

To the Graduate Council:

I am submitting herewith a thesis written by Jo Ann Jordan entitled "A Numerical Study of the Effects of Curvature on the Fluid Dynamics of Gas Centrifuges." I have examined the final copy of this thesis for form and content and recommend that it be accepted in partial fulfillment of the requirements for the degree of Master of Science, with a major in Mathematics.

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A NUMERICAL STUDY OF THE EFFECTS OF
CURVATURE ON THE FLUID DYNAMICS OF
GAS CENTRIFUGES

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ABSTRACT

A finite element method for the approximate solution of the flow in rapidly rotating gas centrifuges is presented. The Onsager model, as amended by Maslen, is used in deriving the model equations to be discretized. The pancake effects are not assumed in the model, i.e., curvature terms are retained. To show the effects of these terms on the hydrodynamics of a gas centrifuge, numerical examples done with and without these curvature terms are presented and compared. Two flow models are used for the examples, one for flow driven by a linear temperature gradient along the wall and the other for flow driven by axial mass fluxes through the end caps of the centrifuge.

PREFACE

The primary use of the gas centrifuge today is to enrich uranium to produce fuel for light water reactors. Naturally occurring uranium consists of essentially two isotopes, U-238 and U-235. The U-235 isotope is the fissionable isotope which makes uranium suitable for use as fuel. Before natural uranium can be used as reactor fuel it must be enriched to 3 to 4 percent by weight in the U-235 isotope. (Only 0.711 percent of natural uranium is U-235.) Although there are several methods for enriching uranium, for more than 40 years gaseous diffusion was the only method used by the uranium enrichment industry. Now, because of the substantial energy savings of the gas centrifuge process over the gaseous diffusion process, the gas centrifuge has been chosen as the method to provide new uranium enrichment capacity. A centrifuge facility is under construction in Portsmouth, Ohio, and other centrifuge programs are being developed throughout the world. For more details see Olander [1981].

Von Halle [1977] describes a typical gas centrifuge as basically a long hollow cylinder inside an evacuated casing. The cylinder or rotor is centered by a magnetic bearing assembly at the top and a needle bearing at the bottom. The rotor is spun about its axis at a high angular velocity by an electric motor located at the rotor bottom. Gaseous uranium hexafluoride is fed into the centrifuge via a stationary tube at the axis of the rotor. The withdrawal of enriched and depleted streams is also done through stationary tubes with stationary scoops extending

into the gas at the rotor ends. Figure A-1, Appendix, shows a model of the gas centrifuge.

The uranium hexafluoride gas inside the centrifuge spins at about the same angular velocity as that of the rotor. The high angular velocity of the gas causes the light isotope to be slightly enriched near the axis and the heavy isotope near the rotor wall. A countercurrent circulation, as shown by the arrows in Figure A-1, Appendix, is induced in the gas to produce large end-to-end separation effects. This countercurrent circulation may be driven by temperature differences along the rotor wall and/or end caps or by the mechanical action of the stationary bottom scoop. The top scoop is shielded to prevent a reverse effect on the circulation. Another way to drive a countercurrent circulation is axial mass fluxes through the end caps. Since all these drives are assumed to perturb the flow only slightly from solid body rotation, the equations of motion are linearized enabling the flow contributions of each of the drives to be analyzed separately and superimposed to obtain the total hydrodynamic velocity.

TABLE OF CONTENTS

SECTION	PAGE
I. INTRODUCTION.	1
II. THE ONSAGER-MASLEN MODEL.	3
III. BOUNDARY CONDITIONS OF THE ONSAGER-MASLEN MODEL	10
IV. WEAK FORMULATION.	19
V. THE FINITE ELEMENT ALGORITHM.	23
VI. EXAMPLES.	28
LIST OF REFERENCES.	40
APPENDIX.	42
VITA.	44

LIST OF FIGURES

FIGURE		PAGE
1	Basis Functions for Finite Element Spaces.	25
2	Linear Wall Temperature Drive with a Peripheral Speed of 400 m/s	31
3	Linear Wall Temperature Drive with a Peripheral Speed of 500 m/s	32
4	Linear Wall Temperature Drive with a Peripheral Speed of 700 m/s	33
5	Mass Throughput with a Peripheral Speed of 400 m/s . . .	34
6	Mass Throughput with a Peripheral Speed of 500 m/s . . .	35
7	Mass Throughput with a Peripheral Speed of 700 m/s . . .	36
A-1	Model Gas Centrifuge	43

I. INTRODUCTION

In recent years the gas centrifuge has become a viable, energy efficient method for meeting the demands of the nuclear industry for enriched uranium. The efficiency of the centrifuge in enriching uranium in the U-235 isotope is determined largely by the hydrodynamics of the fluid flow within the centrifuge. This internal flow may be described by the Onsager "pancake" model equations which were derived in an unpublished report by Onsager [1965] on the pancake approximation. Several methods of solving these equations have been presented previously, including eigenfunction expansion [Wood and Morton, 1980] and finite element methods [Gunzburger and Wood, 1983] and [Gunzburger, Wood, and Jordan, To Be Published (TBP)]. These methods have assumed a rotational speed high enough so that most of the gas is located very near the wall of the cylinder (or rotor), i.e., the perpendicular distance from the axis of rotation is set equal to the radius of the cylinder wherever it appears algebraically in the equations. In this thesis this assumption is not made. Instead, a finite element method is described for solving the Onsager pancake equations without the pancake approximation effects, i.e., curvature effects are retained. Here the Onsager model, as amended by Maslen [1979], in the absence of mass, momentum, and energy sources is considered; the perturbation flow is driven by boundary conditions such as wall temperature differences or mass flow through the boundaries.

In Sections II and III of this thesis, the Onsager model equations are derived taking into account curvature effects. A weak formulation, suitable for finite element discretization, is derived in Section IV, and a Galerkin finite element algorithm is discussed in Section V. Numerical results are presented in Section VI.

II. THE ONSAGER-MASLEN MODEL

The Onsager-Maslen model is a linear, steady-state approximation of the compressible viscous flow inside a gas centrifuge. To start the derivation let (r, θ, z) denote the cylindrical coordinates with the origin located at the bottom of the centrifuge, which we take to be a right circular cylinder, and on the axis of rotation. The z -axis coincides with the axis of rotation while the r -coordinate measures the perpendicular distance from the axis. In the case of isothermal fluid flow and solid body rotation, the velocity components in the (r, θ, z) direction are given, respectively, by Wood and Morton [1980]

$$(2.1) \quad U = 0, V = \Omega r, \text{ and } W = 0$$

and the pressure distribution is given by

$$(2.2) \quad \hat{p} = p_w \exp\left\{-A^2\left(1 - \frac{r^2}{a^2}\right)\right\}$$

where Ω is the angular velocity of the cylinder, p_w is the pressure on the wall of the cylinder, a is the radius of the cylinder, $A = a\Omega/(2RT_0)^{1/2}$, R is the specific gas constant, and T_0 is the uniform gas temperature.

The derivation of the governing equations for the flow in the centrifuge follows that of Maslen [1979]. Three assumptions are made in the derivation of this model. The first assumption is that the flow in the centrifuge is a small perturbation from the isothermal solid body rotation described by (2.1)-(2.2). By this assumption we are able to linearize the equations of viscous compressible flow. The second assumption involves the Ekman layers adjacent to the top and bottom of

the cylinder. We assume outside of these Ekman layers that axial diffusion effects are negligible. Third, we assume that the flow is axially symmetric.

The first order equations for conservation of mass, momentum, energy, and state for steady axisymmetric perturbation about an isothermal solid body rotation are given by [Wood and Morton, 1982]

$$(2.3) \quad \frac{1}{r}(\hat{\rho}u')_r + \hat{\rho}w'_z = 0,$$

$$(2.4) \quad -2\Omega\hat{\rho}v' - r\Omega^2\rho' = -p'_r + \frac{4}{3}\mu\left[\frac{1}{r}(ru'_r)_r - \frac{u'}{r^2}\right] + \mu u'_{zz} + \frac{\mu}{3}w'_{zr},$$

$$(2.5) \quad 2\Omega\hat{\rho}u' = \mu\left[\frac{1}{r}(rv'_r)_r + v'_{zz} - \frac{v'}{r^2}\right],$$

$$(2.6) \quad p'_z = \mu\left[\bar{\Delta}w' - \frac{\Omega^2 r}{3RT_0}u'_z\right],$$

$$(2.7) \quad 0 = r\Omega^2\hat{\rho}u' + k\left[\frac{1}{r}(rT'_r)_r + T'_{zz}\right],$$

$$(2.8) \quad p' = \hat{\rho}RT' + \rho'RT_0,$$

where $\hat{\rho}$ is the density of the solid body rotation, $\bar{\Delta}$ is the Laplace operator, k is the thermal conductivity, and μ is the viscosity, and where the bulk viscosity has been taken to be 0. In (2.3)-(2.8) u' , v' , w' , ρ' , p' , T' are, respectively, the perturbed velocity components in the (r, θ, z) direction, the perturbed density, the perturbed pressure, and the perturbed temperature. From the equation of state we have that $\hat{\rho} = \hat{p}/RT_0$.

