To the Graduate Council:

I am submitting herewith a dissertation written by Michael J. Roberts entitled "Transmission Lines with Distributed Sources and Some Applications in Active Filtering." I recommend that it be accepted in partial fulfillment of the requirements for the degree of Doctor of Philosophy, with a major in Electrical Engineering.

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TRANSMISSION LINES WITH DISTRIBUTED SOURCES AND SOME APPLICATIONS IN ACTIVE FILTERING

A Dissertation

Presented for the

Doctor of Philosophy

Degree

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Michael J. Roberts

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ABSTRACT

Methods were developed for the analysis of two-wire transmission lines with distributed independent and dependent current and voltage sources. An exact closed form analytical method using Green's functions, for uniform time-invariant lines, was derived. Two numerical solution methods were designed and programmed. The first of these is a frequency response approach which is useful in analysis of nonuniform lines with arbitrary terminating impedances, length, and source distributions. The second numerical method is a direct solution of the basic partial differential equations describing the local behavior of the voltage and current on a transmission line. This second method is applicable to nonuniform, time-variant lines with arbitrary source distributions and terminating resistances. The programming procedures are explained and a copy of the programs for each numerical method is included in the appendices. Two synthesis methods are developed. The first is a direct time domain synthesis of a desired impulse response. Implementation of this idea using transversal filters is explored. The second synthesis method is a frequency domain method which allows the designer to choose several points for the frequency response to pass through at certain frequencies. Design ideas for notch filters and a low-pass filter and experimental verification of these are presented.

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CHAPTER I

INTRODUCTION

The field of distributed parameter circuit theory has been investigated from many aspects and the characteristics and uses of distributed parameter circuit elements like the delay line and the thin film resistor are well known. The spatial dependence of distributed parameter circuits offers an extra degree of freedom which makes them inherently more versatile than lumped parameter circuits. Another reason to study distributed parameter circuits is the fact that all circuits are inevitably distributed because they have size. Thus, circuits which may be considered lumped by the uninitiated, may not act like lumped circuits at all in some frequency range. As in any endeavor, every difficulty in analyzing distributed parameter circuits is an opportunity in disguise. The converse holds true also. That is the extra versatility is accompanied by an extra degree of difficulty in analysis.

Probably the most familiar distributed parameter circuit is the uniform, passive, two-wire transmission line with incremental series resistance and/or inductance per unit length and incremental shunt conductance and/or capacitance per unit length. An incremental section model is shown in Figure I-1. As Δx approaches zero yielding a true distributed parameter circuit, the two familiar describing equations, called the Telegrapher's Equations, can be derived.

$$\frac{\partial v}{\partial x} = - ri(x,t) - l \frac{\partial i(x,t)}{\partial t} \qquad (1)$$

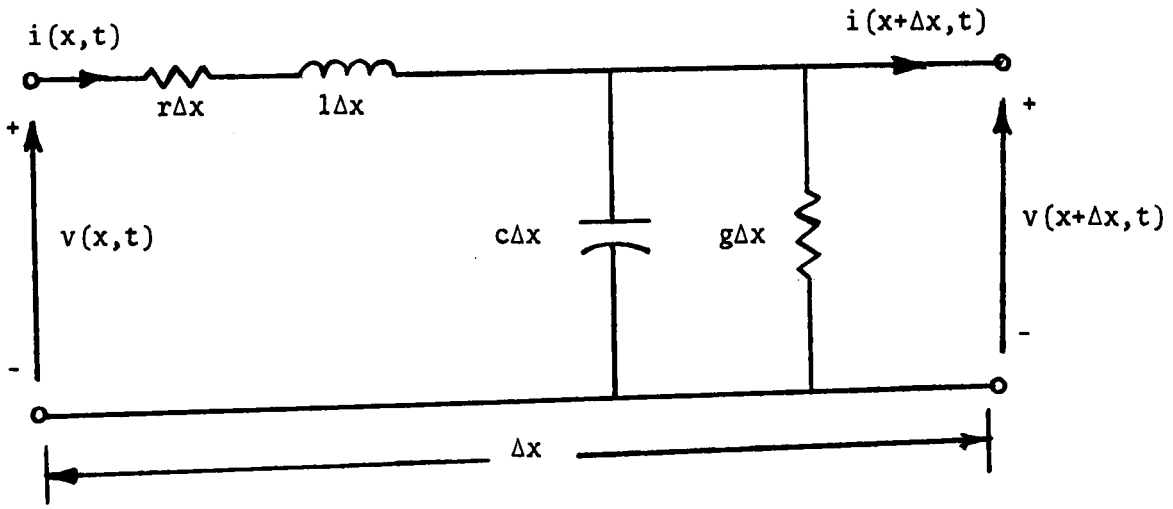


Figure I-1. Incremental transmission line model.

$$\frac{\partial i}{\partial x} = -gv(x,t) - c\frac{\partial v(x,t)}{\partial t} . \qquad (2)$$

In these equations, r, l, g, and c are constants. These are two first order, linear, homogeneous, constant-coefficient, coupled, partial differential equations. They can always be solved in closed form and a specific solution can always be found by considering the boundary conditions imposed by the terminations at each end. These equations, although more complicated than lumped parameter circuit equations, are quite simple and easily solved.

The problem is made significantly more complicated by letting r, l, g, and c be functions of the single space dimension, x. This is the case of the nonuniform, two-wire transmission line, or tapered line. Actually the term tapered implies a monotonic spatial dependence, not implied by the term nonuniform. Four special tapers can be solved in closed form. The rest usually have to be solved by an infinite series solution or numerically[1]. Tapered lines have an added degree of freedom and have been developed to realize novel impedances and voltage transfer functions.

Rather than being space dependent the parameters, r, l, g, and c could be time dependent. This idea has led to applications such as the travelling wave tube which parametrically amplifies a wave travelling down it longitudinally.

Still another variation on the basic transmission line is the extension to three or more wire models.

All these different kinds of distributed parameter circuits can be approximated by lumped element circuits. This idea leads to the area of

so-called artificial transmission lines. These lumped approximations may be much easier to physically realize and still be close enough to the distributed parameter model to be useful. An example is the distributed amplifier which is not distributed at all, but made up of a lumped l-c line driven at several points by controlled current sources. These current sources are controlled by a voltage propagating down another lumped l-c line. Over a restricted frequency range this type of circuit acts very much like the model of a true distributed circuit.

Another variation on the basic transmission line, and the one to which this dissertation speaks, is the addition to the basic model of distributed current and voltage sources, of both the independent and dependent type[2]. If an independent voltage source is placed in series in the incremental model, it does not change the natural response of the line. The same holds true for an independent shunt current source. Some physical manifestations of transmission lines with distributed sources follow:

1. Thermocouples have a distributed series independent voltage source caused by the heat of their environment.

2. When two wire coil transmission lines are placed in close proximity their fluxes link each other and each induces in the other a series distributed voltage.

3. Photodiodes with extreme aspect ratios have practically one-dimensional current flow. When light strikes their photosensitive surface it generates a distributed transverse current flow.

Also, lumped approximations can be built with current and/or voltage sources. On these lumped lines it is feasible to place, not only independent, but dependent sources.

Scope and Objectives of the Dissertation

The objectives of this dissertation are to develop very general analysis techniques for all kinds of two-wire transmission lines and investigate and develop new filter design techniques using the added degree of freedom of the distributed source.

Organization of the Dissertation

An exact closed form solution technique for uniform, time-invariant transmission lines with independent and dependent sources and arbitrary incremental parameters and terminations is developed in Chapter II. This is done using Green's functions and the Laplace transform.

Two numerical techniques are developed in Chapter III for solution of nonuniform or time-variant transmission lines. One is a frequency domain approach; the other is a time domain approach. The advantages and limitations of each are explored.

Some of the programming techniques used in implementation of the numerical methods of Chapter III are explained in Chapter IV. A special effort is made to explain the way in which the fast fourier transform is used to find impulse responses from frequency responses.

In Chapter V two approaches are taken toward approximating an ideal filter. The first is a time domain or impulse response technique. The second is a frequency domain method. Also, the similarity between transmission lines with distributed sources and transversal filters is discussed.

Chapter VI presents some specific design examples, some with experimental data supporting them.

Many detailed calculations and two computer programs are put in appendices to preserve the continuity of the material in the text.

CHAPTER II

AN EXACT ANALYTICAL SOLUTION METHOD

The incremental model of a uniform two-wire transmission line with independent and dependent current and voltage source distributions is shown in Figure II-1. If Δx is allowed to approach zero, the following two equations describe the voltage and current along the line:

$$\frac{\partial v(x,t)}{\partial x} = v_S(x,t) + \mu v(x,t) - ri(x,t) - l\,\frac{\partial i(x,t)}{\partial t} \qquad (1)$$

$$\frac{\partial i(x,t)}{\partial x} = i_S(x,t) + \rho i(x,t) - gv(x,t) - c\,\frac{\partial v(x,t)}{\partial t}\,. \qquad (2)$$

The symbols used in Figure II-1 and Eqs. (1) and (2) are defined below:

$v(x,t)$ - the transverse voltage at position x at time t

$i(x,t)$ - the longitudinal current at position x at time t

$v_S(x,t)$ - the incremental longitudinal independent voltage source value at position x at time t

$i_S(x,t)$ - the incremental transverse independent current source value at position x at time t

r - the longitudinal resistance per unit length

l - the longitudinal inductance per unit length

g - the transverse conductance per unit length

c - the transverse capacitance per unit length

μ - the dimensionless proportionality constant relating the incremental longitudinal dependent voltage source value at position x and time t to the transverse voltage at position x and time t

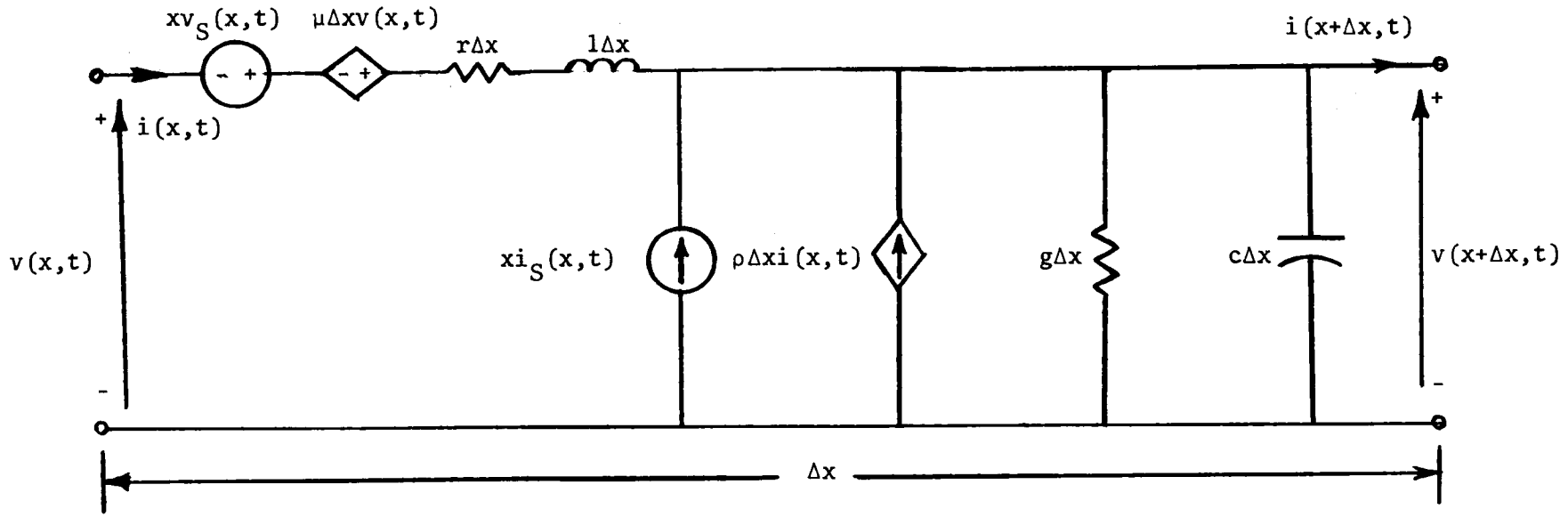


Figure II-1. Distributed source transmission line model.

ρ - the dimensionless proportionality constant relating the incremental transverse dependent current course value at position x and time t to the longitudinal current at position x and time t.

Suppressing the functional arguments, x and t, in Eqs. (1) and 2), and using the notation,

$$\frac{\partial f(x,t)}{\partial x} = f_x \quad \text{and} \quad \frac{\partial f(x,t)}{\partial t} = f_t \ ,$$

Eqs. (1) and (2) may be compactly rewritten as,

$$v_x = v_S + \mu v - ri - li_t \tag{3}$$

$$i_x = i_S + \rho i - gv - cv_t \ . \tag{4}$$

These equations are two coupled first order, linear, constant coefficient, inhomogeneous, partial differential equations.

A standard technique for solution of the homogeneous equations which are written for uniform lines without independent sources is the use of a Laplace transformation of the time variable to obtain ordinary differential equations. These can then be solved by classical techniques. An extension of this technique to the inhomogeneous case is accomplished by the use of Green's functions[3].

Laplace transforming the time variable in (1) and (2) yields these equations:

$$\frac{\partial V(x,s)}{\partial x} = V_S(x,s) + \mu V(x,s) - rI(x,s) - slI(x,s) \tag{5}$$

$$\frac{\partial I(x,s)}{\partial x} = I_S(x,s) + \rho I(x,s) - gV(x,s) - scV(x,s) \ . \qquad (6)$$

In (5) and (6),

$$V(x,s) = L[v(x,t)] \qquad (7)$$

$$I(x,s) = L[i(x,t)] \qquad (8)$$

$$V_S(x,s) = L[v_S(x,t)] \qquad (9)$$

$$I_S(x,s) = L[i_S(x,t)] \ , \qquad (10)$$

and the operator, L, indicates Laplace transformation. All initial conditions are assumed to be zero. Rewriting (5) and (6) in compact notation,

$$V_x = V_S + \mu V - rI - slI \qquad (11)$$

$$I_x = I_S + \rho I - gV - scV \ . \qquad (12)$$

The usual notation of,

$$Z(s) = r + ls \qquad (13)$$

$$Y(s) = g + cs \ , \qquad (14)$$

will be used here to produce these equations,

$$V_x = V_S + \mu V - ZI \qquad (15)$$

$$I_x = I_S + \rho I - YV \ . \qquad (16)$$

Equations (15) and (16) form the starting point in the process of applying the Green's function method of solution. Conceptually, the Green's function method consists of considering each driving function, V_S or I_S, as being a sum of impulses whose weights vary according to the spatial variation of the driving function. When these impulses are increased in number and more densely spaced, their distribution approaches, in the limit, the driving function. Then the principle of superposition in linear systems is invoked and the sum of the responses to the individual impulses is the response to the driving function. In the limit as the number of impulses approaches infinity, the sum approaches an integral. The specific steps in the solution to follow are as follows:

1. Solve the equations (15) and (16) without the driving functions to obtain the complementary solution form.
2. Solve (15) and (16) with a point current source at one specific location, x_0, on the line. This solution will be the Green's function for the case of a distributed current source only.
3. Solve (15) and (16) with a point voltage source at the point, x_0. This solution will be the Green's function for the case of a distributed voltage source only.
4. The total solution is then formed by summing all the impulse responses in an integral over the length of the line including the Green's functions for both voltage and current.

The first step is to solve these equations:

$$V_x - \mu V + ZI = 0 \qquad (17)$$

$$I_x - \rho I + YV = 0 \ . \tag{18}$$

These two equations may be combined into one equation in terms of voltage only:

$$V_{xx} - (\rho + \mu) V_x + (\rho\mu - YZ) V = 0 \ . \tag{19}$$

Making the substitutions,

$$-(\rho + \mu) = \lambda \tag{20}$$

and

$$YZ - \rho\mu = \gamma^2 \ , \tag{21}$$

the complementary solution is,

$$V(x,s) = A(s)e^{a_1 x} + B(s)e^{a_2 x} \ , \tag{22}$$

where

$$a_1 = \frac{-\lambda + \sqrt{\lambda^2 + 4\gamma^2}}{2} \ , \tag{23}$$

and

$$a_2 = \frac{-\gamma - \sqrt{\lambda^2 + 4\gamma^2}}{2} \ . \tag{24}$$

The second step toward a complete solution is to obtain the solutions to a line excited by a point source of voltage and by a point source of current. The point voltage and current sources can be represented mathematically by Dirac delta functions or impulses in one-dimensional space:

$$V_S(x,s) = f^V(s)\delta(x-x_0) \tag{25}$$

$$I_S(x,s) = f^I(s)\delta(x-x_0) \ . \tag{26}$$

Here the superscripts, V and I, refer to voltage and current, respectively. These equations are a mathematical way of saying that the voltage or current source is infinitely densely distributed at x_0 and has no density of distribution elsewhere. The strength of the spatial impulse if f^V or f^I. These are the Laplace transforms of the time variations. Since (15) and (16) are linear, the response of the line to a combination of point voltage and current sources is the sum of the responses to the individual sources. In the discussion to follow, V^V and V^I are the voltage responses to a point voltage source and a point current source, respectively.

For the case of a point current source only,

$$V_x^I - \mu V^I + ZI^I = 0 \tag{27}$$

and

$$I_x^I - \rho I^I + YV^I = f^I\delta(x-x_0) \ . \tag{28}$$

Combining (27) and (28) into one equation in V^I,

$$V_{xx}^I + \lambda V_x^I - \gamma^2 V^I = -Zf^I\delta(x-x_0) \ . \tag{29}$$

For this second order equation, two boundary conditions will be required to solve for the constants in the complementary solution. These conditions are obtained from the terminations at the two ends of the line. For the sake of generality an arbitrarily left and right end termination

will be assumed (Figure II-2). Also the line length, D, will be left general. The impedances Z_1 and Z_2, can be any function of s. Boundary conditions are then described by,

$$\frac{- V^I(0,s,x_0)}{I^I(0,s,x_0)} = Z_1(s) \ , \tag{30}$$

and

$$\frac{V^I(D,s,x_0)}{I^I(D,s,x_0)} = Z_2(s) \ . \tag{31}$$

The point source position, x_0, is included as a functional argument because the voltages and currents depend on the position of the source. The point source divides the line into two parts. In each part this equation holds:

$$V^I_{xx} + \lambda V^I_x - \gamma^2 V^I = 0 \ . \tag{32}$$

The solution, from (22), is,

$$V^I(x,s,x_0) = A(s,x_0)\, e^{a_1 x} + B(s,x_0)\, e^{a_2 x} \tag{33}$$

where $0 \leq x \leq x_0$.

In the region, $x_0 \leq x \leq D$,

$$V^I(x,s,x_0) = C(s,x_0)\, e^{a_1 x} + E(s,x_0)\, e^{a_2 x} \ . \tag{34}$$

In (33) and (34), A, B, C, and E must be chosen to satisfy the boundary conditions and voltage and current continuity conditions at $x = x_0$.

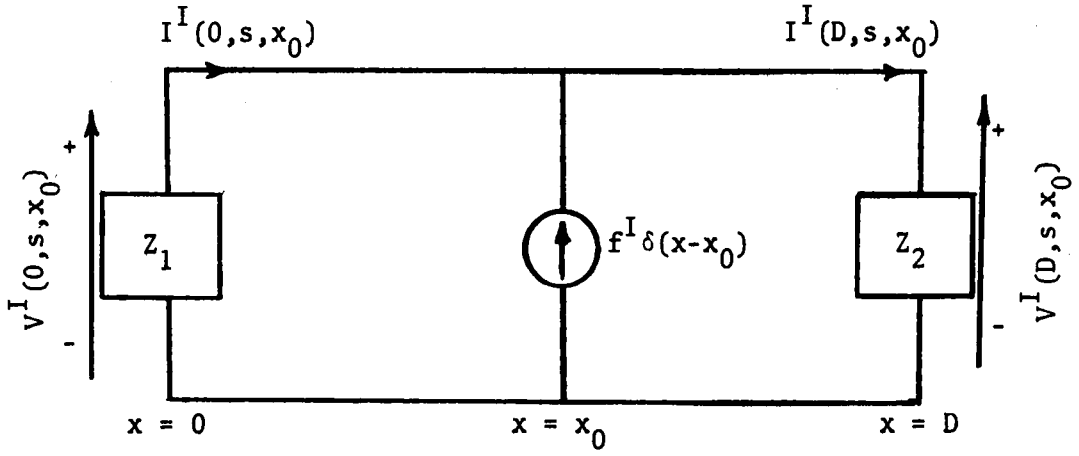


Figure II-2. Transmission line with terminations.

Fitting the left end boundary condition (at x = 0) one obtains,

$$A(s,x_0)\left[1 + \frac{Z_1}{Z}(\mu - a_1)\right] = -B(s,x_0)\left[1 + \frac{Z_1}{Z}(\mu - a_2)\right] . \quad (35)$$

Fitting the right end boundary condition (at x = D) one obtains,

$$C(s,x_0)\, e^{a_1 D}\left[1 - \frac{Z_2}{Z}(\mu - a_1)\right] = E(s,x_0)\, e^{a_2 D}\left[\frac{Z_2}{Z}(\mu - a_2) - 1\right] . \quad (36)$$

The fact that the voltage must be continuous at $x = x_0$ imposes this relation among A, B, C, and E:

$$A(s,x_0)\, e^{a_1 x_0} + B(s,x_0)\, e^{a_2 x_0} = C(s,x_0)\, e^{a_1 x_0} + E(s,x_0)\, e^{a_2 x_0} . \quad (37)$$

The current is discontinuous at $x = x_0$ because of the presence of the point current source. Summing currents at this point yields this equation:

$$\lim_{\varepsilon \to 0}\left[I^I(x_0 - \varepsilon, s, x_0) - I^I(x_0 + \varepsilon, s, x_0)\right] = -f^I(s) . \quad (38)$$

This relationship produces the following equation relating A, B, C, and E:

$$e^{a_1 x_0}(\mu - a_1)\left[A - C\right] + e^{a_2 x_0}(\mu - a_2)\left[B - E\right] = -Zf^I(s) . \quad (39)$$

The four equations (35), (36), (37), and (39) are sufficient to determine A, B, C, and E in any specific problem. Once these are determined, the solution, V^I, may be written in the following form:

$$V^I = \begin{cases} A\, e^{a_1 x} + B\, e^{a_2 x}, & 0 \le x \le x_0 \\ C\, e^{a_1 x} + E\, e^{a_2 x}, & x_0 \le x \le D \end{cases} . \tag{40}$$

Using the relationship between V^I and I^I in (27) the following expression for I^I can be derived:

$$I^I = \frac{1}{Z} \begin{cases} A\, e^{a_1 x} (\mu - a_1) + B\, e^{a_2 x} (\mu - a_2), & 0 \le x < x_0 \\ C\, e^{a_1 x} (\mu - a_1) + E\, e^{a_2 x} (\mu - a_2), & x_0 < x \le D \end{cases} . \tag{41}$$

Following a development dual to that used for the point current source, the following equations can be derived for the case of a point voltage source:

$$I^V = \begin{cases} F\, e^{a_1 x} + G\, e^{a_2 x}, & 0 \le x \le x_0 \\ H\, e^{a_1 x} + J\, e^{a_2 x}, & x_0 \le x \le D \end{cases} . \tag{42}$$

In (42) F, G, H, and J are related by the following four equations:

$$F\left(1 + \frac{\rho - a_1}{Z_1 Y}\right) = -G\left(1 + \frac{\rho - a_2}{Z_1 Y}\right) \tag{43}$$

$$H\, e^{a_1 D}\left(1 - \frac{\rho - a_1}{Z_2 Y}\right) = -J\left(1 - \frac{\rho - a_2}{Z_2 Y}\right) e^{a_2 D} \tag{44}$$

$$F\, e^{a_1 x_0} + G\, e^{a_2 x_0} = H\, e^{a_1 x_0} + J\, e^{a_2 x_0} \tag{45}$$

$$e^{a_1 x_0}(\rho - a_1)(F - H) + e^{a_2 x_0}(\rho - a_2)(G - J) = -Yf^V(s) \quad . \tag{46}$$

Also,

$$V^V = \frac{1}{Y}\begin{cases} F\, e^{a_1 x}(\rho - a_1) + G\, e^{a_2 x}(\rho - a_2) \ , \ 0 \leq x < x_0 \\ \\ H\, e^{a_1 x}(\rho - a_1) + J\, e^{a_2 x}(\rho - a_2) \ , \ x_0 < x \leq D \end{cases} \quad . \tag{47}$$

The total solution in the frequency domain may now be written as in the following superposition integrals:

$$V(x,s) = \int_0^D \left[h^V(x_0) V^V(x,s,x_0) + h^I(x_0) V^I(x,s,x_0) \right] dx_0 \ , \tag{48}$$

$$I(x,s) = \int_0^D \left[h^V(x_0) I^V(x,s,x_0) + h^I(x_0) I^I(x,s,x_0) \right] dx_0 \ . \tag{49}$$

In these integrals, $h^V(x)$ and $h^I(x)$, describe the spatial variation of the distributed voltage and current sources, respectively,

These solutions, (48) and (49), are limited to uniform, time-invariant lines, but have the advantage of being exact and in closed form. To extend the analysis to nonuniform and/or time-variant lines, one must resort to numerical methods unless the nonuniformity is of one of the four types which can be solved in closed form. These numerical methods will be discussed in Chapter III.

The solution integrals, (48) and (49), have such great implicit generality that it is hard to gain an intuitive feel for their

significance. In the discussion to follow they will be explored and progressively narrowed to more familiar forms in transmission line theory.

Recall the definitions of a_1 and a_2 from (23) and (24:

$$a_1 = \frac{-\lambda + \sqrt{\lambda^2 + 4\gamma^2}}{2} \tag{50}$$

$$a_2 = \frac{-\lambda - \sqrt{\lambda^2 + 4\gamma^2}}{2} . \tag{51}$$

In the special case of a transmission line without dependent sources, λ is zero and a_1 and a_2 reduce to,

$$a_1 = \gamma \text{ and } a_2 = -\gamma. \tag{52}$$

The quantities, γ and $-\gamma$, are commonly given the name propagation constant. Therefore, a_1 and a_2 could be called the more general form of the propagation constant. It is then apparent that the presence of dependent sources has a significant effect on the natural response of a transmission line. If one now narrows the solutions down to the special case of transmission lines without dependent sources he obtains the following solutions:

$$V^I = \begin{cases} A \sinh \gamma x + B \cosh \gamma x, & 0 \leq x \leq x_0 \\ C \sinh \gamma x + E \cosh \gamma x, & x_0 \leq x \leq D \end{cases} \tag{53}$$

$$I^I = \frac{-1}{Z_0} \begin{cases} A \cosh \gamma x + B \sinh \gamma x, & 0 \leq x < x_0 \\ C \cosh \gamma x + E \sinh \gamma x, & x_0 < x \leq D \end{cases} \tag{54}$$

$$I^V = \begin{cases} F \sinh \gamma x + G \cosh \gamma x, & 0 \leq x \leq x_0 \\ \\ H \sinh \gamma x + J \cosh \gamma x, & x_0 \leq x \leq D \end{cases} \tag{55}$$

$$V^V = -Z_0 \begin{cases} F \cosh x + G \sinh \gamma x, & 0 \leq x < x_0 \\ \\ H \cosh \gamma x + J \sinh \gamma x, & x_0 < x \leq D \end{cases} . \tag{56}$$

The symbol, Z_0, is given the name characteristic impedance and is defined by,

$$Z_0 = \sqrt{\frac{Z}{Y}} . \tag{57}$$

The relationships among A, B, C, and E and F, G, H, and J are given by the following eight equations:

$$AZ_1 = BZ_0 \tag{58}$$

$$(Z_0E + Z_2C) \cosh \gamma D = -(Z_0C + Z_2E) \sinh \gamma D \tag{59}$$

$$(B - E) \cosh \gamma x_0 = (C - A) \sinh \gamma x_0 \tag{60}$$

$$(B - E) \sinh \gamma x_0 + (A - C) \cosh \gamma x_0 = Z_0 f^I(s) \tag{61}$$

$$Z_0F = Z_1G \tag{62}$$

$$(Z_2J + Z_0H) \cosh \gamma D = -(Z_0J + Z_2H) \sinh \gamma D \tag{63}$$

$$(F - H) \sinh \gamma x_0 = (J - G) \cosh \gamma x_0 \tag{64}$$

$$(F - H) \cosh \gamma x_0 + (G - J) \sinh \gamma x_0 = \frac{f^V(s)}{Z_0} . \tag{65}$$

These point source responses are noticeably simpler than before but the character of the solutions is still obscure. The next simplification, to a matched line, will remove this obscurity. For a matched line,

$$Z_0 = Z_1 = Z_2 . \tag{66}$$

The point source responses now become the following:

$$V^I = \frac{Z_0}{2} f^I(s) \begin{cases} e^{\gamma(x - x_0)}, & 0 \leq x \leq x_0 \\ e^{\gamma(x_0 - x)}, & x_0 \leq x \leq D \end{cases} \tag{67}$$

$$I^I = \frac{1}{2} f^I(s) \begin{cases} -e^{\gamma(x - x_0)}, & 0 \leq x < x_0 \\ +e^{\gamma(x_0 - x)}, & x_0 < x \leq D \end{cases} \tag{68}$$

$$I^V = \frac{1}{2Z_0} f^V(s) \begin{cases} e^{\gamma(x - x_0)}, & 0 \leq x \leq x_0 \\ e^{\gamma(x_0 - x)}, & x_0 \leq x \leq D \end{cases} \tag{69}$$

$$V^V = \frac{1}{2} f^V(s) \begin{cases} -e^{\gamma(x - x_0)}, & 0 \leq x < x_0 \\ +e^{\gamma(x_0 - x)}, & x_0 < x \leq D \end{cases} . \qquad (70)$$

These point source responses are now still simpler and in explicit form. They have been expressed here in travelling wave form. For example, in V^I, one can see that there are two travelling waves. In the region to the left of the point current source, there is a left propagating wave and in the region to the right of the point current source there is a right propagating wave. Also notice the coefficient of both travelling waves. The characteristic impedance is there because the voltage is in response to an applied current source, hence the response is a trans-impedance. The division by two occurs because the current from the point current source splits in half; each half propagating in a different direction. The $f^I(s)$ is a coefficient because the propagating waveform is the same as the applied current waveform except that it is modified by the characteristic impedance and propagation constant. For the case of a lossless line the characteristic impedance is a constant and the propagation constant introduces pure delay so that the waveform shape does not change as it travels down the line. For the case of an r-c line the characteristic impedance is a function of s and the propagation constant introduces diffusion instead of delay so the waveform is significantly distorted as it propagates down the line.

A special case will now be examined to serve as an example of a filter design technique which will be expanded upon in Chapter V.

Consider a lossless matched line with a distributed independent current source. First let the current source distribution be a train of impulses in space and let all the time responses of these spatial impulses be impulses also. Then the voltage response at each end of the line could be called the impulse response because the time variations are all impulses. The impulse response at the left end, for example, would be an impulse train in time which looks exactly like the current source distribution in space looks. The impulse response at the right end would look like the time inverse of the left end impulse response. Now let the distribution become continuous space. The left end impulse response will now be described exactly by the spatial variation of the current source, modified by a constant multiplier perhaps. Since there is a one to one correspondence between impulse response and frequency response through the Fourier transform, this property just described leads to the thought of time domain filter synthesis. All one needs to do is to choose the desired frequency response, take the inverse Fourier transform of it, and directly synthesize the impulse response which results. There are limitations on this method. Some desired frequency responses have noncausal impulse responses. These would have to be truncated at t equals zero to make them causal. As will be shown in Chapter V, this makes the frequency response somewhat different. Also, the impulse response will be of finite duration in time. This also introduces a distortion in the frequency response if the desired impulse response is of infinite time duration.

This design idea is based on a matched lossless line. If one now tries to extend the idea to other types of lines he quickly runs into

difficulty. If the line is lossless but not matched, he has to contend with reflections in the impulse response. This is a way of making the impulse response of infinite time duration, however. If the line is matched but not lossless, he now has the problem of diffusion to contend with. The impulses which propagate only a short distance diffuse only slightly; the ones which propagate longer diffuse more. Deciding how to modify the current source distribution to correct for this behavior is a difficult mathematical problem. Another problem is in realizing a distributed current source, or even a lumped approximation, which can be controlled to give the desired spatial variation. All these problems will be considered in more detail in Chapter V.

CHAPTER III

NUMERICAL SOLUTION METHODS

In this chapter two numerical approaches to the solution of general two-wire transmission line problems, each with its own limitations and advantages, will be presented. The first is a phasor or frequency domain method. This method will be limited to time-invariant lines, but has the advantage of easy programming and high speed on a computer. The second method is a direct numerical solution of the basic partial differential equations describing the transmission line. It can, in principle at least, find an approximate solution to any time-variant, nonuniform transmission line problem. Its disadvantages are that it is limited in the type of terminations that can be put on the line and it requires more storage and time on a computer than the frequency domain method.

