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I am submitting herewith a dissertation written by John James Iskra entitled Decidability in Algebraic Geometry. I have examined the final electronic copy of this dissertation for form and content and recommend that it be accepted in partial fulfillment of the requirements for the degree of Doctor of Philosophy, with a major in Mathematics.

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(Original signatures are on file with official student records.)

Decidability In Algebraic Geometry

A Dissertation

Presented for the

Doctor of Philosophy Degree

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John James Iskra

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Acknowledgments

Any work which involved so much time and anxiety owes its existence to many many people. I think of the many mathematics teachers I've had: My parents first, of course, but each of my teachers has contributed in some way to whatever knowledge I was able to bring to bear while working on my dissertation problem. I only wish now that I had paid better attention. Many of the faculty at the University of Tennessee showed great patience with my ignorance. In particular, the members of my committee Professors David Anderson and Yasayuki Kachi. Both often dropped whatever they were doing - without complaint - to consider whatever crazy question I might ask them. My advisor S. Mulay provided crucial and substantial help related to the fourth chapter of this dissertation. My fellow graduate students were a source of comfort, and served as sources of both mathematical inspiration and technical assistance. For the latter I especially benefitted from Mike Saum and Jason Howard. I hesitate to say I *learned* any mathematics as a consequence. John von Neumann once admitted only gradually getting comfortable with the subject, never learning it. The most profound debt of gratitude is that which I owe to my family for their preternatural patience and fortitude. My wife Betty Jo and my sons Nathaniel and Theo were vital, not only to my efforts to complete

this document, but - most importantly - to maintaining my tenuous hold on sanity while I did so. Without them neither would have been possible.

Abstract

The central theme of our investigation is the concept of Decidability in Algebra / Algebraic Geometry. To the best of our knowledge this seems to be novel in the sense that there is no work known to isolate or to focus on the concept of Decidability in the context of Commutative Algebra. Decidability is more restrictive than Grothendieck's concept of *formally unramified*, but weaker than the concept of *étale*. In this article we study these relationships by characterizing Decidability for ring-extensions of essentially finite type. In the absence of essential finiteness we can only show, at present, that a separable algebraic extension of fields is indeed Decidable.

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Chapter 1

Decidability

1 Preliminaries

We begin by defining the field of discourse:

Definition 1.1. We define **Crng** to be the category whose objects are commutative rings with identity and ring homomorphisms which preserve it. We include in this category the ring with one element: $0 = 1$ and call it the **zero ring**. We further set **Calg**(A), (where A is a commutative ring) and **Calg**(k) to be the subcategories of A –, and k – algebras respectively.

Remark: In **Crng**, **Calg**(k) and **Calg**(A), (where A is a commutative ring) it is well known that push forwards exist. They are just the tensor product over the integers Z , the field k , and the ring A respectively.

(See [3]Chapter IV section 7)

Definition 1.2. *Let*

$$\begin{array}{ccc} B \otimes_A C & \xleftarrow{\tilde{f}} & C \\ \tilde{g} \uparrow & & \uparrow g \\ B & \xleftarrow{f} & A \end{array}$$

be the pushforward diagram. We will refer to $\tilde{g}$ as the pushforward of g by f

Definition 1.3. *Let $f : A \rightarrow B$ be a ring homomorphism. Let*

$$\begin{array}{ccc} B \otimes_A B & \xleftarrow{i} & B \\ j \uparrow & & \uparrow f \\ B & \xleftarrow{f} & A \end{array}$$

be the pushforward diagram. Then, by the definition of pushforward there exists a unique map μ_f so that

$$\begin{array}{c} B \xleftarrow{1_B} B \\ \mu_f \swarrow \quad \nwarrow 1_B \\ B \otimes_A B \xleftarrow{i} B \\ j \uparrow \quad \quad \uparrow f \\ B \xleftarrow{f} A \end{array}$$

commutes. We will refer to μ_f as the codiagonal or multiplication map associated with f , since it is well-known that $\mu : b \otimes b' \mapsto bb'$. We may leave μ undecorated by f if the context ensures that no confusion will

result.

Definition 1.4. In this paper we will distinguish between *epimorphisms* and surjective maps, although the latter is an example of the former. We define a *ring epimorphism* f to be any map which is “right cancelable”. That is, f is an epimorphism if whenever $gf = hf$ it must be that $g = h$.

Example: The canonical map carrying a domain into its quotient field is an epimorphism, though generally not surjective.

Lemma 1.1. Suppose $f : A \rightarrow B$ is a ring epimorphism. Then μ_f is an isomorphism.

Proof. Consider the pushforward diagram

$$\begin{array}{ccccc}
 & & B & \xleftarrow{1_B} & B \\
 & \mu \swarrow & & \searrow i & \\
 B & \xleftarrow{1_B} & B \otimes_A B & \xleftarrow{i} & B \\
 & \uparrow j & & \uparrow f & \\
 & B & \xleftarrow{f} & A &
 \end{array}$$

Recall that the pushforward of an epimorphism is again an epimorphism. But since $\mu i = 1_B$ we have that $i \mu i = i = 1_{B \otimes_A B} i$. The fact that i is an epimorphism ensures that $i \mu = 1_{B \otimes_A B}$. That is that μ is an isomorphism with $\mu^{-1} = i$. $\square$

2 Pushforwards

Lemma 2.1. *Suppose $f : C \rightarrow B$ and $g : D \rightarrow C$ are ring homomorphisms. Then, there exists a map ϕ so that*

$$\begin{array}{ccc} B & \xleftarrow{\mu_f} & B \otimes_C B \\ \uparrow 1_B & & \uparrow \phi \\ B & \xleftarrow{\mu_{fg}} & B \otimes_D B \end{array}$$

is a pushforward diagram.