For convenience we rewrite these equations using the auxiliary variable

$$(2.9) \quad \phi = \frac{T'}{T_0} - \frac{2v'}{\Omega r}.$$

Eliminating the pressure between (2.4) and (2.6) and using (2.8), we get

$$(2.10) \quad -\Omega^2 r \phi_z = \left[\frac{\mu}{\rho} (\Delta w' - \frac{\Omega^2 r}{3RT_0} u'_z) \right]_r \\ - \left[\frac{\mu}{\rho} \left(\frac{4}{3} \left(\frac{1}{r} (ru'_r)_r - \frac{u'}{r^2} \right) + u'_{zz} + \frac{1}{3} w_{zr} \right) \right]_z.$$

Also we can eliminate u' between (2.5) and (2.7) to get

$$(2.11) \quad \frac{\Omega \mu r}{2k} \left[\frac{1}{r} (rv'_r)_r + v'_{zz} - \frac{v'}{r^2} \right] = - \left[\frac{1}{r} (rT'_r)_r + T'_{zz} \right].$$

Now recall we are assuming that in the regions away from the ends of the cylinder that the axial diffusion terms are negligible. Also, we assume that the radial diffusion of radial momentum terms is small. Thus

(2.10) and (2.11) can be simplified to read

$$(2.12) \quad -\Omega^2 r \phi_z = \left[\frac{\mu}{\rho} \left(w'_{rr} + \frac{w'}{r} \right) \right]_r$$

and

$$(2.13) \quad \frac{\Omega \mu}{2k} [r^2 v'_r - rv']_r = -(rT'_r)_r.$$

Integrating (2.13) and setting the constants to zero to satisfy boundedness conditions at the axis $r = 0$, we get

$$(2.14) \quad rT'_r + \frac{\Omega u}{2k}(r^2 v'_r - rv') = 0.$$

We now consider the linear operator

$$(2.15) \quad L_1 = \frac{d}{dr} \left\{ \frac{r^3}{1 + S \frac{r^2}{a^2}} \frac{d}{dr}(\cdot) \right\}$$

where $S = \frac{\mu \Omega^2 a^2}{4kT_0} = \frac{(\gamma - 1)}{2\gamma} Pr A^2$, $Pr = C_p \mu / k$ is the Prandtl number, and C_p is the specific heat at constant pressure. Observe that

$$(2.16) \quad L_1 \left(\frac{v'}{\Omega r} - \frac{T'}{2T_0} \right) = \frac{(r^2 v'_r - rv')_r}{\Omega(1 + S \frac{r^2}{a^2})} - \frac{2S \frac{r^2}{a^2}}{\Omega(1 + S \frac{r^2}{a^2})^2} (rv'_r - v') \\ - \frac{1}{2T_0} \left[\frac{r^2}{(1 + S \frac{r^2}{a^2})} (rT'_r)_r + \frac{2r}{(1 + S \frac{r^2}{a^2})^2} (rT'_r) \right]$$

or using equations (2.5), (2.13), and (2.14)

$$(2.17) \quad L_1 \left(\frac{v'}{\Omega r} - \frac{T'}{2T_0} \right) = \frac{2\hat{\rho} r^2 u'}{\mu}.$$

Now by (2.9)

$$(2.18) \quad L_1 \left(\frac{v'}{\Omega r} - \frac{T'}{2T_0} \right) = -\frac{1}{2} L_1(\phi)$$

or by (2.15), (2.17), and (2.18)

$$(2.19) \quad \left\{ \frac{r^3}{1 + S \frac{r^2}{a^2}} \phi_r \right\}_r = \frac{-4\hat{\rho} r^2 u'}{\mu}.$$

Equations (2.3), (2.12), and (2.19) form a system of three equations for u' , w' , and ϕ . We rewrite these equations in dimensionless form by letting $\eta = r/a$, $y = z/a$, $Re = \rho_w \Omega a^2 / \mu$, $u = u' / \Omega a$, $w = w' / \Omega a$, $\omega = v' / \Omega r$, $\rho_0 = \hat{\rho} / \rho_w$, $\rho = \rho' / \rho_w$, $p = p' / p_w$, and $\theta = T' / T_0$ where $\rho_w = p_w / RT_0$ is the density at the vertical wall of the cylinder. Equations (2.3), (2.12), and (2.19) become, respectively,

$$(2.20) \quad (\eta \rho_0 u)_\eta + \eta \rho_0 w_y = 0,$$

$$(2.21) \quad \phi_y = \frac{-1}{Re \eta \rho_0} \left[\frac{1}{\eta} \frac{(\eta w_\eta)_\eta}{\eta} \right]_\eta,$$

$$(2.22) \quad \left\{ \frac{\eta^3}{1 + S\eta^2} \phi_\eta \right\}_\eta = -4Re \rho_0 \eta^2 u,$$

where ϕ is now defined by

$$(2.23) \quad \phi = \theta - 2\omega.$$

Our next goal is to reduce this system of equations to a single partial differential equation which will be valid in regions away from the ends of the cylinder. We introduce the radial variable $x = A^2(1 - \eta^2)$ which measures distance from the rotor wall in scale heights (e-folding heights) of the ambient density which is now given by $\hat{\rho} = \rho_w e^{-x}$. Our system of equations can be written as

$$(2.24) \quad -2A^2(\eta e^{-x} u)_x + e^{-x} w_y = 0,$$

$$(2.25) \quad \phi_y = \frac{8A^6}{Re} [e^x (\eta^2 w_x)_x]_x,$$

$$(2.26) \quad \frac{A^4}{Re} \left\{ \frac{\eta^4}{1 + S\eta^2} \phi_x \right\}_x = -\eta e^{-x} u.$$

We define the stream function by the relations

$$(2.27) \quad ne^{-x}u = -\psi_y$$

and

$$(2.28) \quad e^{-x}w = -2A^2\psi_x$$

so that (2.24) is identically satisfied. Introducing (2.27) and (2.28) into (2.25) and (2.26) we get

$$(2.29) \quad \phi_y = \frac{-16A^8}{Re}(e^x(\eta^2(e^x\psi_x)_x)_x)_x$$

and

$$(2.30) \quad \left\{ \frac{\eta^4}{1 + S\eta^2} \phi_x \right\}_x = \frac{Re\psi_y}{A^4}.$$

We eliminate ϕ from these last two equations to get

$$(2.31) \quad \left\{ \frac{\eta^4}{1 + S\eta^2} (e^x(\eta^2(e^x\psi_x)_x)_x)_{xx} \right\}_x = \frac{-Re^2}{16A^{12}} \psi_{yy}.$$

We introduce Onsager's "master potential" χ defined by

$$(2.32) \quad \psi = -2A^2\chi_x.$$

Substituting (2.32) into (2.31) then yields

$$(2.33) \quad \left\{ \frac{\eta^4}{1 + S\eta^2} (e^x(\eta^2(e^x\chi_{xx})_x)_x)_{xx} \right\}_x = \frac{-Re^2}{16A^{12}} \chi_{xyy}.$$

Integrating both sides of the equation with respect to x and absorbing the integration factor into χ we have

$$(e^{\chi}(\eta^2(e^{\chi}\chi_{xx})_{\chi})_{\chi})_{xx} = \frac{-Re^2(1 + S\eta^2)}{16A^{12}\eta^4} \chi_{yy}$$

or

$$(2.34) \quad (e^{\chi}(\eta^2(e^{\chi}\chi_{xx})_{\chi})_{\chi})_{xx} + \frac{B^2(1 + S\eta^2)}{\eta^4} \chi_{yy} = 0$$

$$\text{for } 0 < x < x_T \quad \text{and } 0 < y < y_T$$

where $B = Re/4A^6$ and y_T is the location of the top of the cylinder.

Equation (2.34) is the Onsager pancake equation without the pancake approximation effects. This equation is equivalent to that given by Maslen [1979]. Also, if η is set equal to 1 this equation is equivalent to the pancake equation given by Wood and Morton [1980].

III. BOUNDARY CONDITIONS OF THE ONSAGER-MASLEN MODEL

We examine the boundary conditions which the master potential χ should satisfy. At $x = x_T$, "the top of the atmosphere," i.e., a distance in scale heights from the rotor wall at which the gas becomes rarefied, we assume that $u = \omega_x = w_x = \theta_x = 0$ [Wood and Morton, 1980], i.e., that the radial velocity and the normal derivative of the azimuthal velocity, the axial velocity, and the temperature vanish. Then, since $w_x = 0$, by (2.28) and (2.32) we have that

$$(3.1) \quad L_3 \chi(x_T, y) = 0 \text{ for } 0 < y < y_T$$

where

$$(3.2) \quad L_3 \chi = (e^x \chi_{xx})_x.$$

By (2.23) since $\theta_x = \omega_x = 0$, we have that $\phi_x(x_T, y) = 0$. But (2.29), (2.32), and (2.34) combine to

$$(3.3) \quad \phi_{yx} = \frac{-32A^{10}B^2(1 + S\eta^2)}{Re\eta^4} \chi_{yy}.$$

Integrating with respect to y yields

$$(3.4) \quad \phi_x(x, y) - \phi_x(x, 0) = \frac{-32A^{10}B^2(1 + S\eta^2)}{Re\eta^4} [\chi_y(x, y) - \chi_y(x, 0)].$$

Evaluating at $x = x_T$ yields

$$(3.5) \quad \chi_y(x_T, y) = k_1$$

where k_1 is a constant. Integrating with respect to y for a second time gives

$$(3.6) \quad \chi(x_T, y) = k_1 y + k_2$$

where k_2 is a constant. Without loss of generality, we may take $k_2 = 0$ since all variables of interest only involve derivatives of the master potential χ . We may take $k_1 = 0$ since all variables of interest except ϕ_x involve an x derivative of χ and by (3.4) ϕ_x depends only on the difference $\chi_y(x, y) - \chi_y(x, 0)$ which again is independent of k_1 . Therefore the second boundary condition at $x = x_T$ is

$$(3.7) \quad \chi(x_T, y) = 0 \text{ for } 0 < y < y_T.$$

The assumption, $u = 0$ at $x = x_T$, and (2.27) yield

$$(3.8) \quad \psi_y(x_T, y) = 0$$

or

$$(3.9) \quad \psi(x_T, y) = k_3$$

where k_3 is a constant. Setting the stream function to zero at the top of the atmosphere yields $k_3 = 0$ so that with (2.32) we have

$$(3.10) \quad \chi_x(x_T, y) = 0 \text{ for } 0 < y < y_T,$$

the third boundary condition at the top of the atmosphere.

At the wall of the cylinder, i.e., $x = 0$, we assume that the velocity components vanish and the temperature gradient along the wall is prescribed. From (2.29) and (2.32) we have

$$(3.11) \quad \phi_y(0, y) = \frac{32A^{10}}{\text{Re}} L_5 \chi(0, y)$$

where

$$(3.12) \quad L_5 x = (e^x (n^2 (e^x x_{xx})_x)_x)_x.$$

Furthermore, since $\phi = \theta - 2\omega$ and $\omega(0,y) = 0$, we have

$$(3.13) \quad \phi_y(0,y) = \theta_y(0,y)$$

which yields from (3.11) the fourth boundary condition

$$(3.14) \quad L_5 x(0,y) = \frac{\text{Re}}{32A^{10}} \theta_y(0,y)$$

where $\theta_y(0,y)$ is the prescribed wall temperature gradient. Since $w(0,y) = 0$, (2.28) and (2.32) imply that

$$(3.15) \quad x_{xx}(0,y) = 0 \text{ for } 0 < y < y_T.$$

Finally, we have that $u(0,y) = 0$ so by (2.27) and (2.32)

$$(3.16) \quad x_{xy}(0,y) = 0.$$

Integrating with respect to y from 0 to y yields

$$(3.17) \quad x_x(0,y) = x_x(0,0).$$

Since we have previously set $x_x(x_T,y) = 0$, then $x_x(0,y)$ is proportional to the net mass flow through the bottom end cap. We assume that there is no net mass flow through the bottom so that $x_x(0,0) = 0$. Thus by (3.17)

$$(3.18) \quad x_x(0,y) = 0$$

which is the last boundary condition at $x = 0$.

