}_1^\infty$, we conclude that $\sum_1^\infty \alpha_n\phi_n$ is universal. ///

We now list three corollaries whose proofs clearly follow from the previous work.

3.4.9. Corollary. Let X be a metrizable topological vector space whose dual is trivial. If $\{\phi_n\}_1^\infty$ is total in X, then there exists scalars $\{\alpha_n\}_1^\infty$ such that $\sum_1^\infty \alpha_n\phi_n$ is universal.

3.4.10. Corollary. Let $\{\phi_n\}_1^\infty$ be total in $L^p[0,1]$, $0 < p < 1$. Then there exists scalars $\{\alpha_n\}_1^\infty$ such that $\sum_1^\infty \alpha_n\phi_n$ is universal.

3.4.11. Corollary. There exists scalars $\{\alpha_n\}_1^\infty$ such that $\sum_1^\infty \alpha_n x^n$ is universal in $L^p[0,1]$, $0 < p < \infty$.

We will now show that the Goffman-Waterman result [7] concerning deletions in M_0, the collection of finite almost everywhere measurable functions on [0,1], can also be obtained for certain L_d spaces. Goffman and Waterman showed that if the convergence in measure topology is placed on M_0, then the dual of M_0 is trivial. Restricting ourselves again to those proper function metrics d such that L_d is a complete topological vector space, we see that L_d is a subspace of M_0. Hence if we place the convergence in measure topology on L_d, it might occur that the dual is no longer trivial. Our next proposition shows that this is generally not the case and that the Goffman-Waterman result holds.

3.4.12. Proposition. Let μ be Lebesgue measure on [0,1] and let $1 \in L_d$. Let $\{\phi_n\}_1^\infty$ be a sequence which satisfies M; i.e., let $\{\phi_n\}_1^\infty$ be total in measure. Then $\{\phi_n\}_k^\infty$ also satisfies M for any k. Moreover, by making judicious choices, we may delete infinitely many elements from $\{\phi_n\}_1^\infty$ and still retain M.

Proof. Let τ_M denote the convergence in measure topology on L_d. By Corollary 3.4.4, it suffices to show that $(L_d,\tau_M)^*$ is trivial. While it is not necessary for the proof of this theorem, it is worth mentioning that τ_M is a First Axiom topology which has an equivalent metric given by $d^*(f,0) = \int_0^1 \frac{|f|}{1+|f|}\, d\mu$ (see Hewitt and Stromberg [8, p. 183]).

We claim that $(L_d,\tau_M)^* = \{0\}$. For let $F \in (L_d,\tau_M)^*$ and suppose that $F(f) \neq 0$ for some $f \in L_d$. Pick simple functions s_n such that $s_n \to f$ in measure. Since $1 \in L_d$, it follows that $s_n \in L_d$ for all n. Let $s_n = \sum_1^{k_n} \alpha_{n,i}\chi_{n,i}$ where $\chi_{n,i}$ is the indicator function on some set $A_{n,i}$. Without loss of generality, we may assume that $\mu(A_{n,i}) \leq \frac{1}{n}$. Now since $F(f) \neq 0$ and since F is continuous, there exists N such that $n \geq N$ implies $F(s_n) \neq 0$. Thus, relabeling our sets $A_{n,i}$, there exist sets A_n such that $F(\chi_n) \neq 0$ for $n \geq N$, where $\mu(A_n) \leq \frac{1}{n}$. Hence there exists scalars β_n such that $F(\beta_n\chi_n) = 1$. But since $\mu(A_n) \leq \frac{1}{n}$, it follows that $\beta_n\chi_n \to 0$ in measure. Hence $F(\beta_n\chi_{A_n}) \to 0$, a contradiction. Therefore $F \equiv 0$ and $(L_d,\tau_M)^*$ is consequently trivial.

Since the dual of L_d under the convergence in measure topology is trivial, we now see by Corollary 3.4.4 that $\{\phi_n\}_k$ is total in measure for any k and that by making the appropriate choices we may delete infinitely many elements and still retain totality in measure. ///

Not only does this result hold when we use statement M, but we may also obtain a similar result for statement BP or statement T. Precisely, we have the following corollary.

3.4.13. Corollary. Under the same hypotheses as in Proposition 3.4.12, we may replace M by

(1) T if C holds and d possesses a homogeneity function h,

(2) BP if in addition L_d possesses scalar operators.

Proof. This is a direct application of Theorems 3.3.5 and 3.3.3. ///

It is worth noting that Corollary 3.4.13 says that in $L^p[0,1]$, $0 < p < \infty$, a stronger result than the original Boas-Pollard result [3] concerning finite deletions is true. Precisely, if $\{\phi_n\}_1^\infty$ is any total sequence in $L^p[0,1]$, $0 < p < \infty$, then finitely many, or certain infinitely many, terms may be deleted from $\{\phi_n\}_1^\infty$ and the resulting sequence may be restored to totality by multiplication of a bounded, nonnegative, measurable function.

BIBLIOGRAPHY

BIBLIOGRAPHY

1. Bade, W., Weak and Strong Limits of Spectral Operators, Pacific J. Math. 4 (1954), 393-413.

2. Berkson, E., Semigroups of Scalar Type Operators and a Theorem of Stone, Illinois J. Math. 10 (1966), 345-352.

3. Boas, R. and Pollard, H., The Multiplicative Completion of Sets of Functions, Bull. Amer. Math. Soc. 54 (1948), 518-522.

4. Braun, B., Completion of Certain Sequences in L^p, $1 < p < \infty$, Trans. Amer. Math. Soc. 176 (1973), 499-508.

5. Dunford, N. and Schwartz, J., Linear Operators, New York: Wiley-Interscience, 1971.

6. Foguel, S., Sums and Products of Commuting Spectral Operators, Ark. Mat. 3 (1958), 449-461.

7. Goffman, C. and Waterman, D., Basic Sequences in the Space of Measurable Functions, Proc. Amer. Math. Soc. 11 (1960), 211-213.

