Partition functions and Fourier coefficients of modular forms

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For my daughter, Natalie, and the memory of my father, Terry.

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Abstract

The theories of integer partitions and modular forms are intimately connected subjects in modern number theory, with many deep questions lying in the intersection of these areas. The goal of this work is to investigate a collection of problems in this direction, particularly on various methods for proving partition congruences using modular forms. We also study properties of partitions and modular forms in their own right, including elementary congruence relations and the p-adic behavior of certain cusp forms.

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Chapter 1

Introduction

A partition of a natural number n is a non-increasing sequence $\lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_r$ of positive integers whose sum is n. We call $\lambda_1, \ldots, \lambda_r$ the parts of the partition, and the partition function $p(n)$ counts the number of partitions of n. Despite its simple and innocuous definition, the partition function has been shown, perhaps surprisingly, to possess several rich properties. For instance, the celebrated Ramanujan congruences for $p(n)$ assert that, for all non-negative integers n and all integers $j \geq 1$,

$$\begin{aligned} p(5^j n + \beta_5(j)) &\equiv 0 \pmod{5^j}, \\ p(7^j n + \beta_7(j)) &\equiv 0 \pmod{7^{\lfloor j/2 \rfloor + 1}}, \\ p(11^j n + \beta_{11}(j)) &\equiv 0 \pmod{11^j}, \end{aligned} \tag{1.1}$$

where $24\beta_\ell(j) \equiv 1 \pmod{\ell^j}$ for $\ell \in \{5, 7, 11\}$. The study of such arithmetic properties is a flourishing area of modern research, for the partition function and its generalizations and refinements alike. Several methods exist to prove partition congruences, ranging from elementary q-series techniques to complex-analytic methods. In particular, the theory of modular forms is a powerful tool within the theory of partition congruences, and the interplay between these areas has led to many deep and beautiful results.

In this work, the main focus is to investigate various methods for proving partition congruences using modular forms. In addition, we also study properties of partitions and modular forms in their own right, such as the p-adic behavior of a special class of cusp forms.

Chapter 2

Background

In this chapter, we recall several classical facts about modular forms and partition functions. Unless otherwise mentioned, proofs of most of these results can be found in a standard text, such as [And76, Ono04, DS05].

2.1 q-Series and partitions

A q-series is a formal power series inn the variable q. Given a sequence $(a_n)_{n=0}^{\infty}$ of natural numbers, we are often interested in its generating function, which is the q-series

$$\sum_{n=0}^{\infty} a(n)q^n.$$

The generating functions for interesting arithmetic sequences often have expressions as infinite products, usually built from the q-series f_r given by

$$f_r := \prod_{n=1}^{\infty}(1-q^{rn}).$$

For example, the generating function for the partition function $p(n)$ can be written as

$$\sum_{n=0}^{\infty} p(n)q^n = \frac{1}{f_1} = \prod_{n=1}^{\infty}\frac{1}{1-q^n}, \tag{2.1}$$

a result known even to Euler in the 18th century.

Aside from the partition function itself, its many generalizations and refinements are of great interest in modern number theory and combinatorics. In particular, here we focus on the following related partition functions:

i) $a_c(n)$, which counts the number of generalized cubic partitions of n; these are partitions in which the even parts may be independently chosen in one of c colors ($c \geq 1$ an integer),

ii) $\mathrm{ped}(n)$, which counts the number of PED partitions of n, those whose even parts are distinct,

iii) $\mathrm{pend}(n)$, which counts the number of PEND partitions of n, those whose even parts are not distinct,

iv) $\mathrm{pod}(n)$, which counts the number of POD partitions of n, those whose odd parts are distinct,

v) $\mathrm{pond}(n)$, which counts the number of POND partitions of n, those whose odd parts are not distinct,

vi) $\overline{p}(n)$, which counts the number of overpartitions of n, those in which the first occurrence of each part may be tagged, and

vii) $\overline{p_o}(n)$, which counts the number of overpartitions of n into odd parts.

The generating functions for these functions are given by

$$\sum_{n=0}^{\infty} a_c(n)q^n = \frac{1}{f_1 f_2^{c-1}}, \qquad \sum_{n=0}^{\infty} \mathrm{ped}(n)q^n = \frac{f_4}{f_1}, \qquad \sum_{n=0}^{\infty} \mathrm{pend}(n)q^n = \frac{f_2 f_{12}}{f_1 f_4 f_6},$$

$$\sum_{n=0}^{\infty} \mathrm{pod}(n)q^n = \frac{f_2}{f_1 f_4}, \qquad \sum_{n=0}^{\infty} \mathrm{pond}(n)q^n = \frac{f_4 f_6^2}{f_2^2 f_3 f_{12}}, \qquad \sum_{n=0}^{\infty} \overline{p}(n)q^n = \frac{f_2}{f_1^2},$$

$$\sum_{n=0}^{\infty} \overline{p_o}(n)q^n = \frac{f_2^3}{f_1^2 f_4}.$$

2.2 Modular forms

The theory of partitions is deeply connected with the theory of modular forms, as we shall shortly see.

Denote by $\mathrm{SL}_2(\mathbb{Z})$ the group of 2×2 matrices over $\mathbb{Z}$ with determinant 1, i.e.

$$\mathrm{SL}_2(\mathbb{Z}) \coloneqq \left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} : ad - bc = 1, (a, b, c, d) \in \mathbb{Z}^4 \right\}.$$

We are primarily concerned with the congruence subgroups $\Gamma_0(N) \leq \mathrm{SL}_2(\mathbb{Z})$, defined by

$$\Gamma_0(N) \coloneqq \left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in \mathrm{SL}_2(\mathbb{Z}) : c \equiv 0 \pmod{N} \right\}.$$

Let $\mathbb{H} \coloneqq \{z \in \mathbb{C} : \mathrm{Im}(z) > 0\}$ be the complex upper half-plane. The group $\mathrm{SL}_2(\mathbb{Z})$ and its subgroups $\Gamma_0(N)$ act on $\mathbb{H}$ via linear fractional transformations: for $\gamma = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in \mathrm{SL}_2(\mathbb{Z})$ and $\tau \in \mathbb{H}$,

$$\gamma\tau \coloneqq \frac{a\tau + b}{c\tau + d}.$$

The orbits of $\mathbb{P}^1(\mathbb{Q}) \coloneqq \mathbb{Q} \cup \{\infty\}$ under the action of $\Gamma_0(N)$ are called the cusps of $\Gamma_0(N)$. For example, $\mathrm{SL}_2(\mathbb{Z}) = \Gamma_0(1)$ has one cusp, usually denoted by ∞.

Definition 2.1. *The modular curve $X_0(N)$ is the set of orbits of $\hat{\mathbb{H}} \coloneqq \mathbb{H} \cup \mathbb{P}^1(\mathbb{Q})$ under the action of $\Gamma_0(N)$.*

Classically, the modular curve $X_0(N)$ can be given the structure of a compact Riemann surface.

In order to define modular forms, we first introduce the notion of the slash operator.

Definition 2.2. *Let $\gamma = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in \mathrm{GL}_2^+(\mathbb{Q})$, the group of 2×2 matrices of positive determinant with entries in $\mathbb{Q}$. For $k \in \mathbb{Z}$, the weight k slash operator is defined on functions $f : \mathbb{H} \to \mathbb{C}$ by*

$$(f \mid_k \gamma)(\tau) \coloneqq (\det \gamma)^{k/2} (c\tau + d)^{-k} f(\gamma\tau).$$

Definition 2.3. *Let $N \in \mathbb{N}$. A holomorphic function $f : \mathbb{H} \to \mathbb{C}$ is called a weakly holormophic modular form of weight k and level $\Gamma_0(N)$ (or simply N) if the following properties hold:*

1. *f is invariant under the weight k slash operator for matrices in $\Gamma_0(N)$, i.e. for all $\gamma = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in \Gamma_0(N)$, we have $f \mid_k \gamma = f$,*

2. *$f(\tau)$ is meromorphic at the cusps of $\Gamma_0(N)$, which means that for all $\gamma = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in SL_2(\mathbb{Z})$ we have an expansion*

$$(f \mid_k \gamma)(\tau) = \sum_{n \geq n_\gamma} a_\gamma(n) q_N^{n \cdot \gcd(c^2, N)}$$

in the variable $q_N \coloneqq \exp(2\pi i \tau / N)$. When $\gamma = I$ is the identity matrix, we simply write

$$f(\tau) = \sum_{n \geq n_0} a(n) q^n$$

for $q \coloneqq q_N^N = \exp(2\pi i \tau)$, called the Fourier expansion of $f(\tau)$ (at ∞). Here n_γ, n_0 are assumed to be the smallest integers for which $a(n) \neq 0$.

If $n_\gamma \geq 0$, we say that f is holomorphic at the cusp $[a/c]_N$. If $n_\gamma < 0$, we say that f has a pole of order n_γ at the cusp $[a/c]_N$, and its principal part is

$$\sum_{n=n_\gamma}^{-1} a_\gamma(n) q_N^{n \cdot \gcd(c^2, N)}.$$

In either case, we denote the order of f at the cusp $[a/c]_N$ by $\mathrm{ord}_{a/c}^N(f) \coloneqq n_\gamma$.

The $\mathbb{C}$-vector space of weakly holomorphic modular forms of weight k and level N is denoted by $M_k^!(\Gamma_0(N))$. If $[a/c]_N$ is a cusp of $\Gamma_0(N)$, then the subspace $M_k^{a/c}(\Gamma_0(N)) \subseteq M_k^!(\Gamma_0(N))$ denotes the set of weakly holomorphic modular forms that are holomorphic at all cusps of $\Gamma_0(N)$ except possibly at $[a/c]_N$. The set of forms in $M_k^{a/c}(\Gamma_0(N))$ that vanish at all cusps except possibly $[a/c]_N$ is given by $S_k^{a/c}(\Gamma_0(N))$. The set $M_k(\Gamma_0(N))$ denotes the

collection of all modular forms of weight k and level N that are holomorphic at all cusps of $\Gamma_0(N)$, and we refer to these functions simply as modular forms of weight k and level N. A modular form that vanishes at all cusps is referred to as a cusp form, and the space of all cusp forms of weight k and level N is denoted by $S_k(\Gamma_0(N))$. Finally, in the special case $k = 0$, we write $\mathcal{M}$ instead of M_0 and we call such functions modular functions. For instance, the space of all modular functions of level 10 whose only pole (if any) lies at ∞ is written $\mathcal{M}^\infty(\Gamma_0(10))$.

Remark. Meromorphic functions on $X_0(N)$ with poles supported at the cusps are in one-to-one correspondence with modular functions of level $\Gamma_0(N)$.

For $k \in \mathbb{Z}$, the spaces $M_k(\mathrm{SL}_2(\mathbb{Z}))$ are finite-dimensional vector spaces over $\mathbb{C}$. In fact, we have

$$\dim(M_k(\mathrm{SL}_2(\mathbb{Z}))) = \begin{cases} \left\lfloor \frac{k}{12} \right\rfloor, & k \geq 2 \text{ even and } k \equiv 2 \pmod{12}, \\ \left\lfloor \frac{k}{12} \right\rfloor + 1, & k > 2 \text{ even and } k \not\equiv 2 \pmod{12}, \\ 0, & k \leq 0 \text{ or } k \geq 1 \text{ odd}. \end{cases}$$

From Definition 2.3, it is clear that the space of all weakly holomorphic modular forms has the structure of a graded ring, whose grading is given by weight. That is, if $M^!(\Gamma_0(N))$ denotes the set of all weakly holomorphic modular forms (of arbitrary weight), then

$$M^!(\Gamma_0(N)) = \bigoplus_{k \in \mathbb{Z}} M_k^!(\Gamma_0(N)).$$

The family of Eisenstein series provide classical examples of modular forms. For integers $k \geq 2$, the Eisenstein series E_{2k} is given by

$$E_{2k} := 1 - \frac{4k}{B_{2k}} \sum_{n=0}^{\infty} \sigma_{2k-1}(n) q^{2n},$$

where B_n is the nth Bernoulli number and $\sigma_r(n) = \sum_{d|n} d^r$ is the extended sum-of-divisors function. It is well-known that $E_{2k} \in M_{2k}(\mathrm{SL}_2(\mathbb{Z}))$ for $k \geq 2$.

Finally, we recall a famous theorem of Sturm [Stu87], which allows one to prove that two modular forms are congruent (i.e., their Fourier coefficients are pairwise congruent) by checking the congruence for only finitely many terms.

Theorem 2.4. *If $f(\tau) = \sum_{n\geq 0} a(n)q^n$ and $g(\tau) = \sum_{n\geq 0} b(n)q^n$ are elements of $M_k(\Gamma_0(N))$ with integer Fourier coefficients that satisfy $a(n) \equiv b(n) \pmod{M}$ for all*

$$n \leq B(k,N) := \left\lfloor \frac{kN}{12} \prod_{p|N} \left(1 + \frac{1}{p}\right) \right\rfloor,$$

then $f(\tau) \equiv g(\tau) \pmod{M}$. Moreover, if $a(n) = b(n)$ for all $n \leq B(n,K)$, then $f(\tau) = g(\tau)$.

Remark. The integer $B(k,N)$ above is called the Sturm bound for the space $M_k(\Gamma_0(N))$.

2.3 Operators on spaces of modular forms

An extremely important operator on spaces of modular functions is Atkin's U_d-operator, defined by its action on Fourier expansions.

Definition 2.5. *For $d \in \mathbb{N}$, define U_d by*

$$\left(\sum_{n\geq n_0} a(n)q^n\right) \mid U_d := \sum_{dn\geq n_0} a(dn)q^n. \tag{2.2}$$

Remark. We shall occasionally denote the action of U_d on a function f as $U_d(f)$, rather than $f \mid U_d$.

Clearly U_d is $\mathbb{C}$-linear. The following two lemmas record key mapping properties of the U_d-operator.

Lemma 2.6 (Lemmas 6 and 7 of [AL70])**.** *Suppose d, N are positive integers such that $d \mid N$. Then*

$$U_d : \mathcal{M}(\Gamma_0(N)) \to \mathcal{M}(\Gamma_0(N)).$$

Moreover, if $d^2 \mid N$ *then*

$$U_d : \mathcal{M}(\Gamma_0(N)) \to \mathcal{M}\left(\Gamma_0\left(\frac{N}{d}\right)\right).$$

Lemma 2.7 (Lemma 7.1 of [BS25])**.** *If* $f \in \mathcal{M}^\infty(\Gamma_0(50))$*, then* $f \mid U_5 \in \mathcal{M}^\infty(\Gamma_0(10))$.

We shall also require the following well-known result that can be used to simplify the application of the U_d-operator on functions of a particular form.

Lemma 2.8. *Let* $F(q)$ *and* $G(q)$ *be* q*-series and take* $d \in \mathbb{N}$*. Then*

$$\big(F(q^d)G(q)\big) \mid U_d = F(q) \cdot G(q) \mid U_d.$$

Now, for integers $d \geq 1$ define the V_d-operator by

$$\left(\sum_{n \geq n_0} a(n)q^n\right) \mid V_d \coloneqq \sum_{n \geq n_0} a(n)q^{dn}.$$

If $p \nmid N$ is prime and $n \geq 1$, the Hecke operator $T_k(p^n)$ on $M_k^!(\Gamma_0(N))$ is defined by

$$f(z) \mid T_k(p^n) = \sum_{j=0}^{n} p^{(k-1)j} f \mid U(p^{n-j}) \mid V(p^j).$$

In particular, for $p \nmid N$ prime,

$$f(z) \mid T_k(p) = f \mid U(p) + p^{k-1} f \mid V(p).$$

It is well-known that

$$T_k(p^n) : M_k^\infty(\Gamma_0(N)) \to M_k^\infty(\Gamma_0(N))$$

for all primes $p \nmid N$ and all $n \geq 1$. Moreover, $T_k(p^n)$ restricts to the subspace $S_k^\infty(\Gamma_0(N))$, i.e. $f(\tau) \mid T_k(n) \in S_k^\infty(\Gamma_0(N))$ whenever $f(\tau) \in S_k^\infty(\Gamma_0(N)$.

We also define the differential operator Θ, given by

$$\Theta \coloneqq \frac{1}{2\pi i}\frac{d}{d\tau} = q\frac{d}{dq}.$$

While Θ alone is not an operator on spaces of modular forms, a well-known result known as Bol's Identity asserts that powers of Θ do map modular forms to modular forms.

Lemma 2.9 (Bol's Identity)**.** *We have*

$$\Theta^{k-1} : M^{\infty}_{2-k}(\Gamma_0(N)) \to S^{\infty}_{k}(\Gamma_0(N)).$$

Finally, given a divisor $q \mid N$ with $\gcd(q, N/q) = 1$, we define the matrix

$$W_q \coloneqq \begin{bmatrix} qA & B \\ NC & qD \end{bmatrix},$$

for some $A, B, C, D \in \mathbb{Z}$ such that $\det(W_q) = q$. This matrix gives rise to an involution on the space $\mathcal{M}(\Gamma_0(N))$ by $f \mapsto f \mid_0 W_q$, called an Atkin-Lehner involution. Note that this action is unique, regardless of the choice of A, B, C, D.

In certain circumstances, the Atkin-Lehner involution and the operator U_d commute. More precisely, we have the following classical result of Atkin and Lehner [AL70].

Theorem 2.10. *If* $\gcd(d, q) = 1$*, then* W_q *and* U_d *commute as operators. That is,*

$$f \mid U_d \mid W_q = f \mid W_q \mid U_d.$$

2.4 The Dedekind eta-function

Many of the modular functions that are of interest to us are eta-quotients, which are functions of the form

$$\prod_{\delta \mid N} \eta(\delta\tau)^{r_\delta}$$

for integers N, r_δ. Here $\eta(\tau)$ is Dedekind's eta-function, defined by

$$\eta(\tau) \coloneqq q^{1/24} \prod_{n=1}^{\infty} (1 - q^n).$$

The following well-known theorem of Newman allows one to easily prove that a given eta-quotient is a weakly holomorphic modular form.

Lemma 2.11 (Theorem 1.64 of [Ono04]). *Suppose that the eta-quotient* $\prod_{\delta|N} \eta(\delta\tau)^{r_\delta}$ *satisfies*

1. $\frac{1}{2}\sum_{\delta|N} r_\delta = k \in \mathbb{Z}$,
2. $\sum_{\delta|N} \delta r_\delta \equiv 0 \pmod{24}$,
3. $\sum_{\delta|N} \frac{N}{\delta} r_\delta \equiv 0 \pmod{24}$,
4. $\prod_{\delta|N} \delta^{|r_\delta|}$ *is a perfect square.*

Then $\prod_{\delta|N} \eta(\delta\tau)^{r_\delta} \in M_k^!(\Gamma_0(N))$.

Furthermore, the order of vanishing of an eta-quotient at a cusp of $\Gamma_0(N)$ is straightforward to compute, thanks to the following result typically credited to Ligozat.

Lemma 2.12 (Theorem 1.65 of [Ono04]). *The order of vanishing of the eta-quotient* $\prod_{\delta|N} \eta(\delta\tau)^{r_\delta} \in M_k^!(\Gamma_0(N))$ *at the cusp* $[a/c]_N$ *of* $\Gamma_0(N)$ *is given by*

$$\frac{N}{24}\sum_{\delta|N} \frac{\gcd(c,\delta)^2 r_\delta}{\gcd(c,\frac{N}{c})c\delta}.$$

Remark. The Dedekind eta-function provides a glimpse into the deep connection between the partition function and the theory of modular forms. Indeed, from (2.1) one can write

$$\sum_{n=0}^{\infty} p(n)q^n = \frac{q^{1/24}}{\eta(\tau)}.$$

While $\eta(\tau)$ is not a modular form in the exact notion of the definition above, it does still satisfy a remarkable transformation law with respect to the action of $\mathrm{SL}_2(\mathbb{Z})$.

Lemma 2.13 (Chapter 3 of [Apo90]). *For all* $\begin{bmatrix} a & b \\ c & d \end{bmatrix} \in SL_2(\mathbb{Z})$,

$$\eta\left(\frac{a\tau+b}{c\tau+d}\right) = (-i(c\tau+d))^{1/2} \cdot \varepsilon(a,b,c,d) \cdot \eta(\tau).$$

Here, $\varepsilon(a,b,c,d)$ is the 24th root of unity defined by

$$\varepsilon(a,b,c,d) \coloneqq \begin{cases} \exp\left(\frac{b\pi i}{12}\right), & c=0, d=1, \\ \exp\left(\pi i\left(\frac{a+d}{12c} - s(d,c) - \frac{1}{4}\right)\right), & c>0, \end{cases}$$

where $s(\cdot,\cdot)$ is the usual Dedekind sum.

The following lemma provides lower bounds on the orders of vanishing of certain modular functions built from $\eta(\tau)$.

Lemma 2.14 (Theorem 39 of [Rad15])**.** *Let M be a positive integer and define integers $b(n)$ by*

$$\sum_{n=0}^{\infty} b(n)q^n \coloneqq \prod_{\delta|M} f_\delta^{r_\delta}.$$

Suppose that t, m, N are non-negative integers with $0 \le t < m$ and that

$$g \coloneqq \prod_{\lambda|N} \eta(\lambda\tau)^{s_\lambda} \sum_{n=0}^{\infty} b(mn+t)q^n \in \mathcal{M}(\Gamma_0(N)).$$

Then

$$\operatorname{ord}_{a/c}^{N}(g) \ge \frac{N}{24\gcd(c^2,N)} \left(\min_{0\le \ell \le m-1} \sum_{\delta|M} r_\delta \frac{\gcd(\delta(a+\ell c\cdot \gcd(m^2-1,24)), mc)^2}{\delta m} + \sum_{\lambda|N} \frac{s_\lambda \gcd(\lambda,c)^2}{\lambda} \right).$$

2.5 MacMahon's partition analysis

MacMahon's partition analysis is a powerful computational method designed to compute generating functions for objects which satisfy linear Diophantine inequalities (and equations). This technique is often useful in the study of certain families of partition functions, particularly those whose parts satisfy linear inequality relations.