To complete the specification of the problem, we need to determine the boundary conditions at the top and bottom of the cylinder. Since we have neglected all axial diffusion terms in deriving equation (2.34), this equation does not adequately describe the flow in the Ekman layers adjacent to the top and bottom of the cylinder. A relation among the flow variables has been developed by Carrier and Maslen [1962] which replaces the details of the flow within the Ekman layer. Although this relation holds outside the Ekman layer, we may impose, in a manner consistent with our linearization assumption, the Carrier-Maslen relation at a physical boundary, e.g., $y = 0$.

At the bottom end cap, the Carrier-Maslen relation takes the form [Maslen, 1979]

$$(3.19) \quad \left\{ \frac{\overline{u}'}{\Omega r} - \frac{\overline{\phi}}{2(1 + S \frac{r^2}{a^2})^{1/2}} - \sqrt{\frac{4\hat{\rho}\Omega a^2}{\mu}} \frac{(1 + S \frac{r^2}{a^2})^{1/4}}{\rho_0 r^2} a^2 \overline{\psi} \right\} \\ = \left\{ \frac{-\phi}{2(1 + S \frac{r^2}{a^2})^{1/2}} - \sqrt{\frac{4\hat{\rho}\Omega a^2}{\mu}} \frac{(1 + S \frac{r^2}{a^2})^{1/4}}{\rho_0 r^2} a^2 \psi \right\}_{y=0}$$

where the overbars denote the prescribed boundary values. For convenience, we consider this equation in dimensionless form as done in our problem formulation by calling $\eta = r/a$, $Re = \rho_w \Omega a^2 / \mu$, $u = u' / \Omega a$, and $\rho_0 = \hat{\rho} / \rho_w$. Then (3.19) becomes

$$\begin{aligned}
(3.20) \quad & \left\{ \frac{\bar{u}}{\eta} - \frac{\bar{\phi}}{2(1 + S\eta^2)^{1/2}} - \frac{2\text{Re}^{1/2}(1 + S\eta^2)^{1/4}}{\sqrt{\rho_0}\eta^2} \bar{\psi} \right\} \\
& = \left\{ \frac{-\phi}{2(1 + S\eta^2)^{1/2}} - \frac{2\text{Re}^{1/2}(1 + S\eta^2)^{1/4}}{\sqrt{\rho_0}\eta^2} \psi \right\}_{y=0}
\end{aligned}$$

Changing to the variable x where $x = A^2(1 - \eta^2)$ and $\rho_0 = e^{-x}$, our equation becomes

$$\begin{aligned}
(3.21) \quad & \left\{ \frac{\bar{u}(x)}{\eta} - \frac{\bar{\phi}(x)}{2(1 + S\eta^2)^{1/2}} - \frac{2\text{Re}^{1/2}}{\eta^2}(1 + S\eta^2)^{1/4} e^{x/2} \bar{\psi}(x) \right\} \\
& = \left\{ \frac{-\phi(x,0)}{2(1 + S\eta^2)^{1/2}} - \frac{2\text{Re}^{1/2}}{\eta^2}(1 + S\eta^2)^{1/4} e^{x/2} \psi(x,0) \right\}
\end{aligned}$$

or

$$\begin{aligned}
(3.22) \quad & - \frac{4\text{Re}^{1/2}}{\eta^2}(1 + S\eta^2)^{3/4} e^{x/2} \{\psi(x,0) - \bar{\psi}(x)\} \\
& = -\bar{\phi}(x) + \phi(x,0) + \frac{2}{\eta}(1 + S\eta^2)^{1/2} \bar{u}(x).
\end{aligned}$$

From (2.28) we have

$$\begin{aligned}
(3.23) \quad & \bar{\psi}(x) = \bar{\psi}(x_T) + \frac{1}{2A^2} \int_x^{x_T} e^{-\xi} \bar{w}(\xi) d\xi \\
& = \frac{1}{2A^2} \int_x^{x_T} e^{-\xi} \bar{w}(\xi) d\xi
\end{aligned}$$

since we have set $\psi(x_T, y) = 0$. Using (3.23), (3.22) is rewritten as

$$(3.24) \quad -\frac{4\text{Re}}{\eta^2} (1 + S\eta^2)^{3/4} e^{x/2} \left\{ \psi(x, 0) - \frac{1}{2A^2} \int_x^{x_T} e^{-\xi} \overline{w}(\xi) d\xi \right\} \\ = -\overline{\phi}(x) + \phi(x, 0) + 2 \frac{(1 + S\eta^2)^{1/2}}{\eta} \overline{u}(x).$$

We need to express (3.24) in terms of the master potential χ . By (2.30) and (2.32) we have

$$(3.25) \quad \left\{ \frac{\eta^4}{1 + S\eta^2} \phi_x \right\}_x = \frac{-2\text{Re}}{A^2} \chi_{xy}.$$

Integrating with respect to x over the interval x to x_T and evaluating at $y = 0$ yields

$$(3.26) \quad \left\{ \frac{\eta^4(x)}{1 + S\eta^2(x)} \phi_x(x, 0) - \frac{\eta^4(x_T)}{1 + S\eta^2(x_T)} \phi_x(x_T, 0) \right\} \\ = \frac{-2\text{Re}}{A^2} \{ \chi_y(x, 0) - \chi_y(x_T, 0) \}.$$

The top of the atmosphere, i.e., $x = x_T$, boundary conditions are derived in part by assuming that $\phi_x(x_T, 0) = 0$ and $\chi_y(x_T, 0) = 0$. Thus (3.26) yields

$$(3.27) \quad \phi_x(x, 0) = \frac{-2\text{Re}}{A^2 \eta^4} (1 + S\eta^2) \chi_y(x, 0).$$

Differentiating (3.24) with respect to x and combining with (3.27) yields

$$\begin{aligned}
(3.28) \quad & -4\text{Re}^{1/2} \left\{ \frac{(1 + S\eta^2)^{3/4}}{\eta^2} e^{x/2} \psi(x,0) \right\}_x \\
& + \frac{2\text{Re}^{1/2}}{A^2} \left\{ \frac{(1 + S\eta^2)^{3/4}}{\eta^2} e^{x/2} \int_x^{x/2} e^{-\xi \overline{w}(\xi)} d\xi \right\}_x \\
& = -\overline{\phi}_x(x) - \frac{2(1 + S\eta^2)}{A^2 \eta^4} \text{Re} \chi_y(x,0) + 2 \left\{ \frac{(1 + S\eta^2)^{1/2}}{\eta} \overline{u}(x) \right\}_x
\end{aligned}$$

or using (2.32)

$$\begin{aligned}
(3.29) \quad & -8\text{Re}^{1/2} A^2 \left\{ \frac{(1 + S\eta^2)^{3/4}}{\eta^2} e^{x/2} \chi_x(x,0) \right\}_x \\
& - \frac{2\text{Re}^{1/2}}{A^2} \left\{ \frac{(1 + S\eta^2)^{3/4}}{\eta^2} e^{x/2} \int_x^{x_T} e^{-\xi \overline{w}(\xi)} d\xi \right\}_x \\
& - \overline{\phi}_x(x) + 2 \left\{ \frac{(1 + S\eta^2)^{1/2}}{\eta} \overline{u}(x) \right\}_x = \frac{2(1 + S\eta^2)}{A^2 \eta^4} \text{Re} \chi_y(x,0).
\end{aligned}$$

For simplification, let $B = \text{Re}/4A^6$ so that (3.29) becomes

$$\begin{aligned}
B^2 \frac{(1 + S\eta^2)}{\eta^4} \chi_y(x,0) &= -2AB^{3/2} \left\{ \frac{(1 + S\eta^2)^{3/4}}{\eta^2} e^{x/2} \chi_x(x,0) \right\}_x \\
& - \frac{B^{3/2}}{2A^3} \left\{ \frac{(1 + S\eta^2)^{3/4}}{\eta^2} e^{x/2} \int_x^{x_T} e^{-\xi \overline{w}(\xi)} d\xi \right\}_x \\
& - \frac{\text{Re}}{32A^{10}} \overline{\phi}_x(x) + \frac{B}{4A^4} \left\{ \frac{(1 + S\eta^2)^{1/2}}{\eta} \overline{u}(x) \right\}_x
\end{aligned}$$

or

$$\begin{aligned}
 (3.30) \quad & B^2 \frac{(1 + S\eta^2)}{\eta^4} x_y(x, 0) + 2AB^{3/2} \left\{ \frac{(1 + S\eta^2)^{3/4}}{\eta^2} e^{x/2} x_x(x, 0) \right\}_x \\
 & = g_0(x) \\
 & \equiv - \frac{Re}{32A^{10}} \bar{\phi}_x(x) + \frac{B}{4A^4} \left\{ \frac{(1 + S\eta^2)^{1/2}}{\eta} \bar{u}(x) \right\}_x \\
 & - \frac{B^{3/2}}{2A^3} \left\{ \frac{(1 + S\eta^2)^{3/4}}{\eta^2} e^{x/2} \int_x^{x_T} \bar{w}(\xi) d\xi \right\}_x \quad \text{for } 0 \leq x \leq x_T
 \end{aligned}$$

where $\bar{\phi}$, $\bar{u}$, and $\bar{w}$ are the prescribed combination of temperature and angular velocity, radial velocity, and axial velocity components, respectively, at the bottom of the cylinder.

A similar boundary condition may be derived for the top end cap. This is

$$\begin{aligned}
 (3.31) \quad & B^2 \frac{(1 + S\eta^2)}{\eta^4} x_y(x, y_T) - 2AB^{3/2} \left\{ \frac{(1 + S\eta^2)^{3/4}}{\eta^2} e^{x/2} x_x(x, y_T) \right\}_x \\
 & = g_1(x) \\
 & \equiv - \frac{Re}{32A^{10}} \tilde{\phi}_x(x) + \frac{B}{4A^4} \left\{ \frac{(1 + S\eta^2)^{1/2}}{\eta} \tilde{u}(x) \right\}_x \\
 & + \frac{B^{3/2}}{2A^3} \left\{ \frac{(1 + S\eta^2)^{3/4}}{\eta^2} e^{x/2} \int_x^{x_T} \tilde{w}(\xi) d\xi \right\}_x \quad \text{for } 0 \leq x \leq x_T.
 \end{aligned}$$

where the tilde denotes the prescribed boundary values at the top end cap.