8. Hewitt, E. and Stromberg, K., Real and Abstract Analysis, Berlin: Springer-Verlag, 1969.

9. Ionescu Tulcea, C., Spectral Operators in Locally Convex Spaces, Bull. Amer. Math. Soc. 67 (1961), 125-128.

10. Köthe, G., Topological Vector Spaces I, New York: Springer-Verlag, 1969.

11. Luxemburg, W. and Zaanen, A., Riesz Spaces, Amsterdam: North Holland Publishing Company, 1971.

12. Peressini, A., Ordered Topological Vector Spaces, New York: Harper and Row, 1967.

13. Price, J., Topics in Orthogonal Functions, Amer. Math. Monthly 82 (1975), 594-609.

14. Price, J. and Zink, R., On Sets of Functions That Can Be Multiplicatively Completed, Ann. of Math. (2) 82 (1965), 139-145.

15. Rudin, W., Functional Analysis, New York: McGraw-Hill, 1973.

16. Schaefer, H., Halbgeordnete lokalkonvexe Vektorräume II, Math. Ann. 138 (1959), 259-286.

17. Schaefer, H., Spectral Measures in Locally Convex Algebras, Acta. Math. 107 (1962), 125-173.

18. Schaefer, H., Topological Vector Spaces, New York: Springer-Verlag, 1971.

19. Simpson, J., On Limits of Scalar Operators, Trans. Amer. Math. Soc. 122 (1966), 163-176.

20. Talalyan, A., The Representation of Measurable Functions by Series, Russian Math. Surveys 15 (1960), No. 5, 75-136.

21. Walsh, B., Structure of Spectral Measures on Locally Convex Spaces, 120 (1965), 295-326.

22. Zaanen, A., Linear Analysis, New York: Interscience Publishers, 1953.

23. Zaanen, A., Integration, Amsterdam: North Holland Publishing Company, 1967.

APPENDICES

APPENDIX I

ELEMENTARY TOPICS FROM TOPOLOGICAL VECTOR SPACES

A.1.1. Topological Vector Spaces

It is assumed that the reader is familiar with the definition of a Hilbert space, Banach space, Frechet space, and topological vector space. For specific definitions see Rudin [15]. A p-normed space, $0 < p \leq 1$, is a topological vector space which possesses an equivalent translation invariant metric with the property that

$$d(\alpha x,0) = |\alpha|^p d(x,0)$$

for all x in X and for all scalars α. See Kothe [10, p. 160] for details. The spaces $L^p[0,1]$, $0 < p \leq 1$, are examples of p-normed spaces.

Let H and K be vector spaces over a scalar field F. Let $f : H \times K \to F$ be bilinear. The $\langle H, K, f \rangle$ is a duality if

(1) $f(x_0,y) = 0$ for all $y \in K$ implies $x_0 = 0$;

(2) $f(x,y_0) = 0$ for all $x \in H$ implies $y_0 = 0$.

We write $\langle H,K \rangle$ instead of $\langle H,K,f \rangle$ and $\langle x,y \rangle$ instead of $f(x,y)$. On H we place the weakest topology such that $\langle \cdot,y \rangle$ is continuous for each $y \in K$. This topology is denoted by $\sigma(H,K)$. Naturally, $x_\alpha \to x$ in the $\sigma(H,K)$ topology if and only if $\langle x_\alpha,y \rangle \to \langle x,y \rangle$ for all $y \in K$.

If X is a topological vector space, then the collection of all continuous linear functions from X into the scalar field is called the dual of X and is denoted by X^*. If X happens to be locally convex, then $\langle X, X^* \rangle$ is a duality and $\sigma(X,X^*)$ is called the weak topology on X. Of course, X^* could be trivial as $L^p[0,1]$, $0 < p < 1$, illustrates (see Rudin [15, p. 35]).

If X is a topological vector space with closed subspaces Y and Z such that $X = Y + Z$ and $Y \cap Z = \{0\}$, then we say X is the direct sum of Y and Z and write $X = Y \oplus Z$.

A.1.2. Operators

We recall that $L(X)$ denotes the collection of continuous, linear operators on X (see section 1.3). We wish to place various topologies on $L(X)$ which convert $L(X)$ into a topological vector space.

If X is a Banach space, we endow $L(X)$ with

(1) the uniform topology by creating a norm given by

$$||A|| = \sup_{||x|| \leq 1} ||Ax||,$$

(2) the strong topology by saying $A_\alpha \xrightarrow{S} A$ if and only if $A_\alpha x \to Ax$ for all $x \in X$,

(3) the weak topology by saying $A_\alpha \xrightarrow{\omega} A$ if and only if $\langle A_\alpha x, y^* \rangle \to \langle Ax, y^* \rangle$ for all x in X and y^* in X^*, the dual of X.

Parts (2) and (3) of this classical definition can naturally be extended to an arbitrary locally convex topological vector space. In this general setting we say $A_\alpha \to A$ in the uniform topology if and only if

$A_\alpha x \to Ax$ uniformly on bounded subsets of X. Under these topologies $L(X)$ will be a locally convex topological vector space. For details see Schaefer [18].

We will need to know the dual of $L(X)$. To find this dual, we need first to examine tensor products. Let H and K be vector spaces over F. Let B(H,K) be the family of bilinear functions from $H \times K$ into F and let B(H,K)' denote the algebraic dual of B(H,K). For $(x,y) \in H \times K$, define the tensor of x and y to be the element $x \otimes y$ of B(H,K)' such that $x \otimes y(f) = f(x,y)$. The linear span of $\{x \otimes y : x \in H, y \in K\}$ is the tensor product of H and K and is denoted by $H \otimes K$. Any element of $H \otimes K$ can be written as a finite sum $\sum \lambda_i (x_i \otimes y_i)$; however, this sum is not unique. The zero element of $H \otimes K$ can be written as 0 or as $x \otimes 0$ or as $0 \otimes y$.

