Various Topics on Graphical Structures Placed on Commutative Rings

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I would like to dedicate this work to my amazing, supportive, and patient wife, Melissa, as well as to my smart and beautiful daughter, Natalie. Thank you for sacrificing so much to join me on this journey.

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Abstract

In this dissertation, we look at two types of graphs that can be placed on a commutative ring: the zero-divisor graph and the ideal-based zero-divisor graph. A zero-divisor graph is a graph whose vertices are the nonzero zero-divisors of a ring and two vertices are connected by an edge if and only if their product is 0. We classify, up to isomorphism, all commutative rings without identity that have a zero-divisor graph on 14 or fewer vertices.

An ideal-based zero-divisor graph is a generalization of the zero-divisor graph where for a ring R and ideal I the vertices are $\{x \in R \setminus I \mid \text{there exists } y \in R \setminus I \text{ such that } xy \in I\}$, and two vertices are connected by an edge if and only if their product is in I. We consider cut-sets in the ideal-based zero-divisor graph. A cut-set is a set of vertices that when they and their incident edges are removed from the graph, separate the graph into several connected components. We will describe all cut-sets in the ideal-based zero-divisor graph for commutative rings with identity.

We also give some additional results about two other graphical structures, as well as include a classification of realizable zero-divisor graphs that have a specified girth and diameter for commutative rings with and without identity.

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Chapter 1

Introduction

1.1 Basic Ring Definitions

Commutative Ring Theory is part of the larger field of Algebra. Throughout this dissertation, we will assume that our rings are commutative. For a general algebra reference see [31]. We will not, however, assume that our rings have a multiplicative identity.

We will largely be focused on the zero-divisors of a ring. A **zero-divisor** of a ring R is an element x where there exists a nonzero element $y \in R$ such that $xy = 0$. Notice that 0 is a zero-divisor in any nonzero ring. We denote the set of all zero-divisors by $Z(R)$. When looking at the zero-divisor graph, we will mostly be concerned with the nonzero zero-divisors, which we will denote $Z(R)^*$. A ring with identity that has no nonzero zero-divisors is called an **integral domain**. If in a ring with identity every nonzero element has a multiplicative inverse, then that ring is said to be a **field**. A **local** ring is a ring that has a unique maximal ideal. An **Artinian** ring is a ring that satisfies the descending chain condition on ideals. The **total quotient ring** (or **total ring of fractions**) of a ring R is the localization of R at the multiplicatively closed set $S = R \setminus Z(R)$.

An element of a ring that has a multiplicative inverse is called a **unit**. If an element of a ring is neither a zero-divisor nor a unit, then we call it a **regular** element. Every element in a finite commutative ring is either a zero-divisor or a unit. An element x is called **idempotent** if $x^2 = x$, and if there exists an integer $n \geq 1$ such that $x^n = 0$, then we call x **nilpotent**. The **annihilator** of an element x is the set $\text{ann}_R(x) = \{y \in R \mid xy = 0\}$. When the ring is clear, we often shorten $\mathrm{ann}_R(x)$ to just $\mathrm{ann}(x)$. It is easy to see that $\mathrm{ann}(x)$ for any element $x \in R$ forms an ideal. As such, we can refer to $\mathrm{ann}(x)$ as an **annihilator ideal**.

In Chapter 3, we can consider our ring to be a zero-divisor semigroup. A **semigroup** is an algebraic set that has an associative binary operation. If every element of our semigroup is a zero-divisor, then we call our semigroup a **zero-divisor semigroup**. For a general reference on semigroups, see [30]. If a semigroup has an identity, then it is called a **monoid**. Since we are mainly interested in the multiplicative structure of commutative rings, we can consider the rings in this dissertation to be semigroups. If the ring has a multiplicative identity, then the multiplication of that ring can be viewed as a monoid. The zero-divisor graph of commutative semigroups have been extensively studied, including in [27]. For a survey of such results, see [5].

An **equivalence relation** is a relation $\sim$ on a set X such that for $a, b, c \in X$ the relation is:

Reflexive $a \sim a$,

Symmetric $a \sim b$ if and only if $b \sim a$, and

Transitive $a \sim b$ and $b \sim c$ implies that $a \sim c$.

This equivalence relation partitions the elements of X into equivalence classes, which we will denote $[a]_\sim$ (or $[a]$ when no confusion on the relation arises). In other words, $[a]_\sim = [b]_\sim$ if and only if $a \sim b$.

1.2 Basic Graph Definitions

Since this dissertation includes a major graph theoretic component, we include some of the basic definitions here. A basic reference for graph theory is [25]. A **graph** $G = (V(G), E(G))$ is an ordered pair of a set of vertices, $V(G)$, and a set of edges, $E(G)$. The set of edges is made up of two-element subsets of $V(G)$. An **undirected graph** is a graph whose edges are bidirectional. Two graphs are **isomorphic** if there is an edge-preserving one-to-one correspondence between the vertex sets of the graphs. We say H is a **subgraph** of the graph G if $V(H) \subseteq V(G)$ and $E(H) \subseteq E(G)$.

If two vertices, x and y, are connected by an edge (i.e., $(x, y) \in E(G)$), then we say that x and y are **adjacent** and write $x — y$. If there is an edge from x to x, then we call that edge a **loop**. A graph that has no loops is called **simple**. The **degree** of a vertex x is the number of edges it is incident on (or the number of vertices it is adjacent to). If a vertex has degree one, then we call that vertex an **end**.

A **path** between two vertices x and y is a sequence of edges and vertices that link x to y. If there exists a path between any two vertices in a graph, then the graph is called **connected**. For graphs that are not connected, we can consider its **connected components**, maximally connected subgraphs of the original graph. We define the **distance** between two vertices x and y to be the number of edges in a shortest path from x to y, and denote it $\mathrm{d}(x, y)$. Note that $\mathrm{d}(x, x) = 0$ and $\mathrm{d}(x, y) = \infty$ if there is not a path from x to y. The **diameter** of a graph G is $\mathrm{diam}(G) = \sup\{\mathrm{d}(x, y) \mid x, y \in V(G)\}$.

A **cycle** is a path that begins and ends on the same vertex without repeating edges or vertices. For example, we call the path $x — a_2 — a_3 — \cdots — a_n — x$ an n**-cycle** since it has n edges (or n (distinct) vertices). Often a 3-cycle is called a **triangle** and a 4-cycle is a **square**. The **girth** of a graph G is the length of a shortest cycle. If the graph does not have a cycle, then the girth is said to be infinite. We denote the girth of a graph G by $\mathrm{gr}(G)$.

If any two distinct vertices in a graph are connected by an edge, then that graph is called **complete**. We denote complete graphs by K^m, where m is the number of vertices (possibly infinite) in the graph. A **bipartite** graph is a graph that can be partitioned into two subgraphs where vertices in one partition are only connected to vertices in the other partition. A **complete bipartite** graph is a bipartite graph where all of the vertices in one partition are connected to all of the vertices in the other partition. We denote complete bipartite graphs by $K^{m,n}$, where m and n are the number of vertices in each of the partitions. If $m = 1$, then $K^{m,n} = K^{1,n}$ is called a **star graph**. We can generalize the complete bipartite graph to a **complete** r**-partite** graph, where r is the number of partitions. If $r = 2$, then this is the complete bipartite graph.

1.3 History

The **zero-divisor graph** is a graph whose vertices are the nonzero zero-divisors of a commutative ring and two vertices are connected by an edge if and only if their product is 0. The zero-divisor graph was first introduced by Istvan Beck in 1988 ([21]), where his definition differed from the standard definition today. He allowed all elements of the ring to be vertices in the graph. His major focus was on the colorings of the graph, e.g., how many colors were necessary to color the vertices such that no two adjacent vertices were colored the same. His work was continued by Daniel D. Anderson and Muhammad Naseer in 1993 in [1] where, among other results, they found a counterexample to one of Beck's conjectures.

The third paper published on zero-divisor graphs came in 1999 in a paper by David F. Anderson and his masters student Philip Livingston ([10]). In this paper, which came from Livingston's Master's thesis [33], they redefined the zero-divisor graph to the definition we accept today and proved some surprising results about the graph. It was in this paper that the focus on zero-divisor graph research shifted to investigate the interplay of the graph-theoretic and ring-theoretic properties. Among other results, they showed the diameter of the zero-divisor graph for a commutative ring is always less than or equal to 3.

Theorem 1.1 ([10, Theorem 2.3])**.** *Let R be a commutative ring. Then $\Gamma(R)$ is connected and $diam(\Gamma(R)) \leq 3$.*

It is well known in graph theory that the girth of a graph G is bounded above by $2\ \mathrm{diam}(G) + 1$. Since the diameter of the zero-divisor graph is always less than or equal to 3, we have that the girth is bounded above by 7. However, we can do better for zero-divisor graphs.

Theorem 1.2 ([35, (1.4)], [28, Theorem 1.6])**.** *Let R be a commutative ring. Then $gr(\Gamma(R)) \in \{3, 4, \infty\}$.*

Since [10], hundreds of papers have been published on zero-divisor graphs. For a recent survey, see [4]. In Section 3.3, we classify all commutative rings without identity that have a zero-divisor graph on 14 or fewer vertices. A similar result exists for rings with identity and was completed by Shane Redmond in [38].

Often in the research on zero-divisor graphs, we consider the zero-divisor graph to be a simple graph. However, in this dissertation, we will allow loops in our graph as we believe loops reveal important information. To see an example of their importance, see Section 2.3. Since considering loops can change results in certain situations, we will use $\Gamma^*(R)$ for the zero-divisor graph with loops when the results for the two graphs differ. Otherwise, we use $\Gamma(R)$ when including loops does not alter the outcome.

The ideal-based zero-divisor graph is a generalization of the zero-divisor graph developed by Shane Redmond in his dissertation, [36], and in [37] in 2003. He laid some of the foundational work for the ideal-based zero-divisor graph, which we give as the next two theorems.

Theorem 1.3 ([37, Theorem 2.4])**.** *Let I be an ideal of a commutative ring R. Then $\Gamma_I(R)$ is connected with diam$(\Gamma_I(R)) \leq 3$.*

Theorem 1.4 ([37, Lemma 5.1])**.** *Let I be an ideal of a commutative ring R. Then gr$(\Gamma_I(R)) \leq$ gr$(\Gamma(R/I))$. In particular, if $\Gamma(R/I)$ contains a cycle, then so does $\Gamma_I(R)$, and therefore gr$(\Gamma_I(R)) \leq$ gr$(\Gamma(R/I)) \leq 4$.*

Both of these theorems remain true when loops are included. The ideal-based zero-divisor graph has been studied further in several papers including [15] and [24], and has been the subject of doctoral dissertations such as [40] (and later published in [41]).

Several variations on the zero-divisor graph have also been considered and investigated. The compressed zero-divisor graph was first considered in [35], and was further investigated in [42], [6], and [7]. The zero-divisor lattice was proposed by Nicholas Baeth and first investigated in [43]. Ayman Badawi first defined the annihilator graph in [18], and Mahmood Behboodi and Zahra Rakeei first defined the annihilating-ideal graph in [22]. We also have the extended zero-divisor graph developed by Driss Bennis, Jilali Mikram, and Fouad Taraza in [23]. Many of these graphs can be considered as a congruence-based zero-divisor graph for specific types of congruence relations, which was developed Elizabeth Lewis and David F. Anderson in [32] and [9].

1.4 Dissertation Structure

This dissertation touches on a few topics relating to zero-divisor graphs. In Chapter 2, we look at describing all cut-sets for the ideal-based zero-divisor graph of commutative rings with identity. In Chapter 3, we classify all commutative rings without identity that have zero-divisor graphs on 14 vertices or fewer. We then compare these results to the results that Redmond obtained for commutative rings with identity in [38]. We also give a classification of the realizable diameters and girths of zero-divisor graphs on rings with and without identity. In Chapter 4, we give some miscellaneous results of note on compressed zero-divisor graphs and zero-divisor lattices that do not fit into the other chapters. We finish in Chapter 5 with a brief summary and some open questions related to the topics in this dissertation. The appendices have some examples of the Mathematica programs that were written for this dissertation and this area of research.

Chapter 2

Cut-Sets in the Ideal-Based Zero-Divisor Graph

Cut-vertices have been studied for several years in Graph Theory, but were brought into the field of zero-divisor graphs in 2009 in [16]. A **cut-vertex** a is a vertex in a connected graph G such that G can be expressed as a union of subgraphs X and Y such that $E(X) \neq \emptyset$, $E(Y) \neq \emptyset$, $E(X) \cup E(Y) = E(G)$, $V(X) \cup V(Y) = V(G)$, $V(X) \cap V(Y) = \{a\}$, $V(X) \setminus \{a\} \neq \emptyset$, and $V(Y) \setminus \{a\} \neq \emptyset$. Another way of describing the cut-vertex is that if you remove it and its incident edges, then you separate the graph into multiple connected components. Cut-vertices have been studied in the context of zero-divisor graphs in many papers, such as [12], [16], and [26].

In [16], the authors gave a few results on cut-vertices. They showed that if a was a cut-vertex with partitions X and Y such that $X \setminus \{a\}$ was a complete subgraph of $\Gamma(R)$, then $V(X) \cup \{0\}$ forms an ideal in the ring ([16, Theorem 4.2]). They also showed that the cut-vertex and 0 form an ideal in the ring ([16, Theorem 4.4]). This result was generalized to cut-sets for nonlocal rings in [26, Theorem 3.4]. Axtell et al. in [12] were able to show many ring properties based on the existence of a cut-vertex. For example, if $\Gamma(R)$ of a finite commutative ring R contains a cut-vertex that does not isolate any vertices, then R is a local ring ([12, Corollary 3.7]). They also showed that for finite local rings with maximal ideal M, if a is a cut-vertex, then $M^{k-1} = \{0, a\}$, $|R| = 2^{n+1}$, and $|M| = 2^n$ ([12, Theorem 4.1, Theorem 4.4]).

In [26], cut-vertices were generalized to cut-sets. Cut-sets are similarly defined as a cut-vertex, but involve a set of vertices instead of only one. The set $A = \{a_1, a_2, \ldots, a_n\}$ is a **cut-set** in a connected graph G if you can express G as a union of subgraphs X and Y such that $E(X) \neq \emptyset$, $E(Y) \neq \emptyset$, $E(X) \cup E(Y) = E(G)$, $V(X) \cup V(Y) = V(G)$, $V(X) \cap V(Y) = A$, $V(X) \setminus A \neq \emptyset$, $V(Y) \setminus A \neq \emptyset$, and no proper subset of A could satisfy the same properties. Coté et al. in [26] gave an alternate definition to the cut-set that mirrors its graphical interpretation. A set of vertices A is a cut-set if there exists $c, d \in V(G) \setminus A$ such that every path from c to d involves at least one vertex in A, and no proper subset of A satisfies the same condition.

Cut-sets have been studied in [13], [26], and [43], as well as others. Coté et al. in [26, Theorem 3.2] demonstrated that for any commutative ring R (not necessarily with identity), $\Gamma(R)$ has a cut-set if and only if $\Gamma(R)$ is not complete. They also classified all cut-sets for finite nonlocal commutative rings, which was then generalized in [13]. We give a restatement of those theorems here as it will be referenced later in this dissertation:

Theorem 2.1. *Let $R \cong R_1 \times \cdots \times R_n$ be a nonlocal Artinian ring such that $R \ncong \mathbb{Z}_2 \times F$, where F is a field. Then A is a cut-set of $\Gamma(R)$ if and only if $A = \{(0, \ldots, 0, r_i, 0, \ldots, 0) \mid r_i \in \operatorname{ann}(r)^*\}$, where $\operatorname{ann}(r)$ is a minimal annihilator ideal.*

In [13, Corollary 2.9], Axtell et al. showed that cut-sets absorb ring multiplication, meaning that cut-sets have the multiplicative structure of an ideal. We now look at cut-sets in the ideal-based zero-divisor graph.

2.1 Definitions

The **ideal-based zero-divisor graph** of a ring R and ideal I is an undirected graph denoted $\Gamma_I(R)$ with vertices $\{x \in R \setminus I \mid xy \in I \text{ for some } y \in R \setminus I\}$, and two vertices x and y are connected by an edge if and only if $xy \in I$. This graph was introduced by Shane Redmond in [36] and [37]. Let $\{a_\lambda\}_{\lambda \in \Lambda} \subseteq R$ be a set of coset representatives of R/I. Then we call the subset $a_\lambda + I$ a **column** of $\Gamma_I(R)$. If $a_\lambda^2 \in I$, then $a_\lambda + I$ is a **connected column**. Notice that the ideal-based zero-divisor graph is a generalization of the usual zero-divisor graph

(Set $I = 0$). Redmond detailed the construction of $\Gamma_I(R)$ based on the zero-divisor graph $\Gamma(R/I)$:

1. Stack $|I|$ copies of the graph $\Gamma(R/I)$.
2. Connect the vertices between two columns $a_\alpha + I$ and $a_\beta + I$ if and only if $a_\alpha + I$ is connected to $a_\beta + I$ in $\Gamma(R/I)$.
3. If $a_\lambda^2 \in I$, then connect the vertices within the connected column $a_\lambda + I$.

Redmond demonstrated a strong connection between $\Gamma(R/I)$ and $\Gamma_I(R)$ in [37]. We highlight one of those results here.

Proposition 2.2 ([37, Proposition 2.2])**.** *Let R be a commutative ring with identity and proper ideal I.*

1. *If $I = (0)$, then $\Gamma_I(R) = \Gamma(R)$.*
2. *Let I be a nonzero ideal of R. Then $\Gamma_I(R) = \emptyset$ if and only if I is a prime ideal of R.*

Cut-vertices in the context of ideal-based zero-divisor graphs were initially studied briefly in Shane Redmond's dissertation, [36], and in the subsequent paper [37], where he used the terminology cut-point. He found that $\Gamma_I(R)$ has no cut-vertices. Cut-sets in ideal-based zero-divisor graphs have been considered in [15]. We give a few key results:

Theorem 2.3 ([15, Theorem 3.1])**.** *Let I be an ideal of R. If A is a cut-set in $\Gamma_I(R)$, then A is a column or a union of columns.*

Theorem 2.4 ([15, Theorem 3.2])**.** *If A is a cut-set in $\Gamma(R/I)$, then $B = \{a + i \mid a + I \in A, i \in I\}$ is a cut-set in $\Gamma_I(R)$.*

Theorem 2.5 ([15, Theorem 3.4])**.** *If a cut-set A in $\Gamma_I(R)$ is a union of n columns and $|Z(R/I)^*| - n \geq 2$, then $B = \{a + I \mid a \in A\}$ is a cut-set in $\Gamma(R/I)$.*

Theorem 2.6 ([15, Theorem 3.5])**.** *Let I be an ideal of R such that R/I is nonlocal, let A be a cut-set in $\Gamma(R/I)$, and let $B = \{a + i \mid a + I \in A, i \in I\}$. Then $B \cup I$ is an ideal of R.*

2.2 Describing Cut-Sets in Ideal-Based Zero-Divisor Graphs

In this section, we work to describe cut-sets in the ideal-based zero-divisor graph. We expand on the previous results from [15] to describe all cut-sets in the ideal-based zero-divisor graph using its relationship to $\Gamma(R/I)$ in order to leverage the work that has been done in classifying cut-sets for the zero-divisor graph. We start with a result about the zero-divisor graph when its one and only cut-set is a cut-vertex.

Lemma 2.7. *Let R be a commutative ring. Then a is a vertex in $\Gamma(R)$ such that $V(\Gamma(R)) \setminus \{a\}$ does not contain a cut-set if and only if a is connected to every other vertex in $\Gamma(R)$.*

Proof. Let R be a commutative ring. If $\Gamma(R)$ is complete, then we are done. So suppose that $\Gamma(R)$ is not complete.

To show the "only if" statement, let a be a vertex in $\Gamma(R)$ such that $V(\Gamma(R)) \setminus \{a\}$ does not contain a cut-set. Assume that there exists a vertex b such that a and b are not connected. Consider the set $A = V(\Gamma(R)) \setminus \{a, b\}$. Since a and b are not connected, the removal of A separates a from b. Thus, either A is a cut-set or it contains a cut-set. This is a contradiction since $A \subsetneq V(\Gamma(R)) \setminus \{a\}$, which does not contain a cut-set. Therefore, a must be connected to b, and thus connected to every other vertex in $\Gamma(R)$.

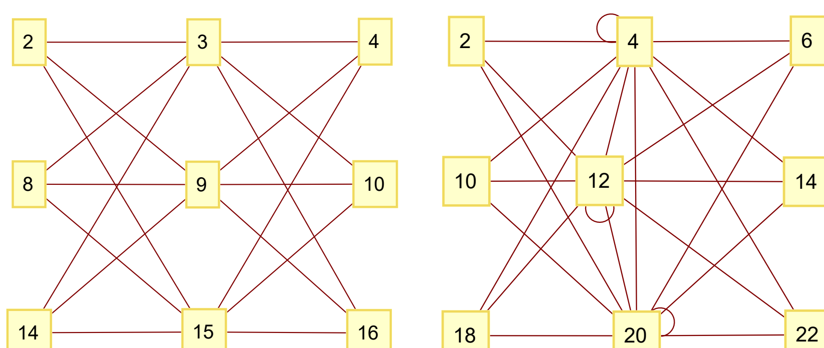
Conversely, let a be connected to every other vertex in $\Gamma(R)$. Assume the set $V(\Gamma(R)) \setminus \{a\}$ contains a cut-set B. Then the removal of B separates some vertices b and c. However, since $a \notin B$, a is not removed with the removal of B. Thus, a remains in the graph and since it is connected to every other vertex in $\Gamma(R)$, it is connected to b and c, a contradiction. So $V(\Gamma(R)) \setminus \{a\}$ does not contain a cut-set. □

Corollary 2.8. *Let R be a commutative ring with cut-vertex a of $\Gamma(R)$ such that it is the only cut-set of $\Gamma(R)$. Then a is connected to every vertex in $\Gamma(R)$.*

Proposition 2.9. *Let R be a commutative ring with ideal I. Let $a + I$ be a vertex in $\Gamma(R/I)$ such that $V(\Gamma(R/I)) \setminus \{a + I\}$ has no cut-sets and $a^2 \notin I$. Then the columns of $V(\Gamma(R/I)) \setminus \{a + I\}$ is a cut-set of $\Gamma_I(R)$.*

Proof. Let R be a commutative ring with ideal I, and let $a+I$ be a vertex in $\Gamma(R/I)$ such that $a^2 \notin I$ and $V(\Gamma(R/I)) \setminus \{a+I\}$ has no cut-sets. Since $a+I$ is not a connected column, removing the columns of $V(\Gamma(R/I)) \setminus \{a+I\}$ will disconnect the graph as it isolates the vertices of the column $a+I$. We denote the set of vertices in the columns of $V(\Gamma(R/I)) \setminus \{a+I\}$ by A' and show it is a minimal set that disconnects the graph. By Theorem 2.3, a vertex is in a cut-set if and only if its associated column is in the cut-set. Let $b+I \in V(\Gamma(R/I)) \setminus \{a+I\}$. By Lemma 2.7, $a+I$ is connected to $b+I$ in $\Gamma(R/I)$ since $V(\Gamma(R/I)) \setminus \{a+I\}$ has no cut-sets. Therefore, by the construction of $\Gamma_I(R)$, the columns of $a+I$ and $b+I$ are connected. Thus, the column of $b+I$ is needed in A' in order for A' to disconnect the graph when it is removed. Hence, A' is minimal and therefore a cut-set of $\Gamma_I(R)$. □

Example 2.10. If $a^2 \in I$, then $a+I$ is a connected column and the columns of $V(\Gamma(R/I)) \setminus \{a+I\}$ is not a cut-set as the removal of the columns of $V(\Gamma(R/I)) \setminus \{a+I\}$ would leave the complete graph of the connected column of $a+I$. In Figure 2.1(a), we see that the columns of $2+\langle 6\rangle$ and $4+\langle 6\rangle$ form a cut-set as it isolates the vertices 3, 9, and 15. In Figure 2.1(b), the columns of $2+\langle 8\rangle$ and $6+\langle 8\rangle$ do not form a cut-set since the column of $4+\langle 8\rangle$ is a connected column.



(a) $\Gamma_{\langle 6\rangle}(\mathbb{Z}_{18})$ **(b)** $\Gamma_{\langle 8\rangle}(\mathbb{Z}_{24})$

Figure 2.1: Ideal-based zero-divisor graphs without, (2.1a), and with, (2.1b), a connected column

From Lemma 6.2(1) in [37], we know that if $\Gamma(R/I)$ only has two vertices, then $\Gamma_I(R)$ either has no or exactly two connected columns. We use this fact in the next result.

Proposition 2.11. *Let R be a commutative ring with identity and ideal I, and let $V(\Gamma(R/I)) = \{a + I, b + I\}$.*

1. *If $a^2 \in I$, then $\Gamma_I(R)$ is a complete graph.*

2. *If $a^2 \notin I$, then $\Gamma_I(R)$ has exactly two cut-sets, namely the columns associated with $a + I$ and $b + I$.*

Proof. Let R be a commutative ring with ideal I and let $V(\Gamma(R/I)) = \{a + I, b + I\}$.

1. Let $a^2 \in I$. Then $b^2 \in I$ by Lemma 6.2(1) in [37]. By construction of the ideal-based zero-divisor graph, we already have that $\Gamma_I(R)$ is a complete bipartite graph where the columns of $a + I$ and $b + I$ are the two partitions. To show that this graph is complete, we need that each column is a connected column, which it is since $a^2, b^2 \in I$. Thus, $\Gamma_I(R)$ is complete.

2. Let $a^2 \notin I$. Then by Lemma 6.2(1) in [37], $b^2 \notin I$. Then the columns of $a + I$ and $b + I$ are not connected. Again, $\Gamma_I(R)$ is a complete bipartite graph and thus has two cut-sets, one for each of the partitions. Hence, the column of $a + I$ and the column of $b + I$ are the cut-sets of $\Gamma_I(R)$.

□

Theorem 2.5 is a partial converse to Theorem 2.4. These theorems, along with Propositions 2.9 and 2.11, describe all cut-sets in $\Gamma_I(R)$ and their relationship with cut-sets in $\Gamma(R/I)$.

Theorem 2.12. *Theorems 2.4 and 2.5 and Propositions 2.9 and 2.11 describe all cut-sets in $\Gamma_I(R)$ for a commutative ring R and ideal I.*

1. *If A is a cut-set in $\Gamma(R/I)$, then $\{a + i \mid a + I \in A, i \in I\}$ is a cut-set in $\Gamma_I(R)$.*

2. *If $a+I$ is a vertex in $\Gamma(R/I)$ such that $a^2 \notin I$ and $V(\Gamma(R/I)) \setminus \{a+I\}$ has no cut-sets, then $V(\Gamma_I(R)) \setminus \{a + i \mid i \in I\}$ is a cut-set of $\Gamma_I(R)$.*

3. *If $V(\Gamma(R/I)) = \{a + I, b + I\}$ and neither vertex is looped, then $\{a + i \mid i \in I\}$ and $\{b + i \mid i \in I\}$ are cut-sets in $\Gamma_I(R)$.*

4. *If A is a cut-set in $\Gamma_I(R)$ and A is a union of n columns with $|Z(R/I)^*| - n \geq 2$, then $\{a + I \mid a \in A\}$ is a cut-set in $\Gamma(R/I)$.*

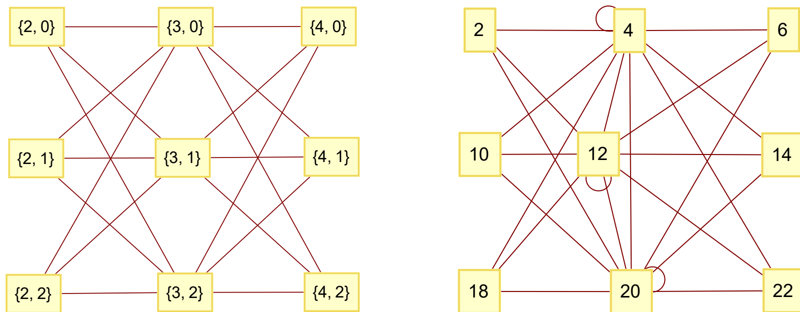
Proof. Let A be a set of vertices in $\Gamma(R/I)$ such that $A' = \{a + i \mid a + I \in A, i \in I\}$ is a cut-set of $\Gamma_I(R)$. Theorem 2.4 describes A' when A is a cut-set in $\Gamma(R/I)$. So suppose that A does not form a cut-set in $\Gamma(R/I)$. Then clearly A does not contain a cut-set as that would contradict the minimality of A' being a cut-set of $\Gamma_I(R)$ by Theorem 2.4. By Theorem 2.5, since A is not a cut-set of $\Gamma(R/I)$, then either A' is not a union of n columns or $|Z(R/I)^*| - n \leq 1$ for some $n \geq 1$. However, Theorem 2.3 gives us that A' is a union of columns, say n columns. Thus, we must have that $|Z(R/I)^*| - n \leq 1$.