Proof. Let

$$\begin{array}{ccc} B \otimes_C B & \xleftarrow{p} & B \\ \uparrow q & & \uparrow f \\ B & \xleftarrow{f} & C \end{array}$$

and

$$\begin{array}{ccc} B \otimes_D B & \xleftarrow{r} & B \\ \uparrow s & & \uparrow fg \\ B & \xleftarrow{fg} & D \end{array}$$

be the pushforward diagrams. Then there exists a unique ϕ so that

$$\begin{array}{ccccc} B \otimes_C B & & & & \\ & \nwarrow p & & & \\ & & B \otimes_D B & \xleftarrow{r} & B \\ & \nearrow \phi & \uparrow s & & \uparrow fg \\ & & B & \xleftarrow{fg} & D \\ & \nwarrow q & & & \end{array}$$

commutes. Now, μ_{fg} is the unique map so that $\mu_{fg}r = 1_B = \mu_{fg}s$. But, $\mu_f\phi r = \mu_f p = 1_B = \mu_{fg}q = \mu_f\phi s$, whence,

$$\begin{array}{ccc} B & \xleftarrow{\mu_f} & B \otimes_C B \\ \uparrow 1_B & & \uparrow \phi \\ B & \xleftarrow{\mu_{fg}} & B \otimes_D B \end{array}$$

commutes. Suppose now that

$$\begin{array}{ccc} Z & \xleftarrow{x} & B \otimes_C B \\ \uparrow y & & \uparrow \phi \\ B & \xleftarrow{\mu_{fg}} & B \otimes_D B \end{array}$$

commutes. Then, $x\phi = y\mu_{fg}$, thus, $x\phi r = y\mu_{fg}r = y$. Thus, $xp = y$. Similarly $xq = y$. Now, consider the commutative diagram

$$\begin{array}{ccc} Z & \xleftarrow{xp} & B \\ \uparrow xq & & \uparrow f \\ B \otimes_C B & \xleftarrow{p} & B \\ \uparrow q & & \uparrow f \\ B & \xleftarrow{f} & C \end{array}$$

We thus have a unique map ψ so that

$$\psi p = xp = y,$$

and

$$\psi q = xq = y.$$

Since substituting either x or $y\mu_f$ for ψ makes these equations true, we must have that $y\mu_f = x$. In other words,

$$\begin{array}{ccccc}
 & & Z & & \\
 & \swarrow y & & \nwarrow x & \\
 & B & \xleftarrow{\mu_f} & B \otimes_C B & \\
 & \uparrow 1_B & & \uparrow \phi & \\
 & B & \xleftarrow{\mu_f g} & B \otimes_D B &
 \end{array}$$

commutes. y is clearly unique. □

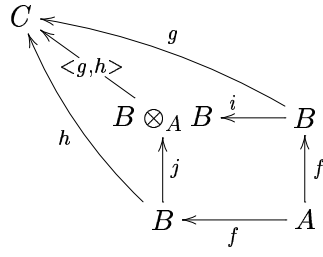
Recall that the coequalizer of two ring homomorphisms $f, g : A \rightarrow B$ is the ring B/J and projection $p : B \rightarrow B/J$, where J is the ideal generated by $\{f(x) - g(x) | x \in A\}$.

Lemma 2.2. *Let $f : A \rightarrow B$ be a ring homomorphism. Then, every A -algebra coequalizer of a pair of maps with domain B is the pushforward of μ_f*

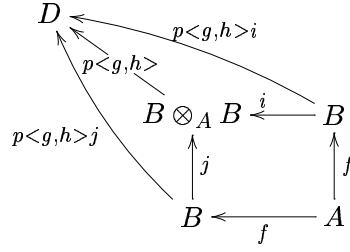
Proof. Consider the commutative diagram

$$\begin{array}{ccccc}
 B & \xrightarrow{g} & C & \xrightarrow{p} & D \\
 & \searrow h & & & \\
 & A & \xrightarrow{gf=hf} & &
 \end{array}$$

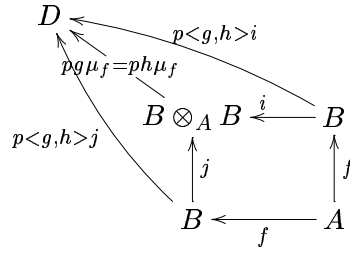
where $p : C \rightarrow D$ is the coequalizer of g and h . Denote by $\langle g, h \rangle$, the unique map so that



commutes. Now, both

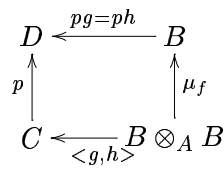


and



commute.

Thus, so does



Suppose

$$\begin{array}{ccccc}
 & & R & & \\
 & \swarrow \psi & & \searrow \phi & \\
 & C & \xleftarrow{\langle g, h \rangle} & B \otimes_A B & \\
 & \uparrow p & & \uparrow \mu_f & \\
 & D & \xleftarrow{pg=ph} & B & \\
 & & & &
 \end{array}$$

commutes. Thus, we have $\psi \langle g, h \rangle = \phi \mu_f$, which implies $\psi g = \phi = \psi h$. Since D is a coequalizer, there exists a unique map $r : D \rightarrow R$ so that $rp = \psi$. Thus, r is the unique map so that

$$\begin{array}{ccccc}
 & & R & & \\
 & \swarrow \psi & & \searrow \phi & \\
 & C & \xleftarrow{\langle g, h \rangle} & B \otimes_A B & \\
 & \uparrow p & & \uparrow \mu_f & \\
 & D & \xleftarrow{pg=ph} & B & \\
 & & & &
 \end{array}$$

(A red arrow labeled r points from D to R in the above diagram.)

commutes. Thus p is the pushforward of μ_f .