The Frequency Domain Method

The incremental model of a uniform, two-wire transmission line with uniformly distributed independent sources, in phasor form, is shown in Figure III-1. If Δx is allowed to approach zero, these equations can be derived:

$$V_x(x,\omega) = V_S(\omega) + \mu(\omega)V(x,\omega) - Z(\omega)I(x,\omega) \qquad (1)$$

$$I_x(x,\omega) = I_S(\omega) + \rho(\omega)I(x,\omega) - Y(\omega)V(x,\omega) \ . \qquad (2)$$

The symbols used in Figure III-1 and Eqs. (1) and (2) are defined below:

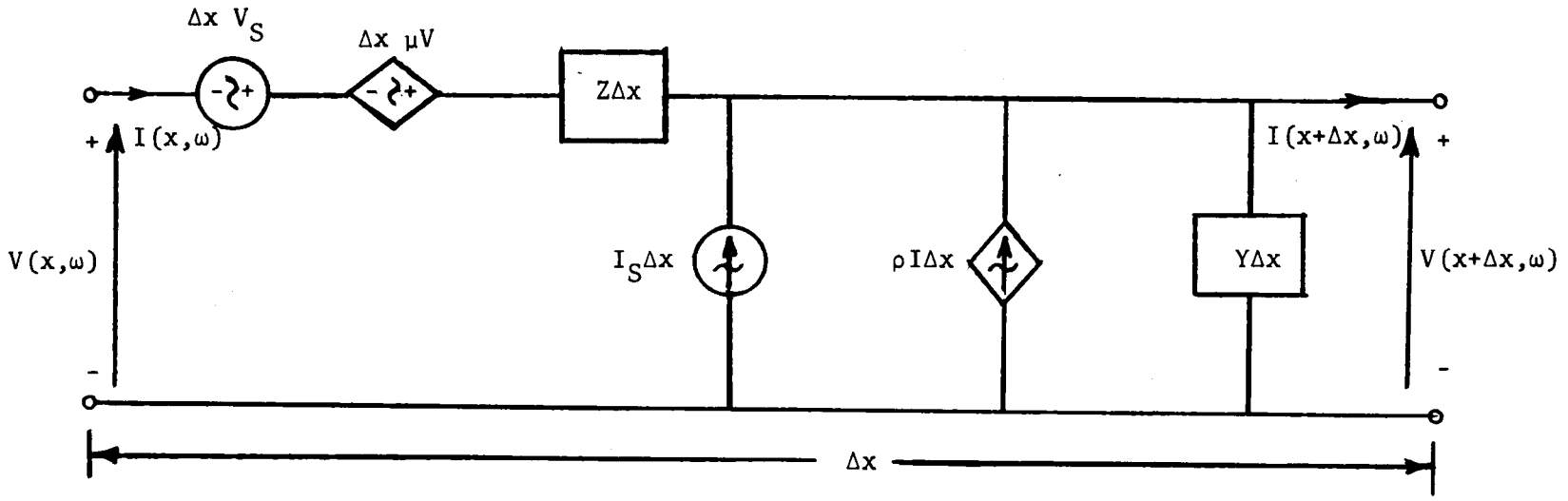


Figure III-1. Frequency domain transmission line model.

$V(x,\omega)$ - the transverse phasor voltage at position x at angular frequency ω.

$I(x,\omega)$ - the longitudinal phasor current at position x at angular frequency ω.

$V_S(\omega)$ - the incremental longitudinal independent voltage source value at angular frequency ω.

$I_S(\omega)$ - the incremental transverse independent current source value at angular frequency ω.

$Z(\omega)$ - the longitudinal impedance per unit length at angular frequency ω.

$Y(\omega)$ - the transverse admittance per unit length at angular frequency ω.

$\mu(\omega)$ - the dimensionless proportionality constant relating the incremental longitudinal dependent voltage source value at position x and angular frequency ω to the transverse voltage at position x and angular frequency ω.

$\rho(\omega)$ - the dimensionless proportionality constant relating the incremental transverse dependent current source value at position x and angular frequency ω to the longitudinal current at position x and angular frequency ω.

Equations (1) and (2) relate the voltage across and current through a uniform transmission line with uniform independent current and voltage sources. To solve for the voltage across and current through a nonuniform line with arbitrary source distributions, one can approximate the line with short sections of line which are cascade connected. The cascade parameters of the sections can be successively solved for and successively

multiplied, progressing from the left to the right end of the line. Then the end conditions can be considered and the solution for the voltages and currents at a single frequency can be solved for by solving a 4 by 4 system of equations. This process can then be repeated at successively higher frequencies to obtain a frequency response graph. This sampled form of the frequency response can then, by use of the digital fourier transform (fast fourier transform), be converted to an impulse response.

Solving for the cascade parameters is a matter of rearranging the equations (1) and (2) into a form in which the right end voltage and current are expressed in terms of the left end voltage and current. To make the solution process general one must allow for the possibility that one or more of the parameters in these equations may be zero all along or only in some region of the transmission line. If one considers all possibilities there are 18 different sets of cascade parameters possible.

Consider the most general possibility:

$$V_x = V_S + \mu V - ZI \tag{3}$$

$$I_x = I_S + \rho I - YV \tag{4}$$

where Z and Y are nonzero.

Combining (3) and (4) into one equation in V, one obtains this equation:

$$V_{xx} + \lambda V_x - \gamma^2 V = -ZI_S - \rho V_S \ . \tag{5}$$

In (5),

$$\lambda = -(\mu + \rho) \quad \text{and} \quad \gamma^2 = YZ - \rho\mu \ . \tag{6}$$

At a constant frequency, equation (5) is a second order, linear, constant coefficient, ordinary inhomogeneous differential equation. The solution form depends upon whether λ or γ^2 or both are zero. To keep the solution most general they will be assumed nonzero. The solution is then given by,

$$V = A\, e^{a_1 x} + B\, e^{a_2 x} - \frac{(ZI_S + \rho V_S)}{\gamma^2}, \tag{7}$$

where

$$a_1 = \frac{-\lambda + \sqrt{\lambda^2 + 4\gamma^2}}{2} \quad \text{and} \quad a_2 = \frac{-\lambda - \sqrt{\lambda^2 + 4\gamma^2}}{2}. \tag{8}$$

Using the left end voltage and current as boundary conditions one can now solve for the voltage and current at any point on the line in terms of the left end voltage and current. These solutions are given in the following equations:

$$V(x) = k_{11}V(0) + k_{12}I(0) + u_V \tag{9}$$

$$I(x) = k_{21}V(0) + k_{22}I(0) + u_I . \tag{10}$$

In (9) and (10),

$$k_{11} = \frac{(a_2 - \mu)e^{a_1 x} - (a_1 - \mu)\, e^{a_2 x}}{a_2 - a_1} \tag{11}$$

$$k_{12} = \frac{Z}{a_2 - a_1}\,(e^{a_1 x} - e^{a_2 x}) \tag{12}$$

$$k_{21} = \frac{(\mu - a_1)(a_2 - \mu)(e^{a_1 x} - e^{a_2 x})}{Z(a_2 - a_1)} \tag{13}$$

$$k_{22} = \frac{(\mu - a_1)\, e^{a_1 x} - (\mu - a_2)\, e^{a_2 x}}{a_2 - a_1} \tag{14}$$

$$u_V = \frac{(V_S + a_1 H)\, e^{a_2 x} - (V_S + a_2 H)\, e^{a_1 x}}{a_2 - a_1} + H \tag{15}$$

$$u_I = \frac{1}{Z}\left[V_S + \mu H + \frac{(V_S + a_1 H)(\mu - a_2)\, e^{a_2 x} - (V_S + a_2 H)(\mu - a_1)\, e^{a_1 x}}{a_2 - a_1}\right], \tag{16}$$

and

$$H = \frac{ZI_S + \rho V_S}{\gamma^2}. \tag{17}$$

Notice that these equations are valid only for a_2 not equal to a_1. Equations (9) and (10) can be rewritten in this matrix form:

$$\begin{bmatrix} V(x) \\ I(x) \end{bmatrix} = \begin{bmatrix} k_{11} & k_{12} \\ k_{21} & k_{22} \end{bmatrix} \begin{bmatrix} V(0) \\ I(0) \end{bmatrix} + \begin{bmatrix} u_V \\ u_I \end{bmatrix}. \tag{18}$$

This form is similar to the ABCD parameters of two-port network theory[4]. The differences between this cascade parameter form and the ABCD parameters are that I(0) is positive and that a forcing function column vector has been added. The convenient property of the ABCD parameters was that they can be multiplied to yield the overall cascade parameters of two or more

cascaded two-port networks. The same is true of these new cascade parameters. A diagram of two cascaded sections of transmission line with distributed sources is shown in Figure III-2. The following equations can be written describing the relationships among the terminal voltages and currents:

$$\begin{bmatrix} V_2 \\ I_2 \end{bmatrix} = \begin{bmatrix} (k_{11})_1 & (k_{12})_1 \\ (k_{21})_1 & (k_{22})_1 \end{bmatrix} \begin{bmatrix} V_1 \\ I_1 \end{bmatrix} + \begin{bmatrix} u_{V1} \\ u_{I1} \end{bmatrix} \tag{19}$$

$$\begin{bmatrix} V_3 \\ I_3 \end{bmatrix} = \begin{bmatrix} (k_{11})_2 & (k_{12})_2 \\ (k_{21})_2 & (k_{22})_2 \end{bmatrix} \begin{bmatrix} V_2 \\ I_2 \end{bmatrix} + \begin{bmatrix} u_{V2} \\ u_{I2} \end{bmatrix} \tag{20}$$

$$\begin{bmatrix} V_3 \\ I_3 \end{bmatrix} = \begin{bmatrix} k_{11} & k_{12} \\ k_{21} & k_{22} \end{bmatrix} \begin{bmatrix} V_1 \\ I_1 \end{bmatrix} + \begin{bmatrix} u_V \\ u_I \end{bmatrix} \tag{21}$$

where

$$\begin{bmatrix} k_{11} & k_{12} \\ k_{21} & k_{22} \end{bmatrix} = \begin{bmatrix} (k_{11})_2 & (k_{12})_2 \\ (k_{21})_2 & (k_{22})_2 \end{bmatrix} \begin{bmatrix} (k_{11})_1 & (k_{12})_1 \\ (k_{21})_1 & (k_{22})_1 \end{bmatrix} \tag{22}$$

$$\begin{bmatrix} u_V \\ u_I \end{bmatrix} = \begin{bmatrix} (k_{11})_2 & (k_{12})_2 \\ (k_{21})_2 & (k_{22})_2 \end{bmatrix} \begin{bmatrix} u_{V1} \\ u_{I1} \end{bmatrix} + \begin{bmatrix} u_{V2} \\ u_{I2} \end{bmatrix} . \tag{23}$$

This process can be performed iteratively on a computer until the cascade parameters for the whole transmission line are found. Once they are found, a system of four equations in four unknowns can be written by including the equations describing the terminations.

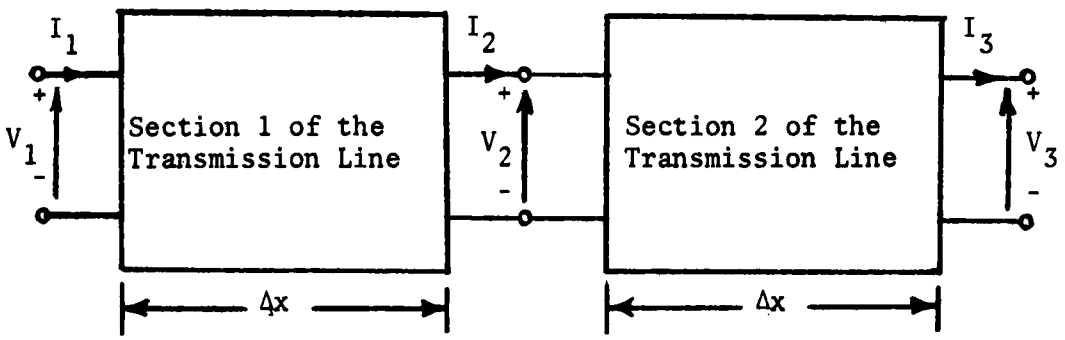


Figure III-2. Two cascaded sections of transmission line.

General left and right end terminations are shown in Figure III-3. The following equations describe the relations among the terminal voltages and currents:

$$V_i = E_L + \mu_L V_i - I_i Z_L \tag{24}$$

$$I_o + \rho_R I_o + I_R = V_o Y_R \ . \tag{25}$$

The cascade parameters for the entire line yield equations of the form,

$$\begin{pmatrix} V_o \\ I_o \end{pmatrix} = \begin{pmatrix} k_{11} & k_{12} \\ k_{21} & k_{22} \end{pmatrix} \begin{pmatrix} V_i \\ I_i \end{pmatrix} + \begin{pmatrix} u_V \\ u_I \end{pmatrix} . \tag{26}$$

Combining (24), (25), and (26), the following equation may be written:

$$\begin{pmatrix} -k_{11} & -k_{12} & 1 & 0 \\ -k_{21} & -k_{22} & 0 & 1 \\ (1 - \mu_L) & Z_L & 0 & 0 \\ 0 & 0 & -Y_R & (1 + \rho_R) \end{pmatrix} \begin{pmatrix} V_i \\ I_i \\ V_o \\ I_o \end{pmatrix} = \begin{pmatrix} u_V \\ u_I \\ E_L \\ -I_R \end{pmatrix} \tag{27}$$

Therefore at any single frequency an equation like (27) may be solved to obtain the left and right end voltage and current.

The cascade parameters for each of the special cases are derived in Appendix A. The conditions for each special case are listed below:

I. $Z = 0, Y \neq 0$

1. $\mu \neq 0 = \rho$

2. $\mu = 0 \neq \rho$

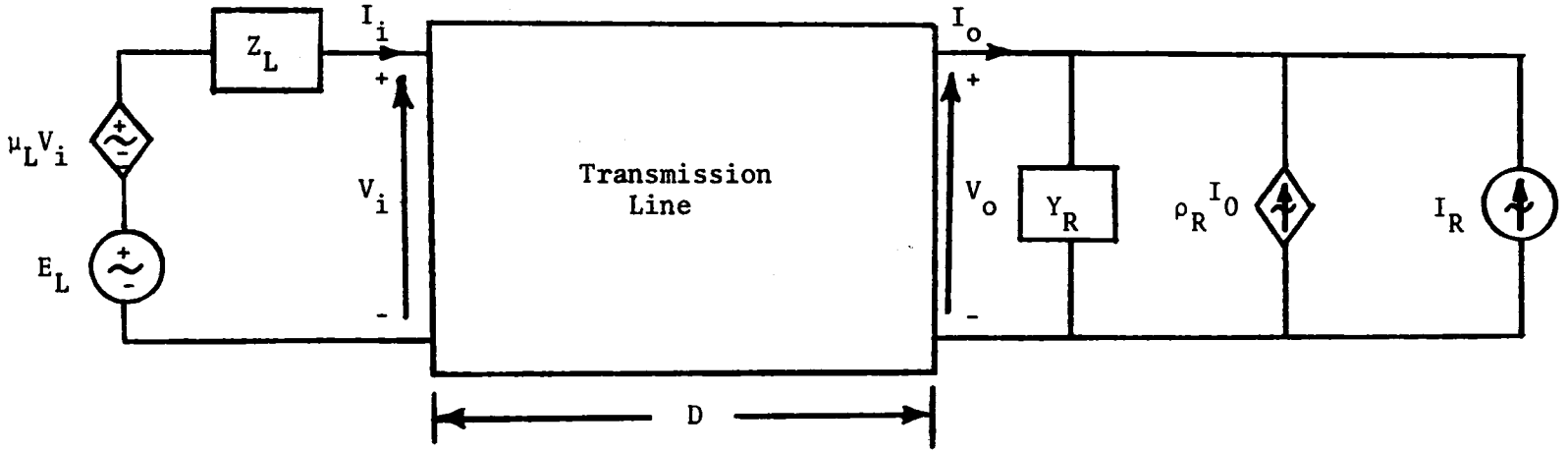


Figure III-3. Transmission line with general terminations.

3. $\mu = 0 = \rho$

4. $\mu = \rho \neq 0$

5. $0 \neq \mu \neq \rho \neq 0$

II. $Z \neq 0, Y = 0$

1. $\mu \neq 0 = \rho$

2. $\mu = 0 \neq \rho$

3. $\mu = 0 = \rho$

4. $\mu = \rho \neq 0$

5. $0 \neq \mu \neq \rho \neq 0$

III. $Z = Y = 0$

1. $\mu \neq 0 = \rho$

2. $\mu = 0 \neq \rho$

3. $\mu = 0 = \rho$

4. $0 \neq \mu \neq \rho \neq 0$

IV. $Z \neq 0, Y \neq 0$

1. $\lambda \neq 0 = \gamma^2$

2. $\gamma^2 = \lambda = 0$

3. $\gamma^2 \neq 0, \lambda \neq 0$

   a. $a_1 \neq a_2$

   b. $a_1 = a_2$

The Time Domain Method

The time domain method consists of a direct numerical solution to the partial differential equations describing the relation between voltage and current in an incremental section[5]. From Chapter II, Equations (3) and (4),

$$v_x = v_s + \mu v - ri - li_t \tag{28}$$

$$i_x = i_s + \rho i - gv - cv_t \ . \tag{29}$$

In Chapter II, the line considered was uniform. That is, the parameters, μ, ρ, r, l, g, and c, were constants. In this numerical method to be described these parameters may be functions of both position and time. Equations (28) and (29) can be rewritten in the following form:

$$\underline{f}_x = [A]\ \underline{f} + [B]\ \underline{f}_t + \underline{u} \ . \tag{30}$$

In (30),

$$\underline{f} = \begin{pmatrix} v \\ i \end{pmatrix} \tag{31}$$

$$\underline{u} = \begin{pmatrix} v_s \\ i_s \end{pmatrix} \tag{32}$$

$$[A] = \begin{pmatrix} \mu & -r \\ -g & \rho \end{pmatrix} \tag{33}$$

$$[B] = \begin{pmatrix} 0 & -1 \\ -c & 0 \end{pmatrix} \ . \tag{34}$$

The approach taken to the solution of (30) is to approximate the functions and derivatives at discrete points in time-position space with finite differences. A typical region in time-position space is shown in Figure III-4. The following symbology will be used to denote the value of a continuous function at a typical discrete point:

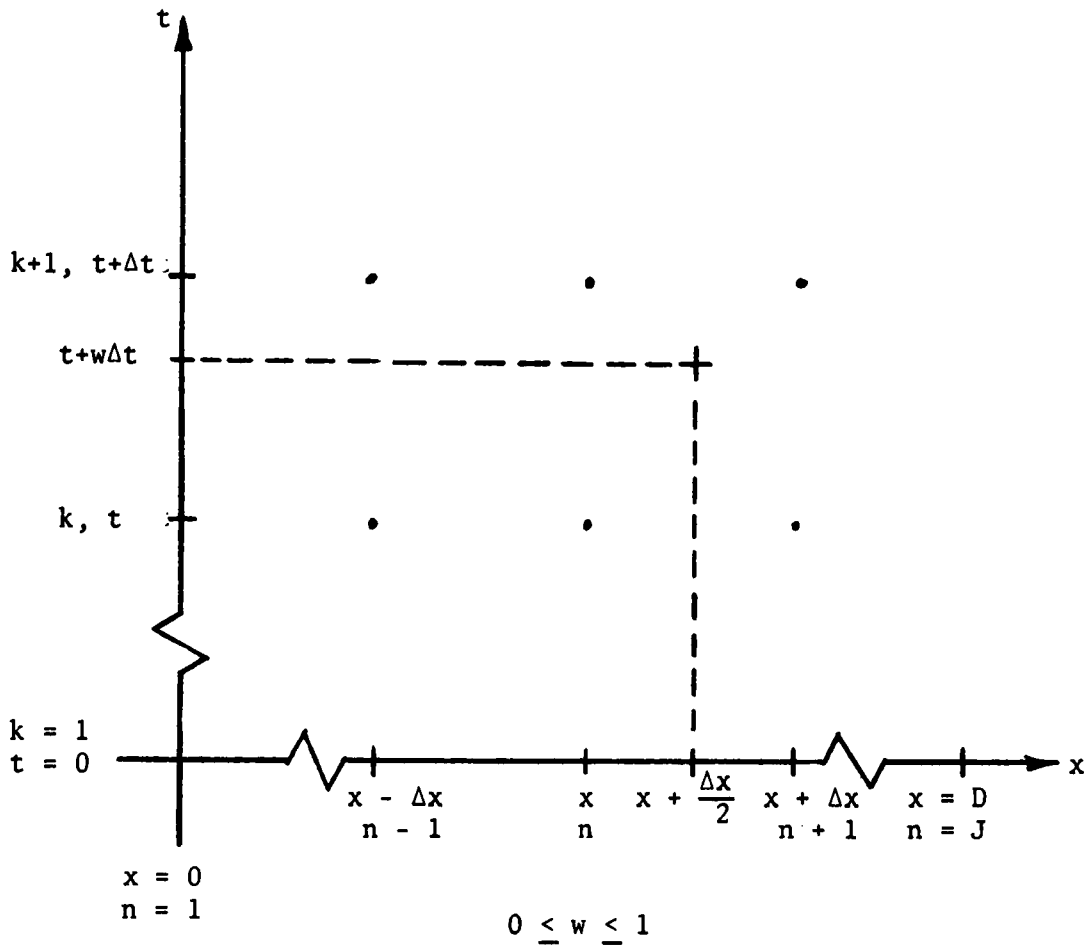


Figure III-4. Typical region in time-position space.

$$h(x,t) = h[(n-1)\Delta x,(k-1)\Delta t] = h_{n,k} \ . \tag{35}$$

A function or its derivative can be approximated at points other than those corresponding to integer values of n and k by the following approximations:

$$h_x(x^+,t^+) \cong \frac{w}{\Delta x}\left[h_{n+1,k+1} - h_{n,k+1}\right] + \frac{1-w}{\Delta x}\left[h_{n+1,k} - h_{n,k}\right] , \tag{36}$$

$$h(x^+,t^+) \cong \frac{w}{2}\left[h_{n+1,k+1} + h_{n,k+1}\right] + \frac{1-w}{2}\left[h_{n+1,k} + h_{n,k}\right] , \tag{37}$$

$$h_t(x^+,t^+) \cong \frac{1}{2\Delta t}\left[h_{n+1,k+1} + h_{n,k+1} - h_{n+1,k} - h_{n,k}\right] , \tag{38}$$

where

$$x^+ = x + \frac{\Delta x}{2} \quad \text{and} \quad t^+ = t + w\Delta t \ . \tag{39}$$

These approximations, when substituted into (30), yield the following:

$$K_1 \underline{f}_{n+1,k+1} + K_2 \underline{f}_{n,k+1} = K_3 \tag{40}$$

where,

$$K_1 = \frac{w}{\Delta x} - \frac{w}{2}\left[A(x^+,t^+)\right] - \frac{1}{2\Delta t}\left[B(x^+,t^+)\right] \tag{41}$$

$$K_2 = -\frac{w}{\Delta x} - \frac{w}{2}\left[A(x^+,t^+)\right] - \frac{1}{2\Delta t}\left[B(x^+,t^+)\right] \tag{42}$$

$$\begin{aligned} K_3 = {} & \underline{u}(x^+,t^+) + \frac{w-1}{\Delta x}\left[\underline{f}_{n+1,k} - \underline{f}_{n,k}\right] \\ & + \frac{1-w}{2}\left[\left[A(x^+,t^+)\right]\left(\underline{f}_{n+1,k} + \underline{f}_{n,k}\right)\right] \\ & - \frac{1}{2\Delta t}\left[\left[B(x^+,t^+)\right]\left(\underline{f}_{n+1,k} + \underline{f}_{n,k}\right)\right] \ . \end{aligned} \tag{43}$$

Equation (40) is written in the form, $\underline{f}$ at some future time is dependent upon the value of $\underline{f}$ at some previous time. Therefore, if $\underline{f}$ can be described at some time, usually t = 0, iterative application of (40) can yield its values at later times.

Equation (40) applies only to an interior section of the transmission line. The boundary conditions must be included for a complete solution. The forms of the left and right end terminations are shown in Figures III-5 and III-6.

The left end boundary condition is described by this approximate equation:

$$w[\mu_L(t^+) - 1]\, v_{1,k+1} - w[R_L(t^+)]\, i_{1,k+1} = -E_L(t^+)$$

$$+ [(1-w)\, R_L(t^+) i_{1,k} + [1 - \mu_L(t^+)]\, v_{1,k}]. \quad (44)$$

The right end boundary condition is described by this approximate equation:

$$w[1 + \rho_R(t^+)]\, i_{J,k+1} - w[G_R(t^+)]\, v_{J,k+1} = -I_R(t^+)$$

$$+ (w-1)\, \left[[1 + \rho_R(t^+)]\, i_{J,k} - G_R(t^+)\, v_{J,k}\right]. \quad (45)$$

From Figure III-4, the number of sections into which the line is divided is J-1. An equation of the form of (40) can be written at each interior point. There are J-1 interior points at each time level. Each equation of the form of (40) is a matrix equation which actually represents two individual equations. Including the boundary condition

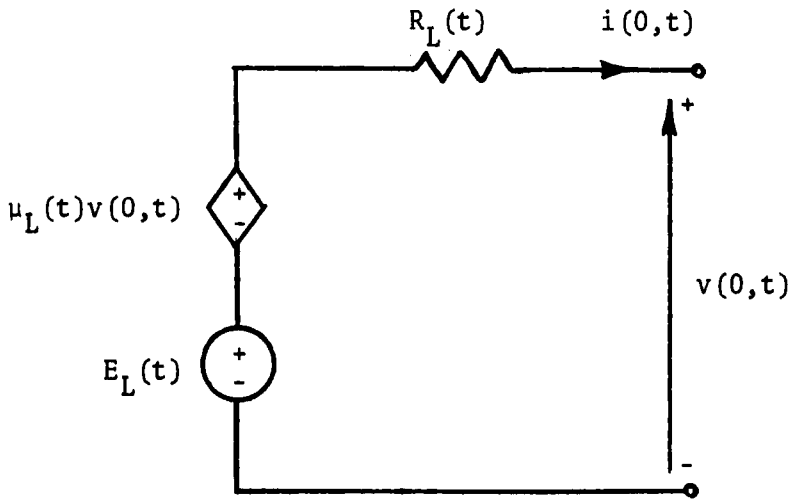


Figure III-5. Left end termination

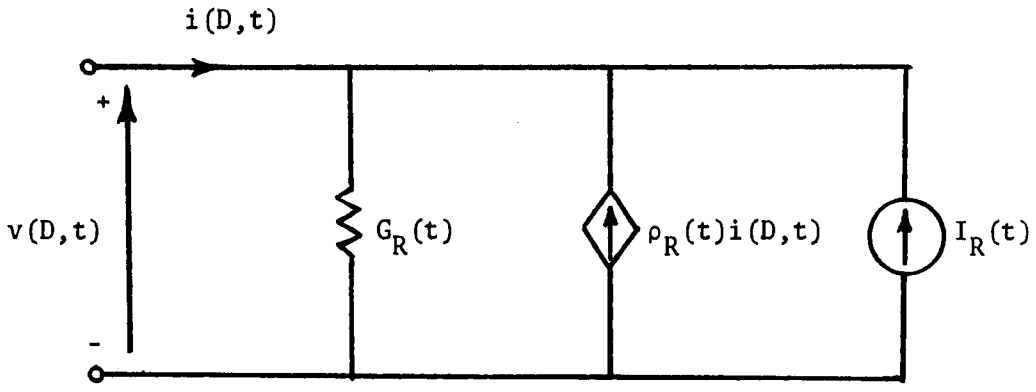


Figure III-6. Right end termination.

equations, this makes a total of 2J equations which can be written at each time level. These can be expressed in the form of one matrix equation as shown in Figure III-7.

The elements in the coefficient matrix and the forcing vector are defined below:

$$K_{1V} = \frac{w}{\Delta x} - \frac{w}{2}\left[\mu(x^+,t^+)\right] \tag{46}$$

$$K_{1I} = \frac{w}{2}\left[r(x^+,t^+)\right] + \frac{1}{2\Delta t}\left[l(x^+,t^+)\right] = K_{2I} \tag{47}$$

$$K_{2V} = -\frac{w}{\Delta x} - \frac{w}{2}\left[\mu(x^+,t^+)\right] \tag{48}$$

$$K_3 = v_S(x^+,t^+) + \frac{w-1}{\Delta x}\left[v_{n+1,k} - v_{n,k}\right] + \frac{1-w}{2}\left[\left[\mu(x^+,t^+)\right]\left[v_{n+1,k} + v_{n,k}\right]\right] + \left[\frac{w-1}{2}\left[r(x^+,t^+)\right] + \frac{1}{2\Delta t}\left[l(x^+,t^+)\right]\right]\left[i_{n+1,k} + i_{n,k}\right]. \tag{49}$$

$$K_{4V} = \frac{w}{2}\left[g(x^+,t^+)\right] + \frac{1}{2\Delta t}\left[c(x^+,t^+)\right] = K_{5V} \tag{50}$$

$$K_{4I} = \frac{w}{\Delta x} - \frac{w}{2}\left[\rho(x^+,t^+)\right] \tag{51}$$

$$K_{5I} = -\frac{w}{\Delta x} - \frac{w}{2}\left[\rho(x^+,t^+)\right] \tag{52}$$

$$
\begin{pmatrix}
K_{7V} & K_{7I} & 0 & 0 & 0 & 0 & 0 \cdots\cdots 0 \\
K_{2V} & K_{2I} & K_{1V} & K_{1I} & 0 & 0 & 0 \cdots\cdots 0 \\
K_{5V} & K_{5I} & K_{4V} & K_{4I} & 0 & 0 & 0 \cdots\cdots 0 \\
0 & 0 & K_{2V} & K_{2I} & K_{1V} & K_{1I} & 0 \cdots\cdots 0 \\
0 & 0 & K_{5V} & K_{5I} & K_{4V} & K_{4I} & 0 \cdots\cdots 0 \\
\vdots & & \ddots & \ddots & \ddots & \ddots & \vdots \\
0 \cdots\cdots 0 & & & K_{2V} & K_{2I} & K_{1V} & K_{1I} \quad 0 \\
0 \cdots\cdots 0 & & & K_{5V} & K_{5I} & K_{4V} & K_{4I} \quad 0 \\
0 \cdots\cdots 0 & & & 0 & 0 & K_{2V} & K_{2I} \quad K_{1V} \quad K_{1I} \\
0 \cdots\cdots 0 & & & 0 & 0 & K_{5V} & K_{5I} \quad K_{4V} \quad K_{4I} \\
0 \cdots\cdots 0 & & & 0 & 0 & 0 & 0 \quad K_{9V} \quad K_{9I}
\end{pmatrix}
\begin{pmatrix} v_1 \\ i_1 \\ v_2 \\ i_2 \\ \vdots \\ v_{J-1} \\ i_{J-1} \\ v_J \\ i_J \end{pmatrix}
=
\begin{pmatrix} K_8 \\ K_3 \\ K_6 \\ K_3 \\ \vdots \\ K_6 \\ K_3 \\ K_6 \\ K_{10} \end{pmatrix}
$$

Figure III-7. Matrix equation at time level, k+1.

$$K_6 = i_S(x^+,t^+) + \frac{w-1}{\Delta x}\left[i_{n+1,k} - i_{n,k}\right]$$

$$+ \frac{1-w}{2}\left[\left[\rho(x^+,t^+)\right]\left[i_{n+1,k} + i_{n,k}\right]\right]$$

$$+ \left[\frac{w-1}{2}\left[g(x^+,t^+)\right] + \frac{1}{2\Delta t}\left[c(x^+,t^+)\right]\right]\left[v_{n+1,k} + v_{n,k}\right] , \qquad (53)$$

$$K_{7V} = w[\mu_L(t^+) - 1] \qquad (54)$$

$$K_{7I} = -w[R_L(t^+)] \qquad (55)$$

$$K_8 = (1-w)\left[R_L(t^+)i_{1,k} + [1 - \mu_L(t^+)]\, v_{1,k}\right] - E_L(t^+) , \qquad (56)$$

$$K_{9V} = -w[G_R(t^+)] \qquad (57)$$

$$K_{9I} = w[1 + \rho_R(t^+)] \qquad (58)$$

$$K_{10} = (w-1)\left[[1 + \rho_R(t^+)]\, i_{J,k} - G_R(t^+)\, v_{J,k}\right] - I_R(t^+) . \qquad (59)$$

This five-diagonal system is solved at each time level for all the voltages and currents and these solutions are used as previous values for the solution at the next time level.

CHAPTER IV

NUMERICAL AND PROGRAMMING TECHNIQUES

Two programs were written, corresponding to the numerical analyses presented in Chapter III. Both were written in FORTRAN IV and tested on an IBM 360 computer. They were written with the thought in mind that they should be versatile and easy to use so that they could be retained by The University of Tennessee Department of Electrical Engineering as standard programs for students and faculty to use in solving transmission line problem. The programs themselves are presented in Appendices B and C.

Much of the programming is straightforward and follows the numerical analysis very directly. This part of the programming will not be discussed since anyone with a working knowledge of FORTRAN IV programming could follow it. Several techniques were used, however, which from a cursory look at the programs would not be obvious. These techniques will be discussed in the text to follow.

Step Function Approximation of Continuous Functions

In the frequency response numerical technique, nonuniform lines are approximated by the cascade connection of many short uniform line sections. Mathematically, this means that the parameters, Z, Y, ρ, and μ as well as the independent current and voltage source spatial variations are approximated by a sum of step functions. In the transient response numerical technique, the parameters, r, l, g, c, μ, and ρ, were functions

of space and time, as were the independent current and voltage source distributions. The parameters, R_L, μ_L, G_R, and ρ_R, and the functions E_L and I_R were functions of time only. The functions of space and time are approximated by products of step functions in space and time. The approximation can be visualized as a surface above time-position space which consists of flat rectangles parallel to the time-position plane. The functions of time alone are approximated along the time axis in the same way as the nonuniformities in the frequency response technique are.

