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SOME NONSTANDARD - OVERDETERMINED BOUNDARY VALUE PROBLEMS FOR THE BIHARMONIC OPERATOR

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ABSTRACT

In the early seventies, Serrin considered an overdetermined boundary value problem and determined that if a solution exists, then the domain must be a ball. Many authors have then extended this result to other overdetermined problems. In this work, we first examine an overdetermined problem considered by Bennett and then consider three fourth order boundary value problems for the differential equation $\Delta^2 u = f(r)$ in Ω. Two of the three auxiliary conditions are assumed to hold on the surface of a ball which is completely contained in Ω. Each auxiliary condition will be of a different type. We also show that Ω must be a ball. We then determine an integral representation of the solution for these problems.

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CHAPTER I

INTRODUCTION

In the early seventies, Serrin [9] considered a problem posed by Professor R.L. Fosdick concerning the flow of a viscous imcompressible fluid in straight parallel streamlines through a straight pipe of a given cross section. In general, the problem is formulated as follows. Let Ω be a bounded domain in Euclidean space R^N with a smooth boundary $\partial\Omega$. Suppose there exists a function $u = u(x) = u(x_1,\ldots x_N)$ satisfying the boundary value problem for the Poisson differential equation in Ω,

$$\Delta u = -1 \text{ in } \Omega, \quad u=0, \ \partial u/\partial n = c \text{ on } \partial\Omega, \tag{1.1}$$

where c is a constant, $\Delta = \partial^2/\partial x_1^2 + \ldots + \partial^2/\partial x_N^2$, and $\partial u/\partial n$ is the normal derivative. Must Ω be a ball? Serrin found that if u satisfies (1.1), then Ω is a ball with radius b and $u = (b^2 - r^2)/2N$. Serrin's reflection-in-moving planes argument can also be extended to more general elliptic equations and slightly more general boundary conditions. Weinberger [11] presented an elementary proof of Serrin's basic result by using a Rellich identity and a maximum principle.

Using Serrin's moving planes method, Alessandrini [1] was able to prove a symmetry theorem for condensers in a capacity problem. Troy [10] also adapted this method to a system of semilinear elliptic equations. Serrin's technique has also been generalized to ring domains by Willms, Gladwell, and Siegel [12].

Some years later, Bennett [3] extended Weinberger's method to the following fourth order boundary value problem.

$$\Delta(\Delta u) = -1 \ in \ \Omega, \qquad u = \partial u/\partial n = 0, \ \Delta u = c \ on \ \partial\Omega, \tag{1.2}$$

where c is a constant. His result states that Ω must be an open ball and u must be radially symmetric about the center of Ω. Weinberger's method was also extended to more general second order partial differential equations by Garofalo and Lewis in [4].

In [7], Payne and Schaefer introduced an alternate method for some overdetermined problems that depended on the establishment of an integral identity which was equivalent to the overdetermined problem. They were able to apply their method to some second order, fourth order, and $2m^{th}$ order elliptic overdetermined boundary value problems as well as to some problems involving the Green's function for the Laplace and biharmonic operators. In further research [8], Payne and Schaefer considered several overdetermined boundary value problems for the equation $\Delta^2 u = 1$ with one or more auxiliary condition specified on an interior surface. A particular solution of an associated problem, namely the torsion function, was used to obtain the result in these cases. The torsion function Ψ is the solution to the Dirichlet problem

$$\Delta\Psi = -1 \text{ in } \Omega, \quad \Psi = 0 \text{ on } \partial\Omega.$$

Some other overdetermined problems were studied in [5], where Payne considered various classes of problems in classical elasticity and determined the geometry of the underlying domain. Also, Payne and Philippin [6] obtained spherical symmetry results for an overdetermined Stekloff eigenvalue problem and Alessandrini and Magnanini [2] answered a conjecture raised in [6] about the geometry of the domain in R^2 when a nonstandard condition was imposed.

In the present work, we will examine Bennett's argument in Chapter II since his problem constituted a starting point for this thesis. We will then consider three fourth order overdetermined boundary value problems in Chapter III for the more general differential equation $\Delta^2 u = f(r)$ in Ω, where r is the distance from a fixed point in Ω. We specify three auxiliary conditions for our differential equation. Two of these conditions are assumed to hold on the surface of a ball which is completely contained in Ω. Each auxiliary condition will be of a different type, i.e., we will specify the values of u, $\partial u/\partial n$, and Δu. Our results state that Ω must be a ball. For these problems, we also determine an integral representation of the solution whereas in the special case when $f(r) = 1$, we exhibit an explicit answer.

CHAPTER II

A BASIC OVERDETERMINED PROBLEM

Since our work is somewhat related to problem (1.2), Bennett's paper will be examined in this chapter. We use the comma and repeated index conventions, i.e., $u,_i = \partial u / \partial x_i$ and repeated indices in a term denotes summation from 1 to N. Also we let dx denote the element of volume and dS represent the element of surface area.

Bennett's paper can be divided into three lemmas and a theorem.

Lemma 1: If u depends only on the distance r from a fixed point and solves $\Delta^2 u = -1$, then

$$\Delta^2 (r \frac{\partial u}{\partial r}) = -4.$$

Lemma 2: If u solves (1.2), then $\int_\Omega u \, dx = \frac{-N}{N+4} c^2 |\Omega|$, where $|\Omega|$ denotes the volume of Ω.

Lemma 3: If u solves (1.2), then

$$\Phi = \frac{N-4}{N+2} u + \frac{N-4}{2(N+2)} (\Delta u)^2 + \sum_{i,j=1}^{N} (u,_{ij})^2 - u,_i (\Delta u),_i = const.$$

Theorem: If Ω is a bounded domain in R^N with $\partial\Omega \in C^{4+\varepsilon}$ and $u \in C^4(\overline{\Omega})$ satisfies (1.2), then Ω is a ball of radius $\left[N(N+2)|c|\right]^{1/2}$ and $u = \frac{-1}{2N}\left\{\frac{1}{4}(N+2)(Nc)^2 + \frac{1}{2}Ncr^2 + \frac{1}{4(N+2)} r^4\right\}$.

We assume u depends only on the distance r from a fixed point in Ω. Some equations we will need in the following pages are

$$\Delta u = u_{rr} + \frac{N-1}{r} u_r \tag{2.1}$$

$$\Delta(\Delta u) = \Delta^2 u = \frac{\partial^2 \Delta u}{\partial r^2} + \frac{N-1}{r} \frac{\partial \Delta u}{\partial r} \tag{2.2}$$

$$\frac{\partial (r \frac{\partial}{\partial r}(\Delta u))}{\partial r} = \frac{\partial}{\partial r}(\Delta u) + r \frac{\partial^2 (\Delta u)}{\partial r^2} \tag{2.3}$$

$$\frac{\partial^2 (r \frac{\partial}{\partial r}(\Delta u))}{\partial r^2} = 2 \frac{\partial^2 (\Delta u)}{\partial r^2} + r \frac{\partial^3 (\Delta u)}{\partial r^3} \tag{2.4}$$

We begin by proving Lemma 1. From Weinberger [11], we have

$$\Delta(r\frac{\partial u}{\partial r}) = \left[r\frac{\partial}{\partial r}(\Delta u) + 2\Delta u \right]. \tag{2.5}$$

We take the Laplacian of (2.5) and use (2.1) to obtain

$$\Delta^2 (r\frac{\partial u}{\partial r}) = \Delta\left[r\frac{\partial}{\partial r}(\Delta u) + 2\Delta u \right]$$

$$= \frac{\partial^2 (r\frac{\partial}{\partial r}(\Delta u))}{\partial r^2} + \frac{(N-1)}{r}\frac{\partial(r\frac{\partial}{\partial r}(\Delta u))}{\partial r} + 2\frac{\partial^2 (\Delta u)}{\partial r^2} + \frac{2(N-1)}{r}\frac{\partial(\Delta u)}{\partial r}.$$

Using (2.3), and (2.4) and simplifying the above equation, we write

$$\Delta^2 (r\frac{\partial u}{\partial r}) = 4\frac{\partial^2 (\Delta u)}{\partial r^2} + \frac{3(N-1)}{r}\frac{\partial(\Delta u)}{\partial r} + r\frac{\partial}{\partial r}(\frac{\partial^2 (\Delta u)}{\partial r^2}) + (N-1)\frac{\partial^2 (\Delta u)}{\partial r^2} \tag{2.6}$$

We now use

$$\frac{1}{r}\frac{\partial^2 \Delta u}{\partial r^2} = \frac{\partial}{\partial r}(\frac{1}{r}\frac{\partial \Delta u}{\partial r}) + \frac{1}{r^2}\frac{\partial \Delta u}{\partial r}$$

in (2.6) and combine terms to find that

$$\begin{aligned}\Delta^2 (r\frac{\partial u}{\partial r}) &= 4(\frac{\partial^2 (\Delta u)}{\partial r^2} + \frac{(N-1)}{r}\frac{\partial}{\partial r}(\Delta u)) + r\frac{\partial}{\partial r}(\frac{\partial^2 \Delta u}{\partial r^2} + \frac{(N-1)}{r}\frac{\partial \Delta u}{\partial r}) \\ &= 4[\Delta(\Delta u)] + r\frac{\partial}{\partial r}[\Delta(\Delta u)]\end{aligned} \tag{2.7}$$

via (2.2). Lastly, using our hypothesis, u solves $\Delta^2 u = -1$, we get $\Delta^2 (r\frac{\partial u}{\partial r}) = -4$.

