

Phase Dynamics of LOCSET Control Methodology

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ABSTRACT

Single-mode fiber amplifiers produce diffraction-limited beams very efficiently. Maximum beam intensity requires that an array of these amplifiers have their beams coherently combined at the target. Optical path differences and noise adversely affect beam quality. An existing closed loop phase control methodology, called the locking of optical coherence by single-detector electronic-frequency tagging (LOCSET), corrects phase errors in real time by electronically detecting path length differences and sending signals to lithium niobate phase adjusters. Broadening the line-width using “jitter” of the input signal can increase the output power of an individual amplifier by suppressing nonlinearity. The system dynamics of LOCSET are derived in consideration of laser line-width, phase jitter for SBS mitigation, path length errors, and noise. It is shown that the system dynamics satisfies differential equations of Kuramoto type when the response dynamics are low. Stability analysis is applied to this model to describe the region of stable operation.

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CHAPTER 1

PUBLICATION NOTE

This thesis includes a version of material being prepared for publication (Bochove, Neschke et al. 2015). The publication's authors are Erik J. Bochove, Brendan Neschke, Niketh Nair, Paul M. Delgado and Yehuda Braiman. My main contributions to this publication are in Section 2.3 entitled "Finding Steady State Solutions" and Section 2.4 entitled "Temporal Evolution and Stability". Dr. Bochove created an early version of the derivations for the system's dynamic equations. I assisted him in refining the derivations and elaborating on the assumptions that make up Section 2.2 of the publication. The derivations are presented in detail in Chapter 3 of this thesis.

I have presented oral presentations of the material in this thesis on several occasions. Twice, I presented details of my work and its conclusions in front of groups at the Air Force Research Laboratory in Albuquerque, NM. I also presented this material at the Poster Conference organized by the AFRL Summer Scholar program.

CHAPTER 2

INTRODUCTION

2.1 Coherent Single-Mode Fiber Arrays

Single-mode fiber laser amplifiers produce diffraction-limited beams very efficiently and compactly, making them a desirable choice for many high-power laser applications. Despite these qualities, chemical and bulk solid-state lasers outperform fiber lasers in total output power. The intensity available from single-mode optical fibers is limited by optical surface damage, thermal loads, and nonlinear optical effects (Agrawal 2007, Flores, Pulford et al. 2013). Arranging arrays of these amplifiers to coherently combine can compensate for this disadvantage. The motivation for this thesis is to increase the power and radiance of this class of lasers by phasing. This section reviews how coherent combination of the beams from individual single-mode fiber amplifiers realizes greater radiance, defined as the power per unit solid angle spread per unit area (Fan 2005).

Nonlinear and thermal effects limit a fiber laser's potential radiance. The features that make single-mode fiber amplifiers attractive also inhibit their performance at high power. Stimulated Brillouin Scattering (SBS) is a third-order nonlinear interaction where optical power is transferred from the laser field to the backscattered Stokes light. This detracts from the amplification and may irreversibly damage the amplifier through pulsation. One common way of achieving higher levels of power from an individual single-mode amplifier is to use a broadening of the signal's line-width to counter the effects of SBS, and thus increase the potential power available (Agrawal 2007). Applying a phase jitter to the incoming signal is a common method of

increasing its line-width (Agrawal 2007). Despite this, the output power from individual amplifiers is too small to meet the demands of directed energy applications (Flores, Pulford et al. 2013).

The goal of greater radiance requires using several single-mode amplifiers at once. In the unsophisticated case, one can direct N independently operating amplifiers at the target. The sources would be mutually incoherent and the resulting radiance would be no greater than that of any individual element (Fan 2005). This is the same as N times greater intensity. To achieve the highest levels, an array of the fiber amplifiers should coherently combine in the far-field. Coherent laser beam combining can be used to increase optical radiance: in an ideal implementation there is a factor of N greater radiance, or N^2 times greater intensity (Fan 2005).

Coherence is difficult because the phases of signals change over time. Control strategies can correct for the varying optical phase states of each individual beam in the system. Noise and other influences affect the optical phase differences (OPD) dynamically over time causing destructive interference (Fan 2005).

In laser physics, phase synchronization of an array to enhance beam brightness is by convention classified into passive (Fan 2005) and active methods (Goodno, Asman et al. 2007, Redmond, Creedon et al. 2011). Both passive and active systems are closed-loop: one utilizes inherent physics and the other uses control systems via internal electronics. In passive methods, coherence of the phases occurs by optically coupling the dynamics of individual array elements. The underlying physics provides the mechanism for the phasing. Applying feedback to each array element from a common external cavity allows the signal of each element to influence the

others (Fan 2005). Examples, as shown in Figure 1, include fiber ring (Bochove and Shakir 2009) and self-Fourier cavity (Corcoran, Durville et al. 2007).

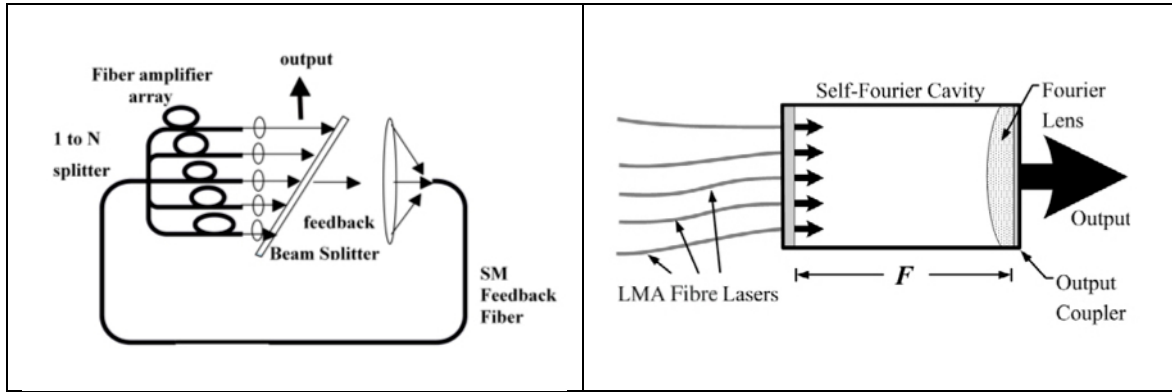


Figure 1. Schematics of passive coherent beam combining techniques.

The left panel shows the schematic for a fiber ring design: Figure 1 from (Bochove, Aceves et al. 2011). The right panel shows the schematic for a self-Fourier cavity: Figure 1 from (Corcoran, Durville et al. 2007).

In active phasing methods, the feedback comes from electronic signals that result from measurements of the array's output field. The control signals activate mechanical or electro-optic phase modulators located on the array. The active control methods have the potential to scale to very large numbers of amplifiers. There are many variations on the array design, but most follow the master-oscillator power-amplifier (MOPA) scheme, where the array of amplifiers receives identical signals from a common master oscillator (Goodno, Asman et al. 2007). Each signal receives the appropriate phase correction prior to amplification to ensure that the signals are coherently combined in the far-field. Figure 2 displays a general MOPA configuration.

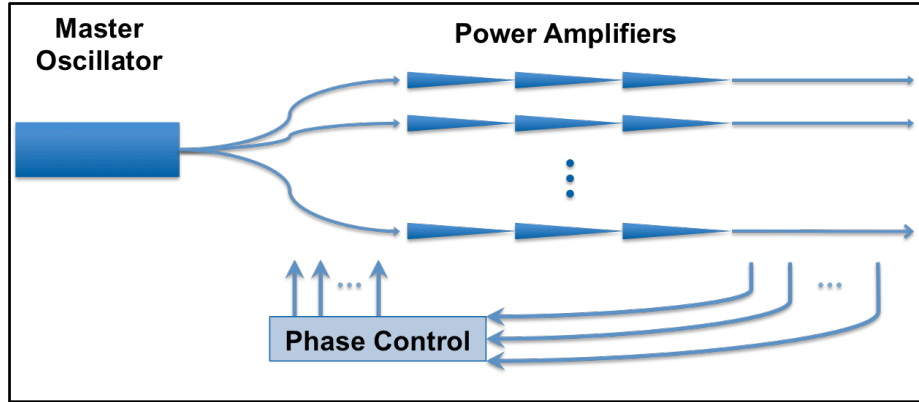


Figure 2. Active phase control using MOPA scheme.

The master oscillator sends identical signals to the power amplifiers. Phase errors are corrected by electronic feedback signals.

2.2 LOCSET Active Phase Control System

2.2.1 Overview of LOCSET

The LOCSET (locking of optical coherence by single-detector electronic-frequency tagging) model achieves active control of the optical phase lengths by changing the phase of the signals entering the amplifier array according to the MOPA configuration. Adaptive phase control was discussed generically by O'Meara (1977) and then for fiber lasers by Shay, who introduced the LOCSET terminology (2006). In the O'Meara work, sinusoidal multi-phase dithering techniques tried to compensate for atmospheric turbulence that distorted the different beams' paths (O'Meara 1977). In LOCSET, an array of fiber amplifiers receives the output of a master oscillator (MO) after it is split into N separate beams. Each beam is phase-modulated ("dithered") at distinct radio frequency (RF) tag frequencies before amplification. The active control mechanism measures and controls the phases in an efficient closed-loop control system (Shay 2006).

