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K. C. Reddy, Major Professor

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4

A FINITE DIFFERENCE SCHEME FOR POISSON'S EQUATION USING GAUSSIAN QUADRATURE

A Thesis

Presented for the

Master of Science

Degree

The University of Tennessee, Knoxville

Frances Caroline Wilson

August 1976

1288118

ACKNOWLEDGMENTS

The author sincerely expresses her gratitude to Dr. K. C. Reddy for his help and guidance in preparing this work. Sincere appreciation and thanks go to Dr. Terry Feagin, Dr. Kenneth Kimble, Mr. Prasad Mikkilineni, and Mr. F. C. Loper for their valuable suggestions. A special thank you goes to Mr. W. C. Armstrong for supervising the work of a Research Assistant.

The author would especially like to express her sincere appreciation to ARO, Inc. and the officials of the Air Force of Arnold Engineering Development Center, Arnold Air Force Station, Tennessee, for their encouragement, assistance, and use of the computer.

The work reported herein was sponsored by Arnold Research Organization, Inc. and Arnold Engineering Development Center, Air Force Systems Command, under Contract Number ARO 76-7-TD/SE with The University of Tennessee Space Institute. Reproduction as required by the Government is authorized.

ABSTRACT

Finite difference methods are often used for solving partial differential equations such as Poisson's equation. The accuracy of these schemes depends on the number of grid points. The present thesis is an exploratory work to investigate a new finite difference scheme which is derived by the use of the weighted residual method and Gauss-Lobatto quadratures. The objective is to obtain a high degree of accuracy in the difference scheme with a given number of grid points chosen.

Poisson's equation has been investigated in two forms; one, a direct formulation of the problem, and two, a Cauchy-Riemann type of system. Both Dirichlet problems and Neumann problems are considered. Three different classes of forcing functions have been used to evaluate and test the proposed method. Comparisons have been made between solutions obtained by this method and solutions obtained from a standard five point finite difference scheme. In most cases, the combination of weighted residuals and Gauss-Lobatto quadrature yield a more accurate solution for a given number of points than the five point finite difference scheme.

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CHAPTER I

INTRODUCTION

Partial differential equations, such as Poisson's equation, are often solved by finite difference schemes. In these schemes, the domain is usually divided into an appropriate system of grid points, usually equally spaced. The partial differential equation is approximated at these points by difference equations, and the problem is reduced to solving a system of algebraic equations. The accuracy of these finite difference schemes depends on the grid size. In order to achieve a desired accuracy, a large number of points must often be used so that a small enough grid can be produced. These numerous points required for the small grids may produce storage problems. The desired accuracy cannot always be attained by using a reasonable number of points. A method is presented here, which by using the abscissas of Gaussian quadrature, requires less points than the traditional finite difference scheme for the same accuracy.

In the proposed scheme, the principal idea is the replacement of an equally spaced grid by a suitable non-uniform grid. Difference equations are formulated for the problem by the use of the weighted residual method and Gaussian quadrature formulae. It is well known that for definite integrals, the Gaussian quadrature formulae

generally produce more accurate solutions than the Newton-Cotes formulae for the same number of points. We propose that difference schemes based on Gaussian quadrature formulae be used as a basis for solving partial differential equations with the expectation that this should produce better accuracy for the same number of points than the classical finite difference scheme.

Chandrasekhar (1)[1] has used this idea of Gaussian quadratures extensively in solving integral equations and has indicated a technique for integro-differential equations where an integral and a derivative occur with respect to the same variable. Crawford (2) has extended Chandrasekhar's work to devise highly accurate finite difference schemes for the solution of ordinary differential equations. Using this method, she expresses derivatives of a function in terms of the values of that function at the Gaussian quadrature points. These difference formulae, derived from Gaussian quadrature, are in turn used to improve the finite difference schemes used in solving differential equations. Crawford has investigated a number of second order equations and compared their solutions to the solutions obtained from the traditional equal-interval central difference approximations. She has also investigated the different types of Gaussian quadrature formulae for optimizing the finite

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[1]Numbers in parentheses refer to similarly numbered references in the Bibliography.

difference schemes for different types of problems. Among the problems she has studied are one-dimensional ordinary differential equations, a nonlinear ordinary differential equation, and turning point problems.