The problem statement governing the master potential χ is complete. The differential equation (2.34) and the boundary conditions (3.1), (3.7), (3.10), (3.14), (3.15), (3.18), (3.30), and (3.31) are satisfied by χ . By setting $\eta = 1$ wherever it appears algebraically, we recover the Onsager pancake model used by Wood and Morton [1980], Gunzburger and Wood [1982], and Gunzburger, Wood, and Jordan [TBP], which does not account for the effects of curvature. From the solution χ , we will be able to determine ψ , u , and w by differentiating χ , i.e., $\psi = -2A^2\chi_\chi$, $w = 4A^4\chi_{\chi\chi}$, and $u = 2A^2\chi_{\chi y}$.

To recover ω and θ , an auxiliary second order elliptic equation must be solved. See Wood and Morton [1980] and Gunzburger, Wood, and Jordan [TBP]. We do not consider this in this thesis.

IV. WEAK FORMULATION

The problem is put into a weak (or Galerkin) form to prepare for a finite element solution. To achieve this, we multiply the differential equation by a smooth function $\tilde{\chi}$ and integrate the results over the domain $D = \{0 < x < x_T, 0 < y < y_T\}$. This gives us

$$(4.1) \quad \int_0^{y_T} \int_0^{x_T} \{ (e^x (\eta^2 (e^x \chi_{xx})_x)_x)_{xx} + \frac{B^2(1 + S\eta^2)}{\eta^4} \chi_{yy} \} \tilde{\chi} dx dy = 0$$

where $\eta^2 = 1 - x/A^2$. Integrating by parts the first integral three times with respect to x and the second integral once with respect to y yields

$$(4.2) \quad \int_0^{y_T} \int_0^{x_T} \{ \eta^2 (L_3 \chi) (L_3 \tilde{\chi}) + \frac{B^2(1 + S\eta^2)}{\eta^4} \chi_y \tilde{\chi}_y \} dy dx \\ - \int_0^{y_T} \{ \tilde{\chi} L_5 \chi - e^x \tilde{\chi}_x L_4 \chi + e^x \tilde{\chi}_{xx} \eta^2 L_3 \chi \} \Big|_0^{x_T} dy \\ - \int_0^{x_T} \{ B^2 \frac{(1 + S\eta^2)}{\eta^4} \chi_y \tilde{\chi} \} \Big|_0^{y_T} dx = 0$$

where L_3 and L_5 are defined in Sections 2 and 3 and L_4 is

$$(4.3) \quad L_4 \chi = (\eta^2 (e^x \chi_{xx})_x)_x.$$

We substitute the natural boundary conditions (3.1), (3.14), (3.30), and (3.31) into (4.2) and require that the test function will satisfy the

essential boundary conditions, i.e., $\tilde{\chi}_x(0,y) = \tilde{\chi}_{xx}(0,y) = \tilde{\chi}(x_T,y) = \tilde{\chi}_x(x_T,y) = 0$. This results in

$$\begin{aligned}
 (4.4) \quad & \int_0^{y_T} \int_0^{x_T} \left\{ n^2 (L_3 \chi) (L_3 \tilde{\chi}) + \frac{B^2 (1 + S n^2)}{n^4} x_y \tilde{\chi}_y \right\} dx dy \\
 & - 2AB^{3/2} \int_0^{x_T} \left\{ \frac{(1 + S n^2)^{3/4}}{n^2} e^{x/2} x_x(x, y_T) \right\}_x \tilde{\chi}(x, y_T) dx \\
 & - 2AB^{3/2} \int_0^{x_T} \left\{ \frac{(1 + S n^2)^{3/4}}{n^2} e^{x/2} x_x(x, 0) \right\}_x \tilde{\chi}(x, 0) dx \\
 & - \int_0^{x_T} \{ g_1(x) \tilde{\chi}(x, y_T) - g_0(x) \tilde{\chi}(x, 0) \} dx \\
 & + \int_0^{y_T} \tilde{\chi}(0, y) f(y) dy = 0
 \end{aligned}$$

where $f(y) \equiv \frac{\text{Re}}{32A^{10}} \theta_y(0, y)$. Integrating the second and third integrals by parts with respect to x and recalling $x_x(x_T, y) = x_x(0, y) = \tilde{\chi}(x_T, y) = 0$ yield

$$\begin{aligned}
 (4.5) \quad & \int_0^{y_T} \int_0^{x_T} \left\{ n^2 (L_3 \chi) (L_3 \tilde{\chi}) + \frac{B^2 (1 + S n^2)}{n^4} x_y \tilde{\chi}_y \right\} dx dy \\
 & + 2AB^{3/2} \int_0^{x_T} \left\{ \frac{(1 + S n^2)^{3/4}}{n^2} e^{x/2} x_x(x, y_T) \right\} \tilde{\chi}_x(x, y_T) dx \\
 & + 2AB^{3/2} \int_0^{x_T} \left\{ \frac{(1 + S n^2)^{3/4}}{n^2} e^{x/2} x_x(x, 0) \right\} \tilde{\chi}_x(x, 0) dx \\
 & = \int_0^{x_T} \{ \tilde{\chi}(x, y_T) g_1(x) - \tilde{\chi}(x, 0) g_0(x) \} dx \\
 & - \int_0^{y_T} \tilde{\chi}(0, y) f(y) dy.
 \end{aligned}$$

We define a class of functions in which we will seek our solution x and in which we will choose our test function $\tilde{x}$. We let $H(D)$ denote the space of functions with three square integrable derivatives in the x -direction and one square integrable derivative in the y -direction, i.e., if $x \in H(D)$ then

$$\int_D \int \left(\frac{\partial^j x}{\partial x^j} \right)^2 dx dy < \infty \text{ for } j = 0, 1, 2, 3$$

and

$$\int_D \int \left(\frac{\partial^k x}{\partial y^k} \right)^2 dx dy < \infty \text{ for } k = 0, 1.$$

The space $H(D)$ is an anisotropic Sobolev space in the sense of Nikol'skii [Kufner, John, and Fucik, 1977]. We introduce the subspace $H^0(D)$ defined by

$$H^0(D) = \{x \in H(D); x_x(0, y) = x_{xx}(0, y) = x(x_T, y) = x_x(x_T, y) = 0\},$$

i.e., $H^0(D)$ consists of those functions in $H(D)$ which satisfy the essential boundary conditions of the problem. We define the bilinear form

$$\begin{aligned} (4.6) \quad B(x, \tilde{x}) = & \int_D \int \left\{ \eta^2 (L_3 x)(L_3 \tilde{x}) + \frac{B^2(1 + S\eta^2)}{\eta^4} x_y \tilde{x}_y \right\} dx dy \\ & + 2AB^{3/2} \int_0^{x_T} e^{x/2} \frac{(1 + S\eta^2)^{3/4}}{\eta^2} \{ x_x(x, y_T) \tilde{x}_x(x, y_T) \\ & + x_x(x, 0) \tilde{x}_x(x, 0) \} dx \end{aligned}$$

and the linear functional

$$(4.7) \quad F(\tilde{\chi}) = -\int_0^y \tilde{\chi}(0,y) f(y) dy \\ + \int_0^x \tilde{\chi}(x,y_T) g_1(x) - \tilde{\chi}(x,0) g_0(x) dx.$$

Then the weak formulation is given by the following problem. We seek a function $\tilde{\chi} \in H(D)$ such that (4.5) holds for all $\tilde{\chi} \in H(D)$, i.e., we seek $\tilde{\chi} \in H(D)$ such that

$$(4.8) \quad B(\chi, \tilde{\chi}) = F(\tilde{\chi}) \text{ for all } \tilde{\chi} \in H(D).$$

If χ satisfies (4.8) and is smooth enough, then by retracing the steps from (2.34) to (4.5) χ satisfies (2.34) and all the boundary conditions. In other words, χ is a classical solution. There are solutions to (4.8), however, that are not smooth enough to be classical solutions of (2.34). These are called weak solutions which are important in the fact that they allow us to find generalized solutions to (2.34) when the data $f(y)$, $g_1(x)$, and $g_0(x)$ are not smooth enough for classical solutions to exist. The existence, uniqueness, and approximation of a related problem was considered by Eastham [1981].

V. THE FINITE ELEMENT ALGORITHM

To begin our finite element algorithm, we define sequences of finite dimensional subspaces S_h^0 of $H^0(D)$ parametrized by a parameter h such that h tends to zero. In our case, h is determined by partitioning the domain $\bar{D} = \{0 < x < x_T, 0 < y < y_T\}$ into rectangular elements. The interval $0 < x < x_T$ is divided into M contiguous subintervals $[x_i, x_{i+1}]$, $i = 0, 1, \dots, M-1$ where $0 = x_0 < x_1 < \dots < x_M = x_T$ and the interval $0 < y < y_T$ is divided into N contiguous subintervals $[y_j, y_{j+1}]$, $j=0, 1, \dots, N-1$, where $0 = y_0 < y_1 < \dots < y_N = y_T$ so that $\bar{D}$ is now partitioned into MN rectangles $[x_i, x_{i+1}] \times [y_j, y_{j+1}]$ for $i = 0, \dots, M-1$ and $j = 0, \dots, N-1$. With this partitioning of $\bar{D}$, we define h as

$$(5.1) \quad h = \max (h_1, h_2)$$

where

$$(5.2) \quad h_1 = \max_{0 \leq i \leq M-1} |x_{i+1} - x_i| \text{ and } h_2 = \max_{0 \leq j \leq N-1} |y_{j+1} - y_j|.$$

Because of the requirement that x will possess three L^2 derivatives in x and one in y , we require $H^0(D)$ to consist of $C^2(0, x_T)$ functions in x and $C^0(0, y_T)$ functions in y . We, therefore, choose to approximate with cubic splines in x and linear splines in y . The space of linear splines $L(0, y_T)$, with respect to the partitioning of the interval $0 < y < y_T$, is defined by

$$(5.3) \quad L(0, y_T) = \{\ell(y) \in C^0(0, y_T); \ell(y) \text{ is a linear polynomial} \\ \text{in each subinterval } [y_j, y_{j+1}], j=0, \dots, N-1\}.$$

The basis set for $L(0, y_T)$, consisting of functions with minimal support, is the set $\ell_j(y)$ of hat functions depicted in Figure 1a.