The central operator in partition analysis is MacMahon's Omega operator $\Omega_{\ge}$, which we now define.

Definition 2.15. *The operator $\Omega_{\geq}$ is given by*

$$\underset{\geq}{\Omega} \sum_{s_1=-\infty}^{\infty} \cdots \sum_{s_r=-\infty}^{\infty} A_{s_1,\ldots,s_r} \lambda_1^{s_1} \cdots \lambda_r^{s_r} := \sum_{s_1=0}^{\infty} \cdots \sum_{s_r=0}^{\infty} A_{s_1,\ldots,s_r},$$

where the domain of the $A_{s_1,\ldots,s_r}$ is the field of rational functions over $\mathbb{C}$ in several complex variables and the λ_i are restricted to a neighborhood of the circle $|\lambda_i| = 1$. In addition, the $A_{s_1,\ldots,s_r}$ are required to be such that any of the series involved is absolutely convergent within the domain of the definition of $A_{s_1,\ldots,s_r}$.

We will always have $\Omega_{\geq}$ operate on variables denoted by letters λ, μ from the Greek alphabet (so that letters from the Latin alphabet will be unaffected by $\Omega_{\geq}$).

The benefit of using the MacMahon operator comes from our ability to compute it quickly using elimination formulae. MacMahon [Mac04, pp. 102-103] gave a list of some of these, e.g.,

$$\underset{\geq}{\Omega} \frac{1}{(1-\lambda x)(1-\lambda^{-1} y)} = \frac{1}{(1-x)(1-xy)}. \tag{2.3}$$

Chapter 3

A Congruence Family Modulo Powers of 5 for Generalized Cubic Partitions via the Localization Method

The content of this chapter can also be found in [Doc26].

3.1 Introduction

Recall from Chapter 1 the famous Ramanujan congruences for the partition function $p(n)$:

$$\begin{aligned} p(5^j n + \beta_5(j)) &\equiv 0 \pmod{5^j}, \\ p(7^j n + \beta_7(j)) &\equiv 0 \pmod{7^{\lfloor j/2 \rfloor + 1}}, \\ p(11^j n + \beta_{11}(j)) &\equiv 0 \pmod{11^j}, \end{aligned}$$

where $24\beta_\ell(j) \equiv 1 \pmod{\ell^j}$ for $\ell \in \{5, 7, 11\}$. Using q-series techniques, Ramanujan was able to prove the case $j = 1$ of the first two congruences above. However, proofs of the general congruences modulo powers of 5 and 7 are attributed to Watson [Wat38]. Watson's proof relied on special sequences $(L_{\alpha,\ell})_{\alpha \geq 1}$ of modular functions over the modular curves $X_0(\ell)$ ($\ell \in \{5, 7\}$), as he showed that $L_{\alpha,\ell}$ naturally encodes the partition numbers $p\left(\ell^\alpha n + \beta_\ell(\alpha)\right)$. Precisely, Watson proved the Ramanujan congruences modulo powers of ℓ by showing that

$L_{\alpha,\ell}$ have integral Fourier coefficients that are all divisible by the appropriate power of ℓ. A proof of the partition congruences for $\ell = 11$ was given by Atkin [Atk67] nearly 30 years after Watson's work, using a suitably modified version of Watson's approach. The main challenge in this case lies in the fact that the modular curve $X_0(11)$ has genus 1, whereas $X_0(5)$ and $X_0(7)$ have genus 0.

Since Watson's and Atkin's seminal work, the method of constructing sequences of modular functions to prove infinite congruence families has seen much success. In fact, many such proofs are very similar to those of Watson and Atkin, both in structure and in difficulty. However, others are notably harder, and recent work of Smoot [Smo23] suggests that the number of cusps of $X_0(N)$ plays a major role in the complexity of constructing proofs for such congruence families. When the modular curve $X_0(N)$ has two cusps, such as in all three cases of the Ramanujan congruences for $p(n)$ or, more generally, when N is prime, the spaces of modular functions with a single pole at a given cusp are quite easy to describe. In fact, for modular curves of genus 0, these spaces are isomorphic to $\mathbb{C}[t]$, for some Hauptmodul t. When the cusp count exceeds 2, the modular functions may have poles at various cusps, and so representing such spaces of functions becomes much more difficult. One tool for tackling congruences over these modular curves is the so-called localization method, recently developed by Banerjee and Smoot [BS25, BS23, Smo23]. Their key insight is to express the modular functions of interest not as polynomials in some given reference functions (such as a Hauptmodul), but rather as rational polynomials in the reference functions with predictable denominators.

Here, our goal is to use the localization method to prove an infinite congruence family modulo powers of 5 for the function $a_3(n)$, where $a_3(n)$ is a member of the family known as generalized cubic partitions (see Section 2.1). In particular, we prove the following.

Theorem 3.1. *For all alpha* $\geq 1, n \geq 0$, *the congruence*

$$a_3\left(5^{2\alpha}n + \frac{19 \cdot 5^{2\alpha} + 5}{24}\right)$$

holds.

Remark. Note that $a_1(n) = p(n)$ and $a_2(n) = a(n)$, where $a(n)$ denotes the number of cubic partitions of n, as studied by Chan [Cha10] and named for their connection to Ramanujan's continued cubic fraction. For a more detailed study of $a(n)$ and, more generally, $a_d(n)$, we refer to [ASS25, Cha10, CT10, CD17, Gua25].

Theorem 3.1 is a corollary of a more technical theorem, Theorem 3.4 (see below); Theorem 3.4 involves a special sequence of modular functions $(L_\alpha)_{\alpha\geq 1}$ that encodes $a_3(5^{2\alpha}n + \gamma_\alpha)$. Importantly, these modular functions of interest are over the modular curve $X_0(10)$, which is the same modular curve as is studied by Banerjee and Smoot in [BS25]. As such, our analysis here is quite convenient, as we are able to immediately apply several of their results, and many others follow analogously. Our aim is to summarize arguments of Banerjee and Smoot in order to keep statements in context; however, for a more detailed account, we refer the reader to [BS25].

The remainder of this chapter is organized as follows: in Section 3.2, we define the aforementioned sequence of modular functions $(L_\alpha)_{\alpha\geq 1}$ and present some helpful facts related to this sequence. In Section 3.3, we prove several preliminary facts that are critical to our proof of Theorem 3.4, which we provide in Section 3.4.

3.2 Sequence of modular functions

Throughout, we follow the notation and technique of Banerjee and Smoot [BS25]. Our first task is to construct the sequence $(L_\alpha)_{\alpha\geq 1}$. To this end, let

$$\Phi := \frac{\eta(25\tau)\eta(50\tau)^2}{\eta(\tau)\eta(2\tau)^2} = q^5\frac{f_{25}f_{50}^2}{f_1 f_2^2} \in \mathcal{M}(\Gamma_0(50)).$$

Define the operators $U^{(0)}$, $U^{(1)}$ by

$$U^{(0)}(f) = U_5(\Phi \cdot f) \text{ and } U^{(1)}(f) = U_5(f),$$

where U_5 is as in (2.2). Now, set $L_0 \coloneqq 1$, and for $n \geq 0$ let

$$L_{\alpha+1} \coloneqq \begin{cases} U^{(0)}(L_\alpha), & \alpha \text{ even}, \\ U^{(1)}(L_\alpha), & \alpha \text{ odd}. \end{cases} \tag{3.1}$$

We now show that this sequence naturally contains values of $a_3(n)$ in the arithmetic progressions of interest. First note that the generating function for $a_3(n)$ is given by

$$\sum_{n=0}^{\infty} a_3(n) q^n = \frac{1}{f_1 f_2^2}.$$

Lemma 3.2. *For all $\alpha \geq 1$,*

$$L_{2\alpha} = q f_1 f_2^2 \sum_{n=0}^{\infty} a_3\left(5^{2\alpha} n + \gamma_\alpha\right) q^n.$$

Proof. We prove this by induction. Using Lemma 2.8, it follows that

$$\begin{aligned} L_1 = U^{(0)}(1) = U_5(\Phi) &= U_5\left(q^5 \frac{f_{25} f_{50}^2}{f_1 f_2^2}\right) \\ &= q f_5 f_{10}^2 U_5\left(\sum_{n=0}^{\infty} a_3(n) q^n\right) = q f_5 f_{10}^2 \sum_{n=0}^{\infty} a_3(5n) q^n, \end{aligned}$$

Thus

$$L_2 = U^{(1)}(L_1) = U_5\left(f_5 f_{10}^2 \sum_{n=0}^{\infty} a_3(5n) q^{n+1}\right) = q f_1 f_2^2 \sum_{n=0}^{\infty} a_3(25n + 20) q^n.$$

Now suppose that the conclusion holds for some $\alpha \geq 1$. Again thanks to Lemma 2.8, we see

$$\begin{aligned} L_{2\alpha+1} = U^{(0)}(L_{2\alpha}) &= U_5\left(q^6 f_{25} f_{50}^2 \sum_{n=0}^{\infty} a_3\left(5^{2\alpha} n + \gamma_\alpha\right) q^n\right) \\ &= q f_5 f_{10}^2 U_5\left(\sum_{n=0}^{\infty} a_3\left(5^{2\alpha} n + \gamma_\alpha\right) q^{n+1}\right) \\ &= q f_5 f_{10}^2 \sum_{n=1}^{\infty} a_3\left(5^{2\alpha+1} n - 5^{2\alpha} + \gamma_\alpha\right) q^n, \end{aligned}$$

Hence

$$\begin{aligned} L_{2(\alpha+1)} = U^{(1)}(L_{2\alpha+1}) &= U_5\left(qf_5f_{10}^2\sum_{n=1}^{\infty}a_3\left(5^{2\alpha+1}n - 5^{2\alpha} + \gamma_\alpha\right)q^n\right) \\ &= qf_1f_2^2\sum_{n=0}^{\infty}a_3\left(5^{2(\alpha+1)}n + 5^{2\alpha+2} - 5^{2\alpha+1} - 5^{2\alpha} + \gamma_\alpha\right)q^n \\ &= qf_1f_2^2\sum_{n=0}^{\infty}a_3\left(5^{2(\alpha+1)}n + \gamma_{\alpha+1}\right)q^n, \end{aligned}$$

as

$$\begin{aligned} 5^{2\alpha+2} - 5^{2\alpha+1} - 5^{2\alpha} + \gamma_\alpha &= 19\cdot 5^{2\alpha} + \left(20 + \frac{19\cdot 25(25^{\alpha-1}-1)}{24}\right) \\ &= 20 + \frac{19\cdot 25(25^{\alpha}-1)}{24} \\ &= \gamma_{\alpha+1}. \end{aligned}$$

This completes the proof. □

By Lemmas 2.11 and 2.12 we verify that $\Phi \in \mathcal{M}(\Gamma_0(50))$, and so by Lemma 2.6 we know $L_1 = U_5(\Phi) \in \mathcal{M}(\Gamma_0(10))$. Using Lemma 2.14, we compute

$$\operatorname{ord}_{\infty}^{10}(L_1) \geq 1,\ \operatorname{ord}_{1/5}^{10}(L_1) \geq 1,\ \operatorname{ord}_{1/2}^{10}(L_1) \geq -5,\ \operatorname{ord}_{0}^{10}(L_1) \geq -4.$$

Now let

$$z = z(\tau) \coloneqq \frac{\eta(2\tau)^5\eta(5\tau)}{\eta(\tau)^5\eta(10\tau)} = \frac{f_2^5f_5}{f_1^5f_{10}}.$$

Again using Lemmas 2.11 and 2.12, we see that z is a Hauptmodul at $[0]_{10}$ with positive order at $[1/2]_{10}$. That is,

$$\operatorname{ord}_{\infty}^{10}(z) = 0,\ \operatorname{ord}_{1/5}^{10}(z) = 0,\ \operatorname{ord}_{1/2}^{10}(z) = 1,\ \operatorname{ord}_{0}^{10}(z) = -1.$$

So, $z^5 L_1$ is a modular function over $\Gamma_0(10)$ whose only pole is at $[0]_{10}$, and thus we may write $z^5 L_1$ as a polynomial in z. As in [BS25], however, the polynomial obtained in this way has rational coefficients with denominators that are divisible by large powers of 5:

$$z^5 L_1 = -\frac{1}{625} - \frac{6}{625} z + \dots.$$

Thus, we shift our attention to the function

$$x = x(\tau) := \frac{\eta(2\tau)\eta(10\tau)^3}{\eta(\tau)^3\eta(5\tau)} = q\frac{f_2 f_{10}^3}{f_1^3 f_5},$$

which is closely related to z as shown by the following identity.

Lemma 3.3 ([BS25, Lemma 4.2])**.** *We have*

$$z = 1 + 5x.$$

It will turn out that our modular functions L_α are not expressible as polynomials in x, but rather they are expressible as rational polynomials in x whose denominators are powers of $1 + 5x$. For example,

$$\begin{aligned} L_1 = \frac{1}{(1+5x)^5}(x + 40x^2 + 794x^3 + 9125x^4 + 64475x^5 + 286000x^6 \\ + 7800000x^7 + 1200000x^8 + 800000x^9). \end{aligned} \tag{3.2}$$

More generally, we have the following.

Theorem 3.4. *Let $\alpha \geq 1$, and set*

$$\psi := \psi(\alpha) = \left\lfloor \frac{5^{\alpha+2}}{24} \right\rfloor + 1 - \gcd(\alpha, 2).$$

Then

$$\frac{(1+5x)^\psi}{5^{\lfloor \alpha/2 \rfloor}} L_\alpha \in \mathbb{Z}[x].$$

Remark. Since $1+5x \equiv 1 \pmod 5$, the presence of $(1+5x)^{\psi}$ does not affect the 5-divisibility of L_α. This observation, along with Lemma 3.2, justifies our claim that Theorem 3.1 is a direct corollary of 3.4.

3.3 Preliminary results

We begin by defining the spaces of rational polynomials in which the members of our sequence $(L_\alpha)_{\alpha\geq 1}$ lie. Set

$$\theta_0(m) \coloneqq \begin{cases} 0, & 1 \leq m \leq 4, \\ \left\lfloor \frac{5m-1}{7} \right\rfloor - 2, & m \geq 5, \end{cases}$$
$$\theta_1(m) \coloneqq \begin{cases} 0, & 1 \leq m \leq 7, \\ \left\lfloor \frac{5m-2}{7} \right\rfloor - 5, & m \geq 8. \end{cases}$$

Definition 3.5. *Let $s : \mathbb{Z}_{\geq 1} \to \mathbb{Z}$ be an arbitrary discrete function. Then for $n \in \mathbb{Z}$ we set*

$$\mathcal{V}_n^{(0)} \coloneqq \left\{ \frac{1}{(1+5x)^n} \sum_{m\geq 1} s(m) \cdot 5^{\theta_0(m)} \cdot x^m \right\},$$
$$\hat{\mathcal{V}}_n \coloneqq \left\{ \frac{1}{(1+5x)^n} \sum_{m\geq 1} s(m) \cdot 5^{\theta_1(m)} \cdot x^m \right\},$$

$$\mathcal{V}_n^{(1)} \coloneqq \left\{ f \in \hat{\mathcal{V}}_n : \begin{pmatrix} s(1)+s(2)+s(3)+2s(4)+s(5) \\ 4s(4)+s(6)+s(7)+s(8) \end{pmatrix} \equiv \begin{pmatrix} 0 \\ 0 \end{pmatrix} \pmod 5 \right\}. \tag{3.3}$$

Next, we recall useful modular equations over $\Gamma_0(N)$ that are satisfied by x and z. To state these identities, define

$$a_0(\tau) \coloneqq -(x + 20x^2 + 150x^3 + 500x^4 + 625x^5),$$
$$a_1(\tau) \coloneqq -(15x + 305x^2 + 2325x^3 + 7875x^4 + 10000x^5,$$
$$a_2(\tau) \coloneqq -(85x + 1750x^2 + 13525x^3 + 46500x^4 + 60000x^5),$$

$$a_3(\tau) := -(215x + 4475x^2 + 35000x^3 + 122000x^4 + 160000x^5),$$
$$a_4(\tau) := -(205x + 4300x^2 + 34000x^3 + 120000x^4 + 160000x^5),$$

and

$$b_0(\tau) := -z^5,$$
$$b_1(\tau) := 1 + 5z + 5z^2 + 5z^3 + 5z^4 - 16z^5,$$
$$b_2(\tau) := -4 - 15z + 10z^2 + 35z^3 + 60z^4 - 96z^5,$$
$$b_3(\tau) := 6 + 15z - 35^2 + 40z^3 + 240z^4 - 256z^5,$$
$$b_4(\tau) := -4 - 5z + 20z^2 - 80z^3 + 320z^4 - 256z^5,$$
$$b_5(\tau) := 1.$$

Lemma 3.6 ([BS25, Theorem 4.3 and Theorem 4.4])**.** *We have the relations (correcting the coefficient of x^3 in $a_2(\tau)$ and the leading term of the second equation)*

$$x^5 + \sum_{j=0}^{4} a_j(5\tau)x^j = 0 \tag{3.4}$$

and

$$z^5 + \sum_{k=0}^{4} b_k(5\tau)z^k = 0. \tag{3.5}$$

In light of the form of L_1 given above in (3.2) and the way in which subsequent functions L_α are constructed as defined by (3.1), we aim to understand the functions $U^{(i)}(x^m/(1+5x)^n)$ for $i \in \{0, 1\}$. Using the modular equations (3.4) and (3.5), we construct recurrence relations for these functions, stated below.

Lemma 3.7. *For $m, n \in \mathbb{Z}$ and $i \in \{0, 1\}$, we have*

$$U^{(i)}\left(\frac{x^m}{(1+5x)^n}\right) = -\frac{1}{(1+5x)^5}\sum_{j=0}^{4}\sum_{k=1}^{5} a_j(\tau)b_k(\tau) \cdot U^{(i)}\left(\frac{x^{m+j-5}}{(1+5x)^{n-k}}\right).$$

Proof. The proof follows, mutatis mutandis, as in the proof of [BS25, Lemma 5.1]. □

Our next goal is to use Lemma 3.7 to prove explicit formulas for $U^{(i)}(x^m/(1+5x)^n)$ as rational polynomials in x whose numerators have coefficients with predicable 5-adic valuations, as seen in the following theorem.

Definition 3.8. *For $m, r, \ell \in \mathbb{Z}$, let*

$$\pi_0(m,r) \coloneqq \max\left(0, \left\lfloor \frac{5r-m+2}{7} \right\rfloor - 5\right),$$
$$\pi_1(m,r) \coloneqq \left\lfloor \frac{5r-m}{7} \right\rfloor.$$

Recall that a function $h : \mathbb{Z}^n \to \mathbb{Z}$ is called a discrete array if $h(m_1, m_2, \ldots, m_n)$ has finite support as a function of m_n for fixed $m_1, m_2, \ldots, m_{n-1}$.

Theorem 3.9. *There are discrete arrays $h_0, h_1 : \mathbb{Z}^3 \to \mathbb{Z}$ such that for $m, n \in \mathbb{N}_0$,*

$$U^{(0)}\left(\frac{x^m}{(1+5x)^n}\right) = \frac{1}{(1+5x)^{5n+5}} \sum_{r \geq \lceil (m+4)/5 \rceil} h_0(m,n,r) \cdot 5^{\pi_0(m,r)} \cdot x^r,$$
$$U^{(1)}\left(\frac{x^m}{(1+5x)^n}\right) = \frac{1}{(1+5x)^{n}} \sum_{r \geq \lceil m/5 \rceil} h_1(m,n,r) \cdot 5^{\pi_1(m,r)} \cdot x^r.$$

Proof. The identity for $i = 1$ is proved as part of [BS25, Theorem 5.3], and for $i = 0$ we proceed by induction on m and n. First, we consider $0 \leq m, n \leq 4$. However, we need not explicitly prove all 25 of these base cases, as they can be constructed algorithmically from only five such equations. Indeed, by the Binomial Theorem

$$\begin{aligned} U^{(0)}\left(\frac{x^m}{(1+5x)^n}\right) &= \frac{1}{5^m} U^{(0)}\left(\frac{(z-1)^m}{z^n}\right) \\ &= \frac{1}{5^m} \sum_{r=0}^{n} (-1)^{m-r} \binom{m}{r} \cdot U^{(0)}(z^{r-n}). \end{aligned}$$

So, if we are able to compute $U^{(0)}(z^n)$ for all integers n, then we can quickly compute our 25 initial relations. Taking the modular equation (3.5) and multiplying both sides by $z^n/b_0(5\tau)$,

we see

$$z^n = -\frac{1}{b_0(5\tau)} \sum_{k=1}^{5} b_k(5\tau) z^{n+k}.$$

Consequently, by Lemma 2.8,

$$\begin{aligned} U^{(0)}(z^n) &= -\sum_{k=1}^{5} U^{(0)}\left(\frac{b_k(5\tau) z^{n+k}}{b_0(5\tau)}\right) \\ &= -\sum_{k=1}^{5} \frac{b_k(\tau)}{b_0(\tau)} U^{(0)}(z^{n+k}) \end{aligned} \tag{3.6}$$

Since $n+k > n$ for $1 \le k \le 5$, we only need to consider $U^{(0)}(z^n)$ for positive powers n by inductively applying (3.6). But for $n \ge 1$,

$$\begin{aligned} U^{(0)}(z^n) = U^{(0)}((1+5x)^n) &= U^{(0)}\left(\sum_{i=0}^{n} \binom{n}{i} (5x)^r\right) \\ &= \sum_{r=0}^{n} \binom{n}{r} 5^r \cdot U^{(0)}(x^r). \end{aligned}$$

Finally, note that for any $n \ge 1$ we may express $U^{(0)}(x^n)$ in terms of $U^{(0)}(x^k)$ for $0 \le k \le 4$ by applying (3.4) as necessary. In Appendix A, we provide explicit expressions for these five functions as rational polynomials in x with denominator $(1+5x)^5$, from which the remaining base cases may be quickly settled as described above.