The method proposed here is a combination of the method of weighted residuals and Gaussian quadratures. In the weighted residual method, a trial solution is taken, substituted into the differential equation, and the residual is integrated with some weight functions. If the trial function were the exact solution, the residual would be zero. As it is, however, weighted integrals of the residual are set to zero which, in turn, produce algebraic equations for the parameters in the trial function. The weighting functions can be chosen in many ways and each choice produces a different form of the method of weighted residuals. For example, in the collocation method, certain Dirac delta functions are the weighting functions. According to Finlayson (3), one of the best known approximation methods is due to Galerkin, who uses the basis functions of the trial functions as the weighting functions. In the method proposed here, integrals of the Legendre polynomials are chosen as the weighting functions. However, in this method there is no need to write an explicit trial function.

We approximate the weighted residual equations by a Gaussian quadrature formula. The resulting difference equation is a function of all the points of the grid. Enough weighting functions are taken so that the proper

number of algebraic equations is formed. This system of algebraic equations is then solved. This is the method which is presented in this work.

In solving Poisson's equation with this method two formulations of the problem have been made. The first formulation is a direct application of the method of weighted residuals to Poisson's equation. In the second formulation Poisson's equation is rewritten as a system of first order equations, and then the weighted residual method is applied. Both forms are then approximated by the Gaussian quadratures and the system of algebraic equations is solved. Though a number of different types of Gaussian quadrature formulae exist, the same formula, Gauss-Lobatto, is used throughout this work, because of the simplicity in applying boundary conditions with Gauss-Lobatto quadrature.

Many test problems are solved with this scheme. A number of different functions are utilized for the forcing function. Polynomials of successively higher degrees have been used for testing the scheme and also estimating the order of accuracy of the scheme. Exponential forms are also used as well as combinations of the cosine functions. Both formulations of Poisson's equation are solved for all these test cases. These test cases are all chosen such that exact solutions are known. Solutions are also obtained from the traditional five point finite difference scheme in all the cases, and the solutions obtained by the present scheme and

five point difference scheme are compared with exact solutions.

In Chapter II, a general mathematical analysis is presented including the derivations of the problem, error analysis, and the problem of linear dependency. Chapter III presents some specific examples and the results obtained, as well as comparisons with the regular five point finite difference scheme. It is shown that the proposed method produces a better accuracy for the same number of points in almost all cases than the five point finite difference scheme.

CHAPTER II

MATHEMATICAL ANALYSIS OF THE PROBLEM

I. FORMULATION OF THE PROBLEM

The difference equations for the present scheme, which are a combination of the weighted residual method and the Gauss-Lobatto quadrature, are formed in the following manner.

Let $\phi(x,y)$ be a function with continuous derivatives in the rectangular domain $-1 \leq x \leq 1$, $-1 \leq y \leq 1$. Let $P_L(x)$ be the Legendre polynomial of order L and let $Q_L(x)$ be defined by the relations

$$dQ_L/dx = -P_L(x), \quad Q_L(\pm 1) = 0 \tag{1}$$

Using the properties of Legendre polynomials it can be shown that

$$Q_L(x) = 1/(2L + 1)\ [P_{L-1}(x) - P_{L+1}(x)] \tag{2}$$

Let us further define

$$Q_{m,n}(x,y) = Q_m(x)Q_n(y) \tag{3}$$

Again using the properties of Legendre polynomials and Equation 1 it can be shown

$$\partial Q_{m,n}/\partial x = -P_m(x)Q_n(y), \quad \partial Q_{m,n}/\partial y = -Q_m(x)P_n(y) \tag{4}$$

The equation we study in this work is Poisson's equation.

$$\nabla^2\phi = F(x,y) \tag{5}$$

For our first formulation let us rewrite Equation 5 into a system of two first order equations. Letting

$$U = \phi_x,\ V = \phi_y \tag{6}$$

we have

$$U_x + V_y = F(x,y) \tag{7.1}$$

$$U_y - V_x = 0 \tag{7.2}$$

as our system of first order equations. We call this a Cauchy-Riemann formulation. For Equation 7.1 the weighted residual method is applied as follows. Multiply Equation 7.1 by $Q_{m,n}(x,y)$, the weighting function, and integrate over the square domain $(-1 \le x \le 1) \times (-1 \le y \le 1)$.