A.1.2.1. Proposition. Let X be a locally convex topological vector space. Let $f : L(X) \times (X \otimes X^*) \to \mathbb{C}$ be given by $f(A, \sum_1^n x_i \otimes y_i) = \sum_1^n \langle Ax_i, y_i \rangle$. Then $\langle L(X), X \otimes X^* \rangle$ is a duality.

Proof. We first must show that f is well defined. Suppose that $\sum_1^n x_i \otimes y_i = \sum_1^m z_i \otimes w_i$. Then for all g in $B(X,X^*)$, $(\sum_1^n x_i \otimes y_i)g = (\sum_1^m z_i \otimes w_i)g$. Hence $\sum_1^n g(x_i,y_i) = \sum_1^m g(z_i,w_i)$ for all g in B(X,X'). But for A in $L(X)$, the map $\langle A\cdot,\cdot \rangle$ lies in $B(X,X^*)$. Hence $\sum_1^n \langle Ax_i, y_i \rangle = \sum_1^m \langle Az_i, w_i \rangle$. Thus f is well defined.

Since the map f is clearly bilinear, we need only show that each space separates points in the other. Suppose that $f(A_0, \sum_1^n x_i \otimes y_i) = 0$ for all $\sum_1^n x_i \otimes y_i$. Then for a fixed $x_0 \in X$ we see that $\langle A_0x_0, y \rangle = 0$ for all $y \in X^*$. Hence $A_0x_0 = 0$. Since x_0 was arbitrary, it follows that $A_0 = 0$. On the other hand, suppose that $f(A, \sum_1^n x_i \otimes y_i) = 0$ for all $A \in L(X)$. Now if $\sum_1^n x_i \otimes y_i \neq 0$, then $x_{i_0} \neq 0$ and $y_{i_0} \neq 0$ for some i_0. Since $y_{i_0} \neq 0$, there is some $z_{i_0} \in X$ such that $\langle z_{i_0}, y_{i_0} \rangle \neq 0$. Pick $A \in L(X)$ such that $Ax_{i_0} = z_{i_0}$ and $Ax_j = 0$ for all other j. (Without loss of generality the vectors $\{x_i\}_1^n$ and $\{y_i\}_1^n$ are linearly independent.) For this A we see that

$$f(A, \sum_1^n x_i \otimes y_i) = \langle Ax_{i_0}, y_{i_0} \rangle = \langle z_{i_0}, y_{i_0} \rangle \neq 0,$$

a contradiction. Hence $\sum_1^n x_i \otimes y_i = 0$. ///

From the duality given by this proposition, it is possible to place the $\sigma(L(X), X \otimes X^*)$-topology on L(X). By definition, this topology is the weak topology on L(X). It can be shown that not only $(L(X), \tau_w) = X \otimes X^*$, but also $(L(X), \tau_s) = X \otimes X^*$, where τ_w and τ_s denote the weak and strong topologies respectively. For details see Schaefer [18].

APPENDIX II

SCALAR OPERATORS

A basic result from matrix theory is that if an $n \times n$ matrix A commutes with its adjoint (i.e., if A is normal), then A is diagonalizable with the entries on the main diagonal being the eigenvalues of A. Thus A can be written as a sum of eigenvalues times projections. The extension of this result to an arbitrary Hilbert space X states that an operator $A \in L(X)$ is normal if and only if there is a resolution of the identity E on $(\sigma(A),\mathcal{B})$ where $\sigma(A)$ is the spectrum of A and $\mathcal{B}$ the Borel sets of $\sigma(A)$ (see sections 1.3 and 1.4) such that $A = \int_{\sigma(A)} \lambda E(d\lambda)$. Since $E(\omega)$ is a projection for all $\omega \in \mathcal{B}$, it should be clear that this is an extension of the finite dimensional result. A readable account of this can be found in Rudin [15]. This concept is extended to a Banach space X by defining an operator $A \in L(X)$ to be <u>scalar</u> if there is an equicontinuous resolution of the identity E on some measurable space $(S,\mathcal{F})$ such that $A = \int_S f(\lambda)E(d\lambda)$ for some bounded, measurable, complex valued function f. The most natural measurable space to take for $(S,\mathcal{F})$ is the spectrum $(\sigma(A),\mathcal{B})$. A standard reference is Dunford and Schwartz [5].

In the Banach space setting, and hence in the Hilbert space setting, the integral mentioned above is defined as a limit of Riemann sums. Due to the structure inherent in a Banach space, the properties of an equicontinuous resolution of the identity guarantee that these sums converge when f is bounded. More precisely, if $f \in M^{\infty}$, then we pick

simple functions s_n such that $s_n \to f$ pointwise. Naturally, $\int_S s_n(\lambda)E(d\lambda) = \sum_1^{k_n} \alpha_i E(\omega_i)$. It can be shown now that $\|\int_S s_n(\lambda)E(d\lambda)\| \leq K\|s_n\|_\infty$, where K is a constant depending on E only. Hence the sums converge in the uniform topology on $L(X)$ to some operator A since $L(X)$ is complete.

Unfortunately, in a general locally convex topological vector space, it is not clear that such a desirable chain of events will occur. While it is clear how to define the integral for simple functions, it is not clear that a Cauchy sequence of simple functions will give rise to a Cauchy sequence of operators in any of the possible topologies for $L(X)$ which were listed in Appendix A.1. To investigate this situation further we need to examine real ordered topological vector spaces (see Ionescu Tulcea [9] for the existence of scalar operators when X is a complex, locally convex, barreled, quasicomplete topological vector space).

While the succeeding material is known and comes directly from Peressini [12], Schaefer [18], and Walsh [21], we include it here for completeness and for the reader's convenience since this material is not generally considered to be standard information from functional analysis.

Let X be a real vector space and $\leq$ be a transitive, antisymmetric relation on X which satisfies

(1) if $x, y, z \in X$ and $x \leq y$, then $x + z \leq y + z$;

(2) if $x, y \in X$ and $\alpha > 0$, then $x \leq y$ implies $\alpha x \leq \alpha y$.