For the input into the controller, a beam divider splits off a fraction of the array's output beams into a photo-detector. The resulting photocurrent contains a sum of terms that are modulated at the tag frequencies. Interference beat notes from the dithering, measured from the output of the array, contain information about phase changes. The sine of the phase errors is obtained using an electronic heterodyning technique and proportionally converted into voltages. Lithium niobate phase correctors built into the amplifiers correct the phases. The simplicity of the control system gives a lot of promise for its potential to operate at large numbers of amplifiers, $N \sim 100$. There is only one input to the controller. Also, the N outputs all act independently on the signals: the influence of one control signal, defined in the next section, does not immediately influence other beams until the closed-loop round trip is finished (Pulford 2011).

2.2.2 Details of LOCSET Operation

The LOCSET feedback control setup for an amplifier array is illustrated in Figure 3. All of the mathematical and descriptive symbols in this thesis are defined in Table 1 on the next page. The single-mode master oscillator (MO) sends identical signals $E^{in}(t)$ to each of N elements the array. The frequency of the signal is centered about ω_0 and has a line-width generated from two sources: the natural variation $\phi(t)$ of the MO phase and a deliberately imposed phase jitter $\psi_J(t)$. These are defined in Equation (1) where the jitter has amplitude m_J and frequency ω_J . The natural phase variation is assumed to follow a Gaussian diffusion process.

$$\phi(t) = \text{Gaussian phase diffusion process}, \quad \psi_J(t) = m_J \sin \omega_J t \quad (1)$$

Table 1. List of symbols.

<i>Symbol</i>	<i>Description</i>
α_n	phase shift of the n^{th} tag signal
$C_{j\ell}(t)$	phase modulation difference function from signals j and ℓ
γ	natural line-width of the master oscillator
δ_n	phase adjustment applied at the n^{th} phase modulator
E_0	amplitude of input signal
$E^{\text{ff}}(t)$	on-axis far-field signal
$E^{\text{in}}(t)$	input signal from the master oscillator
$E_n^{\text{out}}(t)$	output signal from the n^{th} amplifier
ε_n	phase adjustments, shifted by the mean of $\omega_0 T_n$, for signal n
f_J	jitter related terms in the coupling coefficients
g_n	signal gain from the n^{th} amplifier
η_n	variation from the mean optical delay through amplifier n
$I(t)$	photocurrent measured by output photodetector
$I_d(t)$	terms in photocurrent that may have frequency components v_d
J	Jacobian of the linearized system
$J_n(x)$	first order Bessel functions
$\mathbf{K}_{\ell n}$	coupling coefficients of the differential equations
m_J	amplitude of jitter applied to the master oscillator
μ	variation from the mean frequency of the master oscillator
M_n	modulation depth of the n^{th} tag signal, sign is the polarity of the beam
N	number of amplifiers in the array
$N_{\ell n}$	noise terms with indices ℓ and n
v_n	frequency of the n^{th} tag signal
PE_d	phase error found from demodulation of the d^{th} beam
$p_n(t)$	tag phase modulation (dither) of signal in n^{th} amplifier
R	scaling factor between photocurrent and far-field intensity
$S_{d\ell}(t)$	product of line shape function and phase jitter function
$S'_{d\ell}, S''_{d\ell}$	real and imaginary coefficients of $S_{d\ell}(t)$, respectively
T_n	time delay through the optical path in the n^{th} amplifier
τ	period of integration in the demodulation
$\phi(t)$	natural variation of the master oscillation phase
$\Phi_{j\ell}(t)$	line shape function from signals j and ℓ
$\psi_J(t)$	jitter applied to the phase of the master oscillator
$\Psi_{j\ell}(t)$	phase jitter function from signals j and ℓ
ω_0	mean central frequency of the master oscillator
ω_J	frequency of jitter applied to the master oscillator

The jitter is applied before injection into the array for the purpose of broadening the spectrum further to suppress the nonlinear conversion processes of Stimulated Raman Scattering (SRS) and SBS (Agrawal 2007). Both of these terms are included in the input signal defined in Equation (2). As the line-width increases, however, the coherence length decreases (Agrawal 2007). Thus, the controller would have less margin of error to have synchrony. The central frequency is ω_0 .

$$E^{in}(t) = E_0 e^{i(\phi(t) + \psi_J(t))} e^{-i\omega_0 t} \quad (2)$$

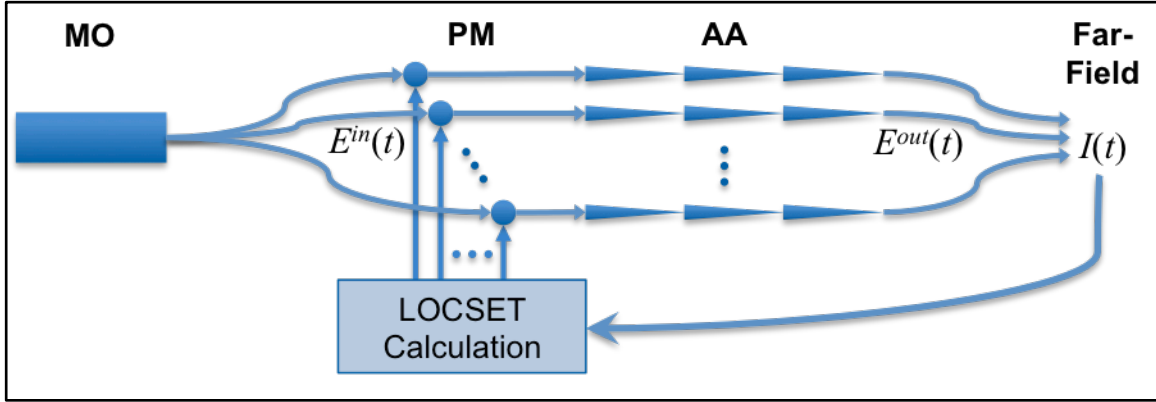


Figure 3. Closed-loop control of fiber amplifier array using LOCSET.

Phase modulators (PM) adjust the signals $E^{in}(t)$ from the master oscillator (MO) before they pass through the amplifier array (AA). The far-field intensity is used in determining phase adjustments.

Before the signals $E^{in}(t)$ from the MO are amplified by the N elements, the phase modulators (PM) adjust the phases in two ways: they apply unique tag phase modulations $p_n(t)$ and phase adjustments δ_n . Separate devices do this in practice. The tag phase modulation $p_n(t)$ (or “dither”) is sinusoidal with frequencies ν_n in the radiofrequency range. The modulation depth M_n (or amplitude of the phase modulation) of the dithering is very low, as shown below.

$$p_n(t) = M_n \cos(v_n t - \alpha_n), \quad |M_n| = M \ll 1, \quad n = 1, \dots, N \quad (3)$$

The phase adjustments δ_n are derived from the phase error signals as discussed in the next chapter. The phase errors are measured from the beat notes of the dithering from different beams. The signals generated by small phase errors are approximated by the lower order terms in the Taylor expansion of cosine functions. As a result, the signal applied to each phase shifter is proportional to the difference of the beam's phase on the detector and the average phase of the entire array. The feedback mechanism directs the array's output phases towards their instantaneous collective mean value. The controlling one of the beam's phases will result in an updated collective mean once that signal has propagated down the array. The proportional control discussed later ensures all phases closely follow the collective mean, so their phase differences will all be close to zero. This creates an ideally synchronous state, even if the collective mean is dynamic in time. Enhanced phase errors, from adverse effects discussed in Section 2.2.3, can break the linear assumption (Bochove, Neschke et al. 2015).

This thesis follows the model of the self-synchronous LOCSET system. To obtain the equations for the self-referenced LOCSET system, the modulations applied to the first amplifier, δ_1 and M_1 , should be set to zero (Shay, Benham et al. 2007). The amplifier array (AA) contributes to each signal a gain g_n and time delay T_n , which depends on time varying optical path lengths. The resulting output signal from each amplifier is $E_n^{out}(t)$.

$$E_n^{out}(t) = E_0 g_n e^{i(\phi(t-T_n) + \psi_J(t-T_n))} e^{-i\omega_0(t-T_n)} e^{i\delta_n + iM_n \cos(v_n(t-T_n) - \alpha_n)} \quad (4)$$

The α_n values, inside the phase modulation, are tuned before operation to cancel out the $v_n T_n$ terms. The tuning however is limited to the output magnitude of the cosine function, so the

value $v_n T_n + \alpha_n$ becomes either 0 or π . This results in an unknown factor of ± 1 outside the cosine function. This factor is included in the M_n terms for simplicity. The M_n may have different signs but their magnitudes are the same. The sign of any M_n could be switched if the polarity at the phase modulator is flipped. The possible changes in T_n over time are not dramatic enough to affect the quantity of $v_n T_n$ with respect to π . Thus, the expression for the output signal for each amplifier is now simplified below.

$$E_n^{out}(t) = E_0 g_n e^{i(\phi(t-T_n) + \psi_J(t-T_n))} e^{-i\omega_0(t-T_n) + i\delta_n} e^{iM_n \cos v_n t} \quad (5)$$

The on-axis far-field signal $E^{ff}(t)$ is the sum of the output signals if one considers the detected beam signals to be linearly independent and well aligned.

$$E^{ff}(t) = \sum_{n=1}^N E_0 g_n e^{i(\phi(t-T_n) + \psi_J(t-T_n))} e^{-i\omega_0(t-T_n) + i\delta_n} e^{iM_n \cos v_n t} \quad (6)$$

The far-field intensity is read by a photodetector. The current $I(t)$ measured at the photodetector is proportional to the total incident radiant flux, or the product of the far-field signal with its complex conjugate as shown below. The sampling rate of the current is sufficiently large compared to the timescale of the stochastic phase fluctuations $\phi(t)$ that the photodetector measurements reflect their time averaging. The factor R converts the intensity value into the current.

$$I(t) = R \sum_{j=1}^N \sum_{\ell=1}^N E_0^2 g_j g_\ell e^{i(\phi(t-T_j) - (\phi(t-T_\ell) + \psi_J(t-T_j) - \psi_J(t-T_\ell)))} e^{i\omega_0(T_j - T_\ell) + i(\delta_j - \delta_\ell)} e^{i(M_j \cos v_j t - M_\ell \cos v_\ell t)} \quad (7)$$

Over a period τ , the photocurrent from the far-field intensity is demodulated in Equation (8). The current is multiplied by the cosine of one of the tag frequencies and integrated. This is

equivalent to finding the Fourier series coefficients of the tag frequencies in the photocurrent. The result of this operation determines the phase error for the d^{th} beam, PE_d , which is used to update the phase adjustments δ_d . It is performed concurrently for all beams at the end of each period τ .

$$PE_d = \frac{1}{\tau} \int_{-\tau/2}^{\tau/2} I(t) \cos v_d t dt \quad (8)$$

In order to define this phase error, the tag frequency components inside the photocurrent must be isolated. Section 3.1 uses a series of expansions to find the relevant frequency components of the current $I(t)$: those with frequency v_d .