$$\int_{-1}^{1}\int_{-1}^{1}\left(\frac{\partial U}{\partial x} + \frac{\partial V}{\partial y}\right)Q_{m,n}(x,y)\,dxdy = \int_{-1}^{1}\int_{-1}^{1}F(x,y)Q_{m,n}(x,y)\,dxdy$$

Integration by parts gives us

$$\int_{-1}^{1}\left((U\ Q_{m,n})_{x=-1}^{x=1} - \int_{-1}^{1}U\ \frac{\partial Q_{m,n}}{\partial x}\,dx\right)dy + \int_{-1}^{1}\left((V\ Q_{m,n})_{y=-1}^{y=1}\right.$$

$$\left. - \int_{-1}^{1}V\ \frac{\partial Q_{m,n}}{\partial y}\,dy\right)dx = \int_{-1}^{1}\int_{-1}^{1}F(x,y)Q_{m,n}(x,y)\,dxdy$$

Since $Q_m(\pm 1) = Q_n(\pm 1) = 0$, we have

$$-\int_{-1}^{1}\int_{-1}^{1} U \frac{\partial Q_{m,n}}{\partial x}\, dxdy - \int_{-1}^{1}\int_{-1}^{1} V \frac{\partial Q_{m,n}}{\partial y}\, dxdy$$

$$= \int_{-1}^{1}\int_{-1}^{1} F(x,y) Q_{m,n}(x,y)\, dxdy$$

Using the properties (Equation 4), this equation becomes

$$\int_{-1}^{1}\int_{-1}^{1} U(x,y) P_m(x) Q_n(y)\, dxdy + \int_{-1}^{1}\int_{-1}^{1} V(x,y) Q_m(x) P_n(y)\, dxdy$$

$$= \int_{-1}^{1}\int_{-1}^{1} F(x,y) Q_m(x) Q_n(y)\, dxdy \qquad (8.1)$$

Similarly, we apply this procedure to Equation 7.2 and arrive at

$$\int_{-1}^{1}\int_{-1}^{1} U(x,y) Q_m(x) P_n(y)\, dxdy - \int_{-1}^{1}\int_{-1}^{1} V(x,y) P_m(x) Q_n(y)\, dxdy = 0 \qquad (8.2)$$

Now we approximate the integral Equations 8.1 and 8.2 by a Gauss-Lobatto quadrature and our equations become:

$$\sum_{j=1}^{J}\sum_{i=1}^{I} w_i w_j\, U(x_i,y_j) P_m(x_i) Q_n(y_j) + \sum_{j=1}^{J}\sum_{i=1}^{I} w_i w_j\, V(x_i,y_j)$$

$$\times\, Q_m(x_i) P_n(y_j) = \sum_{j=1}^{J}\sum_{i=1}^{I} w_i w_j\, F(x_i,y_j) Q_m(x_i) Q_n(y_j) \qquad (9.1)$$

$$\sum_{j=1}^{J} \sum_{i=1}^{I} w_i w_j \, U(x_i,y_j) Q_m(x_i) P_n(y_j)$$

$$- \sum_{j=1}^{J} \sum_{i=1}^{I} w_i w_j \, V(x_i,y_j) P_m(x_i) Q_n(y_j) = 0 \qquad (9.2)$$

where $w_1, w_2, \ldots, w_I$ and $w_1, w_2, \ldots, w_J$ are the weighting coefficients of the Gauss-Lobatto quadrature and $x_1, x_2, \ldots, x_I$ and $y_1, y_2, \ldots, y_J$ are the abscissas of Gauss-Lobatto quadrature available in any standard table. In the present exploratory work, we restrict ourselves to a square domain, thus the weights and quadrature points are the same in both directions. We allow m and n to range from one to N where N is the number of abscissas of the particular Gauss-Lobatto quadrature used. Thus, we will have $2N^2$ linear algebraic equations in $2N^2$ variables, since at each quadrature point both U and V must be evaluated. These two equations are the difference equations for the first derivatives of ϕ in terms of all the points in the grid.