Similarly, the space of cubic splines, $S(0, x_T)$, with respect to the portion of the interval $0 < x < x_T$, is defined by

$$(5.4) \quad S(0, x_T) = \{s(x) \in C^2(0, x_T); s(x) \text{ is a cubic polynomial in} \\ \text{each subinterval } [x_i, x_{i+1}], i=0, \dots, M-1\}.$$

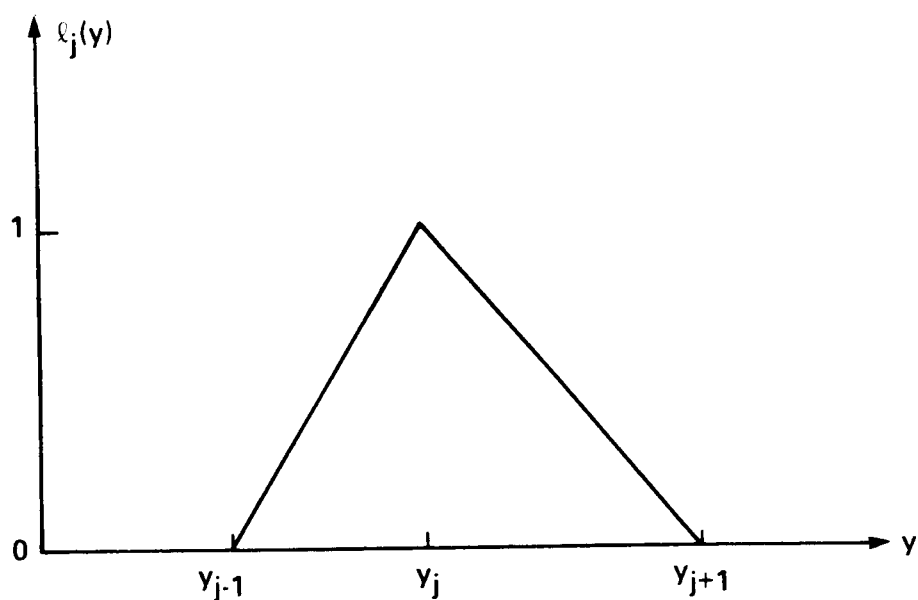
Figure 1b depicts a basis set for $S(0, x_T)$ consisting of functions of as small support as possible. These are the B-spline functions $s_i(x)$. For this set of functions we add six fictitious nodes outside the interval $[0, x_T]$, three on each end. [A node is a point (x_i, y_j) , $0 < i < M$, $0 < j < N$.] Details on the construction of both the hat and the B-spline basis functions can be found in DeBoor [1978]. Note that $\dim(L(0, y_T)) = N+1$ and $\dim(S(0, x_T)) = M+3$.

The basic finite element space we use to approximate (4.8) is the tensor product of the one-dimensional spaces $L(0, y_T)$ and $S(0, x_T)$. Specifically we let

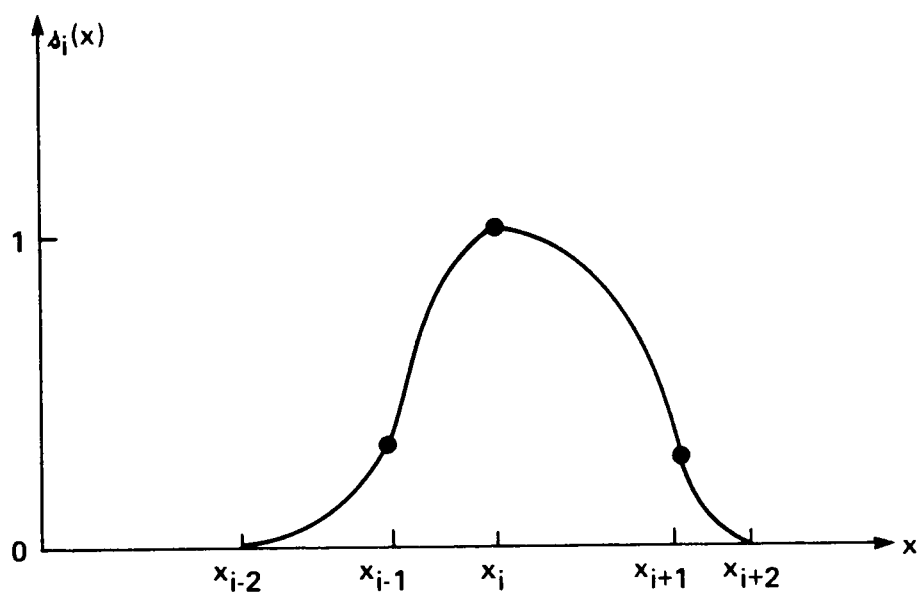
$$(5.5) \quad S^h = \{x^h(x, y); x^h(x, y) = \sum_{i=-1}^{M+1} \sum_{j=0}^N c_{ij} \ell_j(y) s_i(x)\}$$

where $\ell_j(y) \in L(0, y_T)$ is one of the hat functions, $s_i(x) \in S(0, x_T)$ is one of the B-spline functions, and c_{ij} is a set of coefficients.

Eastham [1981] shows that S^h is a subspace of $H(D)$. We then have that $\dim(S^h) = (M+3)(N+1)$ and that the set



(a) The hat function $\ell_j(y) \in C^0(0, y_T)$



(b) The B-spline function $s_i(x) \in C^2(0, x_T)$

Figure 1. Basis functions for finite element spaces.

$$(5.6) \quad x_k^h(x,y) = \{\ell_j(y) \delta_i(x)\}, \quad k = j(M+3) + (i+2)$$

for $-1 < i < M+1$ and $1 < j < N$

is a set of basis functions for S^h . We now define the subspace

$$(5.7) \quad S^{0h} = \{x^h \in S^h; x_x^h(0,y)=0, x_{xx}^h(0,y)=0, x^h(x_T,y)=0, x_x^h(x_T,y)=0\}.$$

The basis functions (5.6) can be combined into new basis functions $\hat{x}_k^h$ so that $\{\hat{x}_k^h\}$ for $k = j(M+3) + (i+2)$, $1 < i < M-1$ and $0 < j < N$, are a basis set for S^{0h} [Gunzburger and Wood, 1982]. We have that $\dim(S^{0h}) = (M-1)(N+1)$.

The approximate problem is given by: seek a $x^h \in S^{0h}$ such that

$$(5.8) \quad B(x^h, \tilde{x}^h) = F(\tilde{x}^h) \quad \forall \tilde{x}^h \in S^{0h}.$$

It suffices to choose $\tilde{x}^h = \hat{x}_\ell^h$ for $\ell = j(M+3) + (i+2)$ for $1 < i < M-1$ and $0 < j < N$. Then (5.8) is equivalent to the matrix problem

$$(5.9) \quad \vec{Ac} = \vec{b}$$

where

$$(5.10) \quad A_{rs} = B(\hat{x}_k^h, \hat{x}_\ell^h) \quad 1 < r, s < (M-1)(N+1)$$

with $k = j(M+3) + (i+2)$, $r = j(M-1) + i$, $\ell = j'(M+3) + (i'+2)$ and $s = j'(M-1) + i'$. Here i, j and i', j' are node counters as are k and ℓ while r, s count only those nodes whose associated coefficient c_{ij} is not determined by the essential boundary conditions to vanish. The

$(M-1)(N+1)$ vector $\vec{c}$ has components $c_s = c_{i,j}$, while the $(M-1)(N+1)$ vector $\vec{b}$ has components $b_{r_0} = F(x_\ell^h)$. Since the bilinear form $B(x, \tilde{x})$ is symmetric and coercive on $H(D)$, the matrix A is symmetric and positive definite and thus the existence and uniqueness of the approximate solution is assured. In (5.6) we have used a node numbering system wherein we sweep in the x -direction first then in the y -direction. Hence, the half bandwidth of the matrix A is given by $(M+2)$ while A has $(M-1)(N+1)$ rows and columns.

VI EXAMPLES

In this section, the numerical results obtained by using a computer code which was written to implement the algorithm described in Section V are presented. This finite element code has the capability to solve for an approximate solution to the Onsager pancake equation both with and without curvature effects, i.e., η is a function of radius or set equal to 1 respectively. To show curvature effects on hydrodynamics within the centrifuge, numerical examples were calculated with both choices of η for two different methods of inducing a countercurrent flow. The first method is for the countercurrent flow induced by a linear temperature distribution along the cylinder wall, with the temperature on the two end caps being constant and equal to the corresponding temperature of the cylinder wall. The end-to-end temperature difference is 1 K and, the hotter end cap is on the bottom. This case was studied by both Wood and Morton [1980] and Gunzburger and Wood [1982]. The second method is for the countercurrent flow induced by introducing and removing mass through the end caps. This type of countercurrent drive has been studied by Gunzburger and Wood [1982] and by Soubbaramayer [1979]. A uniform mass flux of unit amplitude is introduced in the interval $7 < x < 8$ on the bottom end cap and in the interval $1 < x < 2$ on the top end cap and is removed in the interval $1 < x < 2$ on the bottom and $7 < x < 8$ on the top. In this case, isothermal boundaries are assumed.

The centrifuge parameters which are described in Table 1 are those of a hypothetical centrifuge and were obtained from the paper of Gunzburger and Wood [1982]. These are also the same parameters used in the papers of Durivault and Louvet [1976] and May [1977]. These parameters are used in the calculations for both flow models.

Table 1
CENTRIFUGE PARAMETERS

Length	335.3 cm
Radius	9.145 cm
Wall Pressure	13.3 kPa (100 torr)
Peripheral Speed	400, 500, 700 m/s
Average Temperature	300 K

The grid spacing for all calculations was determined by the transformations $x = \bar{x}^2/x_T$ and $y = y_T \sin^2(\pi\bar{y}/2y_T)$ with even spacing in $\bar{x}$ and $\bar{y}$. Calculations for the linear wall temperature drives were performed using a mesh of 24 X 24 and with $x_T = 8$. The cases involving axial mass fluxes at the ends were performed using a mesh of 35 radial grid points and 20 axial grid points and with $x_T = 11$. These values of x_T were large enough so that any further increase in their magnitude did not produce any appreciable change in the solution.

One measure of the countercurrent flow is provided by the spatial distribution of the axial mass flux. Figures 2-7 show the axial mass flux ($\text{gm}^{-2}\text{s}^{-1}$) as a function of radius measured in scale heights at the axial location $y_T/4$. Each figure displays two curves, one with curvature effects and one without. Figures 2-4 are linear wall temperature drive cases at peripheral speeds of 400, 500, and 700 m/s, respectively, and Figures 5-7 are the cases involving mass flow through the end caps at the same peripheral speeds.