□

Lemma 2.3. *Let $f : A \rightarrow B$ be a ring homomorphism. Let*

$$\begin{array}{ccc}
 B \otimes_A B & \xleftarrow{i} & B \\
 \uparrow j & & \uparrow f \\
 B & \xleftarrow{f} & A
 \end{array}$$

be the pushforward diagram. Then, μ_f is the coequalizer of i and j .

Proof. By definition, $\mu_f i = \mu_f j$. Suppose that

$$\begin{array}{ccc}
B & \xRightarrow[i]{j} & B \otimes_A B \xrightarrow{\mu} B \\
& & \downarrow t \\
& & T
\end{array}$$

is a commutative diagram of ring homomorphisms. Then, so is

$$\begin{array}{ccccc}
& & & & T \\
& & & \nearrow ti & \\
& & B \otimes_A B & \xleftarrow{i} & B \\
& \nearrow tj & \uparrow j & & \uparrow f \\
& & B & \xleftarrow{f} & A
\end{array}$$

and so is

$$\begin{array}{ccccc}
& & & & T \\
& & & \nearrow ti & \\
& & B \otimes_A B & \xleftarrow{i} & B \\
& \nearrow tj & \uparrow j & & \uparrow f \\
& & B & \xleftarrow{f} & A
\end{array}$$

$ti\mu = tj\mu$

Thus, by uniqueness, $t = ti\mu = tj\mu$. In other words,

$$\begin{array}{ccc}
B & \xRightarrow[i]{j} & B \otimes_A B \xrightarrow{\mu} B \\
& & \downarrow t \quad \nearrow ti \\
& & T
\end{array}$$

commutes. Since μ is surjective ti is unique. □

Lemma 2.4. *Suppose*

$$\begin{array}{ccc}
 T & \xleftarrow{e} & A \\
 g \uparrow & \nearrow f & \\
 S & &
 \end{array}$$

is a commutative diagram with e an epimorphism. Then

$$\begin{array}{ccccc}
 F & & \xleftarrow{a} & & T \\
 \uparrow b & & & \nearrow e & \uparrow g \\
 & P & \xleftarrow{p} & A & \\
 & \uparrow q & & \uparrow f & \\
 & A & \xleftarrow{f} & S & \\
 & \nearrow e & & \nearrow g & \\
 T & & & &
 \end{array}$$

commutes, where the red and blue diagrams are pushforwards. By definition we then have a unique ϕ so that

$$\begin{array}{ccccc}
 F & & \xleftarrow{a} & & T \\
 \uparrow b & & & \nearrow e & \uparrow g \\
 & P & \xleftarrow{p} & A & \\
 & \uparrow q & & \uparrow f & \\
 & A & \xleftarrow{f} & S & \\
 & \nearrow e & & \nearrow g & \\
 T & & & &
 \end{array}$$

commutes. We then have that

$$\begin{array}{ccc}
 T & \xleftarrow{e} & A \\
 \mu_g \uparrow & & \uparrow \mu_f \\
 F & \xleftarrow{\phi} & P
 \end{array}$$

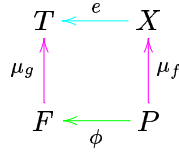
is a pushforward diagram.

Proof. First note that

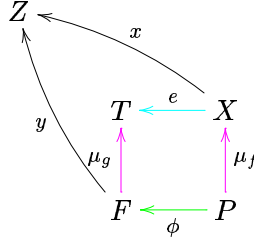
$$ae\mu_f p = ae1_X = ae$$

$$be\mu_f q = be1_X = be.$$

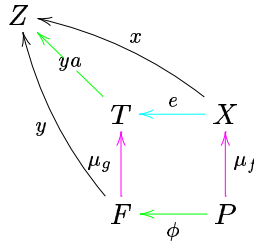
By the uniqueness of ϕ we have that $ae\mu_f = \phi$. Now, $\phi p = ae$ thus $\mu_g \phi p = \mu_g ae$. Thus, by definition of μ_g , $\mu_g \phi p = e$, which implies that $\mu_g \phi p \mu_f = e\mu_f$. We conclude that $\mu_g ae\mu_f = e\mu_f$ and so $\mu_g \phi = e\mu_f$. Thus, the diagram



commutes. Suppose



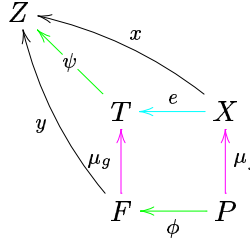
commutes. I claim then that



also commutes. To see this note that $x\mu_f = y\phi$ which implies that $x\mu_f p = y\phi p$ and since $\mu_f p = 1_X$, $x = y\phi p = yae$. Again,

$$ya\mu_g\phi = yae\mu_f = x\mu_f = y\phi.$$

Now, ϕ is an epimorphism since e is. Indeed, if $w\phi = v\phi$, we have that $w\phi p = v\phi p$, so $wae = vbe$. Since e is epi, $wa = vb$. But, since F is a pushforward there is a unique map c so that $ca = cb$. Thus $w = v$, and so ϕ is an epimorphism. Thus, $ya\mu_g = y$. If