Three methods of approximating a function of one variable with step functions are shown in Figure IV-1. The first two are not as accurate as the third because they have significant error in the area under the curve. The third was chosen for these programs. To closely approximate the area under the continuous function by step functions, the area under the continuous function is first approximated using Simpson's rule in each interval of the independent variable. The area under each section of the curve is then divided by the interval length of the independent variable to obtain an average height for the approximating step in that interval. Using this technique, the area under the approximating curve is very close to the actual area under the continuous curve as long as the continuous curve varies no more rapidly than a fourth degree curve in each interval. This technique then yields a step function approximation which is optimal in the sense that the integral of the error is minimized in each interval.

For those functions which are functions of both time and position a two-dimensional extension of Simpson's rule is used. Simpson's rule

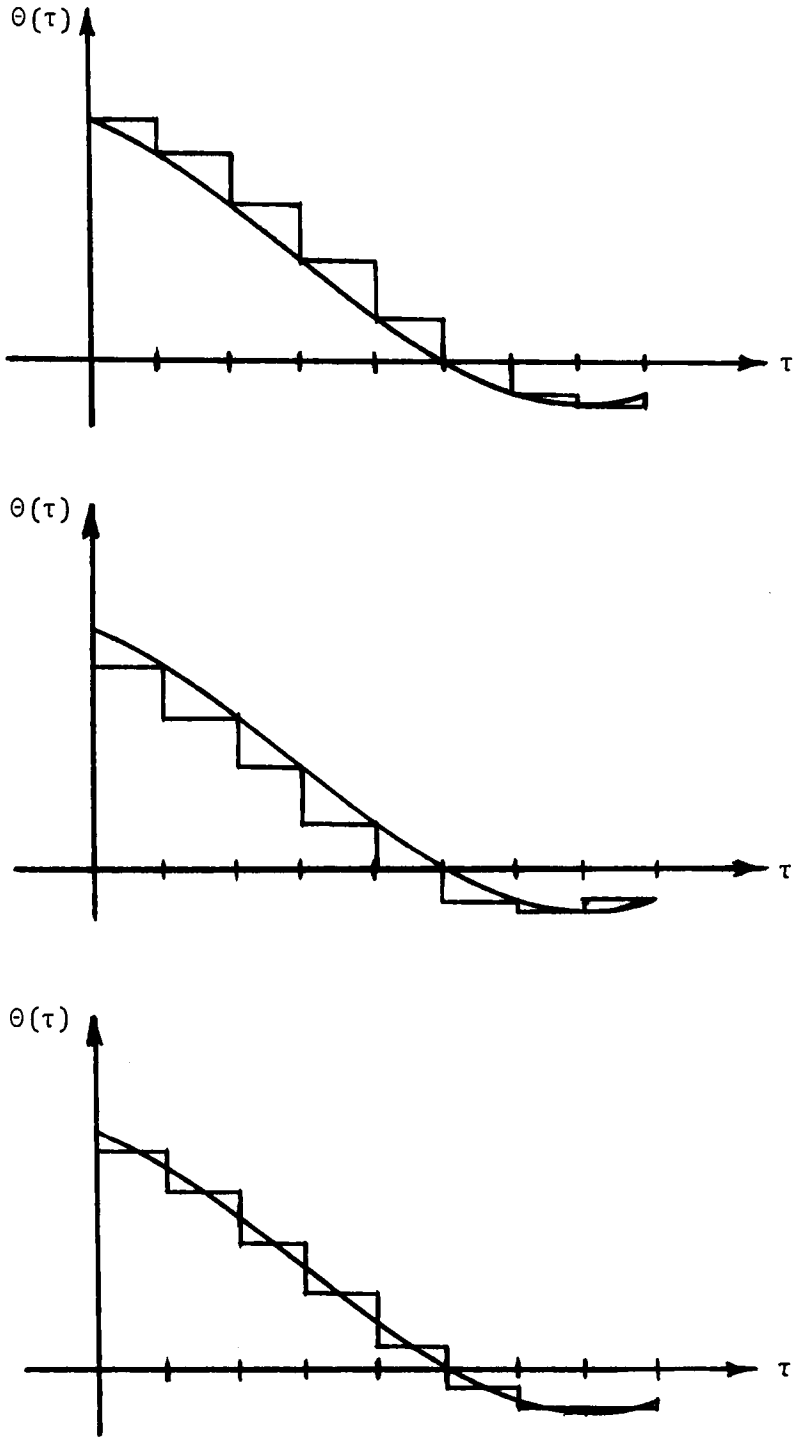


Figure IV-1. Three methods of approximating a continuous function with step functions.

in one dimension approximates a curve by assuming it is parabolic and by sampling its values at three equally spaced points the area under the curve can be calculated. The two-dimensional extension of Simpson's rule assumes that the surface above time-position space is of the following functional form:

$$\Theta(\lambda,\tau) = K_1\tau^2\lambda^2 + K_2\tau^2\lambda + K_3\tau\lambda^2 + K_4\tau^2 + K_5\lambda^2 + K_6\tau + K_7\lambda + K_8\tau\lambda + K_9 \quad . \tag{1}$$

Then the volume under the surface can be calculated by choosing the nine points shown in Figure IV-2. This volume is then divided by the area in time-position space it covers to yield an average height of the surface above the time-position plane. This is the height of the level surface used to approximate the function in the region in question. This height, $\overline{\Theta}$, is given by the following expression:

$$\overline{\Theta} = \frac{1}{36}\left[16\,\Theta(0,0) + 4\,\Theta(\Delta\lambda,0) + 4\,\Theta(-\Delta\lambda,0) + 4\,\Theta(0,\Delta\tau) + 4\,\Theta(0,-\Delta\tau) + \Theta(\Delta\lambda,\Delta\tau) + \Theta(\Delta\lambda,-\Delta\tau) + \Theta(-\Delta\lambda,\Delta\tau) + \Theta(-\Delta\lambda,-\Delta\tau)\right] \quad . \tag{2}$$

Technique for Dealing with Lines with Widely Varying Parameters

To allow for the possibility that a transmission line may have one or more parameters which change several orders of magnitude over the length of the line, a technique was devised to reduce the chance of

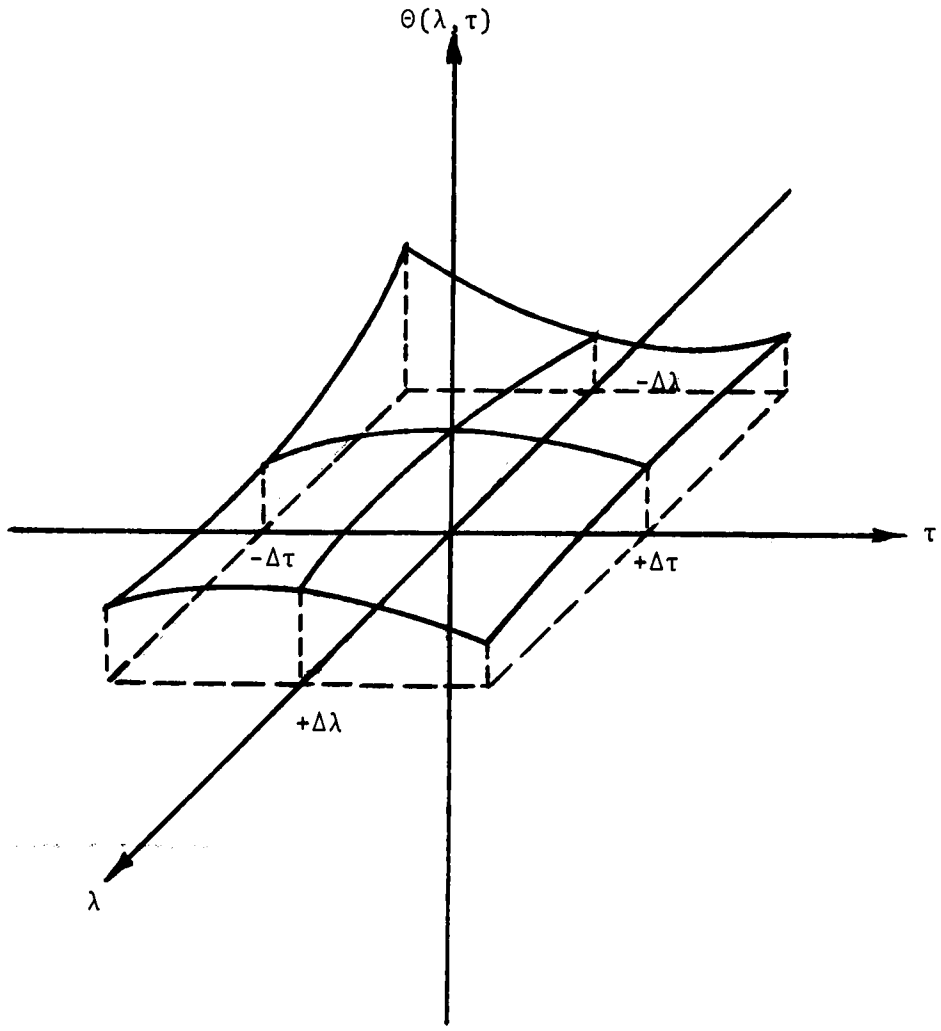


Figure IV-2. A typical surface above two-dimensional space.

numerical error which might result from operating on numbers with significantly different magnitudes. In the frequency response program there is a routine which searches out the largest absolute magnitude of each of the four functions, Z, Y, μ, and ρ, in any increment along the line. Then when calculations are made in each increment, the absolute value of each of these four functions is compared with its maximum on the line. If it is less than or equal to one-millionth of the maximum it is considered to be zero. This changes the type of calculation because, as discussed in Chapter III, if one of these parameters is zero the form of the solution changes. The small parameter than does not enter into the calculation, thus avoiding the problem of widely divergent magnitudes.

Solution of Systems of Linear Equations

In both the frequency response and transient response programs systems of linear equations must be solved[6]. In the frequency response program a system of four equations with complex coefficients must be solved at each frequency. These complex coefficients may have widely divergent absolute magnitudes at some frequencies. Because of this fact great care must be taken in the solution of these systems of equations. First the coefficients are expressed in complex double precision form to keep as many significant digits as possible in the calculations. Then in solving the system a technique of upper triangularization and back substitution is used. The technique used is similar to the Gauss-Jordan method for diagonalizing a matrix. The coefficient matrix is always arranged so that the element with the

largest absolute magnitude is placed at the upper left-hand corner. Then the first equation is divided by this element, placing a one in the upper left-hand corner. Then the first equation is successively multiplied by the leftmost coefficient of each succeeding equation and subtracted from that equation. This process creates all zeros in the first column except for the upper left-hand corner element. Then this process is repeated on the remaining three by three matrix which is the cofactor of the upper left-hand corner element. This process is repeated on each successively smaller matrix until an upper triangular matrix remains. A check is then made to see if any of the diagonal elements is zero. If any is, the system of equations is unsolvable because the coefficient matrix is singular. If the coefficient matrix is nonsingular, the variables are solved for by back substitution.

In the transient program, the system of equations may be considerably larger than four by four. Also, the coefficient matrix is always real and five-diagonal in form. Because of this sparsity, a general solution technique would be inefficient. Again, an upper-triangularization technique is used, but the program is written to take advantage of the five-diagonal character of the coefficient matrix.

Transforming Frequency Response to Impulse Response

The transformation from frequency response to impulse response is accomplished by using the digital fourier transform. The algorithm for implementing the digital fourier transform is the fast fourier transform algorithm developed by Cooley and Tukey. The fast fourier transform (FFT) is a numerical procedure which takes an array of data in sampled

form and returns an array of data which has a well-defined relation to the fourier transform of the original data[7,8].

The digital nature of the fast fourier transform can cause problems when one tries to relate it to the continuous Fourier transform. The continuous Fourier transform takes some function which, in general, is infinite in extent and transforms it into another function of another variable which is also continuous and infinite in extent. The fast fourier transform takes an array of data which is, of necessity, discrete and quantized in nature and finite in extent and returns another array of data which has the same characterization. The problem for the user of the fast fourier transform is to relate these two processes in some meaningful and useful way.

The first problem is to convert the function to be transformed from continuous to sampled form. To obtain an accurate representation of the function, the distance between samples should be chosen to be less than or equal to some maximum distance which is the reciprocal of the minimum sampling frequency usually called the Nyquist frequency. This is a direct result of the implications in Shannon's sampling theorem. Since the data must be finite in extent, a decision must be made about how many points to take. The number of points should be large enough to sample the function over all its significant range. If the function does not asymptotically approach zero for large values of its argument, it cannot be adequately sampled. However, this is no real limitation since the continuous Fourier Transform of a function which does not asymptotically approach zero does not exist anyway. In the fast fourier

transform algorithm, the number of points must be some integer power of two.

Another property of the fast fourier transform that a user should be aware of is that it assumes the input data represent the "aliased" form of the function and the data returned represent the "aliased" form of the transform. The term aliased means that the function has been made periodic by the following process:

$$f_p(\lambda) = f(\lambda) + f(\lambda+T) + f(\lambda-T) + f(\lambda+2T) + f(\lambda-2T) + \ldots \qquad (3)$$

If the function, $f(\lambda)$, asymptotically approaches zero and T is chosen large enough to include all the significant part of $f(\lambda)$, then $f_p(\lambda)$ will be almost the same as $f(\lambda)$. If T is not chosen large enough the parts of $f(\lambda)$ outside the range $0 \leq \lambda \leq T$ will be folded back into this interval in the same way that the frequency components of a signal which are above the Nyquist frequency of a digital filter appear in its output frequency spectrum.

The fast fourier transform assumes that the data given to it is a sampled representation of the aliased form of the function to be transformed in the interval $0 \leq \lambda \leq T$. If the number of data points in this interval is N, the number of sections into which $f_p(\lambda)$ is divided is also N and the data should represent the values of $f_p(\lambda)$ at the beginning of each section. Also, if there is a discontinuity in $f_p(\lambda)$, the sampled representation of $f_p(\lambda)$ cannot be a complete description since the highest frequency component is infinitely high. The best way to sample in this case is to choose the average value of the function at the

discontinuity. The data should be in the form of complex numbers and the data returned will also be complex. The data returned will be a sampled representation of the aliased form of the transform of $f_p(\lambda)$.

An example will serve to clarify the discussion. Consider a function which occurs frequently in electrical engineering:

$$v(t) = e^{-t/\tau}\, u(t) \ . \tag{4}$$

If τ is set equal to one, the graph of this function is shown in Figure IV-3. The sampling period is chosen to be 1/2 and the number of points is chosen to be 16, therefore the period of the aliased function, $v_p(t)$, is eight. The data which return when the fast fourier transform has operated on these input data are shown in Figure IV-4. The transform is denoted as $V_p(f)$, the aliased form of the forward transform of $v_p(t)$. The first half of the $|V_p(f)|$ curve will be recognized as the sampled form of the frequency response of a normalized low-pass filter of the R-C, L-section type. The relation between frequency increments and time increments is given by the following:

$$\Delta f = \frac{1}{N\Delta t} = \frac{\Delta\omega}{2\pi} \ . \tag{5}$$

In this case,

$$\Delta f = \frac{1}{8} \ . \tag{6}$$

Since $V_p(f)$ is an aliased function it differs from the true answer, $V(f)$, by being periodically added to itself. This creates what are called aliasing errors. If Δf and N are large enough, and $V(f)$ approaches zero asymptotically, $V_p(f)$ and $V(f)$ will be approximately equal.

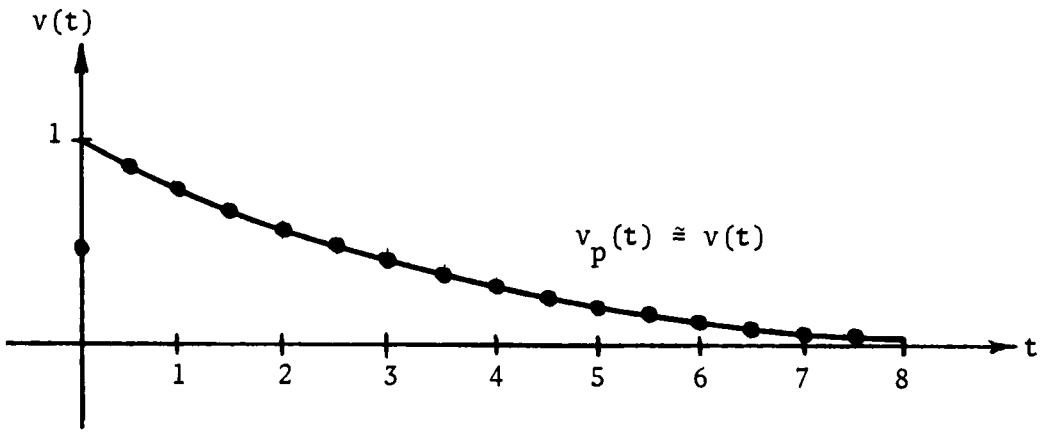


Figure IV-3. Graph of v(t)

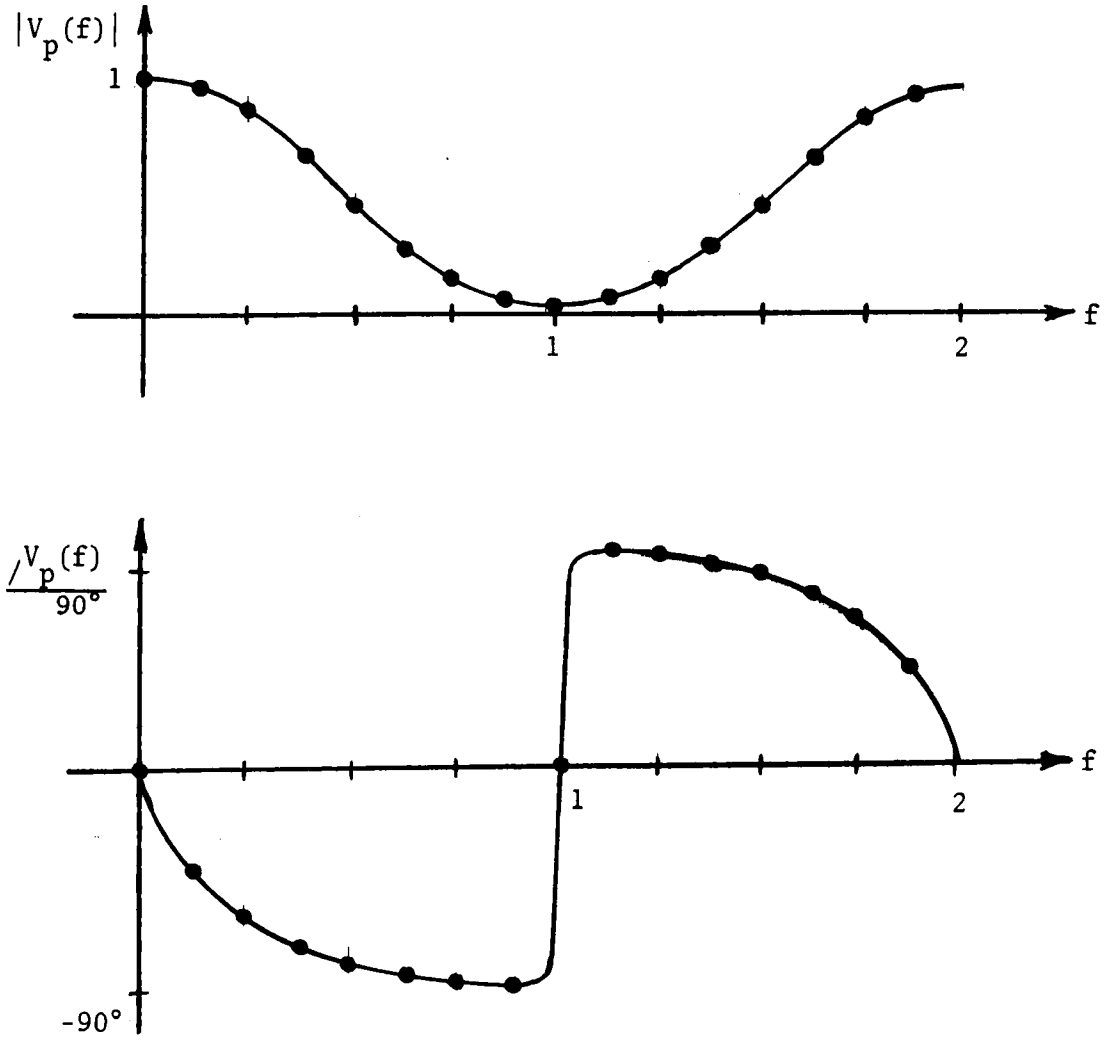


Figure IV-4. Graph of $V_p(f)$.

Otherwise the aliasing effect will cause errors which will be most significant around the mid-point of the output data where V(f) and $V(f-T_f)$ interact most strongly. Here T_f is the period of $V_p(f)$.

All the discussion so far has been about the forward fast fourier transform. The transformation from the frequency response to the impulse response is an inverse transformation. The discussion is still valid, however, because the inverse transform is very similar to the forward transform. To perform inverse transformation only requires that the input data be changed to their complex conjugates and forward transformed. The results are then divided by the number of points. These data are now the inverse fast fourier transform of the original data.

CHAPTER V

ACTIVE FILTER SYNTHESIS USING DISTRIBUTED CIRCUITS

One of the primary goals in network theory is an optimal approximation to the ideal filter. Many techniques of approximation in the frequency domain have been developed. These usually involve some rational approximation in s to the ideal characteristic. This has led to the Butterworth, Chebyshev, and Elliptic filters, among others[4]. These approximations are all optimal in different senses. Another approach which has received attention recently is the idea of time domain or impulse response synthesis, alluded to in Chapter II. The principal result of this idea has been the development of the transversal filter. In principal, any finite time impulse response can be closely approximated by using a tapped delay line and summing amplifiers. In this chapter both approaches will be explored in relation to transmission lines with distributed sources and transmission lines with taps and summing amplifiers.

Time Domain or Impulse Response Synthesis

At the end of Chapter II, the observation was made that on a uniform, lossless, matched transmission line with a distributed independent current source, the impulse response at each end of the line, as a function of time, was either the current source distribution, as a function of line position, or its inverse. This leads to the idea of directly synthesizing an impulse response which corresponds to an

approximately ideal frequency response or any other desired function of frequency which has a realizable impulse response. This idea has been implemented in a slightly different way in the transversal filter. These have been implemented, first with tapped wire coil delay lines, and in the last three or four years with surface wave devices and charge transfer devices[9,10]. On the distributed source transmission line filter the line is fed in a transverse manner by the distributed current source as shown in Figure V-1. For the transversal filter shown in Figure V-2, the signal is taken from the line in a transverse manner, and summed by operational amplifiers. By these methods, any arbitrary impulse response which is finite in duration and causal, can, at least in principle, be approximated.

There are four types of ideal filters which have received great attention in network theory. These are the ideal low-pass, high-pass, band-pass, and band reject. Other strictly band limited filters can be made with combinations of these four. Consider first the ideal low-pass filter whose frequency response is shown in Figure V-3. Its frequency response can be described mathematically by the following:

$$H(\omega) = \begin{cases} Ae^{-jm\omega}, & |\omega| < |\omega_o| \\ 0, & |\omega| > |\omega_o| \end{cases} . \qquad (1)$$

The impulse response corresponding to this frequency response can be found by use of the inverse Fourier transform. Let the impulse response be denoted by h(t). Then the following equation can be written:

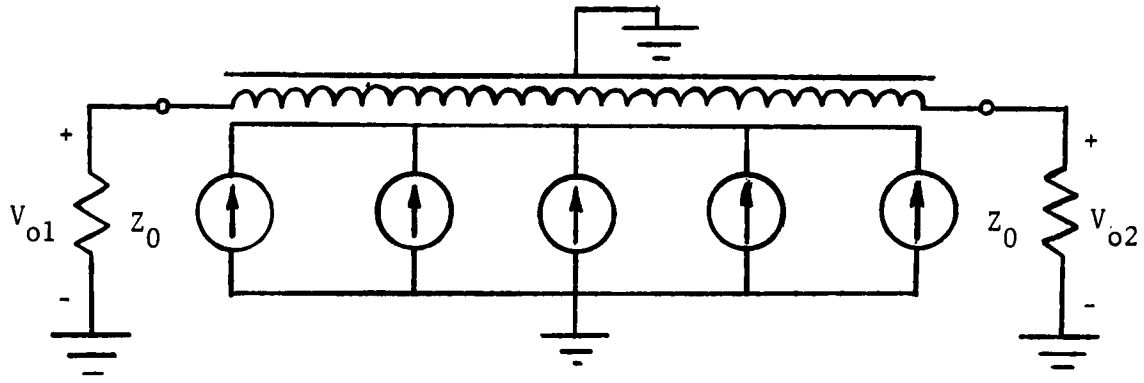


Figure V-1. Transmission line with a distributed independent current source.

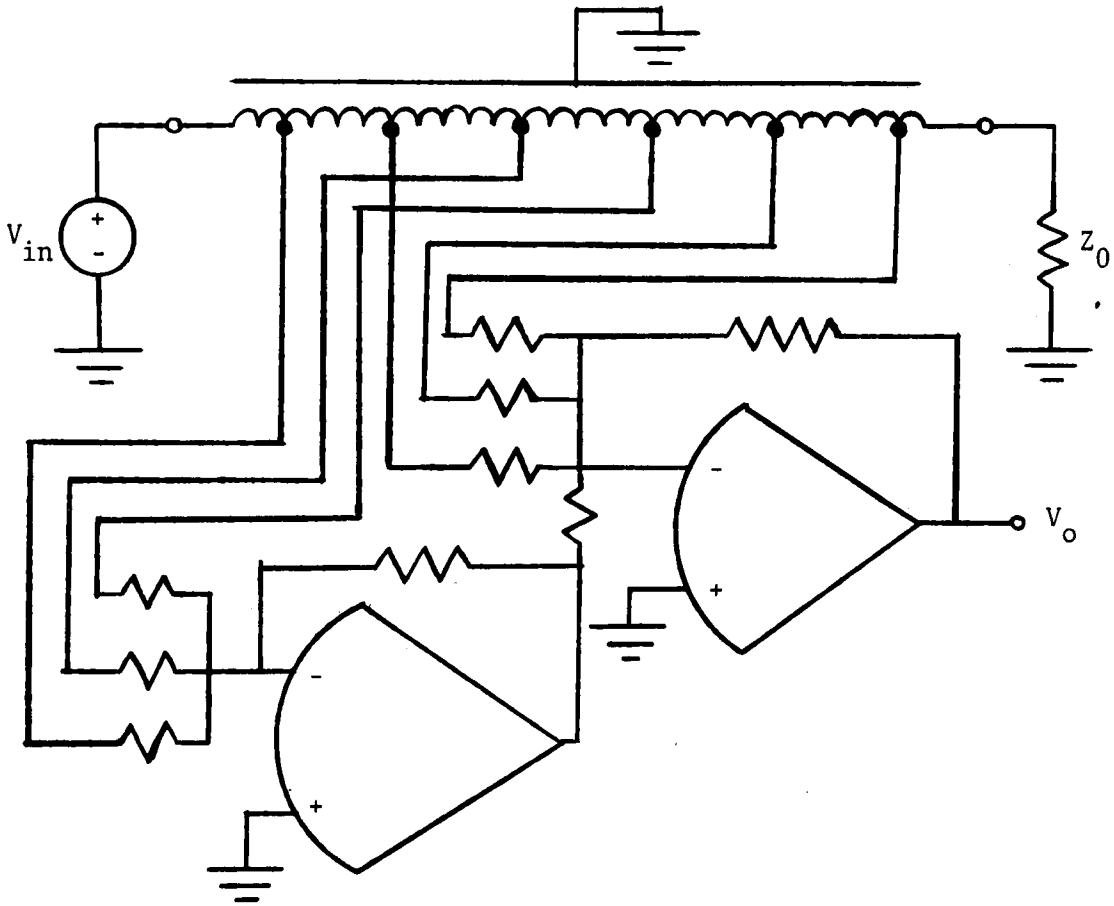


Figure V-2. Transversal filter.

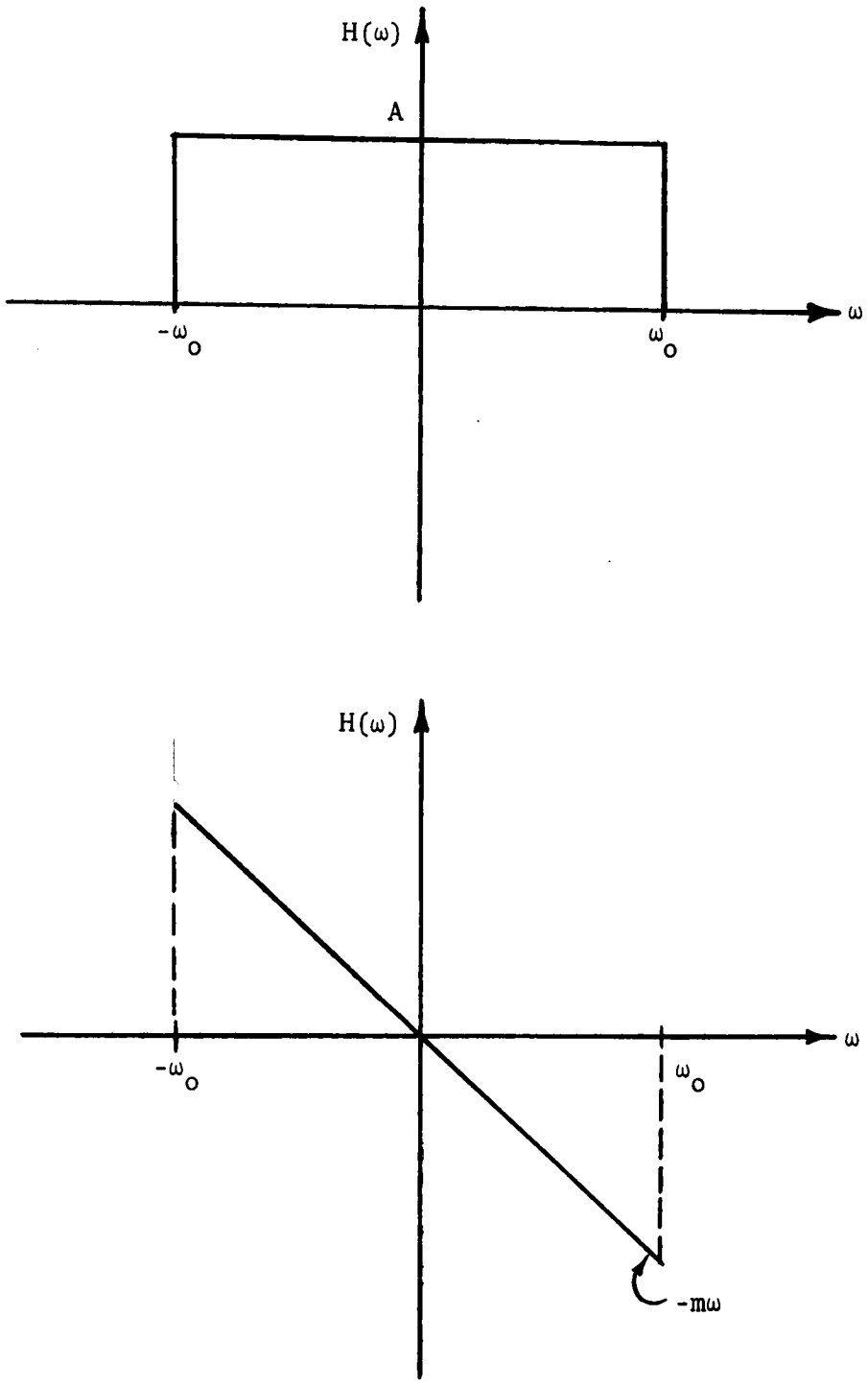


Figure V-3. The ideal low-pass frequency response.

$$h(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} H(\omega)\ e^{j\omega t}\ d\omega \tag{2}$$

$$h(t) = \frac{A}{2\pi} \int_{-\omega_o}^{\omega_o} e^{j\omega(t-m)}\ d\omega \tag{3}$$

$$h(t) = \frac{A\omega_o}{\pi} \left[\frac{\sin\omega_o(t-m)}{\omega_o(t-m)} \right] . \tag{4}$$

A graph of h(t) is shown in Figure V-4.

This impulse response cannot be realized because it is noncausal. However it is useful to try to approximate this response as closely as possible. As the parameter, m, is made larger, less of the significant part of h(t) occurs before t = 0. This means that if the designer can tolerate the large amount of time delay (phase shift in the frequency domain), he can, in principle, make as good a low-pass filter as he wants[11].