Such a space and relation is called an ordered vector space. The associated cone of X is $K = \{x \mid x \geq 0\}$. If X is already a topological vector space and K is closed, we then call X an ordered topological vector space. In general, a cone in a vector space X is any set K such that $\alpha K \subseteq K$ for all $\alpha \geq 0$. A cone K is called proper if $K \cap -K = \{0\}$ and generating if $X = K - K$. Clearly, a set K is a proper, convex cone if and only if

(1) $K \cap -K = \{0\}$;

(2) $K + K \subseteq K$;

(3) $\alpha K \subseteq K$ for all $\alpha > 0$.

Moreover, there is clearly a 1-1 correspondence between proper cones and orders which generate ordered vector spaces.

For a subset U of an ordered vector space X, we let $K(U) = \{\sum_{1}^{n} \lambda_i u_i \mid n \in \mathbb{N}, \lambda_i \geq 0, \text{ and } u_i \in U\}$. Then K(u) is clearly convex cone. We say K(u) is the convex cone generated by U.

For x and y in an ordered vector space X, the order interval from x to y is $[x,y] = \{z \mid x \leq z \leq y\}$. For a subset B of X, $[B] = \bigcup_{x,y \in B} [x,y]$. We say [B] is the full hull of B and that B is full if $B = [B]$. If X is an ordered topological vector space with associated cone K, we say K is normal if X possesses a neighborhood base of zero consisting of full sets.

We recall that a filter F on a set X is a family of subsets of X such that

(1) $\emptyset \notin F$ and $\emptyset \neq F$;

(2) $F \in F$ and $F \subseteq G \subseteq X$ implies that $G \in F$;

(3) $F, G \in \mathcal{F}$ implies that $F \cap G \in \mathcal{F}$.

A collection $\mathcal{B}$ of subsets of X is a filter base if

(3) $\emptyset \notin \mathcal{B}$ and $\emptyset \neq \mathcal{B}$;

(4) if $A, B \in \mathcal{B}$ then there exists $C \in \mathcal{B}$ such that $C \subseteq A \cap B$.

If $\mathcal{F}$ is a filter on X, then $[\mathcal{F}]$ is defined to be the filter on X which has $\mathcal{B} = \{[F] \mid F \in \mathcal{F}\}$ as a filter base. If X is a topological vector space, the neighborhood filter of zero is the filter $\mathcal{U}$ whose filter base is the family of all neighborhoods of zero. If $\mathcal{F}$ is a filter in a topological vector space X, then we say $\lim \mathcal{F} = 0$ provided that $\mathcal{U} \subseteq \mathcal{F}$.

We now show that if K is a normal cone in a topological vector space X, then there is a neighborhood base of zero which consists of full sets.

A.2.1. Proposition. Let X be an ordered topological vector space whose associated cone K is normal. Then $\mathcal{U} = [\mathcal{U}]$, where $\mathcal{U}$ is the neighborhood filter of zero.

Proof. We first claim that $[\mathcal{U}] \subseteq \mathcal{U}$ is always true. For let $V \in [\mathcal{U}]$. Then there exists $U \in \mathcal{U}$ such that $[U] \subseteq V$. Since $U \subseteq [U]$, the assertion is true. Conversely, let $U \in \mathcal{U}$. Then there exists a full set W such that $W \subseteq U$ since K is normal. Thus $[W] = W \subseteq U$. Hence $U \in [\mathcal{U}]$. ///

We now prove a useful characterization of normal cones.

A.2.2. Proposition. Let X be a locally convex, ordered topological vector space with associated cone K. The following are equivalent:

(1) K is normal;

(2) for every filter F such that $\lim F = 0$, $\lim [F] = 0$;

(3) there exists a neighborhood base $\mathcal{B}$ of zero in X such that $V \in \mathcal{B}$ implies $[V \cap K] \subseteq V$;

(4) there is a neighborhood base C of zero in X consisting of convex, circled, full sets;

(5) there exists a generating family P of seminorms on X such that $x, y \in K$ and $p \in P$ implies $p(x) \leq p(x + y)$.

Proof. (1) implies (2): If $\lim F = 0$, then $\mathcal{U} \subseteq F$, where $\mathcal{U}$ is the neighborhood filter of zero. Thus by Proposition A.2.1, $\mathcal{U} = [\mathcal{U}] \subseteq [F]$. Hence $\lim [F] = 0$.

(2) implies (3): Since $\lim \mathcal{U} = 0$, $\lim [\mathcal{U}] = 0$. Thus $[\mathcal{U}]$ is a neighborhood filter of zero. We let $\mathcal{B} = \{[U] \mid U \in \mathcal{U}\}$. Then $\mathcal{B}$ is a neighborhood base of zero. Moreover, if $V \in \mathcal{B}$, then $[V \cap K] \subset [V] = V$.

(3) implies (1): Let $\mathcal{B}$ be as in (2). For $V \in \mathcal{B}$, $[V]$ is a full neighborhood of zero. Since $V \subseteq [V]$, it suffices to show that given a neighborhood of zero U there exists a neighborhood of zero W such that $[W] \subset U$. Pick $V \in \mathcal{B}$ such that $V + V \subseteq U$ and a circled neighborhood of zero W such that $W + W \subseteq V$. Then $[W] = \bigcup_{x,y \in W} [x,y] = \bigcup_{x,y \in W} (x + [0,y - x]) \subseteq W + [(W + W) \cap K] \subseteq V + [V \cap K] \subseteq V + V \subseteq U$.

(1) implies (4): Let $\mathcal{W}$ be the family of all convex, circled neighborhoods of zero. Then $\{[W] \mid W \in \mathcal{W}\}$ is a neighborhood base of zero consisting of convex, circled, full sets.

(4) implies (5): For each [W] as above, let p_W be the Minkowski functional; i.e., let $p_W(x) = \inf \{\lambda > 0 \mid x \in \lambda[W]\}$. Let $\mathcal{P}$ be the family of all such p_W. To see that $\mathcal{P}$ is monotone on K, fix $x, y \in K$ and $p_W \in \mathcal{P}$. If $x + y \in \lambda[W]$, then since $0 \in \lambda[W]$ we see $[0,x+y] \subseteq [\lambda[W]] = \lambda[W]$. But since $0 \leq x \leq x + y$, it follows that $x \in \lambda[W]$. Hence $p_W(x) \leq p_W(x+y)$.