2.2.3 Advantages and Limitations of LOCSET

Benefits of the LOCSET control methodology include its single input and independently acting controllers. These two features make the methodology simple and clean. The only input into the system is the current from the photodetector. The different frequency components within this signal do not interfere with each other and allow for almost no crosstalk. The phase adjustments applied to the system from the controller are independent of each other. One can safely assume that the adjustments of one do not influence the others until the signal has propagated throughout the system. Many control systems lack the ability to independently adjust state variables. This contributes to the scalability of the control system (Flores, Pulford et al. 2013).

If the goal is to maximize brightness from an array of fiber amplifiers, one should account for the adverse effects of laser line-width, phase jitter, path length errors and noise. In many well-established fibers amplifier systems, the effects of these factors are fairly well

understood. Reviewing literature on how the LOCSET model operates under these conditions reveals that little analysis has been performed. The work of Shay et al. (2009) assumes simple low-power inputs and neglects the contributions from the conditions above in its analysis. Their conclusions hold provided that noise and path length differences are low, as demonstrated in the next few chapters. Relevant physical properties of the system such as jitter and line-width, as described in Equation (1), and noisy changes in optical path differences across amplifiers were not fully considered in the analyses by others implementing the LOCSET technique either (O'Meara 1977, Shay 2006, Shay, Benham et al. 2007, Shay, Baker et al. 2009, Seeley 2010, Pulford 2011, Flores, Pulford et al. 2013).

Recent experiments had success implementing the LOCSET model: 32 amplifier elements with $\lambda/71$ average residual phase error was shown. For the high-power case, sixteen 100 W fiber lasers coherently combined allowing the system to attain kilowatt power levels (Flores, Pulford et al. 2013). The aim of this thesis is provide insights into how to improve the performance of the LOCSET technique. Scaling up the number of amplifiers, increasing the output per amplifier, or the robustness against noise and other effects would prove valuable.

2.3 Synchronization in Networks of Oscillators

The LOCSET control theory tries to achieve greater power and radiance by phasing the output of an amplifier array. The results from Chapter 3 reveal that the LOCSET model can be described by dynamic equations of Kuramoto type (Bochove 2014, Bochove, Neschke et al. 2015). In the Kuramoto model a weak sinusoidal coupling can bring heterogeneous oscillators operating at slightly different frequencies in phase with one another. Yoshiki Kuramoto first

developed the model to explain the behavior of biological and chemical systems (Kuramoto 2003). It has since been successfully applied to a wide range of applications. These include the locking of biological clocks, arrays of Josephson junctions, lighting from fireflies, and neural assemblies (Strogatz 2004, Acebrón, Bonilla et al. 2005).

The simplicity of the Kuramoto model leads to tractable analysis, especially when the coupling is global (or mean-field) (Acebrón, Bonilla et al. 2005). In the dynamic equations derived in this thesis, see Equation (42), the coupling of our dynamical equations works out to be a function of the sine of the phase difference from the global mean. Noise manifests inside the sine function. The Kuramoto model has many different variations and is the focus of many active research efforts to better understand synchronization mechanisms (Acebrón, Bonilla et al. 2005).

The LOCSET model exploits this fact and utilizes a proportional control. Each of the phases is corrected towards the global mean phase. The system does not use a reference phase that the system is trying to match. Rather, the observed mean phase over the previous period serves as a dynamic target point. There are solutions where all of the phases move together in unison. This achieves the goal of a fully coherent output, yet the state variables are all changing in time. This is detailed mathematically by the zero eigenvalue of the Jacobian in the later chapters. Potential problems with this approach can include lag within the control system and a mean phase that is moving too dynamically. Selection of a briefer period τ can correct some of these issues by increasing the frequency with which the phases are updated. This however would require shorter computation times and a smaller range of tag frequencies due to the shorter integration time. This is specified in the relation in Equation (20).

CHAPTER 3

DERIVATION OF DYNAMIC EQUATIONS FOR THE SYSTEM

It is shown in this chapter how the LOCSET methodology responds to the contributions of laser line-width, phase jitter, path length errors and noise. The derivations in this chapter were originally done by Bochove (2014). The derivation detailed below includes the contribution to this work in refining the derivations and elaborating on the assumptions. For the model developed in this chapter, group velocity dispersion is neglected.

3.1 Derivation of the Frequency Components of the Current

The terms in the double sum of Equation (7), printed again below, describe the contributions between output beams from amplifiers j and ℓ . The LOCSET control methodology exploits the beat notes created by beams that have different tag frequencies. The tag frequency components inside the photocurrent must first be isolated. This section uses a series of expansions to find the relevant frequency components of the current $I(t)$: all terms with frequency ν_d .

$$I(t) = R \sum_{j=1}^N \sum_{\ell=1}^N E_0^2 g_j g_\ell e^{i(\phi(t-T_j) - (\phi(t-T_\ell) + \psi_j(t-T_j) - \psi_\ell(t-T_\ell)))} e^{i\omega_0(T_j - T_\ell) + i(\delta_j - \delta_\ell)} e^{i(M_j \cos \nu_j t - M_\ell \cos \nu_\ell t)} \quad (7)$$

Below in Equation (9), the terms in the double sum of (7) are characterized by the line shape function $\Phi_{j\ell}(t)$, the phase jitter function $\Psi_{j\ell}(t)$, and the phase modulation difference function $C_{j\ell}(t)$. The functions $\Phi_{j\ell}(t)$ and $\Psi_{j\ell}(t)$ include the time-varying delays T_j and T_ℓ

experienced by the beams over their paths. The line shape function varies very quickly with respect to the sampling rate, so only its time average will be measured, as clarified later.

$$I(t) = R \sum_{j=1}^N \sum_{\ell=1}^N E_0^2 g_j g_\ell \underbrace{e^{i(\phi(t-T_j)-\phi(t-T_\ell))}}_{\Phi_{j\ell}(t)} \underbrace{e^{i(\psi(t-T_j)-\psi(t-T_\ell))}}_{\Psi_{j\ell}(t)} e^{i\omega_0(T_j-T_\ell)+i(\delta_j-\delta_\ell)} \underbrace{e^{i(M_j \cos v_j t - M_\ell \cos v_\ell t)}}_{C_{j\ell}(t)} \quad (9)$$

The double sum can be split up as the sum of symmetric terms ($j=\ell$) and the sum of pairs of asymmetric terms ($j \neq \ell$). The (j, ℓ) and (ℓ, j) terms are complex conjugates (*c.c.*) of each other.

$$I(t) = \sum_{j=1}^N \left(E_0^2 g_j^2 \right) + \sum_{j=2}^N \sum_{\ell=1}^{j-1} \left(E_0^2 g_j g_\ell \Phi_{j\ell}(t) \Psi_{j\ell}(t) e^{i\omega_0(T_j-T_\ell)+i(\delta_j-\delta_\ell)} C_{j\ell}(t) + c.c. \right) \quad (10)$$

The phase modulation difference function $C_{j\ell}(t)$ describes the beating of the tag frequencies v with amplitudes M in the output signal. It is defined in terms of its trigonometric functions.

$$C_{j\ell}(t) = e^{i(M_j \cos v_j t - M_\ell \cos v_\ell t)} = \cos(M_j \cos v_j t - M_\ell \cos v_\ell t) + i \cdot \sin(M_j \cos v_j t - M_\ell \cos v_\ell t) \quad (11)$$

$$C_{j\ell}(t) = \cos(M_j \cos v_j t) \cos(M_\ell \cos v_\ell t) + \sin(M_j \cos v_j t) \sin(M_\ell \cos v_\ell t) + i \cdot \left[\sin(M_j \cos v_j t) \cos(M_\ell \cos v_\ell t) - \cos(M_j \cos v_j t) \sin(M_\ell \cos v_\ell t) \right] \quad (12)$$

The mathematical expansions below, found from the Jacobi-Anger expansion, allow us to simplify $C_{j\ell}(t)$ in terms of Bessel functions of the first kind, $J_n(x)$.

$$\begin{aligned} \cos(M \cos wt) &= J_0(M) + 2 \sum_{n=1}^{\infty} (-1)^n J_{2n}(M) \cos(2n \cdot wt), \\ \sin(M \cos wt) &= -2 \sum_{n=1}^{\infty} (-1)^n J_{2n-1}(M) \cos((2n-1) \cdot wt) \end{aligned} \quad (13)$$

The absolute values of the modulation depths of the tag signals are small, $|M_j|=M \ll 1$.