Consider next an all-pass filter.

$$H(\omega) = A\ e^{-jm\omega}\ , \text{ for all } \omega. \tag{5}$$

$$h(t) = A\ \delta(t-m) . \tag{6}$$

This impulse response is just an impulse of strength, A, and delayed by m seconds. By using this result and the low-pass result one can construct the ideal high-pass impulse response.

$$H_{hp}(\omega) = H_{ap}(\omega) - H_{lp}(\omega) \tag{7}$$

$$h_{hp}(t) = h_{ap}(t) - h_{lp}(t) . \tag{8}$$

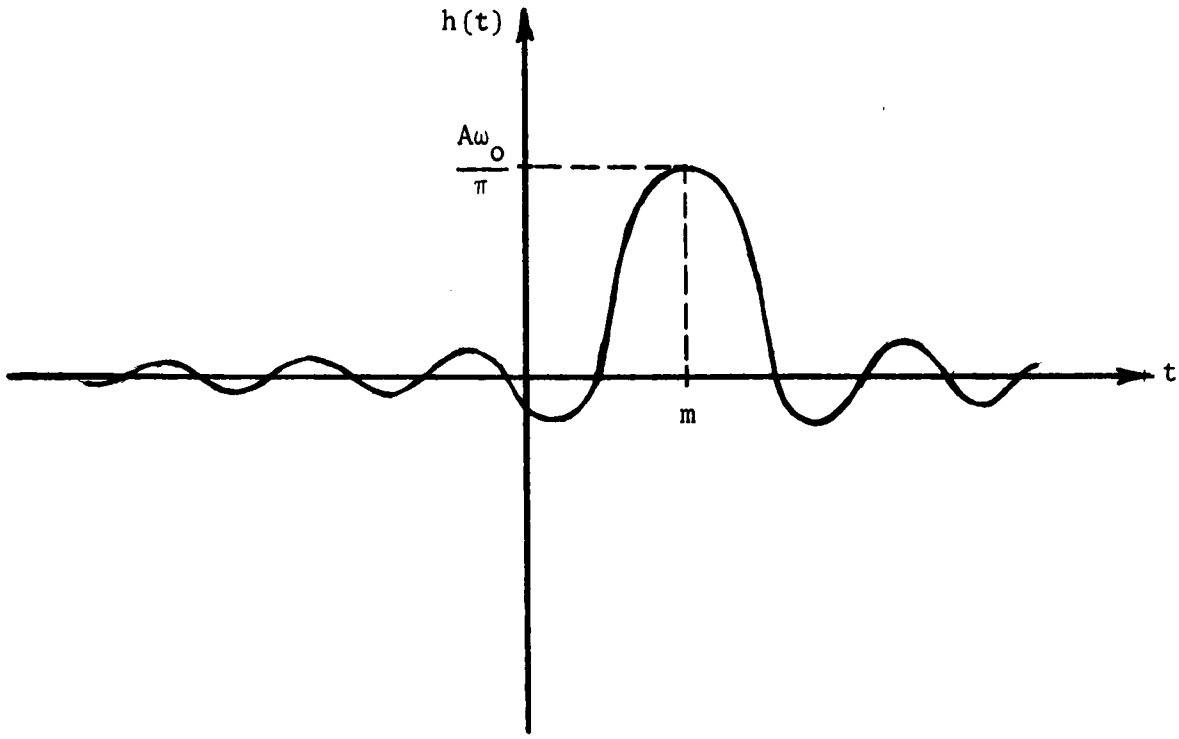


Figure V-4. Impulse response of the ideal low-pass filter.

This response is shown in Figure V-5.

The ideal band-pass and band-reject filters are just a difference and a sum, respectively, of low- and high-pass filters.

The next thing to consider is what kind of frequency response one can expect to get when he truncates an ideal impulse response to make it causal and of finite duration. In other words, what frequency response corresponds to the impulse response shown in Figure V-6?

$$h_T(t) = \begin{cases} h(t) \ , \ 0 \leq t \leq a \\ \\ 0 \ , \ t < 0 \ , \ t > a \end{cases} \tag{9}$$

where

$$h(t) = \frac{\omega_o}{\pi} \frac{\sin\omega_o(t-m)}{\omega_o(t-m)} \tag{10}$$

$$H_T(\omega) = \int_{-\infty}^{\infty} h_T(t)\ e^{-j\omega t}\ dt \tag{11}$$

$$H_T(\omega) = \frac{\omega_o}{\pi} \int_0^a \frac{\sin\omega_o(t-m)}{\omega_o(t-m)}\ e^{-j\omega t}\ dt \ . \tag{12}$$

By making the substitution,

$$\lambda = \omega_o(t-m) \ , \ d\lambda = \omega_o dt \ , \ t = \frac{\lambda}{\omega_o} + m \tag{13}$$

one can write the following equation:

$$H_T(\omega) = \frac{\omega_o}{\pi} \int_{-\omega_o m}^{\omega_o(a-m)} \frac{\sin\lambda}{\lambda}\ e^{-j\omega\left(\frac{\lambda}{\omega_o} + m\right)}\ \frac{d\lambda}{\omega_o} \ . \tag{14}$$

By using Euler's identity and,

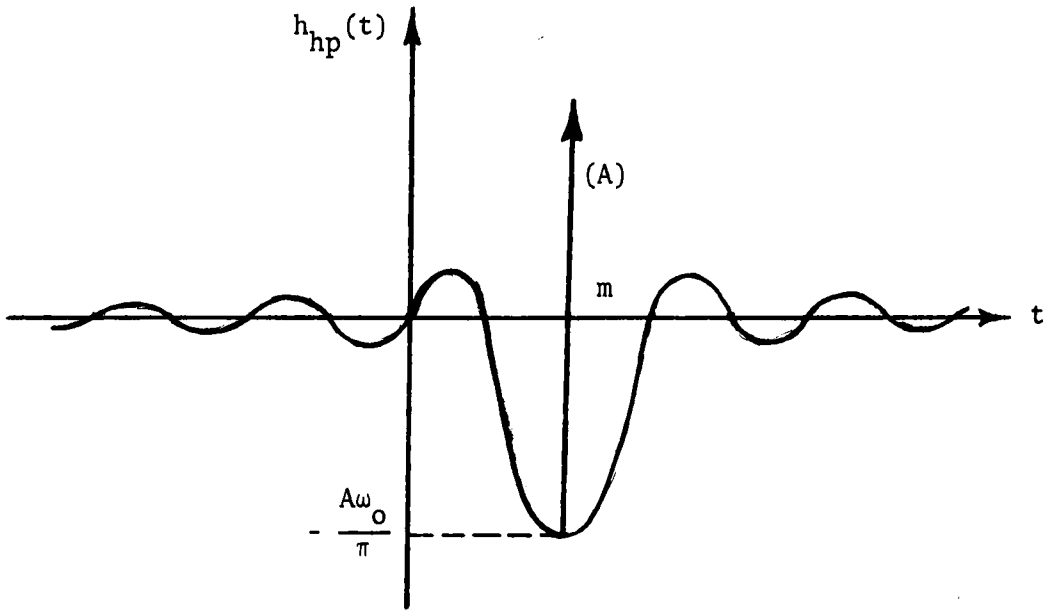


Figure V-5. Impulse response of an ideal high-pass filter.

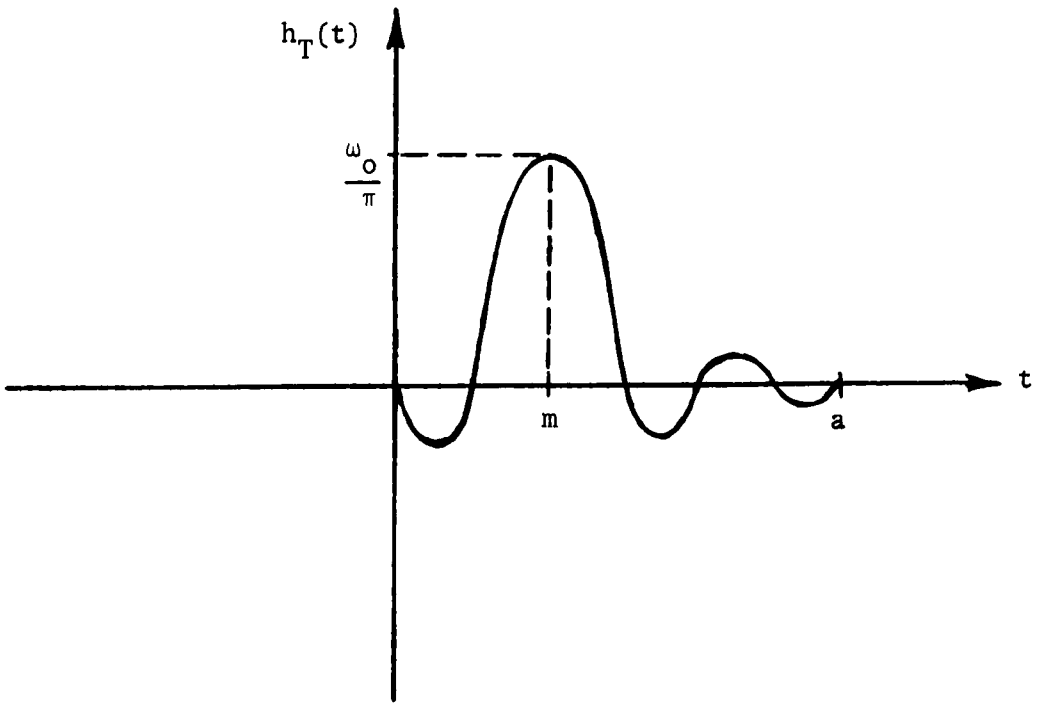


Figure V-6. Truncated ideal low-pass impulse response.

$$\sin x\cos y = \frac{1}{2}[\sin(x+y) + \sin(x-y)] \tag{15}$$

$$\sin x\sin y = \frac{1}{2}[\cos(x-y) - \cos(x+y)] \ , \tag{16}$$

and by dividing the intervals of integration, one may write the following equation:

$$H_T(\omega) = \frac{e^{-j\omega m}}{2\pi}\left[\int_0^{\omega_o m}\frac{\sin(\lambda+\frac{\omega}{\omega_o}\lambda)}{\lambda}d\lambda + \int_0^{\omega_o m}\frac{\sin(\lambda-\frac{\omega}{\omega_o}\lambda)}{\lambda}d\lambda\right.$$

$$+ \int_0^{\omega_o(a-m)}\frac{\sin(\lambda+\frac{\omega}{\omega_o}\lambda)}{\lambda}d\lambda + \int_0^{\omega_o(a-m)}\frac{\sin(\lambda-\frac{\omega}{\omega_o}\lambda)}{\lambda}d\lambda$$

$$-j\int_{\omega_o m}^{\infty}\frac{\cos(\lambda-\frac{\omega}{\omega_o}\lambda)}{\lambda}d\lambda + j\int_{\omega_o m}^{\infty}\frac{\cos(\lambda+\frac{\omega}{\omega_o}\lambda)}{\lambda}d\lambda$$

$$\left.+j\int_{\omega_o(a-m)}^{\infty}\frac{\cos(\lambda-\frac{\omega}{\omega_o}\lambda)}{\lambda}d\lambda - j\int_{\omega_o(a-m)}^{\infty}\frac{\cos(\lambda+\frac{\omega}{\omega_o}\lambda)}{\lambda}d\lambda\right]. \tag{17}$$

One can now make the following change of variable[12]:

In every integral involving $\lambda+\frac{\omega}{\omega_o}\lambda$,

$$\text{let,} \quad \lambda+\frac{\omega}{\omega_o}\lambda = \gamma \ , \quad \text{then } \frac{d\lambda}{\lambda} = \frac{d\gamma}{\gamma} \ , \tag{18}$$

and in every integral involving $\lambda-\frac{\omega}{\omega_o}\lambda$,

$$\text{let,} \quad \lambda-\frac{\omega}{\omega_o}\lambda = \gamma \ , \quad \text{then } \frac{d\lambda}{\lambda} = \frac{d\gamma}{\gamma} \ . \tag{19}$$

Then, using the notation,

$$\mathrm{Si}(x) = \int_0^x \frac{\sin x}{x}\, dx \ , \tag{20}$$

$$\mathrm{ci}(x) = \int_x^\infty \frac{\cos x}{x}\, dx \ , \tag{21}$$

the following equation can be written:

$$\begin{aligned} H_T(\omega) = \frac{e^{-j\omega m}}{2\pi} \Big[& \mathrm{Si}[m(\omega_o+\omega)] + \mathrm{Si}[m(\omega_o-\omega)] \\ & + \mathrm{Si}[(a-m)(\omega_o+\omega)] + \mathrm{Si}[(a-m)(\omega_o-\omega)] \\ & - j\ \mathrm{ci}[m(\omega_o-\omega)] + j\ \mathrm{ci}[m(\omega_o+\omega)] + j\ \mathrm{ci}[(a-m)(\omega_o-\omega)] \\ & - j\ \mathrm{ci}[(a-m)(\omega_o+\omega)] \Big] \ . \end{aligned} \tag{22}$$

In (22), if a and m both approach infinity in such a way that a is greater than m, the bracketed quantity approaches 2π for radian frequencies between $-\omega_o$ and $+\omega_o$, and it approaches zero elsewhere. Then $H_T(\omega)$ would have an ideal low-pass magnitude characteristic and a linear phase characteristic with infinite negative slope. Infinite negative slope on the phase characteristic in the frequency domain translates into infinite time delay in the time domain. Then one could say that perfect low-pass filtering requires an infinite time and is therefore impossible.

If a and m are finite, the function, $H_T(\omega)$, approximates the perfect low-pass function in the same way that a truncated Fourier series approximates a square wave. That is, there is a ripple in the pass and stop bands which is most severe near cutoff and has the characteristic form of the Gibbs phenomenon[13].

Once a satisfactory impulse response has been decided upon, the next problem is its realization. To exactly realize an impulse response would require an ideal delay line with a perfectly distributed voltage or current source or an ideal delay line with a perfectly distributed voltage sensing and summing system along its length. Since ideal deal lines cannot be built and perfectly distributed sources or sensing systems are not attainable, some practical realizable technique must be used to approximate the ideal.

Consider an ideal delay line with an impulse train approximation to a distributed current source. An example of one is illustrated in Figure V-7. If the line is perfectly matched, this distribution is the same in space as the impulse response of the line at the left end in time. Since this impulse train is the sampled form of the desired impulse response, the corresponding frequency response will be the aliased form of the desired frequency response. That is, the frequency response will be periodic in ω, the period being directly proportional to the sampling rate of the impulse train. If the delay line is not perfect, its bandwidth will be finite and the impulse response will be a train of dispersed impulses. The later each of these impulses occurs, the more dispersed it will be. This creates a great analytical difficulty

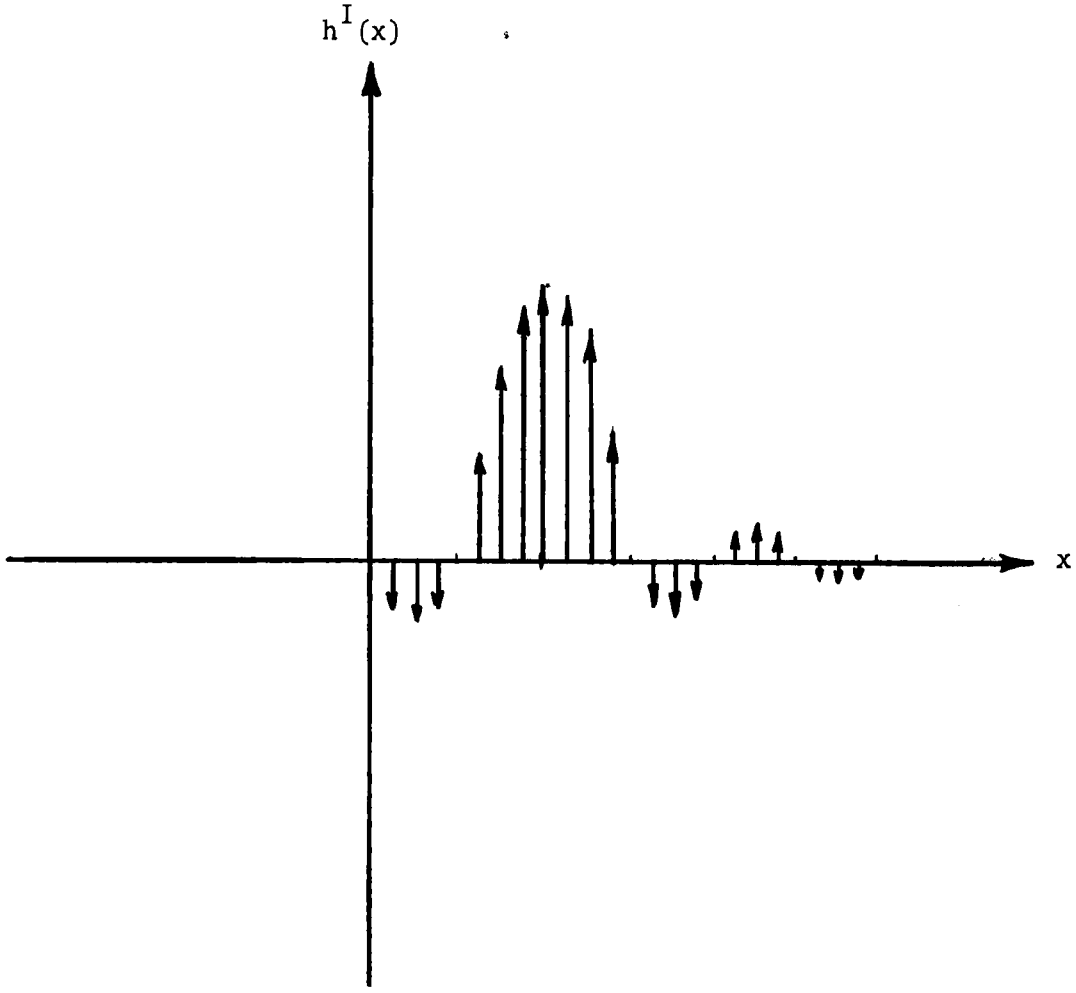


Figure V-7. An impulse train approximation to a current source distribution.

in trying to find the corresponding frequency response. Intuitively, however, one can see that the dispersion of the impulses would cause the approximation to the desired impulse response to be smoother and this would give a closer approximation to the desired frequency response. The aliases in the frequency domain would be attenuated because of the bandwidth limitation of the imperfect delay line. The same results would occur with a transversal filter since discrete tap points would create an impulse train approximation to the desired impulse response.

As previously mentioned, transversal filters are now being implemented with charge transfer devices. This type of realization creates a step function type approximation to the desired impulse response.

Since inductive elements like delay lines are well known in network theory as poor physical approximations to the mathematical model, and bulky and expensive also, consideration will be given here to an approximation technique using an RC transmission line. Since RC networks are easily fabricated in monolithic or thin film or thick film form, if a useful approximation could be found it would be more practical than using a delay line.

Consider the case of a matched uniform RC transmission line with a distributed voltage source. Using the analytical tools of Chapter II, the frequency response at the left end is found to be the following:

$$V(0,s) = -\int_0^D h^V(x_0)\, e^{-\sqrt{rcs}\, x_0}\, dx_0 \,. \qquad (23)$$

Transforming to the time domain yields the following:

$$v(0,t) = -\int_0^D \frac{\sqrt{rc}\, x_0}{2\sqrt{\pi t^3}} e^{-\frac{rcx_0^2}{4t}} h^V(x_0)\, dx_0 \,. \qquad (24)$$

Since the spatial distribution of the voltage source, $h^V(x)$, is to be chosen to form the desired impulse response, $v(0,t)$, this is an integral equation. Also, since $h^V(x)$ must be a real function of x, an exact solution may not exist. For this reason a numerical approximation technique was used to find a close fit to the desired $v(0,t)$.

Since a truly distributed source cannot be modeled exactly numerically, an impulse train is used as an approximation. That reduces the problem to a summation rather than an integral as in the following:

$$v(0,t) \cong -\sum_{k=1}^{K} w_k \frac{\sqrt{rc}\, x_k}{2\sqrt{\pi t^3}} e^{-\frac{rcx_k^2}{4t}} \,. \qquad (25)$$

Here the weights, w_k, are the weights of the impulses in the impulse train which makes up $h^V(x)$. The problem is now how to choose the w_k's optimally. The method used here is the method of least squares which is developed in detail here.

Method of Least Squares[6]

The problem is this:

Given: A function, $y(t)$ to be approximated by a weighted summation of other functions, $\Theta_j(t)$, $(j = 1,2,\ldots J)$, with weights, w_j.

Find: The set of w_j's which minimizes the sum of the squares of the errors at i points (i = 1,2,...I) of the approximating function.

At each of the I points the following equation holds:

$$y(t_i) = \sum_{j=1}^{J} w_j \Theta_j(t_i) + r_i \ . \tag{26}$$

Here r_i is the difference between the final approximation and the desired value of $y(t_i)$. In a similar manner, I equations of the form of (26) could be written. It is desired that the sum of the squares of the residuals, r_i, be a minimum with respect to the weights, w_j. Mathematically that translates to the following:

$$S = \sum_{i=1}^{I} r_i^2 = \sum_{i=1}^{I} (y_i - \sum_{j=1}^{J} w_j \Theta_{ij})^2 \ . \tag{27}$$

Here,

$$y_i = y(t_i) \quad \text{and} \quad \Theta_{ij} = \Theta_j(t_i) \ . \tag{28}$$

By taking the partial derivative of S with respect to w_j for each j, and setting it equal to zero, one obtains J equations in the J unknown weights.

$$\frac{\partial S}{\partial w_j} = -2 \sum_{i=1}^{I} (y_i - \sum_{k=1}^{J} w_k \Theta_{ik}) \Theta_{ij} \ . \tag{29}$$

Now the matrix equation,

$$[p] \underline{w} = \underline{q} \ , \tag{30}$$

can be written. In (30) the matrix elements are defined as follows:

$$p_{kl} = \sum_{i=1}^{I} \Theta_{ik}\Theta_{il} \quad , \quad \text{and} \quad q_k = \sum_{i=1}^{I} y_i\Theta_{ik} \; . \tag{31}$$

The matrix, [p], is J by J; $\underline{w}$ is J by 1; and $\underline{q}$ is J by 1. It is convenient now to establish the notation as follows:

Let $\underline{r}$ be the I by 1 column vector of the residuals.

Let [A] be the I by J matrix of the elements Θ_{ij}.

Let $\underline{y}$ be the I by 1 column vector of the values of y(t) at which the approximation is compared to the ideal.

Equation (26) can now be written as,

$$\underline{y} = [A]\,\underline{w} + \underline{r} \; . \tag{32}$$

$$S = \underline{r}^T\underline{r} = \{\underline{y} - [A]\,\underline{w}\}^T\{\underline{y} - [A]\,\underline{w}\} \; . \tag{33}$$

By comparing elements one can show that,

$$[p] = [A]^T[A] \; , \tag{34}$$

and

$$\underline{q} = [A]^T\underline{y} \; . \tag{35}$$

It then follows that,

$$[A]^T[A]\,\underline{w} = [A]^T\underline{y} \; , \tag{36}$$

and

$$\underline{w} = \{[A]^T[A]\}^{-1}[A]^T\underline{y} \; . \tag{37}$$

This method can be applied to equation (25) by letting,

$$y(t) = v(0,t) \quad \text{and} \quad \Theta(t) = -\frac{\sqrt{rc}\, x_0}{2\sqrt{\pi t^3}} e^{-\frac{rcx_0^2}{4t}} . \tag{38}$$

In using the least squares curve fitting method three main considerations were important:

1. How many impulses will be used in $h^V(x)$ and where will they be placed?
2. How can one be sure that the matrix, $[A]^T[A]$, is nonsingular?
3. The individual RC transmission line impulse responses used to form the desired impulse response are infinite in duration. The approximation is made over a finite interval. How then can a good approximation be insured?

The natural tendency is to use very many impulses in the approximation for $h^V(x)$, spaced uniformly along the line. This would insure a close approximation to a smooth curve. Also, since the sampling rate in the approximation is high the period of the frequency response would also be high. Theoretically, there should be no problem of a singular matrix to invert since the impulse responses of the individual sources are linearly independent. Numerically, however, these two considerations interact and can cause frustrating problems.

Even though the individual responses are theoretically independent of each other, numerically they may be almost identical if they are the responses to two closely spaced sources. If two responses are used which are very similar, an almost singular matrix will be formed. Numerically, this means trouble, since the accuracy of the matrix inversion routine depends on the matrix being not only nonsingular but

not even close to singularity. Therefore, if many impulses are used in the approximation to $h^V(x)$, at least two of them are going to be closely spaced somewhere on the line. Then when $[A]^T[A]$, is formed, it will be almost singular. The inverse matrix returned by the numerical matrix inversion routine, even a very precise one, will be meaningless.

Therefore, to obtain meaningful answers, one must choose only a few points along the line to feed in or tap off energy. Also, the points must be separated by a significant distance so that the impulse responses will be clearly distinguishable, one from another.

The third problem is that of fitting impulse responses which are infinite in duration to a response which is finite in duration. This means that the approximation should quickly approach zero and stay near zero after the desired response has terminated. The way to insure that this will happen is to use only impulse responses which peak within the approximation interval and are asymptotically approaching zero at the end of the interval. Some of the limitations on the approximation can be seen in Figure V-8. The first part of the ideal curve can be closely approximated, but the part after the main peak cannot. This is because the impulse responses which peak in that range of time are very broad and slow while the ideal response still ripples at the cutoff frequency. A typical shape for a least squares approximation is shown in Figure V-9. The corresponding frequency response curve is also shown in Figure V-9. The magnitude frequency response curve is a good approximation to the ideal low-pass filter in the stop band, but is a very poor approximation at the low frequencies of the pass band. A reason for this can be seen in the initial value theorem:

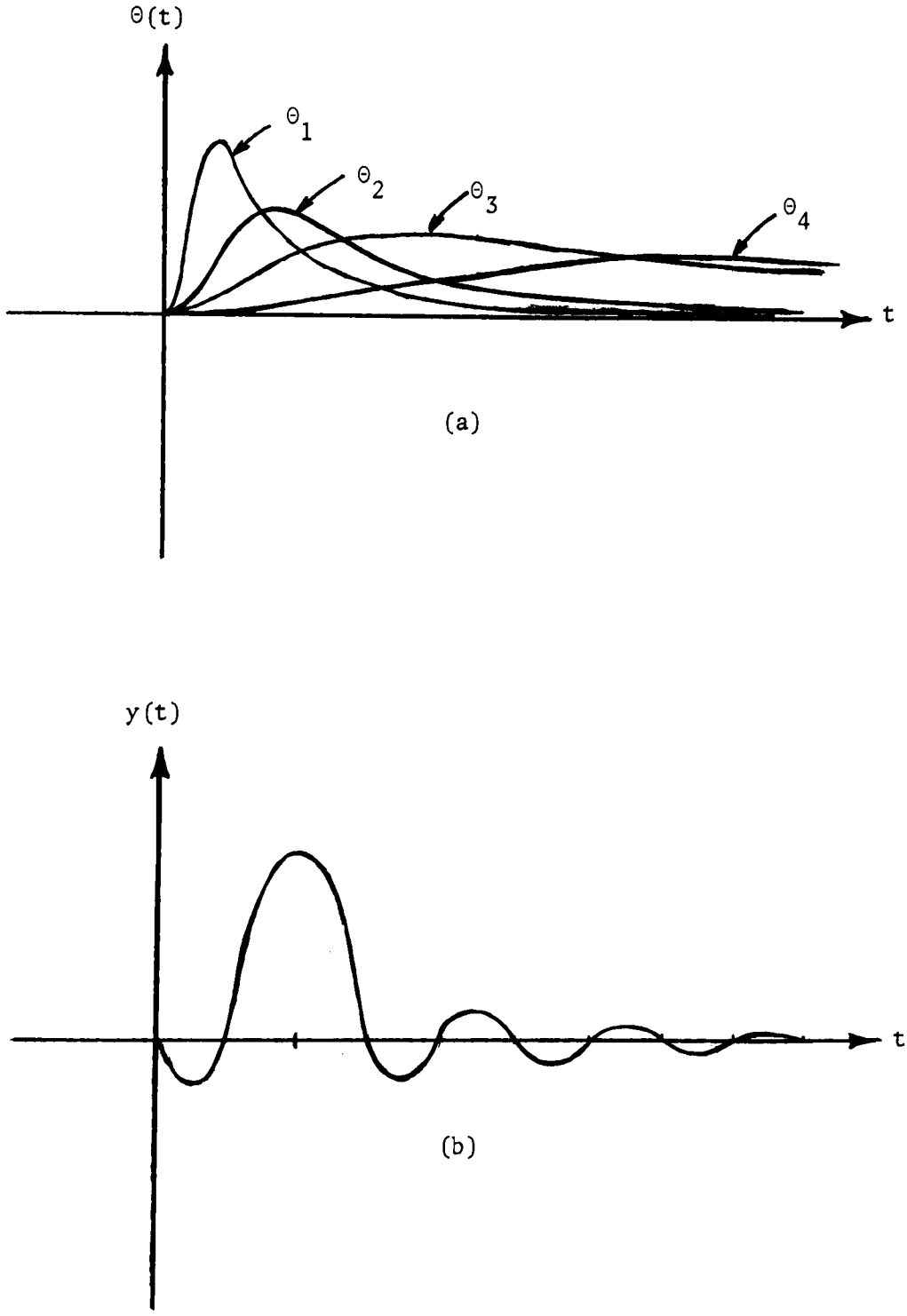


Figure V-8. Typical RC transmission line impulse responses, and (b) typical truncated ideal low-pass impulse response.

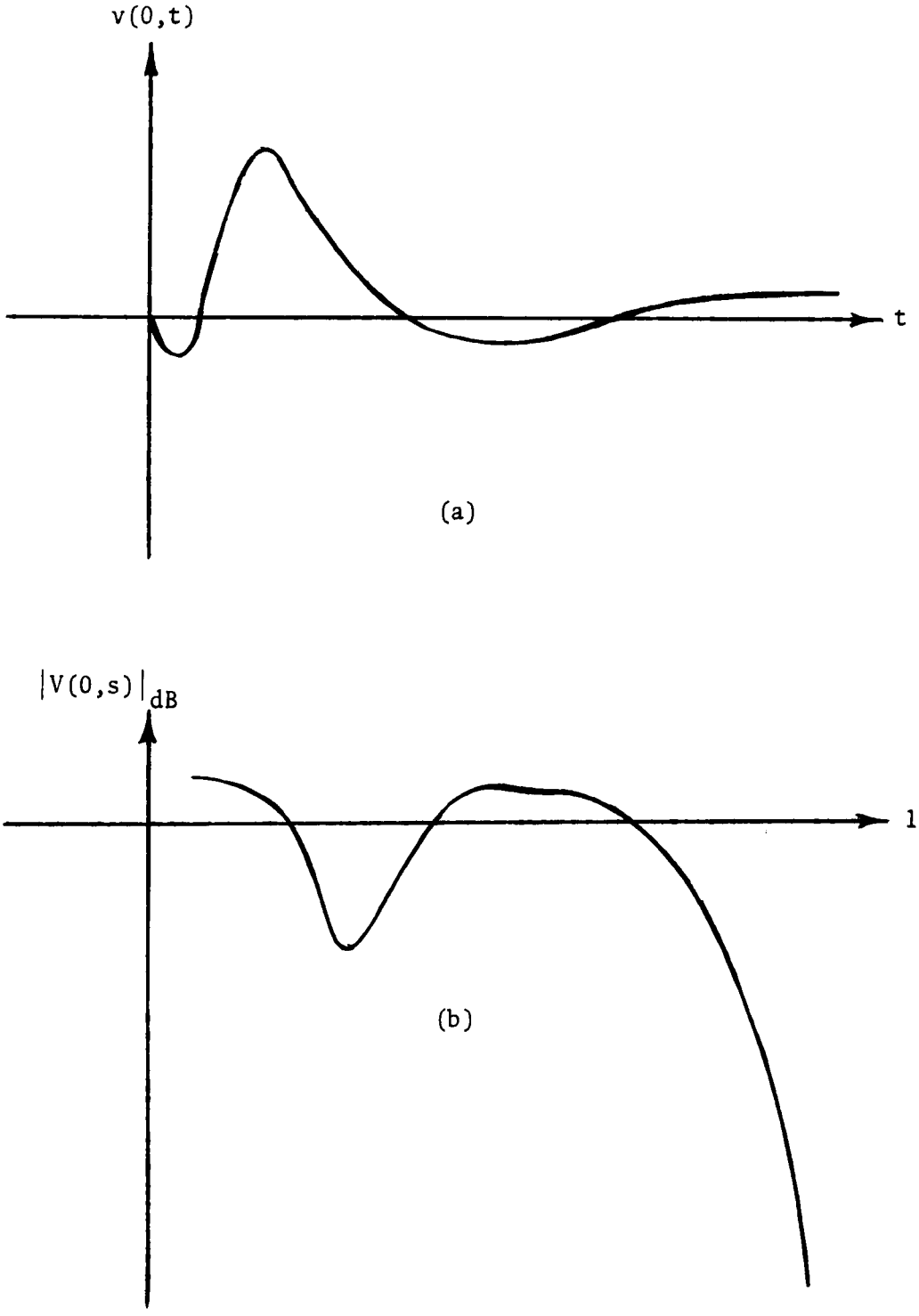


Figure V-9. Least squares approximation curves in (a) the time domain, and (b) in the frequency domain.

$$\lim_{t \to 0} f(t) = \lim_{s \to \infty} sF(s) \ . \tag{39}$$

In the approximation process, those parts of the curve where t is small are closely approximated. Therefore, the parts of the frequency response curve where s is very large will also be closely approximated. Since, for large t the approximation is very poor, the total area under the approximation curve could be significantly in error causing the frequency response at DC to be in error. By the same token, other low frequencies could be in significant error also.