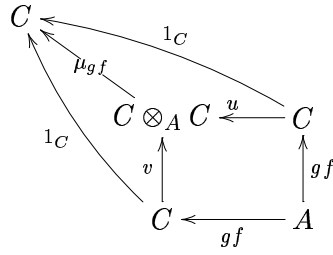
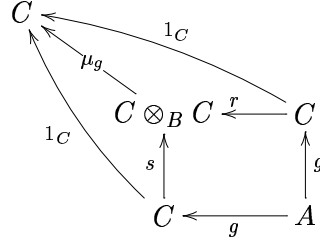
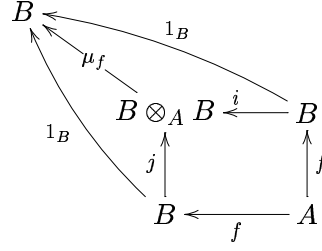
also commutes it would be the case that $yae = \psi e$. Since e is an epimorphism, $ya = \psi$, thus ya is the unique such map. Thus, T is the pushforward of μ_f and ϕ . $\square$

Lemma 2.5. *Let $f : A \rightarrow B$ and $g : B \rightarrow C$ be ring homomorphisms. Then there exists a pushforward of μ_f , call it ϕ , so that the diagram*

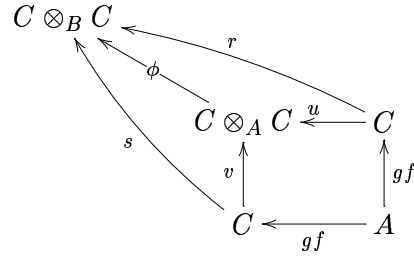
$$\begin{array}{ccc} C \otimes_A C & \xrightarrow{\mu_{gf}} & C \\ & \searrow \phi & \nearrow \mu_g \\ & C \otimes_B C & \end{array}$$

commutes.

Proof. Let



be the pushforward diagrams. Then, there exists a unique ϕ so that



commutes. Now, $\mu_g \phi u = \mu_g r = 1_C$, and $\mu_g \phi v = \mu_g s = 1_C$. By uniqueness of μ_g , we must have that $\mu_g \phi = \mu_{gf}$. In other words,

$$\begin{array}{ccc}
C \otimes_A C & \xrightarrow{\mu_{gf}} & C \\
& \searrow \phi & \nearrow \mu_g \\
& C \otimes_B C &
\end{array}$$

commutes. We have left to show that ϕ is a pushforward of μ_f . Clearly

$$\begin{array}{ccc}
C \otimes_B C & \xleftarrow{rg} & B \\
\uparrow \phi & & \uparrow \mu_f \\
C & \xleftarrow{ug\mu_f} & B \otimes_A B
\end{array}$$

commutes. Suppose now that

$$\begin{array}{ccc}
T & \xleftarrow{\beta} & B \\
& \searrow \alpha & \nearrow \mu_f \\
& C \otimes_B C & \xleftarrow{rg} B \\
& \uparrow \phi & \uparrow \mu_f \\
& C & \xleftarrow{ug\mu_f} B \otimes_A B
\end{array}$$

commutes. Since μ_f is surjective, and thus epi, and because $\beta\mu_f = \alpha u g \mu_f$, it is also true that $\beta = \alpha u g$. Thus, $\alpha u \mu_g r g = \alpha u g = \beta$ and, $\alpha u \mu_g \phi = \alpha u \mu_{gf} = \alpha$. In other words,

$$\begin{array}{ccc}
T & \xleftarrow{\beta} & B \\
& \nearrow \alpha u \mu_g & \nearrow \mu_f \\
& C \otimes_B C & \xleftarrow{rg} B \\
& \uparrow \phi & \uparrow \mu_f \\
& C & \xleftarrow{ug\mu_f} B \otimes_A B
\end{array}$$

commutes. To see that $\alpha u \mu_g$ is the unique such map, observe that if $z\phi = \alpha$, then $z\phi u \mu_g = \alpha u \mu_g$. So, $z r \mu_g = \alpha u \mu_g$. Thus, $z = \alpha u \mu_g$.

□

Lemma 2.6. *Let $f : A \rightarrow S$ and $h : A \rightarrow T$ be ring homomorphisms.*

Let

$$\begin{array}{ccc} S \otimes_A T & \xleftarrow{\tilde{h}} & S \\ \uparrow g & & \uparrow f \\ T & \xleftarrow{h} & A \end{array}$$

be the pushforward of f and h . Then, μ_g is a pushforward of μ_f .

Proof. Let

$$\begin{array}{ccc} S \otimes_A S & \xleftarrow{i} & S \\ \uparrow j & & \uparrow f \\ S & \xleftarrow{f} & A \end{array}$$

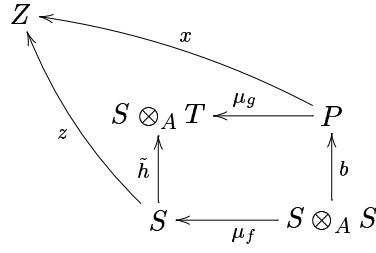
and

$$\begin{array}{ccc} P & \xleftarrow{r} & S \otimes_A T \\ \uparrow s & & \uparrow g \\ S \otimes_A T & \xleftarrow{g} & A \end{array}$$