This allows us to apply the leading term in the expansions of the Bessel functions, below.

$$J_n(M) \approx \frac{1}{n!} \left(\frac{M}{2} \right)^n \quad (14)$$

These are applied to Equation (13) to approximate it up to the second power of M .

$$\begin{aligned} \cos(M \cos wt) &\approx J_0(M) - 2J_2(M) \cos(2wt) \approx 1 - \frac{M^2}{4} \cos(2wt), \\ \sin(M \cos wt) &\approx 2J_1(M) \cos(wt) \approx M \cos(wt) \end{aligned} \quad (15)$$

These are substituted back into Equation (12).

$$\begin{aligned} C_{j\ell}(t) &\approx \left(1 - \frac{M_j^2}{4} \cos(2v_j t) \right) \left(1 - \frac{M_\ell^2}{4} \cos(2v_\ell t) \right) + (M_j \cos(v_j t)) (M_\ell \cos(v_\ell t)) \\ &\quad + i \cdot \left[(M_j \cos(v_j t)) \left(1 - \frac{M_\ell^2}{4} \cos(2v_\ell t) \right) - \left(1 - \frac{M_j^2}{4} \cos(2v_j t) \right) (M_\ell \cos(v_\ell t)) \right] \end{aligned} \quad (16)$$

$$\begin{aligned} C_{j\ell}(t) &\approx \left(1 - \frac{M_j^2}{4} \cos(2v_j t) - \frac{M_\ell^2}{4} \cos(2v_\ell t) + \frac{M_j^2}{4} \frac{M_\ell^2}{4} \cos(2v_j t) \cos(2v_\ell t) \right) \\ &\quad + (M_j M_\ell \cos(v_j t) \cos(v_\ell t)) \\ &\quad + i \cdot \left[\left(M_j \cos(v_j t) - \frac{M_j M_\ell^2}{4} \cos(v_j t) \cos(2v_\ell t) \right) \right. \\ &\quad \left. - \left(M_\ell \cos(v_\ell t) - \frac{M_j^2 M_\ell}{4} \cos(2v_j t) \cos(v_\ell t) \right) \right] \end{aligned} \quad (17)$$

Removing the higher order terms gives us the order M^2 approximation of Equation (12).

$$\begin{aligned}
C_{j\ell}(t) \approx & 1 + i \left(M_j \cos(\nu_j t) - M_\ell \cos(\nu_\ell t) \right) \\
& + M_j M_\ell \cos(\nu_j t) \cos(\nu_\ell t) - \frac{M_j^2}{4} \cos(2\nu_j t) - \frac{M_\ell^2}{4} \cos(2\nu_\ell t)
\end{aligned} \tag{18}$$

Applying this back into Equation (10) yields Equation (19). The first two rows show the terms of order M^0 , the double sum in the third row shows all of the terms of order M^1 , and the double sums in the forth and fifth rows show all of the terms of order M^2 .

$$\begin{aligned}
I(t) = & \sum_{j=1}^N RE_0^2 g_j^2 \\
& + \sum_{j=2}^N \sum_{\ell=1}^{j-1} \left(RE_0^2 g_j g_\ell \Phi_{j\ell}(t) \Psi_{j\ell}(t) e^{i\omega_0(T_j - T_\ell) + i(\delta_j - \delta_\ell)} (1) + c.c. \right) \\
& + \sum_{j=2}^N \sum_{\ell=1}^{j-1} \left(RE_0^2 g_j g_\ell \Phi_{j\ell}(t) \Psi_{j\ell}(t) e^{i\omega_0(T_j - T_\ell) + i(\delta_j - \delta_\ell)} \left(iM_j \cos(\nu_j t) - iM_\ell \cos(\nu_\ell t) \right) + c.c. \right) \\
& + \sum_{j=2}^N \sum_{\ell=1}^{j-1} \left(RE_0^2 g_j g_\ell \Phi_{j\ell}(t) \Psi_{j\ell}(t) e^{i\omega_0(T_j - T_\ell) + i(\delta_j - \delta_\ell)} \left(M_j M_\ell \cos(\nu_j t) \cos(\nu_\ell t) \right) + c.c. \right) \\
& + \sum_{j=2}^N \sum_{\ell=1}^{j-1} \left(RE_0^2 g_j g_\ell \Phi_{j\ell}(t) \Psi_{j\ell}(t) e^{i\omega_0(T_j - T_\ell) + i(\delta_j - \delta_\ell)} \left(-\frac{M_j^2}{4} \cos(2\nu_j t) - \frac{M_\ell^2}{4} \cos(2\nu_\ell t) \right) + c.c. \right) \\
& + O(M^3)
\end{aligned} \tag{19}$$

The phase error is obtained with a demodulation signal applied to the photocurrent. For the d^{th} phase error, the demodulation frequency ν_d is the tag frequency applied to the d^{th} copy of the photocurrent time series. The process is similar to finding the coefficients in a Fourier cosine series. The timespan of the integration τ is long enough to discriminate the contributions from each ν_d , but short enough so the system can respond to dynamic phase errors. Typically, the demodulation frequencies ν_d are chosen to be uniformly spaced in the RF domain.

$$PE_d = \frac{1}{\tau} \int_{-\tau/2}^{\tau/2} I(t) \cos v_d t dt, \quad \text{where } \tau \gg \frac{2\pi}{|v_n - v_d|} \quad (\forall n \neq d) \quad (20)$$

Only the terms in the third row of Equation (19) have frequency components at the various tag frequencies v_n . All of the terms in $I(t)$ relevant to the demodulation above are collected, and labeled $I_d(t)$.

$$I_d(t) = \sum_{\ell \neq d} RE_0^2 g_d g_\ell \Phi_{d\ell}(t) \Psi_{d\ell}(t) e^{i\omega_0(T_d - T_\ell) + i(\delta_d - \delta_\ell)} iM_d \cos(v_d t) + c.c. \quad (21)$$

3.2 Derivation of the Phase Error

In the previous section, the photocurrent was deconstructed to find the terms proportional to $\cos v_d t$ and therefore relevant to the demodulation: $I_d(t)$. In this section the expression in Equation (21) is expressed as simply as possible so the phase error for the d^{th} beam, PE_d , can be evaluated. This leads to a set of equations that can be used to define the closed-loop system dynamics.

To begin, the v_d frequency components of $I_d(t)$ are broken up in terms of the real and imaginary parts. The product of $\Phi_{d\ell}(t)$ and $\Psi_{d\ell}(t)$ is defined to be the function $S_{d\ell}(t)$. Its real and imaginary coefficients are denoted using single and double primes (S' and S'').

$$S_{d\ell}(t) \equiv \Phi_{d\ell}(t) \Psi_{d\ell}(t) = S'_{d\ell}(t) + iS''_{d\ell}(t) \quad (22)$$

$$I_d(t) = RE_0^2 g_d \sum_{\ell \neq d} g_\ell \left(\begin{array}{l} (S'_{d\ell}(t) + iS''_{d\ell}(t)) \\ \cdot \left(\cos(\omega_0(T_d - T_\ell) + (\delta_d - \delta_\ell)) \right) \\ + i \sin(\omega_0(T_d - T_\ell) + (\delta_d - \delta_\ell)) \end{array} \right) + c.c. \quad (23)$$

The imaginary terms cancel their complex conjugates (*c.c.*) to give the result below.

$$I_d(t) = RE_0^2 g_d \sum_{\ell \neq d} g_\ell \left(-2 \left(\begin{aligned} &S''_{d\ell}(t) \cos(\omega_0(T_d - T_\ell) + (\delta_d - \delta_\ell)) M_d \cos \nu_d t \\ &+ S'_{d\ell}(t) \sin(\omega_0(T_d - T_\ell) + (\delta_d - \delta_\ell)) M_d \cos \nu_d t \end{aligned} \right) \right) \quad (24)$$

$$I_d(t) = -2RE_0^2 g_d M_d \cos \nu_d t \sum_{\ell \neq d} g_\ell \left(\begin{aligned} &S''_{d\ell}(t) \cos(\omega_0(T_d - T_\ell) + (\delta_d - \delta_\ell)) \\ &+ S'_{d\ell}(t) \sin(\omega_0(T_d - T_\ell) + (\delta_d - \delta_\ell)) \end{aligned} \right) \quad (25)$$

The $S_{d\ell}(t)$ terms vary according to the statistically varying MO phase ϕ and the deliberately imposed sinusoidal jitter ψ_J , defined in Equation (1). The Gaussian phase diffusion of the natural variation occurs over a small time scale so only the mean value remains when the photodetector samples at the relatively large sampling rate. In Equation (26), γ is the natural line-width of the master oscillator.

$$S_{d\ell}(t) \approx \left\langle e^{i(\phi(t-T_d) - \phi(t-T_\ell))} \right\rangle_t e^{i(\psi_J(t-T_d) - \psi_J(t-T_\ell))} = e^{-\gamma|T_d - T_\ell|} e^{i(\psi_J(t-T_d) - \psi_J(t-T_\ell))} \quad (26)$$

This allows the following representation of the real and imaginary coefficients of S in terms of frequency components.

$$\begin{aligned} S'_{d\ell}(t) &= e^{-\gamma|T_d - T_\ell|} \cos(m_J \sin \omega_J(t - T_d) - m_J \sin \omega_J(t - T_\ell)) \\ S''_{d\ell}(t) &= e^{-\gamma|T_d - T_\ell|} \sin(m_J \sin \omega_J(t - T_d) - m_J \sin \omega_J(t - T_\ell)) \end{aligned} \quad (27)$$