Frequency Domain Synthesis

Consider a terminated uniform RC transmission line of length, D. The voltage transfer ratio from the input port to any point, x, along the line is given by the following:

$$\frac{V_o(s)}{V_i(s)} = T_v = \frac{Z_L \cosh \gamma(D-x) + Z_0 \sinh \gamma(D-x)}{Z_L \cosh \gamma D + Z_0 \sinh \gamma D} \ . \tag{40}$$

For a matched line, $Z_L = Z_0$, and

$$T_v = e^{-\gamma x} \ . \tag{41}$$

Since γ is a function of both complex frequency and line position, so is T_v. For real frequencies, T_v is a complex function of two real variables, x and ω. A graph of this relationship is shown in Figure V-10. The projection of this curve in the T_v plane is shown in Figure V-11. With reference to Figure V-11, a simple synthesis technique can now be demonstrated to synthesize a notch filter. If ω is held constant, the curve

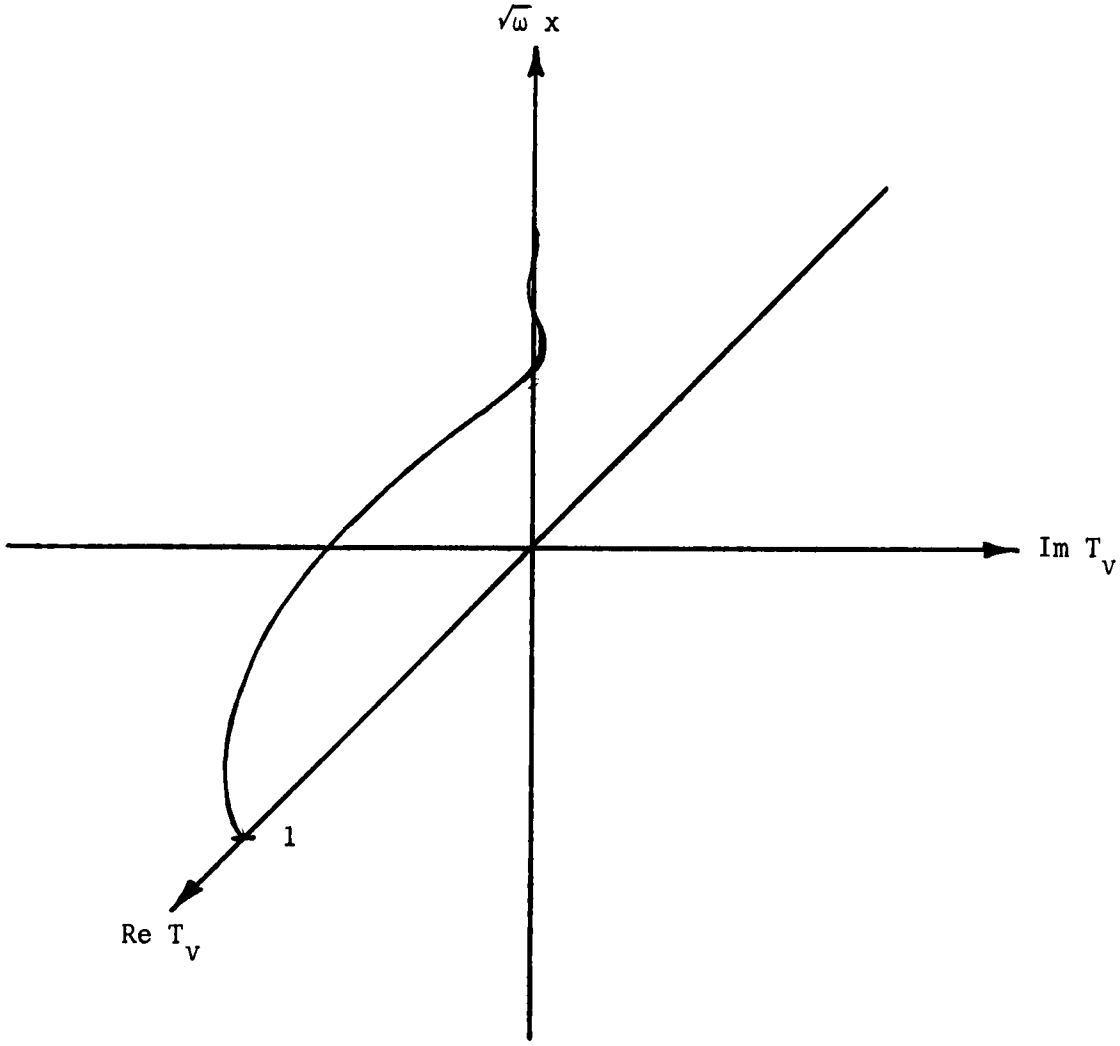


Figure V-10. T_v as a function of $\sqrt{\omega}$ x.

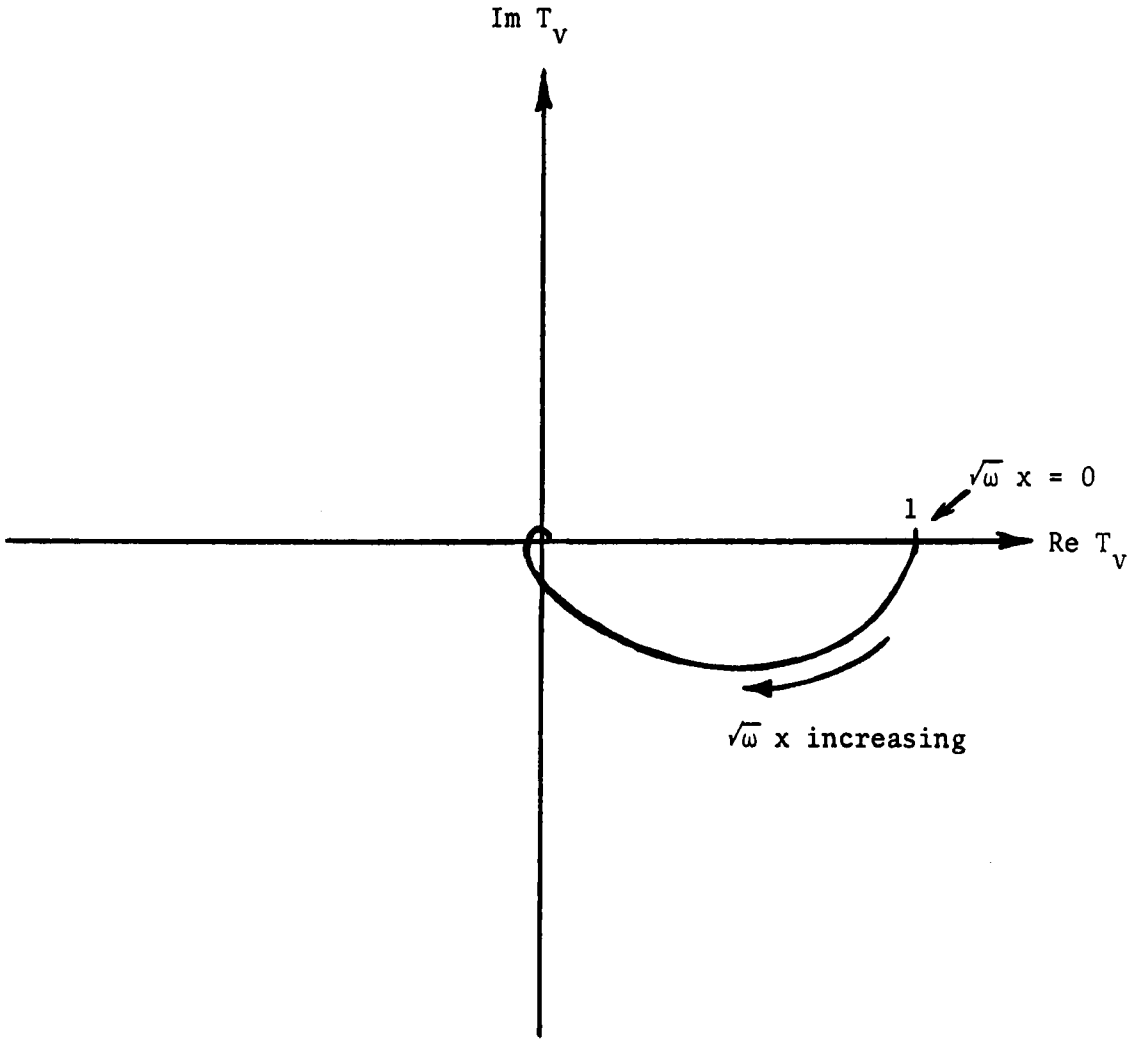


Figure V-11. Projection of Figure V-10 in the T_v plane.

of Figure V-11 is dependent only upon line position x. If one chooses a point, x_1, at which to tap off a voltage and another point, x_2, which is 180° from x_1 at the constant frequency, he can then weight these two voltages so that they exactly cancel at that frequency. This guarantees a notch in the magnitude frequency response curve, but does not design to meet any specified value at any other frequency. By choosing any two points other than the input port, one can fit the frequency response to one point in the complex plane at an arbitrary frequency. The designer can choose two constant weights and satisfy the requirements for a specified real and imaginary part of T_V at one frequency. By choosing a third tap and its weight, he can fit three conditions. These three conditions can be any combination of real and imaginary parts at two or three frequencies. We can now generalize to say that for each tap chosen, the designer may choose to fit a specified real or imaginary part of T_V at any frequency. This does not have to be a real frequency, even though the example was at a real frequency. The frequency of 0Hz is a special case, in that all the responses along the line are purely real at DC and therefore only one tap is required to completely specify the DC frequency response and it must be purely real.

All the design discussion so far has been for a matched line. An RC line cannot in reality be perfectly matched. A perfect match would require a terminating impedance with an irrational functional dependence on s. This can only be satisfied by a matched RC line. This problem is only a small one, however, because an RC line can be so nearly matched as to be considered matched for all practical purposes. One way to approximately match a line is with an identical line. The longer the

second line is, the better the match, regardless of its termination. A long RC line has approximately the same input impedance as a matched line because the end effect is masked by the losses along the line. An open line is a better termination than a shorted one because it is a perfect match at DC. The match at high frequencies is less important because the losses are so great.

Even if no attempt at all is made to match the line, the design procedure above is valid. All one needs to do is use the more general expression for T_V, given in (40), and choose weights accordingly. The results will be similar. The voltage transfer ratio for a shorted line does differ considerably from that for a matched line at low frequencies, but at a middle range of frequencies before the losses have become so great as to make the ratio unusable, the ratio for a shorted line approaches very closely that of a matched line. Therefore, except for a shorted line at low frequencies, the design results for a matched line are approximately valid for any other termination.

CHAPTER VI

NUMERICAL EXAMPLES AND EXPERIMENTAL RESULTS

To test some of the design procedures a lumped approximation to a uniform RC transmission was built. This artificial line was composed of 20 L-sections, each consisting of a 1000 ohm resistor and a 1μF capacitor. The resistors were carbon composition, selected to be within 1% of 1000 ohms. The capacitors were precision nonelectrolytics selected within a 4% tolerance. The frequency response at several points along the line was checked experimentally to see if it was a good approximation to a true distributed line. Except at the open circuited end the responses were so close as to be experimentally indistinguishable from calculations based on a distributed line model.

The first specific design was a time domain synthesis approximation, using the method described in Chapter V, to a low-pass filter with a cut-off frequency of 180 radians/sec or 28.65 Hz. The phase shift in the pass band was chosen to be linear with a slope of -.035 sec. This yields an impulse response which, when truncated at t=0 and t=0.419 sec, looks as shown in Figure VI-1.

By the least squares approximation technique an approximate curve fit was made using six impulse responses from a matched line. The six points where the responses were tapped were x = 5,6,8,10,12, and 14. A graph of the approximation curve is shown in Figure VI-2.

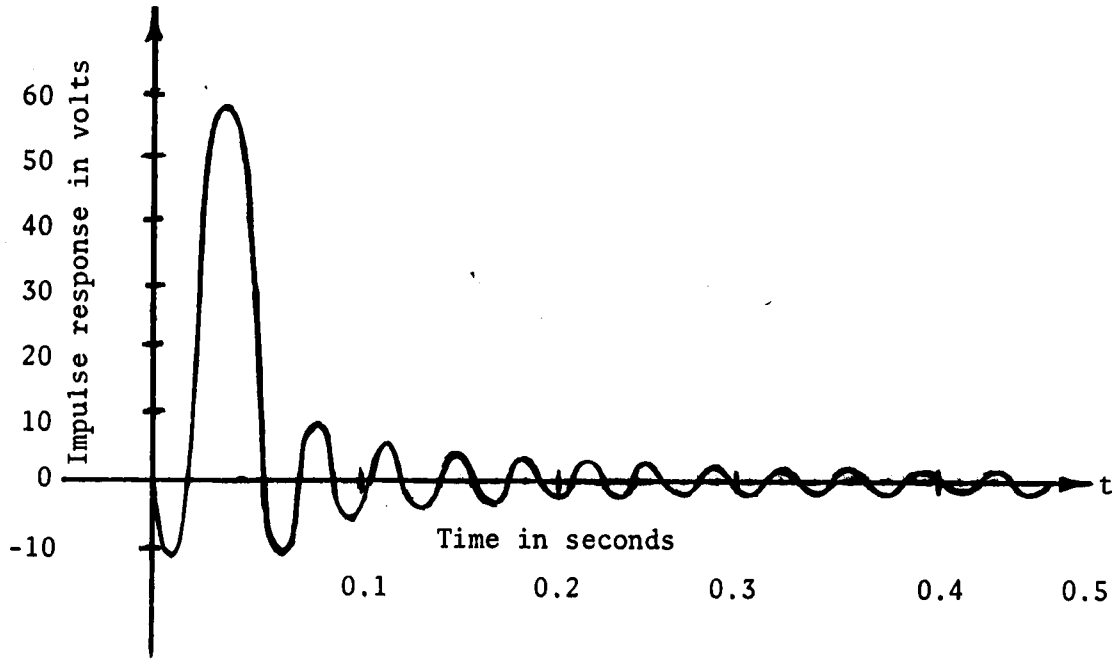


Figure VI-1. Truncated ideal low-pass impulse response.

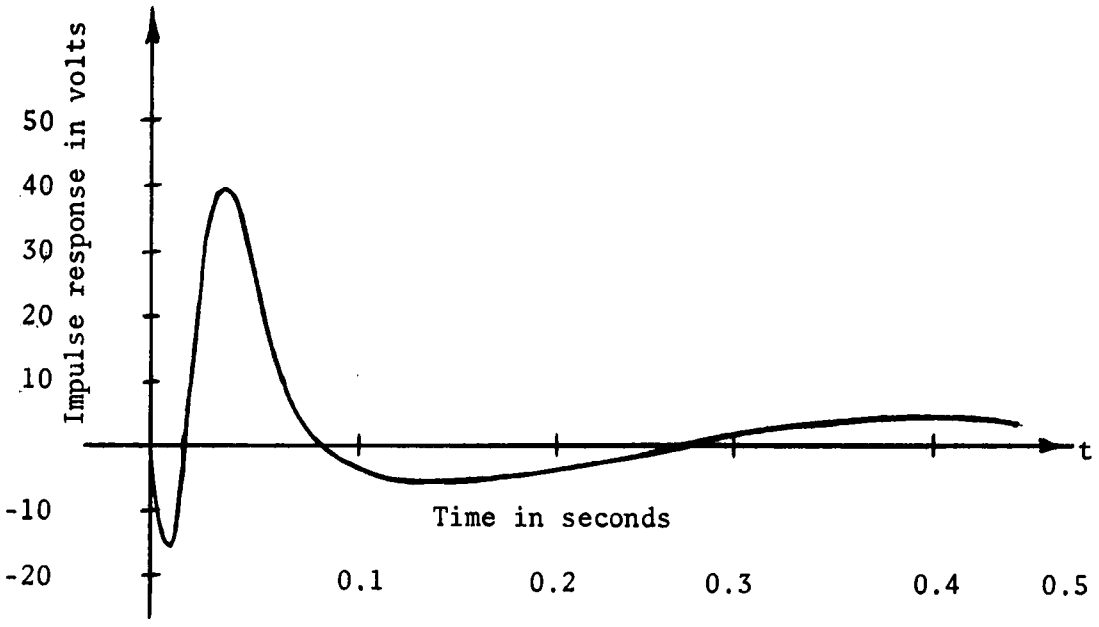


Figure VI-2. Approximate impulse response from least-squares approximation.

A comparison of the magnitude frequency responses corresponding to the approximation and the curve that was being approximated is made in Figure VI-3.

Several frequency domain designs were done. The first one was a simple notch filter. It was designed to have a voltage transfer ratio of zero at a radian frequency of 100 radians/sec. The design was done graphically, with reference to Figure VI-4. Figure VI-4 is a graph, on polar coordinates of the voltage transfer ratio at the radian frequency, ω=100. First, the point x=0 was arbitrarily chosen and its weight was chosen to be one. Next, the point x=1 was chosen with the thought in mind of making its weight approximately minus one. Then the point, x=11, was chosen because its angle looked about right to sum the three vectors to zero. Another reason for these choices was that the DC frequency response and the frequency response at infinity were both near one when the weights were chosen to create a zero at ω=100. Experimental magnitude frequency response results are shown in Figure VI-5.

Several other filters were designed and tested. In Figure VI-6 and in Figure VI-7 are shown magnitude frequency response plots for eight double notch filters, all with a response of 0dB at DC. In Figures VI-8 and VI-9 are shown the magnitude frequency response plots of the experimental filters designed to correspond to Figures VI-7 and VI-7. In Figure VI-8, one of the curves is incomplete because the gains required in the amplifiers were so large that the noise at low levels obscured the signal completely. Also, some of the experimental curves are shifted up or down by a constant dB level. This was done to keep the dynamic signal levels and amplifier gains within reasonable limits.

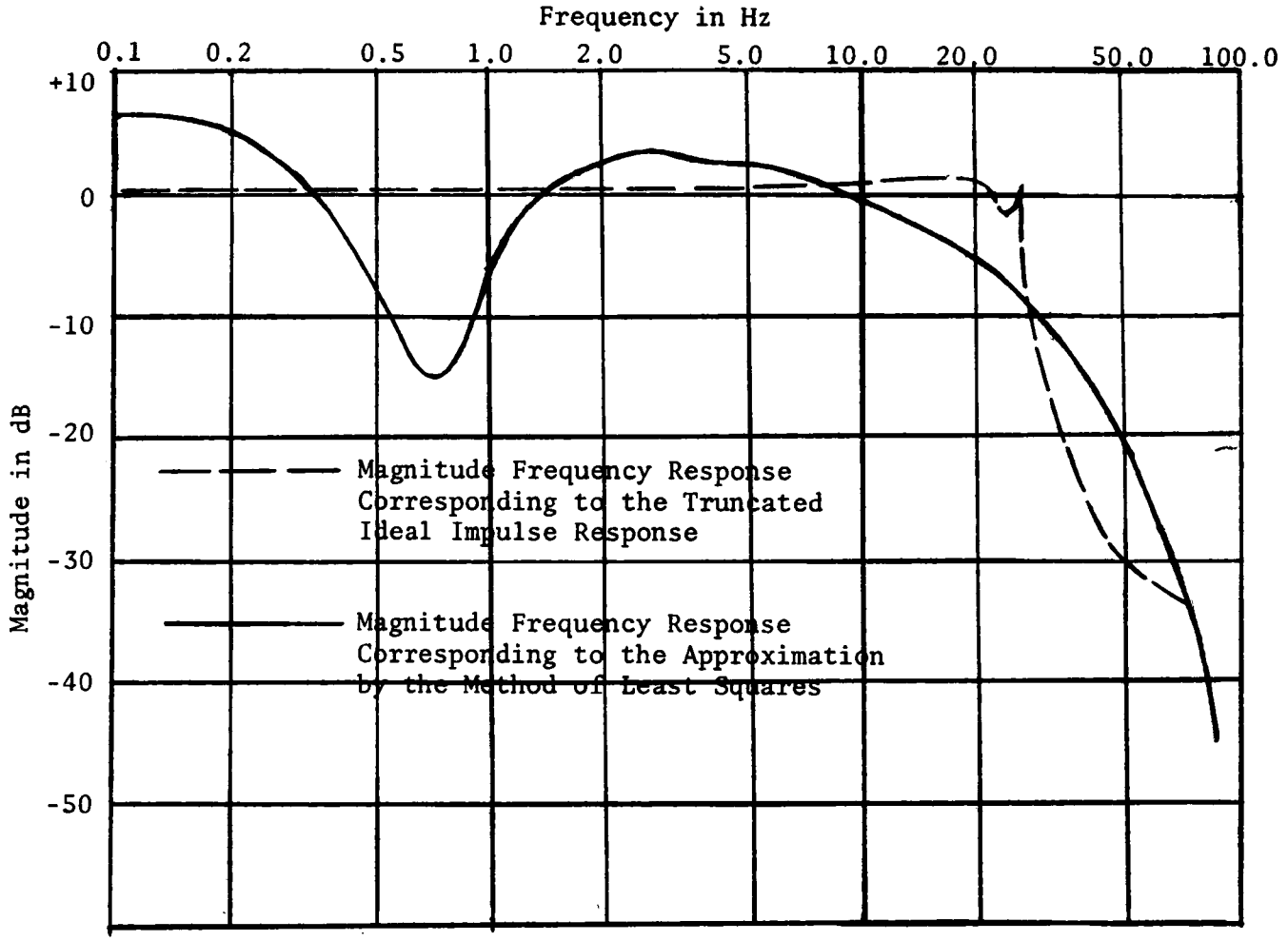


Figure VI-3. A comparison of magnitude frequency responses corresponding to the ideal and approximate impulse responses of Figures VI-1 and VI-2.

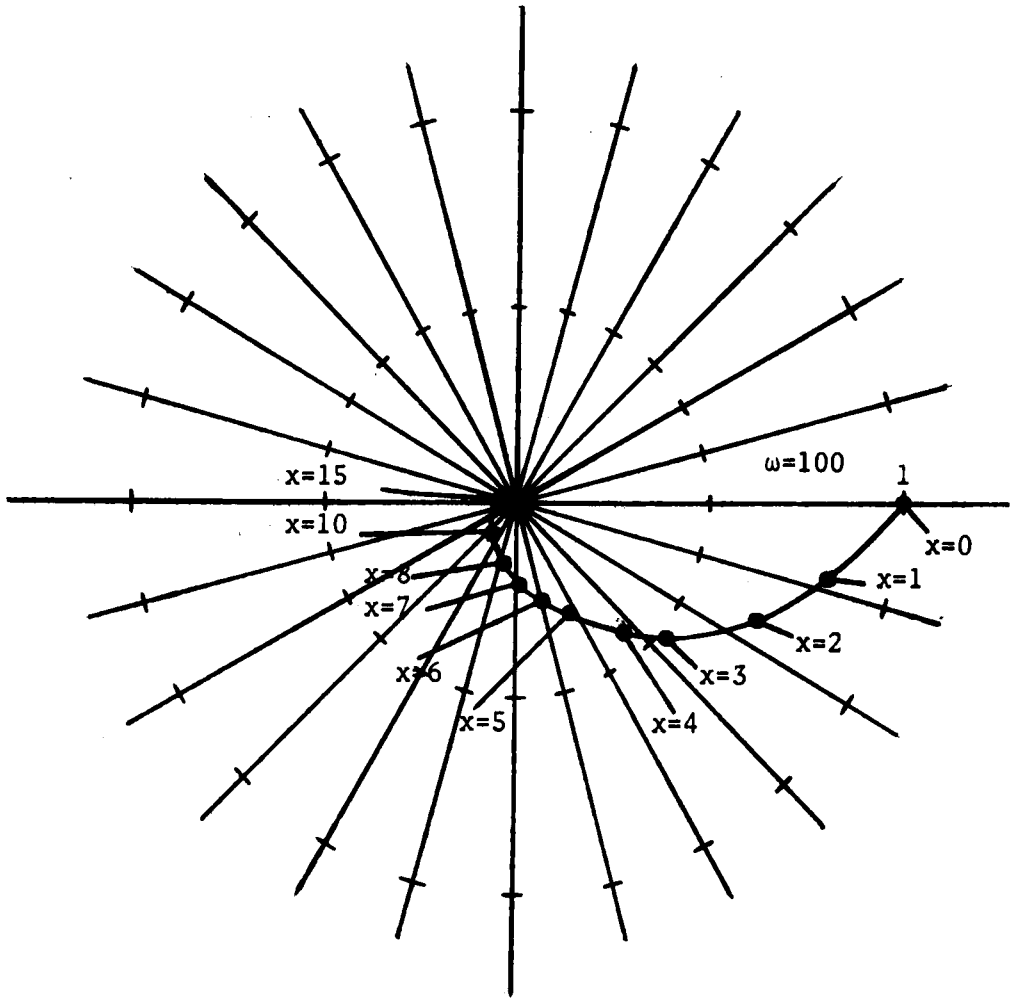


Figure VI-4. A graph of the voltage transfer ratio at $\omega = 100$.

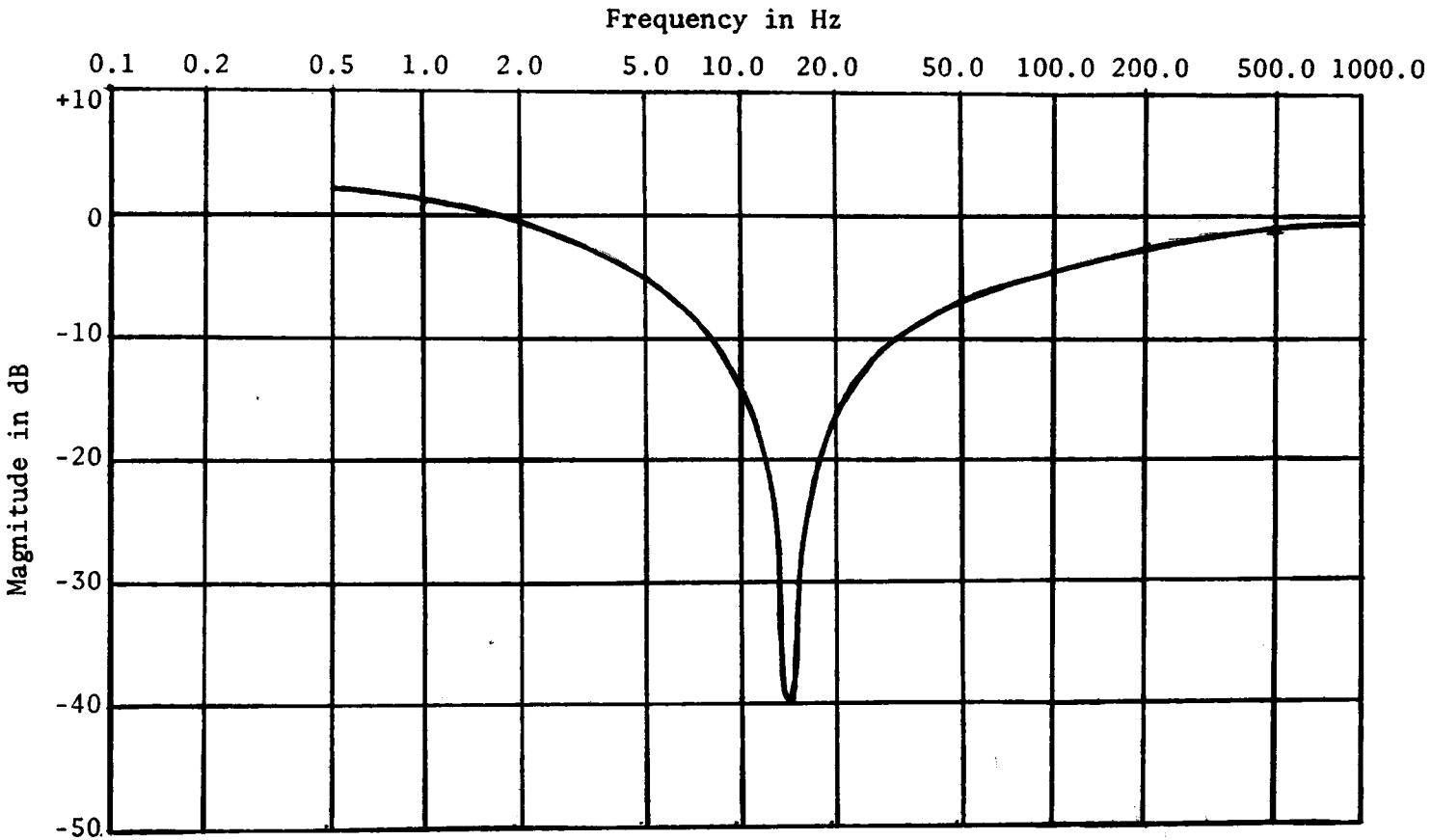


Figure VI-5. Experimental results of a notch filter design.

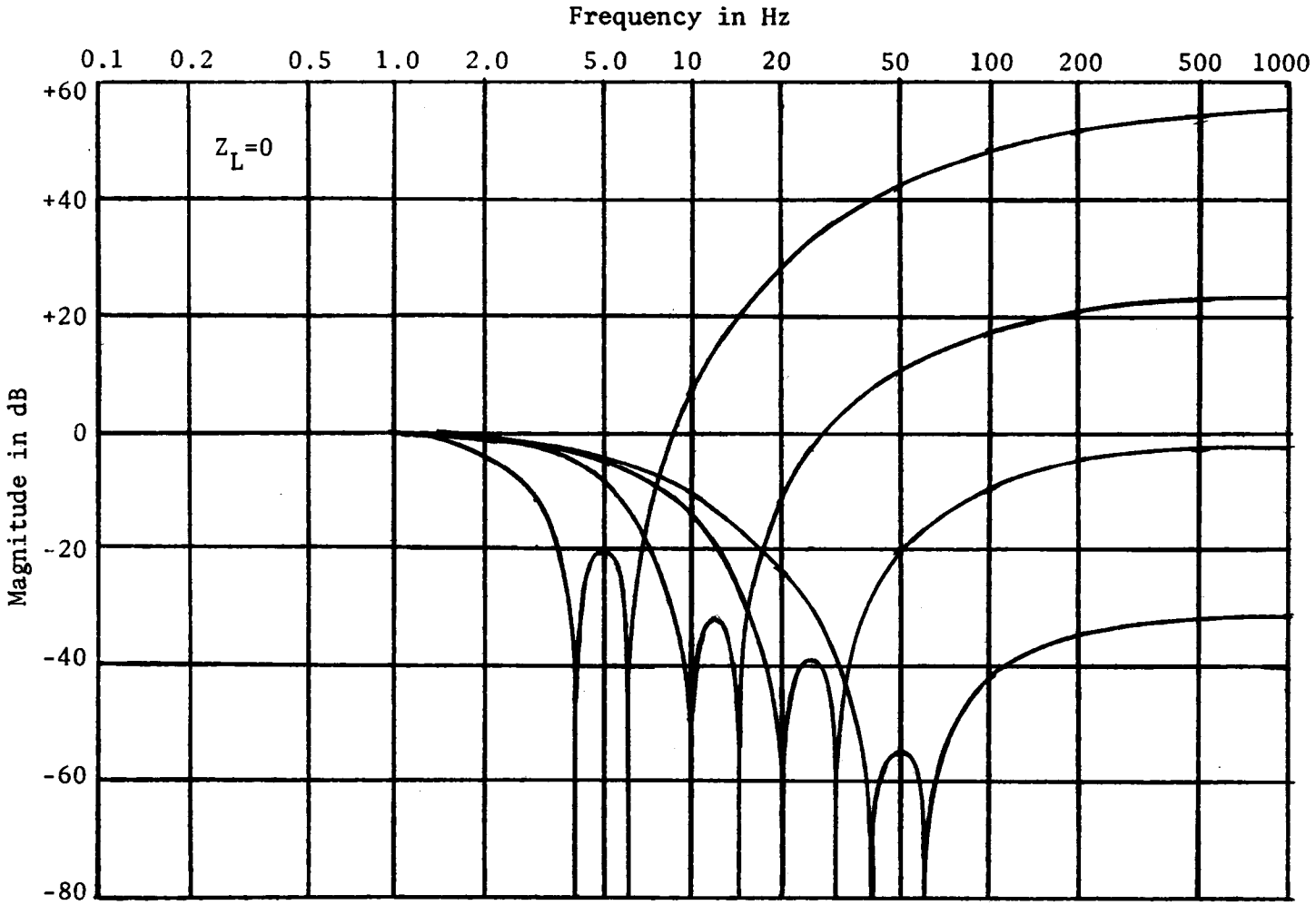


Figure VI-6. Magnitude frequency response of four double notch filters, using a shorted line.