If $n = 1$, then $|Z(R/I)^*| \leq 2$. If $|Z(R/I)^*| = 2$, then we must have no connected columns by Lemma 6.2(1) in [37] and Proposition 2.11(1) since complete graphs do not have cut-sets. Thus, A' must be the column of one of the vertices in $\Gamma(R/I)$ by Proposition 2.11(2). If $|Z(R/I)^*| = 1$, then $\Gamma(R/I)$ is a graph on one vertex and hence is looped. Thus, $\Gamma_I(R)$ is the graph on one connected column and is therefore a complete graph. This contradicts $\Gamma_I(R)$ having a cut-set.

Let $n \geq 2$. Then $|Z(R/I)^*| = n + 1$ since otherwise $A' = V(\Gamma_I(R))$ and hence cannot be a cut-set. Let $b + I$ be the vertex in $V(\Gamma(R/I)) \setminus A$. Since A' is a cut-set of $\Gamma_I(R)$ and $V(\Gamma_I(R)) = A' \cup \{b + i \mid i \in I\}$, the column of $b + I$ is not connected. Thus, $b^2 \notin I$. Therefore, by Proposition 2.9, A' is the vertices in the columns of $V(\Gamma(R/I)) \setminus \{b + I\}$. □

2.3 New Perspective on Relationship between $\Gamma_I(R)$ and $\Gamma(R/I)$

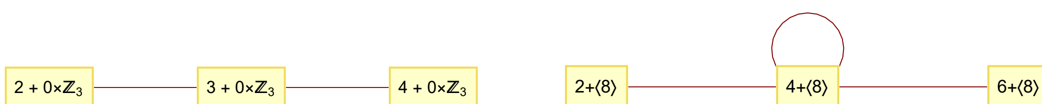
In [37], Shane Redmond considers the ideal-based zero-divisor graph to be a simple graph, i.e., no vertex is looped. Since in this paper we include loops in our graph, it is useful to consider a new perspective on the strong relationship between $\Gamma_I(R)$ and $\Gamma(R/I)$. In Remark 2.3 of [37], Redmond considers the two figures, 2.2a and 2.2b, which are the ideal-based zero-divisor graphs $\Gamma_{0\times\mathbb{Z}_3}(\mathbb{Z}_6 \times \mathbb{Z}_3)$ and $\Gamma_{\langle 8\rangle}(\mathbb{Z}_{24})$, respectively. Clearly, these are nonisomorphic graphs, and Redmond comments that the associated zero-divisor graphs $\Gamma(\mathbb{Z}_6 \times \mathbb{Z}_3/0 \times \mathbb{Z}_3) \cong \Gamma(\mathbb{Z}_6)$ and



(a) $\Gamma_{0\times\mathbb{Z}_3}(\mathbb{Z}_6 \times \mathbb{Z}_3)$ **(b)** $\Gamma_{\langle 8\rangle}(\mathbb{Z}_{24})$

Figure 2.2: Figures 1 and 2 from [37]

$\Gamma(\mathbb{Z}_{24}/\langle 8\rangle) \cong \Gamma(\mathbb{Z}_8)$ are isomorphic. However, when we include looped vertices in our graph, it is clear that $\Gamma^*(\mathbb{Z}_6)$ and $\Gamma^*(\mathbb{Z}_8)$ are not isomorphic graphs, as shown in Figure 2.3.



(a) $\Gamma(\mathbb{Z}_6 \times \mathbb{Z}_3/0 \times \mathbb{Z}_3) \cong \Gamma(\mathbb{Z}_6)$ **(b)** $\Gamma(\mathbb{Z}_{24}/\langle 8\rangle) \cong \Gamma(\mathbb{Z}_8)$

Figure 2.3: The graphs $\Gamma(\mathbb{Z}_6 \times \mathbb{Z}_3/0 \times \mathbb{Z}_6)$ and $\Gamma(\mathbb{Z}_{24}/\langle 8\rangle)$

It is clear to see that this is the difference between $\Gamma_{0\times\mathbb{Z}_3}(\mathbb{Z}_6 \times \mathbb{Z}_3)$ and $\Gamma_{\langle 8\rangle}(\mathbb{Z}_{24})$: the middle vertex should be looped in $\Gamma(\mathbb{Z}_{24}/\langle 8\rangle)$, thus making the middle column of $\Gamma_{\langle 8\rangle}(\mathbb{Z}_{24})$ a connected column. This leads us to consider the relationship between $\Gamma_I(R)$ and $\Gamma(R/I)$ when we allow loops in both graphs, i.e., the relationship between $\Gamma_I^*(R)$ and $\Gamma^*(R/I)$.

To start, we notice a relationship between the number of vertices in the graphs and the size of the proper ideals that uses the following remark from Redmond's initial paper on ideal-based zero-divisor graphs [37].

Remark 2.13 ([37, Remark 2.8])**.** *Let I be an ideal of a ring R. Then $\Gamma_I(R)$ is a graph on a finite number of vertices if and only if R is finite or I is a prime ideal. Moreover, if $\Gamma(R/I)$ is a graph on N vertices, then $\Gamma_I(R)$ is a graph on $N \cdot |I|$ vertices.*

This remark remains true when we substitute $\Gamma_I^*(R)$ and $\Gamma^*(R/I)$ for $\Gamma_I(R)$ and $\Gamma(R/I)$, respectively. Using the "moreover" statement, the following lemma is clear, which we will use in the subsequent theorem. Lemma 2.14 is true regardless of whether we include loops or not, so we state it using $\Gamma_I(R)$ and $\Gamma(R/I)$ notation.

Lemma 2.14. *Any two of the following conditions implies the third for rings R and S with ideals I and J, respectively.*

1. $|V(\Gamma_I(R))| = |V(\Gamma_J(S))|$.

2. $|V(\Gamma(R/I))| = |V(\Gamma(S/J))|$.

3. $|I| = |J|$.

Theorem 2.15. *Let R and S be commutative rings with ideals I and J, respectively, such that $|I| = |J|$. Then $\Gamma_I^*(R) \cong \Gamma_J^*(S)$ if and only if $\Gamma^*(R/I) \cong \Gamma^*(S/J)$.*

Proof. Let R and S be commutative rings with ideals I and J, respectively, such that $|I| = |J|$. Suppose $\Gamma^*(R/I) \cong \Gamma^*(S/J)$. Then clearly by the construction of the ideal-based zero-divisor graphs and the assumption that $|I| = |J|$, we have that $\Gamma_I^*(R) \cong \Gamma_J^*(S)$.

For the "only if" part, let $\Gamma_I^*(R) \cong \Gamma_J^*(S)$. We can consider the subgraphs of $\Gamma^*(R/I)$ in $\Gamma_I^*(R)$ and $\Gamma^*(S/J)$ in $\Gamma_J^*(S)$. Since $|V(\Gamma_I^*(R))| = |V(\Gamma_J^*(S))|$ and $|I| = |J|$, we have by Lemma 2.14 that $|V(\Gamma^*(R/I))| = |V(\Gamma^*(S/J))|$. By the construction of the ideal-based zero-divisor graph, we can restrict the isomorphism $\phi : \Gamma_I^*(R) \to \Gamma_J^*(S)$ to a single copy of the subgraph $\Gamma^*(R/I)$ to get an isomorphism $\phi_0 : \Gamma^*(R/I) \to \Gamma^*(S/J)$. □

This leads us to the following corollary. Theorem 2.15 is only true when allowing loops in the graph. Therefore, Corollary 2.16 is also only true when we allow loops in the graph.

Corollary 2.16. *Let R and S be commutative rings with ideals I and J, respectively. Then any two of the following conditions implies the third.*

1. $|I| = |J|$.

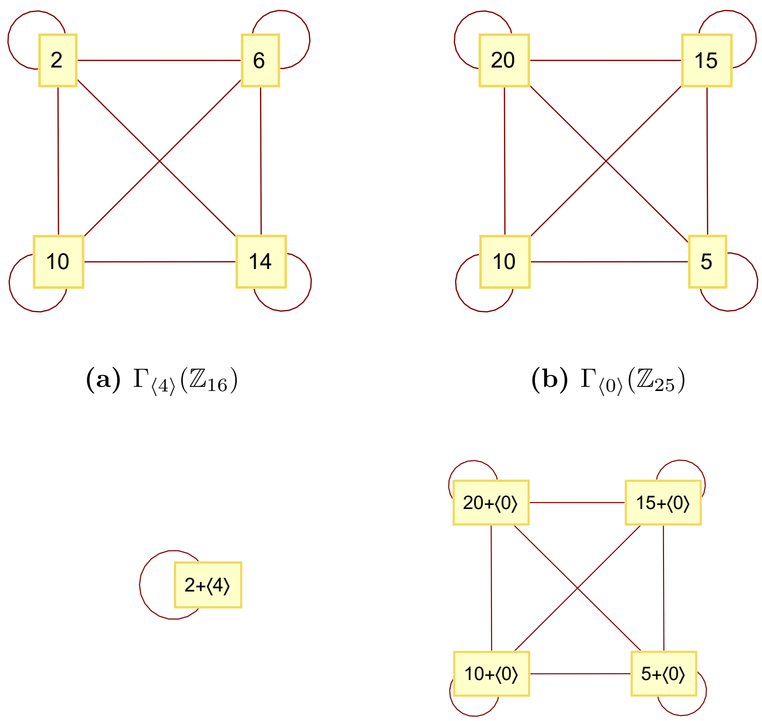
2. $\Gamma_I^*(R) \cong \Gamma_J^*(S)$.

3. $\Gamma^*(R/I) \cong \Gamma^*(S/J)$.

Proof. (1),(2) imply (3), and (1),(3) imply (2) are clear from Theorem 2.15. For (2),(3) imply (1), note that the graph isomorphisms require that the vertex sets in each respective graph be of equal size. Thus, Lemma 2.14 gives us that $|I| = |J|$. $\square$

The assumption that $|I| = |J|$ in Theorem 2.15 is a necessary condition for both directions, and we give counterexamples below. These counterexamples do not depend on loops, and thus we state them using $\Gamma_I(R)$ and $\Gamma(R/I)$.

To see that $\Gamma_I(R) \cong \Gamma_J(S)$, but $\Gamma(R/I) \not\cong \Gamma(S/J)$, consider $R = \mathbb{Z}_{16}$, $I = \langle 4 \rangle$, $S = \mathbb{Z}_{25}$, and $J = \langle 0 \rangle$. Figure 2.4 clearly shows that $\Gamma_{\langle 4 \rangle}(\mathbb{Z}_{16}) \cong \Gamma_{\langle 0 \rangle}(\mathbb{Z}_{25})$. However, $R/I \cong \mathbb{Z}_4$ and $S/J \cong \mathbb{Z}_{25}$, and thus, $\Gamma({}^{\mathbb{Z}_{16}}/_{\langle 4 \rangle}) \not\cong \Gamma({}^{\mathbb{Z}_{25}}/_{\langle 0 \rangle})$. The issue arises because of the differing sizes of $\langle 4 \rangle$ in $\mathbb{Z}_{16}$ and $\langle 0 \rangle$ in $\mathbb{Z}_{25}$.



(a) $\Gamma_{\langle 4 \rangle}(\mathbb{Z}_{16})$ **(b)** $\Gamma_{\langle 0 \rangle}(\mathbb{Z}_{25})$

(c) $\Gamma({}^{\mathbb{Z}_{16}}/_{\langle 4 \rangle}) \cong \Gamma(\mathbb{Z}_4)$ **(d)** $\Gamma({}^{\mathbb{Z}_{25}}/_{\langle 0 \rangle}) \cong \Gamma(\mathbb{Z}_{25})$

Figure 2.4: Counterexample where $\Gamma_I(R) \cong \Gamma_J(S)$, but $\Gamma(R/I) \not\cong \Gamma(S/J)$

To see that $\Gamma(R/I) \cong \Gamma(S/J)$, but $\Gamma_I(R) \not\cong \Gamma_J(S)$, consider $R = \mathbb{Z}_{16}$, $I = \langle 8 \rangle$, $S = \mathbb{Z}_{24}$, and $J = \langle 8 \rangle$. We have that $R/I = {}^{\mathbb{Z}_{16}}/_{\langle 8 \rangle} \cong \mathbb{Z}_8 \cong {}^{\mathbb{Z}_{24}}/_{\langle 8 \rangle} \cong S/J$. Thus, $\Gamma(R/I) \cong \Gamma(S/J)$. However, Figure 2.5 shows that $\Gamma_{\langle 8 \rangle}(\mathbb{Z}_{16}) \not\cong \Gamma_{\langle 8 \rangle}(\mathbb{Z}_{24})$. Again, the problem lies in the size of the ideal. In $\mathbb{Z}_{16}$, the ideal $\langle 8 \rangle$ has size two, and thus two copies of the subgraph $\Gamma({}^{\mathbb{Z}_{16}}/_{\langle 8 \rangle})$ determine $\Gamma_{\langle 8 \rangle}(\mathbb{Z}_{16})$. In $\mathbb{Z}_{24}$, the ideal $\langle 8 \rangle$ has size three, and therefore three copies of the subgraph $\Gamma({}^{\mathbb{Z}_{24}}/_{\langle 8 \rangle})$ determine $\Gamma_{\langle 8 \rangle}(\mathbb{Z}_{24})$.

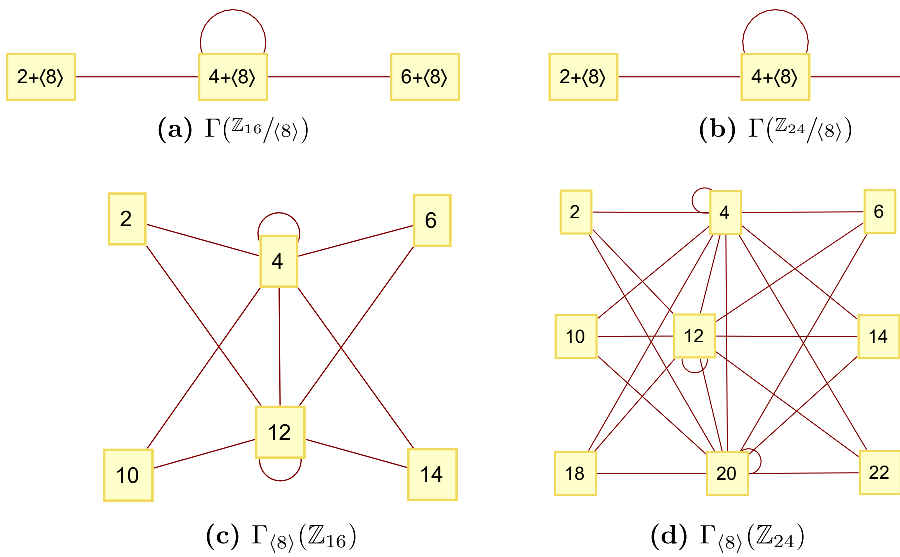


Figure 2.5: Counterexample where $\Gamma(R/I) \cong \Gamma(S/J)$, but $\Gamma_I(R) \ncong \Gamma_J(S)$

Chapter 3

Zero-Divisor Graphs on Commutative Rings Without Identity

We now turn our attention to rings without identity. We begin with investigating finite commutative rings, and then transition to rings R where $R = Z(R)$. Most of the work done on zero-divisor graphs has focused on commutative rings with identity, although there have been a few exceptions. In [14], the authors focus on commutative rings with identity, but specifically mention when the identity is not necessary for their results. It was more explicitly investigated in Section 3 of [16] and in [17].

Even though an identity is usually assumed, many of the results only depend on the semigroup structure of a ring, and thus, the proofs hold for commutative rings without identity. In particular, Theorem 1.1 ($\Gamma(R)$ connected and $\mathrm{diam}(\Gamma(R)) \leq 3$) and Theorem 1.2 ($\mathrm{gr}(\Gamma(R)) \in \{3, 4, \infty\}$) hold for commutative rings without identity, as was stated by Theorems 2.1 and 2.2 in [14].

There are, however, some major differences between the zero-divisor graphs of rings with and without identity. For example, there are many graphs that can only be realized as the zero-divisor graphs of rings with identity or of rings without identity, but not both. The complete graph on 5 vertices, K^5, can be realized as the zero-divisor graph of $\mathbb{Z}_6$ with the trivial multiplication, but cannot be realized as the graph of a commutative ring with identity. In fact, the only finite complete graphs that can be realized as the zero-divisor graph of a commutative ring with identity are K^{p^n-1} for some prime p and integer $n > 1$. Whereas, K^n

for any positive integer n can be realized as the zero-divisor graph of a commutative ring without identity (take $R = \mathbb{Z}_{n+1}$ with the trivial multiplication for a simple example).

Conversely, the complete bipartite graph $K^{2,2}$ can only be realized as the zero-divisor graph of a commutative ring with identity ($R = \mathbb{Z}_3 \times \mathbb{Z}_3$). In fact, graphs on 4 vertices are the smallest such graphs that can be realized as the zero-divisor graph of a commutative ring with identity, but not a commutative ring without identity. Whereas, graphs on 5 vertices are the smallest such graphs to be realized as zero-divisor graphs on commutative rings without identity, but not rings with identity.

Another key difference is in which positive integers n you can find a commutative ring that has a zero-divisor graph on n vertices. For any positive integer n, you can always find a commutative ring without identity that has a zero-divisor graph on n vertices. The simplest example is $\mathbb{Z}_{n+1}$ with the trivial multiplication. However, there exist positive integers n for which there is no commutative ring with identity that has a zero-divisor graph on n vertices. The survey article [4] gives a quick summary of the results in [39], where Redmond used computer calculations to show that there are no commutative rings with identity that have 1210, 3342, or 5466 zero-divisors. Thus, there are no commutative rings with identity that have zero-divisor graphs on 1209, 3341, or 5465 vertices. There is the potential for many more, however, these calculations were only carried out for positive integers up to 7500.

Comparing the list given in Section 5 of [38] to Tables 3.1 through 3.6 will yield several more examples of these differences. Also, Section 3.4 has several tables that will demonstrate the difference in realizable diameters and girths between commutative rings with and without identity.

3.1 Finite Commutative Rings

Lemma 3.1. *Let R be a finite commutative ring. Then every element $x \in R$ is either nilpotent or some x^n is idempotent.*

Proof. Let R be a finite commutative ring and suppose some $0 \neq x \in R$ is not nilpotent. Then $x^m = x^{m+k}$ for some $m, k > 0$. Let $b, i \geq 0$ such that $bk = m + i$ by the division algorithm. Then $(x^{m+i})^2 = x^{2m+2i} = x^{m+i+m+i} = x^{bk+m+i} = x^{m+bk}x^i = x^{m+i}$. Thus, x^{m+i} is idempotent. $\square$

The following lemma first appeared in [2] as Lemma 4.4.

Lemma 3.2 ([2, Lemma 4.4])**.** *Let R be a finite commutative ring. Then $R \cong R_1 \times R_2$, where R_1 has an identity and every element of R_2 is nilpotent.*

Proof. Note that it is possible to have $R = R_1$ or $R = R_2$. Suppose not every element of R is nilpotent. Then by Lemma 3.1 there exists some nonzero idempotent element $e \in R$. Consider $A = Re$ and $B = \{r - re \mid r \in R\}$. Then B is a subring of R and A is a subring of R with identity e. Since for any $r \in R$, $r = re + (r - re)$, we have that $R = A + B$. Also, $A \cap B = \{0\}$ since $re = s - se$ implies that $re = (re)e = (s - se)e = 0$. Thus, $R \cong A \times B$, where our map takes $r \mapsto (re, r - re)$. Now, either every element of B is nilpotent or there exists some nonzero idempotent in B. Continuing the process gives us the conclusion of the lemma. $\square$

Corollary 3.3. *Let R be a finite commutative ring. Then either*

1. *R has an identity,*
2. *R is nilpotent, or*
3. *$R \cong R_1 \times R_2$, where R_1 has an identity and $R_2 \neq \{0\}$ is nilpotent.*

Recall that a zero-divisor semigroup is a semigroup where every element is a zero-divisor. We can treat finite commutative rings without identity as zero-divisor semigroups, as is shown in Lemma 3.4.

Lemma 3.4. *Let R be a finite commutative ring without identity. Then $R = Z(R)$.*

Proof. We show the contrapositive. Let $Z(R) \subsetneq R$ for a finite ring R. Then there exists some $x \in R \setminus Z(R)$ and an injective map $f : R \to R$ such that $f(r) = xr$ for all $r \in R$. Since R is finite, f is also surjective. Thus, there exists an $r \in R$ such that $x = f(r) = xr$. Now, consider $y \in R$. Since f is surjective, there exists an $a \in R$ such that $y = f(a) = xa$. Now, $yr = xar = xa = y$. So r is the identity. $\square$

3.2 Commutative Rings Where $R = Z(R)$

In this section, we look at rings where every element is a zero-divisor. Lemma 3.4 gives us that every finite commutative ring without identity falls into this category. Here, we begin to investigate the properties of the zero-divisor graphs of these types of rings.

Lemma 3.5. *Let R be a finite commutative ring with nil$(R) \subsetneq R = Z(R)$. Then gr$(\Gamma(R)) \in \{3, \infty\}$.*

Proof. From Lemma 3.2, we have that $R = R_1 \times R_2$, where R_1 has an identity and is nonempty (since $\mathrm{nil}(R) \subsetneq R$) and every element of R_2 is nilpotent. Clearly $|R_2| \geq 2$ since $|R_2| \leq 1$ implies that R has an identity, a contradiction on $R = Z(R)$. We consider three cases:

Case 1: $|R_2| = 2$ and R_1 is not an integral domain. Then there are nonzero elements $c, d \in R_1$ such that $cd = 0$ and $R_2 = \{0, a\}$ with $a^2 = 0$. So $(c, 0)$ — (d, a) — $(0, a)$ — $(c, 0)$ is a cycle in $\Gamma(R)$. Thus, $\mathrm{gr}(\Gamma(R)) = 3$.

Case 2: $|R_2| = 2$ and R_1 is an integral domain. Thus, $R = \{(x, 0) \mid 0 \neq x \in R_1\} \cup \{(x, a) \mid 0 \neq x \in R_1\} \cup \{(0, 0), (0, a)\}$. Then $\Gamma(R)$ is a complete bipartite graph with partition $\{(0, a)\}$ and partition $\{(x, y) \mid 0 \neq x \in R_1 \text{ and } y \in \{0, a\}\}$. Thus, $\mathrm{gr}(\Gamma(R)) = \infty$.

Case 3: $|R_2| \geq 3$. Theorem 1.1 gives the existence of nonzero $a, b \in R_2$ such that $ab = 0$ and $a \neq b$. Then $(0, a)$ — $(0, b)$ — $(1, 0)$ — $(0, a)$ is a cycle in $\Gamma(R)$. Thus, $\mathrm{gr}(\Gamma(R)) = 3$.

Thus, in all three cases, we get $\mathrm{gr}(\Gamma(R)) \in \{3, \infty\}$. □

Corollary 3.6. *Let R be a finite commutative ring with nil$(R) \subsetneq R = Z(R)$. Then the following statements are equivalent.*

1. *gr$(\Gamma(R)) \neq 3$.*

2. *gr$(\Gamma(R)) = \infty$.*

3. *$\Gamma(R)$ is complete bipartite.*

4. $\Gamma(R)$ *is a star graph.*

5. $\Gamma(R) = K^{1,2p^n-2}$ *for some prime* p *and integer* $n \geq 1$.

Proof. Since $\mathrm{nil}(R) \subsetneq R$, Lemma 3.5 and its proof give us the equivalence of 1 through 4. To see the equivalence of 5, note that we are in the case where $|R_2| = 2$ and R_1 is an integral domain. In this case, we have that $\Gamma(R)$ is a star graph with center $(0, a)$ and ends $\{(x, y) \mid 0 \neq x \in R_1 \text{ and } y \in \{0, a\}\}$. So $\Gamma(R) \cong K^{1,2k}$, where $k = |R_1| - 1$. Since R_1 is an integral domain (finite field), $|R_1| = p^n$ for some prime p and integer $n \geq 1$. Thus, $\Gamma(R) = K^{1,2p^n-2}$. □

The next result is an analog to Theorem 2.8 in [10] and shows that for complete zero-divisor graphs, every vertex is looped. It also tells us that for complete graphs, every element is nilpotent of order 2. This result is similar to Lemma 2.2 in [17].

Theorem 3.7. *Let* R *be a commutative ring with* $R = Z(R)$. *Then* $\Gamma(R)$ *is complete if and only if* $xy = 0$ *for all* $x, y \in R$.

Proof. The backwards direction is clear. For the forward direction, notice that $|V(\Gamma(R))| = 1$ implies that $x^2 = 0$ for $x \in Z(R)$. Therefore, suppose that $|V(\Gamma(R))| \geq 2$ which implies $|R| \geq 3$. Since $\Gamma(R)$ is complete, $xy = 0$ for every distinct $x, y \in R$. Hence, we need only show that $x^2 = 0$ for every nonzero $x \in R$. Let $y \in R \setminus \{0, x\}$. Then $x + y \neq x$. So $xy = 0 = x(x + y)$. Thus, $x^2 = x^2 + 0 = x^2 + xy = x(x + y) = 0$. □

The next lemma holds for arbitrary commutative rings, but the subsequent corollary gives us a limit on the size of the zero-divisor graph of a ring where every element is nilpotent.

Lemma 3.8. *Let* R *be a commutative ring. If* $x \in R$ *is nilpotent, then* $d(x, y) \leq 2$ *for any* $y \in V(\Gamma(R))$.

Proof. Let R be a commutative ring with $x \in R$ nilpotent. Assume that for some vertex y, $\mathrm{d}(x, y) = 3$. Let $x — a — b — y$ be a shortest path from x to y. Since x is nilpotent and $xb \neq 0$, we have that $x^n b \neq 0$ but $x^{n+1} b = 0$ for some integer $n \geq 1$. Then $x^n b$ is a vertex in our graph and $x — x^n b — y$ is a path, a contradiction. □

Corollary 3.9. *Let $R = nil(R)$ be a commutative ring. Then diam$(\Gamma(R)) \leq 2$.*

Theorem 3.10. *Let $R = Z(R)$ and diam$(\Gamma(R)) = 3$. Then gr$(\Gamma(R)) = 3$.*

Proof. Let $x, y \in V(\Gamma(R))$ such that $\mathrm{d}(x, y) = 3$. Let $a, b \in V(\Gamma(R)) \setminus \{x, y\}$ be distinct vertices such that $x — a — b — y$ is a shortest path in $\Gamma(R)$ between x and y. Since $x \neq y$, we have that $x - y \neq 0$ and thus $t(x - y) = 0$ for some nonzero $t \in R$. This implies that $tx = ty \neq 0$ since $x — t — y$ would be a path in $\Gamma(R)$ and therefore $\mathrm{d}(x, y) = 2$, a contradiction.

We claim that $ta = tb = 0$. To prove this claim, assume $ta \neq 0$. Then $x(ta) = (xa)t = 0$ and $y(ta) = (yt)a = (xt)a = 0$. Then $x — ta — y$ is a path and $\mathrm{d}(x, y) = 2$, a contradiction. Thus, $ta = 0$ and similarly $tb = 0$.

Next, we show that $t \notin \{0, a, b\}$. Clearly, $t \neq 0$. If $t = a$, then $0 = ax = tx = ty = ay$ implying that $x — a — y$ is a path, a contradiction. So $t \neq a$ and similarly $t \neq b$. Thus, $t \in V(\Gamma(R))$ is distinct from a and b, and $t — a — b — t$ is a cycle in $\Gamma(R)$. Therefore, $\mathrm{gr}(\Gamma(R)) = 3$. □

Theorem 3.11. *Let R be a commutative ring such that $R = nil(R)$. Then diam$(\Gamma(R)) \leq 2$ and gr$(\Gamma(R)) \in \{3, \infty\}$. Moreover, if gr$(\Gamma(R)) = \infty$, then $\Gamma(R)$ is either K^1, K^2, or $K^{1,2}$.*

Proof. Let R be a commutative ring such that $R = \mathrm{nil}(R)$. By Corollary 3.9, we know that $\mathrm{diam}(\Gamma(R)) \leq 2$. To show the girth, we consider three cases:

Case 1: Let $a^2 = 0$ for all $a \in R$. If $|R| \leq 3$, then $\Gamma(R)$ is a K^1 or K^2 graph; so suppose that $|R| \geq 4$. Let $a, b, c \in V(\Gamma(R))$ be distinct such that $a — b — c$ is a path in $\Gamma(R)$. Such a path exists since $\Gamma(R)$ is always connected. Consider $d = a + b$. Clearly, $d \neq a, b$. If $d = 0$, then $a = -b$ and thus $ac = (-b)c = 0$. So $a — b — c — a$ is a cycle in $\Gamma(R)$. So suppose that $d \neq 0$. Then $ad = a(a + b) = a^2 + ab = 0$ implying that $a — b — a + b — a$ is a cycle.