We now justify Lemma 2. From Lemma 1 and $\Delta^2 u = -1$, we obtain

$$\int_\Omega (4u - r\frac{\partial u}{\partial r}dx) = \int_\Omega [\, -u\Delta^2 (r\frac{\partial u}{\partial r}) + r\frac{\partial u}{\partial r}\Delta^2 u\,]dx.$$

We use Green's identity for Δ^2, namely,

$$\int_\Omega (u\Delta^2 v - v\Delta^2 u)dx = \int_{\partial\Omega} [\, u\frac{\partial \Delta v}{\partial n} - \Delta v\frac{\partial u}{\partial n} + \Delta u\frac{\partial v}{\partial n} - v\frac{\partial \Delta u}{\partial n}\,]dS$$

replacing u by -u and v by $r\,\partial u/\partial r$. We obtain

$$\int_\Omega \left[-u\Delta^2 (r\frac{\partial u}{\partial r}) + r\frac{\partial u}{\partial r}\Delta^2 u \right]dx$$

$$= \int_{\partial\Omega} \left[-u\frac{\partial}{\partial n}(\Delta(r\frac{\partial u}{\partial r})) + \Delta(r\frac{\partial u}{\partial r})\frac{\partial u}{\partial n} - \Delta u\frac{\partial}{\partial n}(r\frac{\partial u}{\partial r}) + r\frac{\partial u}{\partial r}\frac{\partial \Delta u}{\partial n} \right]dS \tag{2.8}$$

On the boundary of Ω, u=0 and $\partial u/\partial n = 0$. Therefore $u_{x_i} = 0$ on $\partial\Omega$ so that $\partial u/\partial r = 0$ on $\partial\Omega$, and (2.8)

becomes

$$\int_\Omega \left[4u - r\frac{\partial u}{\partial r} \right] dx = -\int_{\partial\Omega} \Delta u \frac{\partial}{\partial n}\left(r\frac{\partial u}{\partial r}\right) dS.$$

Since $\Delta u = c$ and $\frac{\partial u}{\partial r} = \frac{\partial r}{\partial n}\frac{\partial u}{\partial n}$ on $\partial\Omega$, we write

$$\int_\Omega \left[4u - r\frac{\partial u}{\partial r} \right] dx = -c\int_{\partial\Omega} \frac{\partial}{\partial n}\left(r\frac{\partial r}{\partial n}\frac{\partial u}{\partial n}\right) dS. \tag{2.9}$$

Upon differentiating on the right side of (2.9), we obtain

$$\int_\Omega \left[4u - r\frac{\partial u}{\partial r} \right] dx = -c\int_{\partial\Omega} \left[r\frac{\partial r}{\partial n}\frac{\partial^2 u}{\partial n^2} + \frac{\partial u}{\partial n}\frac{\partial}{\partial n}\left(r\frac{\partial r}{\partial n}\right) \right] dS.$$

We use the boundary condition $\partial u/\partial n = 0$ on $\partial\Omega$ and that $\Delta u = \frac{\partial^2 u}{\partial n^2} + \mathrm{K}(N-1)\frac{\partial u}{\partial n} + \frac{\partial^2 u}{\partial s^2} = \frac{\partial^2 u}{\partial n^2}$ on

the boundary, where K is the mean curvature, to show that

$$\int_\Omega \left[4u - r\frac{\partial u}{\partial r} \right] dx = -c\int_{\partial\Omega} r\frac{\partial r}{\partial n}(\Delta u) dS. \tag{2.10}$$

By using $\Delta u = c$ on $\partial\Omega$ and $r\frac{\partial r}{\partial n} = \frac{\partial}{\partial n}\left(\frac{1}{2}r^2\right) = \nabla(\frac{1}{2}r^2)\cdot\vec{n}$, we get

$$-c\int_{\partial\Omega} r\frac{\partial r}{\partial n}(\Delta u) dS = -c^2\int_{\partial\Omega} \nabla(\tfrac{1}{2}r^2)\cdot\vec{n}\, dS.$$

We apply the divergence theorem to the preceding equation and write

$$-c^2\int_{\partial\Omega} \nabla(\tfrac{1}{2}r^2)\cdot\vec{n} = -c^2\int_\Omega div(x) dx = -c^2 N|\Omega|.$$

Therefore, (2.10) becomes

$$\int_\Omega \left[4u - r\frac{\partial u}{\partial r} \right] dx = -c^2 N|\Omega|. \tag{2.11}$$

We now look at $\int_\Omega r\frac{\partial u}{\partial r} dx$ and find that

$$\int_\Omega r\frac{\partial u}{\partial r} dx = \int_\Omega r\nabla u\cdot\vec{r}\, dx = \int_\Omega r\nabla u\cdot\frac{x}{r} dx = \int_\Omega \nabla u\cdot r\frac{x}{r} dx = \int_\Omega \nabla u\cdot\nabla(\tfrac{1}{2}r^2) dx. \tag{2.12}$$

Using the Green identity,

$$\int_\Omega (v\Delta u + \nabla v\nabla u) dx = \int_{\partial\Omega} v\frac{\partial u}{\partial n} dS,$$

(2.12) becomes

$$\int_\Omega r\frac{\partial u}{\partial r}dx = \int_\Omega \nabla u\cdot\nabla(\tfrac{1}{2}r^2)dx = -\int_\Omega u\Delta(\tfrac{1}{2}r^2)dx. \tag{2.13}$$

We note that u=0 on $\partial\Omega$ allowed us to drop $\int_{\partial\Omega} u\frac{\partial}{\partial n}(\tfrac{1}{2}r^2)dS$. After taking the Laplacian of $r^2/2$, we obtain

$$-\int_\Omega u\Delta(\tfrac{1}{2}r^2)dx = -N\int_\Omega u\,dx. \tag{2.14}$$

Therefore, using (2.13) and (2.14), we have

$$\int_\Omega r\frac{\partial u}{\partial r}dx = -N\int_\Omega u\,dx. \tag{2.15}$$

We substitute (2.15) into (2.11) to find

$$\int_\Omega (4u+Nu)dx = -c^2N|\Omega|.$$

and upon rearranging terms have

$$\int_\Omega u\,dx = \frac{-N}{N+4}(c^2|\Omega|). \tag{2.16}$$

Thus Lemma 2 has been proved.

We now proceed to prove Lemma 3. It suffices to show that $\Delta\Phi \geq 0$ in Ω. If $\Delta\Phi \geq 0$ in Ω, then Φ takes its maximum on $\partial\Omega$ and either Φ = constant on $\overline{\Omega}$ or $\Phi <$ constant in Ω. We define Φ as

$$\Phi = \frac{N-4}{N+2}u + \frac{N-4}{2(N+2)}(\Delta u)^2 + \sum_{i,j=1}^{N} u,_{ij}^2 - u,_i(\Delta u),_i\,. \tag{2.17}$$

The Laplacian of Φ is

$$\Delta\Phi = \frac{N-4}{N+2}\Delta u + \frac{N-4}{2(N+2)}2\Delta u\Delta^2 u + \frac{N-4}{2(N+2)}2|\nabla(\Delta u)|^2 \tag{2.18}$$
$$-|\nabla(\Delta u)|^2 - \nabla(\Delta^2 u)\cdot\nabla u + 2u,_{ijk}\,u,_{ijk}.$$

By hypothesis of Lemma 3, we note that $\nabla(\Delta^2 u) = 0$. The first and the second terms cancel and the second to the last term is zero. We combine terms and (2.18) becomes

$$\Delta\Phi = 2u,_{ijk}\,u,_{ijk} - \frac{6}{N+2}|\nabla(\Delta u)|^2. \tag{2.19}$$

To show that $\Delta\Phi$ is nonnegative, we note that for any real number γ

$$\sum_{i,j,k}\left[u,_{ijk} - \gamma\{(\Delta u),_i\,\delta_{jk} + (\Delta u),_j\,\delta_{ik} + (\Delta u),_k\,\delta_{ij}\}\right]^2 \geq 0, \tag{2.20}$$

where δ_{jk} is the Kronecker delta. This implies

$$\sum_{i,j,k}\Big[\, u_{,ijk}^{2} - 2u_{,ijk}\, \gamma\big\{(\Delta u)_{,i}\, \delta_{jk} + (\Delta u)_{,j}\, \delta_{ik} + (\Delta u)_{,k}\, \delta_{ij}\big\}$$
$$+\gamma^2\big\{(\Delta u)_{,i}^{2}\,(\delta_{jk})^2 + (\Delta u)_{,j}^{2}\,(\delta_{ik})^2 + (\Delta u)_{,k}^{2}\,(\delta_{ij})^2 \tag{2.21}$$
$$+2(\Delta u)_{,i}\,(\Delta u)_{,k}\,\delta_{jk}\delta_{ij} + 2(\Delta u)_{,j}\,(\Delta u)_{,k}\,\delta_{ik}\delta_{ij} + 2(\Delta u)_{,i}\,(\Delta u)_{,j}\,\delta_{jk}\delta_{ik}\big\}\Big] \geq 0.$$

We now note that

$$\sum_{i,j,k}^{N} 2u_{,ijk}\, \gamma(\Delta u)_{,i}\, \delta_{jk} = 2u_{,ijj}\, \gamma(\Delta u)_{,i} = 2\gamma\left|\nabla(\Delta u)\right|^2. \tag{2.22}$$

The next two terms are handled in a similar manner. Next we have

$$\sum_{i,j,k}^{N}(\Delta u)_{,i}^{2}\,(\delta_{jk})^2 = \sum_{i}^{N}(\Delta u)_{,i}^{2}\, N = N\sum_{i}^{N}(\Delta u)_{,i}^{2} \tag{2.23}$$

since when j = k, $\delta_{jk} = 1$. The next two terms are handled in a similar manner. We also have

$$\sum_{i,j,k}^{N} 2(\Delta u)_{,j}\,(\Delta u)_{,k}\,\delta_{ik}\delta_{ij} = \sum_{i=1}^{N} 2(\Delta u)_{,i}^{2} \tag{2.24}$$

where only the terms i = j = k remain. Once again the next two terms will be handled similarly.

Using (2.22), (2.23) and (2.24) in (2.21), we have

$$u_{,ijk}\, u_{,ijk} - 6\gamma\left|\nabla(\Delta u)\right|^2 + 3\gamma^2(N+2)\left|\nabla(\Delta u)\right|^2 \geq 0.$$

Therefore the discriminant of this quadratic expression in γ must satisfy

$$36\left|\nabla(\Delta u)\right|^4 - 12(N+2)\left|\nabla(\Delta u)\right|^2 u_{,ijk}\, u_{,ijk} \leq 0. \tag{2.25}$$

By the maximum principle on Δu and the continuity of $\Delta\Phi$ so that $\left|\nabla(\Delta u)\right| \neq 0$, we find (2.25) gives us the negative of $\Delta\Phi$ after dividing by $6(N+2)\left|\nabla(\Delta u)\right|^2$. Therefore $\Delta\Phi \geq 0$ and Φ assumes its maximum value on $\partial\Omega$ since it is a subharmonic function.