In order to prove these identities, consider the function $\Phi(\tau)x(\tau)^k x(5\tau)^{-29} z(5\tau)^5$. Using Lemma 2.12, we compute the orders of each constituent factor at the cusps of $\Gamma_0(50)$ (see Table 3.1). We conclude that for $0 \le k \le 4$ we have

$$\Phi(\tau)x(\tau)^k x(5\tau)^{-29} z(5\tau)^5 \in \mathcal{M}^\infty(\Gamma_0(50)).$$

As such, thanks to Lemma 2.7,

$$U_5\left(\Phi(\tau)x(\tau)^k x(5\tau)^{-29} z(5\tau)^5\right) \in \mathcal{M}^\infty(\Gamma_0(10)).$$

Table 3.1: Modular cusp analysis over $X_0(50)$

Cusp	$\Phi(\tau)$	$x(\tau)$	$x(5\tau)$	$z(5\tau)$	$\Phi(\tau)x(\tau)^k x(5\tau)^{-29} z(5\tau)^5$
$[\infty]_{10}$	5	1	5	0	$k-140$
$[1/25]_{10}$	4	0	0	0	4
$[1/10]_{10}$	0	1	0	1	$k+5$
$[1/5]_{10}$	0	0	-1	-1	4
$[3/10]_{10}$	0	1	0	1	$k+5$
$[2/5]_{10}$	0	0	-1	-1	4
$[1/2]_{10}$	-5	0	0	1	0
$[3/5]_{10}$	0	0	-1	-1	4
$[7/10]_{10}$	0	1	0	1	$k+5$
$[4/5]_{10}$	0	0	-1	-1	4
$[9/10]_{10}$	0	1	0	1	$k+5$
$[0]_{10}$	-4	-5	-1	-1	$20-5k$

Equivalently, by Lemma 2.8,

$$\frac{z^5}{x^{29}} U^{(0)}(x^k) \in \mathcal{M}^\infty(\Gamma_0(10)). \tag{3.7}$$

On the other hand, each of the identities in Appendix A is of the form

$$(1+5x)^5 U^{(0)}(x^k) = p_k(x)$$

for some polynomial $p_k(x) \in \mathbb{Z}[x]$. Note that for each $0 \le k \le 4$,

$$\frac{p_k(x)}{x^{29}} \in \mathbb{Z}[x^{-1}] \subseteq \mathcal{M}^\infty(\Gamma_0(10)). \tag{3.8}$$

Computing the Fourier expansions of the modular functions in (3.7) and (3.8), we see that they have the same principal part, hence their difference must be a modular function over $\Gamma_0(10)$ with no poles. The only such functions are constants, and since the Fourier expansions of these functions have the same constant term, their difference must be 0. This proves the five initial relations in Appendix A, thereby settling the base cases $0 \le m, n \le 4$ of Theorem 3.9. The inductive step follows identically to that of [BS25, Theorem 5.3]. □

The discrete arrays $h_i(m, n, r)$ appearing in Theorem 3.9 satisfy internal congruences in n modulo 5, which play a pivotal role in the proof of Theorem 3.4.

Proposition 3.10. *For all $m, n, r \geq 1$ and $i \in \{0, 1\}$, we have*

$$h_i(m, n, r) \equiv h_i(m, n - 5, r) \pmod 5.$$

Proof. The proof is identical to that of [BS25, Theorem 5.5]. □

By explicitly computing $h_1(m, n, 1)$ and $h_1(m, n, 2)$ for $1 \leq m \leq 8$ and $0 \leq n \leq 4$, we immediately obtain the following congruences.

Corollary 3.11. *For all $n \in \mathbb{Z}$,*

$$\begin{pmatrix} h_1(1,n,1) & h_1(2,n,1) & h_1(3,n,1) & h_1(4,n,1) & h_1(5,n,1) \\ h_1(4,n,2) & h_1(5,n,2) & h_1(6,n,2) & h_1(7,n,2) & h_1(8,n,2) \end{pmatrix}$$
$$\equiv \begin{pmatrix} 1 & 1 & 1 & 2 & 1 \\ 4 & 0 & 1 & 1 & 1 \end{pmatrix} \pmod 5.$$

3.4 Proof of Theorem 3.4

We first consider the application of the $U^{(0)}$-operator to functions in $\mathcal{V}_n^{(0)}$.

Theorem 3.12. *Suppose $f \in \mathcal{V}_n^{(0)}$. Then $U^{(0)}(f) \in \hat{\mathcal{V}}_{5n+5}$.*

Proof. Since $f \in \mathcal{V}_n^{(0)}$, we can write

$$f = \frac{1}{(1+5x)^n} \sum_{m \geq 1} s(m) \cdot 5^{\theta_0(m)} \cdot x^m.$$

From Theorem 3.9,

$$\begin{aligned} U^{(0)}(f) &= \sum_{m \geq 1} s(m) \cdot 5^{\theta_0(m)} \cdot U^{(0)}\left(\frac{x^m}{(1+5x)^n}\right) \\ &= \frac{1}{(1+5x)^{5n+5}} \sum_{m \geq 1} \sum_{r \geq \lceil (m+4)/5 \rceil} s(m) \cdot h_0(m, n, r) \cdot 5^{\theta_0(m) + \pi_0(m,r)} \cdot x^r \\ &= \frac{1}{(1+5x)^{5n+5}} \sum_{r \geq 1} \sum_{m \geq 1} s(m) \cdot h_0(m, n, r) \cdot 5^{\theta_0(m) + \pi_0(m,r)} \cdot x^r. \end{aligned}$$

Therefore, it suffices to show that, for all $m \leq 5r - 4$,

$$\theta_0(m) + \pi_0(m, r) \geq \theta_1(r).$$

This inequality is proved in [BS25, Theorem 6.1]. □

Next, we show that $U^{(1)}$ maps functions in $\mathcal{V}_n^{(1)}$ to $\mathcal{V}_{n'}^{(0)}$ while picking up an extra power of 5.

Theorem 3.13. *Suppose $f \in \mathcal{V}_n^{(1)}$. Then*

$$\frac{1}{5} U^{(1)}(f) \in \mathcal{V}_{5n}^{(0)}.$$

Proof. This is [BS25, Theorem 6.2] with the assumption $n \equiv 1 \pmod 5$ removed; we summarize the proof here for completeness.

Since $f \in \mathcal{V}_n^{(1)}$, we can write

$$f = \frac{1}{(1+5x)^n} \sum_{m \geq 1} s(m) \cdot 5^{\theta_1(m)} \cdot x^m.$$

From Theorem 3.9,

$$\begin{aligned} U^{(1)}(f) &= \sum_{m \geq 1} s(m) \cdot 5^{\theta_1(m)} \cdot U^{(1)}\left(\frac{x^m}{(1+5x)^n}\right) \\ &= \frac{1}{(1+5x)^{5n}} \sum_{m \geq 1} \sum_{r \geq \lceil m/5 \rceil} s(m) \cdot h_1(m, n, r) \cdot 5^{\theta_1(m) + \pi_1(m,r)} \cdot x^r \\ &= \frac{1}{(1+5x)^{5n+5}} \sum_{r \geq 1} \sum_{m \geq 1} s(m) \cdot h_1(m, n, r) \cdot 5^{\theta_1(m) + \pi_1(m,r)} \cdot x^r. \end{aligned}$$

For $r \geq 3$, from [BS25, Equation 6.7] we get

$$\theta_1(m) + \pi_1(m, r) \geq \theta_0(r) + 1. \tag{3.9}$$

However, (3.9) is not true for $r = 1$ with $1 \leq m \leq 4$ and for $r = 2$ with $4 \leq m \leq 8$. For these problematic values of m and r, by Corollary 3.11, we see

$$\sum_{m=1}^{5} s(m) \cdot h_1(m, n, 1) \equiv s(1) + s(2) + s(3) + 2s(4) + s(5) \pmod 5,$$
$$\sum_{m=4}^{8} s(m) \cdot h_1(m, n, 2) \equiv 4s(4) + s(6) + s(7) + s(8) \pmod 5,$$

but $f \in \mathcal{V}_n^{(1)}$ implies that both of these sums vanish modulo 5. Therefore, for $r \in \{1, 2\}$ the missing power of 5 is accounted for in the coefficients $s(m) \cdot h_1(m, n, r)$, completing the proof. □

From Theorems 3.12 and 3.13, we know that

$$\frac{1}{5} U^{(0)} \circ U^{(1)} : \mathcal{V}_n^{(1)} \to \hat{\mathcal{V}}_{25n+5}.$$

In order to complete the proof of Theorem 3.4, we need the image of this operator to lie in $\mathcal{V}_{25n+5}^{(0)}$ rather than just $\hat{\mathcal{V}}_{25n+5}$. We establish this stability of $\frac{1}{5} U^{(0)} \circ U^{(1)}$ in the following theorem.

Theorem 3.14. *Suppose* $n \equiv 0 \pmod 5$ *and* $f \in \mathcal{V}_n^{(1)}$. *Then*

$$\frac{1}{5} U^{(0)} \circ U^{(1)}(f) \in \mathcal{V}_{25n+5}^{(1)}.$$

Proof. Let

$$f = \frac{1}{(1 + 5x)^n} \sum_{m \geq 1} s(m) \cdot 5^{\theta_1(m)} \cdot x^m,$$

so that by Theorem 3.13

$$\begin{aligned} U^{(1)}(f) &= \sum_{m \geq 1} s(m) \cdot 5^{\theta_1(m)} \cdot U^{(1)} \left(\frac{x^m}{(1 + 5x)^n} \right) \\ &= \frac{1}{(1 + 5x)^{5n}} \sum_{m \geq 1} \sum_{r \geq \lceil m/5 \rceil} s(m) \cdot 5^{\theta_1(m)} \cdot h_1(m, n, r) \cdot 5^{\pi_1(m,r)} \cdot x^r \\ &= \frac{1}{(1 + 5x)^{5n}} \sum_{r \geq 1} \sum_{m \geq 1} s(m) \cdot 5^{\theta_1(m) + \pi_1(m,r)} \cdot h_1(m, n, r) \cdot x^r. \end{aligned}$$

Now from Theorem 3.12,

$$\frac{1}{5}U^{(0)} \circ U^{(1)}(f) = \sum_{r\geq 1}\sum_{m\geq 1} s(m)\cdot 5^{\theta_1(m)+\pi_1(m,r)-1}\cdot h_1(m,n,r)\cdot U^{(0)}\left(\frac{x^r}{(1+5x)^{5n}}\right)$$

$$\begin{aligned}
&= \sum_{r\geq 1}\sum_{m\geq 1} s(m)\cdot 5^{\theta_1(m)+\pi_1(m,r)-1}\cdot h_1(m,n,r)\cdot \frac{1}{(1+5x)^{25n+5}}\\
&\times \sum_{w\geq \lceil (r+4)/5\rceil} h_0(r,5n,w)\cdot 5^{\pi_0(r,w)}\cdot x^w\\
&= \frac{1}{(1+5x)^{25n+5}}\sum_{w\geq 1}\sum_{r\geq 1}\sum_{m\geq 1} s(m)\cdot h_1(m,n,r)\cdot h_0(r,5n,w)\\
&\times 5^{\theta_1(m)+\pi_1(m,r)+\pi_0(r,w)-1}\cdot x^w.
\end{aligned}$$

Therefore, we can write

$$\frac{1}{5}U^{(0)} \circ U^{(1)}(f) = \frac{1}{(1+5x)^{25n+5}}\sum_{w\geq 1} t(w)\cdot 5^{\theta_1(w)}\cdot x^w, \tag{3.10}$$

where

$$t(w) \coloneqq \sum_{r=1}^{5w-4}\sum_{m=1}^{5r} s(m)\cdot h_1(m,n,r)\cdot h_0(r,5n,w)\cdot 5^{\theta_1(m)+\pi_1(m,r)+\pi_0(r,w)-\theta_1(w)-1}. \tag{3.11}$$

Thus, all that remains to be shown is that

$$\begin{pmatrix} t(1)+t(2)+t(3)+2t(4)+t(5)\\ 4t(4)+t(6)+t(7)+t(8)\end{pmatrix} \equiv \begin{pmatrix}0\\0\end{pmatrix} \pmod 5. \tag{3.12}$$

Notice that if r, m are such that

$$\theta_1(m)+\pi_1(m,r)+\pi_0(r,w)-\theta_1(w)-1\geq 0,$$

then the summand in (3.11) corresponding to r, m is congruent to 0 modulo 5. As such, when computing $t(w)$ in order to verify (3.12), it is sufficient to only consider r, m for which

$$\theta_1(m) + \pi_1(m, r) + \pi_0(r, w) - \theta_1(w) - 1 < 0. \tag{3.13}$$

Hence, if we define

$$\hat{t}(w) := \sum_{r=1}^{5w-4} \sum_{\substack{m=1 \\ r,m \text{ satisfy (3.13)}}}^{5r} s(m) \cdot h_1(m, n, r) \cdot h_0(r, 5n, w) \cdot 5^{\theta_1(m)+\pi_1(m,r)+\pi_0(r,w)-\theta_1(w)-1},$$

we see $t(w) \equiv \hat{t}(w) \pmod 5$. But by assumption $n \equiv 0 \pmod 5$, so a repeated application of Proposition 3.10 implies

$$\hat{t}(w) \equiv \sum_{r=1}^{5w-4} \sum_{\substack{m=1 \\ r,m \text{ satisfy (3.13)}}}^{5r} s(m) \cdot h_1(m, 0, r) \cdot h_0(r, 0, w) \cdot 5^{\theta_1(m)+\pi_1(m,r)+\pi_0(r,w)-\theta_1(w)-1} \pmod 5.$$

Therefore, it suffices to show that

$$\begin{pmatrix} \hat{t}(1) + \hat{t}(2) + \hat{t}(3) + 2\hat{t}(4) + \hat{t}(5) \\ 4\hat{t}(4) + \hat{t}(6) + \hat{t}(7) + \hat{t}(8) \end{pmatrix} \equiv \begin{pmatrix} 0 \\ 0 \end{pmatrix} \pmod 5.$$

Computing these values of $\hat{t}(w)$, we find $\hat{t}(1) = 0$ and

$$\begin{aligned}
\hat{t}(2) &= \frac{1}{5}\left(4961s(1) + 10406s(2) + 6171s(3) + 4575812s(4) + 2991921s(5)\right) \\
&\quad + 236628s(6) + 58408s(7) + 8848s(8), \\
\hat{t}(3) &= \frac{1}{5}\left(388844s(1) + 815624s(2) + 483684s(3) + 1151708938s(4) + 753195134s(5)\right) \\
&\quad + 59571099s(6) + 14704214s(7) + 2227484s(8), \\
\hat{t}(4) &= 2632405s(1) + 5521630s(2) + 3274455s(3) + 18352874062s(4) \\
&\quad + 12003016215s(5) + 4746698523s(6) + 1171649878s(7) + 177488668s(8), \\
\hat{t}(5) &= 51888370s(1) + 108839020s(2) + 64544070s(3) + 767043871640s(4) \\
&\quad + 501666289570s(5) + 198388915200s(6) + 48969267200s(7) + 7418163200s(8),
\end{aligned}$$

$$\hat{t}(6) = 671367825s(1) + 1408234950s(2) + 835116075s(3) + 202583215529 00s(4)$$
$$+ 13249576649825s(5) + 5239683382500s(6) + 1293335645000s(7) + 195922370000s(8),$$

$$\hat{t}(7) = 6053383500s(1) + 12697341000s(2) + 7529818500s(3) + 371382334240250s(4)$$
$$+ 242896720484750s(5) + 96056103747375s(6) + 23709978986750s(7)$$
$$+ 3591732195500s(8),$$
$$\hat{t}(8) = 39202970000s(1) + 82230620000s(2) + 48764670000s(3) + 4989377344380000s(4)+$$
$$+ 3263230087870000s(5) + 1290479615910000s(6) + 318535141260000s(7)$$
$$+ 48253645560000s(8).$$

Then

$$\hat{t}(1) + \hat{t}(2) + \hat{t}(3) + 2\hat{t}(4) + \hat{t}(5) = 57231941s(1) + 120047486s(2) + 71190951s(3)$$
$$+ 803980876714s(4) + 525823559411s(5)$$
$$+ 207942119973s(6) + 5132732957s(7)$$
$$+ 7775376868s(8) \tag{3.14}$$

and

$$4\hat{t}(4) + \hat{t}(6) + \hat{t}(7) + \hat{t}(8) = 45938250945s(1) + 96358282470s(2) + 57142702395s(3)$$
$$+ 5381091411669398s(4) + 3519424397069435s(5)$$
$$+ 1391794389833967s(6) + 343543142491262s(7)$$
$$+ 52042010080172s(8). \tag{3.15}$$

Let I be the ideal of $\mathbb{Z}[s(1), \ldots, s(8)]$ defined by

$$I \coloneqq \langle s(1) + s(2) + s(3) + 2s(4) + s(5), 4s(4) + s(6) + s(7) + s(8) \rangle .$$

Using a computer algebra software, we see that the right-hand sides of (3.14) and (3.15) reduce to 0 modulo $5I$, therefore establishing (3.12) and completing the proof. □

We are now ready to prove Theorem 3.4.

Proof of Theorem 3.4. . We prove this by induction, showing that $\frac{1}{5^{\lfloor \alpha/2 \rfloor}} \cdot L_\alpha \in \mathcal{V}^{(i)}_{\psi(\alpha)}$, where $i \equiv n \pmod 2$. Indeed, from (3.2) it is clear that $L_1 \in \hat{\mathcal{V}}_5$. To verify that $L_1 \in \mathcal{V}^{(1)}_5$, we just need to check that the coefficients of the numerator in our rational polynomial expression for L_1 satisfy the defining congruences given in (3.3). This is straightforward to see, as

$$1 + 40 + 794 + 2 \cdot 9125 + \frac{64475}{5} \equiv 0 \pmod 5,$$
$$4 \cdot 9125 + \frac{286000}{25} + \frac{7800000}{25} + \frac{800000}{125} \equiv 0 \pmod 5.$$

If $\frac{1}{5^{\lfloor \alpha/2 \rfloor}} \cdot L_\alpha \in \mathcal{V}^{(i)}_{\psi(\alpha)}$ for some odd $\alpha \geq 1$, then Theorems 3.13 and 3.14 immediately imply

$$\frac{1}{5} L_{\alpha+1} = \frac{1}{5} U^{(1)}(L_\alpha) \in \mathcal{V}^{(0)}_{5 \cdot \psi(\alpha)}$$

and

$$\frac{1}{5} L_{\alpha+2} = \frac{1}{5} U^{(0)} \circ U^{(1)}(L_\alpha) \in \mathcal{V}^{(1)}_{25 \cdot \psi(\alpha)+5}.$$

Hence, we are done if we show $\psi(\alpha+1) = 5\psi(\alpha)$ and $\psi(\alpha+2) = 25\psi(\alpha) + 5$ for all odd $\alpha \geq 1$. But this is immediate since, for $\alpha \geq 1$ odd,

$$5\psi(\alpha) = 5 \left\lfloor \frac{5^{\alpha+2}}{24} \right\rfloor = 5 \left(\frac{5^{\alpha+2}}{24} - \frac{5}{24} \right) = \left(\frac{5^{\alpha+3}}{24} - \frac{1}{24} \right) - 1 = \psi(\alpha+1)$$

and

$$\begin{aligned} 25\psi(\alpha) + 5 &= 25 \left\lfloor \frac{5^{\alpha+2}}{24} \right\rfloor + 5 = 25 \left(\frac{5^{\alpha+2}}{24} - \frac{5}{24} \right) + 5 = \frac{5^{\alpha+4}}{24} - \frac{125}{24} + 5 \\ &= \frac{5^{\alpha+4}}{24} - \frac{5}{24} = \psi(\alpha+2). \end{aligned}$$

This completes the proof. □

Chapter 4

Linking Congruences for PED and POD Partitions

The content of this chapter can also be found in [DJ25].

4.1 Introduction

As demonstrated in Chapter 3, sequences of modular functions are a powerful technique to prove congruences for partition functions. However, while this process is straightforward in theory, it is often tedious and difficult in practice; as such, alternate methods for proving partition congruences are desirable. A recent example in this direction stems from the notion of congruence multiplicity. This phenomenon, first observed by Garvan, Sellers, and Smoot [GSS24] in their study of generalized Frobenius partitions, occurs when divisibility properties of a single sequence of modular functions manifests as simultaneous congruence properties for multiple partition functions. The main allure of this technique is that it allows one to prove a congruence family by connecting it to another family existing in the literature, instead of a long direct proof.