Case 2: Suppose there exists an $a \in R$ such that $a^2 \neq 0$, but $a^3 = 0$. Then $a^2 \in V(\Gamma(R))$ and $a^2 — a$ is a path in $\Gamma(R)$. If a is not an end, then there is a $b \in R$ such that $a^2 \neq b$ and b is adjacent to a. Then $ab = 0$, which implies that $a^2 b = 0$. Thus, $a^2 — a — b — a^2$ is a cycle. So suppose that a is an end. Clearly, $a^2 + a \in R \setminus \{0, a, a^2\}$. So $a^2 + a —$

a^2 — a is a path. The existence of $a^2 + a$ shows that there is some $b \in R \setminus \{0, a, a^2\}$ adjacent to a^2. We claim that if $\mathrm{gr}(\Gamma(R)) \neq 3$, then $\Gamma(R)$ must be a star graph. To prove this claim, suppose that $\Gamma(R)$ is not a star graph. Then there exists distinct $b, d \in R \setminus \{0, a, a^2\}$ such that d — b — a^2 — a is a path. However, $\mathrm{d}(a, d) \leq 2$ and a an end implies that d must be adjacent to a^2. Thus, b — d — a^2 — b is a cycle in $\Gamma(R)$. Note that we can have $R = \{0, a, a^2, a^2 + a\}$, in which case $\Gamma(R) = K^{1,2}$.

Case 3: Suppose there is an $a \in R$ such that $a^n = 0$, but $a^{n-1} \neq 0$ for some $n \geq 4$. Then a^{n-1}, a^{n-2}, and $a^{n-1} + a^{n-2}$ are all distinct nonzero elements of R. Thus, a^{n-2} — a^{n-1} — $a^{n-1} + a^{n-2}$ — a^{n-2} is a cycle in $\Gamma(R)$.

To show the "Moreover" statement, let $\mathrm{gr}(\Gamma(R)) = \infty$. From the above, we may assume that there exists an $a \in R$ such that $a^2 \neq 0$, $a^3 = 0$, and a is an end (Case 2). From the proof of this case, we know that $\Gamma(R)$ must be a star graph. To show that $\Gamma(R) \cong K^1$, K^2, or $K^{1,2}$, we consider the size of the ring. If $|R| = 2$, then $|Z^*(R)| = 1$ and $\Gamma(R) \cong K^1$. If $|R| = 3$, then $|Z^*(R)| = 2$ and clearly $\Gamma(R) \cong K^2$. Therefore, we may assume that $|R| \geq 4$. We already have in this case that $\Gamma(R)$ is the graph $a^2 + a$ — a^2 — a. We show that R must be $\{0, a, a^2, a^2 + a\}$.

Notice that a an end forces $a^2 = -a^2$. Suppose there is an end vertex b distinct from a and $a^2 + a$. If $b \neq -a$, then $a + b \neq 0$. We claim that $a + b \notin \{0, a, b, a^2 + a\}$. To see this claim, it is clear that $a + b \neq a$ and $a + b \neq b$. If $a + b = a^2 + a$, then $b = a^2$, a contradiction to b being an end. Thus, $a + b \notin \{0, a, b, a^2 + a\}$. Notice that $(ab)a = a^2 b = 0$ since b is an end and $\Gamma(R)$ a star graph with center a^2. This forces $ab = a^2$. Therefore, $(a + b)(a^2 + a) = a^2 + ab = 2a^2 = 0$ since $a^2 = -a^2$. Then $a + b$ — $a^2 + a$ — a^2 — $a + b$ is a cycle, a contradiction. So we must have that $b = -a$. Then $|V(\Gamma(R))| = 4$ which implies that $|R| = 5$. Since R does not have an identity, this means that R has the trivial multiplication as there are only two nonisomorphic rings of prime order for some prime p: $\mathbb{Z}_p$ and $\mathbb{Z}_p$ with the trivial multiplication. Thus $\Gamma(R) \cong K^4$, a contradiction. So $b = a$ or $b = a^2 + a$, and hence, $R = \{0, a, a^2, a^2 + a\}$. Therefore, $\Gamma(R) \cong K^{1,2}$. $\square$

The case where $R = \{0, a, a^2, a^2{+}a\}$ can be realized as $R = {}^{x\mathbb{Z}[x]}/_{(4x,\, x^2\, -\, 2x)} = \{0, x, 2x, 3x\}$ (with $a = x$). Clearly, $\mathrm{nil}(R) = R$ (as $x^3 = 0$, $(2x)^2 = 0$, and $(3x)^3 = 0$) and $\Gamma(R) = K^{1,2}$.

The graph K^1 can be realized by $R = \mathbb{Z}_2$ under trivial multiplication. The graph K^2 can be realized by $R = \mathbb{Z}_3$, under trivial multiplication.

The next theorem relaxes the condition in Theorem 3.11 that $R = \text{nil}(R)$ to just $R = Z(R)$ while still maintaining a similar result for the girth.

Theorem 3.12. *Let R be a commutative ring with $R = Z(R)$. Then $gr(\Gamma(R)) \in \{3, \infty\}$.*

Proof. Let R be a commutative ring with $R = Z(R)$. It suffices to show that $\Gamma(R)$ contains a 3-cycle when it has a 4-cycle. So suppose $a — b — c — d — a$ is a cycle for nonzero vertices a, b, c, and d such that $ac \neq 0 \neq bd$. Clearly, $a - b \neq 0$ which implies that there exists some nonzero $t \in R$ such that $t(a - b) = 0$.

Case 1: $ta = tb \neq 0$. Then $0 = (ta)a = (ta)b = (ta)c = (ta)d$. Now since $ac \neq 0 \neq bd$, $ta \notin \{a, b, c, d\}$. Thus, $a — ta — b — a$ is a cycle.

Case 2: $ta = tb = 0$. If $t \notin \{a, b\}$, then $a — b — t — a$ is a cycle. So suppose $t \in \{a, b\}$. Then either $a^2 = 0$ or $b^2 = 0$. Notice that if $a^2 = b^2 = 0$, then $a+b \notin \{0, a, b\}$ ($a+b \neq 0$ as this would imply that $a = -b$, and thus $ac = (-b)c = 0$, a contradiction). This would give us the cycle $a — a + b — b — a$. So suppose that only one of $a^2 = 0$ or $b^2 = 0$ holds.

Without loss of generality, let $a^2 = 0$ and $b^2 \neq 0$. Repeating the above argument also gives us, without loss of generality, that $c^2 = 0$ and $d^2 \neq 0$. Now, $0 \neq ac \in V(\Gamma(R))$ and $a(ac) = a^2c = 0$, $b(ac) = 0$, $c(ac) = ac^2 = 0$, and $d(ac) = 0$. Again, $ac \notin \{a, b, c, d\}$ since it would lead to a similar contradiction as above. Thus, $a — ac — d — a$ is a cycle.

Therefore, $\text{gr}(\Gamma(R)) = 3$. □

Our next theorem follows from several previous results, but requires an additional graph-theoretic argument.

Theorem 3.13. *If $R = Z(R)$ and $gr(\Gamma(R)) \neq 3$, then $\Gamma(R)$ is a star graph. In particular, $\Gamma(R)$ is either K^1, $K^2 = K^{1,1}$, $K^{1,2}$, $K^{1,2p^n-2}$, or $K^{1,m}$, where m is an infinite cardinal number.*

Proof. Let $R = Z(R)$ and $\mathrm{gr}(\Gamma(R)) \neq 3$. Then by Theorem 3.12, $\mathrm{gr}(\Gamma(R)) = \infty$. For $R = \mathrm{nil}(R)$ and R finite with $\mathrm{nil}(R) \subsetneq R$, Theorem 3.11 and Corollary 3.6, respectively, give us our conclusion. So we need only worry about the case where $\mathrm{nil}(R) \subsetneq R$ and R is infinite. Then, by Proposition 3.22 and Theorem 3.10, we have that $\mathrm{diam}(\Gamma(R)) = 2$. We show that $\Gamma(R)$ has to be a star graph.

Let x — y — z be a shortest path from x to z. Let $a_0 \in V(\Gamma(R))$. Without loss of generality, let a_0 — x. Then a_0 — y or a_0 — z since $\mathrm{diam}(\Gamma(R)) = 2$. However, this creates a cycle in $\Gamma(R)$, a contradiction on the girth. So a_0 is connected to y and only to y. Therefore, all vertices must be connected to y and only to y. Hence, $\Gamma(R)$ is a star graph. □

Our next corollary gives a quick summary of possible girths for various diameters of the zero-divisor graphs of commutative rings $R = Z(R)$. Tables 3.24 and 3.25 give some examples of the realizable cases.

Corollary 3.14. *Let R be a commutative ring such that $R = Z(R)$.*

1. *If diam$(\Gamma(R)) = 0$, then gr$(\Gamma(R)) = \infty$.*

2. *If diam$(\Gamma(R)) = 1$, then gr$(\Gamma(R)) \in \{3, \infty\}$.*

3. *If diam$(\Gamma(R)) = 2$, then gr$(\Gamma(R)) \in \{3, \infty\}$.*

4. *If diam$(\Gamma(R)) = 3$, then gr$(\Gamma(R)) = 3$.*

Theorem 3.15. *Let $R = R_1 \times R_2$, where R_1 is a commutative ring with identity and $R_2 = nil(R_2) \neq \{0\}$ a commutative ring. Then diam$(\Gamma(R)) = 2$.*

Proof. Let $R = R_1 \times R_2$, where R_1 is commutative with identity and $R_2 = \mathrm{nil}(R_2)$ nontrivial. Then for nonzero $b \in R_2$, let $x = (1, 0)$ and $y = (1, b)$. Clearly, $xy \neq 0$ so $\mathrm{d}(x, y) \geq 2$. Thus, $\mathrm{diam}(\Gamma(R)) \geq 2$. For distinct $x, y \in Z(R) \setminus \{(0, 0)\}$, we consider cases where $a, c \in R_1$ and $b, d \in R_2$ are all nonzero:

Case 1: Let $x = (a, 0)$ and $y = (0, b)$. Then $xy = 0$ and $\mathrm{d}(x, y) = 1$.

Case 2: Let $x = (a, 0)$ and $y = (c, 0)$. Consider $z = (0, b)$. Then $xz = yz = 0$, so $\mathrm{d}(x, y) \leq 2$.

Case 3: Let $x = (0, b)$ and $y = (0, d)$. Then for $z = (1, 0)$ we have $xz = yz = 0$. So $\mathrm{d}(x, y) \le 2$.

Case 4: Let $x = (a, 0)$ and $y = (c, d)$. Let $n \ge 2$ be an integer such that $d^n = 0$ but $d^{n-1} \ne 0$. Then for $z = (0, d^{n-1})$, we have $xz = yz = 0$. Thus, $\mathrm{d}(x, y) \le 2$.

Case 5: Let $x = (0, b)$ and $y = (c, d)$. If $bd = 0$, then $xy = 0$ and $\mathrm{d}(x, y) = 1$. Otherwise, Corollary 3.9 gives us the existence of a nonzero $f \in R_2$ distinct from b and d such that $bf = fd = 0$. Then $z = (0, f)$ implies that $xz = yz = 0$. Therefore, $\mathrm{d}(x, y) = 2$.

Case 6: Let $x = (a, b)$ and $y = (c, d)$. If $b = d$, then there exists an integer $n \ge 2$ where $b^n = 0$, $b^{n-1} \ne 0$, and for $z = (0, b^{n-1})$, $xz = yz = 0$. Thus, $\mathrm{d}(x, y) \le 2$. If $b \ne d$, then Corollary 3.9 gives the existence of a nonzero $f \in R_2$ such that $bf = fd = 0$. Then $z = (0, f)$ gives that $xz = yz = 0$, and hence $\mathrm{d}(x, y) \le 2$.

Thus, $\mathrm{d}(x, y) \le 2$ for all distinct nonzero $x, y \in R$. Therefore, $\mathrm{diam}(\Gamma(R)) \le 2$, which implies that $\mathrm{diam}(\Gamma(R)) = 2$. □

Corollary 3.16. *Let R be a finite commutative ring without identity. Then diam$(\Gamma(R)) \le 2$.*

Through the next two results, we extend the result of Theorem 2.2 from [8] to commutative rings without identity. Let R be a commutative ring not necessarily with identity and total quotient ring $T(R)$ with $S = R \setminus Z(R)$. As in [8], for $x, y \in R$, we can define $x \sim y$ if and only if $\mathrm{ann}(x) = \mathrm{ann}(y)$, which is clearly an equivalence relation.

Lemma 3.17. *Let R be a commutative ring not necessarily with identity and let $S = R \backslash Z(R)$ with $T = T(R)$. Let $\sim_R$ and $\sim_T$ denote the equivalence relations described above on $Z(R)^*$ and $Z(T)^*$, respectively, with their respective equivalence classes being denoted by $[a]_R$ and $[a]_T$. Then the following hold for all $x, y \in Z(R)^*$ and $s, t \in S$.*

1. $\mathrm{ann}_T(x/s) = \mathrm{ann}_R(x)_S$.

2. $\mathrm{ann}_T(x/s) \cap R = \mathrm{ann}_R(x)$.

3. $x/s \sim_T x/t$.

4. $x \sim_R y$ *if and only if* $x/s \sim_T y/s$.

5. $([x]_R)_S = [xs/s]_T$.

6. $[x/s]_T \cap R = [x]_R$.

Proof. Let R be a commutative ring not necessarily with identity and $S = R \setminus Z(R)$ with $T = T(R)$.

1. Let $y/t \in \text{ann}_T(x/s)$. Then $\frac{y}{t} \cdot \frac{x}{s} = 0$, which implies that $s'xy = 0$ for some $s' \in S$. Thus, $s'y \in \text{ann}_R(x)$ and $y/t = \frac{ys'}{ts'} \in \text{ann}_R(x)_S$. For the reverse inclusion, let $y/t \in \text{ann}_R(x)_S$. We may assume that $y \in \text{ann}_R(x)$. So $xy = 0$ and $\frac{x}{s} \cdot \frac{y}{t} = 0$. Thus, $y/t \in \text{ann}_T(x/s)$.

2. Let $y \in \text{ann}_T(x/s) \cap R$. Then $y(x/s) = 0$ implies $s'yx = 0$ for some $s' \in S$. Thus, $xy \in \text{ann}_R(s') = \{0\}$, which means that $xy = 0$. So $y \in \text{ann}_R(x)$. For the reverse inclusion, let $y \in \text{ann}_R(x)$. Then clearly $\frac{ys}{s} = y \in R$ and $\frac{ys}{s} \cdot \frac{x}{s} = 0$ since $xy = 0$. Thus, $y = \frac{ys}{s} \in \text{ann}_T(x/s)$ and we have $\text{ann}_T(x/s) \cap R = \text{ann}_R(x)$.

3. Clear from parts (1) and (2).

4. Clear from parts (1) and (2).

5. We show inclusion both ways. Let $y \in [x]_R$. We need to show that $\text{ann}_T(y/t) = \text{ann}_T(xs/s)$. Let $r/u \in \text{ann}_T(y/t)$. Then $\frac{ry}{ut} = 0$, which implies that $s'ry = 0$. Thus, $s'r \in \text{ann}_R(y) = \text{ann}_R(x)$. So $\frac{r}{u} \cdot \frac{xs}{s} = \frac{rs'xs}{us's} = 0$ and thus $r/u \in \text{ann}_T(xs/s)$. For the reverse inclusion, let $r/u \in \text{ann}_T(xs/s)$. Then $\frac{rxs}{us} = 0$ and hence $s'rxs = 0$. So $s'rs \in \text{ann}_R(x) = \text{ann}_R(y)$. So $\frac{r}{u} \cdot \frac{y}{t} = \frac{ry}{ut} = \frac{rs'sy}{us'st} = 0$. Thus, $r/u \in \text{ann}_T(y/t)$. Therefore, we have that $y/t \in [xs/s]_T$.

   Now, let $y/t \in [xs/s]_T$ and show that $\text{ann}_R(y) = \text{ann}_R(x)$. To do this, let $r \in \text{ann}_R(y)$. Then $\frac{r}{s} \cdot \frac{y}{t} = 0$ for some $s, t \in S$ and hence $r/s \in \text{ann}_T(y/t) = \text{ann}_T(xs/s)$. This means that $rxss' = 0$ for some $s' \in S$ and from part (2) we have that $r = \frac{rs'}{s'} \in \text{ann}_T(xs/s) \cap R = \text{ann}_R(xs) = \text{ann}_R(x)$ since $s \notin Z(R)$, and thus $r \in \text{ann}_R(x)$. To show the reverse inclusion, let $r \in \text{ann}_R(x) = \text{ann}_R(xs)$. Then $\frac{r}{t'} \cdot \frac{xs}{s} = 0$ for some $t' \in S$. So $r/t' \in \text{ann}_T(xs/s) = \text{ann}_T(y/t)$ and hence $rys' = 0$ for some $s' \in S$. Thus,

$r = \frac{rs'}{s'} \in \text{ann}_T(y/t) \cap R = \text{ann}_R(y)$. Hence, we have shown that $y \in [x]_R$, and therefore, $([x]_R)_S = [xs/s]_T$.

6. Clear from part (2).

□

The next three theorems will aide in the classification of some of the diameters and girths of zero-divisor graphs in Section 3.4. The proof and ideas of Theorem 3.18 are very similar to that of Theorem 2.2 in [8] and will use the results from Lemma 3.17.

Theorem 3.18. *Let R be a commutative ring not necessarily with an identity and with total quotient ring $T(R)$. Then $\Gamma(T(R))$ and $\Gamma(R)$ are isomorphic.*

Proof. Let $S = R \setminus Z(R)$ with $T = T(R)$ the total quotient ring. Let $\sim_R$ and $\sim_T$ be the equivalence relations described above with $[x]_R$ and $[x]_T$ the equivalence classes. To show the graphs are isomorphic, we need to show that there is a bijection, ϕ, from the vertices in $\Gamma(R)$ to the vertices in $\Gamma(T)$ and that two vertices a and b in $\Gamma(R)$ are connected if and only if $\phi(a)$ and $\phi(b)$ are connected in $\Gamma(T)$.

From Lemma 3.17, we have that $Z(R) = \cup_{\alpha\in I}[x_\alpha]_R$ and $Z(T) = \cup_{\alpha\in I}[x_\alpha s/s]_T$ for some $s \in S$. Clearly, these are disjoint unions and note that $[xs/s]_T = [xt/t]_T$ for $s, t \in S$ by Lemma 3.17(3). First, we show that $|[x]_R| = |[xs/s]_T|$ for $x \in Z(R)^*$. Suppose $[x]_R$ is finite. Then $[x]_R \subseteq [xs/s]_T$ is clear. Now, for the reverse inclusion let $r \in [xs/s]_T$, where $r = b/t$ with $b \in [xs]_R = [x]_R$ and $t \in S$. Notice that since $\{t^n b \mid n \geq 1\} \subseteq [x]_R$ is finite, we get that $b = t^i b$ for some $i \geq 1$, and therefore, $r = b/t = t^i b/t = t^{i-1}b \in [x]_R$. Now suppose that $[x]_R$ is infinite. Similarly, we clearly have that $|[x]_R| \leq |[xs/s]_T|$. So we need to show the reverse inequality. Let $\approx$ be an equivalence relation on S by $t \approx u$ if and only if $tx = ux$ for $x \in Z(R)^*$. Then we clearly have that $ty = uy$ for all $y \in [x]_R$. Define a map $\alpha : [x]_R \times {}^S/_\approx \to [xs/s]_T$ given by $(b, [t]_\approx) \mapsto b/t$. To see that α is well-defined, let $b = c$ and $[t]_\approx = [u]_\approx$. Then $(b, [t]_\approx) = (c, [u]_\approx)$. Now since $[t]_\approx = [u]_\approx$, $ub = tb = tc \Rightarrow s'(bu - tc) = 0$ for some $s' \in S$. Therefore, $b/t = c/u$ and α is well-defined. It is also easy to see that α is surjective, so we get $|[xs/s]_T| \leq |[x]_R||{}^S/_\approx|$. Now the map ${}^S/_\approx \to [x]_R$ given by $s \mapsto sx$ is clearly well-defined and injective. So $|{}^S/_\approx| \leq |[x]_R|$, and therefore, $|[xs/s]_T| \leq |[x]_R|^2 = |[x]_R|$

since $|[x]_R|$ is infinite. Thus, $|[x]_R| = |[xs/s]_T|$. This gives us that there is a bijection $\phi_\alpha : [x_\alpha]_R \to [x_\alpha s/s]_T$ for each $\alpha \in I$.

Let $\phi : Z(R)^* \to Z(T)^*$ be a map defined by $\phi(a) = \phi_\alpha(a)$ if $a \in [x_\alpha]_R$. Clearly, ϕ is injective since ϕ_α is injective. To show ϕ is surjective, let $t \in Z(T)^*$. Then $t \in [x_\alpha s/s]_T$ for some $\alpha \in I$. Thus, there exists an $a \in [x_\alpha]_R$ such that $\phi_\alpha(a) = t$. Hence, $\phi(a) = t$ for some $a \in Z(R)^*$ and ϕ is a bijection.

We have left to show that a and b are connected in $\Gamma(R)$ if and only if $\phi(a)$ and $\phi(b)$ are connected in $\Gamma(T)$. If we let $a \in [x]_R$, $b \in [y]_R$, $w \in [xs/s]_T$, and $z \in [ys/s]_T$, then it suffices to show that $ab = 0$ if and only if $wz = 0$. By Lemma 3.17, we have $\mathrm{ann}_T(a) = \mathrm{ann}_T(x) = \mathrm{ann}_T(w)$ and $\mathrm{ann}_T(b) = \mathrm{ann}_T(y) = \mathrm{ann}_T(z)$. Thus, $ab = 0 \Leftrightarrow b \in \mathrm{ann}_T(a) = \mathrm{ann}_T(w) \Leftrightarrow bw = 0 \Leftrightarrow w \in \mathrm{ann}_T(b) = \mathrm{ann}_T(z) \Leftrightarrow wz = 0$. Therefore, $\Gamma(R)$ and $\Gamma(T(R))$ are isomorphic as graphs. □

Theorem 3.19. *Let R be a commutative ring without identity that has a non-unit regular element. Then $T(R)$ has an identity and $\Gamma(R) \cong \Gamma(T(R))$.*

Proof. Let R be a commutative ring without identity with a non-unit regular element. Let $s \in S = R \setminus Z(R)$. Then $T(R)$ is the localization of R at S, and has identity s/s. To see that s/s is the identity, notice that for some $t \in S$, $t(xs - xs) = 0$, which implies that $x(s/s) = xs/s = x$. Also, for $t \in S$ not equal to s, we have $t'(st - st) = 0$ for some $t' \in S$, which implies that $s/s = t/t$. The fact of $\Gamma(R) \cong \Gamma(T(R))$ was shown in Theorem 3.18. □

Theorem 3.20. *Let R be a commutative ring with identity that has a non-unit regular element x. Then xR is a commutative ring without identity and $\Gamma(R) \cong \Gamma(xR)$.*

Proof. Let R be a commutative ring with identity, and let $x \in R$ be a non-unit regular element. If xR had an identity, then there exists some element $r \in R$ such that $xr = 1$. This means that x is a unit, which is a contradiction. Thus, xR does not have an identity. Furthermore, $\Gamma(R) \cong \Gamma(xR)$ by the map $\varphi : R \to xR$, where $\varphi(r) = xr$. Clearly, φ maps $Z(R)$ to $Z(xR)$ since x is not a zero-divisor, and φ preserves the edge set since $\mathrm{ann}(xr) = \mathrm{ann}(r)$ for all $r \in Z(R)$. □

Another technique to obtain commutative rings without identity is by looking at a local ring $(R, \mathfrak{M})$ with maximal ideal $\mathfrak{M} = Z(R)$. Since $\mathfrak{M}$ is an ideal, it is a commutative ring, and it does not have an identity. Also, it is easy to see that $\Gamma(R) = \Gamma(Z(R)) = \Gamma(\mathfrak{M})$.

What these results show is that the zero-divisor graph of a commutative ring with a non-unit regular element can be realized by both a ring that has an identity and by a ring that does not have an identity. Note that since these rings have a non-unit regular element, they are necessarily infinite.

3.3 Classification of Small Finite Commutative Rings Without Identity

In this section, we classify all nonisomorphic rings without identity that have a zero-divisor graph on 14 or fewer vertices. Three papers, [19], [20], and [29], were helpful in determining the nonisomorphic rings of order 8. In [29], they also reference results for the nonisomorphic rings of other orders.

For rings whose underlying Abelian group structure is a cyclic group of order k, the commutative rings are $x\mathbb{Z}[x]/(kx, x^2 - k_i x)$, where k_i is a positive divisor of k. All of these rings are commutative, and the only ring of this type that has an identity is $x\mathbb{Z}[x]/(kx, x^2 - x) \cong \mathbb{Z}_k$. Thus, we are interested in the other types of these rings in this paper. The ring $x\mathbb{Z}[x]/(kx, x^2 - kx)$ is the commutative ring of order k with trivial multiplication. We introduce a new notation for this ring: $\mathbb{Z}_k^0$.

For rings that have order p^2 for some prime p, we have eleven nonisomorphic rings: $\mathbb{Z}_{p^2}$, $x\mathbb{Z}[x]/(p^2x, x^2 - px)$, $\mathbb{Z}_{p^2}^0$, $\mathbb{Z}_p \times \mathbb{Z}_p$, $\mathbb{Z}_p \times \mathbb{Z}_p^0$, $\mathbb{Z}_p^0 \times \mathbb{Z}_p^0$, $\mathbb{F}_{p^2}$, $\mathbb{Z}_p[x]/(x^2)$, $x\mathbb{Z}_p[x]/x^3\mathbb{Z}_p[x]$, $A = \{\left[\begin{smallmatrix} a & b \\ 0 & 0 \end{smallmatrix}\right] \mid a, b \in \mathbb{Z}_p\}$, and $B = \{\left[\begin{smallmatrix} 0 & a \\ 0 & b \end{smallmatrix}\right] \mid a, b \in \mathbb{Z}_p\}$. We will be interested in the rings $x\mathbb{Z}[x]/(p^2x, x^2 - px)$, $\mathbb{Z}_{p^2}^0$, $\mathbb{Z}_p \times \mathbb{Z}_p^0$, $\mathbb{Z}_p^0 \times \mathbb{Z}_p^0$, and $x\mathbb{Z}_p[x]/x^3\mathbb{Z}_p[x]$, as these are the rings that are commutative and do not have an identity.

The number of associative rings of order p^3 is given in [29]. We need only worry about the case when $p = 2$ in this paper. Recall Lemma 3.4 as it will aid in finding the nonisomorphic classes of rings.

Tables 3.1, 3.2, 3.3, 3.4, 3.5, and 3.6 give the list of nonisomorphic commutative rings without identity, their Abelian group structure type, and their zero-divisor graph.