We now find the value of Φ on $\partial\Omega$ using the boundary conditions u = 0 and $\Delta u = c$ on $\partial\Omega$. We note that $u_{,ij}\, u_{,ij}$ on $\partial\Omega$ becomes

$$u_{,ij}\, u_{,ij} = \frac{\partial^2 u}{\partial n^2} n^i n^j \frac{\partial^2 u}{\partial n^2} n^i n^j = \left(\frac{\partial^2 u}{\partial n^2}\right)^2 = (\Delta u)^2 = c^2 \tag{2.26}$$

since $\partial^2 u/\partial n^2 = \Delta u$ on $\partial\Omega$. We also have $\nabla u = 0$ on $\partial\Omega$ because $u = 0 = \partial u/\partial n$ on $\partial\Omega$. Substituting into (2.17), we obtain

$$\Phi = \frac{3Nc^2}{2(N+2)} \text{ on } \partial\Omega.$$

Since Φ attains its maximum on the boundary, we know by the maximum principle that

$$\Phi \le \frac{3Nc^2}{2(N+2)} \text{ in } \Omega. \tag{2.27}$$

Next, we show that

$$\int_\Omega \Phi\, dx = \frac{-3(N+4)}{2(N+2)} \int_\Omega u\, dx = \frac{3Nc^2}{2(N+2)}|\Omega|, \tag{2.28}$$

which implies that Φ is constant in $\overline{\Omega}$. Integrating Φ over Ω, we write

$$\int_\Omega \Phi\, dx = \frac{N-4}{N+2}\int_\Omega u\, dx + \frac{N-4}{2(N+2)}\int_\Omega (\Delta u)^2\, dx + \int_\Omega u_{,ij}\, u_{,ij}\, dx - \int_\Omega \nabla u \cdot \nabla(\Delta u) dx. \tag{2.29}$$

Using Green's first identity with v replaced by Δu, we have

$$\int_\Omega \left[(\Delta u)^2 - u\Delta^2 u\right] dx = \int_{\partial\Omega} \left[\Delta u \frac{\partial u}{\partial n} - u \frac{\partial}{\partial n}(\Delta u)\right] dS.$$

Using the boundary condition and $\Delta^2 u = -1$ in Ω, we obtain

$$\int_\Omega (\Delta u)^2\, dx = -\int_\Omega u\, dx. \tag{2.30}$$

We now find an equivalent way to write $\int_\Omega u_{,ij}\, u_{,ij}\, dx$:

$$\begin{aligned}\int_\Omega u_{,ij}\, u_{,ij}\, dx &= \int_\Omega \left[(u_{,ij}\, u_{,i})_{,j} - u_{,ijj}\, u_{,i}\right] dx \\ &= \int_{\partial\Omega} u_{,ij}\, u_{,i}\, n_j dS - \int_\Omega u_{,ijj}\, u_{,i}\, dx = \int_{\partial\Omega} u_{,ij} \frac{\partial u}{\partial n} \cdot n_i n_j dS - \int_\Omega u_{,ijj} \cdot u_{,i}\, dx \\ &= -\int_\Omega \nabla(\Delta u) \cdot \nabla u\, dx\end{aligned} \tag{2.31}$$

by using the divergence theorem and $\partial u / \partial n = 0$ on $\partial\Omega$. We now substitute (2.30) and (2.31) into (2.29) and obtain

$$\int_\Omega \Phi\, dx = \frac{N-4}{N+2}\int_\Omega u\, dx - \frac{N-4}{2(N+2)}\int_\Omega u\, dx - 2\int_\Omega \nabla u \cdot \nabla(\Delta u) dx.$$

The last term is simplified using Green's first identity, that is $\int_\Omega \nabla u \cdot \nabla(\Delta u) dx = \int_\Omega u\, dx$. We substitute and simplify to find

$$\int_\Omega \Phi\, dx = \frac{-3(N+4)}{2(N+2)}\int_\Omega u\, dx. \tag{2.32}$$

After substituting the result of Lemma 2 into (2.32), we see (2.28) has been verified. Therefore it follows that $\Phi = \dfrac{3Nc^2}{2(N+2)}$ in $\overline{\Omega}$ is a constant and Lemma 3 is proved.

We now proceed to prove the theorem. Since Φ is a constant in $\overline{\Omega}$, this implies that $\Delta\Phi$ vanishes identically in $\overline{\Omega}$ and equality holds in (2.20) when $\gamma = \dfrac{1}{(N+2)}$, where γ is obtained using the quadratic formula. When we substitute in for γ, lift off the square term, and differentiate with respect to x_k, we can write

$$\sum_{i,j,k}\left[u,_{ijkk} - \frac{1}{N+2}\left\{(\Delta u),_{ik}\,\delta_{jk} + (\Delta u),_{jk}\,\delta_{ik} + (\Delta u),_{kk}\,\delta_{ij}\right\}\right] = 0.$$

Summing over k and rearranging terms, we obtain after simplification

$$u,_{ijkk} = (\Delta u),_{ij} = \frac{1}{N+2}\left[2(\Delta u),_{ij} - \delta_{ij}\right].$$

We then solve for $(\Delta u),_{ij}$ and have

$$(\Delta u),_{ij} = -\frac{1}{N}\delta_{ij}.$$

Therefore if $i \neq j$, $(\Delta u),_{ij}$ equals zero, and Δu will not have any products of different independent variables. It will also be quadratic in each x_i since $(\Delta u),_{ij} = \frac{-1}{N}$ when $i = j$. This implies Δu is of the form

$$\Delta u = \frac{-1}{2N}(x_1^2 + \ldots + x_n^2 + A_1 x_1 + \ldots + A_n x_n + B),$$

and upon completing the square, we have $\Delta u = \dfrac{1}{2N}(A - |x-a|^2)$ in $\overline{\Omega}$, where $A/2N$ is the maximum value of Δu in $\overline{\Omega}$.

We determine Ω is a ball by using the boundary condition $\Delta u = c$ on $\partial\Omega$. On $\partial\Omega$, we have

$$\Delta u = \frac{1}{2N}(A - |x-a|^2) = c. \tag{2.33}$$

Solving for $|x-a|$, we obtain

$$|x-a| = (A - 2cN)^{1/2}.$$

Thus Ω is a ball and $(A-2cN)^{1/2}$ is the radius of Ω. The center is at a.

We now proceed to find the solution. Using (2.1) for the $N > 2$ case, (2.33) becomes

$$\Delta u = u_{rr} + \frac{N-1}{r}u_r = \frac{1}{2N}(A - r^2)$$

where $r = |x - a|$. We solve the equation using ordinary differential equation methods. Letting $u_r = v$, it follows that $u_{rr} = v_r$, and our equation becomes

$$v_r + \frac{N-1}{r}v = \frac{1}{2N}(A - r^2). \tag{2.34}$$

The solution to this first order equation in v is

$$v = \frac{1}{2N}r^{(-N+1)}(\frac{A}{N}r^N - \frac{1}{N+2}r^{(N+2)}) + Cr^{(-N+1)}.$$

Since $u_r = v$, we integrate again and obtain

$$u = \frac{1}{2N}(\frac{A}{2N}r^2 - \frac{1}{4(N+2)}r^4) + C\frac{1}{-N+2}r^{(-N+2)} + D. \tag{2.35}$$

Since r = 0 is a singular point of the ODE, we impose the implicit boundary condition u (r) finite as $r \to 0^+$. Therefore, C = 0.

We now use the B.C. $\partial u/\partial n = 0$ on $\partial\Omega$. Since Ω is a ball, $\partial u/\partial n = \partial u/\partial r$ and

$$\frac{\partial u}{\partial n} = \frac{1}{2N}(\frac{A}{N}r - \frac{1}{N+2}r^3).$$

Since $\partial u/\partial n = 0$ on $\partial\Omega$ and $r = (A - 2cN)^{1/2}$ on $\partial\Omega$, we obtain

$$\frac{\partial u}{\partial n} = \frac{1}{2N}(\frac{A}{2N}(A - 2cN)^{1/2} - \frac{1}{(N+2)}(A-2cN)^{1/2}(A - 2cN)) = 0.$$

After multiplying the equation by $2N(A - 2cN)^{-1/2}$, we simplify and and find

$$A = -cN^2. \tag{2.36}$$

Therefore, the maximum value of Δu on Ω is $A/2N = -cN/2$. We now use u = 0 and $r = (A - 2cN)^{1/2}$ on $\partial\Omega$ along with $A = -cN^2$ to obtain

$$\frac{1}{2N}(\frac{-cN^2}{2N}(-cN^2 - 2cN) - \frac{1}{4(N+2)}(-cN^2 - 2cN)^2 + D = 0.$$

Upon simplification, we find

$$D = \frac{1}{2N}(\frac{(N+2)(cN)^2}{4}). \tag{2.37}$$

We substitute (2.36) and (2.37) into (2.35) and write

$$u(x) = \frac{-1}{2N}\left\{\frac{1}{4}(N+2)(Nc)^2 + \frac{Nc}{2}r^2 + \frac{1}{4(N+2)}r^4\right\}$$

where r denotes the distance from x to the center of Ω. This completes the proof of the theorem for N > 2.

The case N=2 is handled in a simpler manner and results in

$$u(x) = \frac{-1}{4}\left\{(2c)^2 + cr^2 + \frac{1}{8}r^4\right\}.$$

CHAPTER III

THREE PARTICULAR OVERDETERMINED PROBLEMS

In this chapter we consider three overdetermined boundary value problems with two of the auxiliary conditions specified on the surface of an interior ball. The method used in finding a solution to our problems is based on the method used by Payne and Schaefer in [8] except that now we can not make use of the torsion function since we have a more general partial differential equation. We first solve an associated problem an interior domain and then analytically continue this particular solution to the remainder of Ω.

Let Ω be a bounded domain in Euclidean space R^N with a sufficiently smooth boundary ∂Ω and r denote the distance from a fixed point in Ω which we take to be the origin so $r = |x| = \sqrt{x_1^2 + \ldots + x_N^2}$. Since we assume the right hand side of our equation only depends on r, we assume that the solution u only depends on r.

1. PROBLEM ONE

Our first nonstandard-overdetermined boundary value problem (BVP) is

$$\Delta^2 u = f(r) \text{ in } \Omega,$$
$$u = c_1,\ \partial u / \partial n = c_2, \text{ on } \partial B_a(0) \subset \Omega, \tag{3.1}$$
$$\Delta u = c_3 \text{ on } \partial\Omega,$$

where f (r) is a continuous function of r > 0 that is C^0 and c_1 ,c_2 , and c_3 are constants. The set $B_a(0)$ is a ball of radius a and center 0 contained in Ω. We show that Ω must be a ball. We first consider (3.1) for N = 2 dimensions and then consider the N > 2 case.

For N = 2, the Laplacian can be written

$$\Delta u = \frac{1}{r}\frac{\partial}{\partial r}(r\frac{\partial u}{\partial r}) = u_{rr} + \frac{1}{r}u_r. \tag{3.2}$$

It then follows that

$$\Delta^2 u = r^{-1}\frac{\partial}{\partial r}(r\frac{\partial \Delta u}{\partial r}). \tag{3.3}$$

Hence we consider the equation

$$r^{-1}\frac{\partial}{\partial r}(r\frac{\partial \Delta u}{\partial r}) = f(r), \tag{3.4}$$

multiply by r, and integrate to obtain

$$r\frac{\partial \Delta u}{\partial r} = \int_0^r \rho f(\rho)\,d\rho + k_1,$$

where k_1 is an arbitrary constant. We now multiply by r^{-1} and integrate again so that

$$\Delta u = \int_0^r \sigma^{-1}\int_0^\sigma \rho f(\rho)\,d\rho d\sigma + k_1 \ln r + k_2,$$

for k_2, an arbitrary constant. Since ln r is singular at r = 0, we take $k_1 = 0$. By an interchange of the order of integration, we have

$$\begin{aligned}\Delta u &= \int_0^r\int_\rho^r \sigma^{-1}\rho f(\rho)\,d\sigma d\rho + k_2,\\ &= \int_0^r (\ln r - \ln \rho)\rho f(\rho)\,d\rho + k_2\\ &= \int_0^r F(r,\rho)d\rho + k_2,\end{aligned} \tag{3.5}$$

where we let $F(r,\rho) = (\ln r - \ln \rho)\rho f(\rho)$.