be the pushforward diagrams. Consider the commutative diagram

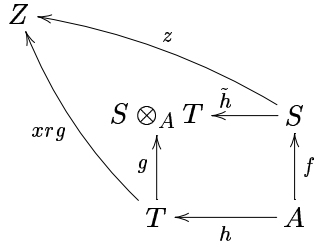
$$\begin{array}{ccccc} S \otimes_A T & \xleftarrow{\mu_g} & P & \xleftarrow{rg} & T \\ \uparrow \tilde{h} & & \uparrow b & & \uparrow h \\ S & \xleftarrow{\mu_f} & S \otimes_A T & \xleftarrow{if} & A \end{array}$$

where we've set $b = s\tilde{h}\mu_f$ for ease of notation. Suppose

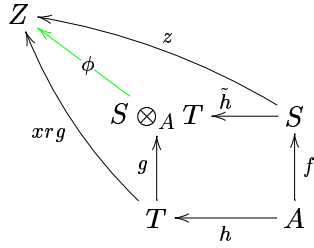


commutes. Since $z\mu_f = xb$ implies that $z\mu_f i f = x b i f$, thus, $z f = x r g h$.

Thus, we have the commutative diagram



. Thus, there exists a unique ϕ so that



commutes. Suppose P is the pushforward of h by $i f$. Then, x is the unique map so that $x r g = x r g$ and $x b = z \mu_f$. But certainly

$$\phi \mu_g r g = \phi \mu_g r g$$

and

$$\phi \mu_g b = \phi \tilde{h} \mu_f = z \mu_f.$$

Thus, $x = \phi\mu_g$. In other words,

$$\begin{array}{ccccc}
 & & Z & & \\
 & \nearrow \phi & & \nwarrow x & \\
 & S \otimes_A T & \xleftarrow{\mu_g} & P & \\
 & \uparrow \tilde{h} & & \uparrow b & \\
 & S & \xleftarrow{\mu_f} & S \otimes_A S & \\
 & \nwarrow z & & &
 \end{array}$$

commutes. The fact that μ_g is surjective insures that ϕ is unique. We now only have left to show that

$$\begin{array}{ccc}
 P & \xleftarrow{rg} & T \\
 \uparrow s\tilde{h}\mu_f & & \uparrow h \\
 S \otimes_A S & \xleftarrow{i_f} & A
 \end{array}$$

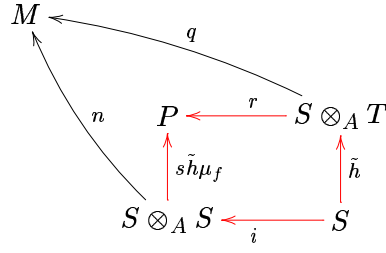
is a pushforward diagram. To see this, first note that the above diagram can be factored into two commuting diagrams:

$$\begin{array}{ccccc}
 P & \xleftarrow{r} & S \otimes_A T & \xleftarrow{g} & T \\
 \uparrow s\tilde{h}\mu_f & & \uparrow \tilde{h} & & \uparrow h \\
 S \otimes_A S & \xleftarrow{i} & S & \xleftarrow{f} & A
 \end{array}$$

The blue diagram is a pushforward, thus we need only show that

$$\begin{array}{ccc}
 P & \xleftarrow{r} & S \otimes_A T \\
 \uparrow s\tilde{h}\mu_f & & \uparrow \tilde{h} \\
 S \otimes_A S & \xleftarrow{i} & S
 \end{array}$$

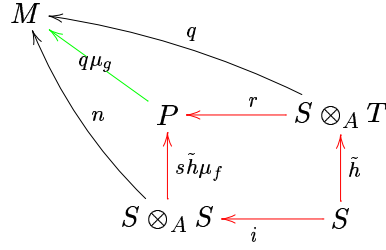
is a pushforward diagram. Suppose that



commutes. Then, since

$$q\mu_g s\tilde{h}\mu_f = q\tilde{h}\mu_f = ni\mu_f = n$$

and $q\mu_g r = q$, we have that



commutes. To see that $q\mu_g$ is the unique such map, suppose that $\alpha r = q$ and that $\alpha s\tilde{h}\mu_f = n$. Note that α is the unique map so that $\alpha r = q$ and $\alpha s = \alpha s$. But, $q\mu_g r = q$ and certainly, $q\mu_g s = q\mu_g s$. Thus, $\alpha = q\mu_g$ proving that P is the pullback of h by if . The proof of the lemma is thus completed.

□

3 Flatness

Definition 3.1. A ring homomorphism is *pre-flat* if the pushforward of a monomorphism by it is a monomorphism.

Definition 3.2. A ring homomorphism is *flat* if any pushforward of it is pre-flat.

Lemma 3.1. A ring homomorphism $f : A \rightarrow B$ is flat as we have defined it if and only if for any monomorphism of A -algebras $C \rightarrow D$, the induced map $B \otimes_A C \rightarrow B \otimes_A D$ is also a monomorphism.

Proof. Suppose $f : A \rightarrow B$ is flat as we have defined it, and $C \rightarrow D$ is a monomorphism of A -algebras. Since both squares of the diagram

$$\begin{array}{ccccc} B \otimes_A D & \longleftarrow & B \otimes_A C & \longleftarrow & B \\ \uparrow & & \uparrow \tilde{f} & & \uparrow f \\ D & \longleftarrow & C & \longleftarrow & A \end{array}$$

are pushforwards, as is the rectangle, the induced map $B \otimes_A C \rightarrow B \otimes_A D$ is also a monomorphism.