The Jacobi-Anger expansion below describes how the cosine and sine terms in the coefficients of S can be represented as a double sum over indices q and r , both going from negative to positive infinity.

$$e^{imsin wt} = \sum_{q=-\infty}^{\infty} J_q(m) e^{iqwt} \quad (28)$$

$$\begin{aligned}
\cos(m_J \sin \omega_J (t - T_d) - m_J \sin \omega_J (t - T_\ell)) &= \sum_q \sum_r J_q(m_J) J_r(m_J) \cos((q-r)\omega_J t + \omega_J (rT_\ell - qT_d)), \\
\sin(m_J \sin \omega_J (t - T_d) - m_J \sin \omega_J (t - T_\ell)) &= \sum_q \sum_r J_q(m_J) J_r(m_J) \sin((q-r)\omega_J t + \omega_J (rT_\ell - qT_d))
\end{aligned} \tag{29}$$

$$\begin{aligned}
S'_{d\ell}(t) &= e^{-\gamma|T_d - T_\ell|} \sum_q \sum_r J_q(m_J) J_r(m_J) \cos((q-r)\omega_J t + \omega_J (rT_\ell - qT_d)), \\
S''_{d\ell}(t) &= e^{-\gamma|T_d - T_\ell|} \sum_q \sum_r J_q(m_J) J_r(m_J) \sin((q-r)\omega_J t + \omega_J (rT_\ell - qT_d))
\end{aligned} \tag{30}$$

The two double sums in Equation (30) need to be simplified. The equation for $S''_{d\ell}(t)$ can be simplified by reorganizing the summands. The double sum is first split into two collections: whether or not the index q equals r .

$$\begin{aligned}
S''_{d\ell}(t) &= e^{-\gamma|T_d - T_\ell|} \sum_q J_q(m_J)^2 \sin(q\omega_J (T_\ell - T_d)) \\
&\quad + e^{-\gamma|T_d - T_\ell|} \sum_q \sum_{r \neq q} J_q(m_J) J_r(m_J) \sin((q-r)\omega_J t + \omega_J (rT_\ell - qT_d))
\end{aligned} \tag{31}$$

All of the q and $-q$ terms in the first row above cancel since the sine function is odd (and zero when $q=0$).

$$S''_{d\ell}(t) = e^{-\gamma|T_d - T_\ell|} \sum_q \sum_{r \neq q} J_q(m_J) J_r(m_J) \sin((q-r)\omega_J t + \omega_J (rT_\ell - qT_d)) \tag{32}$$

All of the remaining components of $S''_{d\ell}(t)$, however, have frequencies that are nonzero multiples of ω_J . These cannot contribute to the demodulation integral since ω_J is much larger than any of the tag frequencies ν_d . Now, the double sum for $S'_{d\ell}(t)$ is split into two parts: whether or not the index q equals r .

$$\begin{aligned}
S'_{d\ell}(t) = & e^{-\gamma|T_d-T_\ell|} \sum_q J_q(m_J)^2 \cos(q\omega_J(T_\ell-T_d)) \\
& + e^{-\gamma|T_d-T_\ell|} \sum_q \sum_{r \neq q} J_q(m_J) J_r(m_J) \cos((q-r)\omega_J t + \omega_J(rT_\ell - qT_d))
\end{aligned} \tag{33}$$

The first sum contains only constant terms while the other sum contains only terms with frequencies that are nonzero multiples of ω_J . If the only relevant contributions from S to the demodulation are constant (no $\sin(k\omega_J t + \theta)$ terms permitted), then the $I_d(t)$ expression in (25) can be rewritten as follows.

$$I_d(t) = -2RE_0^2 g_d M_d \cos v_d t \sum_{\ell \neq d} \left(g_\ell e^{-\gamma|T_d-T_\ell|} \sin(\omega_0(T_d-T_\ell) + (\delta_d - \delta_\ell)) \cdot \sum_q J_q(m_J)^2 \cos(q\omega_J(T_\ell-T_d)) \right) \tag{34}$$

This is entered into the integral in Equation (20) to obtain the d^{th} phase error signal in Equation (35) below. The negative sign has been brought inside the sine function. As mentioned in Section 2.2, this phase error signal is a lower order term within a Taylor expansion of the output signal.

$$PE_d(t) = RE_0^2 g_d M_d \sum_{\ell \neq d} \left(g_\ell e^{-\gamma|T_\ell-T_d|} \sin(\omega_0(T_\ell-T_d) + (\delta_\ell - \delta_d)) \cdot \sum_q J_q(m_J)^2 \cos(q\omega_J(T_\ell-T_d)) \right) \tag{35}$$

3.3 Characterization of the Dynamic Equations

The phase adjustments δ_n made before each signal arrives at the amplifiers can be updated every timespan τ according to the phase error control signal. There is an additional lag in the system due to the response time of the LiNbO₃ crystal that implements the phase

adjustment. The control scheme of the LOCSET model is a proportional control, as defined below. The phase change applied to the signal of the n^{th} beam is proportional to the measured phase error, modulo 2π .

$$\delta_n(t + \tau) - \delta_n(t) \propto \tau P E_n(t) \quad (36)$$

If the timespan τ is sufficiently small compared to the time scale of the fluctuation dynamics, then the N differential equations below govern the system. The scalar R collects gain/rate related terms. The summation index ℓ now spans all amplifiers since the $\ell=n$ terms are zero.

$$\frac{d\delta_n(t)}{dt} = RE_0^2 g_n M_n \sum_{\ell} \left(g_{\ell} e^{-\gamma|T_{\ell}-T_n|} \sin(\omega_0(T_{\ell}-T_n) + \delta_{\ell} - \delta_n) \cdot \sum_q J_q(m_J)^2 \cos(q\omega_J(T_{\ell}-T_n)) \right) \quad (37)$$

These differential equations are still too cumbersome to interpret, so the non-sinusoidal parts are collected into symmetric coefficients $K_{\ell n}$. As an additional step, the time-varying and jitter terms within $K_{\ell n}$ are collected into a jitter-related function f_J in Equation (40). The following three equations define the system.

$$\frac{d\delta_n(t)}{dt} = \sum_{\ell} K_{\ell n} \sin(\omega_0(T_{\ell}-T_n) + \delta_{\ell} - \delta_n) \quad (38)$$

$$K_{\ell n} \equiv RE_0^2 M_n g_{\ell} g_n f_J(m_J, \omega_J, T_{\ell} - T_n) \quad (39)$$

$$f_J(m_J, \omega_J, T_{\ell} - T_n) \equiv e^{-\gamma|T_{\ell}-T_n|} \sum_q J_q(m_J)^2 \cos(q\omega_J(T_{\ell}-T_n)) \quad (40)$$

A shift in variables further simplifies (38). The variables ε_n can be defined as the phase adjustments δ_n added to the mean values of $\omega_0 T_n$ for amplifier n . The remainder of the argument of the sine function is collected into one term $N_{\ell n}$ for simplicity. These are examined further in Equation (50)(50)(50) later. The equations that specify the closed-loop system dynamics are in (42).

$$\varepsilon_n(t) \equiv \delta_n(t) + \bar{\omega}_0 \bar{T}_n, \quad N_{\ell n}(t) = \text{noise} \quad (41)$$

$$\frac{d\varepsilon_n(t)}{dt} = \sum_{\ell} K_{\ell n}(t) \sin(\varepsilon_{\ell}(t) - \varepsilon_n(t) + N_{\ell n}(t)) \quad (42)$$

This set of differential equations takes the form of Kuramoto oscillators. These are a widely studied set of equations often used to describe synchronous behavior in biological systems: locking of biological clocks, neural activity, firefly lighting, etc. (Kuramoto 2003, Strogatz 2004, Acebrón, Bonilla et al. 2005). The next chapter explores these equations and their dynamics in greater detail. The results are related to the theory about Kuramoto oscillators.

The jitter-related function f_J , see (40), inside the coupling coefficients $K_{\ell n}$ includes the influence of the natural variation ϕ and jitter ψ_J on the phase adjustments δ_n . It has contributions from the inputs m_J , the jitter amplitude from Equation (1), and $T_{\ell} - T_n$, the difference in the delay times. The difference in delay times multiplied by the natural line-width γ is not expected to vary too greatly. To provide insights into f_J , its dependence with respect to values of $\omega_J(T_{\ell} - T_n)$ is plotted in Figure 4 for different values of m_J . The y-axis is f_J and the x-axis is $\omega_J(T_{\ell} - T_n)$.