By (3.2) and (3.5), we can write

$$r^{-1}\frac{\partial}{\partial r}(r\frac{\partial u}{\partial r}) = \int_0^r F(r,\rho)d\rho + k_2.$$

Using the same multiplication and integration technique, we obtain

$$r\frac{\partial u}{\partial r} = \int_0^r \sigma\int_0^\sigma F(\sigma,\rho)d\rho d\sigma + \frac{k_2 r^2}{2} + k_3$$

for k_3, an arbitrary constant, and

$$\frac{\partial u}{\partial r} = \int_0^r \frac{\sigma}{r}\int_0^\sigma F(\sigma,\rho)d\rho d\sigma + \frac{k_2 r}{2} + k_3 r^{-1}. \tag{3.6}$$

Since r^{-1} is singular at r = 0, we let $k_3 = 0$. We also note that the first term in (3.6) is not singular because by the L'Hopital's rule the limit is zero as $r \to 0^+$.

We now use the boundary condition (B.C.) $\partial u/\partial n = c_2$ on ∂B_a. Since on ∂B_a, $\partial u/\partial n = \partial u/\partial r$, we write (3.6) as

$$\int_0^a \frac{\sigma}{a}\int_0^\sigma F(\sigma,\rho)d\rho d\sigma + k_2\frac{a}{2} = c_2$$

and thus

$$k_2 = \frac{2c_2}{a} - \int_0^a \frac{2\sigma}{a^2}\int_0^\sigma F(\sigma,\rho)\,d\rho\,d\sigma. \tag{3.7}$$

Consequently,

$$\frac{\partial u}{\partial r} = \int_0^r \frac{\sigma}{r}\int_0^\sigma F(\sigma,\rho)\,d\rho\,d\sigma + \frac{k_2 r}{2} \tag{3.8}$$

where k_2 is defined in (3.7).

We integrate (3.8) with respect to r and obtain

$$u = \int_0^r\int_0^s \frac{\sigma}{s}\int_0^\sigma F(\sigma,\rho)\,d\rho\,d\sigma\,ds + \frac{k_2 r^2}{4} + k_4.$$

Interchanging the order of integration, we have

$$\begin{aligned} u &= \int_0^r\int_\sigma^r \frac{\sigma}{s}\int_0^\sigma F(\sigma,\rho)\,d\rho\,ds\,d\sigma + \frac{k_2 r^2}{4} + k_4 \\ &= \int_0^r (\ln r - \ln\sigma)\sigma\int_0^\sigma F(\sigma,\rho)\,d\rho\,d\sigma + \frac{k_2 r^2}{4} + k_4. \end{aligned}$$

We interchange the dummy variables σ and ρ and write

$$u = \int_0^r (\ln r - \ln\rho)\rho\int_0^\rho F(\rho,\sigma)\,d\sigma\,d\rho + \frac{k_2 r^2}{4} + k_4. \tag{3.9}$$

By the B.C., $u = c_1$ on ∂B_a, we find k_4 to be

$$k_4 = c_1 - \int_0^a (\ln a - \ln\rho)\rho\int_0^\rho F(\rho,\sigma)\,d\sigma\,d\rho + \frac{k_2 a^2}{4}$$

where we recall that k_2 is determined by (3.7). With this k_4 in (3.9), we have

$$\begin{aligned} u = &\int_0^r (\ln r - \ln\rho)\rho\int_0^\rho F(\rho,\sigma)\,d\sigma\,d\rho - \int_0^a (\ln a - \ln\rho)\rho\int_0^\rho F(\rho,\sigma)\,d\sigma\,d\rho \\ &- \frac{r^2 - a^2}{2}\int_0^a \frac{\rho}{a^2}\int_0^\rho F(\rho,\sigma)\,d\sigma\,d\rho + \frac{c_2(r^2 - a^2)}{2a} + c_1. \end{aligned} \tag{3.10}$$

The function u in (3.10) is a unique solution to the problem (via Green identity arguement)

$$\Delta^2 u = f(r) \text{ in } B_a, \quad u = c_1,\ \partial u/\partial n = c_2, \text{ on } \partial B_a(0) \subset \Omega. \tag{3.11}$$

We continue u to the rest of Ω and ask that it satisfy the condition $\Delta u = c_3$ on $\partial\Omega$. From (3.1) and (3.5), we obtain for points on $\partial\Omega$ where $r > a$,

$$\Delta u = \int_0^r F(r,\sigma)\,d\sigma + k_2 = c_3. \tag{3.12}$$

The resulting transcendental equation (3.12) implies that r is a constant and hence that $\partial\Omega$ is a circle.

We can analyze the equation (3.12) to clearly demonstrate conditions under which there is a root $r > a$. We let

$$G(r) := g(r) + k_2 - c_3, \tag{3.13}$$

and seek a solution of $G(r) = 0$, where $g(r) = \int_0^r (\ln r - \ln\sigma)\sigma f(\sigma)\,d\sigma$. We now assume that $f(r) > 0$ and thus note that $g(r) > 0$. This implies $k_2 < c_3$ and from (3.7) we have an inequality constraint

$$c_2 < \int_0^a \frac{\rho}{a}\int_0^\rho (\ln\rho - \ln\sigma)\sigma f(\sigma)\,d\sigma d\rho + \frac{ac_3}{2}. \tag{3.14}$$

We examine $G'(r)$ and obtain

$$G'(r) = \int_0^r \frac{\sigma}{r} f(\sigma)\,d\sigma$$

which is positive since $f(\sigma) > 0$. Since $G'(r)$ is positive, our function $G(r)$ is increasing. Looking at the second derivative we find that

$$\begin{aligned} G''(r) &= -\int_0^r \frac{\sigma}{r^2} f(\sigma)\,d\sigma + f(r) = -\frac{1}{2}f(r) + \int_0^r \frac{\sigma^2}{2r^2} f'(\sigma)\,d\sigma + f(r) \\ &= \frac{1}{2}f(r) + \int_0^r \frac{\sigma^2}{2r^2} f'(\sigma)\,d\sigma. \end{aligned}$$

If $f'(\sigma) \geq 0$, then $G''(r) > 0$ indicating that $G(r)$ is concave up. The conditions $f(\sigma) > 0$ and $f'(\sigma) \geq 0$ ensure we have an $r > a$ which solves the equation (3.13). Let $r = b$ represent the radius of Ω. Then b is a solution of the following equation

$$\int_0^b F(b,\sigma)\,d\sigma + \frac{2c_2}{a} - \int_0^a \frac{2\rho}{a^2}\int_0^\rho F(\rho,\sigma)\,d\sigma d\rho - c_3 = 0. \tag{3.15}$$

We summarize our results in Theorem 1.

Theorem 1: Let Ω be a bounded domain in R^2 with a sufficiently smooth boundary $\partial\Omega$. If the overdetermined problem (3.1) has a solution $u \in C^4(\Omega) \cap C^2(\overline{\Omega})$, then Ω is an open disk and the solution is

$$\begin{aligned} u = &\int_0^r (\ln r - \ln\rho)\rho \int_0^\rho F(\rho,\sigma)\,d\sigma d\rho - \int_0^a (\ln a - \ln\rho)\rho\int_0^\rho F(\rho,\sigma)\,d\sigma d\rho \\ &- \frac{r^2 - a^2}{2}\int_0^a \frac{\rho}{a^2}\int_0^\rho F(\rho,\sigma)\,d\sigma d\rho + \frac{c_2(r^2 - a^2)}{2a} + c_1, \end{aligned}$$

where r denotes the distance from the point (x , y) to the center of Ω and the radius b of Ω is found by equation (3.15).

We now consider the N > 2 case for problem (3.1). Once again since $f(r)$ is only a function of r, we use the radial form of the Laplacian, which for N > 2 is

$$\Delta u = u_{rr} + \frac{N-1}{r} u_r = r^{-(N-1)} \frac{\partial}{\partial r}\left(r^{(N-1)} \frac{\partial u}{\partial r}\right). \tag{3.16}$$

It then follows that

$$\Delta^2 u = r^{-(N-1)} \frac{\partial}{\partial r}\left(r^{(N-1)} \frac{\partial \Delta u}{\partial r}\right). \tag{3.17}$$

Therefore, we consider the equation

$$r^{-(N-1)} \frac{\partial}{\partial r}\left(r^{(N-1)} \frac{\partial \Delta u}{\partial r}\right) = f(r). \tag{3.18}$$

We multiply (3.18) by $r^{(N-1)}$ and integrate to obtain

$$r^{(N-1)} \frac{\partial \Delta u}{\partial r} = \int_0^r \rho^{(N-1)} f(\rho) d\rho + k_1$$

where k_1 is an arbitrary constant. We now multiply by $r^{-(N-1)}$ and integrate so that

$$\Delta u = \int_0^r \sigma^{-(N-1)} \int_0^\sigma \rho^{(N-1)} f(\rho) d\rho d\sigma + k_1 \frac{r^{-(N-2)}}{-N+2} + k_2,$$

where k_2 is an arbitrary constant. Since $r^{-(N-2)}$ is singular at r = 0, we take $k_1 = 0$. By interchanging the order of integration, we have

$$\begin{aligned} \Delta u &= \int_0^r \int_\rho^r \sigma^{-(N-1)} \rho^{(N-1)} f(\rho) d\sigma d\rho + k_2 \\ &= -\frac{1}{(N-2)} \int_0^r \left(\frac{\rho^{(N-1)}}{r^{(N-2)}} - \rho\right) f(\rho) d\rho + k_2 \end{aligned}$$

or

$$\Delta u = \frac{1}{(N-2)} \int_0^r K(r,\rho) d\rho + k_2 \tag{3.19}$$

where we let

$$K(r,\rho) = \left(\rho - \frac{\rho^{N-1}}{r^{N-2}}\right) f(\rho). \tag{3.20}$$

By (3.16) and (3.19), we can write

$$r^{-(N-1)} \frac{\partial}{\partial r}\left(r^{(N-1)} \frac{\partial u}{\partial r}\right) = \frac{1}{(N-2)} \int_0^r K(r,\rho) d\rho + k_2.$$

Using the same multiplication and integration technique, we obtain

$$r^{(N-1)}\frac{\partial u}{\partial r}=\frac{1}{(N-2)}\int_0^r \sigma^{N-1}\int_0^\sigma K(\sigma,\rho)d\rho d\sigma+\frac{k_2 r^N}{N}+k_3$$

for an arbitrary constant k_3 and

$$\frac{\partial u}{\partial r}=\frac{1}{(N-2)}\int_0^r \frac{\sigma^{N-1}}{r^{N-1}}\int_0^\sigma K(\sigma,\rho)d\rho d\sigma+\frac{k_2 r}{N}+k_3 r^{-(N-1)}. \tag{3.21}$$

Since $r^{-(N-1)}$ is singular at r = 0, we let $k_3 = 0$. We also note that the first term in (3.21) is not singular because the limit is zero as $r \to 0^+$.