On the other hand, if given any monomorphism of A -algebras $C \rightarrow D$, the induced map $B \otimes_A C \rightarrow B \otimes_A D$ is also a monomorphism, then given any A -algebra $A \rightarrow C$ and A -algebra monomorphism $C \rightarrow D$ as in the diagram above, $\tilde{f}$ is pre-flat. Thus f is flat according to our definition. $\square$

Lemma 3.2. *The pushforward of a flat ring homomorphism is flat.*

Proof. Suppose $f : S \rightarrow A$ is flat and $b : S \rightarrow B$ is any ring homomorphism. Let

$$\begin{array}{ccc} P & \xleftarrow{p} & A \\ q \uparrow & & \uparrow f \\ B & \xleftarrow{b} & S \end{array}$$

be the pushforward diagram. Let $b : B \rightarrow D$ and $e : D \rightarrow E$ be monomorphisms. Consider the sequence of pushforward diagrams

$$\begin{array}{ccccccc} K & \xleftarrow{\tilde{e}} & T & \xleftarrow{\tilde{d}} & P & \xleftarrow{p} & A \\ \uparrow & & \uparrow \tilde{p} & & \uparrow q & & \uparrow f \\ E & \xleftarrow{e} & D & \xleftarrow{b} & B & \xleftarrow{d} & S \end{array}$$

Now, $\tilde{p}$ is the pushforward of f by bd . By the definition of flat, $\tilde{p}$ must be preflat. Since e is a monomorphism and $\tilde{p}$ is preflat, we have that $\tilde{e}$ is monomorphism. Thus, since the pushforward of q by an arbitrary ring homomorphism is preflat, it follows that q is flat. $\square$

Definition 3.3. *A ring homomorphism $f : A \rightarrow B$ is said to be **faithfully flat** if it makes B a faithfully flat A -module.*

Lemma 3.3. *Suppose that $f : A \rightarrow B$ is a faithfully flat ring homomorphism, and that $g : G \rightarrow D$ is an A -algebra homomorphism. Suppose that $g \otimes 1_B : G \otimes_A B \rightarrow D \otimes_A B$ is a monomorphism. Then, so is g*

Proof. Let K be the kernel of g . Then, we have the exact sequence

$$0 \longrightarrow K \longrightarrow G \xrightarrow{g} D$$

Since B is flat we have the exact sequence

$$0 \longrightarrow K \otimes_A B \longrightarrow G \otimes_A B \xrightarrow{g \otimes 1_B} D \otimes_A B$$

Since $g \otimes 1_B$ is a monomorphism, however, $K \otimes_A B = 0$. Since B is faithful, $K = 0$, that is, g is a monomorphism. $\square$

Lemma 3.4. *Suppose that M is a faithfully flat A -module, and that N is an A -module so that $M \otimes_A N$ is flat. Then, N is also a flat A module.*

Proof. Suppose $h : D \rightarrow B$ is a monomorphism of A -modules. Then, so is $1_M \otimes 1_N \otimes h$ since $M \otimes_A N$ is flat since M is faithfully flat. The last lemma implies then that $1_N \otimes h$ must also be a monomorphism. Thus, N is flat. $\square$

4 Decidability

Definition 4.1. *A ring homomorphism $f : A \rightarrow Y$ is **decidable** if its codiagonal $\mu_f : Y \otimes_A Y \rightarrow Y$ is flat.*

Lemma 4.1. *a. Suppose we have the commutative diagram*

$$\begin{array}{ccc}
T & \xleftarrow{e} & A \\
\uparrow g & \nearrow f & \\
S & &
\end{array}$$

with e an epimorphism and f decidable. Then, g is also decidable.

b. Suppose that f and g are decidable ring homomorphisms. Then gf is decidable.

c. The pushforward of a decidable ring homomorphism is decidable.

d. Any epimorphism is decidable.

e. Suppose fg is decidable. Then, so is f .

f. A map $f : A \rightarrow B$ is decidable if and only if for every pair of A -algebra homomorphisms $g, h : B \rightarrow C$, the coequalizer of g and h is flat over C .

g. A map $f : A \rightarrow B$ is decidable if and only if for every pair of A -algebra homomorphisms $g, h : B \rightarrow C$ with C a quasi-local ring, the coequalizer of g and h is either the identity or the zero map.

Proof. a.: By definition f decidable implies that μ_f , the diagonal of f , is flat. By 2.4 μ_g is a pushforward of μ_f since e is an epimorphism. But, 2.1 then implies that μ_g is also flat. That is, g is decidable.

b.: By 2.6, $\mu_{gf} = \mu_g \phi$ where ϕ is a pushforward of μ_f . Since f and g are decidable, μ_f and μ_g are flat. By 3.2, ϕ is also flat. Since the composition of flat homomorphisms is flat, μ_{gf} is flat. Thus gf is

decidable.

c.: Let g be a pushforward of a ring homomorphism f . Then by 2.5, μ_g is the pushforward of μ_f . By 2.1, μ_g is flat since μ_f is. Thus, by definition, g is decidable.

d.: Suppose $f : A \rightarrow S$ is an epimorphism. Then, by 1.1, μ_f is an isomorphism and thus is flat. Thus, f is decidable.

e: By Lemma 2.1,

$$\begin{array}{ccc} B & \xleftarrow{\mu_f} & B \otimes_C B \\ \uparrow 1_B & & \uparrow \phi \\ B & \xleftarrow{\mu_{fg}} & B \otimes_D B \end{array}$$

is a pushforward diagram. Since the pushforward of a flat map is flat, and μ_{fg} is flat by supposition, we have that μ_f is flat. That is, f is decidable.

f: Follows from Lemmas 2.3 and 2.4.

g: Suppose $f : A \rightarrow B$ is decidable and $g, h : B \rightarrow C$ are A -algebra homomorphisms with coequalizer $p : C \rightarrow D$. Suppose C is quasi-local and $D \neq 0$. Then p is a local map. Since f is decidable, part f of this lemma implies that p is flat. Thus, p is faithfully flat, and, so, it is one-to-one. Since p is also surjective, p is an isomorphism.