After the work of Garvan, Sellers, and Smoot, congruence multiplicity was utilized in the study of PEND and POND partitions by Sellers and Smoot [SS25]. In their work, Sellers

and Smoot considered the congruence families

$$\operatorname{pend}\left(3^{2\alpha+1}n + \frac{17\cdot 3^{2\alpha}-1}{8}\right) \equiv 0 \pmod 3, \tag{4.1}$$

$$\operatorname{pond}\left(3^{2\alpha+1}n + \frac{23\cdot 3^{2\alpha}+1}{8}\right) \equiv 0 \pmod 6, \tag{4.2}$$

($\alpha \geq 1, n \geq 0$) originally proven by Sellers [Sel24]. More specifically, they defined two sequences of modular functions, one enumerating each partition function, and they demonstrated that applying a certain conjugated Atkin-Lehner involution to one sequence yields the other sequence. In their closing remarks, Sellers and Smoot suggested that a similar relationship may hold for PED and POD congruences, in light of their similar combinatorial definitions and the congruence families (see [HS10, AHS10])

$$\operatorname{ped}\left(3^{2\alpha+1}n + \frac{17\cdot 3^{2\alpha}-1}{8}\right) \equiv 0 \pmod 6, \tag{4.3}$$

$$\operatorname{pod}\left(3^{2\alpha+1}n + \frac{23\cdot 3^{2\alpha}+1}{8}\right) \equiv 0 \pmod 3 \tag{4.4}$$

($\alpha \geq 1, n \geq 0$) that bear a striking resemblance to those in (4.1) and (4.2).

Here we address this question of Sellers and Smoot, and perhaps surprisingly, we find that PED and POD are not linked in the same manner as PEND and POND. Rather, PED partitions are linked to overpartitions into odd parts, and POD partitions are linked to overpartitions. Precisely, we connect (4.3)-(4.4) to the congruences

$$\overline{p_o}\left(3^{2\alpha}(3n+1)\right) \equiv 0 \pmod 6, \tag{4.5}$$

$$\overline{p}\left(3^{2\alpha}(3n+2)\right) \equiv 0 \pmod{12} \tag{4.6}$$

($\alpha \geq 1, n \geq 0$) of Hirschhorn and Sellers [HS06, HS05b] using a conjugated Atkin-Lehner involution γ, which we define in Section 4.2. In order to link the congruences (4.3)-(4.6), we first describe them using families of modular functions $L_\alpha^{\text{ped}}, L_\alpha^{\text{pod}}, L_\alpha^{\overline{p_o}}$, and $L_\alpha^{\overline{p}}$, defined explicitly in Section 4.2. We now state the first main theorem of this chapter.

Theorem 4.1. *For all $\alpha \geq 1$, we have*

$$L_\alpha^{\text{ped}} \mid \gamma = (-1)^\alpha \cdot L_\alpha^{\overline{p_o}}$$
$$L_\alpha^{\text{pod}} \mid \gamma = (-1)^{\alpha+1} \frac{1}{4} \cdot L_\alpha^{\overline{p}}.$$

To further strengthen the connection between these partition functions, we also note that they satisfy related internal congruences. For instance, we have

$$\text{ped}(3n+1) \equiv -\text{ped}(27n+10) \pmod 6$$
$$\overline{p_o}(3n) \equiv \overline{p_o}(27n) \pmod{12}$$

for all $n \geq 0$ (and similar internal congruences for $\text{pod}(n)$ and $\overline{p}(n)$ as described in Section 4.3). (In fact, these internal congruences could be used as a tool to prove equations (4.3)-(4.6) using induction.) Then after defining appropriate modular functions $L^{\text{ped}}, L^{\text{pod}}, L^{\overline{p_o}}$, and $L^{\overline{p}}$ and operator γ, we have the following second main theorem, which connects their internal congruences as well.

Theorem 4.2. *For all $\alpha \geq 1$, we have*

$$L^{\text{ped}} \mid \gamma = \frac{1}{2} L^{\overline{p_o}}$$
$$L^{\text{pod}} \mid \gamma = -\frac{1}{8} L^{\overline{p}}.$$

Remark. Although it may seem surprising that congruence multiplicity techniques connect PED/POD partition functions to overpartition functions (instead of to each other), we note that prior work of Lovejoy and Osburn in [LO11] uses different techniques to connect these four partition functions in the same pairs.

4.2 Connecting infinite congruence families

We begin by defining the sequences of modular functions that will enumerate these partition functions of interest.

For $\alpha \geq 1$, let

$$
\begin{aligned}
P_{\alpha}^{\mathrm{ped}} &:= \frac{\eta(6\cdot 3^{2\alpha+1}\tau)^4}{\eta(3^{2\alpha+1}\tau)^2\eta(3\cdot 3^{2\alpha+1}\tau)\eta(12\cdot 3^{2\alpha+1}\tau)}\cdot\frac{\eta(4\tau)}{\eta(\tau)},\\
P_{\alpha}^{\mathrm{pod}} &:= \frac{\eta(2\cdot 3^{2\alpha+1}\tau)^4\eta(4\cdot 3^{2\alpha+1}\tau)^6}{\eta(3^{2\alpha+1}\tau)^4\eta(3\cdot 3^{2\alpha+1}\tau)^3\eta(6\cdot 3^{2\alpha+1}\tau)\eta(12\cdot 3^{2\alpha+1}\tau)}\cdot\frac{\eta(2\tau)}{\eta(\tau)\eta(4\tau)},\\
P_{\alpha}^{\overline{p_o}} &:= \frac{\eta(6\cdot 3^{2\alpha+1}\tau)\eta(12\cdot 3^{2\alpha+1}\tau)}{\eta(3^{2\alpha+1}\tau)^2}\cdot\frac{\eta(2\tau)^3}{\eta(\tau)^2\eta(4\tau)},\\
P_{\alpha}^{\overline{p}} &:= \frac{\eta(2\cdot 3^{2\alpha+1}\tau)^{22}\eta(12\cdot 3^{2\alpha+1}\tau)}{\eta(3^{2\alpha+1}\tau)^{10}\eta(3\cdot 3^{2\alpha+1}\tau)^2\eta(4\cdot 3^{2\alpha+1}\tau)^6\eta(6\cdot 3^{2\alpha+1}\tau)^4}\cdot\frac{\eta(4\tau)}{\eta(2\tau)^2}.
\end{aligned}
$$

Using Lemma 2.11 it is straightforward to verify that $P_{\alpha}^{*} \in \mathcal{M}(\Gamma_0(12\cdot 3^{2\alpha+1}))$ for $* \in \{\mathrm{ped}, \mathrm{pod}, \overline{p_o}, \overline{p}\}$. Hence, by Lemma 2.6 we conclude

$$
L_{\alpha}^{*} := U_3^{2\alpha+1}\left(P_{\alpha}^{*}\right) \in \mathcal{M}(\Gamma_0(12)).
$$

Routine calculations reveal

$$
\begin{aligned}
L_{\alpha}^{\mathrm{ped}} &= \frac{f_6^4}{f_1^2 f_3 f_{12}}\sum_{n=0}^{\infty}\mathrm{ped}\left(3^{2\alpha+1}n+\frac{17\cdot 3^{2\alpha}-1}{8}\right)q^{n+1},\\
L_{\alpha}^{\mathrm{pod}} &= \frac{f_2^4 f_4^6}{f_1^4 f_3^3 f_6 f_{12}}\sum_{n=0}^{\infty}\mathrm{pod}\left(3^{2\alpha+1}n+\frac{23\cdot 3^{2\alpha}+1}{8}\right)q^{n+1},\\
L_{\alpha}^{\overline{p_o}} &= \frac{f_6 f_{12}}{f_1^2}\sum_{n=0}^{\infty}\overline{p_o}\left(3^{2\alpha+1}n+3^{2\alpha}\right)q^{n+1},\\
L_{\alpha}^{\overline{p}} &= \frac{f_2^{22} f_{12}}{f_1^{10} f_3^2 f_4^6 f_6^4}\sum_{n=0}^{\infty}\overline{p}\left(3^{2\alpha+1}n+2\cdot 3^{2\alpha}\right)q^{2n+1}.
\end{aligned}
$$

For example, using Lemma 2.8 we have

$$
\begin{aligned}
L_{\alpha}^{\mathrm{ped}} &= U_3^{2\alpha+1}\left(q^{\frac{7\cdot 3^{2\alpha}+1}{8}}\frac{f_{6\cdot 3^{2\alpha+1}}^4}{f_{3^{2\alpha+1}}^2 f_{3\cdot 3^{2\alpha+1}} f_{12\cdot 3^{2\alpha+1}}}\sum_{n=0}^{\infty}\mathrm{ped}(n)q^n\right)\\
&= \frac{f_6^4}{f_1^2 f_3 f_{12}}U_3^{2\alpha+1}\left(\sum_{n\geq\frac{7\cdot 3^{2\alpha}+1}{8}}\mathrm{ped}\left(n-\frac{7\cdot 3^{2\alpha}+1}{8}\right)q^n\right)
\end{aligned}
$$

$$= \frac{f_6^4}{f_1^2 f_3 f_{12}} \sum_{n=1}^{\infty} \text{ped}\left(3^{2\alpha+1}n - \frac{7\cdot 3^{2\alpha}+1}{8}\right) q^n$$
$$= \frac{f_6^4}{f_1^2 f_3 f_{12}} \sum_{n=0}^{\infty} \text{ped}\left(3^{2\alpha+1}n + \frac{17\cdot 3^{2\alpha}-1}{8}\right) q^{n+1},$$

as desired. Note that we have constructed these modular functions so that the congruences (4.3)-(4.6) are equivalent to the statements that

$$L_\alpha^{\text{ped}} \equiv 0 \pmod 6$$
$$L_\alpha^{\text{pod}} \equiv 0 \pmod 3$$
$$L_\alpha^{\overline{po}} \equiv 0 \pmod 6$$
$$L_\alpha^{\overline{p}} \equiv 0 \pmod{12}.$$

Following recent work of Garvan, Sellers, and Smoot [GSS24, SS25], we now define a so-called "conjugation operator" obtained by conjugating an Atkin-Lehner operator with the map sending $q \mapsto -q$. More precisely, we consider

$$\nu := \begin{bmatrix} 1 & 1/2 \\ 0 & 1 \end{bmatrix} \qquad \text{and} \qquad W := \begin{bmatrix} 4 & -1 \\ 12\cdot 3^{2\alpha+1} & 1-3^{2\alpha+2} \end{bmatrix}$$

and use them to define

$$\nu W \nu = \begin{bmatrix} 4+6\cdot 3^{2\alpha+1} & \frac{3+3^{2\alpha+2}}{2} \\ 12\cdot 3^{2\alpha+1} & 1+3^{2\alpha+2} \end{bmatrix},$$

which reduces to

$$\gamma = \begin{bmatrix} 2+3\cdot 3^{2\alpha+1} & \frac{3+3^{2\alpha+2}}{4} \\ 6\cdot 3^{2\alpha+1} & \frac{1+3^{2\alpha+2}}{2} \end{bmatrix}.$$

Note that here $W = W_4$ is the Atkin-Lehner operator for modular forms of level $N = 12\cdot 3^{2\alpha+1}$ with prime power $q = 4$, as described in Section 2.3.

In order to compute $P_\alpha^{\text{ped}} \mid \gamma$ and $P_\alpha^{\text{pod}} \mid \gamma$, it suffices to compute $\eta(\delta\gamma\tau)$ for the appropriate positive integers δ. However, in order to apply Lemma 2.13, we must factor

the corresponding matrix via

$$\begin{bmatrix} A & B \\ C & D \end{bmatrix} = \begin{bmatrix} A/g & -y \\ C/g & x \end{bmatrix} \begin{bmatrix} g & Bx + Dy \\ 0 & (AD - BC)/g \end{bmatrix}$$

(where $g := \gcd(A, C)$ and $x, y \in \mathbb{Z}$ satisfy $Ax + Cy = g$) in order to obtain a matrix in $\mathrm{SL}_2(\mathbb{Z})$.

For example, if $\delta = 4$ then we have

$$\begin{aligned} 4\gamma(\tau) &= \begin{bmatrix} 8 + 12 \cdot 3^{2\alpha+1} & 3 + 3^{2\alpha+2} \\ 6 \cdot 3^{2\alpha+1} & \frac{1+3^{2\alpha+2}}{2} \end{bmatrix} (\tau) = \begin{bmatrix} 4 + 6 \cdot 3^{2\alpha+1} & \frac{-1-3^{2\alpha+2}}{2} \\ 3^{2\alpha+2} & \frac{1-3^{2\alpha+2}}{4} \end{bmatrix} \begin{bmatrix} 2 & 1 \\ 0 & 2 \end{bmatrix} (\tau) \\ &= \begin{bmatrix} 4 + 6 \cdot 3^{2\alpha+1} & \frac{-1-3^{2\alpha+2}}{2} \\ 3^{2\alpha+2} & \frac{1-3^{2\alpha+2}}{4} \end{bmatrix} (\tau + 1/2) \end{aligned}$$

and so

$$\begin{aligned} \eta(4\gamma\tau) = e^{\pi i/24} \left(-i \left(3^{2\alpha+2} \left(\tau + \frac{1}{2} \right) + \frac{1 - 3^{2\alpha+2}}{4} \right) \right)^{1/2} \\ \cdot \varepsilon \left(4 + 6 \cdot 3^{2\alpha+1}, \frac{-1 - 3^{2\alpha+2}}{2}, 3^{2\alpha+2}, \frac{1 - 3^{2\alpha+2}}{4} \right) \frac{\eta(2\tau)^3}{\eta(\tau)\eta(4\tau)} \end{aligned}$$

by Lemma 2.13, together with the fact that $\eta\left(\tau + \frac{1}{2}\right) = e^{\pi i/24} \frac{\eta(2\tau)^3}{\eta(\tau)\eta(4\tau)}$ (see, for example, [SS25, Lemma 2.1]). Completing this calculation for each factor in P_α^{ped} and P_α^{pod} gives

$$\begin{aligned} P_\alpha^{\mathrm{ped}} \mid \gamma &= (-1)^\alpha \cdot P_\alpha^{\overline{p_o}} \\ P_\alpha^{\mathrm{pod}} \mid \gamma &= (-1)^{\alpha+1} \frac{1}{4} \cdot P_\alpha^{\overline{p}}. \end{aligned}$$

Since the U_3-operator commutes with W by Theorem 2.10 and with ν by inspection, applying $U_3^{2\alpha+1}$ gives the desired result.

4.3 Connecting internal congruences

In this section, we further strengthen the connection between PED (resp. POD) partitions and overpartitions into odd parts (resp. overpartitions) by pairing up the following internal congruences satisfied by these functions.

Proposition 4.3. *The following internal congruences are true.*

1. *For all* $n \geq 0$,
$$\text{ped}(3n+1) \equiv -\text{ped}(27n+10) \pmod 6.$$

2. *For all* $n \geq 0$,
$$\text{pod}(3n+2) \equiv -\text{pod}(27n+17) \pmod 3.$$

3. *For all* $n \geq 0$,
$$\overline{p_o}(3n) \equiv \overline{p_o}(27n) \pmod{12}.$$

4. *For all* $n \geq 0$,
$$\overline{p}(3n) \equiv \overline{p}(27n) \pmod{24}.$$

Proof. 1. This congruence follows immediately from Theorem 3.4 of [AHS10] with $\alpha = 0, 1$.

2. This congruence follows immediately from the work of Hirschhorn and Sellers ([HS10]) in the proof of Theorem 4.1.

3. A proof of this congruence is given in Theorem 3.8 of [HS06].

4. The congruence modulo 3 is Theorem 2.1 of [HS05b]. From the work of Hirschhorn and Sellers in [HS05a], we have
$$\sum_{n=0}^{\infty} \overline{p}(3n)(-q)^n \equiv \frac{\phi(q^9)^{12}}{\phi(q^3)^{13}} \pmod 8,$$
$$\sum_{n=0}^{\infty} \overline{p}(27n)(-q)^n \equiv \phi(q)^8\phi(q^3)^{27} \pmod 8,$$

where $\phi(q) := \dfrac{f_2^5}{f_1^2 f_4^2}$. It is straightforward to verify that the ratio

$$\Phi := \frac{\phi(q)^8 \phi(q^3)^{40}}{\phi(q^9)^{12}} \in M_{18}(\Gamma_0(36))$$

is a (holomorphic) modular form. Let $E_6 \in M_6(\mathrm{SL}_2(\mathbb{Z})) \subseteq M_6(\Gamma_0(36))$ denote the Eisenstein series of weight 6. Since $B_6 = \frac{1}{42}$, we see that

$$E_6 = 1 - 504 \sum_{n=0}^{\infty} \sigma_5(n) q^n \equiv 1 \pmod 8.$$

Then $\Phi - E_6^3 \in M_{18}(\Gamma_0(36))$, and note that this function is identically 0 mod 8. Indeed, the Sturm bound for the space $M_{18}(\Gamma_0(36))$ is 108, so by Theorem 2.4, a brief computation of the Fourier coefficients of $\Phi - E_6^3$ up to q^{108} is sufficient. This implies that $\Phi \equiv 1 \pmod 8$, from which the desired internal congruence is immediate.

□

In order to define modular functions that enumerate these internal congruences, we must first establish generating functions for $\mathrm{ped}(n), \mathrm{pod}(n), \overline{p_o}(n)$, and $\overline{p}(n)$ in certain arithmetic progressions. To this end, we require the following lemma.

Lemma 4.4 (Lemma 2.1 of [Toh12])**.** *We have*

$$\frac{f_2}{f_1 f_4} = \frac{f_{18}^9}{f_3^2 f_9^3 f_{12}^2 f_{36}^3} + q \frac{f_6^2 f_{18}^3}{f_3^3 f_{12}^3} + q^2 \frac{f_6^4 f_9^3 f_{36}^3}{f_3^4 f_{12}^4 f_{18}^3}.$$

Proposition 4.5.

1. *We have*
$$\sum_{n=0}^{\infty} \mathrm{ped}(9n+1) q^n = \frac{f_2^2 f_3^4 f_4}{f_1^5 f_6^2} + 24q \frac{f_2^3 f_3^3 f_4 f_6^3}{f_1^{10}}.$$

2. *We have*
$$\sum_{n=0}^{\infty} \mathrm{pod}(9n+8) q^n = 10 \frac{f_2 f_6^{24}}{f_1^7 f_3^6 f_4^7 f_{12}^6} + 16q \frac{f_2^7 f_3^3 f_6^6 f_{12}^3}{f_1^{10} f_4^{10}} + q^2 \frac{f_2^{13} f_3^{12} f_{12}^{12}}{f_1^{13} f_4^{13} f_6^{12}}.$$

3. *We have*

$$\sum_{n=0}^{\infty}\overline{p_0}(9n)q^n=\frac{f_2^5f_3^4}{f_1^6f_4f_6^2}+24q\frac{f_2^6f_3^3f_6^3}{f_1^{11}f_4}.$$

4. *We have*

$$\sum_{n=0}^{\infty}\overline{p}(9n)q^n=\frac{f_2^{13}f_3^{24}}{f_1^{26}f_6^{12}}+128q\frac{f_2^{10}f_3^{15}}{f_1^{23}f_6^3}+640q^2\frac{f_2^7f_3^6f_6^6}{f_1^{20}}.$$

Proof. 1. This generating function appears as part of Theorem 3.2 of [AHS10].