We now use the boundary condition $\partial u/\partial n = c_2$ on ∂B_a. Since on ∂B_a, $\partial u/\partial n = \partial u/\partial r$, we obtain

$$\frac{1}{(N-2)}\int_0^a \frac{\sigma^{N-1}}{a^{N-1}}\int_0^\sigma K(\sigma,\rho)d\rho d\sigma+\frac{k_2 a}{N}=c_2,$$

and thus

$$k_2=\frac{Nc_2}{a}-\frac{N}{(N-2)}\int_0^a \frac{\sigma^{N-1}}{a^N}\int_0^\sigma K(\sigma,\rho)d\rho d\sigma. \tag{3.22}$$

Consequently,

$$\frac{\partial u}{\partial r}=\frac{1}{(N-2)}\int_0^r \frac{\sigma^{N-1}}{r^{N-1}}\int_0^\sigma K(\sigma,\rho)d\rho d\sigma+\frac{k_2 r}{N} \tag{3.23}$$

with k_2 as found above.

We integrate (3.23) with respect to r and obtain

$$u=\frac{1}{(N-2)}\int_0^r\int_0^s \frac{\sigma^{N-1}}{s^{N-1}}\int_0^\sigma K(\sigma,\rho)d\rho d\sigma ds+\frac{k_2 r^2}{2N}+k_4$$

with k_4 being an arbitrary constant. Interchanging the order of integration, we have

$$\begin{aligned} u &= \frac{1}{(N-2)}\int_0^r\int_\sigma^r \frac{\sigma^{N-1}}{s^{N-1}}\int_0^\sigma K(\sigma,\rho)d\rho ds d\sigma+\frac{k_2 r^2}{2N}+k_4 \\ &= -\frac{1}{(N-2)^2}\int_0^r (r^{-(N-2)}-\sigma^{-(N-2)})\sigma^{(N-1)}\int_0^\sigma K(\sigma,\rho)d\rho d\sigma+\frac{k_2 r^2}{2N}+k_4 \end{aligned} \tag{3.24}$$

where we must exclude N = 4 since

$$\begin{aligned} &\int_0^r (r^{-(N-2)}-\sigma^{-(N-2)})\sigma^{(N-1)}\int_0^\sigma K(\sigma,\rho)d\rho d\sigma \\ &= \int_0^r \Big\{\int_\rho^r (r^{-(N-2)}-\sigma^{-(N-2)})\sigma^{(N-1)}(\rho-\frac{\rho^{N-1}}{\sigma^{N-2}})d\sigma\Big\} f(\rho)d\rho \end{aligned}$$

$$= \int_0^r \left[\frac{2-N}{N}(\rho r^2 - \frac{\rho^{N+1}}{r^{N-2}}) + \frac{N-2}{2(N-4)}(\rho^3 - \frac{\rho^{N-1}}{r^{N-4}}) \right] f(\rho) d\rho \tag{3.25}$$

upon integration. We interchange the dummy variables σ and ρ in (3.24) and write for $N > 2$, $N \neq 4$

$$u = \frac{1}{(N-2)^2} \int_0^r (\rho - \frac{\rho^{N-1}}{r^{N-2}}) \int_0^\rho K(\rho,\sigma) d\sigma d\rho + \frac{k_2 r^2}{2N} + k_4. \tag{3.26}$$

By the boundary condition, $u = c_1$ on ∂B_a, we find k_4 to be

$$k_4 = c_1 - \frac{1}{(N-2)^2} \int_0^a (\rho - \frac{\rho^{N-1}}{a^{N-2}}) \int_0^\rho K(\rho,\sigma) d\sigma d\rho + \frac{k_2 a^2}{2N}.$$

For this k_4 in (3.26), we have

$$\begin{aligned} u = \frac{1}{(N-2)^2} \int_0^r (\rho - \frac{\rho^{N-1}}{r^{N-2}}) \int_0^\rho K(\rho,\sigma) d\sigma d\rho - \frac{1}{(N-2)^2} \int_0^a (\rho - \frac{\rho^{N-1}}{a^{N-2}}) \int_0^\rho K(\rho,\sigma) d\sigma d\rho \\ - \frac{(r^2 - a^2)}{2(N-2)} \int_0^a \frac{\rho^{N-1}}{a^N} \int_0^\rho K(\rho,\sigma) d\sigma d\rho + \frac{c_2 (r^2 - a^2)}{2a} + c_1. \end{aligned} \tag{3.27}$$

The function u in (3.27) is a unique solution to the problem (3.11) for $N > 2$, $N \neq 4$. We continue u to the rest of Ω and ask that it satisfy the condition $\Delta u = c_3$ on $\partial\Omega$. From (3.1) and (3.19), we obtain for points on $\partial\Omega$ where $r > a$,

$$\Delta u = \frac{1}{(N-2)} \int_0^r K(r,\sigma) d\sigma + k_2 = c_3. \tag{3.28}$$

This equation implies that r is a constant for points on $\partial\Omega$, i.e. Ω is a ball.

We could analyze the equation (3.28) to again determine sufficient conditions on $f(r)$ under which there is a positive root $r > a$ as was done in the $N = 2$ case. However, the assumption that a solution exists for (3.1) implies the solvability of (3.28) for an $r > a$.. Letting $r = b$ represent the radius of Ω, b is a solution of the following equation

$$\frac{1}{(N-2)} \int_0^b (\sigma - \frac{\sigma^{N-1}}{b^{N-2}}) f(\sigma) d\sigma - \frac{N}{(N-2)} \int_0^a \frac{\rho^{N-1}}{a^N} \int_0^\rho (\sigma - \frac{\sigma^{N-1}}{\rho^{N-2}}) f(\sigma) d\sigma d\rho + \frac{Nc_2}{a} = c_3. \tag{3.29}$$

The above results for the $N > 2$, $N \neq 4$ case are stated in Theorem 2.

When N=4, we have from (3.24) and (3.25)

$$u = \frac{-1}{4} \int_0^r \{ \int_\rho^r (r^{-2} - \sigma^{-2}) \sigma^3 (\rho - \frac{\rho^3}{\sigma^2}) d\sigma \} f(\rho) d\rho + \frac{k_2 r^2}{8} + k_4.$$

After simplification, we find that

$$u=\frac{-1}{4}\int_0^r\{\int_\rho^r(\frac{\rho\sigma^3}{r^2}-\frac{\rho^3\sigma}{r^2}-\sigma\rho+\frac{\rho^3}{\sigma})d\sigma\}f(\rho)d\rho+\frac{k_2r^2}{8}+k_4$$

and integrate to obtain

$$u=\frac{-1}{4}\int_0^r(\frac{-\rho r^2}{4}+\rho^3(\ln r-\ln\rho)+\frac{\rho^5}{4r^2})f(\rho)d\rho+\frac{k_2r^2}{8}+k_4. \tag{3.30}$$

By the boundary condition, u = c_1 on ∂B_a , we find k_4 to be

$$k_4=c_1+\frac{1}{4}\int_0^r(\frac{-\rho a^2}{4}+\rho^3(\ln a-\ln\rho)+\frac{\rho^5}{4a^2})f(\rho)d\rho-\frac{k_2a^2}{8}. \tag{3.31}$$

For this k_4 in (3.30), we have

$$u=\frac{-1}{4}\int_0^r(\frac{-\rho r^2}{4}+\rho^3(\ln r-\ln\rho)+\frac{\rho^5}{4r^2})f(\rho)d\rho+\frac{1}{4}\int_0^a(\frac{-\rho a^2}{4}+\rho^3(\ln a-\ln\rho)+\frac{\rho^5}{4a^2})f(\rho)d\rho$$
$$+(\frac{4c_2}{a}-2\int_0^a\frac{\sigma^3}{a^4}\int_0^\sigma(\rho-\frac{\rho^3}{\sigma^2})f(\rho)d\rho d\sigma)\frac{(r^2-a^2)}{8}+c_1. \tag{3.32}$$

The function u in (3.32) is a unique solution to (3.11) for N = 4. We continue u to the rest of Ω and ask that it satisfy the condition $\Delta u=c_3$ on $\partial\Omega$. From (3.1) and (3.19), we obtain points on $\partial\Omega$ where r > a,

$$\Delta u=\frac{1}{2}\int_0^r(\rho-\frac{\rho^3}{r^2})f(\rho)d\rho+k_2=c_3. \tag{3.33}$$

The equation implies that r is a constant for points on $\partial\Omega$, i.e, Ω is a ball.

We summarize our results in Theorem 2.

<u>Theorem 2</u>: Let Ω be a bounded domain in R^N , N > 2, with a sufficiently smooth boundary $\partial\Omega$, and suppose that the overdetermined problem (3.1) has a solution $u\in C^4(\Omega)\cap C^2(\overline{\Omega})$. Then Ω is an open ball and the solution when N> 2, N≠ 4 is given by

$$u=\frac{1}{(N-2)^2}\int_0^r(\rho-\frac{\rho^{N-1}}{r^{N-2}})\int_0^\rho(\sigma-\frac{\sigma^{N-1}}{\rho^{N-2}})f(\sigma)d\sigma d\rho-\frac{1}{(N-2)^2}\int_0^a(\rho-\frac{\rho^{N-1}}{a^{N-2}})\int_0^\rho(\sigma-\frac{\sigma^{N-1}}{\rho^{N-2}})f(\sigma)d\sigma d\rho$$
$$-\frac{(r^2-a^2)}{2(N-2)}\int_0^a\frac{\rho^{N-1}}{a^N}\int_0^\rho(\sigma-\frac{\sigma^{N-1}}{\rho^{N-2}})f(\sigma)d\sigma d\rho+\frac{c_2(r^2-a^2)}{2a}+c_1$$

where r denotes the distance from the point $x\in R^N$ to the center of Ω and the radius of Ω is found by (3.29). For N = 4, the solution is

$$u=\frac{-1}{4}\int_0^r(\frac{-\rho r^2}{4}+\rho^3(\ln r-\ln\rho)+\frac{\rho^5}{4r^2})f(\rho)d\rho+\frac{1}{4}\int_0^a(\frac{-\rho a^2}{4}+\rho^3(\ln a-\ln\rho)+\frac{\rho^5}{4a^2})f(\rho)d\rho$$

$$+\left(\frac{4c_2}{a}-2\int_0^a\frac{\sigma^3}{a^4}\int_0^\sigma(\rho-\frac{\rho^3}{\sigma^2})f(\rho)d\rho d\sigma\right)\frac{(r^2-a^2)}{8}+c_1$$

where the radius of Ω is found by (3.33).

We now provide an illustration by finding an explicit solution when $f(r)=1$. We begin with the N= 2 case. Substituting $f(r)=1$ into the solution u in Theorem 1, we obtain

$$u=\int_0^r(\ln r-\ln\rho)\rho\int_0^\rho(\ln\rho-\ln\sigma)\sigma d\sigma d\rho-\int_0^a(\ln a-\ln\rho)\rho\int_0^\rho(\ln\rho-\ln\sigma)\sigma d\sigma d\rho$$
$$-\frac{r^2-a^2}{2}\int_0^a\frac{\rho}{a^2}\int_0^\rho(\ln\rho-\ln\sigma)\sigma d\sigma d\rho+\frac{c_2(r^2-a^2)}{2a}+c_1 \quad . \tag{3.34}$$

To aid in our calculations, we introduce the following lemma.

Lemma 1: $\int_0^\rho(\ln\rho-\ln\sigma)\sigma d\sigma=\frac{\rho^2}{4}$.