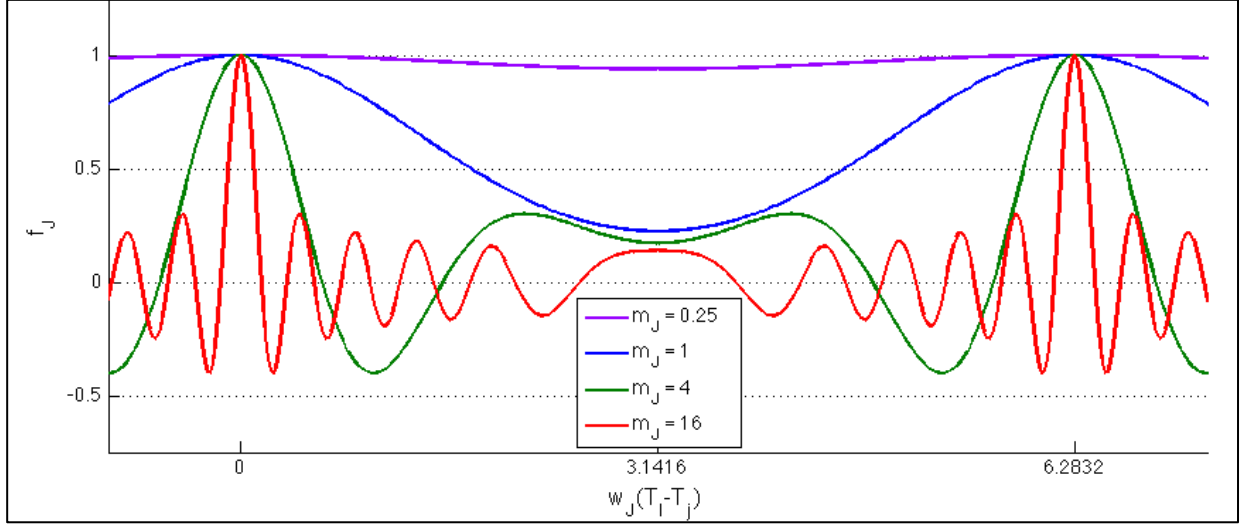


Figure 4. Plots of the jitter-related function for different values of jitter amplitude.

Plots of the jitter-related function f_J (on y-axis) from Equation (40) as a function of $\omega_J(T_\ell - T_n)$ (on x-axis) for the values of jitter amplitude m_J equal to 0.25 (violet), 1 (blue), 4 (green), and 16 (red).

The functions f_J are 2π -periodic in $\omega_J(T_\ell - T_n)$ and are bounded between ± 1 . For very small values of m_J , the function is close to one for all values of $\omega_J(T_\ell - T_n)$. As the size of m_J grows, oscillations emerge, and these cause f_J to have both positive and negative values. For any value of m_J , f_J approaches one as the magnitude of $\omega_J(T_\ell - T_n)$ approaches zero and all multiples of 2π .

The central frequency of the MO and the delay times of the amplifiers can have fluctuations that vary about their means according to Gaussian distributed variables μ and η_n as defined below where the σ 's are the standard deviations of the variables μ and η_n .

$$\omega_0(t) = \bar{\omega}_0 + \mu(t), \quad T_n(t) = \bar{T}_n + \eta_n(t) \quad (43)$$

$$P(\mu) = \frac{e^{-\mu^2/2\sigma_\mu^2}}{\sigma_\mu\sqrt{2\pi}}, \quad P(\eta_n) = \frac{e^{-\eta_n^2/2\sigma_{\eta,n}^2}}{\sigma_{\eta,n}\sqrt{2\pi}} \quad (44)$$

CHAPTER 4

STABILITY ANALYSIS

It is shown in the previous chapter that the system dynamics satisfy differential equations of Kuramoto type (similar to those describing locking of biological clocks, neural assemblies, etc.) (Kuramoto 2003, Strogatz 2004, Acebrón, Bonilla et al. 2005). The following sections report the results of investigations into the equations and their properties. The steady state solutions to the set of equations are explored in the next section. The stable operation region of the LOCSET system is described using linear stability analysis in the following section.

4.1 Steady State Solutions

In the steady, or stationary, state each time derivative term must be zero, as opposed to a solution where all phase adjustments are increasing in unison. In the absence of noise, steady state solutions are possible. An important observation is that the sum of phase adjustment derivatives is zero in the absence of the noise terms. This can be found by summing the derivatives of ε_n . All of the terms vanish since the coefficients are symmetric ($K_{\ell n} = K_{n\ell}$) and the sine function is odd.

$$\sum_n \frac{d\varepsilon_n(t)}{dt} = \sum_n \sum_{\ell} K_{\ell n} \sin(\varepsilon_{\ell}(t) - \varepsilon_n(t)) = 0 \quad (45)$$

This means that the quantity of the total phase adjustment remains constant. This also implies that the system has $N-1$ degrees of freedom, where N is the number of amplifiers in the array. The same conclusion is reached from the fact that only differences in phase appear on the

right-hand side of the equations. Change to the control implementation, the proportional control defined in Equation (36), would likely alter this trait.

One obvious set of steady state solutions is that all of the shifted phases, ε_n , are π multiples apart from each other. This corresponds to either in-phase or anti-phase behavior between every pair of fields from amplifiers ℓ and n , depending on whether the difference $\varepsilon_\ell - \varepsilon_n$ is an even or odd multiple of π . To simplify the steady state discussion, all of the fibers that are in-phase are assigned a phase equal to 0 and all of those that are anti-phase are assigned a phase equal to π . This is appropriate since only the relative phases of the fields are of interest. The conventional set of steady state solutions is defined as all of the possible combinations of in-phase and out-of-phase solutions, of which there are 2^{N-1} many. Only one of them, the case where all phases are in-phase, gives the desired output of a synchronized array. In this case, all phases are coherently combining at the output. In the next section, the stability of these states is analyzed.

It should also be noted that other steady state solutions beyond the conventional set, all combinations of $\varepsilon_n = 0$ or π , might exist when the coupling constants $K_{\ell n}$ are not all positive. The product of the non-zero coupling constants and the sine of $\varepsilon_\ell - \varepsilon_n$ need to cancel out with other non-zero terms. In practice, one would expect almost all coupling terms to be non-negative. There are N sums of these products that need to cancel to ensure the derivatives of ε_n are zero. The issue with these steady state solutions is that small changes in any of the path lengths will alter coupling values and likely break the delicate cancelations needed for these unconventional steady state solutions. In the analysis that follows, only the conventional set of steady state solutions are investigated.

4.2 Linear Stability Analysis

The linear stability analysis for this set of in-phase and anti-phase solutions requires that the eigenvalues of the linearized system be analyzed. The Equations in (42) are linearized with respect to the state variables ε_n in the absence of noise. The result is expressed in matrix form, below. The state variables are collected in a vector, and the linear coefficients are in the matrix $J=[J_{n,m}]$, which is the Jacobian of the system.

$$\frac{d}{dt}\vec{\varepsilon}_n(t) = [J_{nm}]_{n,m=1,\dots,N} \vec{\varepsilon}_n(t) + O(\vec{\varepsilon}_n(t)^2) \quad (46)$$

The Jacobian is evaluated for the set of conventional steady states defined in the previous section. The Jacobian has the following entries in its diagonal.

$$\text{For } n = m, \quad J_{nm} = J_{nn} = \frac{d}{d\varepsilon_n} \sum_{\ell} K_{\ell n} \sin(\varepsilon_{\ell} - \varepsilon_n) = - \sum_{\ell \neq n} K_{\ell n} \cos(\varepsilon_{\ell} - \varepsilon_n) \quad (47)$$

The off-diagonal elements of the Jacobian have these entries.

$$\text{For } n \neq m, \quad J_{nm} = \frac{d}{d\varepsilon_m} \sum_{\ell} K_{\ell n} \sin(\varepsilon_{\ell} - \varepsilon_n) = K_{mn} \cos(\varepsilon_m - \varepsilon_n) \quad (48)$$

The coupling terms are in each entry of the Jacobian. Notice that each coupling term is also multiplied by the cosine of the difference $\varepsilon_m - \varepsilon_n$. The arguments of the cosine function are multiples of π for the set of steady state solutions defined in the previous section, making each cosine function equal to 1 or -1 . All together, the Jacobian assumes the following definition.

$$J = \begin{bmatrix} \sum_{\ell \neq 1} K_{\ell 1} (-1)^{1 + \frac{\varepsilon_\ell - \varepsilon_1}{\pi}} & \cdots & K_{N1} (-1)^{\frac{\varepsilon_N - \varepsilon_1}{\pi}} \\ \vdots & \ddots & \vdots \\ K_{1N} (-1)^{\frac{\varepsilon_1 - \varepsilon_N}{\pi}} & \cdots & \sum_{\ell \neq N} K_{\ell N} (-1)^{1 + \frac{\varepsilon_\ell - \varepsilon_N}{\pi}} \end{bmatrix} \quad (49)$$

The eigenvalues of this matrix can now be examined. As observed earlier in Equation (45), there is a linear dependence in the system of equations. The Jacobian matrix is thus singular since the last row is dependent on previous rows. As explained in Section 2.3, this corresponds to the physical scenario of the phases moving together in unison. The Jacobian is expected to have a zero eigenvalue. This is not an issue however, since a sliding phase solution is still fully phased. Therefore, the zero eigenvalue is tolerated for the following analyses. If all of the eigenvalues, excluding the one zero eigenvalue, of the Jacobian are negative when the system is in one of the conventional steady states, the linearized differential equations will be locally asymptotically stable about that state. This will imply stability for the full system provided the linear assumption is justified. However, that condition may break down due to a number of reasons, such as under the influence of noise or if asymmetries due to path-length error (OPD) are significant, both of which may result in enhanced phase errors (Bochov, Neschke et al. 2015).

The primary steady state under consideration is the one where all phase adjustment differences ε_n are fully phased. If all of the couplings are equal to one, as is the case when the jitter amplitude m_J is very small, then the matrix has all entries equal to 1 except the diagonal, which is equal to $(1-N)$. The eigenvalues for this matrix are all negative except the one zero

eigenvalue. This makes the steady state locally asymptotically stable. A small perturbation from the steady state results in the phase derivatives directing each phase adjustment difference ε_n to the new mean phase.