2. By Lemma 4.4,

$$\sum_{n=0}^{\infty}\operatorname{pod}(n)q^n=\frac{f_2}{f_1f_4}=\frac{f_{18}^9}{f_3^2f_9^3f_{12}^2f_{36}^3}+q\frac{f_6^2f_{18}^3}{f_3^3f_{12}^3}+q^2\frac{f_6^4f_9^3f_{36}^3}{f_3^4f_{12}^4f_{18}^3}.$$

Extracting the terms of the form q^{3n+2} yields

$$\sum_{n=0}^{\infty}\operatorname{pod}(3n+2)q^{3n+2}=q^2\frac{f_6^4f_9^3f_{36}^3}{f_3^4f_{12}^4f_{18}^3},$$

which upon dividing by q^2 and replacing q^3 by q yields

$$\sum_{n=0}^{\infty}\operatorname{pod}(3n+2)q^n=\frac{f_2^4f_3^3f_{12}^3}{f_1^4f_4^4f_6^3}.$$

Another application of Lemma 4.4 implies

$$\sum_{n=0}^{\infty}\operatorname{pod}(3n+2)q^n=\frac{f_3^3f_{12}^3}{f_6^3}\left(\frac{f_{18}^9}{f_3^2f_9^3f_{12}^2f_{36}^3}+q\frac{f_6^2f_{18}^3}{f_3^3f_{12}^3}+q^2\frac{f_6^4f_9^3f_{36}^3}{f_3^4f_{12}^4f_{18}^3}\right)^4.$$

We now expand and again extract the terms of the form q^{3n+2} to find

$$\sum_{n=0}^{\infty}\operatorname{pod}(9n+8)q^{3n+2}=\frac{f_3^3f_{12}^3}{f_6^3}\left(10q^2\frac{f_6^4f_{18}^{24}}{f_3^{10}f_9^6f_{12}^{10}f_{36}^6}+16q^5\frac{f_6^{10}f_9^3f_{18}^6f_{36}^3}{f_3^{13}f_{12}^{13}}+q^8\frac{f_6^{16}f_9^{12}f_{36}^{12}}{f_3^{16}f_{12}^{16}f_{18}^{12}}\right).$$

Again we divide by q^2 and replace q^3 by q to obtain

$$\sum_{n=0}^{\infty}\operatorname{pod}(9n+8)q^n=\frac{f_1^3f_4^3}{f_2^3}\left(10\frac{f_2^4f_6^{24}}{f_1^{10}f_3^6f_4^{10}f_{12}^6}+16q\frac{f_2^{10}f_3^3f_6^6f_{12}^3}{f_1^{13}f_4^{13}}+q^2\frac{f_2^{16}f_3^{12}f_{12}^{12}}{f_1^{16}f_4^{16}f_6^{12}}\right)$$

$$= 10\frac{f_2 f_6^{24}}{f_1^7 f_3^6 f_4^7 f_{12}^6} + 16q\frac{f_2^7 f_3^3 f_6^6 f_{12}^3}{f_1^{10} f_4^{10}} + q^2\frac{f_2^{13} f_3^{12} f_{12}^{12}}{f_1^{13} f_4^{13} f_6^{12}}.$$

3. In the proof of Theorem 3.8 of [HS06], Hirschhorn and Sellers established

$$\begin{aligned}\sum_{n=0}^{\infty}\overline{p_o}(9n)q^n &= \frac{f_2^9 f_3^{12}}{f_1^{14} f_4 f_6^6} + 16q\frac{f_2^6 f_3^3 f_6^3}{f_1^{11} f_4} \\ &= \left(\frac{f_2 f_3^2}{f_1^2 f_6}\right)^4 \frac{f_2^5 f_3^4}{f_1^6 f_4 f_6^2} + 16q\frac{f_2^6 f_3^3 f_6^3}{f_1^{11} f_4},\end{aligned}$$

so it is sufficient to show

$$\frac{f_2^4 f_3^8}{f_1^8 f_6^4} = 1 + 8q\frac{f_2 f_6^5}{f_1^5 f_3}. \tag{4.7}$$

Lemmas 2.11 and 2.12 imply

$$\begin{aligned}\frac{f_2^4 f_3^8}{f_1^8 f_6^4} &= \frac{\eta(2\tau)^4\eta(3\tau)^8}{\eta(\tau)^8\eta(6\tau)^4} \in \mathcal{M}(\Gamma_0(6)), \\ 8q\frac{f_2 f_6^5}{f_1^5 f_3} &= 8\frac{\eta(2\tau)\eta(6\tau)^5}{\eta(\tau)^5\eta(3\tau)} \in \mathcal{M}(\Gamma_0(6)),\end{aligned}$$

and both have simple poles at the cusp 0 and are holomorphic elsewhere. Since $\eta(\tau)^{24} =: \Delta(\tau) \in M_{12}(\Gamma_0(1))$ vanishes at all cusps, it follows that

$$\Phi(\tau) := \Delta(\tau)\left(\frac{\eta(2\tau)^4\eta(3\tau)^8}{\eta(\tau)^8\eta(6\tau)^4} - 8\frac{\eta(2\tau)\eta(6\tau)^5}{\eta(\tau)^5\eta(3\tau)}\right) \in M_{12}(\Gamma_0(6))$$

is a (holomorphic) modular form. We now claim that the above modular form is identically equal to $\Delta(\tau)$. Indeed, the Sturm bound for the space $M_{12}(\Gamma_0(6))$ is 12, and a comparison of the q-expansions of $\Phi(\tau)$ and $\Delta(\tau)$ reveals that they agree up to the coefficient of q^{12} (and thus $\Phi(\tau) = \Delta(\tau)$). The equality of these functions by Theorem 2.4 immediately implies (4.7), completing the proof.

4. This identity quickly follows from the generating function for $\overline{p}(9n)$ found in the proof of Theorem 3 of [HS05a].

□

We now consider the functions

$$
\begin{aligned}
P^{\mathrm{ped}} &:= \frac{\eta(9\tau)^2\eta(12\tau)^3}{\eta(3\tau)^3\eta(36\tau)^2}\left[\frac{\eta(4\tau)}{\eta(\tau)} + \frac{\eta(2\tau)^2\eta(3\tau)^4\eta(4\tau)}{\eta(\tau)^5\eta(6\tau)^2} + 24\frac{\eta(2\tau)^3\eta(3\tau)^3\eta(4\tau)\eta(6\tau)^3}{\eta(\tau)^{10}}\right],\\
P^{\mathrm{pod}} &:= \frac{\eta(6\tau)^4\eta(12\tau)^5\eta(36\tau)^2}{\eta(3\tau)\eta(9\tau)^4\eta(18\tau)^5}\left[\frac{\eta(2\tau)}{\eta(\tau)\eta(4\tau)} + 10\frac{\eta(2\tau)\eta(6\tau)^{24}}{\eta(\tau)^7\eta(3\tau)^6\eta(4\tau)^7\eta(12\tau)^6}\right.\\
&\quad\left. + 16\frac{\eta(2\tau)^7\eta(3\tau)^3\eta(6\tau)^6\eta(12\tau)^3}{\eta(\tau)^{10}\eta(4\tau)^{10}} + \frac{\eta(2\tau)^{13}\eta(3\tau)^{12}\eta(12\tau)^{12}}{\eta(\tau)^{13}\eta(4\tau)^{13}\eta(6\tau)^{12}}\right],\\
P^{\overline{p_o}} &:= \frac{\eta(6\tau)^9\eta(9\tau)^4\eta(36\tau)^2}{\eta(3\tau)^6\eta(12\tau)^3\eta(18\tau)^6}\left[\frac{\eta(2\tau)^3}{\eta(\tau)^2\eta(4\tau)} - \frac{\eta(2\tau)^5\eta(3\tau)^4}{\eta(\tau)^6\eta(4\tau)\eta(6\tau)^2} - 24\frac{\eta(2\tau)^6\eta(3\tau)^3\eta(6\tau)^3}{\eta(\tau)^{11}\eta(4\tau)}\right],\\
P^{\overline{p}} &:= \frac{\eta(6\tau)^{19}\eta(18\tau)}{\eta(3\tau)^6\eta(9\tau)^6\eta(12\tau)^5\eta(36\tau)^2}\left[\frac{\eta(4\tau)}{\eta(2\tau)^2} - \frac{\eta(4\tau)^{13}\eta(6\tau)^{24}}{\eta(2\tau)^{26}\eta(12\tau)^{12}} - 128\frac{\eta(4\tau)^{10}\eta(6\tau)^{15}}{\eta(2\tau)^{23}\eta(12\tau)^3}\right.\\
&\quad\left. - 640\frac{\eta(4\tau)^7\eta(6\tau)^6\eta(12\tau)^6}{\eta(2\tau)^{20}}\right].
\end{aligned}
$$

As before, using Lemmas 2.11 and 2.6 it is straightforward to check that $P^* \in \mathcal{M}(\Gamma_0(36))$ and thus $L^* \in \mathcal{M}(\Gamma_0(12))$, where

$$L^* := U_3\left(P^*\right).$$

Note now that from Proposition 4.5(1),

$$
\begin{aligned}
P^{\mathrm{ped}} &= q^{-1}\frac{f_9^2f_{12}^3}{f_3^3f_{36}^2}\left(\frac{f_4}{f_1} + \frac{f_2^2f_3^4f_4}{f_1^5f_6^2} + 24q\frac{f_2^3f_3^3f_4f_6^3}{f_1^{10}}\right)\\
&= q^{-1}\frac{f_9^2f_{12}^3}{f_3^3f_{36}^2}\left(\sum_{n=0}^{\infty}\mathrm{ped}(n)q^n + \sum_{n=0}^{\infty}\mathrm{ped}(9n+1)q^n\right).
\end{aligned}
$$

In a similar manner, using the generating functions given in Proposition 4.5(2)-(4) we see

$$
\begin{aligned}
P^{\mathrm{pod}} &= q\frac{f_6^4f_{12}^5f_{36}^2}{f_3f_9^4f_{18}^5}\left(\sum_{n=0}^{\infty}\mathrm{pod}(n)q^n + \sum_{n=0}^{\infty}\mathrm{pod}(9n-1)q^n\right),\\
P^{\overline{p_o}} &= \frac{f_6^9f_9^4f_{36}^2}{f_3^6f_{12}^3f_{18}^6}\left(\sum_{n=0}^{\infty}\overline{p_o}(n)q^n - \sum_{n=0}^{\infty}\overline{p_o}(9n)q^n\right),\\
P^{\overline{p}} &= \frac{f_6^{19}f_{18}}{f_3^6f_9^6f_{12}^5f_{36}^2}\left(\sum_{n=0}^{\infty}\overline{p}(n)q^{2n} - \sum_{n=0}^{\infty}\overline{p}(9n)q^{2n}\right).
\end{aligned}
$$

Therefore, by Lemma 2.6,

$$L^{\mathrm{ped}} = \frac{f_3^2 f_4^3}{f_1^3 f_{12}^2} \left(\sum_{n=0}^{\infty} \mathrm{ped}(3n+1) q^n + \sum_{n=0}^{\infty} \mathrm{ped}(27n+10) q^n \right),$$
$$L^{\mathrm{pod}} = \frac{f_2^4 f_4^5 f_{12}^2}{f_1 f_3^4 f_6^5} \left(\sum_{n=0} \mathrm{pod}(3n+2) q^{n+1} + \sum_{n=0}^{\infty} \mathrm{pod}(27n+17) q^{n+1} \right),$$
$$L^{\overline{po}} = \frac{f_2^9 f_3^4 f_{12}^2}{f_1^6 f_4^3 f_6^6} \left(\sum_{n=0}^{\infty} \overline{po}(3n) q^n - \sum_{n=0}^{\infty} \overline{po}(27n) q^n \right),$$
$$L^{\overline{p}} = \frac{f_2^{19} f_6}{f_1^6 f_3^6 f_4^5 f_{12}^2} \left(\sum_{n=0}^{\infty} \overline{p}(3n) q^{2n} - \sum_{n=0}^{\infty} \overline{p}(27n) q^{2n} \right).$$

As in Section 4.2, we can connect these sequences via a conjugated Atkin-Lehner involution. Indeed, let W be the Atkin-Lehner involution with $q = 4$ on $\mathcal{M}(\Gamma_0(36))$ and recall that ν is the map defined by $q \mapsto -q$. Then

$$W = \begin{bmatrix} 28 & 3 \\ 36 & 4 \end{bmatrix}, \quad \nu = \begin{bmatrix} 1 & \frac{1}{2} \\ 0 & 1 \end{bmatrix},$$

and we now define

$$\gamma := \nu W \nu = \begin{bmatrix} 46 & 28 \\ 36 & 22 \end{bmatrix}.$$

Proof of Theorem 4.2. We utilize a similar approach to that of Section 4.2. We compute $P^* \mid \gamma$ by first simplifying $\eta(d\gamma\tau)$ for the appropriate positive integers d. For example, when $d = 2$ we seek to understand

$$\eta(2\gamma\tau) = \eta\left(\begin{bmatrix} 92 & 56 \\ 36 & 22 \end{bmatrix} \tau \right).$$

Notice that

$$\begin{bmatrix} 92 & 56 \\ 36 & 22 \end{bmatrix} = \begin{bmatrix} 23 & 5 \\ 9 & 2 \end{bmatrix} \begin{bmatrix} 4 & 2 \\ 0 & 2 \end{bmatrix}$$

where

$$\begin{bmatrix} 23 & 5 \\ 9 & 2 \end{bmatrix} \in \mathrm{SL}_2(\mathbb{Z}),$$

hence

$$\eta(2\gamma\tau) = \eta\left(\begin{bmatrix} 23 & 5 \\ 9 & 2 \end{bmatrix}(2\tau+1)\right) = (-i(9(2\tau+1)+2)^{1/2} \cdot \varepsilon(23,5,9,2) \cdot \eta(2\tau+1)$$

by Lemma 2.13. The proof now follows in a straightforward but tedious manner by applying γ to each factor and simplifying the resulting expressions as above, recalling the fact that the U_3-operator commutes with γ. □

Chapter 5

d-Fold Partition Diamonds

The content of this chapter can also be found in [DJSW24].

5.1 Introduction

While technical methods exist for proving infinite families of partition congruences, as discussed in Chapters 3 and 4, single "isolated" congruences can often be proven with more elementary and simple means. In this chapter, our goal is demonstrate a few of these techniques in the context of d-fold partition diamonds, combinatorial objects that generalize the notion of plane partition diamonds. As defined by Andrews, Paule, and Riese [APR01], a plane partition diamond is partition whose parts are non-negative integers a_i and b_j that are placed at the nodes of the graph given in Figure 5.1. Note that each directed edge indicates the relation $\geq$.

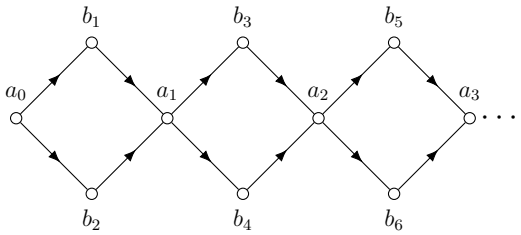


Figure 5.1: Plane partition diamonds ($d = 2$)

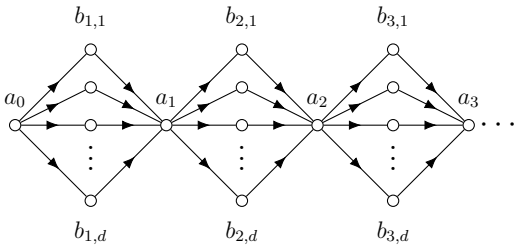


Figure 5.2: d-Fold partition diamonds

More precisely, we define d-fold partition diamonds to be partitions whose parts are non-negative integers a_i and $b_{i,j}$ which are placed at the nodes of the graph given in Figure 5.2, where again each directed edge indicates $\geq$.

Remark. Notice that 1-fold partition diamonds correspond to classical partitions, and 2-fold partition diamonds correspond to plane partition diamonds.

Letting $r_d(n)$ count the number of d-fold partition diamonds of n, the following theorem provides a formula for the generating function of $r_d(n)$.

Theorem 5.1. *The generating function for the number of d-fold partition diamonds is given by*

$$\sum_{n=0}^{\infty} r_d(n)q^n = \prod_{n=1}^{\infty} \frac{F_d(q^{(n-1)(d+1)+1}, q)}{1-q^n},$$

where F_d is a polynomial that is defined recursively in Lemma 5.6.

We also study Schmidt-type d-fold partition diamonds, those partitions whose parts lie on the directed graph in Figure 5.2, but with the $b_{j,k}$'s omitted from the sum. If we define $s_d(n)$ to be the number of Schmidt-type d-fold partition diamonds of n, then the generating function for $s_d(n)$ is given in the following theorem.

Theorem 5.2. *The generating function for the number of Schmidt type d-fold diamond partitions is given by*

$$\sum_{n=0}^{\infty} s_d(n)q^n = \prod_{n=1}^{\infty} \frac{A_d(q^n)}{(1-q^n)^{d+1}},$$

where $A_d(q) = F_d(q, 1)$ is the dth Eulerian polynomial.

Both Theorems 5.1 and 5.2 arise as special cases of a more technical theorem, Theorem 5.4, which will be proved using MacMahon's partition analysis (see Section 2.5). In Section

5.2, we state Theorem 5.4 and provide several preliminary results needed for its proof, which we give in Section 5.3. In Section 5.4, we prove several Ramanujan-like congruences for d-fold partition diamonds using Theorem 5.2.

5.2 Initial computations

We begin by extending MacMahon's elimination formula (2.3).

Proposition 5.3. *For d a positive integer and j an integer, we have that*

$$\underset{\geq}{\Omega} \frac{\lambda^j}{(1-\lambda x_1)\cdots(1-\lambda x_d)(1-\lambda^{-1}y)} = \frac{1}{(1-y)}\left[\frac{1}{(1-x_1)(1-x_2)\cdots(1-x_d)} - \frac{y^{j+1}}{(1-x_1y)(1-x_2y)\cdots(1-x_dy)}\right].$$

Proof. Expanding the left hand side using geometric series and applying the definition of $\Omega_{\geq}$, we obtain

$$\begin{aligned}
\underset{\geq}{\Omega} \frac{\lambda^j}{(1-\lambda x_1)\cdots(1-\lambda x_d)(1-\lambda^{-1}y)} &= \underset{\geq}{\Omega} \sum_{a_1,\ldots,a_{d+1}\geq 0} \lambda^{j+a_1+\cdots+a_d-a_{d+1}} x_1^{a_1}\cdots x_d^{a_d} y^{a_{d+1}} \\
&= \underset{\geq}{\Omega} \sum_{\substack{a_1,\ldots,a_d\geq 0 \\ 0\leq k\leq a_1+\cdots+a_d+j}} \lambda^k x_1^{a_1}\cdots x_d^{a_d} y^{a_1+\cdots+a_d+j-k} \\
&= \sum_{a_1,\ldots,a_d\geq 0} x_1^{a_1}\cdots x_d^{a_d} \sum_{k=0}^{a_1+\cdots+a_d+j} y^k \\
&= \sum_{a_1,\ldots,a_d\geq 0} x_1^{a_1}\cdots x_d^{a_d}\left[\frac{1-y^{a_1+\cdots+a_d+j+1}}{1-y}\right],
\end{aligned}$$

and the result follows by simplifying the geometric series. □

Following the work of Andrews, Paule, and Riese in [APR01], for integers $d, n \geq 1$ we now set

$$D_{d,n} \coloneqq D_{d,n}(q_0, q_1, \ldots, q_n; w) \coloneqq \sum q_0^{a_0}\cdots q_n^{a_n} w^{\sum_{j,k} b_{j,k}}$$

where the (outer) sum for $D_{d,n}$ ranges over all non-negative integers a_i and $b_{j,k}$ which satisfy the inequalities encoded by Figure 5.3. Thus, $D_{d,n}$ is the generating function for all d-fold

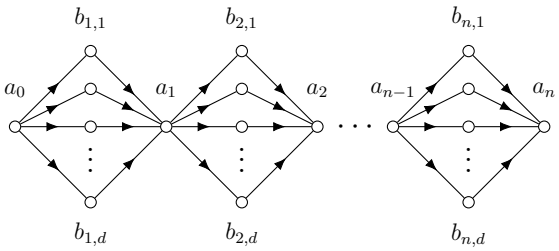


Figure 5.3: d-Fold partition diamond of length n.

partition diamonds of fixed length n, which contain nodes a_i and $b_{j,k}$ for $0 \leq i \leq n$, $1 \leq j \leq n$, and $1 \leq k \leq d$; each part (i.e., vertex of the graph) corresponds to some q_i or w in this generating function.

Using this notation, we are able to state our main technical theorem.

Theorem 5.4. *For $d \geq 1$ and $n \geq 1$,*

$$D_{d,n}(q_0, \ldots, q_n; w) = \left(\prod_{k=0}^{n-1} \frac{F_d(Q_k w^{dk}, w)}{(1 - Q_k w^{dk})(1 - Q_k w^{dk+1}) \cdots (1 - Q_k w^{dk+d})} \right) \frac{1}{1 - Q_n w^{dn}},$$

where F_d is the polynomial that is defined recursively in Lemma 5.6, and $Q_k \coloneqq q_0 q_1 \ldots q_k$ for all $k \geq 0$.

We continue to generalize the objects described in [APR01] as we fix some more notation to describe the crude form of the generating function $D_{d,n}$. For integers $d, n \geq 1$ we set

$$\Lambda_n \coloneqq \Lambda_{d,n} \coloneqq \left(\lambda_{n,1}^{a_{n-1}-b_{n,1}} \lambda_{n,2}^{a_{n-1}-b_{n,2}} \cdots \lambda_{n,d}^{a_{n-1}-b_{n,d}} \right) \left(\mu_{n,1}^{b_{n,1}-a_n} \mu_{n,2}^{b_{n,2}-a_n} \cdots \mu_{n,d}^{b_{n,d}-a_n} \right).$$

Note here that $\Lambda_{d,1}\Lambda_{d,2} \cdots \Lambda_{d,n}$ encodes all of the inequalities that must be satisfied in a d-fold diamond of length n. That is, each inequality (i.e., edge of the graph) corresponds to some $\lambda_{i,j}$ or $\mu_{i,j}$. In other words, we immediately have that

$$D_{d,n} = \underset{\geq}{\Omega} \sum_{a_i \geq 0, b_{j,k} \geq 0} \Lambda_{d,1}\Lambda_{d,2} \cdots \Lambda_{d,n} q_0^{a_0} \cdots q_n^{a_n} w^{\sum_{j,k} b_{j,k}}. \tag{5.1}$$

In the proof of Theorem 5.4, we will also need (for integers $d, n \geq 1$, and $\rho \geq 0$)

$$D_{d,n}^{(\rho)} \coloneqq D_{d,n}^{(\rho)}(q_0, q_1, \ldots, q_n; w) \coloneqq \sum q_0^{a_0} \cdots q_n^{a_n} w^{\sum_{j,k} b_{j,k}}$$

where the sum for $D_{d,n}^{(\rho)}$ ranges over all non-negative integers a_i and $b_{j,k}$ which satisfy the inequalities encoded by $\Lambda_{d,1}\Lambda_{d,2}\cdots\Lambda_{d,n}$ and where $a_n \geq \rho$.

Now we set (for $d, n \geq 1$ and $1 \leq k \leq n$)

$$\begin{aligned} h \coloneqq h_d &\coloneqq \frac{1}{1 - \lambda_{1,1} \cdots \lambda_{1,d}\, q_0} \\ f_k \coloneqq f_{k,d} &\coloneqq \frac{1}{\left(1 - \frac{\mu_{k,1}}{\lambda_{k,1}} w\right) \cdots \left(1 - \frac{\mu_{k,d}}{\lambda_{k,d}} w\right) \cdot \left(1 - \frac{\lambda_{k+1,1}\cdots\lambda_{k+1,d}}{\mu_{k,1}\cdots\mu_{k,d}} q_k\right)} \\ g_n \coloneqq g_{n,d} &\coloneqq \frac{1 - \frac{\lambda_{n+1,1}\cdots\lambda_{n+1,d}}{\mu_{n,1}\cdots\mu_{n,d}} q_n}{1 - \frac{q_n}{\mu_{n,1}\cdots\mu_{n,d}}} \end{aligned}$$

and note that one has the following crude form of the generating functions.