Proof: $\int_0^\rho(\ln\rho-\ln\sigma)\sigma d\sigma=\left(\frac{\sigma^2}{2}\ln\rho-\frac{\sigma^2}{2}\ln\sigma\right)\Big|_0^\rho+\int_0^\rho\frac{\sigma}{2}d\sigma$

$$=\left[\frac{\sigma^2}{2}\ln\rho-\frac{\sigma^2}{2}\ln\sigma+\frac{\sigma^2}{4}\right]_0^\rho=\frac{\rho^2}{4}$$

where $\frac{\sigma^2}{2}\ln\sigma\to 0$ as $\sigma\to 0$.

We use Lemma 1 and write (3.34) as

$$u=\int_0^r(\ln r-\ln\rho)\frac{\rho^3}{4}d\rho-\int_0^a(\ln a-\ln\rho)\frac{\rho^3}{4}d\rho-\frac{r^2-a^2}{2}\int_0^a\frac{\rho^3}{4a^2}d\rho+\frac{c_2(r^2-a^2)}{2a}+c_1.$$

We now combine terms and integrate to obtain

$$u=\left[\frac{\rho^4}{16}\ln r-\frac{\rho^4}{16}\ln\rho+\frac{\rho^4}{64}\right]_0^r-\left[\frac{\rho^4}{16}\ln a-\frac{\rho^4}{16}\ln\rho+\frac{\rho^4}{64}\right]_0^a-\frac{\left(r^2-a^2\right)a^2}{32}+\frac{c_2\left(r^2-a^2\right)}{2a}+c_1.$$

After evaluating at the endpoints, we find that

$$u=\frac{r^4-a^4}{64}+\frac{(16c_2-a^3)(r^2-a^2)}{32a}+c_1. \tag{3.35}$$

For this particular case, we can find b explicitly by solving equation (3.15). Upon doing so, we find

$$b=\sqrt{\frac{a^3+8ac_3-16c_2}{2a}}$$

and note that this will be a real solution if $c_2<\frac{a^3}{16}+\frac{ac_3}{2}$. This constraint on c_2 is in agreement with (3.14). It can easily be verified that u given by (3.35) solves (3.1) when $f(r)=1$.

Similarly using Theorem 2, we find the particular solutions for $f(r) = 1$ when N > 2. This solution is

$$u = \frac{(r^4 - a^4)}{8N(N+2)} + (\frac{c_2}{2a} - \frac{a^2}{4N(N+2)})(r^2 - a^2) + c_1 \tag{3.36}$$

where

$$b = \sqrt{\frac{N(a^3 - 2N(N+2)c_2 + 2a(N+2)c_3)}{a(N+2)}}$$

as can easily be verified. We note, in fact, that (3.36) gives the solution for all $N \geq 2$.

2. PROBLEM TWO

We now consider another problem similar to the previous problem but with auxiliary conditions interchanged. Specifically, we consider

$$\begin{gathered} \Delta^2 u = f(r) \text{ in } \Omega, \\ \partial u/\partial n = c_1, \;\; \Delta u = c_2 \text{ on } \partial B_a(0) \subset \Omega, \\ u = c_3 \text{ on } \partial\Omega, \end{gathered} \tag{3.37}$$

where $f(r)$, c_1, c_2, c_3 and Ω are defined as in the previous problem. We first consider (3.37) for N= 2 dimensions and then consider the N > 2 case.

We follow the procedure used in solving (3.1) up to equation (3.5) where

$$\Delta u = \int_0^r (\ln r - \ln\rho)\rho f(\rho) d\rho + k_2$$

We use the boundary condition $\Delta u = c_2$ on ∂B_a to find k_2 to be

$$k_2 = c_2 - \int_0^a (\ln a - \ln\rho)\rho f(\rho) d\rho.$$

Substituting in for k_2, we obtain

$$\Delta u = \int_0^r (\ln r - \ln\rho)\rho f(\rho) d\rho - \int_0^a (\ln a - \ln\rho)\rho f(\rho) d\rho + c_2. \tag{3.38}$$

An alternate form of (3.38) is

$$\Delta u = -\int_r^a F(r,\rho) d\rho + c_2. \tag{3.39}$$

where $F(r,\rho) = (\ln r - \ln\rho)\rho f(\rho)$. We note that (3.39) is valid since Δu given by (3.39) satisfies the

problem

$$\Delta^2 u = f(r) \text{ in } B_a, \quad \Delta u = c_2 \text{ on } \partial B_a,$$

By (3.2) and (3.39), we can write

$$r^{-1}\frac{\partial}{\partial r}\left(r\frac{\partial u}{\partial r}\right) = -\int_r^a F(r,\rho)d\rho + c_2.$$

Using the same multiplication and integration technique used in the previous problem, we obtain

$$r\frac{\partial u}{\partial r} = -\int_0^r \sigma\int_\sigma^a F(\sigma,\rho)d\rho d\sigma + \frac{c_2 r^2}{2} + k_3,$$

where k_3 is an arbitrary constant, and

$$\frac{\partial u}{\partial r} = -\int_0^r \frac{\sigma}{r}\int_\sigma^a F(\sigma,\rho)d\rho d\sigma + \frac{c_2 r}{2} + \frac{k_3}{r}. \tag{3.40}$$

Since the last term is singular at $r = 0$, we take $k_3 = 0$. We now use the B.C. $\partial u/\partial n = c_1$ on ∂B_a. Since on ∂B_a, $\partial u/\partial n = \partial u/\partial r$, we have from (3.40)

$$-\int_0^a \frac{\sigma}{a}\int_\sigma^a F(\sigma,\rho)d\rho d\sigma + \frac{c_2 a}{2} = c_1, \tag{3.41}$$

which is a constraint on c_1 and c_2 that needs to be satisfied in order to have a solution.

We integrate (3.40) with respect to r and obtain

$$u = -\int_0^r\int_0^s \frac{\sigma}{s}\int_\sigma^a F(\sigma,\rho)d\rho d\sigma ds + \frac{c_2 r^2}{4} + k_4.$$

Interchanging the order of intergration, we have

$$\begin{aligned} u &= -\int_0^r\int_\sigma^r \frac{\sigma}{s}\int_\sigma^a F(\sigma,\rho)d\rho ds d\sigma + \frac{c_2 r^2}{4} + k_4 \\ &= -\int_0^r (\ln r - \ln\sigma)\sigma\int_\sigma^a F(\sigma,\rho)d\rho d\sigma + \frac{c_2 r^2}{4} + k_4. \end{aligned}$$

We interchange the dummy variables σ and ρ and write

$$u = -\int_0^r (\ln r - \ln\rho)\rho\int_\rho^a F(\rho,\sigma)d\sigma d\rho + \frac{c_2 r^2}{4} + k_4. \tag{3.42}$$

The function u in (3.42) satisfies

$$\Delta^2 u = f(r) \text{ in } B_a, \quad \partial u/\partial n = c_1, \ \Delta u = c_2 \text{ on } \partial B_a \tag{3.43}$$

and is a unique solution up to an additive constant due to the type of boundary conditions in (3.43).

We now continue (3.42) to the rest of Ω. We ask that (3.42) satisfy the B.C. $u = c_3$ on $\partial\Omega$ and obtain

$$-\int_0^r (\ln r - \ln\rho)\rho\int_\rho^a F(\rho,\sigma)d\sigma d\rho + \frac{c_2 r^2}{4} + k_4 = c_3. \tag{3.44}$$

This transcendental equation implies that there is a constant solution $r > a$ in which case Ω is a disk.

We summarize our results in Theorem 3.

__Theorem 3:__ Let Ω be a bounded domain in R^2 with sufficiently smooth boundary $\partial\Omega$ and suppose that the overdetermined problem (3.37) has a solution $u \in C^4(\Omega) \cap C(\overline{\Omega})$ where

$$c_1 = -\int_0^a \frac{\sigma}{a}\int_\sigma^a F(\sigma,\rho)d\rho d\sigma + \frac{c_2 a}{2}.$$

Then Ω is an open disk, and the solution is,

$$u = -\int_0^r (\ln r - \ln\rho)\rho\int_\rho^a F(\rho,\sigma)d\sigma d\rho + \frac{c_2 r^2}{4} + k_4,$$

where the radius of Ω is found from (3.44) and k_4 is an arbitrary constant..

We now consider the $N > 2$ case for problem (3.37). Once again we follow the steps used in solving (3.1) up to equation (3.19) where

$$\Delta u = \frac{1}{(N-2)}\int_0^r K(r,\rho)d\rho + k_2$$

with $K(r,\rho)$ defined by (3.20). We now use the B.C. $\Delta u = c_2$ on ∂B_a to find k_2 to be

$$k_2 = c_2 - \frac{1}{(N-2)}\int_0^a K(a,\rho)d\rho. \tag{3.45}$$

Consequently,

$$\Delta u = \frac{1}{(N-2)}\int_0^r K(r,\rho)d\rho - \frac{1}{(N-2)}\int_0^a K(a,\rho)d\rho + c_2.$$

For simplicity, we use

$$\Delta u = \frac{1}{(N-2)}\int_0^r K(r,\rho)d\rho + k_2. \tag{3.46}$$

By (3.16) and (3.46), we can write

$$r^{-(N-1)}\frac{\partial}{\partial r}(r^{N-1}\frac{\partial u}{\partial r}) = \frac{1}{(N-2)}\int_0^r K(r,\rho)d\rho + k_2.$$

Using the same multiplication and integration technique used in the previous problem, we obtain

$$r^{N-1}\frac{\partial u}{\partial r} = \frac{1}{(N-2)}\int_0^r \sigma^{N-1}\int_0^\sigma K(\sigma,\rho)d\rho d\sigma + \frac{k_2 r^N}{N} + k_3$$

where k_3 is an arbitrary constant, and

$$\frac{\partial u}{\partial r} = \frac{1}{(N-2)}\int_0^r \frac{\sigma^{N-1}}{r^{N-1}}\int_0^\sigma K(\sigma,\rho)d\rho d\sigma + \frac{k_2 r}{N} + k_3 r^{-(N-1)}. \tag{3.47}$$

Since $k_3 r^{-(N-1)}$ is singular at r = 0, we let $k_3 = 0$. We now use the B.C. $\partial u/\partial n = c_1$ on ∂B_a and obtain

$$\frac{\partial u}{\partial n} = \frac{\partial u}{\partial r} = \frac{1}{(N-2)}\int_0^a \frac{\sigma^{N-1}}{a^{N-1}}\int_0^\sigma K(\sigma,\rho)d\rho d\sigma + \frac{k_2 a}{N} = c_1. \tag{3.48}$$

The equation (3.48) gives us a constraint for c_1 and c_2 that needs to be satisfied in order to have a solution.