Small differences in the axial mass flux are noticed for the linear wall temperature cases as compared to the larger differences encountered in the mass throughput cases. This is attributed to the fact that the mass throughput cases have significant flow higher in the atmosphere, i.e., further from the wall, than the linear wall temperature cases. The curvature effects do decrease, however, as the peripheral speed is increased due to the increased centrifugal acceleration which forces the flow closer to the wall. The pancake assumption, $n = 1$ wherever it appears algebraically, is expected to become more accurate under these conditions. The results illustrated by Figure 4 compare well with the results presented by Wood and Morton [1980], and the results illustrated by Figure 7 compare well with the results given by Gunzburger and Wood [1982] except the units used by Gunzburger and Wood for their axial mass flux are dimensionless. When the proper scaling factor is applied, the two plots are identical for the no curvature case.

The conclusions drawn from the axial mass flux plots (Figures 2-7) are confirmed when we look at Tables 2 and 3. Tables 2 and 3 contain

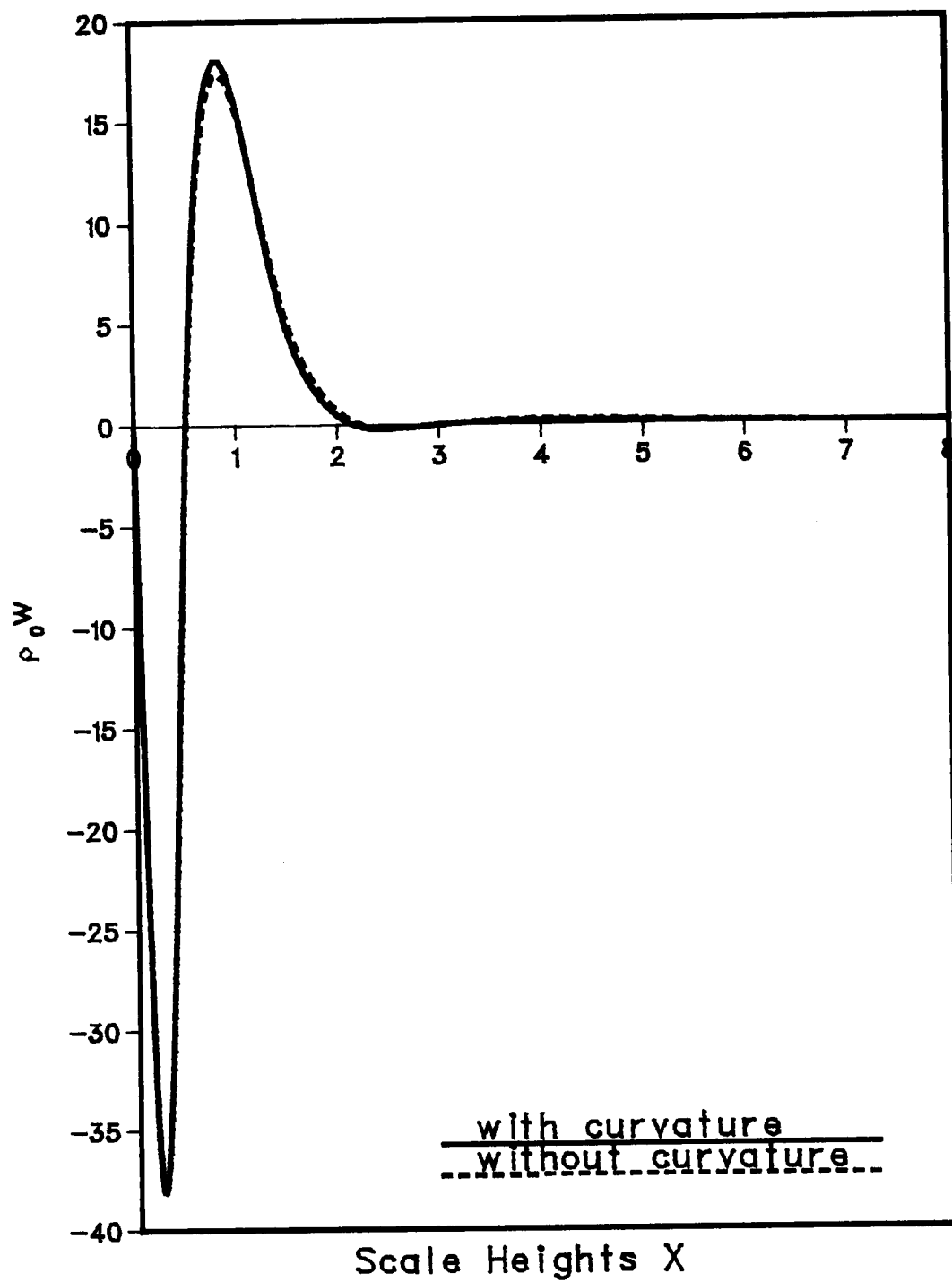


Figure 2. Linear wall temperature drive with a peripheral speed of 400 m/s.

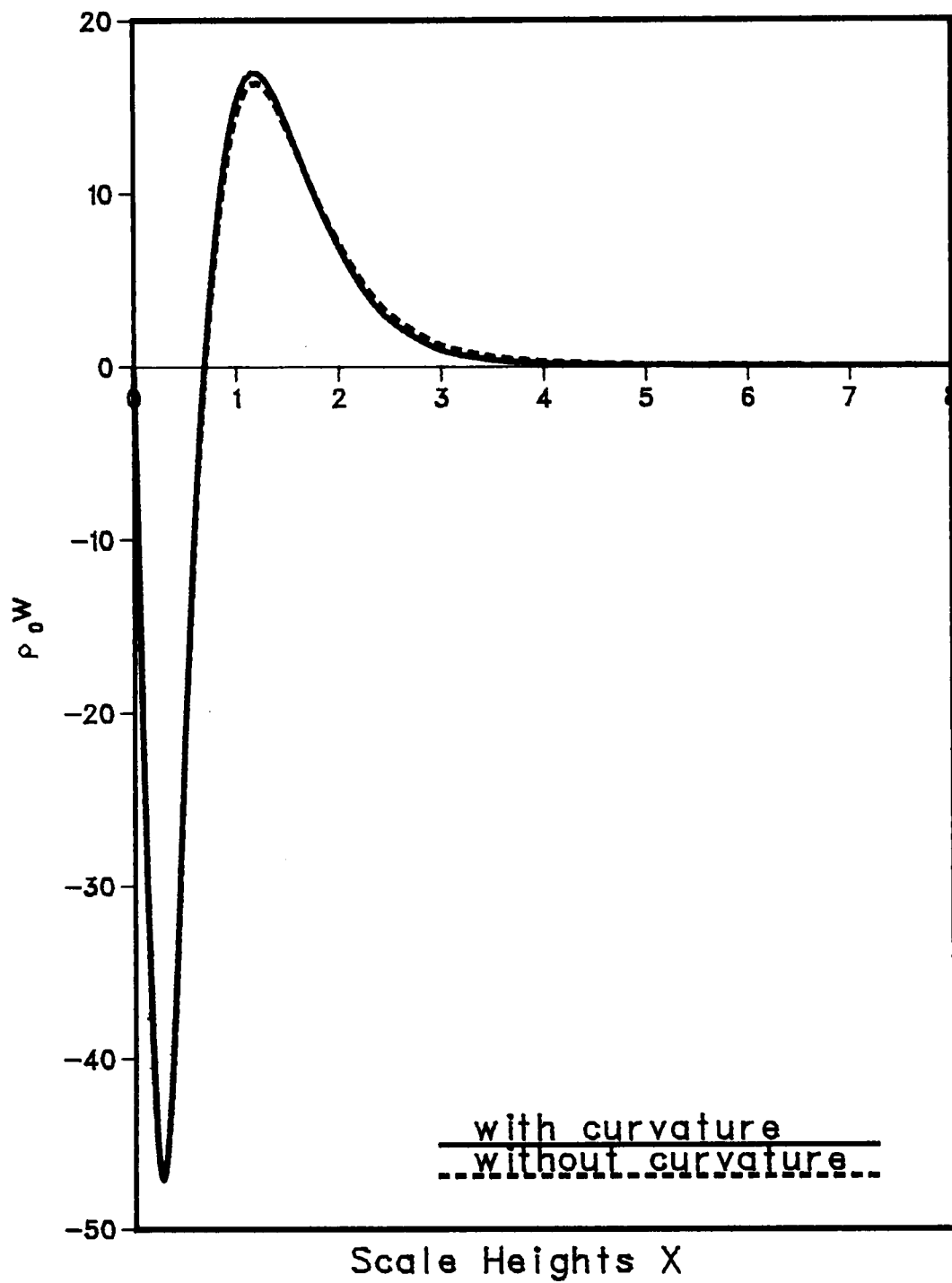


Figure 3. Linear wall temperature drive with a peripheral speed of 500 m/s.