(5) implies (3): Let $\mathcal{B}$ be the family of all finite intersections of sets of the form $V_{p,\varepsilon} = \{x : p(x) \leq \varepsilon\}$ where $p \in \mathcal{P}$ and $\varepsilon > 0$. We claim for $V \in \mathcal{B}$ that $[V \cap K] \subset V$. For let $x \in [V \cap K]$. Then there exists $z, y \in V \cap K$ such that $z \leq x \leq y$. Since $z \leq x$ and $z \in K$, we see that $x \in K$. Hence $p_W(x) \leq p_W(y)$ for all $p_W \in \mathcal{P}$. But since $y \in V$, there exists finitely many $\varepsilon_i > 0$ and sets W_i such that $p_{W_i}(y) \leq \varepsilon_i$. Hence $p_{W_i}(x) \leq \varepsilon_i$ for these ε_i and W_i. Thus $x \in V$ also. Hence $[V \cap K] \subseteq V$. ///

It will be useful when we finally develop the integral to know if a net converges when it is dominated above and below by two other nets which converge to a common value. This is shown to be the case in the presence of a normal cone in our next proposition.

A.2.3. Proposition. Let K be a normal cone in X. If $x_\alpha \leq y_\alpha \leq z_\alpha$ for all $\alpha \in I$ and if $x_\alpha \to 0$ and $z_\alpha \to 0$, then $y_\alpha \to 0$.

Proof. Let $\mathcal{B}$ be a neighborhood base of zero consisting of full sets. Fix $U \in \mathcal{B}$. Then there exists β such that $\alpha \geq \beta$ implies $x_\alpha \in U$ and $z_\alpha \in U$. Thus $[x_\alpha, z_\alpha] \subseteq U = [U]$, since U is full. Hence $y_\alpha \in U$ for all $\alpha \geq \beta$. Thus $y_\alpha \to 0$. ///

Whenever K is a cone in a topological vector space X, we say that the dual cone of K is $K^* = \{x^* \in X^* \mid \langle x, x^* \rangle \geq 0 \text{ for all } x \in K\}$. The dual cone K^* of a proper convex cone K provides information concerning the normality of K, as is shown by the following proposition.

A.2.4. Proposition. Let X be a locally convex topological vector space and K be a proper convex cone in X. Suppose that for each equicontinuous set A in X^* there exists an equicontinuous set B in K^* such that $A \subset B - B$. Then K is normal.

Proof. Let S be the family of all equicontinuous subsets of X^*. It is known that the topology of uniform convergence on sets of S is the given topology and that this topology is generated by the seminorms $\mathcal{Q} = \{q_S\}_{S \in S}$ where $q_S(x) = \sup_{x^* \in S} |\langle x, x^* \rangle|$ (see Schaefer [18] or Köthe [10]). Let $\mathcal{P} = \{p_S\}_{S \in S}$ where $p_S(x) = \sup_{x^* \in S \cap K^*} |\langle x, x^* \rangle|$. We wish to show that $\tau_{\mathcal{P}} = \tau_{\mathcal{Q}}$, where $\tau_{\mathcal{P}}$ and $\tau_{\mathcal{Q}}$ are the topologies generated by $\mathcal{P}$ and $\mathcal{Q}$ respectively.

To begin, we see that since $\{x^*\}$ is equicontinuous, K^* is generating. Hence $\mathcal{P}$ is indeed a family of separating seminorms. Thus $\tau_{\mathcal{P}}$ will indeed be a locally convex Hausdorff topology on X.

Moreover, since $p_S(x) \leq q_S(x)$ for each x and S, we see that $x_\alpha \xrightarrow{Q} 0$ implies $x_\alpha \xrightarrow{P} 0$. Hence $\mathcal{T}_P \subseteq \mathcal{T}_Q$.

Conversely, suppose that $x_\alpha \xrightarrow{P} 0$. Fix $S \in \mathcal{S}$. Then $S \subseteq B - B$ for some $B \subseteq K^*$. Thus for any x_α and any $x^* \in S$,

$$|\langle x_\alpha, x^* \rangle| \leq |\langle x_\alpha, b_1^* \rangle| + |\langle x, b_2^* \rangle|$$

$$\leq 2 \sup_B |\langle x_\alpha, b \rangle|$$

$$= 2p_B(x_\alpha).$$

Therefore we see that $q_S(x_\alpha) \leq 2p_B(x_\alpha)$. Thus given $S \in \mathcal{S}$, there exists $B \subset K^*$ such that $B \in \mathcal{S}$ and such that $q_S(x_\alpha) \leq 2p_B(x_\alpha)$. Since $x_\alpha \xrightarrow{P} 0$, it now follows that $x_\alpha \xrightarrow{Q} 0$. Hence $\tau_Q = \tau_P$.

To finish the proof, we need only show that P is monotone on K. For if this is indeed the case, then by Proposition A.2.2, it follows that K is normal. But observe if $x, y \in K$ and $S \in \mathcal{S}$, then for any $x^* \in K^*$, we see that $0 \leq \langle x, x^* \rangle \leq \langle x + y, x^* \rangle$. Hence $\langle x, x^* \rangle \leq \sup_{S \cap K^*} \langle x + y, x^* \rangle$. Therefore $\sup_{S \cap K^*} \langle x, x^* \rangle \leq \sup_{S \cap K^*} \langle x + y, x^* \rangle$. Hence $p_S(x) \leq p_S(x + y)$. Thus P is monotone on K and the result follows. ///

The duality between K and K^* also characterizes weakly normal cones as the following proposition shows.

A.2.5. Proposition. Let X be a real ordered locally convex topological vector space with associated cone K. Then K is weakly normal if and only if K^* is generating.