Table 3.1: Rings with graphs on 1, 2, and 3 vertices

Vertices	Group Type	R	Graph
1	$\mathbb{Z}_2$	$\mathbb{Z}_2^0$	K^1
2	$\mathbb{Z}_3$	$\mathbb{Z}_3^0$	K^2
3	$\mathbb{Z}_4$	$\mathbb{Z}_4^0$	K^3
	$\mathbb{Z}_2 \times \mathbb{Z}_2$	$\mathbb{Z}_2^0 \times \mathbb{Z}_2$	Figure 3.1
	$\mathbb{Z}_2 \times \mathbb{Z}_2$	$\mathbb{Z}_2^0 \times \mathbb{Z}_2^0$	K^3
	$\mathbb{Z}_4$	$x\mathbb{Z}[x]/(4x, x^2 - 2x)$	Figure 3.1
	$\mathbb{Z}_2 \times \mathbb{Z}_2$	$x\mathbb{Z}_2[x]/x^3\mathbb{Z}_2[x]$	Figure 3.1

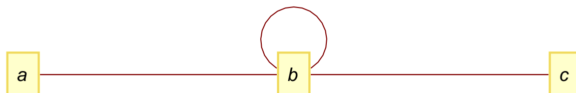


Figure 3.1: 3-vertex graph

Table 3.2: Rings with graphs on 4, 5, and 6 vertices

Vertices	Group Type	R	Graph
4	$\mathbb{Z}_5$	$\mathbb{Z}_5^0$	K^4
5	$\mathbb{Z}_6$	$\mathbb{Z}_6^0$	K^5
	$\mathbb{Z}_6$	$\mathbb{Z}_2^0 \times \mathbb{Z}_3$	Figure 3.2a
	$\mathbb{Z}_6$	$\mathbb{Z}_2 \times \mathbb{Z}_3^0$	Figure 3.2b
6	$\mathbb{Z}_7$	$\mathbb{Z}_7^0$	K^6

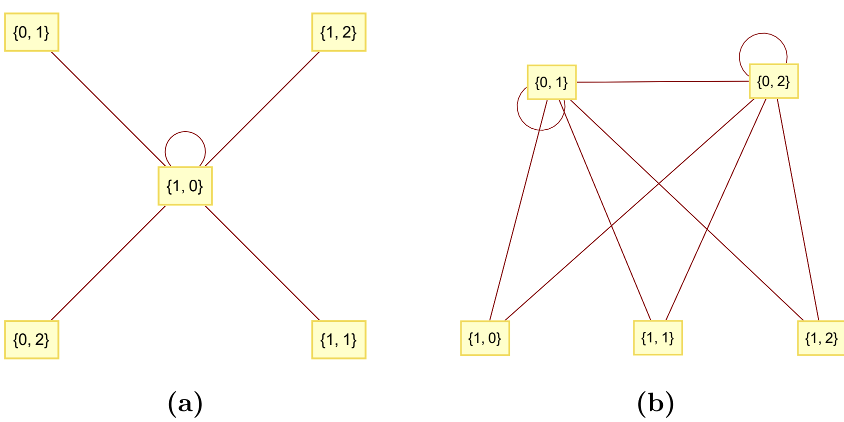


Figure 3.2: 5-vertex graphs

Table 3.3: Rings with graphs on 7 vertices

Vertices	Group Type	R	Graph
7	$\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$	$\mathbb{Z}_2^0 \times \mathbb{Z}_2 \times \mathbb{Z}_2$	Figure 3.3a
	$\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$	$\mathbb{Z}_2^0 \times \mathbb{Z}_2^0 \times \mathbb{Z}_2$	Figure 3.3b
	$\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$	$\mathbb{Z}_2^0 \times \mathbb{Z}_2^0 \times \mathbb{Z}_2^0$	K^7
	$\mathbb{Z}_2 \times \mathbb{Z}_4$	$\mathbb{Z}_2^0 \times \mathbb{Z}_4$	Figure 3.3c
	$\mathbb{Z}_2 \times \mathbb{Z}_4$	$\mathbb{Z}_2 \times \mathbb{Z}_4^0$	Figure 3.3b
	$\mathbb{Z}_2 \times \mathbb{Z}_4$	$\mathbb{Z}_2^0 \times \mathbb{Z}_4^0$	K^7
	$\mathbb{Z}_2 \times \mathbb{Z}_4$	$\mathbb{Z}_2 \times x\mathbb{Z}[x]/(4x, x^2 - 2x)$	Figure 3.3a
	$\mathbb{Z}_2 \times \mathbb{Z}_4$	$\mathbb{Z}_2^0 \times x\mathbb{Z}[x]/(4x, x^2 - 2x)$	Figure 3.3b
	$\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$	$\mathbb{Z}_2^0 \times \mathbb{F}_4$	Figure 3.3d
	$\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$	$\mathbb{Z}_2^0 \times \mathbb{Z}_2[x]/(x^2)$	Figure 3.3c
	$\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$	$\mathbb{Z}_2 \times x\mathbb{Z}_2[x]/x^3\mathbb{Z}_2[x]$	Figure 3.3a
	$\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$	$\mathbb{Z}_2^0 \times x\mathbb{Z}_2[x]/x^3\mathbb{Z}_2[x]$	Figure 3.3b
	$\mathbb{Z}_8$	$\mathbb{Z}_8^0$	K^7
	$\mathbb{Z}_8$	$x\mathbb{Z}[x]/(8x, x^2 - 2x)$	Figure 3.3c
	$\mathbb{Z}_8$	$x\mathbb{Z}[x]/(8x, x^2 - 4x)$	Figure 3.3b
	$\mathbb{Z}_2 \times \mathbb{Z}_4$	Max Ideal of $\mathbb{Z}_4[x]/(x^2)$	Figure 3.3e
	$\mathbb{Z}_2 \times \mathbb{Z}_4$	Max Ideal of $\mathbb{Z}_4[x]/(x^2 - 2x)$	Figure 3.3f
	$\mathbb{Z}_2 \times \mathbb{Z}_4$	Max Ideal of $\mathbb{Z}_4[x]/(x^2 - 2)$	Figure 3.3c
	$\mathbb{Z}_2 \times \mathbb{Z}_4$	Ideal $\langle x, y\rangle$ of $\mathbb{Z}_4[x,y]/(2x, x^2, xy, y^2 - x)$	Figure 3.3b
	$\mathbb{Z}_2 \times \mathbb{Z}_4$	Ideal $\langle x, y\rangle$ of $\mathbb{Z}_4[x,y]/(2x, x^2 - 2y, xy, y^2)$	Figure 3.3b
	$\mathbb{Z}_2 \times \mathbb{Z}_4$	Ideal $\langle x, y\rangle$ of $\mathbb{Z}_4[x,y]/(2x, x^2 - 2y, xy, y^2 - 2y)$	Figure 3.3f
	$\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$	Max Ideal of $\mathbb{Z}_2[x,y]/(x^3, xy, y^2)$	Figure 3.3b
	$\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$	Max Ideal of $\mathbb{Z}_2[x,y]/(x^3, xy, y^2 - x^2)$	Figure 3.3f
	$\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$	Max Ideal of $\mathbb{Z}_2[x,y]/(x^2, y^2)$	Figure 3.3e
	$\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$	$x\mathbb{Z}_2[x]/x^4\mathbb{Z}_2[x]$	Figure 3.3c

Table 3.4: Rings with graphs on 8, 9, and 10 vertices

Vertices	Group Type	R	Graph
8	$\mathbb{Z}_9$	$\mathbb{Z}_9^0$	K^8
	$\mathbb{Z}_3 \times \mathbb{Z}_3$	$\mathbb{Z}_3^0 \times \mathbb{Z}_3$	Figure 3.4
	$\mathbb{Z}_3 \times \mathbb{Z}_3$	$\mathbb{Z}_3^0 \times \mathbb{Z}_3^0$	K^8
	$\mathbb{Z}_9$	$x\mathbb{Z}[x]/(9x, x^2 - 3x)$	Figure 3.4
	$\mathbb{Z}_3 \times \mathbb{Z}_3$	$x\mathbb{Z}_3[x]/x^3\mathbb{Z}_3[x]$	Figure 3.4
9	$\mathbb{Z}_{10}$	$\mathbb{Z}_{10}^0$	K^9
	$\mathbb{Z}_{10}$	$\mathbb{Z}_2^0 \times \mathbb{Z}_5$	Figure 3.5a
	$\mathbb{Z}_{10}$	$\mathbb{Z}_2 \times \mathbb{Z}_5^0$	Figure 3.5b
10	$\mathbb{Z}_{11}$	$\mathbb{Z}_{11}^0$	K^{10}

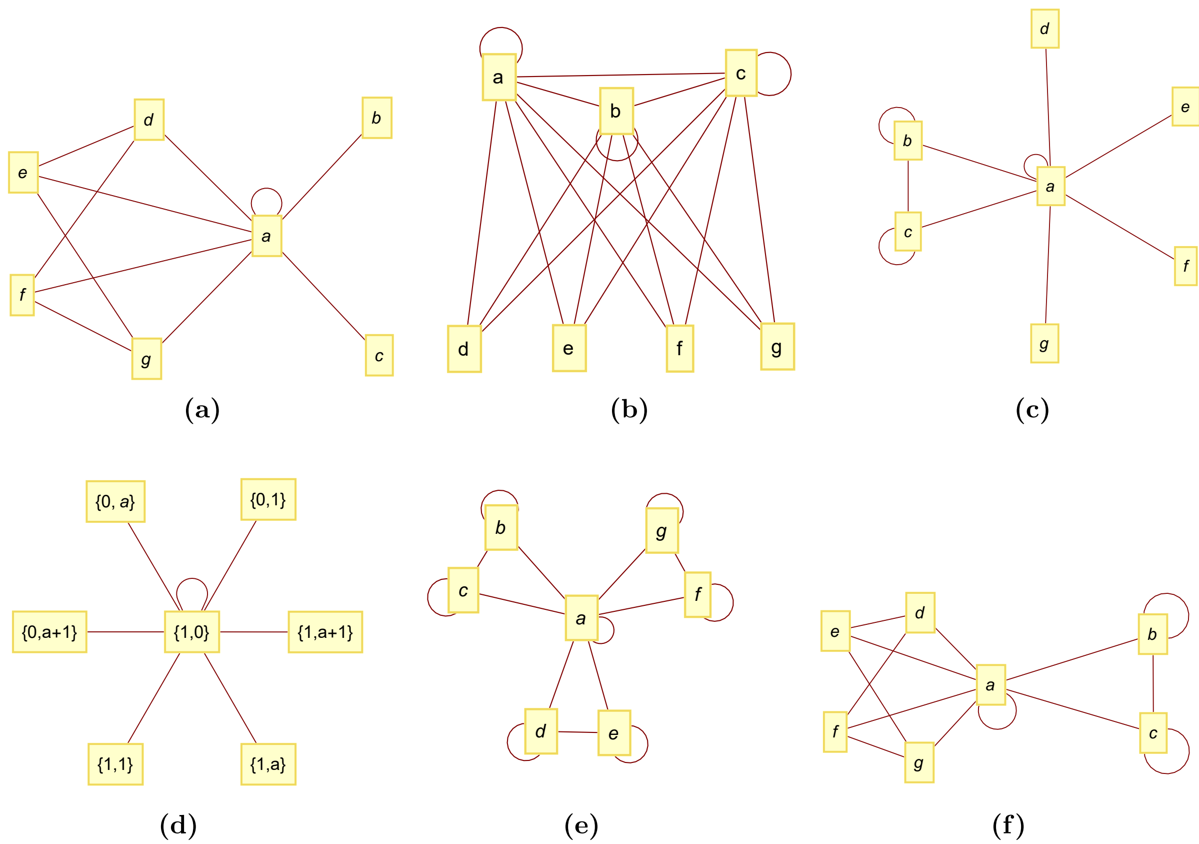


Figure 3.3: 7-vertex graphs

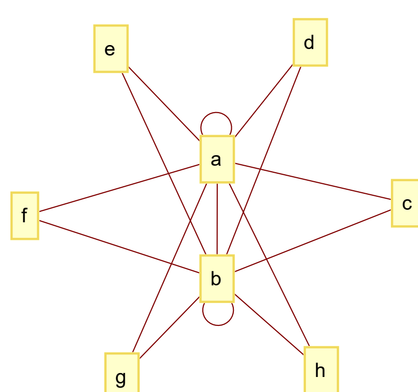
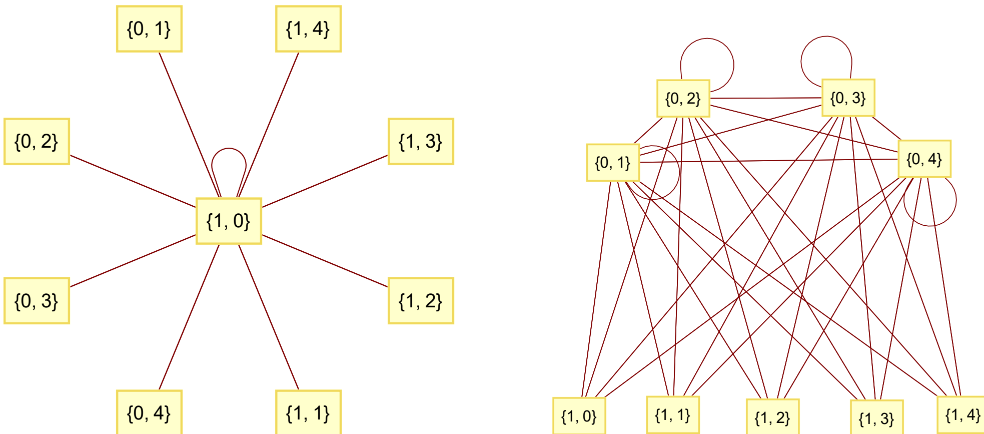


Figure 3.4: 8-vertex graph



(a) (b)

Figure 3.5: 9-vertex graphs

Table 3.5: Rings with graphs on 11 vertices

Vertices	Group Type	R	Graph
11	$\mathbb{Z}_{12}$	$\mathbb{Z}_{12}^0$	K^{11}
	$\mathbb{Z}_2 \times \mathbb{Z}_6$	$\mathbb{Z}_2^0 \times \mathbb{Z}_6$	Figure 3.6a
	$\mathbb{Z}_2 \times \mathbb{Z}_6$	$\mathbb{Z}_2 \times \mathbb{Z}_6^0$	Figure 3.6b
	$\mathbb{Z}_2 \times \mathbb{Z}_6$	$\mathbb{Z}_2^0 \times \mathbb{Z}_6^0$	K^{11}
	$\mathbb{Z}_{12}$	$\mathbb{Z}_3^0 \times \mathbb{Z}_4$	Figure 3.6c
	$\mathbb{Z}_{12}$	$\mathbb{Z}_3 \times \mathbb{Z}_4^0$	Figure 3.6d
	$\mathbb{Z}_2 \times \mathbb{Z}_6$	$\mathbb{Z}_2^0 \times \mathbb{Z}_2^0 \times \mathbb{Z}_3$	Figure 3.6d
	$\mathbb{Z}_2 \times \mathbb{Z}_6$	$\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_3^0$	Figure 3.6e
	$\mathbb{Z}_{12}$	$\mathbb{Z}_3 \times {x\mathbb{Z}[x]}/{(4x, x^2-2x)}$	Figure 3.6a
	$\mathbb{Z}_{12}$	$\mathbb{Z}_3^0 \times {x\mathbb{Z}[x]}/{(4x, x^2-2x)}$	Figure 3.6b
	$\mathbb{Z}_2 \times \mathbb{Z}_6$	$\mathbb{Z}_3 \times {x\mathbb{Z}_2[x]}/{x^3\mathbb{Z}_2[x]}$	Figure 3.6a
	$\mathbb{Z}_2 \times \mathbb{Z}_6$	$\mathbb{Z}_3^0 \times {x\mathbb{Z}_2[x]}/{x^3\mathbb{Z}_2[x]}$	Figure 3.6b

Table 3.6: Rings with graphs on 12, 13, and 14 vertices

Vertices	Group Type	R	Graph
12	$\mathbb{Z}_{13}$	$\mathbb{Z}_{13}^0$	K^{12}
13	$\mathbb{Z}_{14}$	$\mathbb{Z}_{14}^0$	K^{13}
	$\mathbb{Z}_{14}$	$\mathbb{Z}_2^0 \times \mathbb{Z}_7$	Figure 3.7a
	$\mathbb{Z}_{14}$	$\mathbb{Z}_2 \times \mathbb{Z}_7^0$	Figure 3.7b
14	$\mathbb{Z}_{15}$	$\mathbb{Z}_{15}^0$	K^{14}
	$\mathbb{Z}_{15}$	$\mathbb{Z}_3^0 \times \mathbb{Z}_5$	Figure 3.8a
	$\mathbb{Z}_{15}$	$\mathbb{Z}_3 \times \mathbb{Z}_5^0$	Figure 3.8b

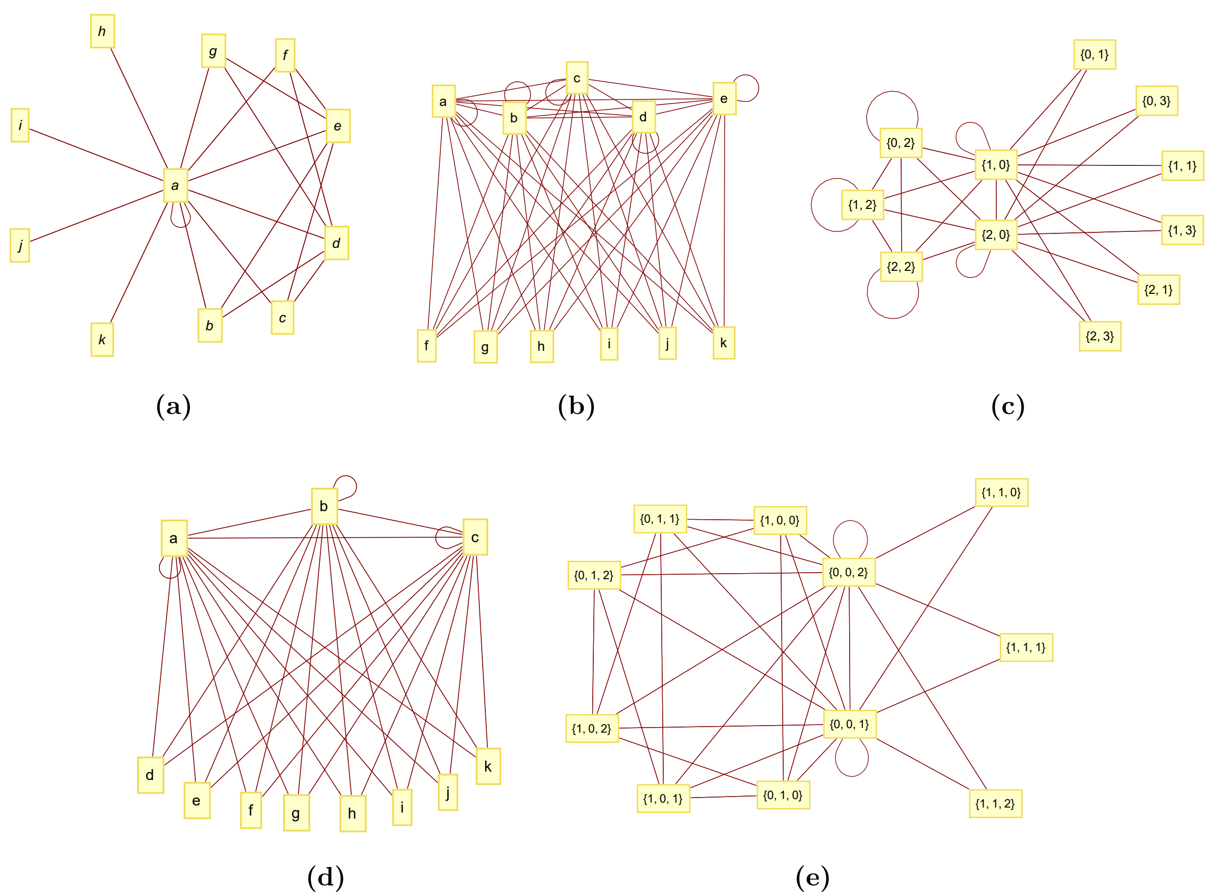


Figure 3.6: 11-vertex graphs

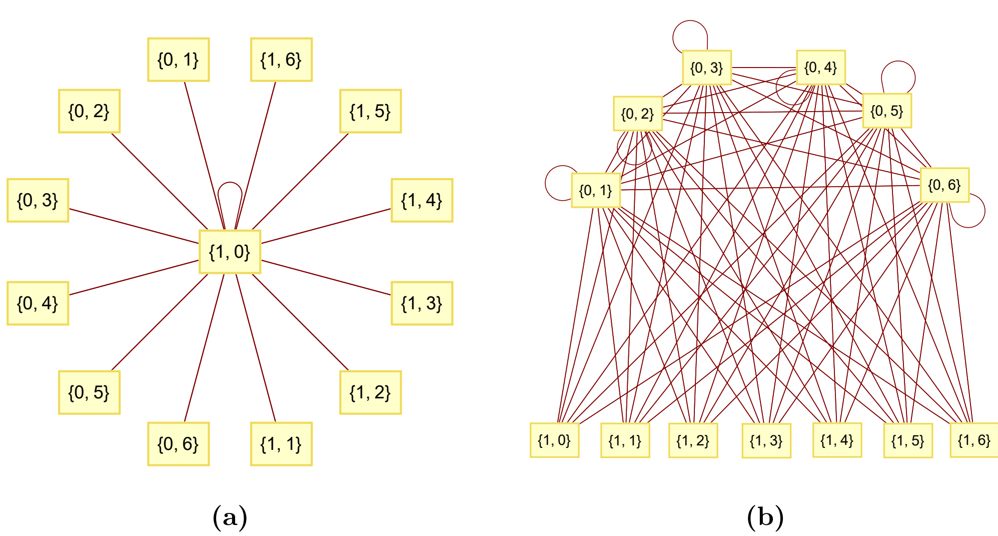
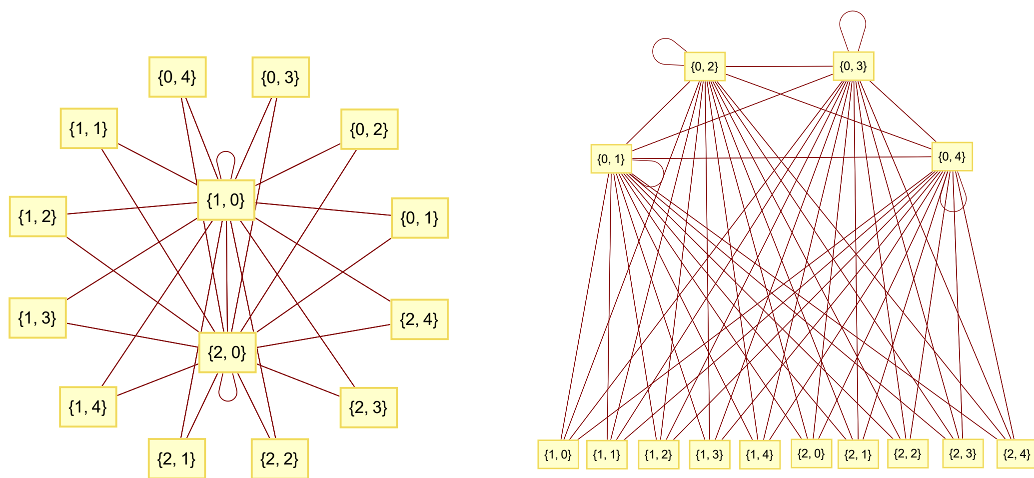


Figure 3.7: 13-vertex graphs



(a) **(b)**

Figure 3.8: 14-vertex graphs

3.3.1 Comparison With Finite Commutative Rings With Identity

The work in Section 3.3 is analogous to the work Redmond did for commutative rings with identity in [38]. As such, it is useful to see a comparison between finite commutative rings with identity and finite commutative rings without identity and the graphs they produce. We look at that comparison here.

Table 3.7: Realizable zero-divisor graphs on 1 vertex

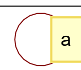
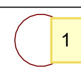
With Identity	
Without Identity	

Table 3.8: Realizable zero-divisor graphs on 2 vertices

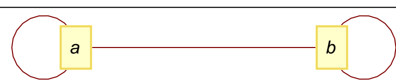
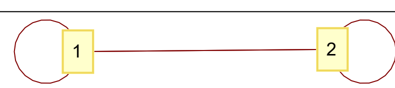
With Identity	 
Without Identity	

Table 3.9: Realizable zero-divisor graphs on 3 vertices

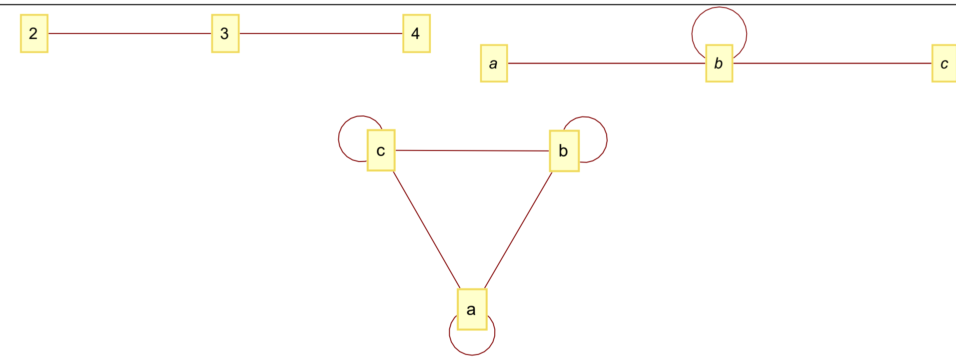
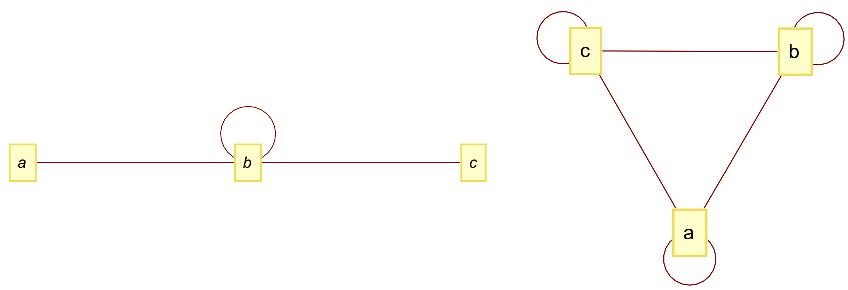
With Identity	
Without Identity	

Table 3.10: Realizable zero-divisor graphs on 4 vertices

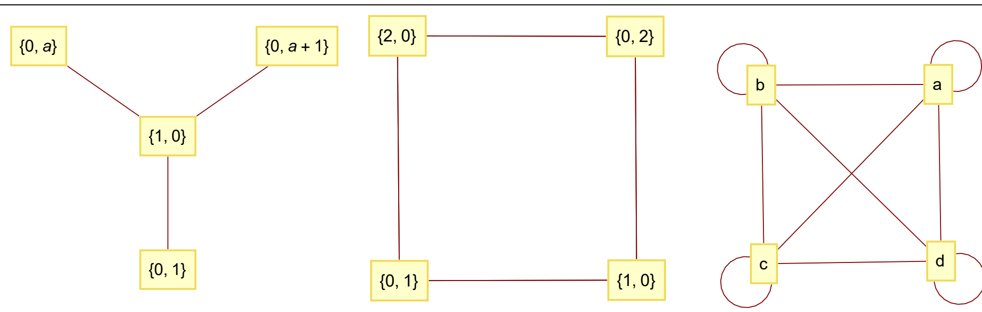
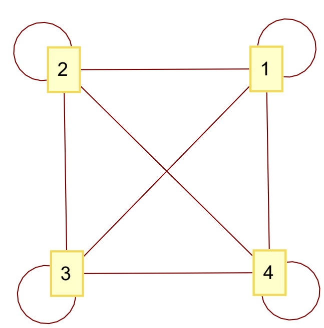
With Identity	
Without Identity	

Table 3.11: Realizable zero-divisor graphs on 5 vertices

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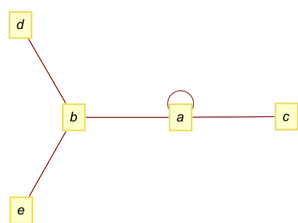

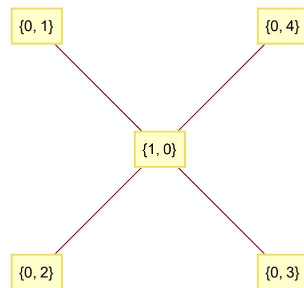

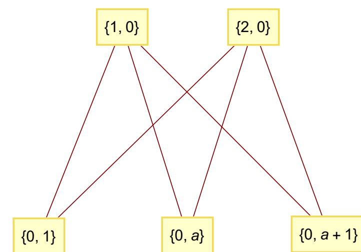

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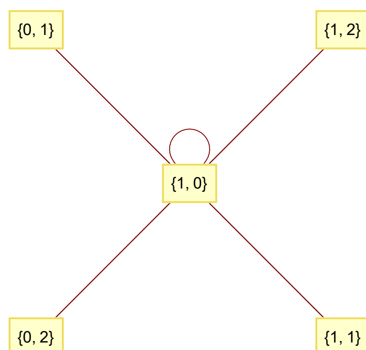

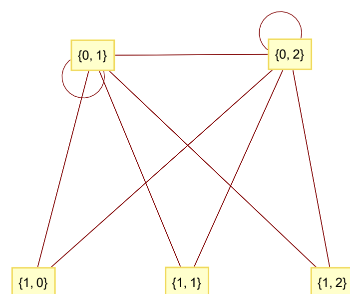

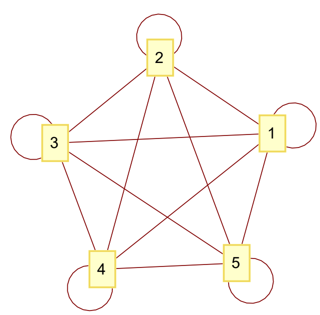

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Table 3.12: Realizable zero-divisor graphs on 6 vertices