The eigenvalues found above apply to only the fully phased steady state. A code in MATLAB has been written to solve for the eigenvalues for example systems with randomly selected optical path lengths T_n . For each system, the stability of the conventional set of steady states is found for different values of jitter amplitude m_J . Equations (39) and (40) are used to obtain the set of K_n and f_J . The sign of f_J determines the sign of K . Figure 4 illustrates that larger values of m_J correspond to more oscillations and a lower mean value of f_J .

Ten example systems are created with 5-element arrays, and ten more example systems are created with 7-element arrays. The “leading” eigenvalue (the eigenvalue furthest to the right on the real number line excluding the one equal to zero) for each system is calculated for many different values of m_J , the jitter amplitude from Equation (1). The system is stable when all of the eigenvalues, except the one zero eigenvalue, are negative. Below in Figure 5, the leading eigenvalue for each system’s Jacobian at the fully coherent steady state is plotted as a function of m_J . When a line rises above the value of 0, the dotted line, the system becomes unstable about the coherent steady state. It shows that all of the systems generated from the random optical path lengths T_n have stable fully coherent states when the value of m_J is below about 1.2. The fully coherent state tends to become unstable, and generally remains that way, when m_J is larger than 1.2. The number of amplifiers in the system does not significantly influence the pattern of the leading eigenvalues. After the value of m_J passes 1.2 and allows negative coupling values to exist, there are still instances where all of the eigenvalues fall back below zero. The function f_J

has a strong sensitivity to small changes in m_J when it is large, so these cases are relatively short lived: the lines soon rise back above the dotted line.

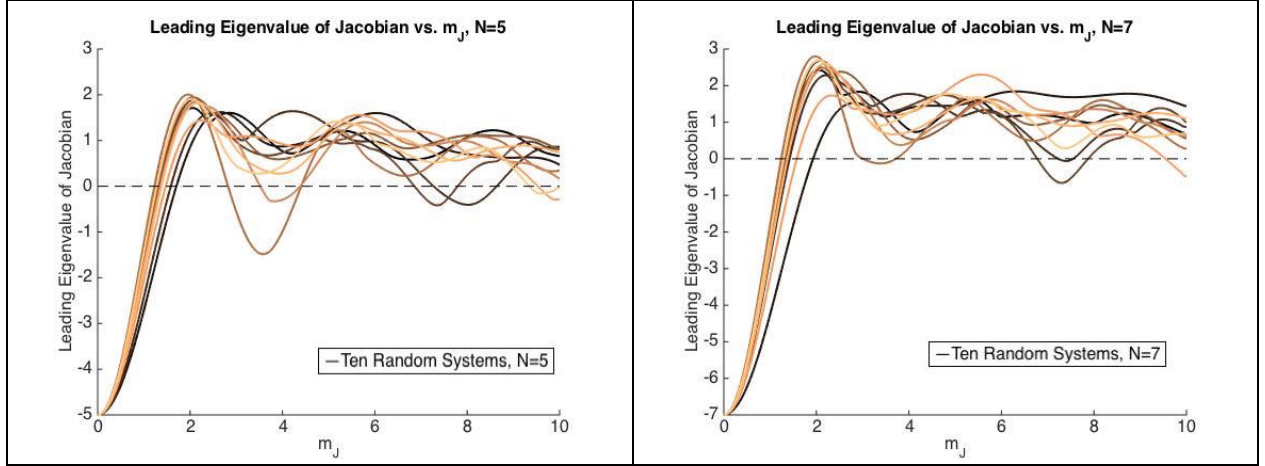


Figure 5. Examples of the evolution of the leading eigenvalue as jitter amplitude increases.

Each curve shows the leading eigenvalue of the Jacobian for the fully coherent state, as a function of the jitter amplitude m_J , for a system made from a set of randomly generated optical path lengths T_n . N values are 5 (left) and 7 (right). The stability threshold is at 0, shown by the horizontal dotted lines.

To analyze the fraction of the linearized systems with a stable fully coherent steady state, eight hundred systems are analyzed, as opposed to the ten systems in Figure 5. In Figure 6, the percentage of eight hundred randomly generated systems whose fully coherent state is stable is plotted for values of m_J between 1.0 and 2.0. Here, a dramatic dip is observed shortly after the value of 1.2. Most of the fully coherent steady states are unstable when the value of m_J is greater than 1.4. The decrease is more pronounced for a system of seven amplifiers as compared to a system of five amplifiers.

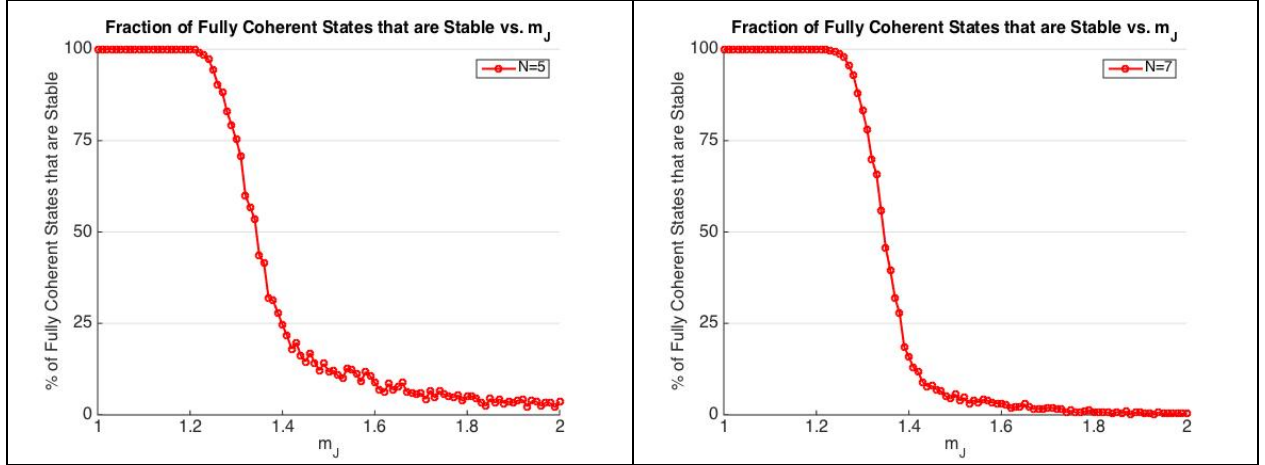


Figure 6. Fraction of systems with stable coherent state with respect to the jitter amplitude.

The percentage of 800 randomly generated systems whose fully coherent state is stable as a function of the jitter amplitude m_J from 1.0 to 2.0. N values are 5 (left) and 7 (right).

We now look at the stability of the rest of the conventional steady states: where the phase adjustments ε_n are multiples of π apart from one another. For each linearized system evaluated in Figure 6, the $2^{N-1}-1$ other steady states are all evaluated for their local stability. The plots in Figure 7 show the percentage of these steady states that are stable out of all possible non-coherent steady states in the eight hundred example systems. The curves act in an opposing manner to those in Figure 6, looking at the stability of only the fully coherent states. When the jitter amplitude is less than 1.2, the only stable steady state is the fully coherent one. It is clear that a few non-coherent states become stable when m_J is larger than 1.2, challenging the fully coherent state. The fraction for the non-coherent states never rises above six percent. This analysis includes only the conventional set of steady states where all phases are in-phase or anti-phase of one another. The end of this section describes the requirements for other steady states to exist.

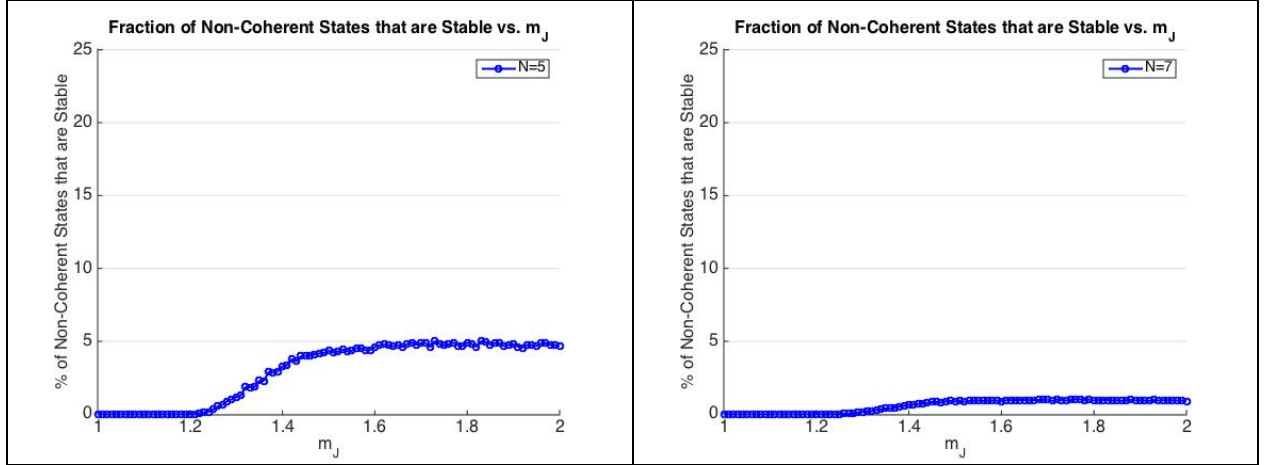


Figure 7. Fraction of stable non-coherent states as a function of the jitter amplitude.

The percent of conventional non-coherent steady states in 800 randomly generated systems that are stable as a function of the jitter amplitude m_J from 1 to 2. N values are 5 (left) and 7 (right). Note the scale is from 0% to 25%.