Proof. See Schaefer [18, p. 220]. ///

We now recall that in section 1.3 we defined a resolution of the identity E on a measurable space $(S,\mathcal{F})$ to be a weakly countably additive map $E : \mathcal{F} \to L(X)$ such that

(1) $E(S) = I$;

(2) $E(\delta \cap \sigma) = E(\delta)E(\sigma)$.

Here X again denotes a locally convex topological vector space. We recall that E is equicontinuous if $\{E(\omega)\}_{\omega\in\mathcal{F}}$ is an equicontinuous subset of $L(X)$. If E is a resolution of the identity, then the range of E will generate a convex cone which is called the cone generated by E whose closure will be denoted by K(E). We will show that this cone is proper and that it is also normal under the strong topology on $L(X)$ which was discussed in section A.1.2. Using this cone and the order it places on $L(X)$, we can then define the desired integral. The next lemma will be useful.

A.2.6. Lemma. Let K be a convex cone in a locally convex topological vector space X. If K^* is generating, then K is proper.

Proof. Let $x \in K \cap -K$. Then given any $k^* \in K^*$, $\langle x,k^* \rangle \geq 0$ since $x \in K$ and $\langle x,k^* \rangle \leq 0$ since $x \in -K$. Hence $\langle x,k^* \rangle = 0$ for all $k^* \in K$. But since K^* is generating, any $x^* \in X^*$ can be written as

$x^* = k_1^* - k_2^*$, $k_i \in K^*$, $i = 1, 2$. Hence $\langle x, x^* \rangle = 0$ for $x^* \in X$. Since X is locally convex, it now follows that $x = 0$. Hence $K \cap -K = \{0\}$ and K is therefore proper. ///

A.2.7. Proposition. Let E be an equicontinuous resolution of the identity and K(E) be the convex cone generated by E. Then E is proper. Moreover, if $L(X)$ is endowed with the strong topology, then E is normal.

Proof. We recall from Appendix A.1 that $(L(X), \tau_s)^* = X \otimes X^*$. To show that K(E) is proper, it suffices therefore to show that for any tensor of the form $x \otimes x^*$, there exists k_1^* and k_2^* in $K(E)^*$ such that $x \otimes x^* = k_1^* - k_2^*$.

To do this, we fix $x \otimes x^*$ and observe that $\langle E(\cdot)x, x^* \rangle : F \to \mathbb{R}$ is a signed measure. Thus by the Hahn Decomposition Theorem, there exists $\omega = \omega(x, x^*) \in F$ such that $\langle E(\delta)x, x^* \rangle = \langle E(\omega \cap \delta)x, x^* \rangle + \langle E(\omega^c \cap \delta)x, x^* \rangle$ where ω is a positive set and ω^c is a negative set. But then

$$\langle E(\delta)x, x^* \rangle = \langle E(\delta)x, E(\omega)^* x^* \rangle + \langle E(\delta)x, E(\omega^c)^* x^* \rangle ,$$

where $E(\omega)^*$ is the adjoint of $E(\omega)$. Therefore $x \otimes x^* = x \otimes E(\omega)^* + x \otimes E(\omega^c)^* x^*$.

We claim that $x \otimes E(\delta)^* x^* \in K(E)^*$. To show this, it suffices to show that for $\delta \in F$, $\langle E(\delta), x \otimes E(\omega)^* x^* \rangle \geq 0$ since the elements of K(E) are limits of finite sums of positive scalars times such

operators. Observe now by definition,

$$\langle E(\delta),\ x \otimes E(\omega)^* x^* \rangle = \langle E(\delta)x, E(\omega)^* x^* \rangle$$

$$= \langle E(\delta \cap \omega)x, x^* \rangle .$$

Since ω is a positive set, it now follows that $x \otimes E(\omega)^* x^* \in K(E)^*$.

By a similar argument, $x \otimes E(\omega^c)^* x^* \in -K(E)^*$. Hence $K(E)^*$ is generating. Therefore $K(E)$ is proper by Lemma A.2.6.

To see that $K(E)$ is normal, fix $A \subset (L(X),\tau_s)^*$. Since $(L(X),\tau_s)^* = X \otimes X^*$, there exist sets $A_1 \subseteq X$ and $A_2 \subseteq X^*$ such that $A = \{\sum x_i \otimes x_i^* \mid x_i \in A_1 \text{ and } x_i^* \in A_2\}$. For each $x_i \otimes x_i^*$, let $\omega(x_i, x_i^*)$ be the positive set which was described above. Let $B = \{\sum x_i \otimes E(\omega(x_i,x_i^*))^* x^*, -\sum x_i \otimes E(\omega(x_i,x_i^*)^c)^* x_i$ where $x_i \in A_1$ and $x_i^* \in A_2\}$. As was seen above, $A \subset B - B$ and $B \subset K(E)^*$.

We claim that B is an equicontinuous subset of $K(E)^*$. For let $\varepsilon > 0$ be given. By equicontinuity of A, there exists a τ_s neighborhood W of zero in $L(X)$ such that $T \in W$ and $\sum x_i \otimes x_i^* \in A$ implies

$$|\langle T, \sum x_i \otimes x_i^* \rangle| = |\sum \langle Tx_i, x_i^* \rangle| \leq \varepsilon. \qquad \text{(A.2.1)}$$

By definition of the strong topology, $W = \{S \in L(X) \mid Sx_i \in U_i\}$ for some finite collection $\{x_i\}_1^n \subseteq X$ and for some finite collection $\{U_i\}_1^n$ of neighborhoods of zero in X. By equicontinuity of E, pick a neighborhood V of zero in X such that $E(\delta)V \subseteq \bigcap_1^n U_i$ for all $\delta \in F$.