On the other hand if the condition holds for all quasi-local C and $g, h : B \rightarrow R$ are a pair of A -algebra homomorphisms with coequal-

izer $p : R \rightarrow D$, we have that every localization of p is either 0 or an isomorphism. In particular, then, every localization of p is flat, and thus, so is p itself.

□

Lemma 4.2. *Consider the pushforward diagram*

$$\begin{array}{ccc} B \otimes_A C & \longleftarrow & C \\ \tilde{g} \uparrow & & \uparrow g \\ B & \xleftarrow{f} & A \end{array}$$

Then, if f is faithfully flat, and $\tilde{g}$ is decidable, so too is g .

Proof. Since $\tilde{g}$ is decidable, the codiagonal map

$$\mu_{\tilde{g}} : B \otimes_A C \rightarrow B \otimes_A C \otimes_C B \otimes_A C$$

is as well. But, clearly, $\mu_{\tilde{g}} \simeq \mu_g \otimes 1_B$. Indeed, if

$$\begin{array}{ccc} B \otimes_A C & \xleftarrow{\tilde{g}} & B \\ h \uparrow & & \uparrow f \\ C & \xleftarrow{g} & A \end{array}$$

is the pushforward, then each rectangle in

$$\begin{array}{ccccccc} B \otimes_A C & \xleftarrow{\mu_{\tilde{g}}} & P & \xleftarrow{j} & B \otimes_A C & \xleftarrow{\tilde{g}} & B \\ h \uparrow & & \uparrow & & \uparrow \tilde{f} & & \uparrow f \\ C & \xleftarrow{\mu_g} & C \otimes_A C & \xleftarrow{p} & C & \xleftarrow{g} & A \end{array}$$

are pushforwards, where $P = B \otimes_A C \otimes_C B \otimes_A C$ and j and p are the pushforwards of f and $\tilde{f}$ by themselves respectively. But, by definition, then, $\mu_{\tilde{g}} = \mu_g \otimes 1_B$. Thus, since B is faithfully flat, we have that μ_g must be flat, and so g is decidable. $\square$

Remark: Since every inclusion of fields is faithfully flat this last lemma implies that if a k -algebra S is decidable under base change $k \rightarrow L$ then it is decidable over k .

Theorem 4.1. *Let I be an ideal of a ring R . Then the following are equivalent.*

- (i) *The canonical epimorphism $h : R \rightarrow R/I$ is flat.*
- (ii) *$aR = aI$ for all $a \in I$.*
- (iii) *There exists $G \subseteq R$ such that $GR = I$ and $gR = gI$ for all $g \in G$.*

Proof. Assume (i). Let J be an ideal of R and let $\iota : J \rightarrow R$ be the natural inclusion. Since h is flat,

$$\iota \otimes 1 : J \otimes_R R/I \rightarrow R \otimes_R R/I$$

is injective. Identifying $J \otimes_R R/I$ with J/IJ and $R \otimes_R R/I$ with R/I we conclude that $I \cap J = JI$. Thus $I \cap J = JI$ for every ideal J of R . In particular consider $a \in I$ and let $J = aR$. Then $aR = I \cap aR = (aR)I = aI$. Evidently (ii) holds.

The implication (ii) $\Rightarrow$ (iii) is obvious.

Now suppose (ii) holds. Then, for each $g \in G$ there exists $b_g \in I$ with $g = b_g g$. Consider a prime ideal P of R and let h_P be the map $R_P \rightarrow R/I \otimes_R R_P$ induced by h . If I is not contained in P , then $R/I \otimes_R R_P = 0$ and hence h_P is flat. Henceforth assume $I \subseteq P$. It follows that $1 - b_g$ is in the multiplicatively closed set $R \setminus P$ for all $g \in G$. Thus the canonical homomorphism $R \rightarrow R_P$ maps I to 0; as a consequence h_P is an isomorphism onto $(R/I)_P$. Flat-ness being a local property, (i) follows. $\square$

Corollary 4.1. *With the above notation, if h is flat, then $I = I^2$.*

Proof: Clearly, $aI \subseteq I^2$ for all $a \in I$. Hence, using (ii), it follows that $a \in I^2$ for all $a \in I$ i.e. $I \subseteq I^2$. $\square$

Corollary 4.2. *We continue to use the notation of Theorem 1. Assume R to be quasilocal and I is a proper ideal. The map h is flat if and only if it is the identity map i.e. $I = 0$.*

Proof. From the proof of Theorem 4.1 it is clear that flat-ness of h is equivalent to h_P being an (onto) isomorphism for all prime ideals P of

R containing I . In particular letting P be the maximal ideal of R our assertion follows. $\square$

Corollary 4.3. *We continue to use the notation of Theorem 4.1. Assume I is a finitely generated ideal of R . Then, the following are equivalent.*

(i) *The canonical epimorphism $h : R \rightarrow R/I$ is flat.*

(ii) *$I = I^2$.*

Moreover, in case (i) holds, I is a principal ideal generated by an idempotent.