The phase dither for any element in the array may be out of phase by π as noted in Chapter 2. This was accounted for by applying a factor of ± 1 to the values of M_n in Equation (3). The sign of could be switched if the polarity is flipped. An unstable fully coherent steady state might become stable by switching the polarity for some subset of M_n . For the same ten systems of each array size shown in Figure 5, the stability of the fully coherent state is examined under all of the polarity combinations, after defining the first polarity to be positive ($M_1 = +1$). In Figure 8, every curve represents the leading eigenvalue for one of the linearized systems. For each value of m_J (x-axis), the percentage of the 2^{N-1} polarity combinations resulting in stability for the fully coherent state in that system is plotted. There is only one polarity combination that is stable for m_J less than 1.2. For larger values of m_J , there are many ways in which the fully coherent state may achieve stability. For systems of size five and seven amplifiers, just under half of the polarity combinations allow the fully coherent state to be stable. This is encouraging, though it may be difficult to implement polarity flipping in practical systems.

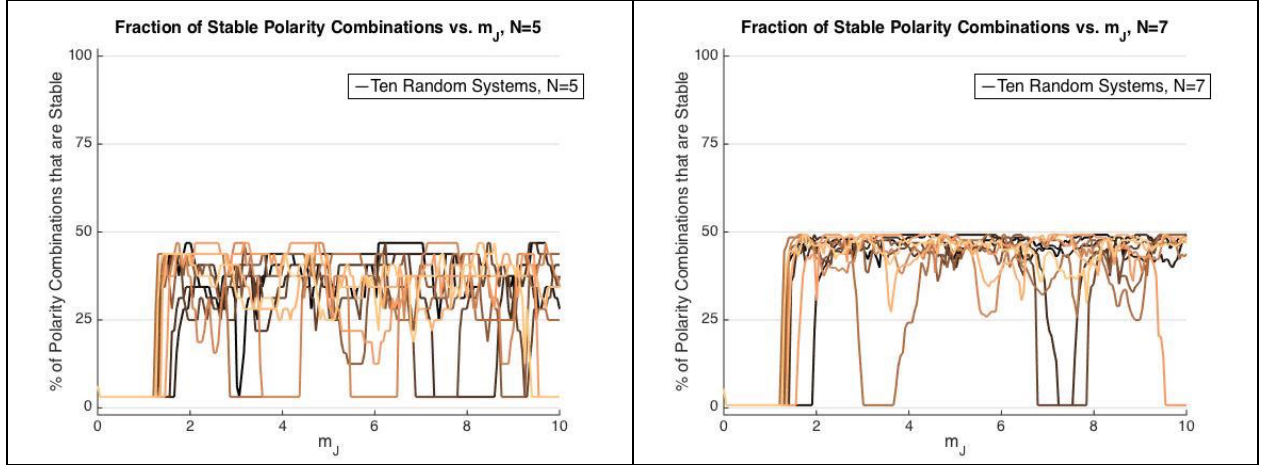


Figure 8. Fraction of stable polarity combinations as a function of the jitter amplitude.

The percentage of polarity combinations resulting in stability for the fully coherent steady state is plotted as a function of the jitter amplitude m_j . Each curve shows the results for one of the randomly generated systems in Figure 5. N values are 5 (left) and 7 (right).

So far the stability analysis is limited to the conventional set of steady states where every pair of amplifiers is either in-phase or anti-phase: $\varepsilon_n = 0$ or π , when ε_1 is defined as 0. For other steady states to exist with at least one of the sine functions in (42) nonzero, there would need to be other cancelling terms. The N^2 values of both $K_{\ell d}$ and $\sin(\varepsilon_\ell - \varepsilon_n)$ have a nonlinear dependence on the N values of T_n and ε_n that vary in time. It seems unlikely these unconventional steady states could exist for very long, especially if N is intended to scale to larger values.

4.3 Effects of First Order Fluctuations (Thermal and Mechanical Noise)

The noise terms in Equation (42) are due to small fluctuations in the central frequency and in the amplifiers' optical lengths. First order variations in Equations (43) yield the following expression for the argument of the sine function in Equation (38). The first row on the right hand side of the equation contains the terms independent of the fluctuations. The second row contains

terms that have first order variations. The third row has the terms with second order variations.

These are neglected since they are much smaller than the first order terms.

$$\begin{aligned}\omega_0(T_\ell - T_n) + \delta_\ell - \delta_n &\approx \varepsilon_\ell - \varepsilon_n \\ &+ \bar{\omega}\eta_\ell(t) + \bar{T}_\ell\mu(t) - \bar{\omega}\eta_n(t) - \bar{T}_n\mu(t) \\ &+ \mu(t)\eta_\ell(t) - \mu(t)\eta_n(t)\end{aligned}\quad (50)$$

$$\omega_0(T_\ell - T_n) + \delta_\ell - \delta_n \approx (\varepsilon_\ell + \bar{\omega}\eta_\ell(t) + \bar{T}_\ell\mu(t)) - (\varepsilon_n + \bar{\omega}\eta_n(t) + \bar{T}_n\mu(t)) \quad (51)$$

The Kuramoto equations in (42) are updated to include the first order fluctuations.

$$\begin{aligned}\frac{d\varepsilon_n(t)}{dt} &= \sum_{\ell} K_{\ell n}(t) \sin(\varepsilon_\ell(t) - \varepsilon_n(t) + \bar{\omega}\eta_\ell(t) + \bar{T}_\ell\mu(t) - \bar{\omega}\eta_n(t) - \bar{T}_n\mu(t)) \\ &= \sum_{\ell} K_{\ell n}(t) \sin(\varepsilon_\ell(t) - \varepsilon_n(t) + \bar{\omega}(\eta_\ell(t) - \eta_n(t)) + (\bar{T}_\ell - \bar{T}_n)\mu(t))\end{aligned}\quad (52)$$

The collection of fluctuations inside the sine function has been labeled $N_{\ell d}$. These vary in time and include changes to both the time delays and the central frequency. The coupling terms $K_{\ell d}$ also have the time delay variations contained within them. The fluctuations $N_{\ell d}$ and the phase differences are separated by the sum of angles identity.

$$\frac{d\varepsilon_n(t)}{dt} = \sum_{\ell} K_{\ell n}(t) \left(\sin(\varepsilon_\ell(t) - \varepsilon_n(t)) \cos(N_{\ell n}(t)) + \cos(\varepsilon_\ell(t) - \varepsilon_n(t)) \sin(N_{\ell n}(t)) \right) \quad (53)$$

The cosine of the fluctuations is close to one since the argument is assumed to be very small. The sine of the fluctuations should be proportional to the fluctuations by the same reasoning. A rough approximation of the equations is shown below, now with the variation outside of the sine function.

$$\frac{d\varepsilon_n(t)}{dt} \approx \sum_{\ell} K_{\ell n}(t) \left(\sin(\varepsilon_\ell(t) - \varepsilon_n(t)) + N_{\ell d}(t) \cos(\varepsilon_\ell - \varepsilon_n) \right) \quad (54)$$

CHAPTER 5

CONCLUSIONS AND RECOMMENDATIONS

The dynamic equations of the LOCSET system reduced to Kuramoto equations provided the use sinusoidal jitter. These well studied equations model synchronization in coupled oscillators, and have been used in many biological systems such as circadian clocks and neuron firing. This makes the system ripe for analysis. The steady-state solutions include the set of all in-phase and anti-phase combinations and others with phases that fall between zero and 2π .

Sinusoidal jitter applied to the master signal has been shown to be compatible with the LOCSET methodology. There is a maximum jitter modulation m_J that leads to stability of the fully coherent steady state solution. The Jacobian of the system is sensitive to the coupling values. Some systems may not favor stable coherence. Flipping the polarity at the site of individual phase modulators can transition the system's eigenvalues into the stable regime even beyond the critical m_J value. This compatibility may not hold for non-sinusoidal jitter types. Also shown by the derivations, the natural line-width, which is stochastically varying, multiplied by the differences in delay times needs to be small in order to limit the coupling values changes in time.

For future work on this topic, an analysis of the system's transient responses would be useful. The work in this thesis focused on the equations. Another code that evolves the system in time could be used to explore the existence of any additional stable steady states beyond the ones studied above. The relative sizes of their attracting domains could be explored, as well as the transitions between equilibria due to noise or other influences. By exploring the system in the

time domain, the limits of the control system could be explored. Parallel computations would be ideal for this approach since the N amplifiers would change independently within every period τ . At the end of each period, demodulation calculations would be used to update control signals for the next interval where the calculations could be distributed again across the processors.

Another avenue to explore in future work is the contribution from a non-sinusoidal jitter function. The influence of the jitter varies if it is not sinusoidal. This document focused on effects of the sinusoidal definition since it allowed the LOCSET model to work properly given a small amplitude m_J . Alternative versions of the jitter, including stochastic variation or the use of multiple sinusoidal functions simultaneously, would need to be explored further.

Other aspects not considered here include contributions from signals that are not linearly independent. If the incident flux recorded by the photodetector occurred after the collimated beams propagated a distance through the atmosphere, the dynamics of one beam might influence that of the others. This would negate the independence and possibly distort parts of the theory. Also, group velocity dispersion could introduce nonlinearities to the dynamics without requiring interaction across beams. It only requires that the power levels are sufficiently high. The paper in preparation by our group handles this assumption in the third chapter (Bochove, Neschke et al. 2015).

In the larger picture, it could be very useful to combine this active control method with passive phasing techniques. It would also be promising to explore including the delay differences from atmospheric contributions during the trip from the amplifier to the far-field target. If the intensity read by the photodetector has come off the target then the atmospheric time delay

differences would be included already. The system would need to be robust enough to account for these noisy factors as well.

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