We integrate (3.47) with respect to r and obtain

$$u = \frac{1}{(N-2)}\int_0^r\int_0^s \frac{\sigma^{N-1}}{s^{N-1}}\int_0^\sigma K(\sigma,\rho)d\rho d\sigma ds + \frac{k_2 r^2}{2N} + k_4'$$

where k_4 is an arbitrary constant. Interchanging the order of integration, we have

$$\begin{aligned} u &= \frac{1}{(N-2)}\int_0^r\int_\sigma^r \frac{\sigma^{N-1}}{s^{N-1}}\int_0^\sigma K(\sigma,\rho)d\rho ds d\sigma + \frac{k_2 r^2}{2N} + k_4 \\ &= \frac{1}{(N-2)^2}\int_0^r (\sigma - \frac{\sigma^{N-1}}{r^{N-2}})\int_0^\sigma K(\sigma,\rho)d\rho d\sigma + \frac{k_2 r^2}{2N} + k_4. \end{aligned}$$

Once again we exclude N = 4 to avoid the difficulty obtained in (3.25). We interchange the dummy variables σ and ρ and write for N> 2, N≠ 4,

$$u = \frac{1}{(N-2)^2}\int_0^r (\rho - \frac{\rho^{N-1}}{r^{N-2}})\int_0^\rho K(\rho,\sigma)d\sigma d\rho + \frac{k_2 r^2}{2N} + k_4. \tag{3.49}$$

The function u in (3.49) satisfies the problem (3.43) and is a unique solution up to an additive constant due to the type of boundary conditions in (3.43).

We now continue our solution to the rest of Ω. We ask that (3.49) satisfy the boundary condition $u = c_3$ on ∂Ω and obtain

$$\frac{1}{(N-2)^2}\int_0^r (\rho - \frac{\rho^{N-1}}{r^{N-2}})\int_0^\rho K(\rho,\sigma)d\sigma d\rho + \frac{k_2 r^2}{2N} + k_4 = c_3. \tag{3.50}$$

The equation (3.50) can theoretically be solved for r in which case Ω is a ball.

When N = 4, we have (3.30) as in Problem One except k_2 is as defined in (3.45). Therefore our function u is

$$u=\frac{-1}{4}\int_0^r\left(\frac{-\rho r^2}{4}+\rho^3(\ln r-\ln\rho)+\frac{\rho^5}{4r^2}\right)f(\rho)d\rho+\frac{k_2r^2}{8}+k_4. \tag{3.51}$$

The function u in (3.51) satisfies the problem (3.43) when N = 4 and is unique up to an additive constant due to the type of boundary conditions in (3.43).

We now continue our solution to the rest of Ω. We ask that (3.51) satisfy the B.C. $u = c_3$ on $\partial\Omega$ and obtain

$$\frac{-1}{4}\int_0^r\left(\frac{-\rho r^2}{4}+\rho^3(\ln r-\ln\rho)+\frac{\rho^5}{4r^2}\right)f(\rho)d\rho+\frac{k_2r^2}{8}+k_4=c_3. \tag{3.52}$$

The equation (3.52) can theoretically be solved for r in which case Ω is a ball.

We can summarize our results in Theorem 4.

<u>**Theorem 4:**</u> Let Ω be a bounded domain in R^N, N > 2, with a sufficiently smooth boundary $\partial\Omega$ and suppose that the overdetermined problem (3.37) has a solution $u\in C^4(\Omega)\cap C(\overline{\Omega})$, where

$$c_1=\frac{1}{(N-2)}\int_0^a\frac{\rho^{N-1}}{a^{N-1}}\int_0^\rho\left(\sigma-\frac{\sigma^{N-1}}{\rho^{N-2}}\right)f(\sigma)d\sigma d\rho-\frac{1}{N-2}\int_0^a\left(\sigma-\frac{\sigma^{N-1}}{a^{N-2}}\right)f(\sigma)d\sigma d\rho+\frac{c_2a}{N}.$$

Then Ω is an open ball and the solution for N > 2, N$\neq$ 4 is

$$\begin{aligned}u=&\frac{1}{(N-2)^2}\int_0^r\left(\rho-\frac{\rho^{N-1}}{r^{N-2}}\right)\int_0^\rho\left(\sigma-\frac{\sigma^{N-1}}{\rho^{N-2}}\right)f(\sigma)d\sigma d\rho\\&-\frac{r^2}{2N(N-2)}\int_0^a\left(\sigma-\frac{\sigma^{N-1}}{a^{N-2}}\right)f(\sigma)d\sigma+\frac{c_2r^2}{2N}+k_4\end{aligned}$$

where the radius of Ω is found by (3.50) and k_4 is an arbitrary constant. For N = 4, the solution is

$$u=\frac{-1}{4}\int_0^r\left(\frac{-\rho r^2}{4}+\rho^3(\ln r-\ln\rho)+\frac{\rho^5}{4r^2}\right)f(\rho)d\rho+\frac{r^2}{8}\left(c_2-\frac{1}{2}\int_0^a\left(\rho-\frac{\rho^3}{a^2}\right)f(\rho)d\rho+k_4\right.$$

and the radius of Ω is found by (3.52) and k_4 is an arbitrary constant.

Similar to the Problem One, we are able to find an explicit solution to (3.37) for the particular case of $f(r)=1$. We state the result for the N = 2 dimension and show the N > 2 case. Using Theorem 3, we substitute $f(r)=1$ into our solution u and evaluate to obtain

$$u=\frac{a^2r^2}{8}(\ln a-\ln r)+\frac{a^2r^2}{16}+\frac{r^4}{64}+\frac{c_2r^2}{4}+k_4 \tag{3.53}$$

and

$$c_1=\frac{a^3}{16}+\frac{c_2a}{2}.$$

We now examine the N > 2 case by using Theorem 4 with $f(r)=1$. We obtain

$$u=\frac{1}{(N-2)^2}\int_0^r(\rho-\frac{\rho^{N-1}}{r^{N-2}})\int_0^\rho(\sigma-\frac{\sigma^{N-1}}{\rho^{N-2}})d\sigma d\rho+\frac{k_2r^2}{2N}+k_4 \tag{3.54}$$

To aid in our calculations, we introduce a second lemma.

<u>**Lemma 2:**</u> $\int_0^\rho(\sigma-\frac{\sigma^{N-1}}{\rho^{N-2}})d\sigma=\frac{(N-2)\rho^2}{2N}.$

Proof: $\int_0^\rho(\sigma-\frac{\sigma^{N-1}}{\rho^{N-2}})d\sigma=\frac{\sigma^2}{2}-\frac{\sigma^N}{N\rho^{N-2}}\Big|_0^\rho=\frac{(N-2)\rho^2}{2N}.$

Using Lemma 2, we can write (3.54) as

$$u=\frac{1}{(N-2)^2}\int_0^r(\rho-\frac{\rho^{N-1}}{r^{N-2}})\frac{(N-2)\rho^2}{2N}d\rho+\frac{k_2r^2}{2N}+k_4$$

We now multiply and integrate to obtain

$$u=\frac{1}{N-2}\left[\frac{r^4}{8N}-\frac{r^4}{2N(N+2)}\right]+\frac{k_2r^2}{2N}+k_4\,.$$

We now combine like terms and evaluate k_2 to find

$$u=\frac{r^4}{8N(N+2)}-\frac{a^2r^2}{4N^2}+\frac{c_2r^2}{2N}+k_4. \tag{3.55}$$

We also find that the constraint on c_1 is

$$c_1=-\frac{a^3}{N^2(N+2)}+\frac{c_2a}{N}.$$

3. PROBLEM THREE

We now consider another problem similar to the Problems One and Two. Specifically, we consider

$$\Delta^2 u = f(r) > 0, \text{ in } \Omega$$

$$u = c_1\,, \quad \Delta u = c_2\,, \text{ on } \partial B_a(0) \subset \Omega \tag{3.56}$$

$$\partial u / \partial n = 0 \text{ on } \partial\Omega$$

where $f(r)$ is a continuous function of $r > 0$. The ball, domain, and their boundaries are as in the previous problems except here for simplicity we assume that the domain Ω is convex. We first consider (3.56) for N=2 dimensions and then consider the $N > 2$ case.

We follow the same steps used in solving (3.37) up to equation (3.40) where

$$\frac{\partial u}{\partial r} = -\int_0^r \frac{\sigma}{r}\int_\sigma^a F(\sigma,\rho)\,d\rho\, d\sigma + \frac{c_2 r}{2}.$$

We note that the boundary condition $\Delta u = c_2$ on ∂B_a has already been used, and we let $k_3 = 0$ to avoid singularities. We now integrate to find that

$$u = -\int_o^r\int_0^s \frac{\sigma}{s}\int_\sigma^a F(\sigma,\rho)\,d\rho\, d\sigma\, ds + \frac{c_2 r^2}{4} + k_4,$$

where k_4 is an arbitrary constant. Interchanging the order of integration, we have

$$u = -\int_o^r (\ln r - \ln\sigma)\sigma\int_\sigma^a F(\sigma,\rho)\,d\rho\, d\sigma + \frac{c_2 r^2}{4} + k_4.$$

We interchange the dummy variables σ and ρ and write

$$u = -\int_o^r (\ln r - \ln\rho)\rho\int_\rho^a F(\rho,\sigma)\,d\sigma\, d\rho + \frac{c_2 r^2}{4} + k_4. \tag{3.57}$$

By the boundary condition $u = c_1$ on ∂B_a, we find k_4 to be

$$k_4 = \int_o^a (\ln a - \ln\rho)\rho\int_\rho^a F(\rho,\sigma)\,d\sigma\, d\rho - \frac{c_2 a^2}{4} + c_1.$$

With this k_4 in (3.57), we have

$$u=-\int_o^r(\ln r-\ln\rho)\rho\int_\rho^a F(\rho,\sigma)d\sigma d\rho+\int_o^a(\ln a-\ln\rho)\rho\int_\rho^a F(\rho,\sigma)d\sigma d\rho+\frac{c_2(r^2-a^2)}{4}+c_1. \qquad (3.58)$$

The function u in (3.58) is a unique solution to the problem

$$\Delta^2 u=f(r)\text{ in }B_a,\quad u=c_1\ ,\quad \Delta u=c_2\text{ on }\partial B_a. \qquad (3.59)$$

We continue u to the rest of Ω and ask that it satisfy the condition $\partial u/\partial n=0$ on $\partial\Omega$. Using the fact that $\partial u/\partial n=\nabla u\cdot\vec{n}$, from (3.57) we obtain for points on $\partial\Omega$ where $r > a$,

$$\frac{\partial u}{\partial n}=\left[\int_0^r\left\{\frac{\rho}{r}\int_\rho^a(\ln\sigma-\ln\rho)\sigma f(\sigma)d\sigma\right\}d\rho+\frac{c_2 r}{2}\right]\frac{x_i}{r}\cdot n_i=0$$

where $\partial r/\partial x_i=x_i/r$. Since Ω is convex, then $x_i n_i > 0$. Therefore

$$\int_0^r\left\{\frac{\rho}{r^2}\int_\rho^a(\ln\sigma-\ln\rho)\sigma f(\sigma)d\sigma\right\}d\rho+\frac{c_2}{2}=0 \qquad (3.60)$$

on $\partial\Omega$. The resulting transcendental equation (3.60) has a positive root $r > a$ and implies Ω is a disk.