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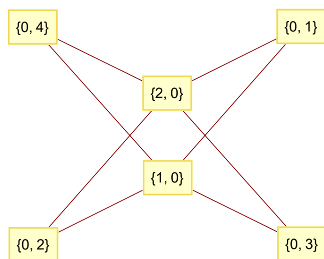

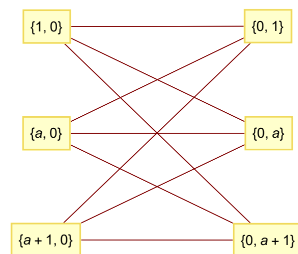

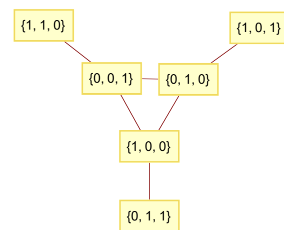

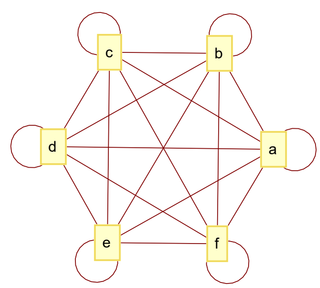

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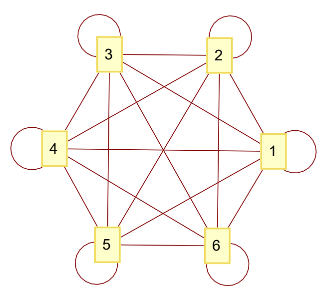

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Table 3.13: Realizable zero-divisor graphs on 7 vertices

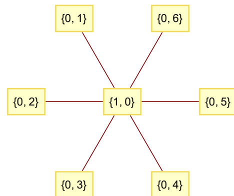
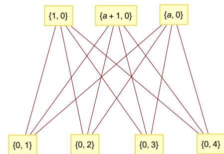
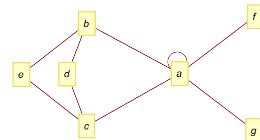
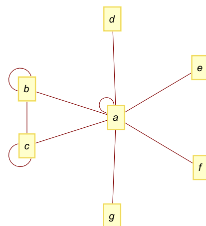
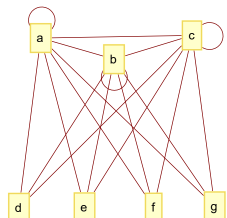
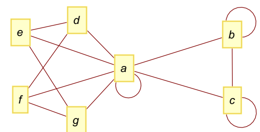
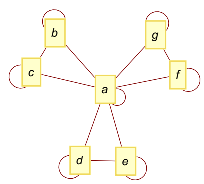
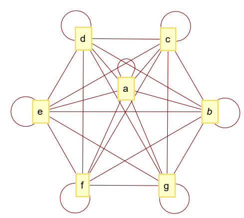
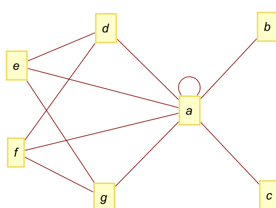
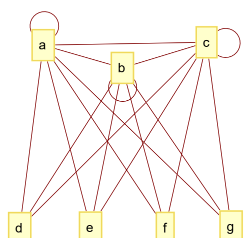
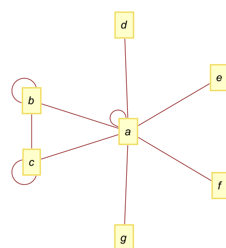
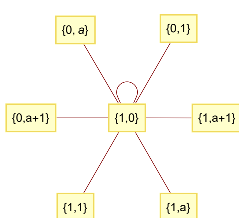
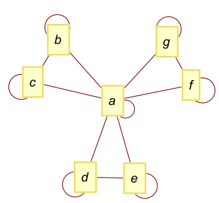
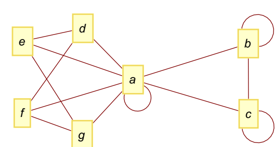
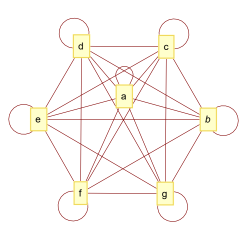
With Identity	       
Without Identity	      

Table 3.14: Realizable zero-divisor graphs on 8 vertices

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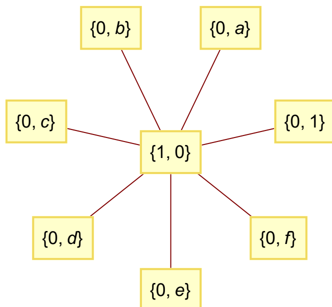

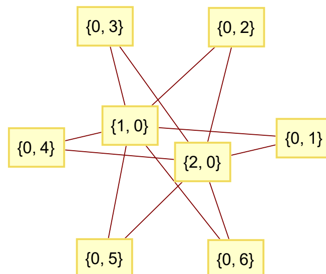

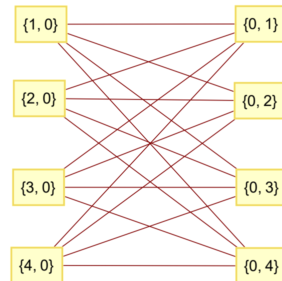

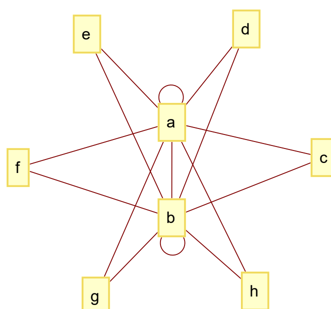

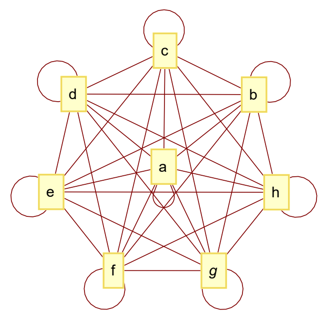

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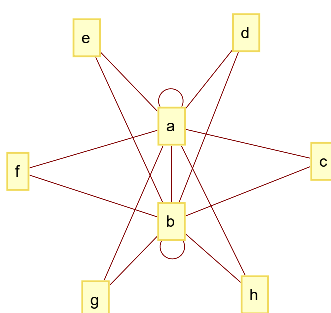

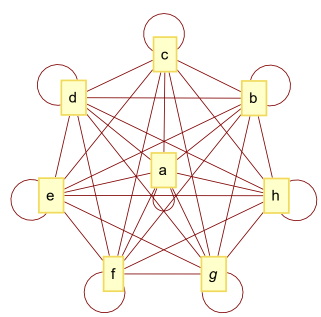

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Table 3.15: Realizable zero-divisor graphs on 9 vertices

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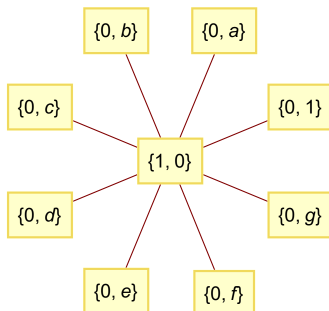

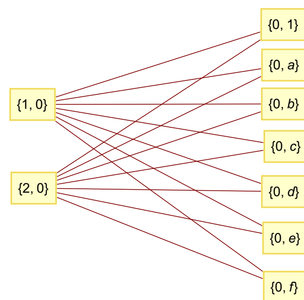

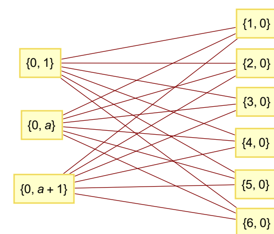

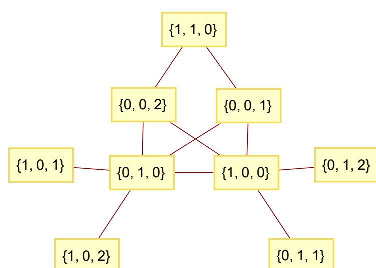

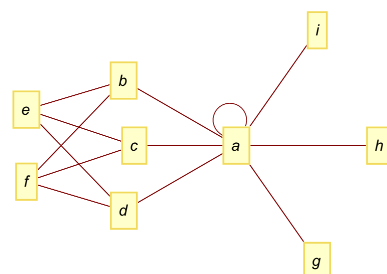

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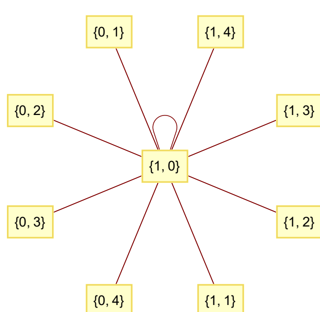

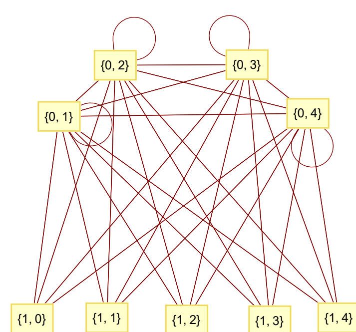

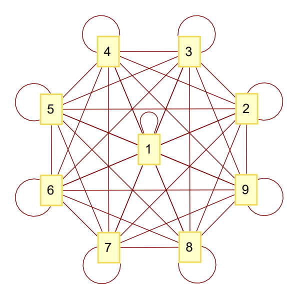

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Table 3.16: Realizable zero-divisor graphs on 10 vertices

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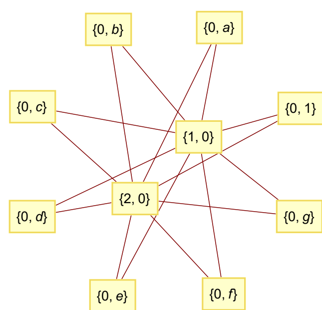

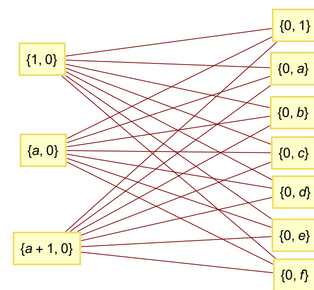

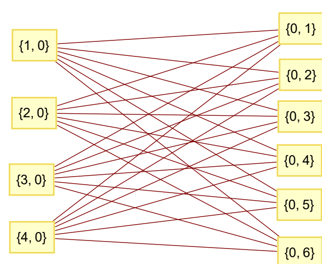

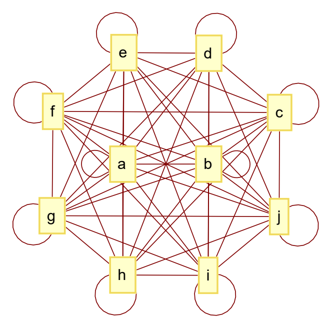

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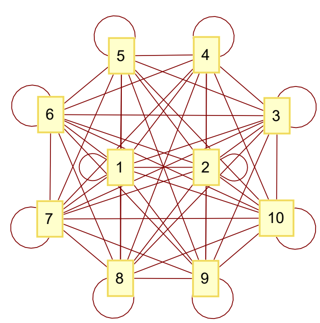

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Table 3.17: Realizable zero-divisor graphs on 11 vertices

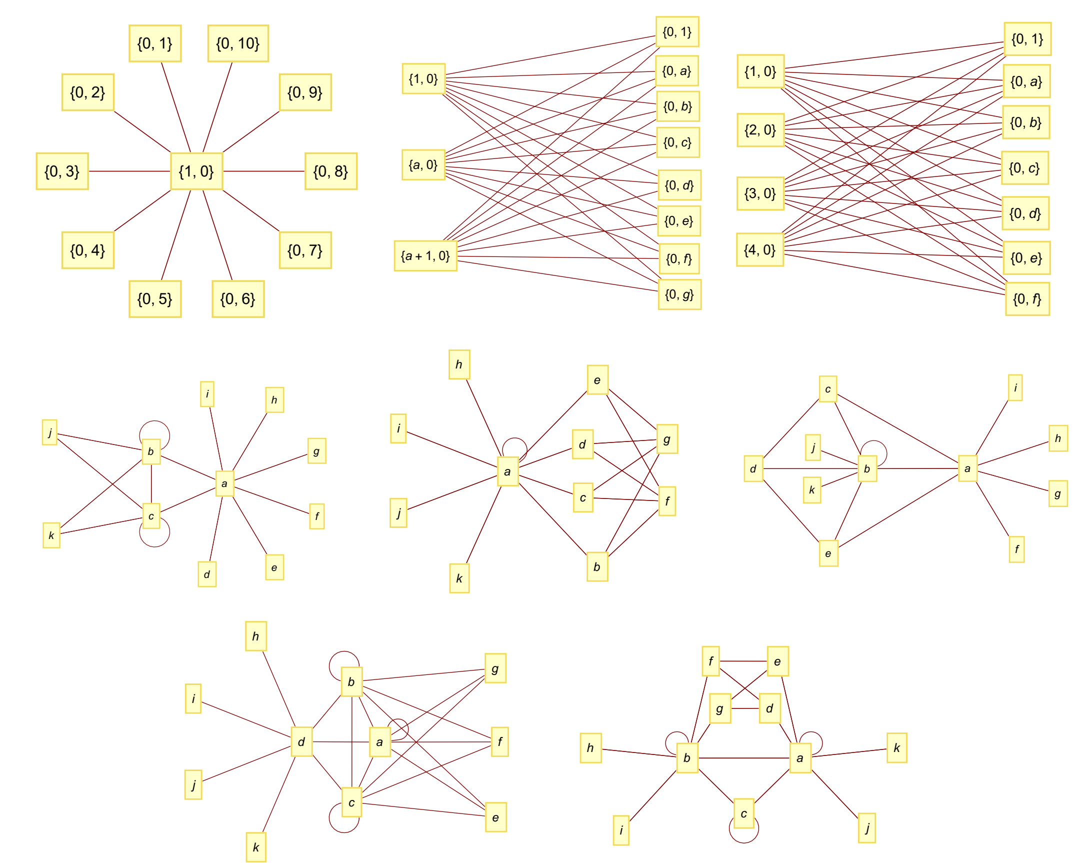
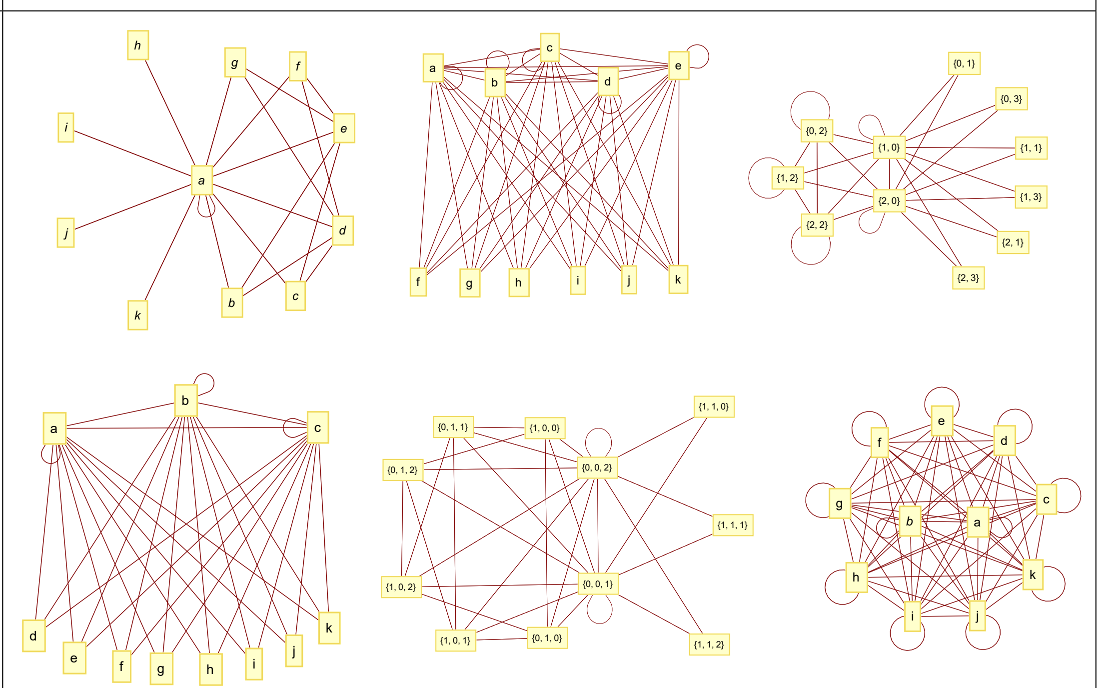
With Identity	
Without Identity	

Table 3.18: Realizable zero-divisor graphs on 12 vertices

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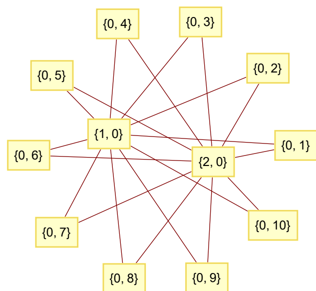

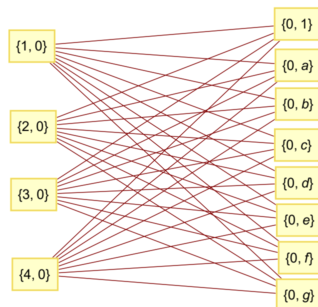

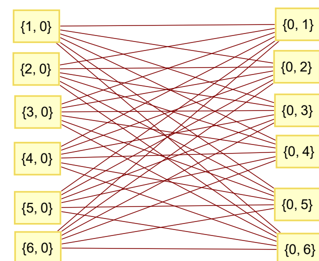

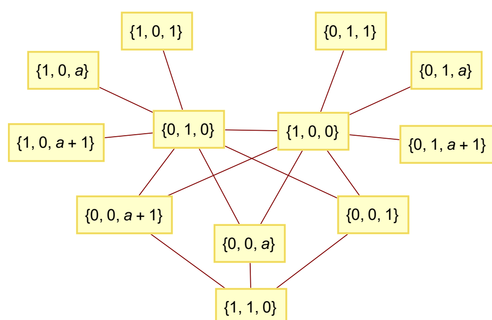

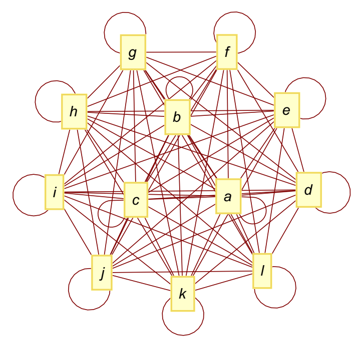

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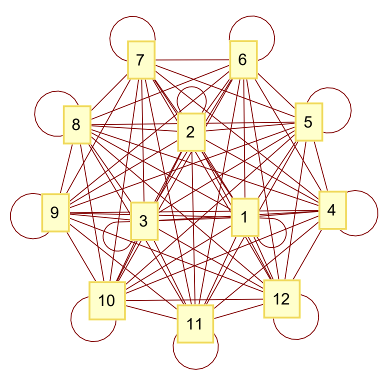

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Table 3.19: Realizable zero-divisor graphs on 13 vertices

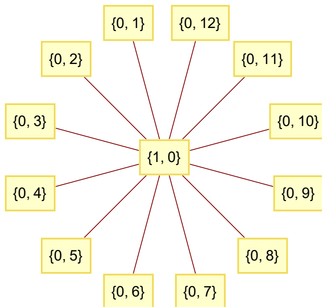
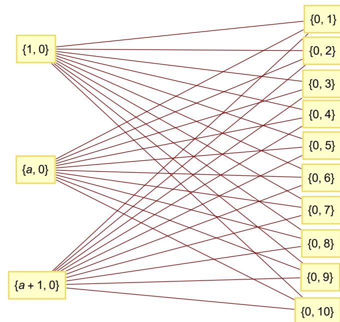
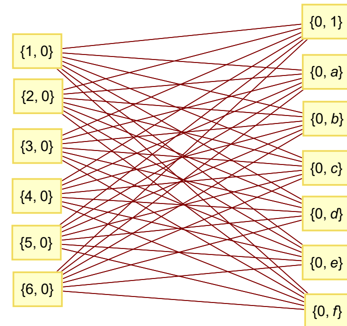
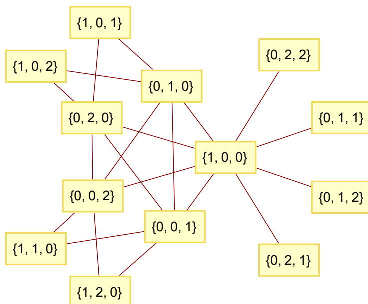
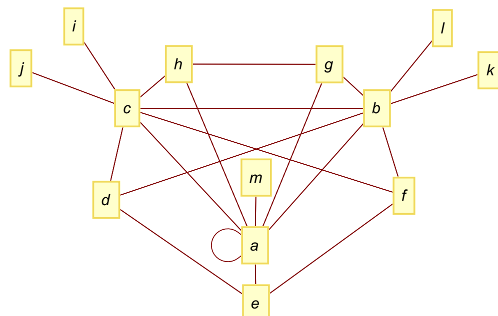
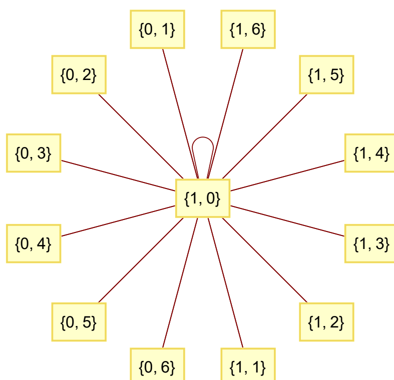
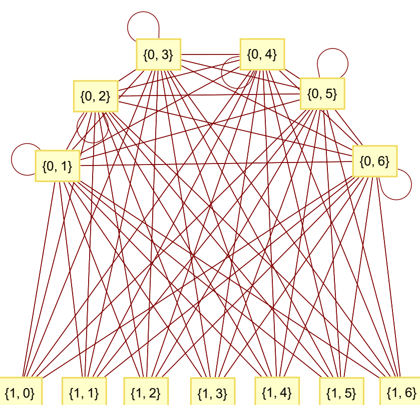
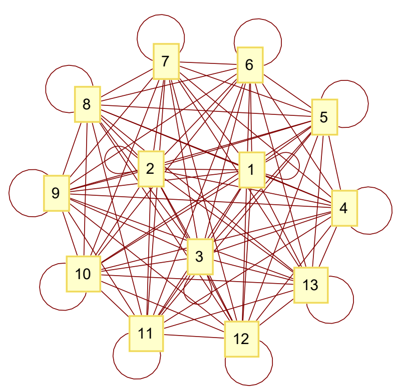
With Identity	    
Without Identity	  

Table 3.20: Realizable zero-divisor graphs on 14 vertices

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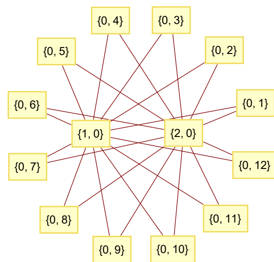

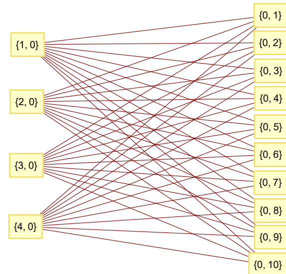

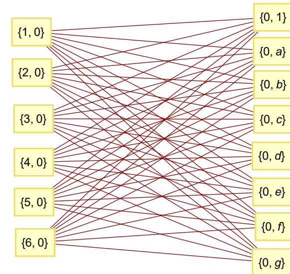

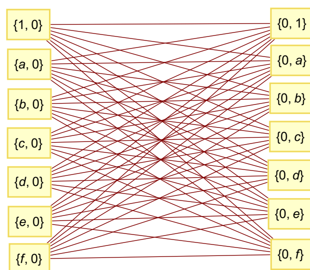

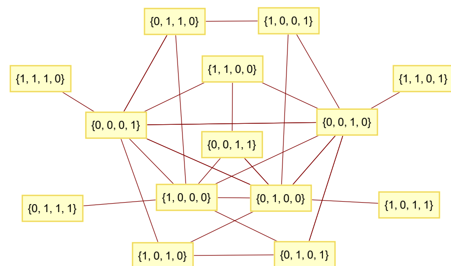

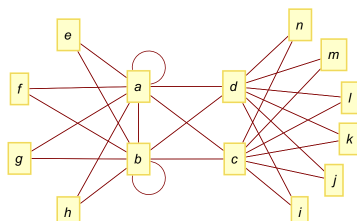

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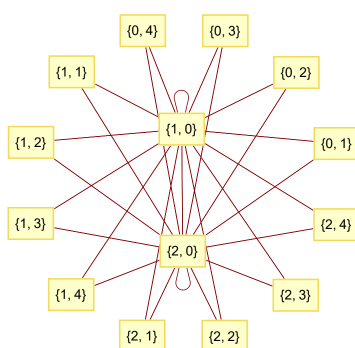

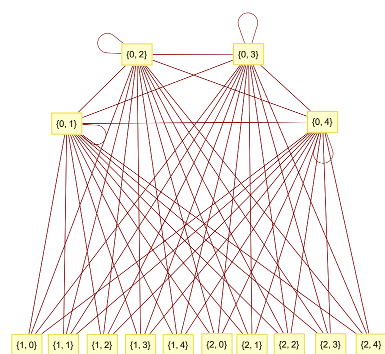

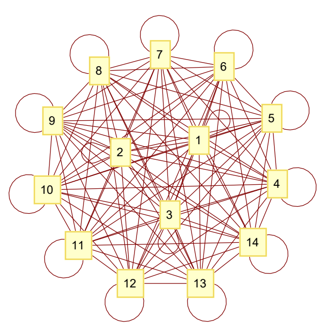

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3.4 Realizable Diameters and Girths of Zero-Divisor Graphs

Recall that a zero-divisor graph has girth 3, 4, or ∞ and diameter 0, 1, 2, or 3. We give a breakdown of whether each combination of girth and diameter can be realized as a zero-divisor graph of some commutative ring. We separate this problem into six different types of rings: finite and infinite rings with identity, where every element is either a zero-divisor or a unit, rings with identity with a non-unit regular element, finite and infinite rings without identity, where every element is a zero-divisor, and rings without identity with a regular element. The following tables give the results, as well as an example of a ring that has the indicated properties.

Note that for a general graph G that contains a cycle, we have that $3 \leq \mathrm{gr}(G) \leq 2\ \mathrm{diam}(G) + 1$. Thus, zero-divisor graphs that have diameter 0 must have infinite girth. Similarly, a zero-divisor graph that has diameter 1 cannot have girth 4.

Table 3.21 gives the breakdown for finite commutative rings with identity, where each element is either a zero-divisor or a unit. Here, we use $\mathbb{F}_4$ to mean the finite field of 4 elements.

Table 3.21: Finite rings with identity (thus $R = Z(R) \cup U(R)$)

diam \ girth	3	4	∞
0	None	None	$\mathbb{Z}_4$
1	$\mathbb{F}_4[x]/(x^2)$	None	$\mathbb{Z}_2 \times \mathbb{Z}_2$
2	$\mathbb{Z}_{16}$	$\mathbb{Z}_3 \times \mathbb{Z}_3$	$\mathbb{Z}_6$
3	$\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$	$\mathbb{Z}_{12}$	$\mathbb{Z}_2 \times \mathbb{Z}_4$

Table 3.22 lists the results for infinite rings R, where each element is either a zero-divisor or a unit. There are several cases that need justification that is provided by Proposition 3.21.

Proposition 3.21. *Let R be an infinite commutative ring with identity.*

1. *There does not exist such an R such that diam$(\Gamma(R)) = 0$.*

2. *There does not exist such an R such that diam$(\Gamma(R)) = 1$ and gr$(\Gamma(R)) = \infty$.*

3. *There does not exist such an R such that diam$(\Gamma(R)) = 3$ and gr$(\Gamma(R)) = \infty$.*

Proof. Let R be a commutative ring such that R has some non-unit regular element. Then R must be infinite.

1. For a graph to have diameter 0, it must be a graph on one vertex. Theorem 2.2 from [10] gives us that it is not possible for $\Gamma(R)$ to have diameter 0 for an infinite ring R.

2. For a zero-divisor graph to have diameter 1 and infinite girth, it must be the complete graph on two vertices (if it was a complete graph on more than two vertices, then the girth would not be infinite. Also, the only way to get a graph of diameter 1 is for every vertex to be connected to every other vertex, i.e., a complete graph). Again, Theorem 2.2 from [10] gives us that this is not possible.

3. Theorem 2.4 from [11] gives us that for a reduced commutative ring R, $\text{gr}(\Gamma(R)) = \infty$ if and only if $\Gamma(R) = K^{1,n}$. Since $\text{diam}(K^{1,n}) = 2$, no such reduced ring has the desired diameter and girth.