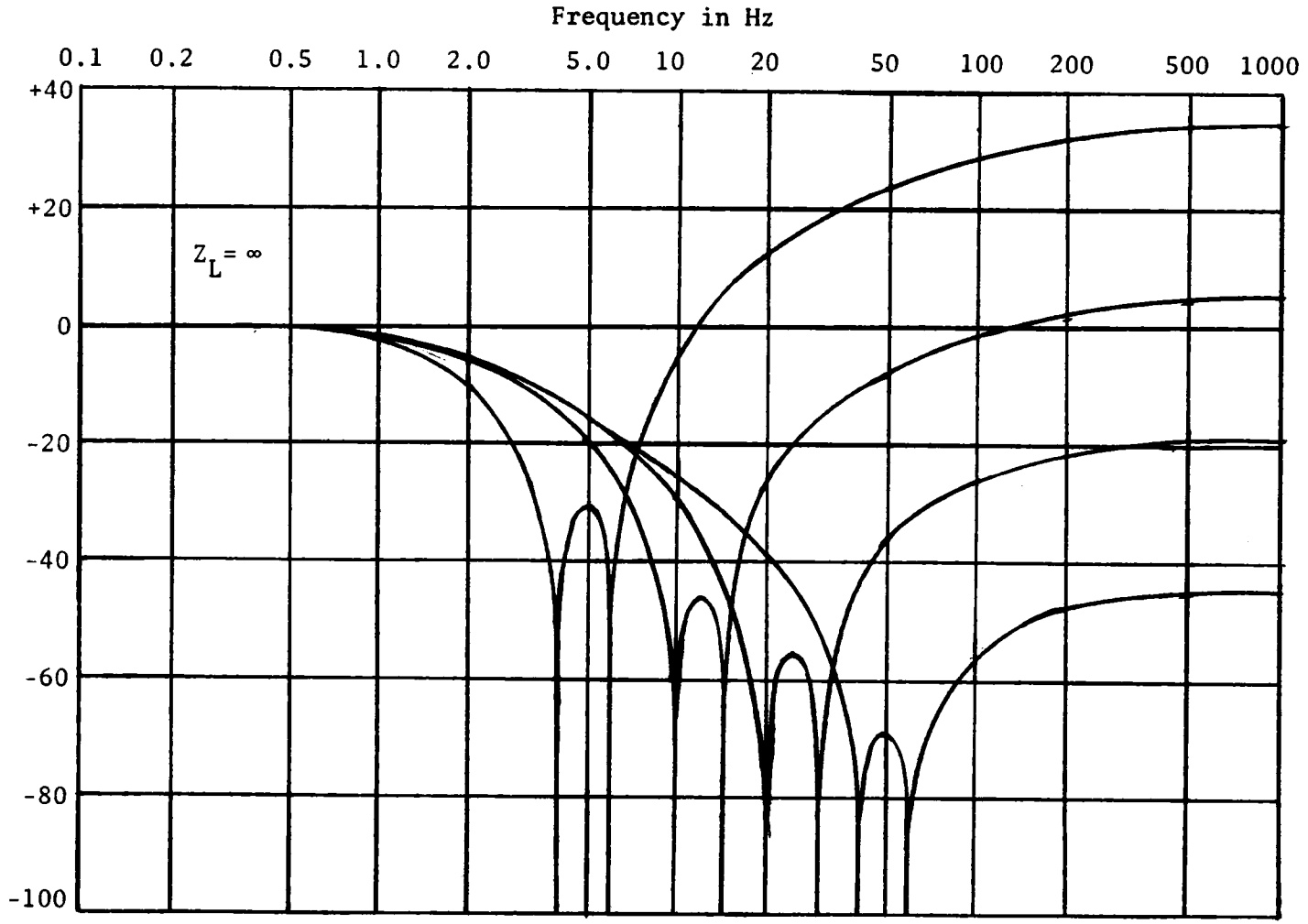


Figure VI-7. Magnitude frequency response of four double notch filters, using an open line.

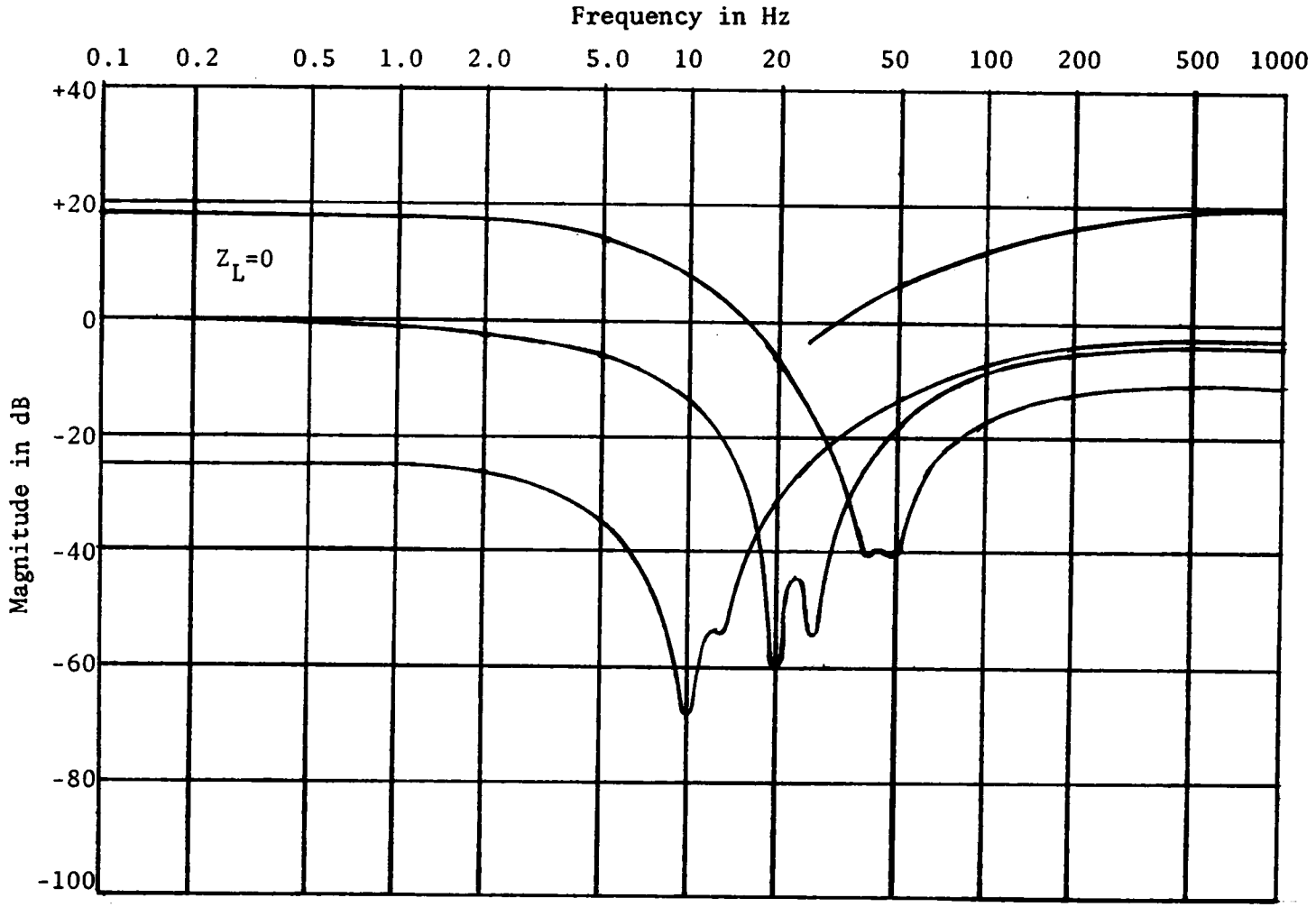


Figure VI-8. Experimental curve corresponding to the theoretical curve of Figure VI-6.

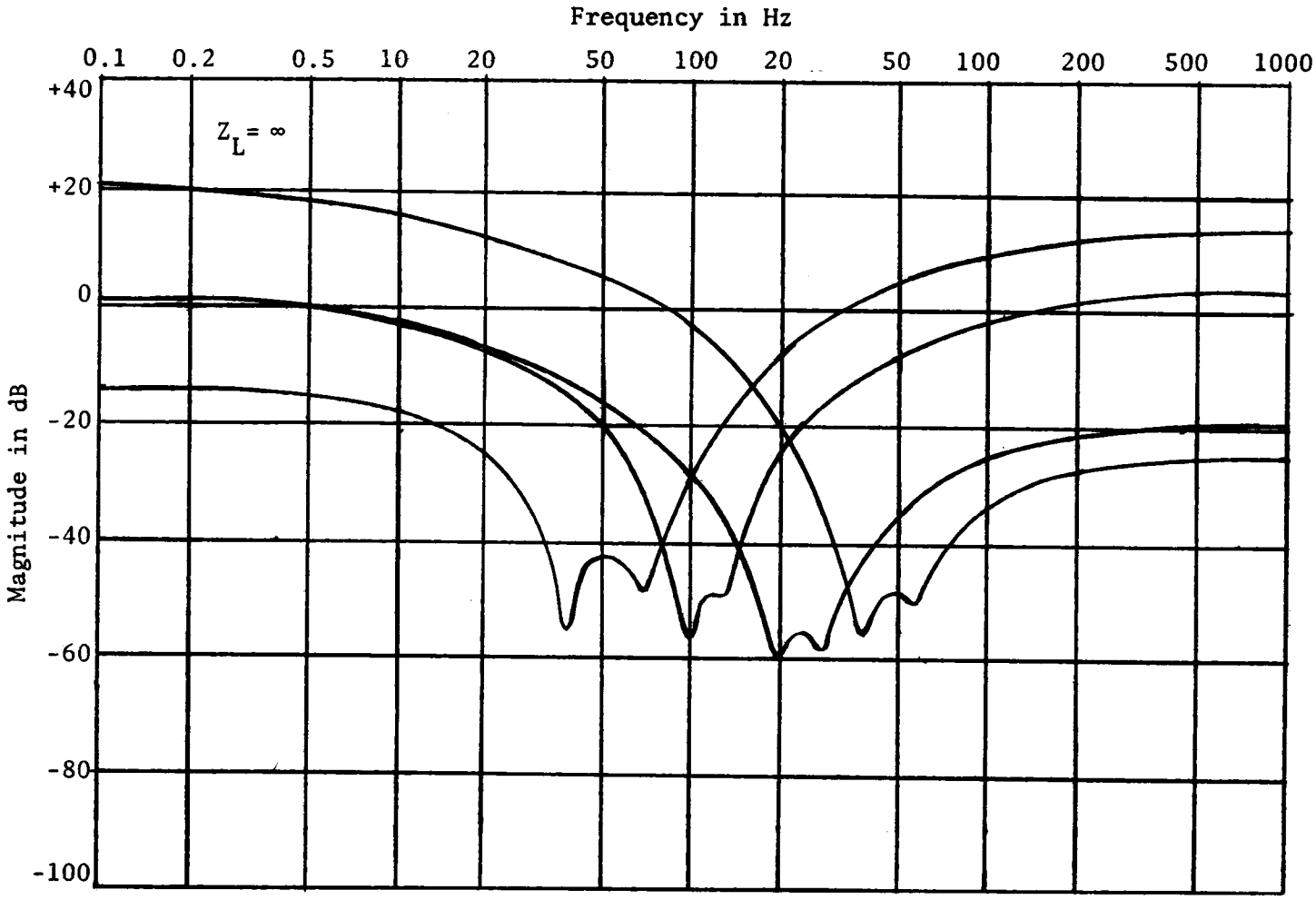


Figure VI-9. Experimental curve corresponding to the theoretical curve of Figure VI-7.

At very large voltage levels the amplifiers saturate and at very low voltage levels the signals are obscured in noise. In Figure VI-10 is shown a diagram of the actual experimental setup.

The experimental results agree very well with theory except at the null points. Here none of the experimental curves goes much below -60 dB because the stray pickup and amplifier noise are that large. The stray pickup came from unshielded patch cords used on the TR-10 analog computer. This component of noise could be minimized in a more expensive, specialized experimental setup. The pickup and noise were most serious in those filters which required large amplifier gains.

If the four filters in Figure VI-6 were cascaded, the overall magnitude frequency response would be that shown in Figure VI-11 as a solid line. This composite filter has a fast rolloff and stays below -37 dB for a decade. It then very quickly returns to approximately +45 dB. The dashed line in Figure VI-11 is the response of this filter cascaded with a three pole rolloff indicated by the dotted line. This three pole rolloff can easily be synthesized by ordinary techniques. The overall filter now rolls off at cutoff almost as fast as before but now stays down below -40 dB to infinity. This is now an approximation to the ideal low-pass filter. The cutoff frequency could be changed to any value desired by adjusting the series resistance per section and the shunt capacitance per section.

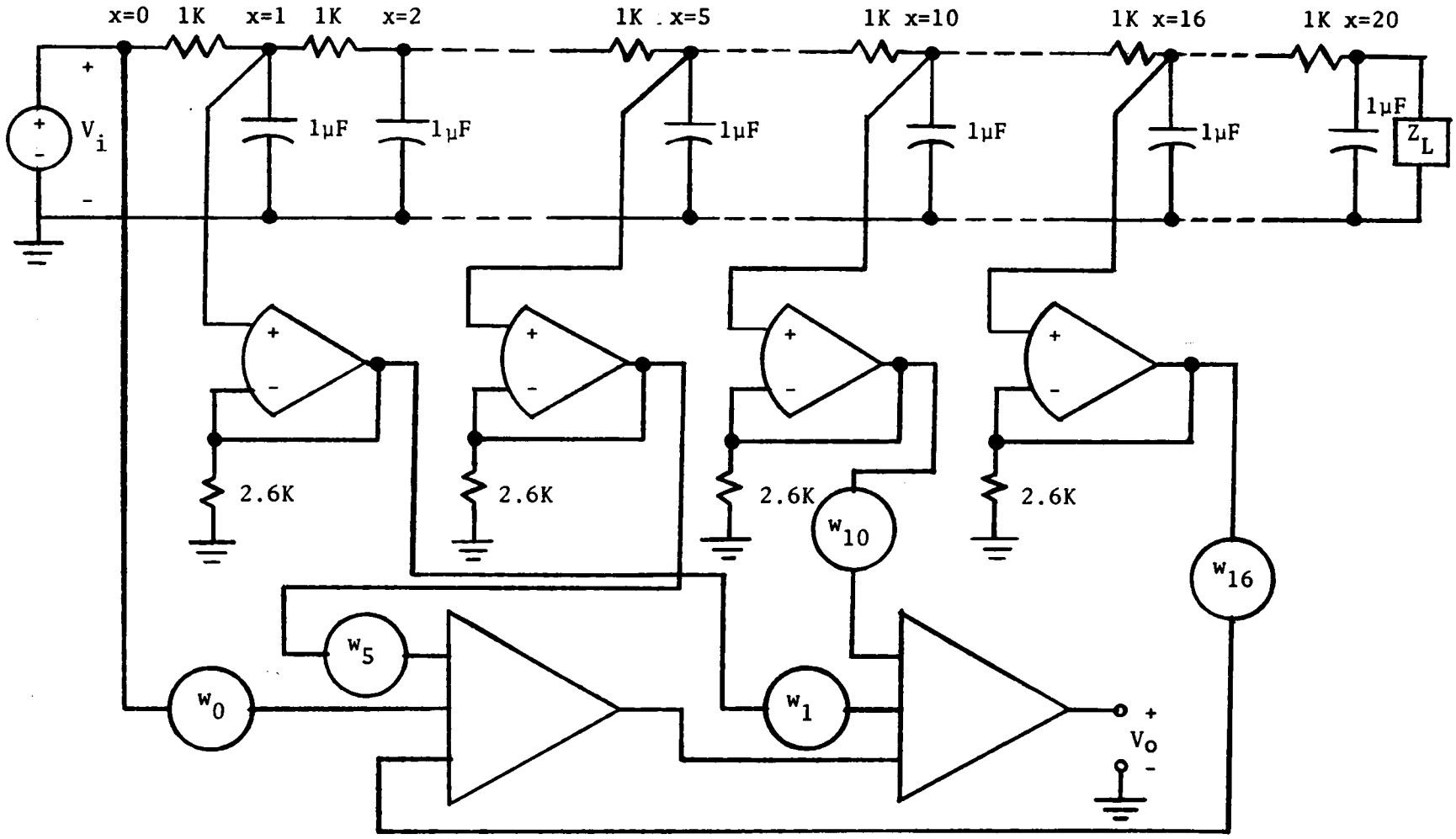


Figure VI-10. A typical experimental setup.

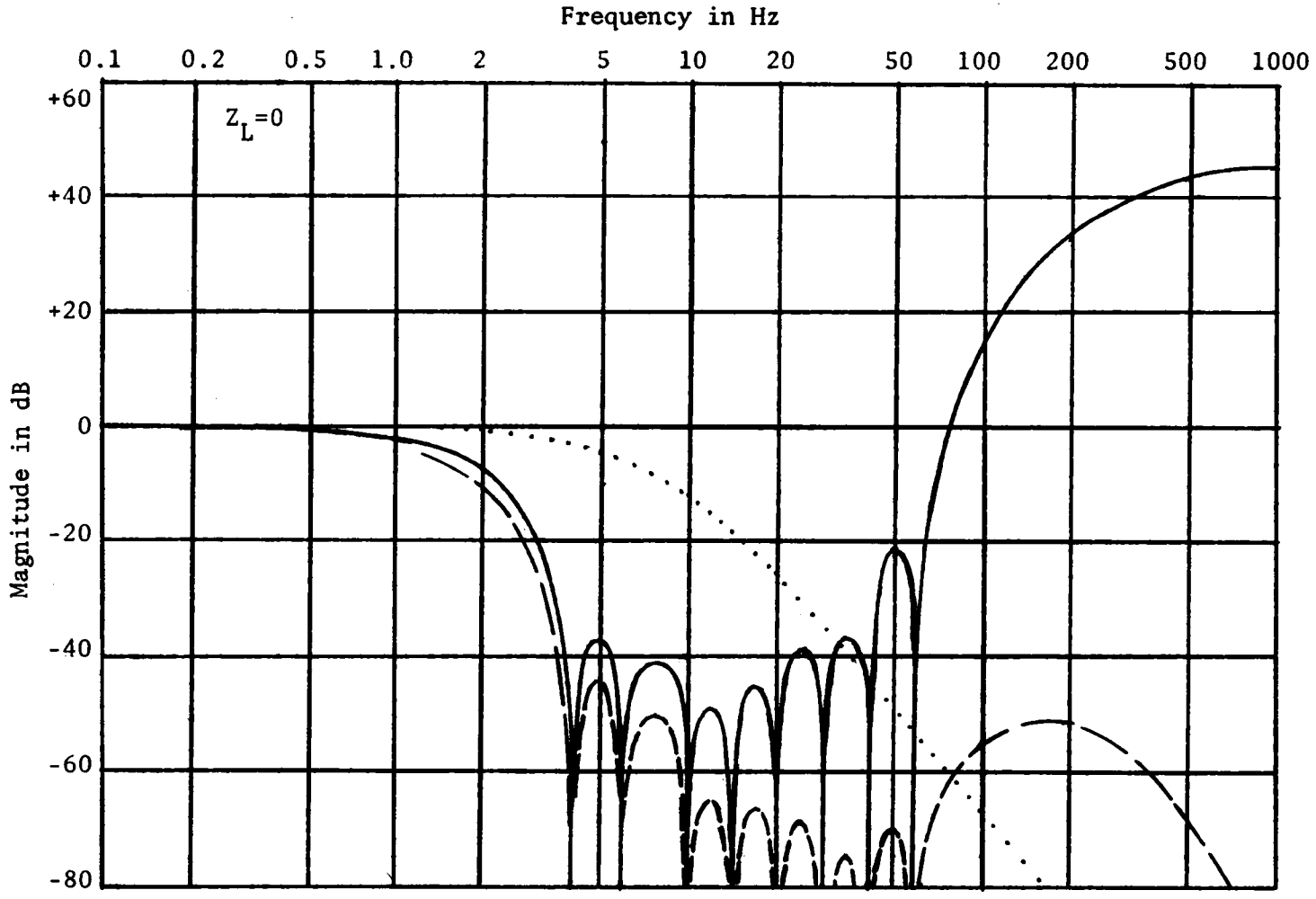


Figure VI-11. Magnitude frequency response of a composite filter.

CHAPTER VII

SUMMARY AND RECOMMENDATIONS

Methods for analyzing and synthesizing transmission lines with distributed sources have been developed and discussed. A complete solution for uniform lines was presented in analytical form. Two numerical analysis methods were described and computer programs implementing them were included in the appendices. The first numerical method solved for the frequency response of nonuniform transmission lines with distributed independent and dependent sources. It then found the impulse response by inverse fast fourier transforming the frequency response. A recommendation for future study is to expand the capability of this program to find the time response to an arbitrary exciting function by convolving this excitation with the impulse response. The second numerical method was designed to solve nonuniform and time variant transmission line problems. The programming techniques were explained with an emphasis on the use of the fast fourier transform.

Two synthesis techniques were presented and discussed. The impulse response technique with least squares curve fitting was found to be inadequate for approximating the ideal low-pass filter. A recommendation for future study is to investigate the possible synthesis of other types of impulse responses, with application to the design of matched filters, which optimize signal-to-noise ratios for particular waveforms.

The frequency response technique was more successfully applied to approximating the ideal low-pass filter. This method was by no means

fully explored. Both the frequency and impulse response methods were based on uniform lines without dependent sources. Research into non-uniform lines could lead to more nearly optimal synthesis techniques, since greater phase shift at the same attenuation can be achieved on a tapered line than on a uniform line. Exploration could also be made into time-variant lines.

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APPENDICES

APPENDIX A

Equations (3) and (4) of Chapter III are repeated here for convenience:

$$V_x = V_S + \mu V - ZI \tag{A-1}$$

$$I_x = I_S + \rho I - YV \ . \tag{A-2}$$

As stated in Chapter III, the solutions, V and I, take on various forms depending on the parameter values. Below is a systematic exposition of all the possible solution forms.

I. $Z = 0, Y \neq 0$

If, $0 \neq \mu \neq \rho \neq 0$,

$$V_x - \mu V = V_S \ . \tag{A-3}$$

The solution is,

$$V(x) = V(0)\ e^{\mu x} + \frac{V_S}{\mu}\ (e^{\mu x} - 1) \ . \tag{A-4}$$

The current equation is,

$$I_x - \rho I + YV = I_S \ . \tag{A-5}$$

The solution is,

$$I(x) = I(0)\ e^{\rho x} + \frac{1}{\rho}\left(I_S + \frac{YV_S}{\mu}\right)\left(e^{\rho x} - 1\right)$$

$$+Y\left(\frac{V(0) + \frac{V_S}{\mu}}{\rho - \mu}\right)\left(e^{\mu x} - e^{\rho x}\right) . \qquad \text{(A-6)}$$

Equations (A-4) and (A-6) can be rearranged to form the following equation in cascade parameter form:

$$\begin{bmatrix} V(x) \\ I(x) \end{bmatrix} = \begin{bmatrix} e^{\mu x} & 0 \\ \frac{Y(e^{\mu x} - e^{\rho x})}{\rho - \mu} & e^{\rho x} \end{bmatrix} \begin{bmatrix} V(0) \\ I(0) \end{bmatrix}$$

$$+ \begin{bmatrix} \frac{V_S}{\mu}\left(e^{\mu x} - 1\right) \\ \frac{e^{\rho x} - 1}{\rho}\left(I_S + \frac{YV_S}{\mu}\right) + \frac{YV_S\left(e^{\mu x} - e^{\rho x}\right)}{\mu(\rho - \mu)} \end{bmatrix} . \qquad \text{(A-7)}$$

If, $\mu = 0 \neq \rho$,

$$V_x = V_S . \qquad \text{(A-8)}$$

The solution is,

$$V(x) = V(0) + xV_S . \qquad \text{(A-9)}$$

The current equation is the same as (A-5), and the solution is,

$$I(x) = I(0)\ e^{\rho x} + \frac{YV_S}{\rho}x + \frac{1}{\rho}\left(Y\left(\frac{V_S}{\rho} + V(0)\right) - I_S\right)\left(1 - e^{\rho x}\right) . \qquad \text{(A-10)}$$

Putting (A-9) and (A-10) in cascade parameter form one obtains,

$$\begin{bmatrix} V(x) \\ I(x) \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ \frac{Y}{\rho}(1 - e^{\rho x}) & e^{\rho x} \end{bmatrix} \begin{bmatrix} V(0) \\ I(0) \end{bmatrix} + \begin{bmatrix} V_S x \\ \frac{1 - e^{\rho x}}{\rho}\left(\frac{YV_S}{\rho} - I_S\right) + \frac{YV_S}{\rho} x \end{bmatrix} . \tag{A-11}$$

If, $\mu = 0 = \rho$, the solution for the voltage is the same as (A-9). The current equation is,

$$I_x + YV = I_S . \tag{A-12}$$

The solution is,

$$I(x) = I(0) + x\,[I_S - YV(0)] - \frac{YV_S}{2} x^2 . \tag{A-13}$$

The solution in cascade parameter form is,

$$\begin{bmatrix} V(x) \\ I(x) \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ -Yx & 1 \end{bmatrix} \begin{bmatrix} V(0) \\ I(0) \end{bmatrix} + \begin{bmatrix} xV_S \\ x(I_S - \frac{YV_S}{2} x) \end{bmatrix} . \tag{A-14}$$

If, $\mu \neq 0 = \rho$, the solution for the voltage is the same as (A-4). The current equation is the same as (A-12), but since the voltage solution is not the same as (A-9), the current solution is not the same as (A-13). The current solution is,

$$I(x) = I(0) + \frac{Y}{\mu}\left(V(0) + \frac{V_S}{\mu}\right)\left(1 - e^{\mu x}\right) + x\left(I_S + \frac{YV_S}{\mu}\right) . \qquad \text{(A-15)}$$

The solution, in cascade parameter form, is,

$$\begin{bmatrix} V(x) \\ I(x) \end{bmatrix} = \begin{bmatrix} e^{\mu x} & 0 \\ \frac{Y}{\mu}(1 - e^{\mu x}) & 1 \end{bmatrix} \begin{bmatrix} V(0) \\ I(0) \end{bmatrix}$$

$$\begin{bmatrix} \frac{V_S}{\mu}(e^{\mu x} - 1) \\ YV_S\left(\frac{1-e^{\mu x}}{\mu^2} + \frac{x}{\mu}\right) \end{bmatrix} . \qquad \text{(A-16)}$$

If, $\mu = \rho \neq 0$, the voltage solution is the same as (A-4). The current equation is the same as (A-5), but the solution, (A-6) is invalid because $\rho = \mu$. The current solution is,

$$I(x) = I(0)\, e^{\mu x} + \frac{1}{\mu}\left(I_S + \frac{YV_S}{\mu}\right)\left(e^{\mu x} - 1\right)$$

$$- xe^{\mu x}\, Y\left(V(0) + \frac{V_S}{\mu}\right) . \qquad \text{(A-17)}$$

The solution, in cascade parameter form, is,

$$\begin{bmatrix} V(x) \\ \\ I(x) \end{bmatrix} = \begin{bmatrix} e^{\mu x} & 0 \\ \\ -xe^{\mu x}\, Y & e^{\mu x} \end{bmatrix} \begin{bmatrix} V(0) \\ \\ I(0) \end{bmatrix}$$

$$+ \begin{bmatrix} \frac{V_S}{\mu}\,(e^{\mu x} - 1) \\ \\ \frac{1}{\mu}\,(I_S + \frac{YV_S}{\mu})(e^{\mu x} - 1) - \frac{xe^{\mu x}\, YV_S}{\mu} \end{bmatrix} . \qquad \text{(A-18)}$$

II. $Z \neq 0,\ Y = 0$

All the solution in this section are dual to those in the previous section, therefore only the final solutions will be presented.

If $0 \neq \mu \neq \rho \neq 0$,

$$\begin{bmatrix} V(x) \\ \\ I(x) \end{bmatrix} = \begin{bmatrix} e^{\mu x} & \frac{Z(e^{\rho x} - e^{\mu x})}{\mu - \rho} \\ \\ 0 & e^{\rho x} \end{bmatrix} \begin{bmatrix} V(0) \\ \\ I(0) \end{bmatrix}$$

$$\begin{bmatrix} \frac{e^{\mu x} - 1}{\mu}\,(V_S + \frac{ZI_S}{\rho}) + \frac{ZI_S\,(e^{\rho x} - e^{\mu x})}{\rho(\mu - \rho)} \\ \\ \frac{I_S}{\rho}\,(e^{\rho x} - 1) \end{bmatrix} . \qquad \text{(A-19)}$$

If, $\mu = 0 \neq \rho$,

$$\begin{bmatrix} V(x) \\ I(x) \end{bmatrix} = \begin{bmatrix} 1 & \frac{Z}{\rho}(1 - e^{\rho x}) \\ 0 & e^{\rho x} \end{bmatrix} \begin{bmatrix} V(0) \\ I(0) \end{bmatrix}$$

$$+ \begin{bmatrix} ZI_S \left(\frac{1 - e^{\rho x}}{\rho^2} + \frac{x}{\rho}\right) \\ \frac{I_S}{\rho}(e^{\rho x} - 1) \end{bmatrix} \qquad \text{(A-20)}$$

If, $\mu = 0 = \rho$,

$$\begin{bmatrix} V(x) \\ I(x) \end{bmatrix} = \begin{bmatrix} 1 & -Zx \\ 0 & 1 \end{bmatrix} \begin{bmatrix} V(0) \\ I(0) \end{bmatrix} + \begin{bmatrix} x(V_S - \frac{ZI_S}{2} x) \\ xI_S \end{bmatrix} . \qquad \text{(A-21)}$$

If, $\mu \neq 0 = \rho$,

$$\begin{bmatrix} V(x) \\ I(x) \end{bmatrix} = \begin{bmatrix} e^{\mu x} & \frac{Z}{\mu}(1 - e^{\mu x}) \\ 1 & 1 \end{bmatrix} \begin{bmatrix} V(0) \\ I(0) \end{bmatrix}$$

$$+ \begin{bmatrix} \frac{1 - e^{\mu x}}{\mu}\left(\frac{ZI_S}{\mu} - V_S\right) + \frac{ZI_S}{\mu} x \\ xI_S \end{bmatrix} . \qquad \text{(A-22)}$$

If, $\mu = \rho \neq 0$,

$$\begin{pmatrix} V(x) \\ I(x) \end{pmatrix} = \begin{pmatrix} e^{\rho x} & -xe^{\rho x} Z \\ 0 & e^{\rho x} \end{pmatrix} \begin{pmatrix} V(0) \\ I(0) \end{pmatrix} + \begin{pmatrix} \frac{e^{\rho x} - 1}{\rho} \left(V_S + \frac{ZI_S}{\rho}\right) - \frac{I_S Z x e^{\rho x}}{\rho} \\ \frac{I_S}{\rho} (e^{\rho x} - 1) \end{pmatrix} . \quad \text{(A-23)}$$

III. $Z = Y = 0$

The individual solutions for voltage and current for each case in this section have all been found in previous cases in the preceding sections. Therefore, only the results in cascade parameter form will be presented here.

If $0 \neq \mu \neq \rho \neq 0$,

$$\begin{pmatrix} V(x) \\ I(x) \end{pmatrix} = \begin{pmatrix} e^{\mu x} & 0 \\ 0 & e^{\rho x} \end{pmatrix} \begin{pmatrix} V(0) \\ I(0) \end{pmatrix} + \begin{pmatrix} \frac{V_S}{\mu} (e^{\mu x} - 1) \\ \frac{I_S}{\rho} (e^{\rho x} - 1) \end{pmatrix} . \quad \text{(A-24)}$$

If, $\mu = 0 \neq \rho$,

$$\begin{pmatrix} V(x) \\ I(x) \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & e^{\rho x} \end{pmatrix} \begin{pmatrix} V(0) \\ I(0) \end{pmatrix} + \begin{pmatrix} xV_S \\ \frac{I_S}{\rho} (e^{\rho x} - 1) \end{pmatrix} . \quad \text{(A-25)}$$

If, $\mu = \rho = 0$,

$$\begin{pmatrix} V(x) \\ I(x) \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} V(0) \\ I(0) \end{pmatrix} + \begin{pmatrix} xV_S \\ xI_S \end{pmatrix} . \tag{A-26}$$

If, $\mu \neq 0 = \rho$,

$$\begin{pmatrix} V(x) \\ I(x) \end{pmatrix} = \begin{pmatrix} e^{\mu x} & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} V(0) \\ I(0) \end{pmatrix} + \begin{pmatrix} \frac{V_S}{\mu} (e^{\mu x} - 1) \\ xI_S \end{pmatrix} . \tag{A-21}$$

IV. $Z \neq 0$, $Y \neq 0$

The solution developed in Chapter III is only valid if $\gamma^2 \neq 0$, $\lambda \neq 0$, and $a_1 \neq a_2$. The other three possibilities will be explored in this section.