Lemma 5.5. *For $d, n \geq 1$, we have that*

$$\begin{aligned} D_{d,n} &= \underset{\geq}{\Omega}\, h \cdot f_1 \cdots f_n \cdot g_n \\ D_{d,n}^{(\rho)} &= \underset{\geq}{\Omega}\, h \cdot f_1 \cdots f_n \cdot g_n \left(\frac{q_n}{\mu_{n,1}\cdots\mu_{n,d}}\right)^{\rho}. \end{aligned}$$

Proof. The proof follows, mutatis mutandis, as in the proofs of Propositions 2.1 and 2.2 in [APR01]. □

5.3 Proof of Theorem 5.4

In order to prove Theorem 5.4, we must be able to apply some elimination formulae in order to simplify the expression for $D_{d,n}$ given in Lemma 5.5. We'll first do this in the case where $n = 1$.

Lemma 5.6. *For $d \geq 1$, we have that*

$$D_{d,1}(q_0, q_1; w) = \frac{F_d(q_0, w)}{(1-q_0)(1-q_0w)\cdots(1-q_0w^d)\cdot(1-q_0q_1w^d)},$$

where $F_d(q_0, w) \in \mathbb{Z}[q_0, w]$ is a polynomial of degree $d-1$ in q_0 that is given by $F_1(q_0, w) = 1$ and

$$F_d(q_0, w) = \frac{(1-q_0w^d)F_{d-1}(q_0, w) - w(1-q_0)F_{d-1}(q_0w, w)}{1-w}.$$

Proof. Proceed by induction on d. The case $d = 1$ corresponds to classical partitions with at most three parts. We may compute the generating function for such partitions using MacMahon's partition analysis: we have

$$\begin{aligned} D_{1,1}(q_0, q_1; w) &= \underset{\geq}{\Omega} \sum_{a_0,a_1,b_1\geq 0} \lambda_1^{a_0-b_1}\mu_1^{b_1-a_1}q_0^{a_0}w^{b_1}q_1^{a_1} = \underset{\geq}{\Omega} \sum_{a_0,a_1,b_1\geq 0} (\lambda_1 q_0)^{a_0}\left(\frac{\mu_1}{\lambda_1}w\right)^{b_1}\left(\frac{q_1}{\mu_1}\right)^{a_1} \\ &= \underset{\geq}{\Omega} \frac{1}{(1-\lambda_1q_0)(1-\lambda_1^{-1}\mu_1w)(1-\mu_1^{-1}q_1)} = \underset{\geq}{\Omega} \frac{1}{(1-q_0)(1-\mu_1q_0w)(1-\mu_1^{-1}q_1)} \\ &= \frac{1}{(1-q_0)(1-q_0w)(1-q_0q_1w)}, \end{aligned}$$

by applying (2.3) to successively eliminate λ_1 and μ_1. Hence, $F_1 = 1$.

Now suppose that the conclusion holds for $d-1$. Then by Lemma 5.5 (and the induction hypothesis) we have

$$\begin{aligned} D_{d,1}(q_0, q_1; w) &= \underset{\geq}{\Omega} \frac{1}{(1-\lambda_1\cdots\lambda_d\, q_0)\cdot\left(1-\lambda_1^{-1}\mu_1w\right)\cdots\left(1-\lambda_d^{-1}\mu_dw\right)\cdot\left(1-\mu_1^{-1}\cdots\mu_d^{-1}q_1\right)} \\ &= \underset{\geq}{\Omega} \frac{D_{d-1,1}(\lambda_dq_0, \mu_d^{-1}q_1; w)}{1-\lambda_d^{-1}\mu_dw} \\ &= \underset{\geq}{\Omega} \frac{F_{d-1}(\lambda_dq_0, w)}{(1-\lambda_dq_0)\cdots(1-\lambda_dq_0w^{d-1})(1-\lambda_d\mu_d^{-1}q_0q_1w^{d-1})(1-\lambda_d^{-1}\mu_dw)}. \end{aligned}$$

Write F_{d-1} as a polynomial in q_0 with coefficients in $\mathbb{Z}[w]$ as

$$F_{d-1}(q_0, w) = \sum_{i=0}^{d-2} a_i(w)q_0^i.$$

Then

$$\begin{aligned} D_{d,1}(q_0, q_1; w) &\underset{\geq}{\Omega} \frac{\sum_{i=0}^{d-2} a_i(w)(\lambda_d q_0)^i}{(1-\lambda_d q_0)\cdots(1-\lambda_d q_0 w^{d-1})(1-\frac{\lambda_d}{\mu_d} q_0 q_1 w^{d-1})(1-\frac{\mu_d}{\lambda_d} w)} \\ &= \sum_{i=0}^{d-2} a_i(w) q_0^i \underset{\geq}{\Omega} \frac{\lambda_d^i}{(1-\lambda_d q_0)\cdots(1-\lambda_d q_0 w^{d-1})(1-\frac{\lambda_d}{\mu_d} q_0 q_1 w^{d-1})(1-\frac{\mu_d}{\lambda_d} w)} \\ &= \sum_{i=0}^{d-2} a_i(w) q_0^i \underset{\geq}{\Omega} \frac{\lambda_d^i}{(1-\lambda_d q_0)\cdots(1-\lambda_d q_0 w^{d-1})(1-\lambda_d^{-1} w)(1-q_0 q_1 w^d)}, \end{aligned}$$

by (2.3). Rearranging,

$$\begin{aligned} D_{d,1}(q_0, q_1; w) &= \frac{\sum_{i=0}^{d-2} a_i(w) q_0^i}{1-q_0 q_1 w^d} \underset{\geq}{\Omega} \frac{\lambda_d^i}{(1-\lambda_d q_0)\cdots(1-\lambda_d q_0 w^{d-1})(1-\lambda_d^{-1} w)} \\ &= \frac{\sum_{i=0}^{d-2} a_i(w) q_0^i}{(1-q_0 q_1 w^d)(1-w)} \left[\frac{1}{(1-q_0)\cdots(1-q_0 w^{d-1})} - \frac{w^{i+1}}{(1-q_0 w)\cdots(1-q_0 w^d)} \right] \\ &= \frac{1}{(1-q_0)\cdots(1-q_0 w^d)(1-q_0 q_1 w^d)} \sum_{i=0}^{d-2} a_i(w) q_0^i \left[\frac{1-q_0 w^d}{1-w} - \frac{w^{i+1}(1-q_0)}{1-w} \right] \end{aligned}$$

by applying Proposition 5.3. Thus, we have shown that

$$\begin{aligned} F_d(q_0, w) &= \sum_{i=0}^{d-2} a_i(w) q_0^i \left[\frac{1-q_0 w^d}{1-w} - \frac{w^{i+1}(1-q_0)}{1-w} \right] \\ &= \frac{1-q_0 w^d}{1-w} \sum_{i=0}^{d-2} a_i(w) q_0^i - \frac{w}{1-w}(1-q_0) \sum_{i=0}^{d-2} a_i(w) q_0^i w^i \\ &= \frac{(1-q_0 w^d) F_{d-1}(q_0, w) - w(1-q_0) F_{d-1}(q_0 w, w)}{1-w}. \end{aligned}$$

To complete the proof, note that $F_d(q_0, w)$ is indeed a polynomial since $w = 1$ is a root of the numerator given above. □

Lemma 5.6 will serve as the base case of an induction argument in the proof of Theorem 5.4. We will also need the following results in order to complete that proof.

Lemma 5.7 (Lemma 2.1 of [APR01]). *For $k \geq 1$, and $y_1, \ldots, y_k, z \neq 0$ distinct elements from an appropriate field, define*

$$p(\mathbf{y}; z) \coloneqq \prod_{i=1}^{k} \left(1 - \frac{y_i}{z}\right),$$

where $\mathbf{y} = (y_1, \ldots, y_k)$. For $1 \leq j \leq k$ define

$$p_j(\mathbf{y}; z) \coloneqq \prod_{\substack{i=1 \\ i \neq j}}^{k} \left(1 - \frac{y_i}{z}\right)^{-1}.$$

Then we have

$$\frac{1}{p(\mathbf{y}; z)} = \sum_{j=1}^{k} \frac{p_j(\mathbf{y}; y_j)}{1 - (y_j/z)}.$$

Lemma 5.8. *Let $d, n \geq 1$ and $\rho \geq 0$. Then*

$$D_{d,n}^{(\rho)}(q_0, \ldots, q_n; w) = (q_0 \cdots q_n)^{\rho} w^{dn\rho} D_{d,n}(q_0, \ldots, q_n; w).$$

Proof. The above equality is given by the following bijection. Given an arbitrary d-fold diamond partition, adding ρ to each part assigns the partition a unique d-fold diamond partition with smallest part at least ρ. Conversely, given an arbitrary d-fold diamond partition with smallest part at least ρ, subtracting ρ from each part assigns the partition a unique d-fold diamond partition. Thus set of d-fold diamond partitions and the set of d-fold diamond partitions with smallest part at least ρ are in bijection, which is encoded by the above equality on their generating functions; this proves the result. □

We are now equipped to prove Theorem 5.4, following the same approach as in the proof of [APR01, Theorem 2.1].

Proof of Theorem 5.4. We proceed by induction on n. The base case (for $n = 1$) has already been discussed in Lemma 5.6. Now, suppose the theorem holds for n and note that by Lemma 5.5 we have

$$
\begin{aligned}
D_{n+1} =& D_{d,n+1}(q_0,\dots,q_{n+1};w) \underset{\geq}{\Omega} h\cdot f_1\cdots f_{n+1}\cdot g_{n+1}\\
=& \underset{\geq}{\Omega} h\cdot f_1\cdots f_{n-1}\frac{1}{\left(1-\frac{\mu_{n,1}}{\lambda_{n,1}}w\right)\cdots\left(1-\frac{\mu_{n,d}}{\lambda_{n,d}}w\right)}\cdot\frac{1}{\left(1-\frac{\lambda_{n+1,1}\cdots\lambda_{n+1,d}}{\mu_{n,1}\cdots\mu_{n,d}}q_n\right)}\\
&\cdot\frac{1}{\left(1-\frac{\mu_{n+1,1}}{\lambda_{n+1,1}}w\right)\cdots\left(1-\frac{\mu_{n+1,d}}{\lambda_{n+1,d}}w\right)}\cdot\frac{1}{\left(1-\frac{1}{\mu_{n+1,1}\cdots\mu_{n+1,d}}q_{n+1}\right)}.
\end{aligned}
$$

Now, we apply Lemma 5.6 to the last $d+2$ factors to eliminate $\lambda_{n+1,1},\dots\lambda_{n+1,d}$ and $\mu_{n+1,1},\dots\mu_{n+1,d}$ and find that

$$
\begin{aligned}
D_{n+1} =& \underset{\geq}{\Omega} h\cdot f_1\cdots f_{n-1}\frac{1}{\left(1-\frac{\mu_{n,1}}{\lambda_{n,1}}w\right)\cdots\left(1-\frac{\mu_{n,d}}{\lambda_{n,d}}w\right)}D_1\left(\frac{q_n}{\mu_{n,1}\dots\mu_{n,d}},q_{n+1};w\right)\\
=& \underset{\geq}{\Omega} h\cdot f_1\cdots f_{n-1}\frac{1}{\left(1-\frac{\mu_{n,1}}{\lambda_{n,1}}w\right)\cdots\left(1-\frac{\mu_{n,d}}{\lambda_{n,d}}w\right)}\cdot\frac{F_d\left(\frac{q_n}{\mu_{n,1}\cdots\mu_{n,d}},w\right)}{p(\mathbf{y};\mu_{n,1}\cdots\mu_{n,d})},
\end{aligned}
$$

where we set $\mathbf{y}=(y_1,\dots,y_{d+2}) \coloneqq (q_n,q_nw,\dots,q_nw^d,q_nq_{n+1}w^d)$. Then, applying Lemma 5.7 and substituting $F_d\left(\frac{q_n}{\mu_{n,1}\cdots\mu_{n,d}},w\right)=\sum_{i=0}^{d-1}a_i(w)\frac{q_n^i}{(\mu_{n,1}\cdots\mu_{n,d})^i}$ (where $a_i(w)\in\mathbb{Z}[w]$ depends on d) gives

$$
\begin{aligned}
D_{n+1} =& \underset{\geq}{\Omega} h\cdot f_1\cdots f_{n-1}\frac{F_d\left(\frac{q_n}{\mu_{n,1}\cdots\mu_{n,d}},w\right)}{\left(1-\frac{\mu_{n,1}}{\lambda_{n,1}}w\right)\cdots\left(1-\frac{\mu_{n,d}}{\lambda_{n,d}}w\right)}\cdot\sum_{j=1}^{d+2}\frac{p_j(\mathbf{y};y_j)}{1-\frac{y_j}{\mu_{n,1}\cdots\mu_{n,d}}}\\
=& \sum_{i=0}^{d-1}a_i(w)q_n^i\sum_{j=1}^{d+2}p_j(\mathbf{y};y_j)\frac{1}{y_j^i}\left[\underset{\geq}{\Omega}\frac{h\cdot f_1\cdots f_{n-1}}{\left(1-\frac{\mu_{n,1}}{\lambda_{n,1}}w\right)\cdots\left(1-\frac{\mu_{n,d}}{\lambda_{n,d}}w\right)}\cdot\frac{1}{1-\frac{y_j}{\mu_{n,1}\cdots\mu_{n,d}}}\right.\\
&\qquad\left.\cdot\left(\frac{y_j}{\mu_{n,1}\cdots\mu_{n,d}}\right)^i\right]\\
=& \sum_{i=0}^{d-1}a_i(w)q_n^i\sum_{j=1}^{d+2}p_j(\mathbf{y};y_j)\frac{1}{y_j^i}D_n^{(i)}(q_0,q_1,\dots,q_{n-1},y_j;w)\\
=& \sum_{i=0}^{d-1}a_i(w)Q_n^iw^{din}\sum_{j=1}^{d+2}p_j(\mathbf{y};y_j)D_n(q_0,q_1,\dots,q_{n-1},y_j;w)
\end{aligned}
$$

$$= F_d(Q_n w^{dn}, w) \sum_{j=1}^{d+2} p_j(\mathbf{y}; y_j) D_n(q_0, q_1, \ldots, q_{n-1}, y_j; w),$$

where we have used Lemmas 5.5 and 5.8. By the induction hypothesis, we may observe that

$$D_n(q_0, q_1, \ldots, q_{n-1}, y_j; w) = \frac{1 - Q_n w^{dn}}{1 - Q_{n-1} y_j w^{dn}} D_n(q_0, q_1, \ldots, q_{n-1}, q_n; w),$$

and thus we have

$$\begin{aligned}
D_{n+1} =& F_d(Q_n w^{dn}, w)(1 - Q_n w^{dn}) D_n(q_0, q_1, \ldots, q_{n-1}, q_n; w) \sum_{j=1}^{d+2} \frac{p_j(\mathbf{y}; y_j)}{1 - (q_0 \ldots q_{n-1}) y_j w^{dn}} \\
=& F_d(Q_n w^{dn}, w)(1 - Q_n w^{dn}) D_n(q_0, q_1, \ldots, q_{n-1}, q_n; w) \frac{1}{p(\mathbf{y}; \frac{q_n}{Q_n w^{dn}})} \\
=& F_d(Q_n w^{dn}, w)(1 - Q_n w^{dn}) D_n(q_0, q_1, \ldots, q_{n-1}, q_n; w) \\
& \cdot \frac{1}{(1 - Q_n w^{dn})(1 - Q_n w^{dn+1}) \cdots (1 - Q_n w^{dn+d})(1 - Q_{n+1} w^{dn+d})}
\end{aligned}$$

by Lemma 5.7 with $z = (Q_{n-1} w^{dn})^{-1}$. Thus we have shown that

$$D_{n+1} = \frac{F_d(Q_n w^{dn}, w)}{(1 - Q_n w^{dn+1}) \cdots (1 - Q_n w^{dn+d})(1 - Q_{n+1} w^{dn+d})} D_n,$$

which gives the desired result. □

5.4 Ramanujan-like congruences for d-fold partition diamonds

We begin this section by noting the following corollary of Theorem 5.2, which gives an alternate form of the generating function for $s_d(n)$.

Corollary 5.9. *We have*

$$\sum_{n=0}^{\infty} s_d(n) q^n = \prod_{n=1}^{\infty} \left(\sum_{j=0}^{\infty} (j+1)^d q^{jn} \right).$$

Proof. This follows immediately from Theorem 5.2, together with the following identity proved by Euler in [Eul68].

$$\sum_{j=0}^{\infty}(j+1)^d q^{jn} = \frac{A_d(q^n)}{(1-q^n)^{d+1}}.$$

□

Remark. One can view Corollary 5.9 combinatorially by building an arbitrary Schmidt type d-fold partition diamond as follows. First, one may choose the nodes $a_0, a_1, \ldots$ by picking the difference, j_n, between a_{n-1} and a_n for each positive integer n (where $j_n = 0$ for sufficiently large n). This completely determines all of the linking nodes $a_0 = \sum_{n\geq 1} j_n$, $a_1 = \sum_{n\geq 2} j_n$, etc., and the $q^{j_n n}$ keeps track of their contribution. Then, there are $(j_n+1)^d$ options for the nodes $b_{n,1}, \ldots, b_{n,d}$ (since there are $j_n + 1$ possibilities for each of those nodes).

We utilize this new view of the generating function for $s_d(n)$ to easily prove a variety of arithmetic properties satisfied by these functions. Prior to doing so, we remind the reader of two well–known results which will also be helpful below:

Lemma 5.10. *(Euler's Pentagonal Number Theorem) We have*

$$\prod_{m=1}^{\infty}(1-q^m) = \sum_{n=-\infty}^{\infty}(-1)^n q^{n(3n+1)/2}.$$

Proof. See [Hir17, Eq. (1.6.1)]. □

Lemma 5.11. *(Jacobi) We have*

$$\prod_{m=1}^{\infty}(1-q^m)^3 = \sum_{n=0}^{\infty}(-1)^n(2n+1)q^{n(n+1)/2}.$$

Proof. See [Hir17, Eq. (1.7.1)]. □

We now prove the following theorem which yields an infinite family of Ramanujan–like congruences modulo arbitrarily large powers of 2.

Theorem 5.12. *For all $d \geq 1$ and all $n \geq 0$, $s_d(2n+1) \equiv 0 \pmod{2^d}$.*

Proof. For fixed $d \geq 1$,

$$\begin{aligned}\sum_{n=0}^{\infty} s_d(n)q^n &= \prod_{n=1}^{\infty}\left(1+2^dq^n+3^dq^{2n}+4^dq^{3n}+5^dq^{4n}+6^dq^{5n}+\dots\right)\\ &\equiv \prod_{n=1}^{\infty}\left(1+0q^n+3^dq^{2n}+0q^{3n}+5^dq^{4n}+0q^{5n}+\dots\right) \pmod{2^d}\\ &= \prod_{n=1}^{\infty}\left(1+3^dq^{2n}+5^dq^{4n}+\dots\right).\end{aligned}$$

In the penultimate line above, we have used the fact that $2^d \mid (2j)^d$ for all $j \geq 1$. The last expression above is a function of q^2, and this immediately implies the result. □

We note, in passing, that the $d = 2$ case of Theorem 5.12 was proven by Andrews and Paule [AP22, Corollary 2].

We next prove the following overarching lemma that will provide us with the machinery needed to prove infinite families of divisibility properties satisfied by $s_d(n)$.

Lemma 5.13. *Let k and r be nonnegative integers and let $m \geq 2$. For all $n \geq 0$, $s_{\phi(m)k+r}(n) \equiv s_r(n) \pmod{m}$ where $\phi(m)$ is Euler's totient function.*

Proof. Note that

$$\begin{aligned}&\sum_{n=0}^{\infty} s_{\phi(m)k+r}(n)q^n\\ =\ &\prod_{n=1}^{\infty}\left(1+2^{\phi(m)k+r}q^n+3^{\phi(m)k+r}q^{2n}+4^{\phi(m)k+r}q^{3n}\right.\\ &\qquad\left.+5^{\phi(m)k+r}q^{4n}+6^{\phi(m)k+r}q^{5n}+\dots\right)\\ =\ &\prod_{n=1}^{\infty}\left(1+(2^{\phi(m)})^k\cdot 2^rq^n+(3^{\phi(m)})^k\cdot 3^rq^{2n}+(4^{\phi(m)})^k\cdot 4^rq^{3n}\right.\\ &\qquad\left.+(5^{\phi(m)})^k\cdot 5^rq^{4n}+(6^{\phi(m)})^k\cdot 6^rq^{5n}+\dots\right)\\ \equiv\ &\prod_{n=1}^{\infty}\left(1+2^rq^n+3^rq^{2n}+4^rq^{3n}+5^rq^{4n}+6^rq^{5n}+\dots\right) \pmod{m}\\ =\ &\sum_{n=0}^{\infty} s_r(n)q^n.\end{aligned}$$

The penultimate line above follows from Euler's generalization of Fermat's Little Theorem. □

We now transition to a consideration of various families of congruences modulo 5. We first prove an infinite family of such congruences thanks to Lemma 5.13 which uses the function $s_1(n)$ as its starting point.