Let Z be the strong neighborhood of zero in $L(X)$ given by $Z = \{T \mid Tx_i \in V,\ i = 1, 2, \ldots, n\}$. Thus for $T \in Z$ and $\delta \in F$, $E(\delta)Tx_i \in U_i$ for $i = 1, 2, \ldots, n$. Hence for all $\delta \in F$,

$$E(\delta)T \in W. \tag{A.2.2}$$

Therefore $| \langle T, \sum x_i \otimes E(\omega(x_i, x_i^*))^* x_i^* \rangle |$

$$= |\sum \langle Tx_i, E(\omega(x_i, x_i^*))^* x_i^* \rangle |$$

$$= |\sum \langle E(\omega(x_i, x_i^*))Tx_i, x_i^* \rangle |$$

$\leq \varepsilon$ by (A.2.1) and (A.2.2). ///

We are now ready to define $\int_S f(\lambda)E(d\lambda)$ for $f \in M^\infty$, where X is a real, locally convex, sequentially complete topological vector space. Before doing this, we note that by the Monotone Convergence Theorem of Dini-Schaefer (Proposition 1.4.9) E is countably additive since it is weakly countably additive (see Schaefer [16]). When X is a Banach space, this result is due to Orlicz (see Dunford and Schwartz [5]). If f is a simple function, then $\int_S f(\lambda)E(d\lambda) = \sum_1^n \alpha_i E(\delta_i)$ is the natural definition for the integral. By definition of $K(E)$, it follows that if f is a positive simple function, then $\int_S f(\lambda)E(d\lambda) \geq 0$ in $L(X)$. Thus the integral preserves order in the sense that $-\alpha \leq f \leq \alpha$ and f simple imply that $-\alpha I \leq \int_S f(\lambda)E(d\lambda) \leq \alpha I$ in the order on $L(X)$ which

is induced by K(E). Thus if f_n is simple and $f_n \to 0$ in M^∞ (i.e., if $||f_n||_\infty \to 0$), then by Proposition A.2.3 we see that $\int_S f_n(\lambda)E(d\lambda) \to 0$. Therefore the map $f \to \int_S f(\lambda)E(d\lambda)$ is a continuous linear map when defined on the simple functions in M^∞. Moreover, this map is multiplicative in the sense that $\int_S f(\lambda)g(\lambda)E(d\lambda) = \int_S f(\lambda)E(d\lambda) \int_S g(\lambda)E(d\lambda)$. We now extend the definition of the integral to all of M^∞ by noting that if $\{f_n\}_1^\infty$ is a Cauchy sequence of simple functions in M^∞, then there exists a constant α such that $-\alpha \leq f_n(\lambda) \leq \alpha$ for all n and λ. Since E is equicontinuous, it follows that $\{\int_S f_n(\lambda)E(d\lambda)\}_{n=1}^\infty$ is also equicontinuous (see Lemma 2.4.7). Therefore since X is sequentially complete, the strong limit of these operators exists and is continuous and linear. Moreover, this limit is dominated by αI. Hence we may define scalar operator in such spaces.

APPENDIX III

STRONG LIMITS OF SCALAR OPERATORS

We wish now to determine when the sum and strong limit of scalar operators is scalar. Foguel [6] showed that in a Banach space the sum of a scalar operator and a scalar multiple of a commuting projection is scalar. Also in a Banach space Bade [1] showed that the strong limit of a net of scalar operators will again be scalar provided that the norms of the operators are uniformly bounded and that the spectra lie in certain sets. Simpson [19] did similar work in a locally convex, barreled, sequentially complete space. By placing an order on a real, weakly sequentially complex, locally convex topological vector space, Schaefer [17] was able to characterize scalar operators as being elements of the strong closure of the linear space generated by $\{A \in L(X) \mid 0 \leq A \leq I\}$. Moreover, Schaefer showed that in the complex case an operator A is scalar if and only if $A = T_1 + iT_2$ where T_1 and T_2 commute and lie in the strong closure of the linear space generated by $\{A \in L(X) \mid 0 \leq A \leq I\}$. We shall restrict ourselves to real topological vector spaces in order to make use of the material in Appendix A.2; however, the work of Schaefer [17] shows that we may extend the results of this section to the complex setting.

We begin by stating a few definitions and a fundamental theorem. A <u>locally convex algebra</u> A is an algebra with unit e such that the underlying vector space is a locally convex topological vector space and such

that multiplication is separately continuous. The topology on A is, of course, Hausdorff. A locally convex algebra A is ordered if the underlying vector space is ordered with an associated cone K which is closed and weakly normal and such that $e \in K$ and such that $a, b \in K$ and $ab = ba$ imply $ab \in K$ (for terminology, see Appendix A.2). By Proposition A.2.5, we see that the closure of a weakly normal cone is also weakly normal.

A.3.1. Theorem. Let S be a locally convex Hausdorff space, and let X be an ordered locally convex topological vector space whose positive cone is weakly normal and weakly sequentially complete. Let $C(S)$ denote the continuous functions from S into $\mathbb{C}$ which have compact support. For each positive linear map $\phi : C(S) \to X$, there exists a unique positive vector valued measure μ on the Baire sets of S such that

$$\phi(f) = \int_S f(\lambda)\mu(d\lambda).$$

Proof. See Schaefer [17]. ///

A.3.2. Proposition. Let E and X be as in Proposition A.2.7. Then the closure of the cone $K(E)$ generated by E in $L(X)$ is a closed, weakly normal, proper cone which converts $L(X)$ into an ordered locally convex algebra.

Proof. By Proposition A.2.7, we see that $\overline{K(E)}$ is closed, weakly normal, and proper. Now $I \in \overline{K(E)}$ and if $A, B \in K(E)$, then A and B are linear combinations of projections $E(\delta)$; since $E(\delta)E(\omega) = E(\delta \cap \omega)$, we

conclude that $AB \in K(E)$. Hence if $A, B \in \overline{K(E)}$, then $AB \in \overline{K(E)}$. ///

Proposition A.3.2 states that under the conditions of Appendix A.2, $L(X)$ is an ordered locally convex algebra with positive cone $\overline{K(E)}$ The next theorem can be stated in terms of a general locally convex algebra A. However, we will restrict our attention to $L(X)$. This theorem is taken from Schaefer [17] and is stated without proof.

A.3.3. Theorem. Let X and E be as in Proposition A.3.2 and let $J = [0,I]$ (see Appendix A.2). Then every $A \in J$ is a scalar.