If, $\lambda \neq 0 = \gamma^2$, the equation, in voltage, to solve is,

$$V_{xx} + \lambda V_x = -ZI_S - \rho V_S . \tag{A-22}$$

The solution is,

$$V(x) = V(0)\, e^{-\lambda x} + \frac{1}{\lambda} [V_S + (\mu + \lambda)V(0) - ZI(0) - \frac{H}{\lambda}][1 - e^{-\lambda x}] + \frac{H}{\lambda} x . \tag{A-23}$$

The current solution is,

$$I(x) = \frac{1}{Z} \{V_S + \frac{H}{\lambda} (\mu x - 1) + (\mu + \lambda)V(0)\, e^{-\lambda x} + \frac{1}{\lambda} [V_S + (\mu + \lambda)V(0) - ZI(0) - \frac{H}{\lambda}][\mu - (\mu + \lambda)\, e^{-\lambda x}]\} . \tag{A-24}$$

The total solution in cascade parameter form is,

$$\begin{bmatrix} V(x) \\ I(x) \end{bmatrix} = \begin{bmatrix} \frac{\mu(1-e^{-\lambda x}) + \lambda}{\lambda} & \frac{Z}{\lambda}(e^{-\lambda x}-1) \\ \frac{(\mu+\lambda)\left[(1-e^{-\lambda x})\mu+\lambda e^{-\lambda x}\right]}{\lambda Z} & \frac{\mu(e^{-\lambda x}-1) + \lambda e^{-\lambda x}}{\lambda} \end{bmatrix} \begin{bmatrix} V(0) \\ I(0) \end{bmatrix}$$

$$\begin{bmatrix} \frac{1-e^{-\lambda x}}{\lambda}\left(V_S - \frac{H}{\lambda}\right) + \frac{H}{\lambda}x \\ \frac{1}{Z}\left[V_S + \frac{H}{\lambda}(\mu x-1) + \frac{V_S-\frac{H}{\lambda}}{\lambda}\left[\mu - (\mu+\lambda)e^{-\lambda x}\right]\right] \end{bmatrix} . \qquad \text{(A-25)}$$

In (A-23), (A-24), and (A-25),

$$H = -ZI_S - \rho V_S . \qquad \text{(A-26)}$$

If, $\lambda = 0 = \gamma^2$,

$$\begin{bmatrix} V(x) \\ I(x) \end{bmatrix} = \begin{bmatrix} \mu x + 1 & -Zx \\ \frac{\mu^2 x}{Z} & 1-\mu x \end{bmatrix} \begin{bmatrix} V(0) \\ I(0) \end{bmatrix} + \begin{bmatrix} x\left(\frac{Hx}{2} + V_S\right) \\ \frac{1}{Z}\left[\mu x V_S + Hx\left(\frac{\mu x}{2} - 1\right)\right] \end{bmatrix} . \qquad \text{(A-27)}$$

If, $\lambda \neq 0$, and $\gamma^2 = 0$, and $a_1 = a_2$,

$$\begin{bmatrix} V(x) \\ \\ I(x) \end{bmatrix} = \begin{bmatrix} e^{-\frac{\lambda x}{2}} x(\mu + \frac{\lambda}{2}) + 1 & -xZe^{-\frac{\lambda x}{2}} \\ \\ \frac{x}{Z}(\mu + \frac{\lambda}{2})^2 e^{-\frac{\lambda x}{2}} & e^{-\frac{\lambda x}{2}}[1 - (\mu + \frac{\lambda}{2})x] \end{bmatrix} \begin{bmatrix} V(0) \\ \\ I(0) \end{bmatrix}$$

$$+ \begin{bmatrix} e^{-\frac{\lambda x}{2}} \left(x(V_S + \frac{\lambda H}{2\gamma^2}) + \frac{H}{\gamma^2}\right) - \frac{H}{\gamma^2} \\ \\ \frac{1}{Z}\left(V_S - \frac{H}{\gamma^2}\left(\mu - e^{-\frac{\lambda x}{2}}(\mu + \frac{\lambda}{2})\right) + e^{-\frac{\lambda x}{2}}\left((\mu + \frac{\lambda}{2})x - 1\right)\left(V_S + \frac{\lambda H}{2\gamma^2}\right)\right) \end{bmatrix}$$