   If the ring has a nonzero nilpotent element, then Theorem 2.5 from [11] says that $\text{gr}(\Gamma(R)) = \infty$ if and only if either $\Gamma(R)$ is a singleton, $\Gamma(R) = \overline{K}^{1,3}$, or $\Gamma(R) = K^{1,n}$ (where $\overline{K}^{m,3}$ is the graph joining $K^{m,3}$ to the graph $K^{1,m}$ by identifying the center of $K^{1,m}$ with a point in the size 3 partition of $K^{m,3}$). The first two cases are finite graphs, and thus cannot be the graphs of an infinite ring. The last case has diameter 2.

Therefore, no such infinite ring exists to produce the stated diameters and girths. □

Table 3.22: Infinite rings with identity where $R = Z(R) \cup U(R)$

diam \ girth	3	4	∞
0	None	None	None
1	$\mathbb{Q}[x]/(x^2)$	None	None
2	$\mathbb{Q}[x]/(x^3)$	$\mathbb{Q} \times \mathbb{Q}$	$\mathbb{Z}_2 \times \mathbb{Q}$
3	$\mathbb{Q} \times \mathbb{Q} \times \mathbb{Q}$	$\mathbb{Z}_4 \times \mathbb{Q}$	None

Table 3.23 breaks down the case where R is a commutative ring with identity and has some non-unit regular element. Note that R must be infinite in this case. Several cases must be justified here as well and are covered by Proposition 3.21.

Table 3.23: Rings with identity where $Z(R) \cup U(R) \subsetneq R$

diam \ girth	3	4	∞
0	None	None	None
1	$\mathbb{Z}[x]/(x^2)$	None	None
2	$\mathbb{Z}[x]/(x^3)$	$\mathbb{Z} \times \mathbb{Z}$	$\mathbb{Z}_2 \times \mathbb{Z}$
3	$\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}$	$\mathbb{Z}_4 \times \mathbb{Z}$	None

Next, we give the results for a commutative ring without identity where every element is a zero-divisor. Table 3.24 lists those results for finite rings. From Corollary 3.16, we know that $\mathrm{diam}(\Gamma(R)) \leq 2$. Corollary 3.14 gives us that the only possible girths are 3 and ∞.

Table 3.24: Finite rings without identity (thus $R = Z(R)$)

diam \ girth	3	4	∞
0	None	None	$\mathbb{Z}_2^0$
1	$\mathbb{Z}_4^0$	None	$\mathbb{Z}_3^0$
2	$\mathbb{Z}_2 \times \mathbb{Z}_3^0$	None	$\mathbb{Z}_2^0 \times \mathbb{Z}_2$
3	None	None	None

Table 3.25 gives the results for infinite rings where every element is a zero-divisor. Again, Corollary 3.14 gives us that 3 and ∞ are the only possible girths. Proposition 3.22 eliminates the infinite girth for graphs of diameter 0 and 1. Theorem 3.10 gives us that there is no such ring $R = Z(R)$ such that $\mathrm{diam}(\Gamma(R)) = 3$ and $\mathrm{gr}(\Gamma(R)) = \infty$. For the case where the diameter and girth are both 3, we use a ring that was mentioned in Example 5.1 of [34], but where the details were covered in Example 3.13 of [3]. As the ring is a bit complicated, we restate Example 3.13 from [3] in Example 3.23 to explain the ring.

Proposition 3.22. *Let R be an infinite commutative ring without identity.*

1. *There does not exist such an R such that diam$(\Gamma(R)) = 0$.*
2. *There does not exist such an R such that diam$(\Gamma(R)) = 1$ and gr$(\Gamma(R)) = \infty$.*

Proof. For both assertions, the proof of Proposition 3.21 gives us our results as the proof was not dependent on R having an identity. Note that the proof of Theorem 2.2 in [10] also does not require R to have an identity. $\square$

Table 3.25: Infinite rings without identity where $R = Z(R)$

diam \ girth	3	4	∞
0	None	None	None
1	$\mathbb{Z}^0$	None	None
2	$\mathbb{Z}_4 \times \mathbb{Z}^0$	None	$\mathbb{Z}_2^0 \times \mathbb{Z}$
3	Example 3.23	None	None

Example 3.23 ([3, Example 3.13])**.** Let $D = \mathbb{Q}[x, y]_{(x,y)}$, which is a two-dimensional local Unique Factorization Domain with maximal ideal $Q = (x, y)_{(x,y)}$. Let $\mathcal{P}$ be the set of height-one prime ideals of D, and let $I = \mathcal{P} \times \mathbb{N}$. For each $i = (P, n) \in I$, let $Q_i = Q/P$. Let $R = D \times (\oplus_{i \in I} Q_i)$ with coordinate-wise addition and multiplication defined by $(a, (a_i))(b, (b_i)) = (ab, (ab_i + ba_i + a_i b_i))$. This is a reduced commutative ring with maximal ideal $M = Q \times (\oplus_{i \in I} Q_i)$. The maximal ideal, M, is the ring of interest. It is infinite without identity and $M = Z(M)$. Finally, as shown in [3, Example 3.13], $\mathrm{diam}(\Gamma(M)) = 3$ and by Corollary 3.14, $\mathrm{gr}(\Gamma(M)) = 3$.

Lastly, we look at commutative rings without identity where there is a non-zero-divisor element, i.e., $Z(R) \subsetneq R$. Note that R must be infinite in this case. As such, similar results from Proposition 3.22 also apply. The last case that needs justification is when the diameter is 3 and girth is infinite, which is given in Proposition 3.24. Table 3.26 lists the results.

Proposition 3.24. *Let R be a commutative ring without identity such that $Z(R) \subsetneq R$. If $\mathrm{diam}(\Gamma(R)) = 3$, then $\mathrm{gr}(\Gamma(R)) \in \{3, 4\}$.*

Proof. Let R be a commutative ring without identity such that $Z(R) \subsetneq R$ and $\mathrm{diam}(\Gamma(R)) = 3$. Let $S = R \setminus Z(R)$ (i.e., S is the set of regular elements). Localizing R at S gives us the total quotient ring $T(R)$, which has an identity. By Theorem 3.18, we need only look at $\Gamma(T(R))$. Since $T(R)$ is an infinite ring with identity, Proposition 3.21 gives us that $\mathrm{gr}(\Gamma(T(R))) \in \{3, 4\}$. Therefore, if $\mathrm{diam}(\Gamma(R)) = 3$, then $\mathrm{gr}(\Gamma(R)) \in \{3, 4\}$. $\square$

Table 3.26: Rings without identity where $Z(R) \subsetneq R$

diam \ girth	3	4	∞
0	None	None	None
1	$x\mathbb{Z}_4[x]$	None	None
2	$x\mathbb{Z}_8[x]$	$\mathbb{Z} \times 2\mathbb{Z}$	$\mathbb{Z}_2 \times 2\mathbb{Z}$
3	$\mathbb{Z}_2 \times \mathbb{Z}_2 \times 2\mathbb{Z}$	$2\mathbb{Z} \times \mathbb{Z}_4$	None

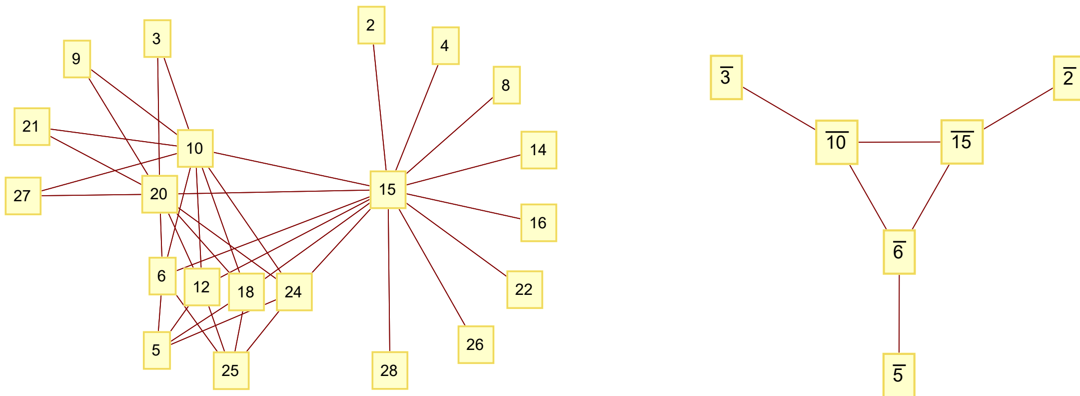
Chapter 4

Miscellaneous Results

The following are additional results that do not fit into the two previous chapters. First, we give a brief result on the compressed zero-divisor graph. Next, we give some results that extend the work in [43] on zero-divisor lattices, a relatively new structure on the zero-divisors of a ring. Both graphs have the same vertex set. Define a relation on the elements of R by $x \sim y$ if and only if $\operatorname{ann}(x) = \operatorname{ann}(y)$. This is clearly an equivalence relation, and we can define $R_E = \{[x]_R \mid x \in R\}$. It is observed in [6] that R_E is a commutative monoid, and they investigate the zero-divisor graphical structure of these commutative monoids.

The **compressed zero-divisor graph**, denoted $\Gamma_E(R)$, is a graph on the nonzero zero-divisors of R_E, where two vertices, $[x]_R$ and $[y]_R$, are joined by an edge if and only if $[x]_R[y]_R = [0]_R$ if and only if $xy = 0$. Figure 4.1 gives a comparison of the zero-divisor graph and compressed zero-divisor graph of $\mathbb{Z}_{30}$.

The compressed zero-divisor graph was first studied in [42], and it was stated in [6] that $\Gamma_E(R) \cong \Gamma(R_E)$. The idea of studying these equivalence classes was first mentioned by Mulay in [35]. The compressed zero-divisor graph has been studied in several papers, including [6], [7], [42], and [43]. Our first result gives a basic example of how the compressed zero-divisor graph can condense a ring's graphical structure. It also gives us that realizable compressed zero-divisor graphs on commutative rings with identity can be realized by compressed zero-divisor graphs on commutative rings without identity.



(a) $\Gamma(\mathbb{Z}_{30})$ **(b)** $\Gamma_E(\mathbb{Z}_{30})$

Figure 4.1: Zero-divisor and compressed zero-divisor graphs

Proposition 4.1. *Let R be a commutative ring and $R' = R \times \mathbb{Z}_2^0$, where the multiplication in the second component is the zero multiplication. Then $\Gamma_E(R) \cong \Gamma_E(R')$.*

Proof. Let R be a commutative ring and $R' = R \times \mathbb{Z}_2^0$, where multiplication in the second component is the zero multiplication. Let $0 \neq r \in R$. Then $\text{ann}((r, 0)) = \text{ann}((r, 1))$ since multiplication in the second component is always zero. Thus, $(r, 0)$ and $(r, 1)$ would be the same vertex in $\Gamma_E(R')$. So $|V(\Gamma_E(R))| = |V(\Gamma_E(R'))|$. Now, for vertices r and s in $\Gamma(R)$, r and s are the same vertex in $\Gamma_E(R)$ if and only if $(r, 0)$ and $(s, 0)$ are the same vertex in $\Gamma_E(R')$. Similarly, there is an edge between r and s in $\Gamma_E(R)$ if and only if there is an edge between $(r, 0)$ and $(s, 0)$ in $\Gamma_E(R')$. Therefore, $\Gamma_E(R) \cong \Gamma_E(R')$. $\square$

The next few results investigate the zero-divisor lattice. A **lattice** is a partially ordered set where every two elements have a unique supremum (called a **join**) and a unique infimum (called a **meet**). We consider a lattice structure on commutative rings and put the partially ordered set into a graphical structure in the following way.

We can put a partial order on the elements of R_E by defining $[x]_R < [y]_R$ if $\text{ann}(x) \subsetneq \text{ann}(y)$. The **zero-divisor lattice** is a partial lattice, denoted $\Lambda(R)$, with vertices the nonzero zero-divisors of R_E, and a directed edge $[y]_R \to [x]_R$ if and only if $[x]_R < [y]_R$ and there does not exist a $[z]_R$ with $[x]_R < [z]_R < [y]_R$. Figure 4.2 gives the zero-divisor lattice of $\mathbb{Z}_{30}$. We can have a join and meet for this lattice, however, the join would be $[0]_R$ and the

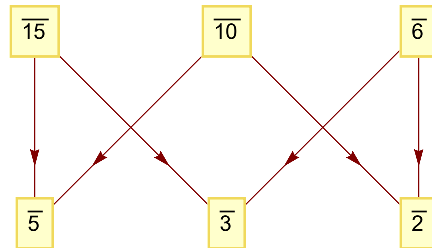


Figure 4.2: Zero-divisor lattice of $\mathbb{Z}_{30}$

meet would be $[1]_R$. Since we do not include these vertices in our graph, we really treat the zero-divisor lattice as a partial lattice.

A **root** of the zero-divisor lattice is a vertex x such that for every other comparable vertex y, we have $y < x$ (two distinct vertices are comparable if either $x < y$ or $y < x$). As a result, the root of a lattice is a maximal element and a lattice can have multiple roots. A **leaf** of the zero-divisor lattice is a vertex x such that for every other comparable vertex y, we have $x < y$. Conversely to the root of a lattice, the leaf of a lattice is a minimal element. Similarly, a lattice can have multiple leafs.

The idea for the zero-divisor lattice came from Nicholas Baeth, but was first investigated in [43] where the interplay between the zero-divisor graph, compressed zero-divisor graph, and zero-divisor lattice was researched. We say a lattice is **connected** if, when the edges are considered undirected, we can get from any vertex to any other vertex by traveling along edges. Of particular note is that the zero-divisor lattice of a finite commutative ring R is connected if and only if $R \ncong F_1 \times F_2$, where F_1 and F_2 are fields. The next several results continue some of the ideas from [43].

Lemma 4.2. *Let R be a nonlocal commutative Artinian ring. Then either each leaf of $\Lambda(R)$ is isolated by some cut-set in $\Gamma_E(R)$ or $\Gamma_E(R)$ does not have a cut-set.*

Proof. Notice that in the case of $R \cong F_1 \times F_2$, $\Gamma_E(R)$ has two vertices, and thus no cut-sets. So let R be a nonlocal commutative Artinian ring with $R \ncong F_1 \times F_2$ for fields F_1 and F_2. By Theorem 2.1, all cut-sets are annihilator ideals. If $|V(\Lambda(R))| = 1$, then $\Gamma_E(R)$ has only one vertex, and thus no cut-set. So assume $|V(\Lambda(R))| \geq 2$. Let $x \in V(\Lambda(R))$ be a leaf. Then for any distinct $y \in V(\Lambda(R))$, either $\operatorname{ann}(x) \subsetneq \operatorname{ann}(y)$ or $\operatorname{ann}(x)$ and $\operatorname{ann}(y)$ are not comparable. Thus, either $\operatorname{ann}(x)$ is a cut-set of $\Gamma_E(R)$ since it separates x from the rest of the graph and is the smallest such set, or $\Gamma_E(R)$ has no cut-set (i.e., $\text{ann}(x) = V(\Lambda(R)) \cup \{0\}$ or $\text{ann}(x) = V(\Lambda(R)) \cup \{0\} \setminus \{x\}$). $\square$

Conjecture 4.3. *Let R be a commutative ring. Then either each leaf of $\Lambda(R)$ is isolated by some cut-set in $\Gamma_E(R)$ or $\Gamma_E(R)$ does not have a cut-set.*

The proof for Lemma 4.2 is a good start to the proof of this conjecture. We run into trouble when we consider the case where R is local, which means that a cut-set in $\Gamma_E(R)$ may not isolate any vertices and thus may not be an annihilator ideal. Therefore, we may have that $\text{ann}(x)$ is a minimal annihilator ideal, but we have to consider the case where there exists a set $A \subsetneq \text{ann}(x)$ that is a cut-set.

Lemma 4.4. *Let R be a commutative Artinian ring. Then the roots of $\Lambda(R)$ are connected in $\Gamma_E(R)$.*

Proof. Let R be a commutative Artinian ring and let $x, y \in V(\Lambda(R))$ be distinct roots. Then $\text{ann}(x)$ and $\text{ann}(y)$ are maximal annihilator ideals. Since R is Artinian, they are maximal ideals of R. Assume $x \notin \text{ann}(y)$. Then $0 \neq xy \in Z(R)$. Now, $\text{ann}(x) \subseteq \text{ann}(xy)$ and $\text{ann}(y) \subseteq \text{ann}(xy)$. Clearly, $\text{ann}(xy) \subsetneq R$ and thus, since $\text{ann}(x)$ and $\text{ann}(y)$ are maximal ideals, we must have that $\text{ann}(x) = \text{ann}(xy) = \text{ann}(y)$. However, this would mean that x and y are the same vertex in $\Lambda(R)$, a contradiction. Thus, $x \in \text{ann}(y)$ and hence x and y are connected in $\Gamma_E(R)$. $\square$

In [26], they defined a **hub** of a graph to be an intersection of cut-sets. Note that it is possible for a hub to be empty. However, their main focus was on **minimal hubs**: a hub $\bigcap_{i=1}^n A_i \neq \emptyset$, where each A_i is a cut-set and n is large enough such that for any other distinct cut-set B, we have $(\bigcap_{i=1}^n A_i) \cap B = \emptyset$. Figure 4.3 gives the zero-divisor graph of ${}^{\mathbb{Z}_8[x]}/_{(x^2)}$. This graph has three cut-sets: $\{2x, 4x, 6x\}$, $\{4, 4x, 4x+4\}$, and $\{4x, 2x+4, 6x+4\}$. However, it has only one minimal hub: $\{4x\}$.

In [26], they showed that the number of minimal hubs in $\Gamma(R)$ for a finite commutative ring R is equal to the number of maximal ideals in R. Theorem 4.5 generalizes those results (i.e., Theorem 3.9 and Corollary 3.10 from [26]).

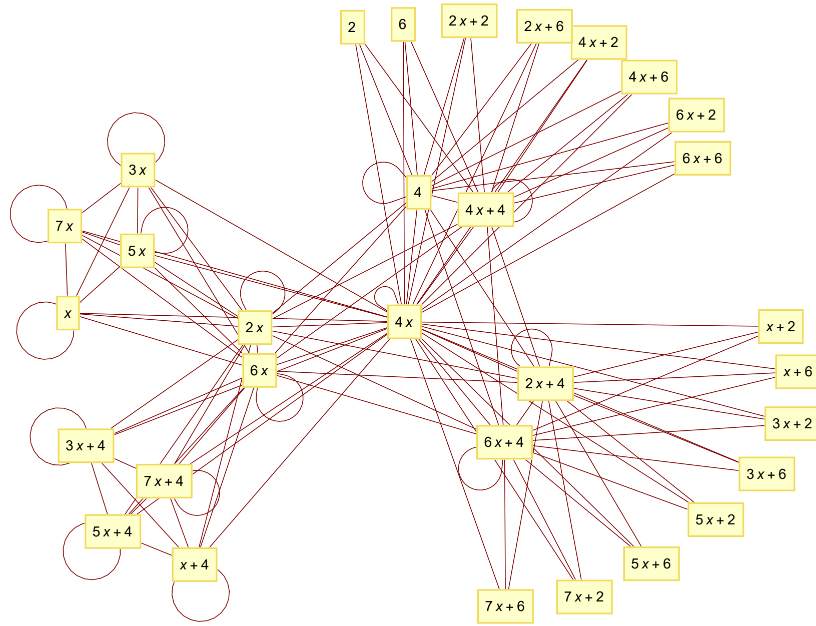


Figure 4.3: Zero-divisor graph of $\mathbb{Z}_8[x]/(x^2)$

Theorem 4.5. *Let R be an Artinian ring, where we can write $R \cong R_1 \times \cdots \times R_n$ for R_i local. Then the number of minimal hubs is equal to n and hence equal to the number of maximal ideals of R.*

Proof. Let R be an Artinian ring. We can write $R \cong R_1 \times \cdots \times R_n$, where each R_i is local. Suppose R is local. Then by Theorem 2.3 of [16], $\Gamma(R)$ is star-shaped reducible. Thus, the intersection of all cut-sets is a minimal hub since every cut-set must contain all zero-divisors in the center of $\Gamma(R)$. If R is not local, then Theorem 2.1 gives a single minimal hub corresponding to each R_i. $\square$

Chapter 5

Conclusion

5.1 Summary

This dissertation was split between two major topics (cut-sets in the ideal-based zero-divisor graph and commutative rings without identity) and some minor results in other topics (compressed zero-divisor graph and zero-divisor lattice). In Chapter 2, we were able to describe all of the cut-sets for commutative rings with identity in the ideal-based zero-divisor graph. We also considered a new viewpoint on the relationship between $\Gamma_I(R)$ and $\Gamma(R/I)$. Loops played a central role in this viewpoint, so much so that the results are not true if you consider these graphs to be simple.

In Chapter 3, we looked at commutative rings without identity. We were able to show several results about the diameter and girth of the zero-divisor graph of rings where every element is a zero-divisor and every element is nilpotent. We also classified all commutative rings without identity that have zero-divisor graphs on 14 or fewer vertices. This mimicked the list from Shane Redmond for commutative rings with identity in [38]. Lastly, in this chapter we also gave examples of commutative rings that have zero-divisor graphs of specified diameter and girth. We did this for finite and infinite commutative rings with and without identity, as well as rings with and without regular elements.

In our final chapter, we gave a few results on the compressed zero-divisor graph and a relatively new structure: the zero-divisor lattice. The appendices are Mathematica programs that the author wrote to aid in zero-divisor graph research by generating many examples very quickly. They were also used to generate several of the graphs displayed in this dissertation. There are far too many programs for all to be listed, so only a few are given to give the reader an idea of the code.

5.2 Open Questions

There are many questions left open in all of the topics covered in this dissertation. The work on cut-sets is just beginning and much more can be done with cut-sets in various other graphical structures. In terms of the ideal-based zero-divisor graph, we were only able to describe the cut-sets. That leaves the following question:

Question 5.1. *What are the cut-sets for an arbitrary ring R with ideal I? In other words, can we get a classification of the cut-sets similar to those in [26] and [13]?*

One way of attacking this problem was hinted at in this dissertation. Since we have a close relationship between $\Gamma_I(R)$ and $\Gamma(R/I)$, can we use the classification of the cut-sets in $\Gamma(R/I)$ to get a classification of the cut-sets in $\Gamma_I(R)$?

As with any classification of nonisomorphic rings that depend on the size of the ring, can we go further?

Question 5.2. *Can we extend the results in Section 3.3 to commutative rings without identity on zero-divisor graphs with more than* 14 *vertices?*

Of course, this kind of classification does not have to be limited to zero-divisor graphs. We can also look at other types of graphs and classify the nonisomorphic commutative rings that obtain those graphs on certain sizes. For example,

Question 5.3. *What nonisomorphic commutative rings, with or without identity, have compressed zero-divisor graphs on n or fewer vertices for some n?*

Question 5.4. *What nonisomorphic commutative rings, with or without identity, have zero-divisor lattices on n or fewer vertices for some n?*

We can also look at realizable graphs for the different types of graphical structures. For example,

Question 5.5. *Are there realizable compressed zero-divisor graphs of commutative rings without identity that are not realizable as compressed zero-divisor graphs of commutative rings with identity?*

Chapter 4 leaves many open questions. For example, we can classify the realizable diameters and girths for compressed zero-divisor graphs of various types of rings. We can also do this for the ideal-based zero-divisor graph.

The original motivation behind the development of the zero-divisor lattice was to investigate cut-sets in the graphs of local rings. Since the classification of cut-sets only holds for nonlocal Artinian rings, the natural question is what happens in the local case?

Question 5.6. *What are the cut-sets in the zero-divisor graph of a local ring?*

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Appendices

A Mathematica Notebooks for Zero-Divisor Graphs

Below are the Mathematica notebooks created and/or updated for this dissertation to generate several examples and pictures to study and illustrate for various types of rings. Almost all of the graphs in this dissertation were produced using these programs or slight alterations to these programs. These Mathematica programs generate the zero-divisor graphs of the following types of rings: $\mathbb{Z}_n$, $\mathbb{Z}_m \times \mathbb{Z}_n$, $\mathbb{Z}_m \times \mathbb{Z}_n \times \mathbb{Z}_q$, $\mathbb{Z}_m \times \mathbb{Z}_n \times \mathbb{Z}_q \times \mathbb{Z}_r$, ${}^{\mathbb{Z}_m[x]}/_{(f_1(x),\ldots,f_k(x))}$, and $\mathbb{Z}_m \times {}^{\mathbb{Z}_n[x]}/_{(f_1(x),\ldots,f_k(x))}$ for integers m, n, q, and r, and polynomials $f_i(x)$. Below is the code for the Mathematica notebooks to generate the zero-divisor graphs of $\mathbb{Z}_n$, $\mathbb{Z}_m \times \mathbb{Z}_n$, and $\mathbb{Z}_m \times {}^{\mathbb{Z}_n[x]}/_{(f_1(x),\ldots,f_k(x))}$. An earlier version of these programs appeared in [44], but have since been adapted or updated.