Proof. See Schaefer [17]. ///

In a Hilbert space, the sum and product of commuting normal operators is again normal. In the proof of Theorem 2.3.1, we used the fact that a linear combination of projections is scalar. We wish to verify this now. The proposition below is an extension of a result by Foguel [6] to locally convex spaces.

A.3.4. Proposition. Let X be a locally convex topological vector space which possesses scalar operators. Let T be a scalar operator with a resolution of the identity E on the Borel sets of the complex plane $\mathbb{C}$. Let P be a projection which commutes with T. Then $T + cP$ is scalar for any $c \in \mathbb{C}$ and the resolution of the identity for $T + cP$ is given by

$$G(\omega) = E(\omega - c)P + E(\omega)(I - P).$$

Proof. Clearly, $G(\mathbb{C}) = I$ and G is weakly countably additive. Now

$$G(\delta)G(\omega) = [E(\delta-c)P + E(\delta)(I-P)][E(\omega-c)P + E(\omega)(I-P)]$$

$$= E(\delta-c)E(\omega-c)P + E(\delta)E(\omega)(I-P)$$

$$= E((\delta \wedge \omega)-c)P + E(\delta \wedge \omega)(I-P)$$

$$= G(\delta \wedge \omega).$$

Hence G is a resolution of the identity.

Now $\int_{\mathbb{C}} \lambda G(d\lambda) = \int_{\mathbb{C}} \lambda E(d\lambda-c)P + \int_{\mathbb{C}} \lambda E(d\lambda)(I-P)$

$$= \int_{\mathbb{C}} (\lambda+c)E(d\lambda)\, P + T(I-P)$$

$$= TP + cP + T - TP = T + cP. \quad ///$$

A.3.5. Proposition. Let X be as in Proposition A.2.7 and let $P = \{P_n\}_1^\infty$ be a nested equicontinuous family of commuting projections such that $\bigcup_1^\infty P_n X$ is dense. Then there exists an equicontinuous resolution of the identity E such that $0 \leq P_n \leq I$ for all n, where $\leq$ is the order induced on $L(X)$ by E.

Proof. Let $S = \mathbb{N}$ and let F be the power set of $\mathbb{N}$. Define E by

$$E(\{n\}) = P_n - P_{n-1},$$

where $P_0 = 0$ by definition. We extend E to $\mathcal{F}$ by letting

$$E(\omega) = \sum_{n\in\omega} P_n - P_{n-1},$$

where the sum is the strong limit. We first must show that this limit does exist. Observe if $x \in \bigcup_1^\infty P_nX$, then $x \in P_NX$ for some N. Since $\mathcal{P}$ is nested, $(P_n - P_{n-1})x = 0$ for $n > N$. Hence $E(\omega)x$ exists on $\bigcup_1^\infty P_nX$. Since this set is dense in X and since $\mathcal{P}$ is equicontinuous, $E(\omega)x$ does exist.

Now $E(\{n\})$ is clearly a projection since $\mathcal{P}$ is nested. Moreover, the fact that $\mathcal{P}$ is nested shows that $E(\{n,m\}) = P_n - P_{n-1} + P_m - P_{m-1}$ is also a projection. A short induction on the number of elements in a finite set ω shows that $E(\omega)$ is a projection when ω is finite. If ω is infinite, $E(\omega)$ can be shown to be a projection by decomposing ω into a countable union of finite subsets and passing to a limit. Hence $E(\omega)$ is a projection for each $\omega \in \mathcal{F}$.

Now $E(S)x = (\sum_1^\infty P_n - P_{n-1})x = \lim_{k\to\infty} \bigcup_1^k (P_n - P_{n-1})x = \lim_{k\to\infty} P_kx = x$. Hence $E(S) = I$.

By definition, E is clearly strongly countably additive and hence weakly countably additive.

Finally, since each $E(\omega)$ is a projection it follows that $E(\omega \cap \delta) = E(\omega)E(\delta)$. Thus since $\mathcal{P}$ is equicontinuous, E is an equicontinuous resolution of the identity. By definition, $0 \leq P_n \leq I$.

///

A.3.6. Theorem. The operator A constructed in the proof of Theorem 2.3.1 is scalar.

Proof. By Proposition A.3.5, we see that $0 \leq P_n \leq I$ for all n. Now for $\overline{A}_n$, we have

$$0 \leq (\prod_{1}^{n} c_i)I + (1-c_1)P_1 + \sum_{i=1}^{n} (\prod_{j=1}^{i-1} c_j - \prod_{j=1}^{i} c_j)P_i$$

$$\leq (\prod_{1}^{n} c_i)I + (1-c_1)I + \sum_{i=1}^{n} (\prod_{j=1}^{i-1} c_j - \prod_{j=1}^{i} c_j)I$$

$$= I.$$

Hence $0 \leq \overline{A}_n \leq I$. Therefore $0 \leq A \leq I$. Consequently, by Proposition A.3.3, A is scalar. ///

The following example shows that the strong limit of scalar operators need not necessarily be scalar. To guarantee that this limit is scalar, it is necessary to place restrictions on the spectra of the operators and on the size of the resolutions of the identity which are involved as was done by Bade [1] or to require that all operators lie inside a common ordered cone as was done by Schaefer [17]. This example is due to Berkson [2].

A.3.7. Example. Let X be the Hilbert space ℓ^2 and define T_n by

$$T_n x = (x_n, x_1, x_2, \ldots, x_{n-1}, x_{n+1}, x_{n+2}, \ldots).$$

Let T be the right shift operator. Then $T_n x \to Tx$ for all x. Hence $T_n \to T$ strongly.

The adjoint of T is the left shift and the adjoint of T_n is

$$T_n^* x = (x_2, x_3, \ldots, x_n, x_1, x_{n+1}, x_{n+2}, \ldots).$$

A short calculation shows that $T_n T_n^* = T_n^* T_n$ but that $TT^* \neq T^* T$. Hence T_n is normal (therefore scalar) but T is not.

VITA

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