(A-28)

```
C.....................................................................
C.....                                                          .....
C.....     FREQUENCY RESPONSE ANALYSIS OF A TRANSMISSION        .....
C.....           LINE WITH DISTRIBUTED SOURCES                  .....
C.....                                                          .....
C.....................................................................
C
C
      REAL LINE(81),A(128),MINUS
      COMPLEX Z,Y,ELM,IRM,ZL,YR,VG,CG,CMPLX,DET,ZT,YT,VGT,CGT,ZTM,YTM,
     1BTM,EMX,EBX,VGTM,CGTM,MUT,BETAT,MU,BETA,MUL,ALP,GAM,A1,A2,AD,EA1X,
     2EA2X,A1MM,A2MM,C,EALX,  RA,VARS,       CEXP,CSQRT,VRC(128),VLC(128
     3),CONJG,BETAR,MUM
      DATA BLANK/' '/,MINUS/'-'/,PLUS/'+'/
      COMPLEX*16 A11P,A12P,A21P,A22P,A11T,A12T,A21T,A22T,A11,A12,A21,A22
     1,UVP,UIP,UV,UI,B(4),COEF(4,4),VAR(4)
      CALL ERRSET (208,256,-1,1,1)
      READ(5,10) D,XI1,XI2,XV1,XV2,FH,NSEC
   10 FORMAT (F12.0,5F10.0,I5)
      PRINT 399
  399 FORMAT(1H1)
      PRINT 400,D
  400 FORMAT (1X,24HTHE TRANSMISSION LINE IS,1PE15.7,12HMETERS LONG.)
      PRINT 401,FH
  401 FORMAT (1X,58HTHE FREQUENCY RESPONSES ARE COMPUTED OVER THE RANGE
     10HZ TO,1PE15.7,3HHZ.)
      PRINT 402,NSEC
  402 FORMAT (1X,28HTHE LINE IS APPROXIMATED BY ,I5,56H CASCADE CONNECTE
     1D SECTIONS OF UNIFORM TRANSMISSION LINE/4X,67HWITH CONSTANT INDEPE
     2NDENT CURRENT AND VOLTAGE SOURCE DISTRIBUTIONS.)
      PRINT 403,XI1,XI2
  403 FORMAT (1X,93HTHE INDEPENDENT CURRENT SOURCE DISTRIBUTION DESCRIBE
     1D BY COMPLEX FUNCTION CG LIES BETWEEN X =,1PE15.7/4X,14HMETERS AND
     2 X =,1PE15.7,7HMETERS.)
      PRINT 404,XV1,XV2
  404 FORMAT (1X,93HTHE INDEPENDENT VOLTAGE SOURCE DISTRIBUTION DESCRIBE
```

```
     1D BY COMPLEX FUNCTION VG LIES BETWEEN X =,1PE15.7/4X,14HMETERS AND
     2 X =,1PE15.7,7HMETERS.)
      DELTX = D/FLOAT(NSEC)
      INDEX = 7
      NP = 2**INDEX
      NPP1 = NP+1
      DF = FH/FLOAT(NP-1)
      DT = 1./(FLOAT(NP)*DF)
      DO 20 K = 1,NPP1
      W = 6.283185308*DF*FLOAT(K-1)
      BIGZ  = 0.
      BIGY = 0.
      BIGMU = 0.
      BIGBTA = 0.
      DO 40 KA = 1,NSEC
      XP2 = FLOAT(KA)*DELTX
      X = XP2-(DELTX/2.)
      XM2 = XP2-DELTX
      IF(CABS(ZT(X,W,XM2,XP2)).GT.BIGZ) BIGZ= CABS(ZT(X,W,XM2,XP2))
      IF(CABS(YT(X,W,XM2,XP2)).GT.BIGY)BIGY=CABS(YT(X,W,XM2,XP2))
      IF(CABS(MUT(X,W,XM2,XP2)).GT.BIGMU)BIGMU=CABS(MUT(X,W,XM2,XP2))
      IF(CABS(BETAT(X,W,XM2,XP2)).GT.BIGBTA)BIGBTA=CABS(BETAT(X,W,XM2,
     1XP2))
   40 CONTINUE
      A11P = CMPLX(1.,0.)
      A12P = CMPLX(0.,0.)
      A21P = A12P
      A22P = A11P
      UVP = CMPLX(0.,0.)
      UIP = UVP
      DO 30 N = 1,NSEC
      XP2 = FLOAT(N)*DELTX
      X = XP2 - (DELTX/2.)
      XM2 = XP2-DELTX
      MUM = MUT(X,W,XM2,XP2)
      ZTM = ZT(X,W,XM2,XP2)
```

```
      YTM = YT(X,W,XM2,XP2)
      BTM = BETAT(X,W,XM2,XP2)
      EMX = CEXP(MUM*DELTX)
      EBX = CEXP(BTM*DELTX)
      VGTM = VGT(X,W,XV1,XV2,DELTX,XM2,XP2)
      CGTM = CGT(X,W,XI1,XI2,DELTX,XM2,XP2)
      IF(CABS(ZTM).LE.(BIGZ/(10.**6)).AND.CABS(YTM).LE.(BIGY/(10.**6)) )
     1 GO TO 50
      IF(CABS(ZTM).LE.(BIGZ/(10.**6))) GO TO 60
      IF(CABS(YTM).LE.(BIGY/(10.**6))) GO TO 70
      ALP = 0.-MUM-BTM
      GAM = (YTM*ZTM) - (MUM*BTM)
      IF(CABS(ALP).EQ.0..AND.CABS(GAM).EQ.0.) GO TO 80
      IF(CABS(GAM).EQ.0.) GO TO 90
      A1 = (CSQRT((ALP**2)+(4.*GAM))-ALP)/2.
      A2 = 0.-(CSQRT((ALP**2)+(4.*GAM))+ALP)/2.
      AD = A2-A1
      EA1X = CEXP(A1*DELTX)
      EA2X = CEXP(A2*DELTX)
      A1MM = A1-MUM
      A2MM = A2-MUM
      C = ((ZTM*CGTM)+(BTM*VGTM))/GAM
      IF(CABS(AD).LT.(CABS(A1)/(10.**6)).OR.CABS(AD).EQ.0.) GO TO 140
      A11T = ((A2MM*EA1X)-(A1MM*EA2X))/AD
      A12T = ZTM*(EA1X-EA2X)/AD
      A21T = A1MM*A2MM*(EA2X-EA1X)/(AD*ZTM)
      A22T = ((A2MM*EA2X)-(A1MM*EA1X))/AD
      UV = (A11T*UVP)+(A12T*UIP)+((((VGTM+(A1*C))*EA2X)-((VGTM+(A2*C))
     1*EA1X))/AD)+C
      UI = (A21T*UVP)+(A22T*UIP)+((VGTM+(MUM*C)+(((VGTM+(A2*C))*A1MM*
     1EA1X)-((VGTM+(A1*C))*A2MM*EA2X))/AD)/ZTM)
      GO TO 150
  140 A11T = EA1X*(1.-(A1MM*DELTX))
      A12T = 0.-(EA1X*DELTX*ZTM)
      A21T = (A1MM**2)*EA1X*DELTX/ZTM
      A22T = EA1X*(1.+(A1MM*DELTX))
```

```
      UV = (A11T*UVP)+(A12T*UIP)+(EA1X*((DELTX*(VGTM+(A1*C)))-C))+C
      UI = (A21T*UVP)+(A22T*UIP)+((VGTM+(C*(MUM+(EA1X*A1MM))))-(EA1X*
     1((1.+(A1MM*DELTX))*(VGTM+(A1*C)))))/ZTM)
      GO TO 150
   90 EALX = CEXP(0.-(ALP*DELTX))
      A11T = ((MUM*(1.-EALX))+ALP)/ALP
      A12T = ZTM*(EALX-1.)/ALP
      A21T = (MUM+ALP)*(((1.-EALX)*MUM)+(ALP*EALX))/(ALP*ZTM)
      A22T = ((MUM*(EALX-1.))+(ALP*EALX))/ALP
      UV = (A11T*UVP)+(A12T*UIP)+((1.-EALX)*(VGTM+(((ZTM*CGTM)+(BTM*VGTM
     1))/ALP))/ALP)-(((ZTM*CGTM)+(BTM*VGTM))*DELTX/ALP)
      UI = (A21T*UVP)+(A22T*UIP)+((VGTM+((1.-(MUM*DELTX))*((ZTM*CGTM)
     1+(BTM*VGTM))/ALP)+((VGTM+(((ZTM*CGTM)+(BTM*VGTM))/ALP))*(MUM-
     2((MUM+ALP)*EALX))/ALP))/ZTM)
      GO TO 150
   80 A11T = (MUM*DELTX)+1.
      A12 T = 0.-(ZTM*DELTX)
      A21 T = (MUM**2)*DELTX/ZTM
      A22T = 1.-(MUM*DELTX)
      UV = (A11T*UVP)+(A12T*UIP)+(DELTX*(VGTM-(((ZTM*CGTM)+(BTM*VGTM))*
     1DELTX/2.)))
      UI = (A21T*UVP)+(A22T*UIP)+(((MUM*DELTX*VGTM)+((1.-(MUM*DELTX/2.))
     1*DELTX*((ZTM*CGTM)+(BTM*VGTM))))/ZTM)
      GO TO 150
   70 IF(CABS(BTM).LE.(BIGBTA/(10.**6)) .AND.CABS(MUM).LE.(BIGMU/(10.
     1**6))) GO TO 160
      IF (CABS(BTM).LE.(BIGBTA/(10.**6))) GO TO 170
      IF(CABS(MUM).LE.(BIGMU/(10.**6))) GO TO 180
      IF(CABS(MUM-BTM).LT.(CABS(MUM)/(10.**6)))  GO TO 190
      A11T = EMX
      A12T = ((EBX-EMX)*ZTM)/(MUM-BTM)
      A21 T = CMPLX(0.,0.)
      A22T = EBX
      UV = (A11T*UVP)+(A12T*UIP)+((EMX-1.)*(VGTM+((ZTM*CGTM)/BTM))/MUM)
     1+(ZTM*CGTM/BTM)*(EBX-EMX)/(MUM-ETM)
      UI = ((CGTM/BTM)*(EBX-1.))+(A21T*UVP)+(A22T*UIP)
```

```
      GO TO 150
  190 A11T = EBX
      A12T = 0.-(DELTX*ZTM*EBX)
      A21T = CMPLX(0.,0.)
      A22T = EBX
      UV = (A11T*UVP)+(A12T*UIP)+(((EBX-1.)/BTM)*(VGTM+(ZTM*CGTM/BTM)))
     1-(CGTM*ZTM*DELTX*EBX/BTM)
      UI = (A21T*UVP)+(A22T*UIP)+((CGTM/BTM)*(EBX-1.))
      GO TO 150
  180 A11T = CMPLX(1.,0.)
      A12T = (ZTM/BTM)*(1.-EBX)
      A21T = CMPLX(0.,0.)
      A22T = EBX
      UV = (A11T*UVP)+(A12T*UIP)+((ZTM*CGTM)*((DELTX/BTM)+((1.-EBX)
     1/(BTM**2))))
      UI = (A21T*UVP)+(A22T*UIP)+((CGTM/BTM)*(EBX-1.))
      GO TO 150
  170 A11T = EMX
      A12T = ZTM*(1.-EMX)/MUM
      A21T = CMPLX(0.,0.)
      A22T = CMPLX(1.,0.)
      UV = (A11T*UVP)+(A12T*UIP)+((1.-EMX)*((ZTM*CGTM/MUM)-VGTM)/MUM)+
     1(ZTM*CGTM*DELTX/MUM)
      UI = (A21T*UVP)+(A22T*UIP)+(CGTM*DELTX)
      GO TO 150
  160 A11T = CMPLX(1.,0.)
      A12T = 0.-(ZTM*DELTX)
      A21T = CMPLX(0.,0.)
      A22T = CMPLX(1.,0.)
      UV = (A11T*UVP)+(A12T*UIP)+(DELTX*(VGTM-(ZTM*CGTM*DELTX/2.)))
      UI = (A21T*UVP)+(A22T*UIP)+(CGTM*DELTX)
      GO TO 150
   60 IF(CABS(BTM).LE.(BIGBTA/(10.**6)).AND.CABS(MUM).LE.(BIGMU/(10.
     1**6))) GO TO 200
      IF(CABS(BTM).LE.(BIGBTA/(10.**6))) GO TO 210
      IF(CABS(MUM).LE.(BIGMU/(10.**6))) GO TO 220
```

```
      IF(CABS(MUM-BTM).LT.(CABS(MUM)/(10.**6))) GO TO 230
      A11T = EMX
      A12T = CMPLX(0.,0.)
      A21T = YTM*(EMX-EBX)/(BTM-MUM)
      A22T = EBX
      UV = (VGTM*(EMX-1.)/MUM)+(A11T*UVP)+(A12T*UIP)
      UI = (A21T*UVP)+(A22T*UIP)+((EBX-1.)*(CGTM+(YTM*VGTM/MUM))/BTM)
     1+(YTM*VGTM*(EMX-EBX)/(MUM*(BTM-MUM)))
      GO TO 150
  230 A11T = EMX
      A12T = CMPLX(0.,0.)
      A21T = 0.-(DELTX*EMX*YTM)
      A22T = EMX
      UV = (A11T*UVP)+(A12T*UIP)+(VGTM*(EMX-1.)/MUM)
      UI = (A21T*UVP)+(A22T*UIP)+((EMX-1.)*(CGTM+(YTM*VGTM/MUM))/MUM)-
     1(DELTX*EMX*YTM*VGTM/MUM)
      GO TO 150
  220 A11T = CMPLX(1.,0.)
      A12T = CMPLX(0.,0.)
      A21T = YTM*(1.-EBX)/BTM
      A22T = EBX
      UV = (A11T*UVP)+(A12T*UIP)+(VGTM*DELTX)
      UI = (A21T*UVP)+(A22T*UIP)+((1.-EBX)*((YTM*VGTM/BTM)-CGTM)/BTM
     1)+(YTM*VGTM*DELTX/BTM)
      GO TO 150
  210 A11T = EMX
      A12T = CMPLX(0.,0.)
      A21T = YTM*(1.-EMX)/MUM
      A22T = CMPLX(1.,0.)
      UV = (A11T*UVP)+(A12T*UIP)+(VGTM*(EMX-1.)/MUM)
      UI = (A21T*UVP)+(A22T*UIP)+(YTM*VGTM*(((1.-EMX)/(MUM**2))+(DELTX/M
     1UM)))
      GO TO 150
  200 A11T = CMPLX(1.,0.)
      A12T = CMPLX(0.,0.)
      A21T = 0.-(DELTX*YTM)
```

```
      A22T = CMPLX(1.,0.)
      UV = (A11T*UVP)+(A12T*UIP)+(DELTX*VGTM)
      UI = (A21T*UVP)+(A22T*UIP)+(DELTX*(CGTM-(YTM*VGTM*DELTX/2.)))
      GO TO 150
   50 IF(CABS(BTM).LE.(BIGBTA/(10.**6)).AND.CABS(MUM).LE.(BIGMU/(10.
     1**6))) GO TO 240
      IF(CABS(BTM).LE.(BIGBTA/(10.**6))) GO TO 250
      IF(CABS(MUM).LE.(BIGMU/(10.**6))) GO TO 260
      A11T = EMX
      A12T = CMPLX(0.,0.)
      A21T = CMPLX(0.,0.)
      A22T = EBX
      UV = (A11T*UVP)+(A12T*UIP)+(VGTM*(EMX-1.)/MUM)
      UI = (A21T*UVP)+(A22T*UIP)+(CGTM*(EBX-1.)/BTM)
      GO TO 150
  260 A11T = CMPLX(1.,0.)
      A12T = CMPLX(0.,0.)
      A21T = CMPLX(0.,0.)
      A22T = EBX
      UV = (A11T*UVP)+(A12T*UIP)+(VGTM*DELTX)
      UI = (A21T*UVP)+(A22T*UIP)+(CGTM*(EBX-1.)/BTM)
      GO TO 150
  250 A11T = EMX
      A12T = CMPLX(0.,0.)
      A21T = CMPLX(0.,0.)
      A22T = CMPLX(1.,0.)
      UV = (A11T*UVP)+(A12T*UIP)+(VGTM*(EMX-1.)/MUM)
      UI = (A21T*UVP)+(A22T*UIP)+(CGTM*DELTX)
      GO TO 150
  240 A11T = CMPLX(1.,0.)
      A12T = CMPLX(0.,0.)
      A21T = CMPLX(0.,0.)
      A22T = CMPLX(1.,0.)
      UV = (A11T*UVP)+(A12T*UIP)+(DELTX*VGTM)
      UI = (A21T*UVP)+(A22T*UIP)+(DELTX*CGTM)
  150 A11 = (A11T*A11P)+(A12T*A21P)
```

```
      A12 = (A11T*A12P)+(A12T*A22P)
      A21 = (A21T*A11P)+(A22T*A21P)
      A22 = (A21T*A12P)+(A22T*A22P)
      UVP = UV
      UIP = UI
      A11P = A11
      A12P =A12
      A21P = A21
   30 A22P = A22
      COEF(1,1) = CMPLX(1.,0.)
      COEF(1,2) = CMPLX(0.,0.)
      COEF(1,3) = 0.-A11
      COEF(1,4) = 0.-A12
      COEF(2,1) = CMPLX(0.,0.)
      COEF(2,2) = CMPLX(1.,0.)
      COEF(2,3)  = 0.-A21
      COEF (2,4) = 0.-A22
      COEF(3,1) = CMPLX(0.,0.)
      COEF (3,2) = CMPLX(0.,0.)
      COEF (3,3) = 1.-MUL(W)
      COEF(3,4) = ZL(W)
      COEF(4,1) = 0.-YR(W)
      COEF(4,2) = 1.+BETAR(W)
      COEF(4,3) = CMPLX(0.,0.)
      COEF(4,4) = CMPLX(0.,0.)
      B(1) = UV
      B(2) = UI
      B(3) = ELM(W)
      B(4) = 0.-IRM(W)
      CALL SOLVE (COEF,B,VAR,DETER)
      IF(DETER.EQ.0.) GO TO 1000
      IF(K.EQ.129) GO TO 20
      VRC(K) = VAR(1)
      VLC (K) = VAR(3)
   20 CONTINUE
      DO 600 II = 1,128
```

```
600 A(II) = CABS(VLC(II))
    PRINT 1
  1 FORMAT (1H1)
    PRINT 2
  2 FORMAT (1X,113HGRAPH OF THE MAGNITUDE OF THE VOLTAGE AT THE LEFT E
   1ND OF THE TRANSMISSION LINE, IN VOLTS, VERSUS FREQUENCY, IN HZ)
    CALL GRAPHM(A  ,DF,128,BLANK,MINUS,PLUS,LINE)
    DO 610 IJ = 1,128
    IF(CABS(VLC(IJ)).EQ.0.) GO TO 605
    A(IJ) = ATAN2(AIMAG(VLC(IJ)),REAL(VLC(IJ)))*360./(2.*3.14159265)
    GO TO 610
605 A(IJ) = 0.
610 CONTINUE
    PRINT 3
  3 FORMAT (1H1)
    PRINT 4
  4 FORMAT(1X,111HGRAPH OF THE PHASE OF THE VOLTAGE AT THE LEFT END OF
   1 THE TRANSMISSION LINE, IN DEGREES, VERSUS FREQUENCY, IN HZ)
    CALL GRAPH (A  ,DF,128,BLANK,MINUS,PLUS,LINE,0)
    DO 620 IK = 1,128
620 A (IK) = CABS(VRC(IK))
    PRINT 5
  5 FORMAT (1H1)
    PRINT 6
  6 FORMAT (1X,114HGRAPH OF THE MAGNITUDE OF THE VOLTAGE AT THE RIGHT
   1END OF THE TRANSMISSION LINE, IN VOLTS, VERSUS FREQUENCY, IN HZ)
    CALL GRAPHM(A  ,DF,128,BLANK,MINUS,PLUS,LINE)
    DO 630 IM = 1,128
    IF(CABS(VRC(IM)).EQ.0.) GO TO 625
    A(IM) = ATAN2(AIMAG(VRC(IM)),REAL(VRC(IM)))*360./(2.*3.14159265)
    GO TO 630
625 A(IM) = 0.
630 CONTINUE
    PRINT 7
  7 FORMAT (1H1)
    PRINT 8
```

```
  8 FORMAT(1X,112HGRAPH OF THE PHASE OF THE VOLTAGE AT THE RIGHT END O
   1F THE TRANSMISSION LINE, IN DEGREES, VERSUS FREQUENCY, IN HZ)
    CALL GRAPH (A  ,DF,128,BLANK,MINUS,PLUS,LINE,0)
    DO 640 JI = 2,65
    MJI = 130-JI
    VLC(JI) = VLC(JI)+CONJG(VLC(MJI))
    VLC(MJI) = CONJG(VLC(JI))
    VRC(JI) = VRC(JI)+CONJG(VRC(MJI))
640 VRC(MJI) = CONJG(VRC(JI))
    VARS = VAR(3)
    VLC(1) = VLC(1)+CONJG(VARS)
    VARS = VAR(1)
    VRC(1) = VRC(1)+CONJG(VARS)
    CALL FFT(VLC,INDEX,NP)
    CALL FFT (VRC,INDEX,NP)
    DO 650 JK = 1,128
    VLC(JK) = CONJG(VLC(JK))/(128.*DT)
650 VRC(JK) = CONJG(VRC(JK))/(128.*DT)
    DO 660 JL = 1,128
660 A(JL) = REAL(VLC(JL))
    PRINT 11
 11 FORMAT (1H1)
    PRINT 12
 12 FORMAT(1X,105HGRAPH OF THE IMPULSE RESPONSE AT THE LEFT END OF THE
   1 TRANSMISSION LINE, IN VOLTS, VERSUS TIME, IN SECONDS)
    CALL GRAPH (A,DT,128,BLANK,MINUS,PLUS,LINE,1)
    DO 670 JM = 1,128
670 A(JM) = REAL(VRC(JM))
    PRINT 13
 13 FORMAT (1H1)
    PRINT 14
 14 FORMAT (1X,106HGRAPH OF THE IMPULSE RESPONSE AT THE RIGHT END OF T
   1HE TRANSMISSION LINE, IN VOLTS, VERSUS TIME, IN SECONDS)
    CALL GRAPH (A,DT,128,BLANK,MINUS,PLUS,LINE,1)
    STOP
1000 PRINT 999
```

```
 999 FORMAT (1H1)
     FREQ = W/6.283185308
     PRINT 1001,FREQ
1001 FORMAT(1X,65HA SINGULAR MATRIX HAS BEEN ENCCUNTERED AT A CYCLICAL
    1FREQUENCY OF,1PE15.7,3HHZ.)
     PRINT 1002
1002 FORMAT(1X,89HPROBABLE CAUSE IS A RESCNANCE CR A FREQUENCY SO LOW C
    1R SO HIGH THAT RCUNDOFF ERRCR CAUSES/1X,43HTHE MATRIX TC SEEM SING
    2ULAR WHEN IT IS NCT.)
     STOP
     END
     SUBROUTINE SOLVE (A,B,V,DET)
     COMPLEX*16 A(4,4),B(4),V(4),DUM
     DIMENSION IS(4)
     DET = 1.
     BIG = 0.
     DO 10 I = 1,4
     DO 10 J = 1,4
     IF(CDABS(A(I,J)).LE.BIG) GO TC 10
     IR = I
     IC = J
     BIG = CDABS(A(I,J))
  10 CONTINUE
     IS(1) = 1
     IS(2) = 2
     IS(3) = 3
     IS(4) = 4
     IF(IR.EQ.1) GO TO 30
     DO 20 KA = 1,4
     DUM = A(IR,KA)
     A(IR,KA) = A(1,KA)
  20 A(1,KA) = DUM
     DUM = B(IR)
     B(IR) = B(1)
     B(1) = DUM
  30 IF (IC.EQ.1) GO TO 40
```

```
      DO 50 L = 1,4
      DUM = A(L,IC)
      A(L,IC) = A(L,1)
   50 A(L,1) = DUM
      IDUM = IS(1)
      IS(1) = IS(IC)
      IS(IC) = IDUM
   40 IF(CDABS(A(1,1)).EQ.0.) GO TO 1000
      DO 55 N = 2,4
   55 A(1,N) = A(1,N)/A(1,1)
      B(1) = B(1)/A(1,1)
      A(1,1) = CMPLX(1.,0.)
      DO 60 M = 2,4
      IF(CDABS(A(M,1)).EQ.0.) GO TO 60
      DO 70 IA = 2,4
   70 A(M,IA) = A(M,IA)-(A(1,IA)*A(M,1))
      B(M) = B(M) -(B(1)*A(M,1))
   60 CONTINUE
      BIG = 0.
      DO 80 IB = 2,4
      DO 80 ID = 2,4
      IF(CDABS(A(IB,ID)).LE.BIG) GO TO 80
      IR = IB
      IC = ID
      BIG = CDABS(A(IB,ID))
   80 CONTINUE
      IF(IR.EQ.2) GO TO 90
      DO 100 IE = 2,4
      DUM = A(IR,IE)
      A(IR,IE) = A(2,IE)
  100 A(2,IE) = DUM
      DUM = B(IR)
      B(IR) = B(2)
      B(2) = DUM
   90 IF(IC.EQ.2) GO TO 110
      DO 120 IF = 1,4
```

```
      DUM = A(IF,IC)
      A(IF,IC) = A(IF,2)
120   A(IF,2) = DUM
      IDUM = IS(2)
      IS(2) = IS(IC)
      IS(IC) = IDUM
110    IF(CDABS(A(2,2)).EQ.0.) GO TO 1000
      DO 130 IG = 3,4
130   A(2,IG) = A(2,IG)/A(2,2)
      B(2) = B(2)/A(2,2)
      A(2,2) = CMPLX(1.,0.)
      DO 140 IH = 3,4
      IF(CDABS(A(IH,2)).EQ.0.) GO TO 140
      DO 150 IJ = 3,4
150   A(IH,IJ) = A(IH,IJ)-(A(2,IJ)*A(IH,2))
      B(IH) = B(IH)-(B(2)*A(IH,2))
140   CONTINUE
      BIG = 0.
      DO 160 IK = 3,4
      DO 160 IL = 3,4
      IF(CDABS(A(IK,IL)).LE.BIG) GO TO 160
      IR = IK
      IC = IL
      BIG = CDABS(A(IK,IL))
160    CONTINUE
      IF(IR.EQ.3) GO TO 170
      DO 180 IM = 3,4
      DUM = A(IR,IM)
      A(IR,IM) = A(3,IM)
180    A(3,IM) = DUM
      DUM = B(4)
      B(4) = B(3)
      B(3) = DUM
170   IF(IC.EQ.3) GO TO 190
      DO 200 IN = 1,4
      DUM = A(IN,IC)
```

```
      A(IN,IC) = A(IN,3)
  200 A(IN,3) = DUM
      IDUM = IS(3)
      IS(3) = IS(IC)
      IS(IC) = IDUM
  190 IF(CDABS(A(3,3)).EQ.0.) GO TO 1000
      A(3,4) = A(3,4)/A(3,3)
      B(3) = B(3)/A(3,3)
      A(3,3) = CMPLX(1.,0.)
      IF(CDABS(A(4,3)).EQ.0.) GO TO 220
      A(4,4) = A(4,4)-(A(3,4)*A(4,3))
      B(4) = B(4)-(B(3)*A(4,3))
  220 IF(CDABS(A(4,4)).EQ.0.) GO TO 1000
      IS1 = IS(1)
      IS2 = IS(2)
      IS3 = IS(3)
      IS4 = IS(4)
      V(IS4) = B(4)/A(4,4)
      V(IS3) = (B(3)-(A(3,4)*V(IS4)))/A(3,3)
      V(IS2) = (B(2)-(A(2,4)*V(IS4))-(A(2,3)*V(IS3)))/A(2,2)
      V(IS1) = (B(1)-(A(1,4)*V(IS4))-(A(1,3)*V(IS3))-(A(1,2)*V(IS2)))/
     1A(1,1)
      RETURN
 1000 DET = 0.
      RETURN
      END
      SUBROUTINE GRAPH (Y,CT,I,BLANK,MINUS,PLUS,LINE,NFL)
      REAL Y(I),LINE(81),MINUS
      PRINT 10
   10 FORMAT (1HO)
      DO 20 J = 1,81
   20 LINE(J) = MINUS
      BIG = ABS(Y(1))
      DO 30 K = 2,I
      IF(ABS(Y(K)).GT.BIG) BIG = ABS(Y(K))
   30 CONTINUE
```

```
      IF(BIG.EQ.0.) GO TO 1
      L = IFIX((40.*((Y(1)/BIG)+1.))+1.5)
      GO TO 2
    1 L = 41
    2 LINE(L) = PLUS
      T = 0.
      IF(NFL.EQ.1) GO TO 160
      PRINT 40,T,Y(1),LINE
   40 FORMAT (1X,6HFREQ =,1PE15.7,4X,3HY =,1PE15.7,2X,81A1)
      GO TO 170
  160 PRINT 180 , T,Y(1),LINE
  180 FORMAT (1X,6HTIME =,1PE15.7,4X,3HY =,1PE15.7,2X,81A1)
  170 DO 50 M = 2,I
      DO 60 N = 1,81
   60 LINE(N) = BLANK
      LINE(41) = MINUS
      IF(BIG.EQ.0.) GO TO 3
      L = IFIX((40.*((Y(M)/BIG)+1.))+1.5)
      GO TO 4
    3 L = 41
    4 LINE(L) = PLUS
      IF(L.GE.40.AND.L.LE.42) GO TO 100
      IF (L.LT.41) GO TO 110
      IF(L.EQ.43) GO TO 120
      LM1 = L-1
      DO 80 II = 42,LM1
   80 LINE(II) = MINUS
  120 LINE(42) = MINUS
      GO TO 100
  110 IF(L.EQ.39) GO TO 130
      LP1 = L+1
      DO 90 JJ = LP1,40
   90 LINE(JJ) = MINUS
  130 LINE(40) = MINUS
  100 T = FLOAT(M-1)*DT
      IF(NFL.EQ.1) GO TO 140
```

```
      PRINT 70,T,Y(M),LINE
   70 FORMAT (1X,6HFREQ =,1PE15.7,4X,3HY =,1PE15.7,2X,81A1)
      GO TO 50
  140 PRINT 150,T,Y(M),LINE
  150 FORMAT(1X,6HTIME =,1PE15.7,4X,3HY =,1PE15.7,2X,81A1)
   50 CONTINUE
      RETURN
      END
      SUBROUTINE GRAPHM (Y,DF,I,BLANK,MINUS,PLUS,LINE)
      REAL Y(I),LINE(81),MINUS
      PRINT 10
   10 FORMAT (1H0)
      DO 20 J = 1,81
   20 LINE(J) = MINUS
      BIG = ABS(Y(1))
      DO 30 K = 2,I
      IF(ABS(Y(K)).GT.BIG) BIG = ABS(Y(K))
   30 CONTINUE
      IF(BIG.EQ.0.) GO TO 1
      L = IFIX(80.*(Y(1)/BIG))+1.5
      GO TO 2
    1 L = 1
    2 LINE(L) = PLUS
      F = 0.
      PRINT 40,F,Y(1),LINE
   40 FORMAT (1X,6HFREQ =,1PE15.7,4X,3HY =,1PE15.7,2X,81A1)
      DO 50 M = 2,I
      DO 60 N = 1,81
   60 LINE(N) = BLANK
      LINE(1 ) = MINUS
      IF (BIG.EQ.0.) GO TO 3
      L = IFIX(80.*(Y(M)/BIG))+1.5
      GO TO 4
    3 L = 1
    4 LINE(L) = PLUS
      IF(L.EQ.1..OR.L.EQ.2.) GO TO 100
```

```
      IF(L.EQ.3.) GO TO 90
      LM1 = L-1
      DO 80 II = 2,LM1
   80 LINE(II) = MINUS
   90 LINE(2 ) = MINUS
  100 F = FLOAT(M-1)*DF
      PRINT 70,F,Y(M),LINE
   70 FORMAT (1X,6HFREQ =,1PE15.7,4X,3HY =,1PE15.7,2X,81A1)
   50 CONTINUE
      RETURN
      END
      SUBROUTINE      FFT (FI,NP,NQ)
      COMPLEX FI( NQ),U,W,T,CMPLX
      R=1.0
      PI=3.14159265
      N=2**NP
      M=N/2
      NM1=N-1
      J=1
      DO 3 I=1,NM1
      IF (I .GE. J) GO TO 1
      T=FI(J)
      FI(J)=FI(I)
      FI(I)=T
1     K=M
2     IF (K .GE. J) GO TO 3
      J=J-K
      K=K/2
      GO TO 2
3     J=J+K
      DO 5 L=1,NP
      LE=2**L
      LE1=LE/2
      U=(1.0,0.)
      W=CMPLX(COS(PI/LE1),R*SIN(PI/LE1))
      DO 5 J=1,LE1
```

```
      DO 4 I=J,N,LE
      IP=I+LE1
      T=FI(IP)*U
      FI(IP)=FI(I)-T
    4 FI(I)=FI(I)+T
    5 U=U*W
      RETURN
      END
      COMPLEX FUNCTION ZT (X,W,XM,XP)
      COMPLEX Z
      ZT = (Z(XM,W)+Z(XP,W)+(4.*Z(X,W)))/6.
      RETURN
      END
      COMPLEX FUNCTION YT (X,W,XM,XP)
      COMPLEX Y
      YT = (Y(XM,W)+Y(XP,W)+(4.*Y(X,W)))/6.
      RETURN
      END
      COMPLEX FUNCTION VGT (X,W,XV1,XV2,DELTX,XM,XP)
      COMPLEX CMPLX,VG
      XT1 = X - XV1
      XT2 = XV2 - X
      DXO2 = DELTX/2.
      IF(XT1.LE.(0.-DXO2).OR.XT2.LE.(0.-DXO2)) GO TO 10
      XF = 1.
      IF((ABS(XT1)).LT.DXO2) XF = (DXO2+XT1)/DELTX
      IF((ABS(XT2)).LT.DXO2) XF = (DXO2+XT2)/DELTX
      IF((ABS(XT1)).LT.DXO2.AND.(ABS(XT2)).LT.DXO2) XF = (XT1+XT2)/DELTX
      VGT = (VG(XM,W)+VG(XP,W)+(4.*VG(X,W)))*(XF/6.)
      GO TO 20
   10 VGT = CMPLX(0.,0.)
   20 RETURN
      END
      COMPLEX FUNCTION CGT (X,W,XI1,XI2,DELTX,XM,XP)
      COMPLEX CMPLX,CG
      XT1 = X - XI1
```

```
      XT2 = XI2 - X
      DXO2 = DELTX/2.
      IF(XT1.LE.(0.-DXO2).OR.XT2.LE.(0.-DXO2)) GO TO 10
      XF = 1.
      IF((ABS(XT1)).LT.DXO2) XF = (DXO2+XT1)/DELTX
      IF((ABS(XT2)).LT.DXO2) XF = (DXO2+XT2)/DELTX
      IF((ABS(XT1)).LT.DXO2.AND.(ABS(XT2)).LT.DXO2) XF = (XT1+XT2)/DELTX
     CGT = (CG(XM,W)+CG(XP,W)+(4.*CG(X,W)))*(XF/6.)
      GO TO 20
   10 CGT = CMPLX(0.,0.)
   20 RETURN
      END
      COMPLEX FUNCTION MUT  (X,W,XM,XP)
      COMPLEX MU
      MUT = (MU  (XM,W)+MU  (XP,W)+(4.*MU  (X,W)))/6.
      RETURN
      END
      COMPLEX FUNCTION BETAT(X,W,XM,XP)
      COMPLEX BETA
      BETAT=(BETA(XM,W)+BETA(XP,W)+(4.*BETA(X,W)))/6.
      RETURN
      END
      COMPLEX FUNCTION Z (X,W)
      COMPLEX CMPLX
      R = 0.
      XR = W
      Z = CMPLX(R,XR)
      RETURN
      END
      COMPLEX FUNCTION Y(X,W)
      COMPLEX CMPLX
      G = 0.
      B = W
      Y = CMPLX(G,B)
      RETURN
      END
```

```
      COMPLEX FUNCTION VG(X,W)
      COMPLEX CMPLX
      VGR = 0.
      VGI = 0.
      VG = CMPLX(VGR,VGI)
      RETURN
      END
      COMPLEX FUNCTION CG(X,W)
      COMPLEX CMPLX,CEXP
      CGR = 0.
      CGI = 0.
      CG = CMPLX(CGR,CGI)
      RETURN
      END
      COMPLEX FUNCTION MU  (X,W)
      COMPLEX CMPLX
      REAL MUR,MUI
      MUR    = 0.
      MUI    = 0.
      MU = CMPLX (MUR,MUI)
      RETURN
      END
      COMPLEX FUNCTION BETA(X,W)
      COMPLEX CMPLX
      BETARE = 1.
      BETAI  = 0.
      BETA = CMPLX (BETARE,BETAI)
      RETURN
      END
      COMPLEX FUNCTION ELM(W)
      COMPLEX CMPLX
      REAL ELMR,ELMI
      ELMR = 1.
      ELMI = 0.
      ELM = CMPLX(ELMR,ELMI)
      RETURN
```

```
END
COMPLEX FUNCTION ZL(W)
COMPLEX CMPLX,CSQRT
RL = 0.
XL = 0.
ZL = CMPLX(RL,XL)
RETURN
END
COMPLEX FUNCTION MUL  (W)
COMPLEX CMPLX
REAL MULR,MULI
MULR = 0.
MULI = 0.
MUL = CMPLX (MULR,MULI)
RETURN
END
COMPLEX FUNCTION IRM(W)
COMPLEX CMPLX
REAL IRMR,IRMI
IRMR = 0.
IRMI = 0.
IRM = CMPLX(IRMR,IRMI)
RETURN
END
COMPLEX FUNCTION YR(W)
COMPLEX CMPLX,CSQRT
GR = 1.
BR = 0.
YR = CMPLX(GR,BR)
RETURN
END
COMPLEX FUNCTION BETAR(W)
COMPLEX CMPLX
BETARR = 0.
BETARI = 0.
BETAR = CMPLX (BETARR,BETARI)
```

```
RETURN
END
```

```
C.....................................................................
C.....                                                          .....
C.....    TRANSIENT RESPONSE ANALYSIS OF A TRANSMISSION         .....
C.....          LINE WITH DISTRIBUTED SOURCES                   .....
C.....                                                          .....
C.....................................................................
C
C
      COMMON COEF(42,42),B(42),DUM(42),NZ,DET
      DIMENSION VL( 801),VR( 801)
      REAL LINE(81),MINUS
      DATA BLANK/' '/,MINUS/'-'/,PLUS/'+'/
      CALL ERRSET (208,256,-1,1,1)
      READ(5,10) ENDTIM,D,XI1,XI2,XV1,XV2,TI1,TI2,TV1,TV2,ELT1,ELT2,
     1CRT1,CRT2
   10 FORMAT ((F12.0,5F10.0)/(4F12.0)/(4F10.0))
      CALL SUBCAL (ENDTIM,D,J,DELTX,DT,I)
      CALL RLCG(D,      XI1,XI2,XV1,XV2,TI1,TI2,TV1,TV2,      VL,VR,
     1J,DELTX,DT,I,ELT1,ELT2,CRT1,CRT2)
      PRINT 100,ENDTIM
  100  FORMAT (1X,25HLENGTH OF TIME ALLOWED - ,1PE15.7,1X,5HSECS.)
      PRINT 110,D
  110 FORMAT (1X,14HLINE LENGTH - ,1PE15.7,1X,6HMETERS)
      PRINT 150
  150 FORMAT (1X,121HGRAPH OF THE VOLTAGE, IN VOLTS, AT THE LEFT END OF
     1THE LINE VERSUS THE TIME, IN SECONDS, SINCE THE EXCITATION WAS APP
     2LIED)
      CALL GRAPH (VL,DT,I,BLANK,MINUS,PLUS,LINE)
      PRINT 160
  160 FORMAT (1H1)
      PRINT 170
  170 FORMAT (1X,122HGRAPH OF THE VOLTAGE, IN VOLTS, AT THE RIGHT END OF
     1 THE LINE VERSUS THE TIME, IN SECONDS, SINCE THE EXCITATION WAS AP
     2PLIED)
      CALL GRAPH (VR,DT,I,BLANK,MINUS,PLUS,LINE)
      STOP
```

```
      END
      SUBROUTINE SUBCAL (ENDTIM,D,J,DELTX,DT,I)
      IF(AL(0.,0.).EQ.0.) GO TO 30
      IF(C(0.,0.).EQ.0.)  GO TO 40
      VPL = 1./SQRT(AL(0.,0.)*C(0.,0.))
      GO TO 50
   30 VPL = 6./(D*R(0.,0.)*C(0.,0.))
      GO TO 50
   40 VPL= 6./(D*AL(0.,0.)*G(0.,0.))
   50 IF(AL(D,0.).EQ.0.) GO TO 60
      IF(C(D,0.).EQ.0.) GO TO 70
      VPR = 1./SQRT(AL(D,0.)*C(D,0.))
      GO TO 80
   60 VPR = 6./(D*R(D ,0.)*C(D ,0.))
      GO TO 80
   70 VPR = 6./(D*AL(D ,0.)*G(D ,0.))
   80 IF(VPL.GT.VPR) VP = VPL
      IF(VPL.LE.VPR) VP = VPR
      DELTX = D/20.
      TIMDX = DELTX/VP
      DT = TIMDX/5.
      TSTEPS = ENDTIM/DT
      I1 = IFIX(TSTEPS)
      I2 = I1/5
      I3 = I2*5
      I4 = I1 - I3
      I5 = 5 - I4
      J = I1 + I5
      I = (J/5) + 1
      PRINT 3
    3 FORMAT (1H1)
      PRINT 1, DELTX
    1 FORMAT (1X,8HDELTX = ,1PE15.7,4X,6HMETERS)
      PRINT 2, DT
    2 FORMAT (1X,5HDT = ,1PE15.7,4X,7HSECONDS)
      IF(I.LE. 801) GO TO 10
```

```
      PRINT 20
   20 FORMAT (1X,39HENDTIM TOO LONG TO COMPLETE CALCULATION)
      J =  4000
      I =  801
   10 RETURN
      END
      SUBROUTINE RLCG (       D,      XI1,XI2,XV1,XV2,TI1,TI2,TV1,TV2,
     1      VL,VR,J,DELTX,DT,I,ELT1,ELT2,CRT1,CRT2)
      COMMON COEF(42,42),B(42),DUM(42),NZ,DET
      REAL VL(I),VR(I)
      INDEX = 1
      W = 0.6
      DO 10 K = 1,J
      DO 50 L1 = 1,42
      DO 60 L2 = 1,42
   60 COEF(L1,L2) = 0.
   50 CONTINUE
      IF (K.NE.1) GO TO 30
      IF(AL(0.,0.).EQ.0.) GO TO 1
      IF(C(0.,0.).EQ.0.) GO TO 2
      ZO = SQRT(AL(0.,0.)/C(0.,0.))
      GO TO 3
    1 ZO = 0.
      GO TO 3
    2 ZO = RL*1000000.
    3 IF(AL(D ,0.).EQ.0.) GO TO 4
      IF(C(D ,0.).EQ.0.) GO TO 5
      YO = 1./SQRT(AL(D,0.)/C(D,0.))
      GO TO 6
    4 YO = GR*1000000.
      GO TO 6
    5 YO = 0.
    6 DUM  (1) = EL(0.)*(ZO/(ZO+RL))
      DUM  (42 ) = 0.-(CR(0.)*(YO/(GR+YO)))
      DUM  (41 ) = (DUM  (42 ) + CR(0.))/GR
      DUM  (2) = (EL(0.)-DUM  (1))/RL
```

```
      DO 20 N1 = 3,40
   20 DUM(N1) = 0.
      VL(1) = DUM(1)
      VR(1) = DUM(41)
   30 A1 = W/DELTX
      KM1 = K-1
      T = DT*FLOAT(KM1)
      TW = T + (W*DT)
      TP = T+DT
      DO 36 LD = 2,40,2
      LE = LD/2
      LF = LD + 1
      LH = LD - 1
      LG = LD+2
      X = (FLOAT(LE-1)*DELTX)
      XP= X+(DELTX/2.)
      XPP = X+DELTX
      A2 = AMUT(XP,TW,X,XPP,T,TP)
      A3 = ALT(XP,TW,X,XPP,T,TP)/(2.*DT)
      A4 = 0.-RT(XP,TW,X,XPP,T,TP)
      A5 = BETAT (XP,TW,X,XPP,T,TP)
      A6 = CT (XP,TW,X,XPP,T,TP)/(2.*DT)
      A7 = 0.-GT(XP,TW,X,XPP,T,TP)
      COEF (LD,LG) = 0. - ((W/2.)*A4)+A3
      COEF (LD,LD) = COEF (LD,LG)
      COEF (LD,LF) = A1 - ((W/2.)*A2)
      COEF (LD,LH) = COEF (LD,LF) - (2.*A1)
      COEF (LF,LF) = 0. - ((W/2.)*A7) + A6
      COEF (LF,LH) = COEF (LF,LF)
      COEF (LF,LG) = A1 - ((W/2.)*A5)
      COEF (LF,LD) = COEF (LF,LG) - (2.*A1)
      B(LD) = VGT (XP,TW,XV1,XV2,TV1,TV2,   DT,DELTX,W,D,X,XPP,T,TP)
     1+(((W-1.)/DELTX)*(DUM(LF)-DUM(LH)))+(((1.-W)/2.)*A2*(DUM(LF)+
     2DUM(LH)))+((((1.-W)/2.)*A4)+A3)*(DUM(LG)+DUM(LD))
   36 B(LF) = CGT (XP,TW,XI1,XI2,TI1,TI2,   DT,DELTX,W,D,X,XPP,T,TP)
     1+(((W-1.)/DELTX)*(DUM(LG)-DUM(LD)))+(((1.-W)/2.)*A5*(DUM(LG)
```

```
     2+DUM(LD)))+((((((1.-W)/2.)*A7)+A6)*(DUM(LF)+DUM(LH)))
      A8 = AMULT (TW,T,TP) - 1.
      A9 = 0.-RLT(TW,T,TP)
      A10 = BETART (TW,T,TP) + 1.
      A11 = 0.-GRT(TW,T,TP)
      COEF (1,1) = W*A8
      COEF (1,2) = W*A9
      COEF (42,42) = W*A10
      COEF (42,41) = W*A11
      B(1) = ((W-1.)*((A9*DUM(2))+(A8*DUM(1))))-ELT(TW,DT,W,ELT1,ELT2,
     1T,TP)
      B(42) = ((W-1.)*((A10*DUM(42))+(A11*DUM(41))))-CRT(TW,DT,W,CRT1
     1,CRT2,T,TP)
      NZ = 42
      CALL SCLV5D (COEF,B,DUM,NZ,DET)
      IF(DET.EQ.0.) RETURN
      INDT5 = INDEX*5
      IF(INDT5.EQ.K) GO TO 90
      GO TO 10
   90 INDEX = INDEX + 1
      VL(INDEX) = DUM(1)
      VR(INDEX) = DUM(41)
   10 CONTINUE
      RETURN
      END
      SUBROUTINE SCLV5D (A,B,DUM,N,DET)
      REAL A(N,N),B(N),DUM(N)
      DET = 1.
      NM4 = N-4
      IG = 2
      DO 10 IA = 1,NM4
      IF(IG.EQ.2) IG = 2
      IF(IG.EQ.3) IG = 2
      IF(IG.EQ.4) IG = 3
      IAP2 = IA + 2
      IF(ABS(A(IA,IA)).GE.ABS(A(IAP2,IA))) GO TO 20
```

```
      IG = 4
      B(IA) = B(IA) + B(IAP2)
      DO 30 IB = 1,5
      IC = IA + IB -1
   30 A(IA,IC) = A(IA,IC) + A(IAP2,IC)
   20 DO 40 IK = 1,2
      IE = IA + IK
      IF(A(IE,IA).EQ.0.) GO TO 40
      RA = 0.-(A(IA,IA)/A(IE,IA))
      A(IE,IA) = 0.
      B(IE) = B(IE) + B(IA)/RA
      DO 50 IH = 1,IG
      IJ = IA + IH
   50 A(IE,IJ) = A(IE,IJ) + A(IA,IJ)/RA
   40 CONTINUE
   10 CONTINUE
      NM3 = N-3
      NM2 = N-2
      NM1 = N-1
      IF(ABS(A(NM3,NM3)).GE.ABS(A(NM2,NM3))) GO TO 60
      B(NM3) = B(NM3) + B(NM2)
      A(NM3,NM3) = A(NM3,NM3) + A(NM2,NM3)
      A(NM3,NM2) = A(NM3,NM2) + A(NM2,NM2)
      A(NM3,NM1) = A(NM3,NM1) + A(NM2,NM1)
      A(NM3,N) = A(NM3,N) + A(NM2,N)
   60 IF(A(NM3,NM3).EQ.0.) GO TO 150
      DO 70 IL = 1,2
      IM = NM3 + IL
      IF(A(IM,NM3).EQ.0.) GO TO 70
      RA = 0.-(A(NM3,NM3)/A(IM,NM3))
      A(IM,NM3) = 0.
      B(IM) = B(IM) + B(NM3)/RA
      DO 80 IN = 1,3
      IO = IN + NM3
   80 A(IM,IO) = A(IM,IO) + A(NM3,IC)/RA
   70 CONTINUE
```

```
      IF(ABS(A(NM2,NM2)).GE.ABS(A(N,NM2))) GO TO 90
      A(NM2,NM2) = A(NM2,NM2) + A(N,NM2)
      B(NM2) = B(N)+B(NM2)
      A(NM2,N) = A(NM2,N) + A(N,N)
   90 IF(A(NM1,NM2).EQ.0.) GO TO  100
      RA = 0.-(A(NM2,NM2)/A(NM1,NM2))
      A(NM1,NM2) = 0.
      A(NM1,NM1) = A(NM1,NM1) + A(NM2,NM1)/RA
      A(NM1,N) = A(NM1,N) + A(NM2,N)/RA
      B(NM1) = B(NM1) + B(NM2)/RA
      IF(A(N,NM2).EQ.0.) GO TO 105
  100 RA = 0.-(A(NM2,NM2)/A(N,NM2))
      A(N,NM2) = 0.
      A(N,NM1) = A(N,NM1) + A(NM2,NM1)/RA
      A(N,N) = A(N,N) + A(NM2,N)/RA
      B(N) = B(N) + B(NM2)/RA
  105 IF(ABS(A(NM1,NM1)).GE.ABS(A(N,NM1))) GO TO 110
      A(NM1,NM1) = A(NM1,NM1) + A(N,NM1)
      A(NM1,N) = A(NM1,N) + A(N,N)
      B(NM1) = B(NM1) + B(N)
  110 IF(A(N,NM1).EQ.0.) GO TO 120
      RA = 0.-(A(NM1,NM1)/A(N,NM1))
      A(N,NM1) = 0.
      A(N,N) = A(N,N) + A(NM1,N)/RA
      B(N) = B(N) + B(NM1)/RA
  120 DUM(N) = B(N)/A(N,N)
      DUM(NM1) = (B(NM1)-(DUM(N)*A(NM1,N)))/A(NM1,NM1)
      DUM(NM2) = (B(NM2)-(A(NM2,NM1)*DUM(NM1))-(A(NM2,N)*DUM(N)))/
     1A(NM2,NM2)
      DUM(NM3) = (B(NM3) - (A(NM3,NM2)*DUM(NM2))-(A(NM3,NM1)*DUM(NM1))
     1-(A(NM3,N)*DUM(N)))/A(NM3,NM3)
      DO 130 IP = 5,N
      IQ = N+1-IP
      SUM = 0.
      DO 140 IR = 1,4
      IS = IQ + IR
```

```
140 SUM = SUM + (A(IQ,IS)*DUM(IS))
    IF (A(IQ,IQ).EQ.0.) GO TO 150
130 DUM(IQ) = (B(IQ)-SUM)/A(IQ,IQ)
    GO TO 160
150 DET = 0.
    PRINT 170
170 FORMAT (1X,18HMATRIX IS SINGULAR)
160 RETURN
    END
    SUBROUTINE GRAPH (Y,DT,I,BLANK,MINUS,PLUS,LINE)
    REAL Y(I),LINE(81),MINUS
    PRINT 10
 10 FORMAT (1HO)
    DO 20 J = 1,81
 20 LINE(J) = MINUS
    BIG = ABS(Y(1))
    DO 30 K = 2,I
    IF(ABS(Y(K)).GT.BIG) BIG = ABS(Y(K))
 30 CONTINUE
    IF(BIG.EQ.0.) GO TO 1
    L = IFIX((40.*((Y(1)/BIG)+1.))+1.5)
    GO TO 2
  1 L = 41
  2 LINE(L) = PLUS
    T = 0.
    PRINT 40,T,Y(1),LINE
 40 FORMAT (1X,6HTIME =,1PE15.7,4X,3HY =,1PE15.7,2X,81A1)
    DO 50 M = 2,I
    DO 60 N = 1,81
 60 LINE(N) = BLANK
    LINE(41) = MINUS
    IF(BIG.EQ.0.) GO TO 3
    L = IFIX((40.*((Y(M)/BIG)+1.))+1.5)
    GO TO 4
  3 L = 41
  4 LINE(L) = PLUS
```

```
      IF(L.GE.40.AND.L.LE.42) GO TO 100
      IF (L.LT.41) GO TO 110
      IF(L.EQ.43) GO TC 120
      LM1 = L-1
      DO 80 II = 42,LM1
   80 LINE(II) = MINUS
  120 LINE(42) = MINUS
      GO TO 100
  110 IF(L.EQ.39) GO TC 130
      LP1 = L+1
      DC 90 JJ = LP1,40
   90 LINE(JJ) = MINUS
  130 LINE(40) = MINUS
  100 T = FLOAT(5*(M-1))*DT
      PRINT 70,T,Y(M),LINE
   70 FORMAT (1X,6HTIME =,1PE15.7,4X,3HY =,1PE15.7,2X,81A1)
   50 CONTINUE
      RETURN
      END
      FUNCTION  RT(X,T,XM,XP,TM,TP)
      IF(R(0.,0.).EQ.0.) GO TO 10
      RT = (R(XP,TP)+R(XM,TM)+R(XP,TM)+R(XM,TP)+(4.*R(XM,T))+(4.*R(XP,T)
     1)+(4.*R(X,TM))+(4.*R(X,TP))+(16.*R(X,T)))/36.
      GO TC 20
   10 RT = 0.
   20 RETURN
      END
      FUNCTION ALT(X,T,XM,XP,TM,TP)
      IF (AL(0.,0.).EQ.0.) GO TO 10
      ALT = (AL(XP,TP)+AL(XM,TM)+AL(XM,TP)+AL(XP,TM)+(4.*AL(XM,T))+(4.*A
     1L(XP,T))+(4.*AL(X,TM))+(4.*AL(X,TP))+(16.*AL(X,T)))/36.
      GO TO 20
   10 ALT = 0.
   20 RETURN
      END
      FUNCTION  CT(X,T,XM,XP,TM,TP)
```

```
      IF(C(0.,0.).EQ.0.) GO TO 10
      CT = (C(XP,TP)+C(XM,TM)+C(XM,TP)+C(XP,TM)+(4.*C(XM,T))+(4.*C(XP,T)
     1)+(4.*C(X,TM))+(4.*C(X,TP))+(16.*C(X,T)))/36.
      GO TO 20
   10 CT = 0.
   20 RETURN
      END
      FUNCTION  GT(X,T,XM,XP,TM,TP)
      IF(G(0.,0.).EQ.0.) GO TO 10
      GT = (G(XP,TP)+G(XM,TM)+G(XP,TM)+G(XM,TP)+(4.*G(XM,T))+(4.*G(XP,T)
     1)+(4.*G(X,TM))+(4.*G(X,TP))+(16.*G(X,T)))/36.
      GO TO 20
   10 GT = 0.
   20 RETURN
      END
      FUNCTION VGT(X,T,XV1,XV2,TV1,TV2,    DT,DELTX,W,D,XM,XP,TM,TP)
      DX02 = DELTX/2.
      XT1 = X - XV1
      XT2 = XV2 - X
      TT1 = T-TV1
      TT2 = TV2-T
      IF(XT1.LE.(0.-DXC2).OR.XT2.LE.(C.-DX02).OR.TT1.LE.(0.-((1.-W)*DT)
     1).OR.TT2.LE.(0.-(DT*W))) GO TO 40
      XF = 1.
      IF((ABS(XT1)).LT.DX02) XF = (DXC2+XT1)/DELTX
      IF((ABS(XT2)).LT.DX02) XF = (DXC2+XT2)/DELTX
      IF((ABS(XT1)).LT.DX02.AND.(ABS(XT2)).LT.DX02) XF = (XT1+XT2)/
     1DELTX
      TF = 1.
      IF(TT1.GT.(0.-((1.-W)*DT)).AND.TT1.LT.(W*DT)) TF = (((1.-W)*DT)
     1+TT1)/DT
      IF(TT2.GT.(0.-(W*DT)).AND.TT2.LT.((1.-W)*DT)) TF = ((W*DT)+TT2)/DT
      IF(TT1.GT.(0.-((1.-W)*DT)).AND.TT1.LT.(W*DT).AND.TT2.GT.(0.-(W*DT
     1)).AND.TT2.LT.((1.-W)*DT)) TF = (TT1+TT2)/DT
      VGT = (VG(XP,TP)+VG(XM,TM)+VG(XP,TM)+VG(XM,TP)+(4.*VG(XM,T))+(4.*
     1VG(XP,T))+(4.*VG(X,TM))+(4.*VG(X,TP))+(16.*VG(X,T)))*XF*TF/36.
```

```
      GO TO 30
   40 VGT= 0.
   30 RETURN
      END
      FUNCTION CGT(X,T,XI1,XI2,TI1,TI2,   DT,DELTX,W,C,XM,XP,TM,TP)
      DXC2 = DELTX/2.
      XT1 = X - XI1
      XT2 = XI2 - X
      TT1 = T-TI1
      TT2 = TI2-T
      IF(XT1.LE.(0.-DXC2).OR.XT2.LE.(C.-DXC2).OR.TT1.LE.(0.-((1.-W)*DT)
     1).OR.TT2.LE.(0.-(DT*W))) GO TO 40
      XF = 1.
      IF((ABS(XT1)).LT.DXC2) XF = (CXC2+XT1)/DELTX
      IF((ABS(XT2)).LT.DXC2) XF = (DXC2+XT2)/DELTX
      IF((ABS(XT1)).LT.CXC2.AND.(ABS(XT2)).LT.DXC2) XF = (XT1+XT2)/
     1DELTX
      TF = 1.
      IF(TT1.GT.(0.-((1.-W)*DT)).AND.TT1.LT.(W*DT)) TF = (((1.-W)*DT)
     1+TT1)/DT
      IF(TT2.GT.(0.-(W*DT)).AND.TT2.LT.((1.-W)*DT)) TF = ((W*DT)+TT2)/DT
      IF(TT1.GT.(0.-((1.-W)*DT)).AND.TT1.LT.(W*DT).AND.TT2.GT.(0.-(W*DT
     1)).AND.TT2.LT.((1.-W)*DT)) TF = (TT1+TT2)/DT
      CGT = (CG(XP,TP)+CG(XM,TM)+CG(XM,TP)+CG(XP,TM)+(4.*CG(XM,T))+(4.
     1*CG(XP,T))+(4.*CG(X,TM))+(4.*CG(X,TP))+(16.*CG(X,T)))*XF*TF/36.
      GO TO 30
   40 CGT= 0.
   30 RETURN
      END
      FUNCTION ELT (T,DT,W,ELT1,ELT2,TM,TP)
      TT1 = T-ELT1
      TT2 = ELT2 - T
      IF(TT1.LE.(0.-((1.-W)*DT)).OR.TT2.LE.(0.-(DT*W))) GO TO 10
      TF = 1.
      IF(TT1.GT.(0.-((1.-W)*DT)).AND.TT1.LT.(W*DT)) TF = (((1.-W)*DT)
     1+TT1)/DT
```

```
      IF(TT2.GT.(0.-(W*DT)).AND.TT2.LT.((1.-W)*DT)) TF = ((W*DT)+
     1TT2)/DT
      IF(TT1.GT.(0.-((1.-W)*DT)).AND.TT1.LT.(W*DT).AND.TT2.GT.(0.-(W*DT)
     1).AND.TT2.LT.((1.-W)*DT)) TF = (TT1+TT2)/DT
      ELT = (1./6.)*TF*(EL(TM)+EL(TP)+(4.*EL(T)))
      GO TO 20
   10 ELT = 0.
   20 RETURN
      END
      FUNCTION CRT (T,DT,W,CRT1,CRT2,TM,TP)
      TT1 = T-CRT1
      TT2 = CRT2 - T
      IF(TT1.LE.(0.-((1.-W)*DT)).OR.TT2.LE.(0.-(DT*W))) GO TO 10
      TF = 1.
      IF(TT1.GT.(0.-((1.-W)*DT)).AND.TT1.LT.(W*DT)) TF = (((1.-W)*DT)
     1+TT1)/DT
      IF(TT2.GT.(0.-(W*DT)).AND.TT2.LT.((1.-W)*DT)) TF = ((W*DT)+
     1TT2)/DT
      IF(TT1.GT.(0.-((1.-W)*DT)).AND.TT1.LT.(W*DT).AND.TT2.GT.(0.-(W*DT)
     1).AND.TT2.LT.((1.-W)*DT)) TF = (TT1+TT2)/DT
      CRT = (1./6.)*TF*(CR(TM)+CR(TP)+(4.*CR(T)))
      GO TO 20
   10 CRT = 0.
   20 RETURN
      END
      FUNCTION AMUT(X,T,XM,XP,TM,TP)
      AMUT = (AMU(XM,TM)+AMU(XP,TP)+AMU(XM,TP)+AMU(XP,TM)+(4.*AMU(XM,T))
     1+(4.*AMU(XP,T))+(4.*AMU(X,TM))+(4.*AMU(X,TP))+(16.*AMU(X,T)))/36.
      RETURN
      END
      FUNCTION BETAT (X,T,XM,XP,TM,TP)
      BETAT = (BETA(XP,TP)+BETA(XM,TM)+BETA(XM,TP)+BETA(XP,TM)+(4.*BETA
     1(XM,T))+(4.*BETA(XP,T))+(4.*BETA(X,TM))+(4.*BETA(X,TP))+(16.*BETA
     2(X,T)))/36.
      RETURN
      END
```

```
      FUNCTION RLT (T,TM,TP)
      RLT = (RL(TM)+RL(TP)+(4.*RL(T)))/6.
      RETURN
      END
      FUNCTION AMULT (T,TM,TP)
      AMULT = (AMUL(TM)+AMUL(TP)+(4.*AMUL(T)))/6.
      RETURN
      END
      FUNCTION GR T (T,TM,TP)
      GR T = (GR (TM)+GR (TP)+(4.*GR (T)))/6.
      RETURN
      END
      FUNCTION BETART (T,TM,TP)
      BETART = (BETAR(TM)+BETAR(TP)+(4.*BETAR(T)))/6.
      RETURN
      END
      FUNCTION R(X,T)
      R = 1.
      RETURN
      END
      FUNCTION AL(X,T)
      AL=0.
      RETURN
      END
      FUNCTION C(X,T)
      C = 1.
      RETURN
      END
      FUNCTION G(X,T)
      G = 0.
      RETURN
      END
      FUNCTION AMU (X,T)
      AMU = 0.
      RETURN
      END
```

```
      FUNCTION BETA (X,T)
      BETA = 0.
      RETURN
      END
      FUNCTION VG (X,T)
      VG = 0.
      RETURN
      END
      FUNCTION CG (X,T)
      CG = 1.
      RETURN
      END
      FUNCTION RL  (T)
      RL  = 10000.
      RETURN
      END
      FUNCTION AMUL (T)
      AMUL = 0.
      RETURN
      END
      FUNCTION EL(T)
      EL = 0.
      RETURN
      END
      FUNCTION GR  (T)
      GR  = 0.
      RETURN
      END
      FUNCTION BETAR(T)
      BETAR = 0.
      RETURN
      END
      FUNCTION CR(T)
      CR = 1.
      RETURN
      END
```

VITA

Michael J. Roberts attended The University of Tennessee as an undergraduate from 1964 to 1969. During this time he participated in the Cooperative Engineering Scholarship Program, assigned to Allen and Hoshall, Consulting Engineers in Memphis, Tennessee. After graduation, he worked for Texas Instruments Inc. of Dallas, Texas, on their Summer Development Program. In September, 1969, he became a full-time instructor for The University of Tennessee Department of Electrical Engineering. He obtained the Master of Science degree in June 1972. Since then he has been working as a graduate assistant and pursuing a Doctor of Philosophy degree.