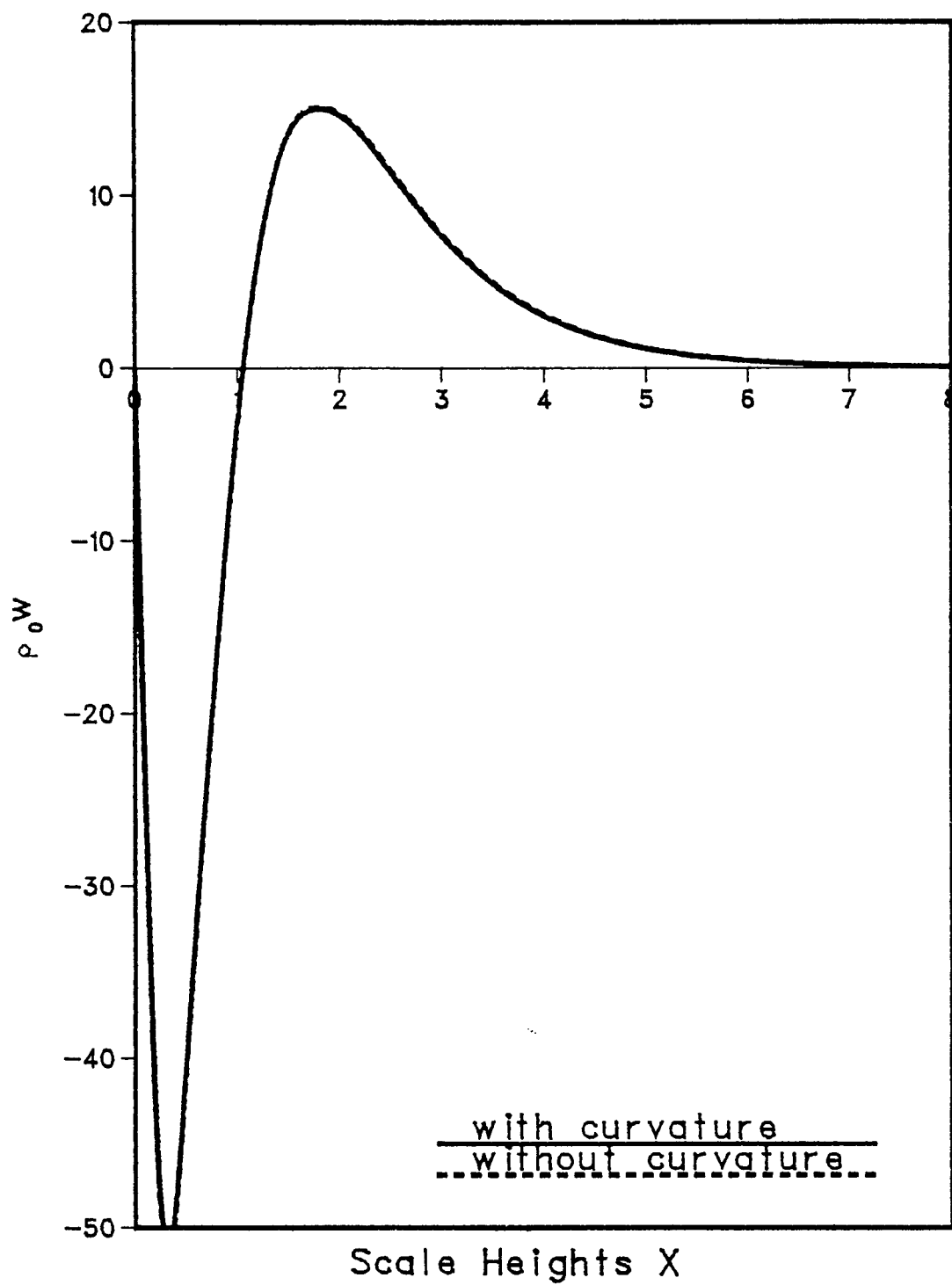


Figure 4. Linear wall temperature drive with a peripheral speed of 700 m/s.

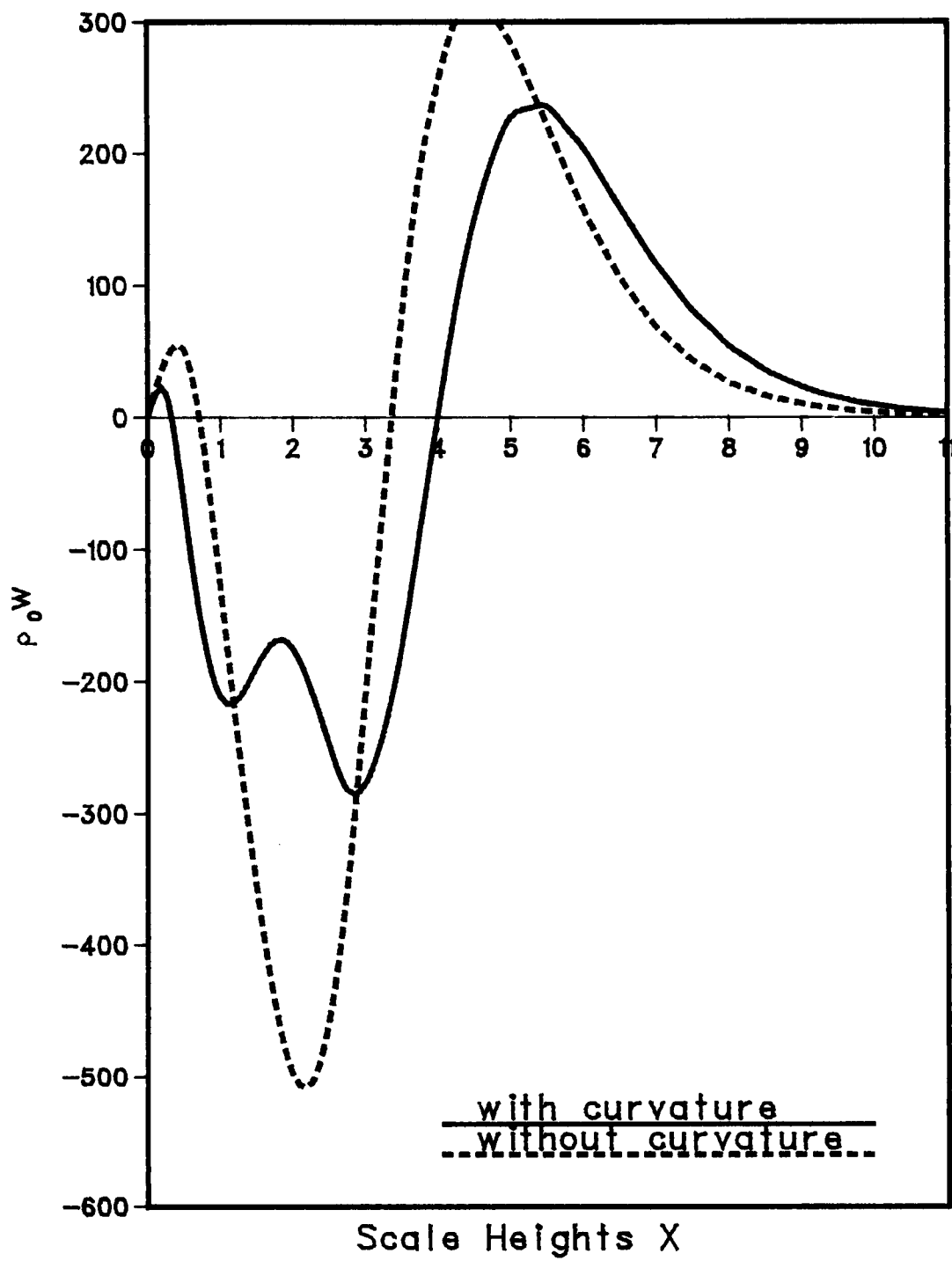


Figure 5. Mass throughput with a peripheral speed of 400 m/s.

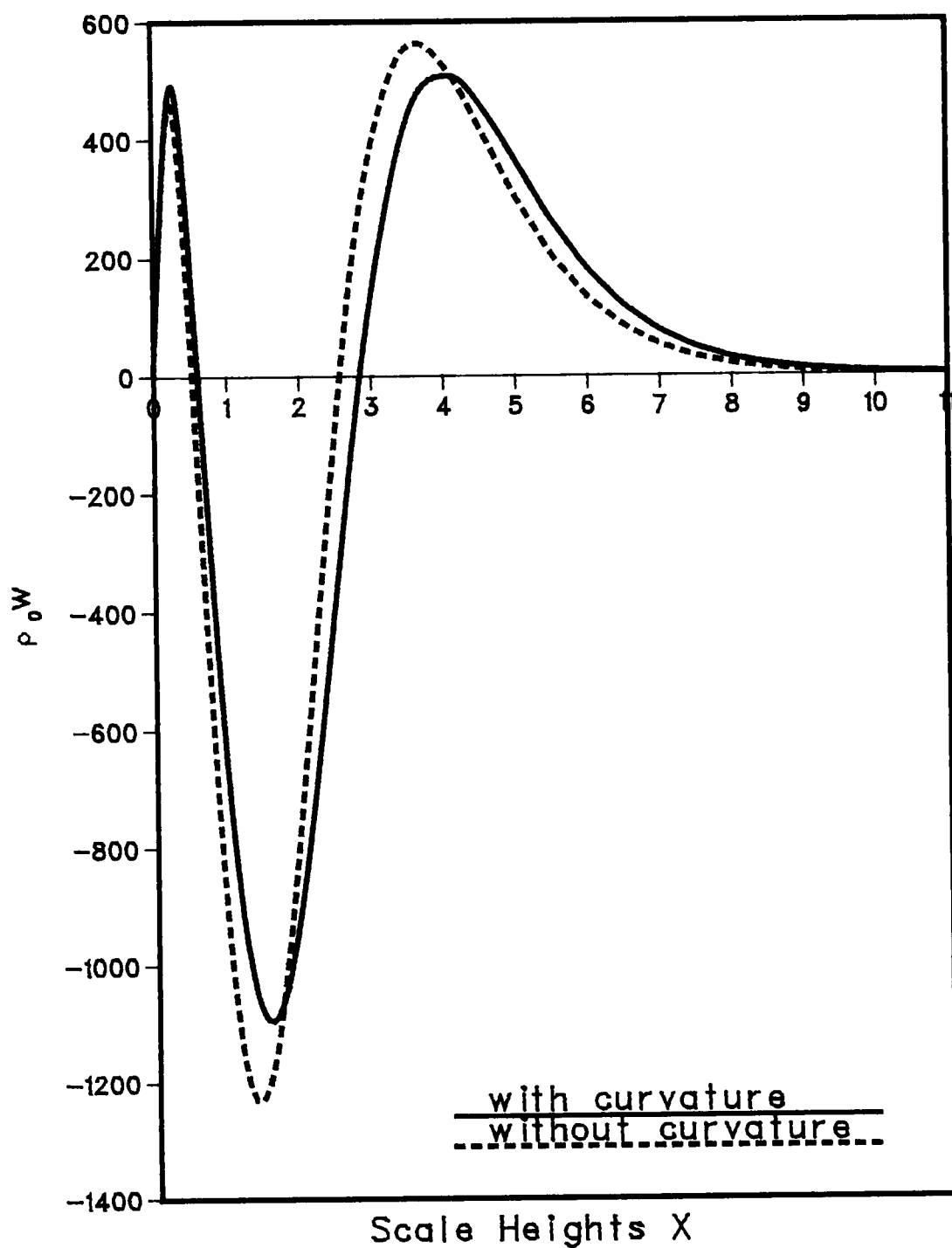


Figure 6. Mass throughput with a peripheral speed of 500 m/s.