A.1 Zero-Divisor Graph of $\mathbb{Z}_n$

```
ZDGraph[n_] :=
  Module[{Loops, Edge, EdgeLoops, LoopedPositions},

   (* This part goes through all the possible nonzero elements of the ring
   and finds the zero-divisor pairs to create the Edge set. If an element
   multiplies to itself to be zero, then it adds it to the Loops set. *)
   EdgeLoops = Last[Reap[Do[If[Mod[Counter1*Counter2, n] == 0,
        Sow[Counter1 -> Counter2, edge];
        If[Counter1 == Counter2,
         Sow[Counter1, loops]]
        ], {Counter1, n - 1}, {Counter2, n - 1}]]];

   (* This section changes the looped vertices (if there are any) to be
   colored red. This way the graph does not have loops making it easier to
   move the vertices around. *)
   If[Length[EdgeLoops] == 2,
    {Edge, Loops} = EdgeLoops;
```

```
  LoopedPositions = Flatten[Last[Reap[Do[
   Sow[Position[Edge, Loops[[i]]], i], {i,Length[Loops]}]]]], 1];
  Do[Edge = ReplacePart[Edge,
   LoopedPositions[[i]] -> Style[Loops[[i]], Red]], {i,Length[Loops]}];,
   Edge = Flatten[EdgeLoops];
  ];

 (* Graph it *)
 GraphPlot[Edge, VertexLabeling -> True, SelfLoopStyle -> False,
  MultiedgeStyle -> False, PlotLabel ->
  \[CapitalGamma][Subscript[\[DoubleStruckCapitalZ],n]]
 ];
```

A.2 Zero-Divisor Graph of $\mathbb{Z}_m \times \mathbb{Z}_n$

```
ZDGraphDoubleCross[m_, n_] :=
 Module[{Loops, Edge, EdgeLoops, LoopedPositions, f},
  f[{a1_, b1_}, {a2_, b2_}] = {Mod[a1*a2, m], Mod[b1*b2, n]};

  (* This part goes through all the possible nonzero elements of the ring
  and finds the zero-divisor pairs to create the Edge set. If an element
  multiplies to itself to be zero, then it adds it to the Loops set. *)
  EdgeLoops =
   Last[Reap[
     Do[If[f[{Counter1, Counter2}, {Counter3, Counter4}] == {0,
          0} && {Counter1, Counter2} != {0, 0} && {Counter3,
          Counter4} != {0, 0},
       Sow[{Counter1, Counter2} -> {Counter3, Counter4}, edge];
       If[{Counter1, Counter2} == {Counter3, Counter4},
        Sow[{Counter1, Counter2}, loops]]
       ], {Counter1, 0, m - 1}, {Counter2, 0, n - 1}, {Counter3, 0,
```

```
    m - 1}, {Counter4, 0, n - 1}]]];

(* This section changes the looped vertices (if there are any) to be
colored red. This way the graph does not have loops making it easier to
move the vertices around. *)
If[Length[EdgeLoops] == 2,
 {Edge, Loops} = EdgeLoops;
 LoopedPositions =
  Flatten[Last[
    Reap[Do[Sow[Position[Edge, Loops[[i]]], i], {i,
       Length[Loops]}]]], 1];
 Do[Edge =
   ReplacePart[Edge,
    LoopedPositions[[i]] -> Style[Loops[[i]], Red]], {i,
   Length[Loops]}];,
 Edge = Flatten[EdgeLoops];
 ];

(* Graph it *)
GraphPlot[Edge, VertexLabeling -> True, SelfLoopStyle -> False,
 MultiedgeStyle -> False(*,PlotLabel\[Rule]Row[{\[CapitalGamma],
 Style["(",Larger],Subscript[\[DoubleStruckCapitalZ], m],"\[Times]",
 Subscript[\[DoubleStruckCapitalZ], n],Style[")",Larger]," "}]*)]
]
```

A.3 Zero-Divisor Graph of $\mathbb{Z}_m \times {}^{\mathbb{Z}_n[x]}/_{(f_1(x),\ldots,f_k(x))}$

```
ZDGraphCrossFactor[m_, n_, func_] :=
 Module[{NumberOfTerms, CoefficientMatrix, Ring, Edge, PartRing,
   ZeroElement, Loops, EdgeLoops, LoopedPositions},
  NumberOfTerms = Max[Exponent[func, x]];
```

```
(* This line creates a set that has all of the necessary coefficients in
order to make every element of the factor ring. CoefficientMatrix looks
like: {{0,0,0},{0,0,1},{0,0,2},...,{0,1,0},{0,1,1},...,{n-1,n-1,n-1}}
(where the length of each element is based on NumberOfTerms). *)
CoefficientMatrix =
 Tuples[Flatten[
   Last[Reap[Do[Sow[j], {j, 0, n - 1}]]]], {NumberOfTerms + 1}];

(* These lines create the elements of the factor ring Subscript[
\[DoubleStruckCapitalZ], n][x]/(Subscript[f, 1](x),Subscript[f,2](x),...,
Subscript[f,  k](x)) by multiplying CoefficientMatrix to a matrix that
looks like this {{1},{x},{x^2},...,{x^(NumberOfTerms)}}. *)
PartRing =
 Flatten[CoefficientMatrix.Table[{x^i}, {i, 0, NumberOfTerms}]];
PartRing =
 DeleteDuplicates[
  PolynomialMod[
   PolynomialMod[PartRing, func, CoefficientDomain -> Integers], n]];

(* This section takes the elements of PartRing and adds the Subscript[
\[DoubleStruckCapitalZ], m] elements to create the cross product. The
ring element (0,0) is deleted from Ring. *)
Ring = Flatten[
  Last[Reap[
    Do[Sow[{j, PartRing[[i]]}], {j, 0, m - 1}, {i,
      Length[PartRing]}]]]], 1];
ZeroElement = Position[Ring, {0, 0}];
Ring = Delete[Ring, ZeroElement];
```

```
(* This section finds which elements are zero-divisors and creates the
edge set to graph. If an element multiplies to itself to be zero, then it
adds that element to a loops set. *)
EdgeLoops =
 Last[Reap[
   Do[If[Mod[Ring[[i, 1]]*Ring[[j, 1]], m] == 0 &&
      PolynomialMod[
        PolynomialMod[Ring[[i, 2]]*Ring[[j, 2]], func,
         CoefficientDomain -> Integers], n] === 0,
     Sow[Ring[[i]] -> Ring[[j]], edge];
     If[Ring[[i]] === Ring[[j]],
      Sow[Ring[[i]], loops]]
     ], {i, Length[Ring]}, {j, Length[Ring]}]]];

(* This section changes the looped vertices (if there are any) to be
colored red. This way the graph does not have loops making it easier to
move the vertices around. *)
If[Length[EdgeLoops] == 2,
 {Edge, Loops} = EdgeLoops;
 LoopedPositions =
  Flatten[Last[
    Reap[Do[Sow[Position[Edge, Loops[[i]]], i], {i,
       Length[Loops]}]]]], 1];
 Do[Edge =
   ReplacePart[Edge,
    LoopedPositions[[i]] -> Style[Loops[[i]], Red]], {i,
   Length[Loops]}];,
 Edge = Flatten[EdgeLoops];
 ];
```

```
(* Graph it *)
GraphPlot[Edge, MultiedgeStyle -> False, SelfLoopStyle -> False,
 VertexLabeling -> True,
 PlotLabel ->
  Row[{\[CapitalGamma], Style["(", Larger],
    Subscript[\[DoubleStruckCapitalZ], m], "\[Times]",
    Subscript[\[DoubleStruckCapitalZ], n], Style["["], x,
    Style["]"], Style["/", Larger], Style["("], func, Style[")"],
    Style[")", Larger]}, " "]]
]
```

B Mathematica Notebooks for Ideal-Based Zero-Divisor Graphs

Below are the Mathematica notebooks created for this dissertation to generate several examples and pictures to study and illustrate for various types of rings. Almost all of the graphs in this dissertation were produced using these programs or slight alterations to these programs. These Mathematica programs generate the ideal-based zero-divisor graphs of the following types of rings: $\mathbb{Z}_n$, $\mathbb{Z}_m \times \mathbb{Z}_n$, $\mathbb{Z}_m \times \mathbb{Z}_n \times \mathbb{Z}_q$, ${}^{\mathbb{Z}_n[x]}/_{(f_1(x),\ldots,f_k(x))}$, and $\mathbb{Z}_m \times {}^{\mathbb{Z}_n[x]}/_{(f_1(x),\ldots,f_k(x))}$ for integers m, n, and q, and polynomials $f_i(x)$. Below is the code for the Mathematica notebooks to generate the ideal-based zero-divisor graphs of $\mathbb{Z}_n$, $\mathbb{Z}_m \times \mathbb{Z}_n \times \mathbb{Z}_q$, and ${}^{\mathbb{Z}_n[x]}/_{(f_1(x),\ldots,f_k(x))}$.

B.1 Ideal-Based Zero-Divisor Graph of $\mathbb{Z}_n$

```
IDGraph[n_, i_] :=
 Module[{Edge, Loops, EdgeLoops, ideal, LoopedPositions},

  (* Generate the ideal *)
  ideal =
   DeleteDuplicates[
    Mod[Flatten[Last[Reap[Do[Sow[Counter1*i], {Counter1, n}]]]], n]];

  (* Create the edge set *)
  EdgeLoops =
   Last[Reap[
     Do[If[! MemberQ[ideal, Counter1] && ! MemberQ[ideal, Counter2] &&
         MemberQ[ideal, Mod[Counter1*Counter2, n]],
       Sow[Counter1 -> Counter2, edge];
       If[Counter1 == Counter2, Sow[Counter1, loops]]], {Counter1,
       n - 1}, {Counter2, n - 1}]]];
```

```
(* This section changes the looped vertices (if there are any) to be
colored red. This way the graph does not have loops making it easier to
move the vertices around. *)
If[Length[EdgeLoops] == 2,
 {Edge, Loops} = EdgeLoops;
 LoopedPositions =
  Flatten[Last[
    Reap[Do[Sow[Position[Edge, Loops[[j]]], j], {j,
       Length[Loops]}]]], 1];
 Do[Edge =
   ReplacePart[Edge,
    LoopedPositions[[j]] -> Style[Loops[[j]], Red]], {j,
   Length[Loops]}];,
 Edge = Flatten[EdgeLoops];
 ];

(* Graph it *)
GraphPlot[Edge, VertexLabeling -> True, SelfLoopStyle -> False,
 MultiedgeStyle -> False,
 PlotLabel ->
  Row[{Subscript[\[CapitalGamma],
     Row[{Style["\[LeftAngleBracket]"], i,
       Style["\[RightAngleBracket]"]}]], Style["("],
    Subscript[\[DoubleStruckCapitalZ], n], Style[")"]}, " "]]
]
```

B.2 Ideal-Based Zero-Divisor Graph of $\mathbb{Z}_m \times \mathbb{Z}_n \times \mathbb{Z}_q$

```
IDGraphTripleCross[{m_, n_, q_}, {i_, j_, k_}] :=
 Module[{Edge, Loops, EdgeLoops, ideal, LoopedPositions, f},
```

```
f[{a1_, b1_, c1_}, {a2_, b2_, c2_}] = {Mod[a1*a2, m], Mod[b1*b2, n],
   Mod[c1*c2, q]};

(* Generate the ideal *)
ideal =
 DeleteDuplicates[
  Flatten[Last[
    Reap[Do[Sow[{Mod[Counter1*i, m], Mod[Counter2*j, n],
        Mod[Counter3*k, q]}], {Counter1, m}, {Counter2,
       n}, {Counter3, q}]]], 1]];

(* Create the edge set *)
EdgeLoops =
 Last[Reap[
   Do[If[! MemberQ[ideal, {Counter1, Counter2, Counter3}] && !
       MemberQ[ideal, {Counter4, Counter5, Counter6}] &&
      MemberQ[ideal,
       f[{Counter1, Counter2, Counter3}, {Counter4, Counter5,
         Counter6}]],
     Sow[{Counter1, Counter2, Counter3} -> {Counter4, Counter5,
        Counter6}, edge];
     If[{Counter1, Counter2, Counter3} == {Counter4, Counter5,
        Counter6},
      Sow[{Counter1, Counter2, Counter3}, loops];
      ]
     ], {Counter1, 0, m - 1}, {Counter2, 0, n - 1}, {Counter3, 0,
     q - 1}, {Counter4, 0, m - 1}, {Counter5, 0, n - 1}, {Counter6,
     0, q - 1}]]];

(* This section changes the looped vertices (if there are any) to be
```

```
colored red. This way the graph does not have loops making it easier to
move the vertices around. *)
If[Length[EdgeLoops] == 2,
 {Edge, Loops} = EdgeLoops;
 LoopedPositions =
  Flatten[Last[
    Reap[Do[Sow[Position[Edge, Loops[[Counter1]]],
       Counter1], {Counter1, Length[Loops]}]]], 1];
 Do[Edge =
   ReplacePart[Edge,
    LoopedPositions[[Counter1]] ->
     Style[Loops[[Counter1]], Red]], {Counter1, Length[Loops]}];,
 Edge = Flatten[EdgeLoops];
 ];

(* Graph it *)
GraphPlot[Edge, VertexLabeling -> True, SelfLoopStyle -> False,
 MultiedgeStyle -> False,
 PlotLabel ->
  Row[{Subscript[\[CapitalGamma],
     Row[{Style["\[LeftAngleBracket]("], i, ",", j, ",", k,
       Style[")\[RightAngleBracket]"]}]], Style["(", Larger],
    Subscript[\[DoubleStruckCapitalZ], m], "\[Times]",
    Subscript[\[DoubleStruckCapitalZ], n], "\[Times]",
    Subscript[\[DoubleStruckCapitalZ], q], Style[")", Larger], " "}]]
]
```

B.3 Ideal-Based Zero-Divisor Graph of $\mathbb{Z}_n[x]/(f_1(x), \ldots, f_k(x))$

```
IDGraphFactor[n_, i_, func_] :=
 Module[{NumberOfTerms, CoefficientMatrix, Ring, Edge, Loops,
```

```
 EdgeLoops, ideal, LoopedPositions},
IDGraphMonic::elem =
 "The element `1` is not in the ring \!\(\*SubscriptBox[\(\
\[DoubleStruckCapitalZ]\), \(`2`\)]\)[x]/(`3`)"; (*
This is an error message in case i is not in the ring *)
NumberOfTerms = Max[Exponent[func, x]];

(* This part creates a set that has all of the necessary coefficients in
order to make every element of the factor ring. CoefficientMatrix looks
like: {{0,0,0},{0,0,1},{0,0,2},...,{0,1,0},{0,1,1},...,{n-1,n-1,n-1}}
(where the length of each element is based on NumberOfTerms). *)
CoefficientMatrix =
 Tuples[Flatten[
   Last[Reap[Do[Sow[j], {j, 0, n - 1}]]]], {NumberOfTerms + 1}];

(* These lines create the elements of the factor ring Subscript[
\[DoubleStruckCapitalZ], n][x]/(Subscript[f, 1](x),Subscript[f,2](x),...,
Subscript[f,k](x)) by multiplying CoefficientMatrix to a matrix that looks
like this {{1},{x},{x^2},...,{x^(NumberOfTerms)}}. *)
Ring = Flatten[
  CoefficientMatrix.Table[{x^j}, {j, 0, NumberOfTerms}]];
Ring = DeleteDuplicates[
  PolynomialMod[
   PolynomialMod[Ring, func, CoefficientDomain -> Integers], n]];

(* The following If[...] checks to make sure that the element i is in the
ring. It prints an error if it is not. If it is, then it runs through the
rest of the program. *)
If[MemberQ[Ring, i],
 (* Generate the ideal *)
```

```
ideal =
 DeleteDuplicates[
  PolynomialMod[
   PolynomialMod[
    Flatten[Last[
      Reap[Do[Sow[Ring[[Counter1]]*i], {Counter1,
         Length[Ring]}]]]]], func, CoefficientDomain -> Integers],
   n]];

(* Create the edge set *)
EdgeLoops =
 Last[Reap[
   Do[If[! MemberQ[ideal, Ring[[Counter1]]] && !
       MemberQ[ideal, Ring[[Counter2]]] &&
      MemberQ[ideal,
       PolynomialMod[
        PolynomialMod[Ring[[Counter1]]*Ring[[Counter2]], func,
         CoefficientDomain -> Integers], n]],
     Sow[Ring[[Counter1]] -> Ring[[Counter2]], edge];
     If[Counter1 == Counter2,
      Sow[Ring[[Counter1]], loops]]], {Counter1,
     Length[Ring]}, {Counter2, Length[Ring]}]]];

(* This section changes the looped vertices (if there are any) to be
colored red.  This way the graph does not have loops making it easier to
move the vertices around. *)
If[Length[EdgeLoops] == 2,
 {Edge, Loops} = EdgeLoops;
 LoopedPositions =
  Flatten[Last[
```

```
    Reap[Do[Sow[Position[Edge, Loops[[j]], 2], j], {j,
       Length[Loops]}]]], 1];
  Do[Edge =
    ReplacePart[Edge,
     LoopedPositions[[j]] -> Style[Loops[[j]], Red]], {j,
    Length[Loops]}];,
  Edge = Flatten[EdgeLoops];
  ];

 (* Graph it *)
 GraphPlot[Edge, VertexLabeling -> True, SelfLoopStyle -> False,
  MultiedgeStyle -> False,
  PlotLabel ->
   Row[{Subscript[\[CapitalGamma],
      Row[{Style["\[LeftAngleBracket]"], i,
        Style["\[RightAngleBracket]"]}]], Style["(", Larger],
     Subscript[\[DoubleStruckCapitalZ], n], Style["["], x,
     Style["]"], Style["/", Larger], Style["("], func, Style[")"],
     Style[")", Larger]}, " "]],
 (Message[IDGraphMonic::elem, i, n, func];);]
]
```

C Mathematica Notebooks for Zero-Divisor Lattices

Below are the Mathematica notebooks created and/or updated for this dissertation to generate several examples and pictures to study and illustrate for various types of rings. Almost all of the graphs in this dissertation were produced using these programs or slight alterations to these programs. These Mathematica programs generate the zero-divisor lattices of the following types of rings: $\mathbb{Z}_n$, $\mathbb{Z}_m \times \mathbb{Z}_n$, $\mathbb{Z}_m \times \mathbb{Z}_n \times \mathbb{Z}_q$, ${}^{\mathbb{Z}_m[x]}/_{(f_1(x),\ldots,f_k(x))}$, and $\mathbb{Z}_m \times {}^{\mathbb{Z}_n[x]}/_{(f_1(x),\ldots,f_k(x))}$ for integers m, n, and q, and polynomials $f_i(x)$. Below is the code for the Mathematica notebooks to generate the zero-divisor lattices of $\mathbb{Z}_m \times \mathbb{Z}_n$, $\mathbb{Z}_m \times \mathbb{Z}_n \times \mathbb{Z}_q$, and $\mathbb{Z}_m \times {}^{\mathbb{Z}_n[x]}/_{(f_1(x),\ldots,f_k(x))}$. An earlier version of these programs appeared in [44], but have since been adapted or updated.

C.1 Zero-Divisor Lattice of $\mathbb{Z}_m \times \mathbb{Z}_n$

```
ZDLatticeDoubleCross[m_, n_] :=
 Module[{f, ZDList, AnnList, Annihilators, ZeroDivisors, duplicateAnn,
    indices, Edge},
  f[{a1_, b1_}, {a2_, b2_}] = {Mod[a1*a2, m], Mod[b1*b2, n]};

  (* This part creates the ZeroDivisors set and Annihilators set. The
  ZeroDivisors set is just a list of the zero-divisors of the ring. The
  Annihilators set is a comprised of sets which are the annihilators of the
  corresponding zero-divisor in ZeroDivisors. For example, in
  Subscript[\[DoubleStruckCapitalZ], 12]:
  ZeroDivisors={2,3,4,6,8,9,10} and
  Annihilators={{6},{4,8},{3,6,9},{2,4,6,8,10},{3,6,9},{4,8},{6}}
   *)
  {ZDList, AnnList} =
   Last[Reap[
     Do[If[f[{i1, j1}, {i2, j2}] == {0, 0} && {i1, j1} != {0,
          0} && {i2, j2} != {0, 0},
```

```
    Sow[{i1, j1}, ZD];
    Sow[{i2, j2}, Ann];
    ], {i1, 0, m - 1}, {j1, 0, n - 1}, {i2, 0, m - 1}, {j2, 0,
    n - 1}]]];

Annihilators = Last[Reap[Do[If[ZDList[[i]] == ZDList[[i + 1]],
    If[i + 1 == Length[ZDList],
      Sow[AnnList[[i]], ToString[ZDList[[i]]]];
      Sow[AnnList[[i + 1]], ToString[ZDList[[i]]]],
      Sow[AnnList[[i]], ToString[ZDList[[i]]]];
      ];,
    Sow[AnnList[[i]], ToString[ZDList[[i]]]];
    If[i + 1 == Length[ZDList],
     Sow[AnnList[[i + 1]], ToString[ZDList[[i + 1]]]]];
    ], {i, Length[ZDList] - 1}]]];
ZeroDivisors = DeleteDuplicates[ZDList];

(*The next part eliminates the repeats of ZeroDivisors and Annihilators.*)
duplicateAnn =
 GatherBy[Range@Length[Annihilators], Annihilators[[#]] &];
indices = duplicateAnn[[All, 1]];
ZeroDivisors = ZeroDivisors[[indices]];
Annihilators = DeleteDuplicates[Annihilators];

(* This next section sorts the Annihilators set from the smallest to the
largest and keeps the relationship between Annihilators and ZeroDivisors
sets intact. *)
ZeroDivisors = ZeroDivisors[[Ordering[Annihilators]]];
Annihilators = Sort[Annihilators];
```

```
(* The next line puts every zero-divisor into the graph as disconnected
vertices. That way we can check to see if paths between the vertices exist
to eliminate the extra paths. *)
Edge = Flatten[
  Last[Reap[
    Do[Sow[OverBar[ZeroDivisors[[i]]] ->
       OverBar[ZeroDivisors[[i]]]], {i, Length[ZeroDivisors]}]]]];

(* The next part populates the Edge set and creates the edges between the
vertices. It determines which annihilator sets are subsets of others. *)
Do[If[Length[Annihilators[[i]]] < Length[Annihilators[[j]]],
   If[Complement[Annihilators[[i]], Annihilators[[j]]] == {},
     If[
       FindShortestPath[Edge, OverBar[ZeroDivisors[[j]]],
         OverBar[ZeroDivisors[[i]]]] == {},
       Edge =
         Join[Edge, {OverBar[ZeroDivisors[[j]]] ->
            OverBar[ZeroDivisors[[i]]]}];
       ];
     ];
   ];, {i, Length[Annihilators] - 1, 1, -1}, {j, i + 1,
  Length[Annihilators]}];

(* Graph it *)
LayeredGraphPlot[Edge, VertexLabeling -> True,
 SelfLoopStyle -> False,
 PlotLabel ->
  Row[{\[CapitalLambda], Style["(", Larger],
    Subscript[\[DoubleStruckCapitalZ], m], "\[Times]",
    Subscript[\[DoubleStruckCapitalZ], n], Style[")", Larger], " "}]]
```

```
]
```

C.2 Zero-Divisor Lattice of $\mathbb{Z}_m \times \mathbb{Z}_n \times \mathbb{Z}_q$

```
ZDLatticeTripleCross[m_, n_, q_] :=
 Module[{f, ZDList, AnnList, Annihilators, ZeroDivisors, duplicateAnn,
    indices, Edge},
  f[{a1_, b1_, c1_}, {a2_, b2_, c2_}] = {Mod[a1*a2, m], Mod[b1*b2, n],
     Mod[c1*c2, q]};

  (* This part creates the ZeroDivisors set and Annihilators set. The
  ZeroDivisors set is just a list of the zero-divisors of the ring. The
  Annihilators set is a comprised of sets which are the annihilators of the
  corresponding zero-divisor in ZeroDivisors. For example, in Subscript[
  \[DoubleStruckCapitalZ], 12]:
  ZeroDivisors={2,3,4,6,8,9,10} and
  Annihilators={{6},{4,8},{3,6,9},{2,4,6,8,10},{3,6,9},{4,8},{6}}
   *)
  {ZDList, AnnList} =
   Last[Reap[
     Do[If[f[{i1, j1, k1}, {i2, j2, k2}] == {0, 0, 0} && {i1, j1,
          k1} != {0, 0, 0} && {i2, j2, k2} != {0, 0, 0},
       Sow[{i1, j1, k1}, ZD];
       Sow[{i2, j2, k2}, Ann];
       ], {i1, 0, m - 1}, {j1, 0, n - 1}, {k1, 0, q - 1}, {i2, 0,
       m - 1}, {j2, 0, n - 1}, {k2, 0, q - 1}]]];

  Annihilators = Last[Reap[Do[If[ZDList[[i]] == ZDList[[i + 1]],
       If[i + 1 == Length[ZDList],
         Sow[AnnList[[i]], ToString[ZDList[[i]]]];
         Sow[AnnList[[i + 1]], ToString[ZDList[[i]]]],
```

```
       Sow[AnnList[[i]], ToString[ZDList[[i]]]];
       ];,
     Sow[AnnList[[i]], ToString[ZDList[[i]]]];
     If[i + 1 == Length[ZDList],
      Sow[AnnList[[i + 1]], ToString[ZDList[[i + 1]]]]];
     ], {i, Length[ZDList] - 1}]]];
ZeroDivisors = DeleteDuplicates[ZDList];

(*The next part eliminates the repeats of ZeroDivisors and Annihilators.*)
duplicateAnn =
 GatherBy[Range@Length[Annihilators], Annihilators[[#]] &];
indices = duplicateAnn[[All, 1]];
ZeroDivisors = ZeroDivisors[[indices]];
Annihilators = DeleteDuplicates[Annihilators];

(* This next section sorts the Annihilators set from the smallest to the
largest and keeps the relationship between Annihilators and ZeroDivisors
sets intact. *)
ZeroDivisors = ZeroDivisors[[Ordering[Annihilators]]];
Annihilators = Sort[Annihilators];

(* The next line puts every zero-divisor into the graph as disconnected
vertices. That way we can check to see if paths between the vertices exist
to eliminate the extra paths. *)
Edge = Flatten[
  Last[Reap[
    Do[Sow[OverBar[ZeroDivisors[[i]]] ->
       OverBar[ZeroDivisors[[i]]]], {i, Length[ZeroDivisors]}]]]];

(* The next part populates the Edge set and creates the edges between the
```

```
vertices. It determines which annihilator sets are subsets of others. *)
Do[If[Length[Annihilators[[i]]] < Length[Annihilators[[j]]],
   If[Complement[Annihilators[[i]], Annihilators[[j]]] == {},
     If[
       FindShortestPath[Edge, OverBar[ZeroDivisors[[j]]],
         OverBar[ZeroDivisors[[i]]]] == {},
       Edge =
         Join[Edge, {OverBar[ZeroDivisors[[j]]] ->
            OverBar[ZeroDivisors[[i]]]}];
       ];
     ];
   ];, {i, Length[Annihilators] - 1, 1, -1}, {j, i + 1,
  Length[Annihilators]}];

(* Graph it *)
LayeredGraphPlot[Edge, VertexLabeling -> True,
 SelfLoopStyle -> False,
 PlotLabel ->
  Row[{\[CapitalLambda], Style["(", Larger],
    Subscript[\[DoubleStruckCapitalZ], m], "\[Times]",
    Subscript[\[DoubleStruckCapitalZ], n], "\[Times]",
    Subscript[\[DoubleStruckCapitalZ], q], Style[")", Larger], " "}]]
]
```

C.3 Zero-Divisor Lattice of $\mathbb{Z}_m \times \mathbb{Z}_n[x]/(f_1(x), \ldots, f_k(x))$

```
ZDLatticeCrossFactor[m_, n_, func_] :=
 Module[{NumberOfTerms, CoefficientMatrix, PartRing, Ring,
   ZeroElement, ZDList, AnnList, Annihilators, ZeroDivisors,
   duplicateAnn, indices, Edge},
  NumberOfTerms = Max[Exponent[func, x]];
```

```
(* This line creates a set that has all of the necessary coefficients in
order to make every element of the factor ring. CoefficientMatrix looks
like: {{0,0,0},{0,0,1},{0,0,2},...,{0,1,0},{0,1,1},...,{n-1,n-1,n-1}}
(where the length of each element is based on NumberOfTerms). *)
CoefficientMatrix =
 Tuples[Flatten[
   Last[Reap[Do[Sow[j], {j, 0, n - 1}]]]], {NumberOfTerms + 1}];

(* These lines create the elements of the factor ring Subscript[
 \[DoubleStruckCapitalZ], n][x]/(Subscript[f, 1](x),Subscript[f,2](x),...,
 Subscript[f,k](x)) by multiplying CoefficientMatrix to a matrix that
 looks like this {{1},{x},{x^2},...,{x^(NumberOfTerms)}}. *)
PartRing =
 Flatten[CoefficientMatrix.Table[{x^i}, {i, 0, NumberOfTerms}]];
PartRing =
 DeleteDuplicates[
  PolynomialMod[
   PolynomialMod[PartRing, func, CoefficientDomain -> Integers], n]];

(* This section takes the elements of PartRing and adds the Subscript[
\[DoubleStruckCapitalZ], m] elements to create the cross product. The ring
element (0,0) is deleted from Ring. *)
Ring = Flatten[
  Last[Reap[
    Do[Sow[{j, PartRing[[i]]}], {j, 0, m - 1}, {i,
      Length[PartRing]}]]]], 1];
ZeroElement = Position[Ring, {0, 0}];
Ring = Delete[Ring, ZeroElement];
```

```
(* This section creates the ZeroDivisors set and Annihilators set. The
ZeroDivisors set is just a list of the zero-divisors of the ring. The
Annihilators set is a comprised of sets which are the annihilators of the
corresponding zero-divisor in ZeroDivisors. For example, in Subscript[
\[DoubleStruckCapitalZ], 12]:
ZeroDivisors={2,3,4,6,8,9,10} and
Annihilators={{6},{4,8},{3,6,9},{2,4,6,8,10},{3,6,9},{4,8},{6}}
 *)
{ZDList, AnnList} =
 Last[Reap[
   Do[If[Mod[Ring[[i, 1]]*Ring[[j, 1]], m] == 0 &&
      PolynomialMod[
        PolynomialMod[Ring[[i, 2]]*Ring[[j, 2]], func,
         CoefficientDomain -> Integers], n] === 0,
     Sow[Ring[[i]], ZD];
     Sow[Ring[[j]], Ann];
     ], {i, Length[Ring]}, {j, Length[Ring]}]]];

Annihilators = Last[Reap[Do[If[ZDList[[i]] === ZDList[[i + 1]],
     If[i + 1 == Length[ZDList],
       Sow[AnnList[[i]], ToString[ZDList[[i]]]];
       Sow[AnnList[[i + 1]], ToString[ZDList[[i]]]],
       Sow[AnnList[[i]], ToString[ZDList[[i]]]];
       ];,
     Sow[AnnList[[i]], ToString[ZDList[[i]]]];
     If[i + 1 == Length[ZDList],
      Sow[AnnList[[i + 1]], ToString[ZDList[[i + 1]]]]]];
     ], {i, Length[ZDList] - 1}]]];
ZeroDivisors = DeleteDuplicates[ZDList];
```

```
(*The next part eliminates the repeats of ZeroDivisors and Annihilators.*)
duplicateAnn =
 GatherBy[Range@Length[Annihilators], Annihilators[[#]] &];
indices = duplicateAnn[[All, 1]];
ZeroDivisors = ZeroDivisors[[indices]];
Annihilators = DeleteDuplicates[Annihilators];

(* This next section sorts the Annihilators set from the smallest to the
largest and keeps the relationship between Annihilators and ZeroDivisors
set intact. *)
ZeroDivisors = ZeroDivisors[[Ordering[Annihilators]]];
Annihilators = Sort[Annihilators];

(* The next line puts every zero-divisor into the graph as disconnected
vertices. That way we can check to see if paths between the vertices
exist to eliminate the extra paths. *)
Edge = Flatten[
  Last[Reap[
    Do[Sow[OverBar[ZeroDivisors[[i]]] ->
       OverBar[ZeroDivisors[[i]]]], {i, Length[ZeroDivisors]}]]]];

(* The next part populates the Edge set and creates the edges between the
vertices. It determines which annihilator sets are subsets of others. *)
Do[If[Length[Annihilators[[i]]] < Length[Annihilators[[j]]],
   If[Complement[Annihilators[[i]], Annihilators[[j]]] == {},
     If[
       FindShortestPath[Edge, OverBar[ZeroDivisors[[j]]],
         OverBar[ZeroDivisors[[i]]]] == {},
       Edge =
         Join[Edge, {OverBar[ZeroDivisors[[j]]] ->
```

```
            OverBar[ZeroDivisors[[i]]]}];
        ];
      ];
    ];, {i, Length[Annihilators] - 1, 1, -1}, {j, i + 1,
   Length[Annihilators]}];

(* Graph it *)
LayeredGraphPlot[Edge, VertexLabeling -> True,
 SelfLoopStyle -> False,
 PlotLabel ->
  Row[{\[CapitalLambda], Style["(", Larger],
    Subscript[\[DoubleStruckCapitalZ], m], "\[Times]",
    Subscript[\[DoubleStruckCapitalZ], n], Style["["], x,
    Style["]"], Style["/", Larger], Style["("], func, Style[")"],
    Style[")", Larger]}, " "]]
]
```

D Mathematica Notebooks for Compressed Zero-Divisor Graphs

Below are the Mathematica notebooks created and/or updated for this dissertation to generate several examples and pictures to study and illustrate for various types of rings. Almost all of the graphs in this dissertation were produced using these programs or slight alterations to these programs. These Mathematica programs generate the compressed zero-divisor graphs of the following types of rings: $\mathbb{Z}_n$, $\mathbb{Z}_m \times \mathbb{Z}_n$, $\mathbb{Z}_m \times \mathbb{Z}_n \times \mathbb{Z}_q$, $\mathbb{Z}_m \times \mathbb{Z}_n \times \mathbb{Z}_q \times \mathbb{Z}_r$, ${\mathbb{Z}_m[x]}/{(f_1(x), \ldots, f_k(x))}$, and $\mathbb{Z}_m \times {\mathbb{Z}_n[x]}/{(f_1(x), \ldots, f_k(x))}$ for integers m, n, q, and r, and polynomials $f_i(x)$. Below is the code for the Mathematica notebooks to generate the compressed zero-divisor graphs of $\mathbb{Z}_n$, $\mathbb{Z}_m \times \mathbb{Z}_n$, and $\mathbb{Z}_m \times {\mathbb{Z}_n[x]}/{(f_1(x), \ldots, f_k(x))}$. An earlier version of these programs appeared in [44], but have since been adapted or updated.