Theorem 5.14. *For all $k \geq 0$ and all $n \geq 0$,*

$$s_{4k+1}(5n+2) \equiv s_{4k+1}(5n+3) \equiv s_{4k+1}(5n+4) \equiv 0 \pmod 5.$$

Proof. Note that

$$\begin{aligned}
\sum_{n=0}^{\infty} s_1(n)q^n &= \prod_{n=1}^{\infty} \frac{1}{(1-q^n)^2} \\
&= \prod_{n=1}^{\infty} \frac{(1-q^n)^3}{(1-q^n)^5} \\
&\equiv \prod_{n=1}^{\infty} \frac{(1-q^n)^3}{(1-q^{5n})} \pmod 5 \\
&= \left(\sum_{j=0}^{\infty} (-1)^j (2j+1) q^{j(j+1)/2}\right) \prod_{n=1}^{\infty} \frac{1}{(1-q^{5n})}
\end{aligned}$$

thanks to Lemma 5.11. Since $\prod_{n=1}^{\infty} \frac{1}{(1-q^{5n})}$ is a function of q^5, the above product will satisfy a congruence modulo 5 in an arithmetic progression of the form $5n+r$, for $0 \leq r \leq 4$, if and only if such a congruence is satisfied by

$$\left(\sum_{j=0}^{\infty} (-1)^j (2j+1) q^{j(j+1)/2}\right).$$

Since $5n+2$ and $5n+4$ are never triangular numbers, this immediately tells us that, for all $n \geq 0$,

$$s_1(5n+2) \equiv s_1(5n+4) \equiv 0 \pmod 5.$$

Moreover, we know that $5n+3 = j(j+1)/2$ if and only if $j \equiv 2 \pmod 5$, and in such cases, $2j+1 \equiv 0 \pmod 5$. Because of the presence of the factor $2j+1$ in the sum above, we then

see that, for all $n \geq 0$,

$$s_1(5n+3) \equiv 0 \pmod 5$$

since $2 \cdot 2 + 1 = 5 \equiv 0 \pmod 5$. The remainder of the proof immediately follows from Lemma 5.13. □

We note, in passing, that Baruah and Sarmah [BS25, Theorem 5.1, equation (5.4)] also provided a proof of the $k = 0$ case of Theorem 5.14.

We next consider the following modulo 5 congruences, with a focus on the case $d = 2$.

Theorem 5.15. *For all $k \geq 0$ and all $n \geq 0$,*

$$s_{4k+2}(25n+23) \equiv 0 \pmod 5.$$

Proof. We begin by noting that

$$\begin{aligned}
\sum_{n=0}^{\infty} s_2(n) q^n &= \prod_{n=1}^{\infty} \frac{1+q^n}{(1-q^n)^3} \\
&= \prod_{n=1}^{\infty} \frac{1-q^{2n}}{(1-q^n)^4} \\
&= \prod_{n=1}^{\infty} \frac{(1-q^{2n})(1-q^n)}{(1-q^n)^5} \\
&\equiv \prod_{n=1}^{\infty} \frac{(1-q^{2n})(1-q^n)}{(1-q^{5n})} \pmod 5 \\
&= \left(\sum_{j,k=-\infty}^{\infty} (-1)^{j+k} q^{2(j(3j+1)/2)+k(3k+1)/2} \right) \prod_{n=1}^{\infty} \frac{1}{(1-q^{5n})}
\end{aligned}$$

thanks to Lemma 5.10. At this point, we consider those terms of the form q^{25n+23} in the power series representation of the last expression above. There are no terms of the form $q^{25n+8}, q^{25n+13}, q^{25n+18}$, or q^{25n+23} in the double sum above, although there are terms of the form q^{25n+3}. Thus, once we multiply the double sum above by

$$\prod_{n=1}^{\infty} \frac{1}{(1-q^{5n})} = \sum_{n=0}^{\infty} p(n) q^{5n}, \tag{5.2}$$

we see that the only way to obtain a term of the form q^{25n+23} which can contribute to $s_2(25n+23)$ modulo 5 is to multiply by a term of the form $q^{5(5m+4)}$ from the power series representation of (5.2). This will then contribute, modulo 5, to the value of $s_2(25n+23)$ by multiplying by the value $p(5m+4)$. Thanks to the Ramanujan congruence $p(5n+4) \equiv 0 \pmod 5$, we then know that, for all $n \geq 0$, $s_2(25n+23) \equiv 0 \pmod 5$. The theorem then follows thanks to Lemma 5.13. □

We can also show that the functions s_{4k+3} satisfy a rich set of congruences modulo 5 via the following:

Theorem 5.16. *For all $k \geq 0$ and all $n \geq 0$,*

$$s_{4k+3}(5n+2) \equiv s_{4k+3}(5n+4) \equiv 0 \pmod 5.$$

Proof. We begin with the following generating function manipulations:

$$\begin{aligned}
\sum_{n=0}^{\infty} s_3(n)q^n &= \prod_{n=1}^{\infty} \frac{1+4q^n+q^{2n}}{(1-q^n)^4} \\
&\equiv \prod_{n=1}^{\infty} \frac{1-q^n+q^{2n}}{(1-q^n)^4} \pmod 5 \\
&= \prod_{n=1}^{\infty} \frac{1+(-q^n)+(-q^n)^2}{(1-q^n)^4} \\
&= \prod_{n=1}^{\infty} \frac{1-(-q^n)^3}{(1-(-q^n))(1-q^n)^4} \\
&= \prod_{n=1}^{\infty} \frac{1+q^{3n}}{(1+q^n)(1-q^n)^4} \\
&= \prod_{n=1}^{\infty} \frac{(1-q^{6n})(1-q^n)^2}{(1-q^{3n})(1-q^{2n})(1-q^n)^5} \\
&\equiv \prod_{n=1}^{\infty} \frac{(1-q^{6n})(1-q^n)^2}{(1-q^{3n})(1-q^{2n})} \prod_{n=1}^{\infty} \frac{1}{(1-q^{5n})} \pmod 5
\end{aligned}$$

We now consider the power series representation of

$$F(q) := \prod_{n=1}^{\infty} \frac{(1-q^{6n})(1-q^n)^2}{(1-q^{3n})(1-q^{2n})}, \tag{5.3}$$

which turns out to be a well–known modular form; indeed, it appears as the seventh modular form in Mersmann's list of the 14 primitive eta–products which are holomorphic modular forms of weight 1/2. See [BvdGHZ08, page 30] and [OEI23, Entry A089812] for additional details. After some elementary calculations (for example, by noting that $qF(q^8)$ appears in [LO13, Theorem 1.2]), we find that

$$F(q) = \sum_{t=0}^{\infty} q^{t(t+1)/2} - 3\sum_{t=0}^{\infty} q^{(3t+1)(3t+2)/2}.$$

As noted in the proof of Theorem 5.14, $5n+2$ and $5n+4$ can never be triangular numbers. Since $\frac{1}{2}(3t+1)(3t+2)$ is always triangular, we conclude that $s_3(5n+2) \equiv 0 \pmod 5$ and $s_3(5n+4) \equiv 0 \pmod 5$. The case where $k > 0$ follows from Lemma 5.13. □

We can also prove the following additional congruence family modulo 5.

Theorem 5.17. *For all $k \geq 0$ and $n \geq 0$,*

$$s_{4k+3}(25n+23) \equiv 0 \pmod 5.$$

Proof. As with the proof of Theorem 5.16 above, we begin by considering $s_3(n)$. Using the notation from above, recall that

$$\begin{aligned} \sum_{n=0}^{\infty} s_3(n)q^n &\equiv \left(\sum_{t=0}^{\infty} q^{t(t+1)/2} - 3\sum_{t=0}^{\infty} q^{(3t+1)(3t+2)/2}\right) \prod_{n=1}^{\infty} \frac{1}{(1-q^{5n})} \pmod 5 \\ &= \left(\sum_{t=0}^{\infty} q^{t(t+1)/2} - 3\sum_{t=0}^{\infty} q^{(3t+1)(3t+2)/2}\right) \left(\sum_{n=0}^{\infty} p(n)q^{5n}\right). \end{aligned} \tag{5.4}$$

We then consider ways that q^{25n+23} can arise when (5.4) is expanded. We note that no numbers of the form $25n+8, 25n+13, 25n+18$, or $25n+23$ can be represented as a triangular number. Thus, the only way to obtain a term of the form q^{25n+23} in (5.4) is to multiply by a term of the form $p(5m+4)q^{5(5m+4)}$ (in the same way as was discussed in the proof of Theorem 5.15). This implies that, for all $n \geq 0$, $s_3(25n+23) \equiv 0 \pmod 5$. The full result follows from Lemma 5.13. □

We close this section with one last infinite family of congruences, this time modulo 11.

Theorem 5.18. *For all* $n \geq 0$,

$$s_{10k+1}(121n + 111) \equiv 0 \pmod{11}.$$

Proof. In the work of Gordon [Gor83, Theorem 2], we set $k = 2$ and $r = 2$, so that $\alpha = 1$. We then see that, if $24n \equiv 2 \pmod{11^2}$, then $12n \equiv 1 \pmod{121}$ or $n \equiv 111 \pmod{121}$. Gordon's work then implies that, for all $n \geq 0$, $s_1(121n+111) \equiv 0 \pmod{11}$. The full result then follows from Lemma 5.13. □

Chapter 6

Cusp Forms Without Complex Multiplication as p-Adic Limits

The content of this chapter can also be found in [Doc25].

6.1 Introduction

In this chapter, we turn our attention from partition congruences to a different phenomenon in the theory of modular forms: the expression of holomorphic modular forms as p-adic limits of weakly holomorphic modular forms. El-Guindy and Ono [EGO10] considered the cusp form with complex multiplication (CM)

$$g(\tau) := \eta(4\tau)^2\eta(8\tau)^2 = \sum_{n=1}^{\infty} a(n)q^n = q - 2q^5 - 3q^9 + 6q^{13} + \dots$$

and the weakly holomorphic modular form

$$F(\tau) := -\frac{\eta(4\tau)^2\eta(16\tau)^6}{\eta(32\tau)^4} = \sum_{n=-1}^{\infty} C(n)q^n = -\frac{1}{q} + 2q^3 + q^7 - 2q^{11} + \cdots.$$

El-Guindy and Ono proved

$$\lim_{m\to\infty} \frac{F \mid U(p^{2m+1})}{C(p^{2m+1})} = g \tag{6.1}$$

as a p-adic limit, for all primes $p \equiv 3 \pmod 4$ with $p \nmid C(p)$. For ease of notation, here and through the remainder of this chapter, we write $F \mid U(p^{2m+1})$ instead of $U_{p^{2m+1}}(F)$. Their proof relied on previous work of Guerzhoy, Kent, and Ono [GKO10] on harmonic Maass forms, studying the p-adic coupling of mock modular forms and their shadows.

Ahlgren and Samart [AS16] strengthened the p-adic limit of El-Guindy and Ono, proving the following theorem.

Theorem 6.1 (Theorem 1.1 of [AS16]). *For all $m \geq 0$ and all primes $p \equiv 3 \pmod 4$, we have*

$$v_p(C(p^{2m+1})) = m, \tag{6.2}$$

$$v_p\left(\frac{F \mid U(p^{2m+1})}{C(p^{2m+1})} - g\right) \geq m+1, \tag{6.3}$$

where $v_p(\cdot)$ denotes the p-adic valuation of $\mathbb{Z}[[q]]$.

Remark. The case $m = 0$ of (6.2) proves that $p \nmid C(p)$ for all primes $p \equiv 3 \pmod 4$, thus removing it as a necessary assumption.

A key ingredient in their proof is the fact that $g(\tau)$ lies in a one-dimensional space of cusp forms. Ahlgren and Samart also proved similar results for two other pairs of forms $(F(\tau), g(\tau))$.

Hanson and Jameson [HJ22] furthered the work of Ahlgren and Samart, proving that p-adic behavior identical to (6.2) and (6.3) holds for all normalized cusp forms (i.e., having leading coefficient 1) of one-dimensional cusp form spaces with trivial character. Specifically, Hanson and Jameson proved the following theorem.

Theorem 6.2 (Theorem 1 of [HJ22]). *Let $S_k(\Gamma_0(N))$ be a one-dimensional space such that the unique normalized cusp form $g(\tau) = \sum_{n\geq 1} a(n)q^n \in S_k(\Gamma_0(N))$ has complex multiplication. Then there exists a corresponding weakly holomorphic modular form $F(\tau) = \sum_{n\geq -1} C(n)q^n \in S_k^{\infty}(\Gamma_0(N))$ such that (6.2) and (6.3) hold for all primes p which are inert in the field of complex multiplication of $g(\tau)$.*

In light of their work, it is natural to investigate those one-dimensional spaces of cusp forms whose normalized form does not have CM. To this end, we now state our main theorem.

Theorem 6.3. *Let $S_k(\Gamma_0(N))$ be a one-dimensional space such that $X_0(N)$ has genus $g_N = 0$, and let $g(\tau) = \sum_{n\geq 1} a(n)q^n \in S_k(N)$ be the unique normalized cusp form. If $g(\tau)$ does not have complex multiplication, then there is a corresponding weakly holomorphic modular form $F(\tau) = \sum_{n\geq -1} C(n)q^n$ such that*

$$v_p(C(p^m)) = v_p(C(p)) \tag{6.4}$$

$$v_p\left(\frac{F \mid T_k(p^m)}{C(p^m)} - g\right) = (k-1)m - v_p(C(p)) \tag{6.5}$$

for all $m \geq 1$ and all primes $p \nmid N$ with $p \nmid a(p)$ and $v_p(C(p)) \leq k-1$. In particular, if $p \nmid N$ is prime such that $p \nmid a(p)$ and $p \nmid C(p)$, then for all integers $m \geq 1$,

$$v_p(C(p^m)) = 0,$$

$$v_p\left(\frac{F \mid T_k(p^m)}{C(p^m)} - g\right) = (k-1)m.$$

Remark. In striking contrast to (6.2), by (6.4) we see that p-adic valuations of the coefficients $C(p^m)$ are completely determined by that of $C(p)$. Because of this disparity, without CM the functions $g(\tau)$ are no longer p-adic limits of $F(\tau)$ under applications of the $U(p)$-operator. However, replacing the $U(p)$-operator with the Hecke operator $T_k(p^m)$ yields a p-adic limit analogous to (6.1), namely

$$\lim_{m\to\infty} \frac{F \mid T_k(p^m))}{C(p^m)} = g. \tag{6.6}$$

Remark. Similar p-adic limits to (6.6) exist in the context of Weierstrass mock modular forms. Specifically, we refer the interested reader to Theorem 1.3 of [AGOR15].

Our first task is to identify all possible pairs (k, N) satisfying the hypotheses of Theorem 6.3, i.e. such that $S_k(N)$ is a one-dimensional space whose normalized form does not have CM and such that $g_N = 0$. This is done in Section 6.2, along with the construction of two families ϕ_n and F_m of forms for each weight and level. These functions are pivotal to the proof of Theorem 6.3, which we provide in Section 6.3.

6.2 Preliminary results

As shown by Hanson and Jameson ([HJ22]), if $\dim(S_k(\Gamma_0(N))) = 1$ then it must be the case that $N \leq 241.5$; in addition, we note that also $k \leq 10$ if $N > 1$ and $k \leq 26$ if $N = 1$. Using Sage to quickly inspect these possibilities, we find that there are 29 spaces for which $\dim(S_k(\Gamma_0(N))) = 1$. Table 6.1 below lists the pairs (k, N) of weights and levels of these spaces, respectively, along with the unique normalized cusp form $g(\tau) \in S_k(N)$ and the genus g_N of the modular curve $X_0(N)$. Here $\Delta(\tau) \coloneqq {}^{E_4^3(\tau)-E_6^2(\tau)}/_{1728}$ is the modular discriminant of weight 12 and $j(\tau) \coloneqq {}^{E_4^3(\tau)}/_{\Delta(\tau)}$ is Klein's j-invariant.

Henceforth, we refer to pairs (k, N) for which the normalized $g(z) \in S_k(\Gamma_0(N))$ does not have CM as non-CM pairs. Notice that the spaces in Table 6.1 of genus 1 are exactly those of weight 2, so spaces $S_k(\Gamma_0(N))$ satisfying the hypotheses of Theorem 6.3 are equivalent to non-CM pairs with $k \geq 4$.

The following proposition establishes, for each non-CM pair (k, N) with $k \geq 4$, the existence of two modular families F_m and ϕ_n. Once these families are constructed, the proof of Theorem 6.3 may be carried out in a general way, so as to handle all such cases simultaneously.

Proposition 6.4. *Let (k, N) be a non-CM pair with $k \geq 4$.*

1. *For all integers $n \geq 2$, there exists $\phi_n \in M_{2-k}^{\infty}(\Gamma_0(N)) \cap \mathbb{Z}((q))$ of the form*

$$\phi_n = q^{-n} + \sum_{m=-1}^{\infty} A_n(m) q^m.$$

2. *For all integers $m \geq -1$, there exists a unique $F_m \in S_k^{\infty}(\Gamma_0(N)) \cap \mathbb{Z}((q))$ of the form*

$$F_m = -q^{-m} + \sum_{n=2}^{\infty} C_m(n) q^n.$$

3. *Let $F \coloneqq F_1 = \sum_{n=-1}^{\infty} C(n) q^n$ and $p \nmid N$ be prime. Then*

$$F \mid T_k(p^n) = p^{(k-1)n} F_{p^n} + C(p^n) g$$

Table 6.1: List of (k, N) with $\dim(S_k(\Gamma_0(N))) = 1$

(k, N)	normalized $g \in S_k(\Gamma_0(N))$	CM?	g_N
$(2, 11)$	$\eta(\tau)^2\eta(11\tau)^2 = q - 2q^2 - q^3 + 2q^4 + \dots$	no	1
$(2, 14)$	$\eta(\tau)\eta(2\tau)\eta(7\tau)\eta(14\tau) = q - q^2 - 2q^3 + q^4 + \dots$	no	1
$(2, 15)$	$\eta(\tau)\eta(3\tau)\eta(5\tau)\eta(15\tau) = q - q^2 - q^3 - q^4 + \dots$	no	1
$(2, 17)$	$q - q^2 - q^4 - 2q^5 + 4q^7 + \dots$	no	1
$(2, 19)$	$q - 2q^3 - 2q^4 + 3q^5 - q^7 + \dots$	no	1
$(2, 20)$	$\eta(2\tau)^2\eta(10\tau)^2 = q - 2q^3 - q^5 + 2q^7 + \dots$	no	1
$(2, 21)$	$q - q^2 + q^3 - q^4 - 2q^5 + \dots$	no	1
$(2, 24)$	$\eta(2\tau)\eta(4\tau)\eta(6\tau)\eta(12\tau) = q - q^3 - 2q^5 + q^9 + \dots$	no	1
$(2, 27)$	$\eta(3\tau)^2\eta(9\tau)^2 = q - 2q^4 - q^7 + 5q^{13} + \dots$	yes	1
$(2, 32)$	$\eta(4\tau)^2\eta(8\tau)^2 = q - 2q^5 - 3q^9 + 6q^{13} + \dots$	yes	1
$(2, 36)$	$\eta(6\tau)^4 = q - 4q^7 + 2q^{13} + 8q^{19} + \dots$	yes	1
$(2, 49)$	$q + q^2 - q^4 - 3q^8 - 3q^9 + \dots$	yes	1
$(4, 5)$	$\eta(\tau)^4\eta(5\tau)^4 = q - 4q^2 + 2q^3 + 8q^4 + \dots$	no	0
$(4, 6)$	$\eta(\tau)^2\eta(2\tau)^2\eta(3\tau)^2\eta(6\tau)^2 = q - 2q^2 - 3q^3 + 4q^4 + \dots$	no	0
$(4, 7)$	$\frac{\eta(\tau)^5\eta(7\tau)^5}{\eta(2\tau)\eta(14\tau)} + 4\eta(\tau)^2\eta(2\tau)^2\eta(7\tau)^2\eta(14\tau)^2 = q - q^2 - 2q^3 - 7q^4 + \dots$	no	0
$(4, 8)$	$\eta(2\tau)^4\eta(4\tau)^4 = q - 4q^3 - 2q^5 + 24q^7 + \dots$	no	0
$(4, 9)$	$\eta(3\tau)^8 = q - 8q^4 + 20q^7 - 70q^{13} + \dots$	yes	0
$(6, 3)$	$\eta(\tau)^6\eta(3\tau)^6 = q - 6q^2 + 9q^3 + 4q^4 + \dots$	no	0
$(6, 4)$	$\eta(2\tau)^{12} = q - 12q^3 + 54q^5 - 88q^7 + \dots$	no	0
$(6, 5)$	$q + 2q^2 - 4q^3 - 28q^4 + 25q^5 + \dots$	no	0
$(8, 2)$	$\eta(\tau)^8\eta(2\tau)^8 = q - 8q^2 + 12q^3 + 64q^4 + \dots$	no	0
$(8, 3)$	$\eta(\tau)^6\eta(3\tau)^4\left(\eta(\tau)^3 + 9\eta(9\tau)^3\right)^2 = q + 6q^2 - 27q^3 - 92q^4 + \dots$	no	0
$(10, 2)$	$\frac{\eta(2\tau)^{28}}{\eta(4\tau)^8} + 16\eta(\tau)^8\eta(2\tau)^4\eta(4\tau)^8 = q + 16q^2 - 156q^3 + 256q^4 + \dots$	no	0
$(12, 1)$	$\Delta(\tau) = q - 24q^2 + 252q^3 - 1472q^4 + \dots$	no	0
$(16, 1)$	$\Delta(\tau)E_4(\tau) = q + 216q^2 - 3348q^3 + 13888q^4 + \dots$	no	0
$(18, 1)$	$\Delta(\tau)E_6(\tau) = q - 528q^2 - 4284q^3 + 147712q^4 + \dots$	no	0
$(20, 1)$	$\Delta(\tau)E_4^2(\tau) = q + 456q^2 + 50652q^3 - 316352q^4 + \dots$	no	0
$(22, 1)$	$\Delta(\tau)E_4(\tau)E_6(\tau) = q - 288q^2 - 128844q^3 - 2014208q^4 + \dots$	no	0
$(26, 1)$	$\Delta(\tau)E_4^2(\tau)E_6(\tau) = q - 48q^2 - 195804q^3 - 33552128q^4 + \dots$	no	0

for all integers $n \geq 0$.