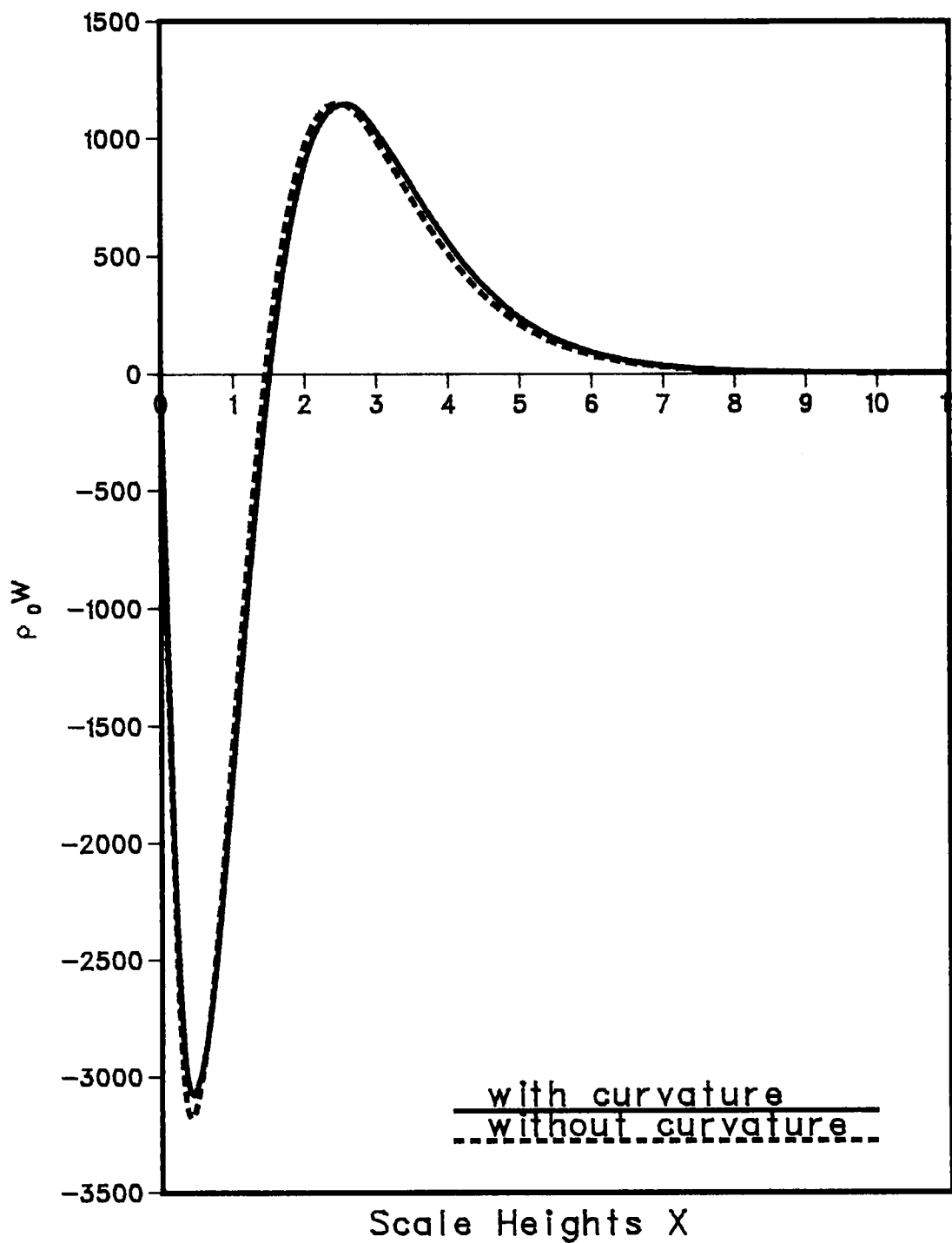


Figure 7. Mass throughput with a peripheral speed of 700 m/s.

Table 2
L² NORMS FOR LINEAR WALL TEMPERATURE CALCULATIONS

Peripheral Speed (m/s)	$\ \psi_c\ _2$	$\ \psi\ _2$	$\frac{\ \psi_c - \psi\ _2}{\ \psi_c\ _2}$
400	5.838E-7	5.996E-7	5.051E-2
500	5.261E-7	5.378E-7	3.923E-2
700	3.771E-7	3.820E-7	1.303E-2

Table 3
L² NORMS FOR MASS THROUGHPUT CALCULATIONS

Peripheral Speed (m/s)	$\ \psi_c\ _2$	$\ \psi\ _2$	$\frac{\ \psi_c - \psi\ _2}{\ \psi_c\ _2}$
400	6.350E-5	6.808E-5	3.008E-1
500	5.735E-5	5.771E-5	1.727E-1
700	3.976E-5	3.843E-5	5.506E-2

the L^2 norms ($\| \cdot \|_2$) of the stream function ψ . The L^2 norm is defined as follows

$$\| \cdot \|_2 = \sqrt{\int_D (\cdot)^2}.$$

In Table 2 we look at $\|\psi_c\|_2$ where ψ_c is the stream function with curvature, $\|\psi\|_2$ where ψ is the stream function without curvature, and $\|\psi_c - \psi\|_2 / \|\psi_c\|_2$ for the linear wall temperature cases of 400, 500, and 700 m/s peripheral speeds. Table 3 is identical to Table 2 except it shows the L^2 norms for the mass throughput cases. Again, these norms confirm the previous conclusion that more curvature effects exist in the mass throughput cases than in the linear wall temperature cases. In addition, from the norms we can see that the pancake assumption is not as accurate when the peripheral speed is decreased for the linear wall temperature cases. This was not as readily apparent from the axial mass flux plots.

Although in high speed centrifuges the assumption of neglecting curvature is usually made, no detailed analysis has been reported in the literature. This thesis is the first attempt to quantify the effects of curvature on the hydrodynamics of a centrifuge. Retaining the curvature terms was found to have some effects on the countercurrent flow. Larger effects were noticed in the mass throughput cases and in the cases where the peripheral speed was decreased. The increased effects are attributed to the fact that the cylindrical boundary layer is radially thicker in these cases, i.e., the countercurrent flow is further from the wall.

Assuming $\eta = 1$ wherever it occurs algebraically will not be as accurate in these cases as for the linear wall temperature cases and the higher speed cases in which the countercurrent flow is closer to the wall. From the results presented in this thesis, the pancake assumption appears to be valid for high speed centrifuge calculations. However, for lower speeds, especially in cases where the countercurrent drives force the flow high in the atmosphere, appreciable differences are evident.

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APPENDIX

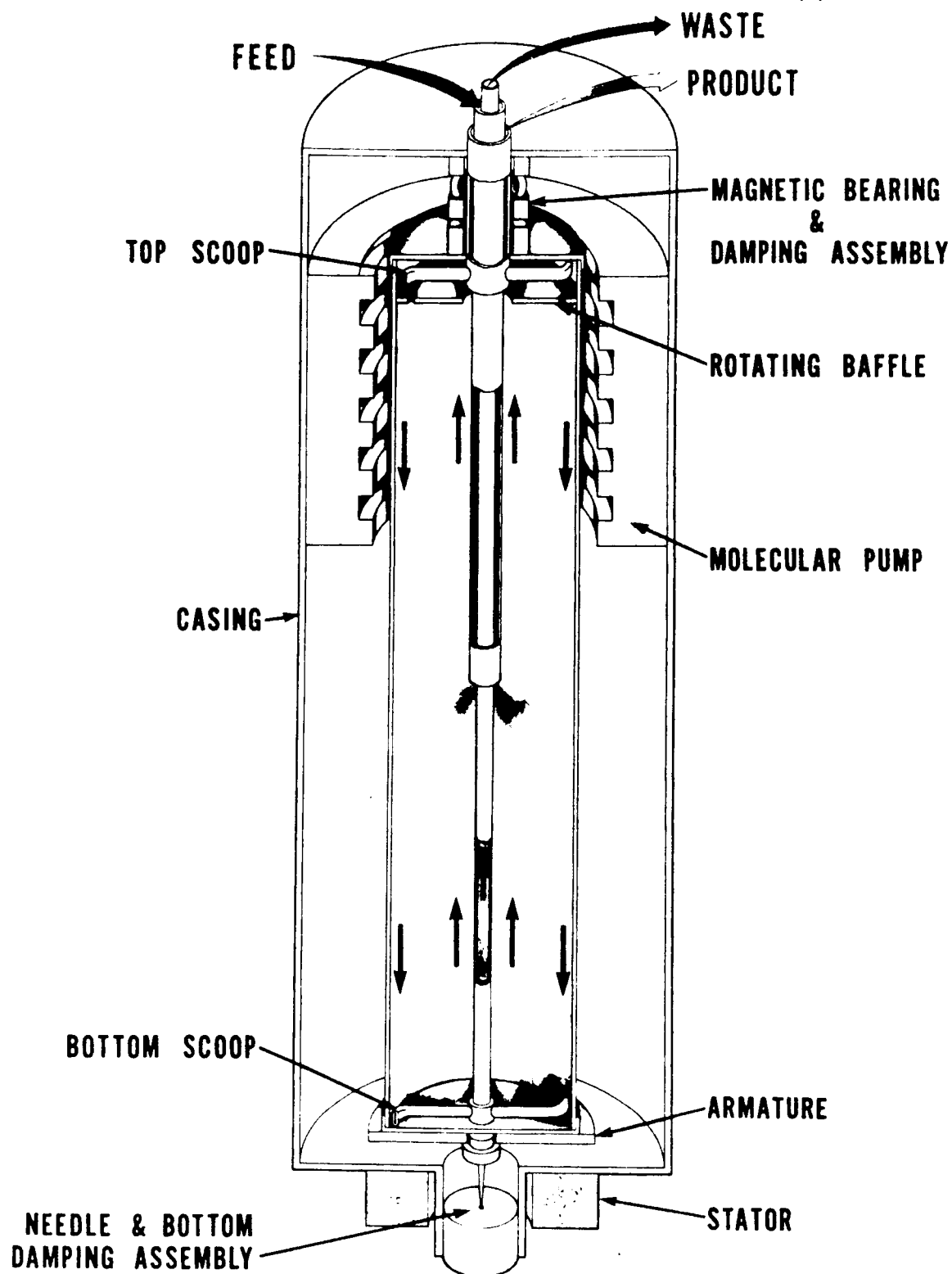


Figure A-1. Model gas centrifuge.

VITA

Jo Ann Jordan was born in Bethesda, Maryland, on February 23, 1956. She spent her early years in Northern Virginia where she attended elementary schools and graduated in June 1974 from Falls Church High School. She entered George Mason University as a full-time student in the fall of 1977 where she recieved a Bachelor's degree in January 1980. That same month, she accepted a Graduate Teaching Assistantship at The University of Tennessee, Knoxville, where she began her work towards a Master's degree. In June 1980, she accepted a position with Union Carbide Corporation, Nuclear Division, in Oak Ridge, Tennessee, as a mathematician with the Advanced Gas Centrifuge (AGC) project. She has continued her studies at The University of Tennessee while working for Union Carbide.

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