D.1 Compressed Zero-Divisor Graph of $\mathbb{Z}_n$

```
ZDGraphCompressed[n_] :=
 Module[{ZDList, AnnList, Annihilators, ZeroDivisors, duplicateAnn,
   indices, EdgeLoops, Edge, Loops, LoopedPositions},

  (* This part creates the ZeroDivisors set and Annihilators set. The
  ZeroDivisors set is just a list of the zero-divisors of the ring. The
  Annihilators set is a comprised of sets which are the annihilators of the
  corresponding zero-divisor in ZeroDivisors. For example, in Subscript[
  \[DoubleStruckCapitalZ], 12]:
  ZeroDivisors={2,3,4,6,8,9,10} and
  Annihilators={{6},{4,8},{3,6,9},{2,4,6,8,10},{3,6,9},{4,8},{6}}
   *)
  {ZDList, AnnList} = Last[Reap[Do[If[Mod[i*j, n] == 0,
       Sow[i, ZD];
       Sow[j, Ann];
```

```
    ], {i, n - 1}, {j, n - 1}]]];

Annihilators = Last[Reap[Do[If[ZDList[[i]] == ZDList[[i + 1]],
    If[i + 1 == Length[ZDList],
      Sow[AnnList[[i]], ZDList[[i]]];
      Sow[AnnList[[i + 1]], ZDList[[i]]],
      Sow[AnnList[[i]], ZDList[[i]]];
      ];,
    Sow[AnnList[[i]], ZDList[[i]]];
    If[i + 1 == Length[ZDList],
     Sow[AnnList[[i + 1]], ZDList[[i + 1]]]];
    ], {i, Length[ZDList] - 1}]]];
ZeroDivisors = DeleteDuplicates[ZDList];

(*The next part eliminates the repeats of ZeroDivisors and Annihilators.*)
duplicateAnn =
 GatherBy[Range@Length[Annihilators], Annihilators[[#]] &];
indices = duplicateAnn[[All, 1]];
ZeroDivisors = ZeroDivisors[[indices]];
Annihilators = DeleteDuplicates[Annihilators];

(* This section creates the Edge set.
If an element multiplies to itself to be zero,
then it adds it to the Loops set. *)
EdgeLoops =
 Last[Reap[Do[If[Mod[ZeroDivisors[[i]]*ZeroDivisors[[j]], n] == 0,
    Sow[OverBar[ZeroDivisors[[i]]] -> OverBar[ZeroDivisors[[j]]],
     edge];
    If[i == j,
     Sow[OverBar[ZeroDivisors[[i]]], loops]]
```

```
    ], {i, Length[ZeroDivisors]}, {j, Length[ZeroDivisors]}]]];

 (* This section changes the looped vertices (if there are any) to be
 colored red. This way the graph does not have loops making it easier to
 move the vertices around. *)
 If[Length[EdgeLoops] == 2,
  {Edge, Loops} = EdgeLoops;
  LoopedPositions =
   Flatten[Last[
     Reap[Do[Sow[Position[Edge, Loops[[i]]], i], {i,
        Length[Loops]}]]], 1];
  Do[Edge =
    ReplacePart[Edge,
     LoopedPositions[[i]] -> Style[Loops[[i]], Red]], {i,
    Length[Loops]}];,
  Edge = Flatten[EdgeLoops];
  ];

 (* Graph it *)
 GraphPlot[Edge, VertexLabeling -> True, SelfLoopStyle -> False,
  MultiedgeStyle -> False, PlotLabel ->
\!\(\*SubscriptBox[\(\[CapitalGamma]\), \("\<E\>"\)]\)[
    Subscript[\[DoubleStruckCapitalZ], n]]]
 ]
```

D.2 Compressed Zero-Divisor Graph of $\mathbb{Z}_m \times \mathbb{Z}_n$

```
ZDGraphDoubleCrossCompressed[m_, n_] :=
 Module[{ZDList, AnnList, Annihilators, ZeroDivisors, duplicateAnn,
   indices, EdgeLoops, Edge, Loops, LoopedPositions, f},
  f[{a1_, b1_}, {a2_, b2_}] = {Mod[a1*a2, m], Mod[b1*b2, n]};
```

```
(* This part creates the ZeroDivisors set and Annihilators set. The
ZeroDivisors set is just a list of the zero-divisors of the ring. The
Annihilators set is a comprised of sets which are the annihilators of the
corresponding zero-divisor in ZeroDivisors. For example, in Subscript[
\[DoubleStruckCapitalZ], 12]:
ZeroDivisors={2,3,4,6,8,9,10} and
Annihilators={{6},{4,8},{3,6,9},{2,4,6,8,10},{3,6,9},{4,8},{6}}
 *)
{ZDList, AnnList} =
 Last[Reap[
   Do[If[f[{i1, j1}, {i2, j2}] == {0, 0} && {i1, j1} != {0,
        0} && {i2, j2} != {0, 0},
     Sow[{i1, j1}, ZD];
     Sow[{i2, j2}, Ann];
     ], {i1, 0, m - 1}, {j1, 0, n - 1}, {i2, 0, m - 1}, {j2, 0,
     n - 1}]]];

Annihilators = Last[Reap[Do[If[ZDList[[i]] == ZDList[[i + 1]],
     If[i + 1 == Length[ZDList],
       Sow[AnnList[[i]], ToString[ZDList[[i]]]];
       Sow[AnnList[[i + 1]], ToString[ZDList[[i]]]],
       Sow[AnnList[[i]], ToString[ZDList[[i]]]];
       ];,
     Sow[AnnList[[i]], ToString[ZDList[[i]]]];
     If[i + 1 == Length[ZDList],
      Sow[AnnList[[i + 1]], ToString[ZDList[[i + 1]]]]];
     ], {i, Length[ZDList] - 1}]]];
ZeroDivisors = DeleteDuplicates[ZDList];
```

```
(*The next part eliminates the repeats of ZeroDivisors and Annihilators.*)
duplicateAnn =
 GatherBy[Range@Length[Annihilators], Annihilators[[#]] &];
indices = duplicateAnn[[All, 1]];
ZeroDivisors = ZeroDivisors[[indices]];
Annihilators = DeleteDuplicates[Annihilators];

(* This section creates the Edge set. If an element multiplies to itself
to be zero, then it adds it to the Loops set. *)
EdgeLoops =
 Last[Reap[
   Do[If[f[{ZeroDivisors[[i, 1]],
        ZeroDivisors[[i, 2]]}, {ZeroDivisors[[j, 1]],
        ZeroDivisors[[j, 2]]}] == {0, 0},
     Sow[OverBar[ZeroDivisors[[i]]] -> OverBar[ZeroDivisors[[j]]],
      edge];
     If[i == j,
      Sow[OverBar[ZeroDivisors[[i]]], loops]]
     ], {i, Length[ZeroDivisors]}, {j, Length[ZeroDivisors]}]]];

(* This section changes the looped vertices (if there are any) to be
colored red. This way the graph does not have loops making it easier to
move the vertices around. *)
If[Length[EdgeLoops] == 2,
 {Edge, Loops} = EdgeLoops;
 LoopedPositions =
  Flatten[Last[
    Reap[Do[Sow[Position[Edge, Loops[[i]]], i], {i,
       Length[Loops]}]]], 1];
 Do[Edge =
```

```
    ReplacePart[Edge,
     LoopedPositions[[i]] -> Style[Loops[[i]], Red]], {i,
    Length[Loops]}];,
  Edge = Flatten[EdgeLoops];
  ];

 (* Graph it *)
 GraphPlot[Edge, VertexLabeling -> True, SelfLoopStyle -> False,
  MultiedgeStyle -> False, PlotLabel -> Row[{
\!\(\*SubscriptBox[\(\[CapitalGamma]\), \("\<E\>"\)]\),
     Style["(", Larger], Subscript[\[DoubleStruckCapitalZ], m],
     "\[Times]", Subscript[\[DoubleStruckCapitalZ], n],
     Style[")", Larger], " "}]]
 ]
```

D.3 Compressed Zero-Divisor Graph of $\mathbb{Z}_m \times {}^{\mathbb{Z}_n[x]}/_{(f_1(x),\ldots,f_k(x))}$

```
ZDGraphCrossFactorCompressed[m_, n_, func_] :=
 Module[{NumberOfTerms, CoefficientMatrix, PartRing, Ring,
   ZeroElement, ZDList, AnnList, Annihilators, ZeroDivisors,
   duplicateAnn, indices, EdgeLoops, Edge, Loops, LoopedPositions},
  NumberOfTerms = Max[Exponent[func, x]];

  (* This line creates a set that has all of the necessary coefficients in
  order to make every element of the factor ring. CoefficientMatrix looks
  like: {{0,0,0},{0,0,1},{0,0,2},...,{0,1,0},{0,1,1},...,{n-1,n-1,n-1}}
  (where the length of each element is based on NumberOfTerms). *)
  CoefficientMatrix =
   Tuples[Flatten[
     Last[Reap[Do[Sow[j], {j, 0, n - 1}]]]], {NumberOfTerms + 1}];
```

```
(* These lines create the elements of the factor ring Subscript[
\[DoubleStruckCapitalZ], n][x]/(Subscript[f, 1](x),Subscript[f,2](x),...,
Subscript[f,k](x)) by multiplying CoefficientMatrix to a matrix that looks
like this {{1},{x},{x^2},...,{x^(NumberOfTerms)}}. *)
PartRing =
 Flatten[CoefficientMatrix.Table[{x^i}, {i, 0, NumberOfTerms}]];
PartRing =
 DeleteDuplicates[
  PolynomialMod[
   PolynomialMod[PartRing, func, CoefficientDomain -> Integers], n]];

(* This section takes the elements of PartRing and adds the Subscript[
\[DoubleStruckCapitalZ], m] elements to create the cross product. The ring
element (0,0) is deleted from Ring. *)
Ring = Flatten[
  Last[Reap[
    Do[Sow[{j, PartRing[[i]]}], {j, 0, m - 1}, {i,
      Length[PartRing]}]]], 1];
ZeroElement = Position[Ring, {0, 0}];
Ring = Delete[Ring, ZeroElement];

(* This section creates the ZeroDivisors set and Annihilators set. The
ZeroDivisors set is just a list of the zero-divisors of the ring. The
Annihilators set is a comprised of sets which are the annihilators of the
corresponding zero-divisor in ZeroDivisors. For example, in Subscript[
\[DoubleStruckCapitalZ], 12]:
ZeroDivisors={2,3,4,6,8,9,10} and
Annihilators={{6},{4,8},{3,6,9},{2,4,6,8,10},{3,6,9},{4,8},{6}}
 *)
{ZDList, AnnList} =
```

```
 Last[Reap[
   Do[If[Mod[Ring[[i, 1]]*Ring[[j, 1]], m] == 0 &&
      PolynomialMod[
        PolynomialMod[Ring[[i, 2]]*Ring[[j, 2]], func,
         CoefficientDomain -> Integers], n] === 0,
     Sow[Ring[[i]], ZD];
     Sow[Ring[[j]], Ann];
     ], {i, Length[Ring]}, {j, Length[Ring]}]]];

Annihilators = Last[Reap[Do[If[ZDList[[i]] === ZDList[[i + 1]],
     If[i + 1 == Length[ZDList],
       Sow[AnnList[[i]], ToString[ZDList[[i]]]];
       Sow[AnnList[[i + 1]], ToString[ZDList[[i]]]],
       Sow[AnnList[[i]], ToString[ZDList[[i]]]];
       ];,
     Sow[AnnList[[i]], ToString[ZDList[[i]]]];
     If[i + 1 == Length[ZDList],
      Sow[AnnList[[i + 1]], ToString[ZDList[[i + 1]]]]]];
     ], {i, Length[ZDList] - 1}]]];
ZeroDivisors = DeleteDuplicates[ZDList];

(*The next part eliminates the repeats of ZeroDivisors and Annihilators.*)
duplicateAnn =
 GatherBy[Range@Length[Annihilators], Annihilators[[#]] &];
indices = duplicateAnn[[All, 1]];
ZeroDivisors = ZeroDivisors[[indices]];
Annihilators = DeleteDuplicates[Annihilators];

(* This section creates the Edge set. If an element multiplies to itself
to be zero, then it adds it to the Loops set. *)
```

```
EdgeLoops =
 Last[Reap[
   Do[If[Mod[ZeroDivisors[[i, 1]]*ZeroDivisors[[j, 1]], m] == 0 &&
      PolynomialMod[
        PolynomialMod[ZeroDivisors[[i, 2]]*ZeroDivisors[[j, 2]],
         func, CoefficientDomain -> Integers], n] === 0,
     Sow[OverBar[ZeroDivisors[[i]]] -> OverBar[ZeroDivisors[[j]]],
      edge];
     If[OverBar[ZeroDivisors[[i]]] === OverBar[ZeroDivisors[[j]]],
      Sow[OverBar[ZeroDivisors[[i]]], loops]]
     ], {i, Length[ZeroDivisors]}, {j, Length[ZeroDivisors]}]]];

(* This section changes the looped vertices (if there are any) to be
colored red. This way the graph does not have loops making it easier to
move the vertices around. *)
If[Length[EdgeLoops] == 2,
 {Edge, Loops} = EdgeLoops;
 LoopedPositions =
  Flatten[Last[
    Reap[Do[Sow[Position[Edge, Loops[[i]]], i], {i,
       Length[Loops]}]]], 1];
 Do[Edge =
   ReplacePart[Edge,
    LoopedPositions[[i]] -> Style[Loops[[i]], Red]], {i,
   Length[Loops]}];,
 Edge = Flatten[EdgeLoops];
 ];

(* Graph it *)
GraphPlot[Edge, VertexLabeling -> True, SelfLoopStyle -> False,
```

```
   MultiedgeStyle -> False, PlotLabel -> Row[{
\!\(\*SubscriptBox[\(\[CapitalGamma]\), \("\<E\>"\)]\),
      Style["(", Larger], Subscript[\[DoubleStruckCapitalZ], m],
      "\[Times]", Subscript[\[DoubleStruckCapitalZ], n], Style["["],
      x, Style["]"], Style["/", Larger], Style["("], func, Style[")"],
       Style[")", Larger]}, " "]]
  ]
```

E Mathematica Notebooks for Annihilator Graphs

Below are the Mathematica notebooks created to generate several examples and pictures to study and illustrate for various types of rings. These Mathematica programs generate the annihilator graphs of the following types of rings: $\mathbb{Z}_n$, $\mathbb{Z}_m \times \mathbb{Z}_n$, $\mathbb{Z}_m \times \mathbb{Z}_n \times \mathbb{Z}_q$, ${}^{\mathbb{Z}_n[x]}/_{(f_1(x),\ldots,f_k(x))}$, and $\mathbb{Z}_m \times {}^{\mathbb{Z}_n[x]}/_{(f_1(x),\ldots,f_k(x))}$ for integers m, n, and q, and polynomials $f_i(x)$. Below is the code for the Mathematica notebooks to generate the annihilator graphs of $\mathbb{Z}_n$, $\mathbb{Z}_m \times \mathbb{Z}_n$, and ${}^{\mathbb{Z}_n[x]}/_{(f_1(x),\ldots,f_k(x))}$.

E.1 Annihilator Graph of $\mathbb{Z}_n$

```
AnnGraph[n_] :=
 Module[{ZDList, AnnList, Annihilators, ZeroDivisors, Edge, union, xy,
    pos},

  (* This part creates the ZeroDivisors set and Annihilators set. The
  ZeroDivisors set is just a list of the zero-divisors of the ring. The
  Annihilators set is a comprised of sets which are the annihilators of the
  corresponding zero-divisor in ZeroDivisors. For example, in Subscript[
  \[DoubleStruckCapitalZ], 12]:
  ZeroDivisors={2,3,4,6,8,9,10} and
  Annihilators={{6},{4,8},{3,6,9},{2,4,6,8,10},{3,6,9},{4,8},{6}}
   *)
  {ZDList, AnnList} = Last[Reap[Do[If[Mod[i*j, n] == 0,
       Sow[i, ZD];
       Sow[j, Ann];
       ], {i, n - 1}, {j, n - 1}]]];

  Annihilators = Last[Reap[Do[If[ZDList[[i]] == ZDList[[i + 1]],
       If[i + 1 == Length[ZDList],
         Sow[AnnList[[i]], ZDList[[i]]];
```

```
       Sow[AnnList[[i + 1]], ZDList[[i]]],
       Sow[AnnList[[i]], ZDList[[i]]];
       ];,
     Sow[AnnList[[i]], ZDList[[i]]];
     If[i + 1 == Length[ZDList],
      Sow[AnnList[[i + 1]], ZDList[[i + 1]]]];
     ], {i, Length[ZDList] - 1}]]];
ZeroDivisors = DeleteDuplicates[ZDList];

(* This next section sorts the Annihilators set from the smallest to the
largest and keeps the relationship between Annihilators and ZeroDivisors
set intact. *)
ZeroDivisors = ZeroDivisors[[Ordering[Annihilators]]];
Annihilators = Sort[Annihilators];

(* The next part populates the Edge set and creates the edges between the
vertices. *)
Edge = Flatten[
  Last[Reap[Do[union = Union[Annihilators[[i]], Annihilators[[j]]];
     xy = Mod[ZeroDivisors[[i]]*ZeroDivisors[[j]], n];
     If[xy == 0,
      Sow[ZeroDivisors[[i]] -> ZeroDivisors[[j]]];,
      pos = Position[ZeroDivisors, xy][[1]][[1]];
      If[Complement[Annihilators[[pos]], union] != {},
       Sow[ZeroDivisors[[i]] -> ZeroDivisors[[j]]];
       ];
      ], {i, Length[ZeroDivisors]}, {j, i, Length[ZeroDivisors]}]]]];

(* Graph it *)
GraphPlot[Edge, VertexLabeling -> True, SelfLoopStyle -> False,
```

```
  MultiedgeStyle -> False,
  PlotLabel -> AG[Subscript[\[DoubleStruckCapitalZ], n]]]
 ]
```

E.2 Annihilator Graph of $\mathbb{Z}_m \times \mathbb{Z}_n$

```
AnnGraphDoubleCross[m_, n_] :=
 Module[{f, ZDList, AnnList, Annihilators, ZeroDivisors, Edge, union,
   xy, pos},
  f[{a1_, b1_}, {a2_, b2_}] = {Mod[a1*a2, m], Mod[b1*b2, n]};

  (* This part creates the ZeroDivisors set and Annihilators set. The
  ZeroDivisors set is just a list of the zero-divisors of the ring. The
  Annihilators set is a comprised of sets which are the annihilators of the
  corresponding zero-divisor in ZeroDivisors. For example, in Subscript[
  \[DoubleStruckCapitalZ], 12]:
  ZeroDivisors={2,3,4,6,8,9,10} and
  Annihilators={{6},{4,8},{3,6,9},{2,4,6,8,10},{3,6,9},{4,8},{6}}
   *)
  {ZDList, AnnList} =
   Last[Reap[
     Do[If[f[{i1, j1}, {i2, j2}] == {0, 0} && {i1, j1} != {0,
          0} && {i2, j2} != {0, 0},
       Sow[{i1, j1}, ZD];
       Sow[{i2, j2}, Ann];
       ], {i1, 0, m - 1}, {j1, 0, n - 1}, {i2, 0, m - 1}, {j2, 0,
       n - 1}]]];

  Annihilators = Last[Reap[Do[If[ZDList[[i]] == ZDList[[i + 1]],
       If[i + 1 == Length[ZDList],
         Sow[AnnList[[i]], ToString[ZDList[[i]]]];
```

```
      Sow[AnnList[[i + 1]], ToString[ZDList[[i]]]],
      Sow[AnnList[[i]], ToString[ZDList[[i]]]];
      ];,
    Sow[AnnList[[i]], ToString[ZDList[[i]]]];
    If[i + 1 == Length[ZDList],
     Sow[AnnList[[i + 1]], ToString[ZDList[[i + 1]]]]];
    ], {i, Length[ZDList] - 1}]]];
ZeroDivisors = DeleteDuplicates[ZDList];

(* This next section sorts the Annihilators set from the smallest to the
largest and keeps the relationship between Annihilators and ZeroDivisors
set intact. *)
ZeroDivisors = ZeroDivisors[[Ordering[Annihilators]]];
Annihilators = Sort[Annihilators];

(* The next part populates the Edge set and creates the edges between the
vertices. *)
Edge = Flatten[
  Last[Reap[Do[union = Union[Annihilators[[i]], Annihilators[[j]]];
    xy = f[ZeroDivisors[[i]], ZeroDivisors[[j]]];
    If[xy == {0, 0},
     Sow[ZeroDivisors[[i]] -> ZeroDivisors[[j]]];,
     pos = Position[ZeroDivisors, xy][[1]][[1]];
     If[Complement[Annihilators[[pos]], union] != {},
      Sow[ZeroDivisors[[i]] -> ZeroDivisors[[j]]];
      ];
     ], {i, Length[ZeroDivisors]}, {j, i, Length[ZeroDivisors]}]]]];

(* Graph it *)
GraphPlot[Edge, VertexLabeling -> True, SelfLoopStyle -> False,
```

```
  MultiedgeStyle -> False,
  PlotLabel ->
   Row[{AG, Style["(", Larger], Subscript[\[DoubleStruckCapitalZ],
     m], "\[Times]", Subscript[\[DoubleStruckCapitalZ], n],
     Style[")", Larger], " "}]]
 ]
```

E.3 Annihilator Graph of $\mathbb{Z}_n[x]/(f_1(x), \ldots, f_k(x))$

```
AnnGraphFactor[n_, func_] :=
 Module[{NumberOfTerms, CoefficientMatrix, Ring, ZeroElement, ZDList,
   AnnList, Annihilators, ZeroDivisors, Edge, union, xy, pos},
  NumberOfTerms = Max[Exponent[func, x]];

  (* This line creates a set that has all of the necessary coefficients in
  order to make every element of the factor ring. CoefficientMatrix looks
  like: {{0,0,0},{0,0,1},{0,0,2},...,{0,1,0},{0,1,1},...,{n-1,n-1,n-1}}
  (where the length of each element is based on NumberOfTerms). *)
  CoefficientMatrix =
   Tuples[Flatten[
     Last[Reap[Do[Sow[j], {j, 0, n - 1}]]]], {NumberOfTerms + 1}];

  (* These lines create the elements of the factor ring Subscript[
  \[DoubleStruckCapitalZ], n][x]/(Subscript[f, 1](x),Subscript[f,2](x),...,
  Subscript[f,k](x)) by multiplying CoefficientMatrix to a matrix that looks
  like this {{1},{x},{x^2},...,{x^(NumberOfTerms)}}. *)
  Ring = Flatten[
    CoefficientMatrix.Table[{x^i}, {i, 0, NumberOfTerms}]];
  Ring = DeleteDuplicates[
    PolynomialMod[
     PolynomialMod[Ring, func, CoefficientDomain -> Integers], n]];
```

```
ZeroElement = Position[Ring, 0];
Ring = Delete[Ring, ZeroElement];

(* This section creates the ZeroDivisors set and Annihilators set. The
ZeroDivisors set is just a list of the zero-divisors of the ring. The
Annihilators set is a comprised of sets which are the annihilators of the
corresponding zero-divisor in ZeroDivisors. For example, in Subscript[
\[DoubleStruckCapitalZ], 12]:
ZeroDivisors={2,3,4,6,8,9,10} and
Annihilators={{6},{4,8},{3,6,9},{2,4,6,8,10},{3,6,9},{4,8},{6}}
 *)
{ZDList, AnnList} =
 Last[Reap[
   Do[If[PolynomialMod[
       PolynomialMod[Ring[[i]]*Ring[[j]], func,
        CoefficientDomain -> Integers], n] === 0,
     Sow[Ring[[i]], ZD];
     Sow[Ring[[j]], Ann];
     ], {i, Length[Ring]}, {j, Length[Ring]}]]];

Annihilators = Last[Reap[Do[If[ZDList[[i]] === ZDList[[i + 1]],
     If[i + 1 == Length[ZDList],
       Sow[AnnList[[i]], ToString[ZDList[[i]]]];
       Sow[AnnList[[i + 1]], ToString[ZDList[[i]]]],
       Sow[AnnList[[i]], ToString[ZDList[[i]]]];
       ];,
     Sow[AnnList[[i]], ToString[ZDList[[i]]]];
     If[i + 1 == Length[ZDList],
      Sow[AnnList[[i + 1]], ToString[ZDList[[i + 1]]]]];
     ], {i, Length[ZDList] - 1}]]];
```

```
ZeroDivisors = DeleteDuplicates[ZDList];

(* This next section sorts the Annihilators set from the smallest to the
largest and keeps the relationship between Annihilators and ZeroDivisors
set intact. *)
ZeroDivisors = ZeroDivisors[[Ordering[Annihilators]]];
Annihilators = Sort[Annihilators];

(* The next part populates the Edge set and creates the edges between the
vertices. *)
Edge = Flatten[
  Last[Reap[Do[union = Union[Annihilators[[i]], Annihilators[[j]]];
     xy =
      PolynomialMod[
       PolynomialMod[ZeroDivisors[[i]]*ZeroDivisors[[j]], func,
        CoefficientDomain -> Integers], n];
     If[xy === 0,
      Sow[ZeroDivisors[[i]] -> ZeroDivisors[[j]]];,
      pos = Position[ZeroDivisors, xy][[1]][[1]];
      If[Complement[Annihilators[[pos]], union] != {},
       Sow[ZeroDivisors[[i]] -> ZeroDivisors[[j]]];
       ];
      ], {i, Length[ZeroDivisors]}, {j, i, Length[ZeroDivisors]}]]]];

(* Graph it *)
GraphPlot[Edge, VertexLabeling -> True, SelfLoopStyle -> False,
 MultiedgeStyle -> False,
 PlotLabel ->
  Row[{AG, Style["(", Larger], Subscript[\[DoubleStruckCapitalZ],
    n], Style["["], x, Style["]"], Style["/", Larger], Style["(",
```

```
    func, Style[")"], Style[")", Larger]}, " "]]
]
```

Vita

Darrin Weber was born and raised in Bloomington, Indiana. Growing up in the Hoosier state gave him a passion for the game of basketball, which led him to play at Millikin University. It was there under the direction of Dr. Joe Stickles, Jr. that he developed a love for mathematics, the teaching of mathematics, and mathematical research. This was further developed at an REU at Wabash College in the summer of 2009 under Dr. Mike Axtell. Coupled with the marriage to his amazing wife, Melissa, and the birth of their daughter, Natalie, Darrin decided to pursue mathematics instead of a coaching career and entered into graduate school at the University of Tennessee in August of 2011. During his time at UT, Darrin added a Master in Statistics and will be joining the Math Department at the University of Evansville upon graduation in August 2017.