Proof. 1. We first define ϕ_2 explicitly for each pair (k, N), as well as a modular function $L \in M_0^\infty(\Gamma_0(N))$ with a simple pole at infinity. We then define $\phi_n, n \geq 3$ inductively by taking appropriate linear combinations of $\phi_{n-1}L, \phi_{n-1}, \ldots, \phi_2$. For example, for the pair $(8, 2)$ we have

$$\phi_3 = \phi_2 L + 104\phi_2 = q^{-3} - 2980q^{-1} + 146432 - 3896490q + \ldots$$
$$\phi_4 = \phi_3 L + 24\phi_3 + 2704\phi_2 = q^{-4} - 71936q^{-1} + 3793608 - 12159360q + \ldots,$$

and so on. Table 6.2 gives an explicit description of all required L and ϕ_2. Most of these functions are eta-quotients, in which case their modular behavior is readily verified by Lemmas 2.11 and 2.12. For the remaining pairs (k, N) the functions $\phi_2 \in M_{2-k}^\infty(\Gamma_0(N))$ are given as products of eta-quotients and sums involving the unique normalized cusp form $g \in S_k(\Gamma_0(N))$ and the Eisenstein series $E_2(z), E_2(Nz), E_4(z)$ and $E_4(Nz)$; as such, the modularity of these functions is also immediate.

2. Similarly, let $F_{-1} \coloneqq -g$ and inductively define F_m for $m \geq 0$ as linear combinations of $F_{n-1}L, F_{n-1}, \ldots, F_{-1}$. For example, in the case of the pair $(8, 2)$

$$F_0 = F_{-1}L + 32F_{-1} = -1 - 224q^2 + 4096q^3 - 31200q^4 + \ldots$$
$$F_1 = F_0 L + 24F_0 - 500F_{-1} = -q^{-1} + 2144q^2 - 98226q^3 + 1817856q^4 + \ldots,$$

and so on.

To show uniqueness, suppose two such F_m exist, say F_m and F'_m. Then $F_m - F'_m$ must be of the form $O(q^2) \in S_k(\Gamma_0(N))$, but $g = q + O(q^2)$ spans $S_k(\Gamma_0(N))$, and thus $F_m - F'_m = 0$.

3. The proof is identical to that of Proposition 3(c) of [HJ22].

□

For non-CM pairs with $k \geq 4$, the coefficients of F_m and ϕ_n satisfy a property known as Zagier duality. Along with the uniqueness of the functions F_m established in Proposition

Table 6.2: Defining L and ϕ_2 for $(k, N), k \geq 4$

(k, N)	$L \in M_0^\infty(\Gamma_0(N))$	$\phi_2 \in M_{2-k}^\infty(\Gamma_0(N))$
$(12, 1)$	$j(\tau) = q^{-1} + 744 + 196884q + \dots$	$\frac{E_4^2(\tau)E_6(\tau)}{\Delta(\tau)^2} = q^{-2} + 24q^{-1} - 196560 + \dots$
$(16, 1)$	$j(\tau) = q^{-1} + 744 + 196884q + \dots$	$\frac{E_4(\tau)E_6(\tau)}{\Delta(\tau)^2} = q^{-2} - 216q^{-1} - 146880 + \dots$
$(18, 1)$	$j(\tau) = q^{-1} + 744 + 196884q + \dots$	$\frac{E_4^2(\tau)}{\Delta(\tau)^2} = q^{-2} + 528q^{-1} + 86184 + \dots$
$(20, 1)$	$j(\tau) = q^{-1} + 744 + 196884q + \dots$	$\frac{E_6(\tau)}{\Delta(\tau)^2} = q^{-2} - 456q^{-1} - 39600 + \dots$
$(22, 1)$	$j(\tau) = q^{-1} + 744 + 196884q + \dots$	$\frac{E_4(\tau)}{\Delta(\tau)^2} = q^{-2} + 288q^{-1} + 14904 + \dots$
$(26, 1)$	$j(\tau) = q^{-1} + 744 + 196884q + \dots$	$\frac{1}{\Delta(\tau)^2} = q^{-2} + 48q^{-1} + 1224 + \dots$
$(8, 2)$	$\frac{\eta(\tau)^{24}}{\eta(2\tau)^{24}} = q^{-1} - 24 + 276q + \dots$	$\frac{\eta(\tau)^{80}}{\eta(2\tau)^{64}} = q^{-2} - 80q^{-1} + 3144 + \dots$
$(10, 2)$	$\frac{\eta(\tau)^{24}}{\eta(2\tau)^{24}} = q^{-1} - 24 + 276q + \dots$	$\frac{\eta(\tau)^{16}}{\eta(2\tau)^{32}} = q^{-2} - 16q^{-1} + 136 + \dots$
$(6, 3)$	$\frac{\eta(\tau)^{12}}{\eta(3\tau)^{12}} = q^{-1} - 12 + 54q + \dots$	$\frac{\eta(\tau)^6(3E_2(3\tau) - E_2(\tau))}{2\eta(3\tau)^{18}} = q^{-2} + 6q^{-1} - 27 + \dots$
$(8, 3)$	$\frac{\eta(\tau)^{12}}{\eta(3\tau)^{12}} = q^{-1} - 12 + 54q + \dots$	$\frac{\eta(\tau)^6}{\eta(3\tau)^{18}} = q^{-2} - 6q^{-1} + q + 28q + \dots$
$(6, 4)$	$\frac{\eta(\tau)^8}{\eta(4\tau)^8} = q^{-1} - 8 + 20q + \dots$	$\frac{\eta(2\tau)^8}{\eta(4\tau)^{16}} = q^{-2} - 8 + 36q^2 - 128q^4 + \dots$
$(4, 5)$	$\frac{\eta(\tau)^6}{\eta(5\tau)^6} = q^{-1} - 6 + 9q + 10q^2 + \dots$	$\frac{\eta(\tau)^2(5E_2(5\tau) - E_2(\tau))}{4\eta(5\tau)^{10}} = q^{-2} + 4q^{-1} + 5 + \dots$
$(6, 5)$	$\frac{\eta(\tau)^6}{\eta(5\tau)^6} = q^{-1} - 6 + 9q + 10q^2 + \dots$	$\frac{\eta(\tau)^2}{\eta(5\tau)^{10}} = q^{-2} - 2q^{-1} - 1 + 2q + \dots$
$(4, 6)$	$\frac{\eta(\tau)^5\eta(3\tau)}{\eta(2\tau)\eta(6\tau)^5} = q^{-1} - 5 + 6q + \dots$	$\frac{\eta(2\tau)^4\eta(3\tau)^6}{\eta(\tau)^2\eta(6\tau)^{12}} = q^{-2} + 2q^{-1} + 1 - 4q + \dots$
$(4, 7)$	$\frac{\eta(\tau)^4}{\eta(7\tau)^4} = q^{-1} - 4 + 2q + 8q^2 + \dots$	$\frac{\eta(\tau)^2}{10\eta(7\tau)^{14}}\left(\frac{E_4(\tau) - E_4(7\tau)}{240} - g\right) = q^{-2} + q^{-1} + \dots$
$(4, 8)$	$\frac{\eta(4\tau)^{12}}{\eta(2\tau)^4\eta(8\tau)^8} = q^{-1} + 4q + 2q^3 + \dots$	$\frac{\eta(4\tau)^4}{\eta(8\tau)^8} = q^{-2} - 4q^2 + 10q^6 - 24q^{10} + \dots$

6.4(2), the duality of these coefficients immediately implies that the functions ϕ_n are also uniquely defined in such cases.

Lemma 6.5. *Let (k, N) be a non-CM pair with $k \geq 4$, and let $\phi_n = q^{-n} + \sum_{m \geq -1} A_n(m) q^m \in M^\infty_{2-k}(\Gamma_0(N))$ and $F_m = -q^{-m} + \sum_{n \geq 2} C_m(n) q^n \in S^\infty_k(\Gamma_0(N))$ be as above. Then for all $n \geq 2$ and all $m \geq -1$, we have*

$$C_m(n) = A_n(m).$$

Proof. Zagier duality for the canonical bases of $S^\infty_k(\Gamma_0(N))$ and $M^\infty_{2-k}(\Gamma_0(N))$ has been established for all N of genus 0. Specifically, we refer to a collection of works by Jenkins and collaborators, such as [JT14], [HJ14],[IJW17], [JT17], and [JM19]. □

We are now equipped to prove our main theorem.

6.3 Proof of Theorem 6.3

Fix a non-CM pair (k, N) with $k \geq 4$, and let $p \nmid N$ be a prime such that $p \nmid a(p)$ and $v_p(C(p)) \leq k - 1$. We first show that (6.4) holds, from which (6.5) will quickly follow.

From Proposition 6.4(a), ϕ_p is of the form

$$\phi_p = q^{-p} + \sum_{n=-1}^{\infty} A_p(n) q^n.$$

Since k is even,

$$\begin{aligned}
\Theta^{k-1}(\phi_p) &= -p^{k-1} q^{-p} - A_p(-1) q^{-1} + A_p(1) q + \sum_{n=2}^{\infty} n^{k-1} A_p(n) q^n \\
&= -p^{k-1} q^{-p} - C_{-1}(p) q^{-1} + C_1(p) q + \sum_{n=2}^{\infty} n^{k-1} A_p(n) q^n \\
&= -p^{k-1} q^{-p} + a(p) q^{-1} + C(p) q + O(q^2),
\end{aligned}$$

by the Zagier duality established in Lemma 6.5. From Proposition 6.4(c), we also have

$$\begin{aligned} F \mid T_k(p) &= p^{k-1}F_p + C(p)g \\ &= -p^{k-1}q^{-p} + C(p)q + O(q^2). \end{aligned}$$

Hence, by Lemma 2.9,

$$F \mid T_k(p) - \Theta^{k-1}(\phi_p) = -a(p)q^{-1} + O(q^2) \in S_k^{\infty}(\Gamma_0(N)),$$

so by the uniqueness established in Proposition 6.4(2), this difference must be

$$a(p)F = a(p)F_1 = -a(p)q^{-1} + O(q^2).$$

Set $a \coloneqq a(p)$ for ease of notation and rearrange so that

$$F \mid U(p) = \Theta^{k-1}(\phi_p) + aF - p^{k-1}F \mid V(p).$$

Since $F \mid V(p) \mid U(p) = F$, we see

$$\begin{aligned} F \mid U(p^2) &= F \mid U(p) \mid U(p) \\ &= \Theta^{k-1}(\phi_p) \mid U(p) + aF \mid U(p) - p^{k-1}F \mid V(p) \mid U(p) \\ &= \Theta^{k-1}(\phi_p) \mid U(p) + a\left[\Theta^{k-1}(\phi_p) + aF - p^{k-1}F \mid V(p)\right] - p^{k-1}F \\ &= \Theta^{k-1}(\phi_p) \mid U(p) + a\Theta^{k-1}(\phi_p) + (a^2 - p^{k-1})F - ap^{k-1}F \mid V(p) \end{aligned}$$

Continuing to apply $U(p)$ this way, we write

$$F \mid U(p^m) = \left[\sum_{i=0}^{m-1} \alpha_{i,m}\Theta^{k-1}(\phi_p) \mid U(p^i)\right] + \beta_m F + \gamma_m F \mid V(p). \tag{6.7}$$

for coefficients $\alpha_{i,m}, \beta_m, \gamma_m$. We now determine $\alpha_{i,m}, \beta_m$, and γ_m, as (6.4) will follow by comparing the coefficients of q in (6.7). Observe

$$F \mid U(p^{m+1}) = F \mid U(p^m) \mid U(p) = \left[\sum_{i=0}^{m-1} \alpha_{i,m}\Theta^{k-1}(\phi_p) \mid U(p^{i+1})\right] + \beta_m F \mid U(p) + \gamma_m F$$
$$= \left[\sum_{i=1}^{m} \alpha_{i-1,m}\Theta^{k-1}(\phi_p) \mid U(p^{i})\right] + \beta_m \left(\Theta^{k-1}(\phi_p) + aF - p^{k-1}F \mid V(p)\right) + \gamma_m F$$
$$= \beta_m\Theta^{k-1}(\phi_p) + \left[\sum_{i=1}^{m} \alpha_{i-1,m}\Theta^{k-1}(\phi_p) \mid U(p^{i})\right] + (a\beta_m + \gamma_m)\, F - p^{k-1}\beta_m F \mid V(p).$$

As such, the coefficients $\alpha_{i,m}, \beta_m$, and γ_m satisfy the recurrences

$$\alpha_{i,m+1} = \begin{cases} \beta_m, & i = 0 \\ \alpha_{i-1,m}, & i \geq 1 \end{cases} \tag{6.8}$$

$$\beta_{m+1} = a\beta_m + \gamma_m \tag{6.9}$$

$$\gamma_{m+1} = -p^{k-1}\beta_m. \tag{6.10}$$

Notice that applying (6.8) inductively yields $\alpha_{i,m+1} = \beta_{m-i}$, and by substituting (6.10) into (6.9) we find

$$\beta_{m+1} = a\beta_m - p^{k-1}\beta_{m-1}.$$

This second-order recurrence can be solved to yield the explicit formula

$$\beta_m = \frac{(a + \sqrt{a^2 - 4p^{k-1}})^{m+1} - (a - \sqrt{a^2 - 4p^{k-1}})^{m+1}}{2^{m+1}\sqrt{a^2 - 4p^{k-1}}}.$$

From the Binomial Theorem,

$$\beta_m = \frac{1}{2^{m+1}\sqrt{a^2 - 4p^{k-1}}} \sum_{i=0}^{m+1} \binom{m+1}{i} a^{m+1-i} \left(\sqrt{a^2 - 4p^{k-1}}^{\,i} - (-1)^i \sqrt{a^2 - 4p^{k-1}}^{\,i}\right)$$
$$= \frac{1}{2^{m}\sqrt{a^2 - 4p^{k-1}}} \sum_{\substack{i=0 \\ i \text{ odd}}}^{m+1} \binom{m+1}{i} a^{m+1-i} \sqrt{a^2 - 4p^{k-1}}^{\,i}$$
$$= \frac{1}{2^m} \sum_{\substack{i=0 \\ i \text{ even}}}^{m} \binom{m+1}{i+1} a^{m-i} (a^2 - 4p^{k-1})^{i/2}$$

$$= \frac{1}{2^m} \sum_{i=0}^{\lfloor \frac{m}{2} \rfloor} \binom{m+1}{2i+1} a^{m-2i} (a^2 - 4p^{k-1})^i$$
$$= \frac{1}{2^m} \sum_{i=0}^{\lfloor \frac{m}{2} \rfloor} \binom{m+1}{2i+1} a^{m-2i} \sum_{j=0}^{i} \binom{i}{j} (a^2)^{i-j} (-4p^{k-1})^j$$
$$= \frac{1}{2^m} \sum_{j=0}^{\lfloor \frac{m}{2} \rfloor} \sum_{i=j}^{\lfloor \frac{m}{2} \rfloor} \binom{m+1}{2i+1} \binom{i}{j} (-4)^j a^{m-2j} p^{(k-1)j}.$$

As such, we may write

$$\beta_m = \sum_{j=0}^{\lfloor \frac{m}{2} \rfloor} c_{m,j} p^{(k-1)j}, \tag{6.11}$$

where

$$c_{m,j} := \frac{(-4)^j a^{m-2j}}{2^m} \sum_{i=j}^{\lfloor \frac{m}{2} \rfloor} \binom{m+1}{2i+1} \binom{i}{j}. \tag{6.12}$$

Now we compare the coefficients of q in (6.7) to obtain

$$C(p^m) = \sum_{i=0}^{m-1} \alpha_{i,m} p^{(k-1)i} A_p(p^i) = \sum_{i=0}^{m-1} \beta_{m-i-1} p^{(k-1)i} A_p(p^i) = \sum_{i=0}^{m-1} \beta_i p^{(k-1)(m-i-1)} A_p(p^{m-i-1})$$
$$= \sum_{i=0}^{m-1} \sum_{j=0}^{\lfloor \frac{i}{2} \rfloor} c_{i,j} p^{(k-1)(m+j-i-1)} A_p(p^{m-i-1}) \tag{6.13}$$

thanks to (6.11).

If $m = 1$, we have nothing to prove, so assume $m \geq 2$. Notice that $m + j - i - 1 \geq 1$ for all $0 \leq i \leq m-1$ and $0 \leq j \leq \lfloor \frac{i}{2} \rfloor$, except for when $i = m-1$ and $j = 0$. Hence, all terms in the double sum of (6.13) are divisible by at least p^{k-1}, other than $c_{m-1,0} A_p(1)$. But by (6.12)

$$c_{m-1,0} A_p(1) = \frac{a^{m-1}}{2^{m-1}} \sum_{i=0}^{\lfloor \frac{m-1}{2} \rfloor} \binom{m}{2i+1} A_p(1) = \frac{a^{m-1}}{2^{m-1}} \cdot 2^{m-1} A_p(1) = a^{m-1} C(p),$$

using Lemma 6.5. Now since $p \nmid a$ by assumption,

$$v_p(c_{m-1,0} A_p(1)) = v_p(a^{m-1} C(p)) = v_p(C(p)) \leq k - 1.$$

Therefore, $v_p(C(p^m)) = v_p(C(p))$, which completes the proof of (6.4).

To establish (6.5), note that from Proposition 6.4

$$\frac{F \mid T_k(p^m)}{C(p^m)} - g = \frac{p^{(k-1)m} F_{p^m}}{C(p^m)}.$$

Thus clearly

$$v_p\left(\frac{F \mid T_k(p^m)}{C(p^m)} - g\right) = v_p\left(\frac{p^{(k-1)m} F_{p^m}}{C(p^m)}\right) = (k-1)m - v_p(C(p^m)) = (k-1)m - v_p(C(p)),$$

thanks to (6.4). Note that equality holds since $p \nmid F_{p^m}$, as the coefficient of q^{p^m} in F_{p^m} is -1.

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Appendix

Appendix A

Initial relations for the proof of Theorem 3.9

$$
\begin{aligned}
U^{(0)}(1) &= \frac{1}{(1+5x)^5}(x + 40x^2 + 794x^3 + 9125x^4 + 64475x^5 + 286000x^6 \\
&\quad + 7800000x^7 + 1200000x^8 + 800000x^9), \\
U^{(0)}(x) &= \frac{1}{(1+5x)^5}(121x^2 + 9484x^3 + 321025x^4 + 6327850x^5 + 81874125x^6 + 738217500x^7 \\
&\quad + 4780850000x^8 + 22488800000x^9 + 76460000000x^{10} + 183600000000x^{11} \\
&\quad + 296000000000x^{12} + 288000000000x^{13} + 128000000000x^{14}) \\
U^{(0)}(x^2) &= \frac{1}{(1+5x)^5}(140x^2 + 35245x^3 + 2808365x^4 + 117376000x^5 + 3100037500x^6 \\
&\quad + 56831205625x^7 + 763507050000x^8 + 7771895500000x^9 + 61182640000000x^{10} \\
&\quad + 376797500000000x^{11} + 1823151200000000x^{12} + 6913681600000000x^{13} \\
&\quad + 20347776000000000x^{14} + 45594240000000000x^15 + 75238400000000000x^{16} \\
&\quad + 86272000000000000x^{17} + 61440000000000000x^{18} + 20480000000000000x^{19})
\end{aligned}
$$

$$U^{(0)}(x^3) = \frac{1}{(1+5x)^5}(64x^2 + 59136x^3 + 10547620x^4 + 850378650x^5 + 40530512250x^6$$
$$+ 1298590915000x^7 + 30103152240625x^8 + 528858099450000x^9$$
$$+ 7262462532500000x^{10} + 79624221710000000x^{11} + 707157357820000000x^{12}$$
$$+ 5136043622800000000x^{13} + 30674128864000000000x^{14}$$
$$+ 150938497280000000000x^{15} + 611049315200000000000x^{16}$$
$$+ 2024736448000000000000x^{17} + 5439398912000000000000x^{18}$$
$$+ 11668643840000000000000x^{19} + 19525324800000000000000x^{20}$$
$$+ 24567808000000000000000x^{21} + 21872640000000000000000x^{22}$$
$$+ 12288000000000000000000x^{23} + 3276800000000000000000x^{24})$$

$$U^{(0)}(x^4) = \frac{1}{(1+5x)^5}(13x^2 + 54342x^3 + 21645560x^4 + 3231134475x^5 + 261994052875x^6$$
$$+ 13648364390000x^7 + 501535624578125x^8 + 13781722427603125x^9$$
$$+ 294509461032250000x^{10} + 5030953041631500000x^{11} + 70082663702580000000x^{12}$$
$$+ 807847198383100000000x^{13} + 7788550590672000000000x^{14}$$
$$+ 63288334721120000000000x^{15} + 435690966505600000000000x^{16}$$
$$+ 2548759030153600000000000x^{17} + 12682694057728000000000000x^{18}$$
$$+ 53629451281920000000000000x^{19} + 192118376960000000000000000x^{20}$$
$$+ 579878492672000000000000000x^{21} + 1462436904960000000000000000x^{22}$$
$$+ 3044539924480000000000000000x^{23} + 5141902131200000000000000000x^{24}$$
$$+ 6869368832000000000000000000x^{25} + 6987448320000000000000000000x^{26}$$
$$+ 5085593600000000000000000000x^{27} + 2359296000000000000000000000x^{28}$$
$$+ 524288000000000000000000000x^{29})$$

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Vita

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