Let $r = b$ represent the radius of Ω, where b is a solution to

$$\int_0^b\left\{\frac{\rho}{b^2}\int_\rho^a(\ln\sigma-\ln\rho)\sigma f(\sigma)d\sigma\right\}d\rho+\frac{c_2}{2}=0. \qquad (3.61)$$

We summarize our result in Theorem 5.

Theorem 5: Let Ω be a bounded domain in R^2 with a sufficiently smooth boundary $\partial\Omega$ and suppose that the overdetermined problem (3.56) has a solution $u\in C^4(\Omega)\cap C^1(\overline{\Omega})$. Then Ω is an open disk and our solution is

$$u=-\int_o^r(\ln r-\ln\rho)\rho\int_\rho^a F(\rho,\sigma)d\sigma d\rho+\int_o^a(\ln a-\ln\rho)\rho\int_\rho^a F(\rho,\sigma)d\sigma d\rho+\frac{c_2(r^2-a^2)}{4}+c_1$$

where r denotes the distance from (x,y) to the center of Ω and the radius of Ω is found by the equation (3.61).

We now consider the $N > 2$ case for (3.56). Once again we follow the same steps used in solving (3.37) for the $N > 2$ case up to equation (3.47) where

$$\frac{\partial u}{\partial r}=\frac{1}{(N-2)}\int_0^r\frac{\sigma^{N-1}}{r^{N-1}}\int_0^\sigma K(\sigma,\rho)d\rho d\sigma+\frac{k_2 r}{N}.$$

We note that the boundary condition $\Delta u = c_2$ on ∂B_a has already been used and k_2 is as in (3.45). We also let $k_3 = 0$ to avoid singularities. We now integrate to find that

$$u = \frac{1}{(N-2)}\int_0^r\int_0^s \frac{\sigma^{N-1}}{s^{N-1}}\int_0^\sigma K(\sigma,\rho)d\rho d\sigma ds + \frac{k_2 r^2}{2N} + k_4, \tag{3.62}$$

where k_4 is an arbitrary constant. Then interchanging the order of integration, (3.62) can be written as

$$\begin{aligned} u &= \frac{1}{(N-2)}\int_0^r\int_\sigma^r \frac{\sigma^{N-1}}{s^{N-1}}\int_0^\sigma K(\sigma,\rho)d\rho ds d\sigma + \frac{k_2 r^2}{2N} + k_4 \\ &= \frac{1}{(N-2)^2}\int_0^r (\sigma - \frac{\sigma^{N-1}}{r^{N-2}}\int_0^\sigma K(\sigma,\rho)d\rho d\sigma + \frac{k_2 r^2}{2N} + k_4. \end{aligned} \tag{3.63}$$

We once again refer back to (3.25) and exclude the N = 4 case. We interchange the dummy variables σ and ρ and write for $N > 2$, $N \neq 4$

$$u = \frac{1}{(N-2)^2}\int_0^r (\rho - \frac{\rho^{N-1}}{r^{N-2}}\int_0^\rho K(\rho,\sigma)d\sigma d\rho + \frac{k_2 r^2}{2N} + k_4. \tag{3.64}$$

We are now able to use the B.C. $u = c_1$ on ∂B_a to find k_4 to be

$$k_4 = \frac{1}{(N-2)^2}\int_0^a (\rho - \frac{\rho^{N-1}}{a^{N-2}}\int_0^\rho K(\rho,\sigma)d\sigma d\rho - \frac{k_2 a^2}{2N} + c_1.$$

For this k_4 in (3.64), we have

$$\begin{aligned} u &= \frac{1}{(N-2)^2}\int_0^r (\rho - \frac{\rho^{N-1}}{r^{N-2}}\int_0^\rho K(\rho,\sigma)d\sigma d\rho \\ &\quad - \frac{1}{(N-2)^2}\int_0^a (\rho - \frac{\rho^{N-1}}{a^{N-2}}\int_0^\rho K(\rho,\sigma)d\sigma d\rho + \frac{k_2 (r^2 - a^2)}{2N} + c_1. \end{aligned} \tag{3.65}$$

The function u in (3.65) is a unique solution to the problem (3.59) when $N > 2$, $N \neq 4$. We continue u to the rest of Ω and ask that it satisfy $\partial u/\partial n = 0$ on $\partial\Omega$. We find that

$$\frac{\partial u}{\partial n} = \nabla u \cdot \vec{n} = [\frac{1}{(N-2)}\int_0^r \frac{\rho^{N-1}}{r^{N-1}}\int_0^\rho K(\rho,\sigma)d\sigma d\rho + \frac{k_2 r}{N}]\frac{x_i}{r}\cdot n_i = 0$$

where $\partial r/\partial x_i = x_i/r$. Since Ω is a convex domain, $x_i n_i > 0$. Therefore ,

$$\frac{1}{(N-2)}\int_0^r \frac{\rho^{N-1}}{r^N}\int_0^\rho K(\rho,\sigma)d\sigma d\rho + \frac{k_2}{N} = 0. \tag{3.66}$$

Theoretically, we are able to solve the equation (3.66) for r. This would imply that Ω is a ball.

When N = 4, we follow Problem Two up to equation (3.51) and have

$$u = \frac{-1}{4}\int_0^r (\frac{-\rho r^2}{4} + \rho^3(\ln r - \ln\rho) + \frac{\rho^5}{4r^2}) f(\rho)d\rho + \frac{k_2 r^2}{8} + k_4, \tag{3.67}$$

where k_2 is as in (3.45). We are now able to use the boundary condition $u = c_1$ on ∂B_a to find k_4 to be

$$k_4 = \frac{1}{4}\int_0^a (\frac{-\rho a^2}{4} + \rho^3(\ln a - \ln\rho) + \frac{\rho^5}{4a^2}) f(\rho)d\rho - \frac{k_2 a^2}{8} + c_1.$$

For this k_4 in (3.67), we have

$$u = \frac{-1}{4}\int_0^r (\frac{-\rho r^2}{4} + \rho^3(\ln r - \ln\rho) + \frac{\rho^5}{4r^2}) f(\rho)d\rho + \frac{1}{4}\int_0^a (\frac{-\rho a^2}{4} + \rho^3(\ln a - \ln\rho) + \frac{\rho^5}{4a^2}) f(\rho)d\rho$$
$$+(c_2 - \frac{1}{2}\int_0^a (\rho - \frac{\rho^3}{a^2}) f(\rho)d\rho)\frac{(r^2 - a^2)}{8} + c_1. \tag{3.68}$$

The function u in (3.68) is a unique solution to the problem (3.59) when $N = 4$. We continue u to the rest of Ω and ask that it satisfy $\partial u/\partial n = 0$ on $\partial\Omega$. We find that

$$\frac{\partial u}{\partial n} = \nabla u \cdot \vec{n} = \left[\frac{-1}{4}\int_0^r (\frac{-\rho r}{2} + \frac{\rho^3}{r} - \frac{\rho^5}{2r^3}) f(\rho)d\rho + \frac{k_2 r}{4}\right]\frac{x_i}{r}\cdot \vec{n}_i = 0$$

where $\partial r/\partial x_i = x_i/r$. Since Ω is a convex domain, $x_i n_i > 0$. Therefore,

$$\frac{-1}{4r}\int_0^r (\frac{-\rho r}{2} + \frac{\rho^3}{r} - \frac{\rho^5}{2r^3}) f(\rho)d\rho + \frac{k_2}{4} = 0. \tag{3.69}$$

Theoretically, we are able to solve (3.69) for r. This would imply that Ω is a ball.

We summarize our results in Theorem 6.

Theorem 6: Let Ω be a bounded domain in R^N with a sufficiently smooth boundary $\partial\Omega$ and suppose that the overdetermined problem (3.56) has a solution $u \in C^4(\Omega) \cap C^1(\overline{\Omega})$. Then Ω is a ball and our solution when $N > 2$, $N \neq 4$ is

$$u = \frac{1}{(N-2)^2}\int_0^r (\rho - \frac{\rho^{N-1}}{r^{N-2}}\int_0^\rho (\sigma - \frac{\sigma^{N-1}}{\rho^{N-2}}) f(\sigma)d\sigma d\rho$$
$$-\frac{1}{(N-2)^2}\int_0^a (\rho - \frac{\rho^{N-1}}{a^{N-2}}\int_0^\rho (\sigma - \frac{\sigma^{N-1}}{\rho^{N-2}}) f(\sigma)d\sigma d\rho$$
$$+\frac{(r^2 - a^2)}{2N}(c_2 - \frac{1}{N-2}\int_0^a (\sigma - \frac{\sigma^{N-1}}{a^{N-2}}) f(\sigma)d\sigma) + c_1.$$

where r denotes the distance from x to the center of Ω and the radius of Ω is found by the equation (3.66) and for $N = 4$, the solution is

$$u = \frac{-1}{4}\int_0^r (\frac{-\rho r^2}{4} + \rho^3(\ln r - \ln\rho) + \frac{\rho^5}{4r^2}) f(\rho)d\rho + \frac{1}{4}\int_0^a (\frac{-\rho a^2}{4} + \rho^3(\ln a - \ln\rho) + \frac{\rho^5}{4a^2}) f(\rho)d\rho$$

$$+(c_2 - \frac{1}{2}\int_0^a (\rho - \frac{\rho^3}{a^2}) f(\rho) d\rho)\frac{(r^2 - a^2)}{8} + c_1$$

and the radius is found by equation (3.69)

Similar to the Problems One and Two, we are able to find explicit solutions to (3.56) for the particular case of $f(r)=1$. Since the calculations are similar to the previous two problems, we just state the results. For the N = 2 case, we use Theorem 5 , substitute $f(r)$ =1 into our solution u , and evaluate to obtain

$$u = \frac{r^4}{64} + \frac{a^2 r^2}{8}(\ln a - \ln r) + \frac{a^2 r^2}{16} - \frac{5a^4}{64} + \frac{c_2 (r^2 - a^2)}{4} + c_1$$

where b, the radius of Ω is found by

$$\frac{a^2}{4}(\ln a - \ln b) + \frac{b^2}{16} + \frac{c_2}{2} = 0.$$

Using Theorem 6, we are able to find an explicit solution for the N > 2 case. After substituting $f(r)=1$ into our solution u and evaluating , we have

$$u = \frac{r^4 - a^4}{8N(N+2)} + \frac{(r^2 - a^2)}{2N}(c_2 - \frac{a^2}{2N}) + c_1$$

where b, the radius of Ω is found by

$$b = \sqrt{\frac{(N+2)a^2}{N} - 2(N+2)c_2}\,.$$

We see once again that c_2 will need to be chosen appropriately to guarantee a b > a.

In this thesis, we have considered three fourth order overdetermined problems. In each case if a solution exists, then Ω is a ball. We have also found that if the necessary conditions in the theorems are not satisfied, then no solution exists, i.e, we obtain nonexistence results. As seen in the particular cases when $f(r)=1$, we note that the constants are not completely arbitrary but may be constrained to guarantee that the radius of Ω is greater than a. Lastly, we note that Problem One and Problem Three have unique solutions but Problem Two has a unique solution only up to an additive constant due to the boundary conditions imposed on the interior problem.

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