Proof. In view of Corollary 4.1, we only need to derive (i) from (ii). Assume (ii) holds. Let $\{z_1, \dots, z_n\}$ be a set of generators of I . Write

$$z_i := \sum_{j=1}^{j=n} a_{ij} z_j \quad \text{for } 1 \leq i \leq n$$

where a_{ij} are elements of I . Then the determinant of $[\delta(i, j) - a_{ij}]$, where δ denotes the Kronecker delta function, is of the form $1 - b$ for some b in I and $(1 - b)z_j = 0$ for $1 \leq j \leq n$. Thus (iii) of Theorem 1 holds. Clearly, $bR = I$ and $b = b^2$. $\square$

Theorem 4.2. *Let $f : A \rightarrow B$ be a homomorphism of rings. For a prime ideal P of B let $Q := f^{-1}(P)$ and let f_P denote the ring-homomorphism $A_Q \rightarrow B_P$ induced by f . Then, f is decidable if and*

only if f_P is decidable for all prime ideals P of B . In other words, decidability is a local property.

Proof. Set $R := B \otimes_A B$ and let $h := \mu_f$ be the codiagonal map associated to f . For a prime ideal P of B , let $\pi := h^{-1}(P)$. From the proof of Theorem 4.1, it is clear that flat-ness of h is equivalent to $h_\pi : R_\pi \rightarrow B_P$ being an isomorphism onto B_P for each such P . Let $Q := f^{-1}(P)$. Note that there is a multiplicative set $T \subseteq R \setminus \pi$ such that $B_P \otimes_{A_Q} B_P$ is (canonically isomorphic to) $T^{-1}R$. Hence flat-ness of μ_{f_P} is equivalent to h_π being an isomorphism onto B_P . $\square$

Definition 4.2. Let $f : A \rightarrow B$ be a homomorphism of rings.

- (1) f is said to be of **essentially finite type** if there exist finitely many elements $b_1, \dots, b_n$ of B and a multiplicatively closed subset T of $f(A)[b_1, \dots, b_n]$ such that $B = T^{-1}f(A)[b_1, \dots, b_n]$.
- (2) f is said to be **unramified** at a prime ideal P of B if upon letting $Q := f^{-1}(P)$, the ring $B_P/f(Q)B_P$ is a field separably algebraic over the field A_Q/QA_Q . f is said to be **unramified** if it is unramified at every prime ideal P of B .

Theorem 4.3. Suppose a homomorphism $f : A \rightarrow B$ of rings is of essentially finite type. Then the following are equivalent.

- (i) f is decidable.

- (ii) $\Omega_{B/A} = 0$ where $\Omega_{B/A}$ denotes the module of Kähler differentials.
- (iii) f is unramified.

Proof. Replacing A by $f(A)$ we may assume, without loss, A to be a subring of B . Set $R := B \otimes_A B$ and let I be the kernel of the codiagonal map μ . Observe that μ is the canonical epimorphism from R to R/I and $\Omega_{B/A} = I/I^2$. Furthermore, by hypothesis B is of essentially finite type over A and hence I is a finitely generated ideal of R . Thus the equivalence of (i) and (ii) follows from Corollary 1.3. Under our hypothesis the equivalence of (ii) and (iii) is well-known; for instance see 17.4.1 of [2]. $\square$

Corollary 4.4. *Let L be a field and let k be a subfield of L such that L is an algebraic field-extension of k . If L is separable over k , then the inclusion map $k \rightarrow L$ is decidable.*

Proof. If $E \subseteq F$ are subfields of L containing k , then observe that the canonical map $E \otimes_k E \rightarrow F \otimes_k F$ is injective and $I_E = I_F \cap (F \otimes_k F)$, where I_E, I_F denote the kernels of the codiagonal maps μ_E, μ_F respectively. It follows that the ring $L \otimes_k L$ is the union (*i.e.* the colimit) of rings $K \otimes_k K$, and I_L is the union (*i.e.* the colimit) of ideals I_K , where K varies over all finite extensions of k contained in L . A finite extension field K of k contained in L is necessarily separable over k and

hence from Theorem 4.3 it follows that the inclusion $k \rightarrow K$ is indeed decidable. In view of (ii) of Theorem 4.1, our assertion is evident. $\square$

Corollary 4.5. *Let F be a function-field over a ground-field k and let L be a finite separable extension of F . Let R be a domain containing k and with quotient-field L . Assume R is of essentially finite type over k and set $S := R \cap F$. Then the inclusion map $S \rightarrow R$ is decidable if and only if it is unramified.*

Proof. Since $k \subseteq S$, it is clear that R is of essentially finite type over S . Hence the assertion follows from Theorem 4.3. $\square$

Corollary 4.6. *Let F, L be as above. Suppose V is a valuation domain with quotient field L and $W := V \cap F$. Assume that the integral closure of W in L is a finite module over W . Then, the inclusion map $W \rightarrow V$ is Decidable if and only if it is unramified.*

Proof. Again, V being of essentially finite type over W , Theorem 4.3 applies. $\square$

Remarks: The integral closure of W in L is well-known to be a finite module over W in the following cases (see for example Chapter VI of [6]).

1. W (and hence also V) is Noetherian.

2. The residue field of W (hence also that of V) is of characteristic 0.
3. L is a normal extension of F and $[L : F]$ is not divisible by the characteristic of F .

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Vita

John Iskra was born in Michigan on June 28, 1964, the fiftieth anniversary of the assassination of Archduke Ferdinand; The event which started the Great War. In between that and the Great War he fought to finish this thesis, not much of international significance happened to him. He graduated in 1987 from the University of Michigan in Ann Arbor with a B.S. in Mathematics. Two years later he moved to Oregon where he taught secondary school and met his wife, Betty Jo. Ten years and two boys later he finished his PhD. He currently lives in Covington, Georgia where he teaches at Emory University's Oxford College.