For the direct formulation, our second method, we have

$$\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} = F(x,y) \qquad (10)$$

and in the weighted residual form

$$\int_{-1}^{1} \int_{-1}^{1} \left(\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} \right) Q_{m,n}(x,y) \, dxdy = \int_{-1}^{1} \int_{-1}^{1} F(x,y) Q_{m,n}(x,y) \, dxdy \qquad (11)$$

Using the same procedure as we did in integrating Equation 7.1, we arrive at an integral equation of the form:

$$-\int_{-1}^{1}\int_{-1}^{1}\phi(x,y)P_m'(x)Q_n(y)\,dxdy - \int_{-1}^{1}\int_{-1}^{1}\phi(x,y)P_n'(y)Q_m(x)\,dxdy$$

$$= \int_{-1}^{1}\int_{-1}^{1}F(x,y)Q_m(x)Q_n(y)\,dxdy + \int_{-1}^{1}[\phi(-1,y)P_m(-1)$$

$$- \phi(1,y)P_m(1)]Q_n(y)\,dy$$

$$+ \int_{-1}^{1}[\phi(x,-1)P_n(-1) - \phi(x,1)P_n(1)]Q_m(x)\,dx \qquad (12)$$

Again, $w_1, w_2 \ldots w_I$ and $w_1, w_2 \ldots w_J$ are the weighting coefficients of the Gauss-Lobatto quadrature and $x_1, x_2 \ldots x_I$ and $y_1, y_2 \ldots y_J$ are the abscissas of the Gauss-Lobatto quadrature. Likewise, again we range m and n from one to N where N is the number of quadrature points in the particular Gauss-Lobatto quadrature used. Thus, here we have a set of N^2 equations in N^2 unknowns.

In the direct formulation, we have ϕ prescribed on the boundary which presents us with a first boundary value problem, or Dirichlet problem. For the system of Equations 7.1 and 7.2, U and V are prescribed on the boundary and thus, we have a Neumann problem. In fact U is prescribed along $x = \pm 1$ and V is prescribed along $y = \pm 1$. For either formulation, once the set of x equations in x unknowns is formed

we apply the boundary conditions, and in the case of the Dirichlet problem the result is a system of N^2 equations in $(N - 2)^2$ unknowns. In the Neumann problem, we have $2N^2$ equations in $2N^2 - 4N$ unknowns. By the algorithm described in the Appendix we select $(N - 2)^2$ linearly independent equations from the N^2 equations for the Dirichlet problem and $2N^2 - 4N$ linearly independent equations from the $2N^2$ equations for the Neumann problem. In either case this gives us a system of $\bar{x}$ linearly independent equations in $\bar{x}$ unknowns. This system is then solved by the solution algorithm given in the Appendix for selecting the linearly independent equations.

In summary, then, two distinct steps exist in the numerical solution of Poisson's equation by the procedure developed here. The first step is the rewriting of the equation, in one of the two formulations, in the weighted residual form with $Q_{m,n}(x,y)$ as the weighting functions. After an integration by parts and an appropriate evaluation, we have an integral equation involving the function, the Legendre polynomials, the related function $Q_{m,n}(x,y)$, and the derivative of the Legendre polynomials. This integral equation is in terms of the values of either the first derivatives of the function or the function itself. The second step involves the approximation of this integral equation by an appropriate Gauss-Lobatto quadrature. The resulting difference equations are in terms of the first derivatives or the function itself at the quadrature points

depending on whether the problem is a Neumann problem or a Dirichlet problem. The resulting system of algebraic equations is solved by the procedure described in the Appendix to get the appropriate results.

Two formulations of Poisson's equation are made in order to easily apply the boundary conditions. For the direct formulation we have expressions in terms of the function itself with boundary conditions also in terms of the function. This presents us with a Dirichlet problem. With the direct formulation we could prescribe the first derivatives on the boundary, which would produce a Neumann problem. However, this creates more work in approximating the boundary conditions. Likewise, for the Cauchy-Riemann formulation, since we are solving for U and V, ϕ_x and ϕ_y respectively, we prescribe the derivatives on the boundary and create a Neumann problem. For this problem we could prescribe the function on the boundary instead of the first derivatives, but again approximating the boundary conditions would be more difficult.

The weighting function, $Q_{m,n}(x,y)$, the integral of the Legendre polynomial is chosen for its property on the boundary. This property simplifies the weighted residual equation after it has been integrated by parts. The boundary terms after integration by parts drop out.

II. LINEAR DEPENDENCY

One of the problems that arises with the N^2 equations in the Dirichlet problem or the $2N^2$ equations in the Neumann problem is that of a row of zeroes. The fact that some equations have coefficients that are all zeroes occurs as the result of the definition of the abscissas of Gauss-Lobatto quadrature and our choice of weight functions. In Gauss-Lobatto quadrature, x_i is the ith zero of $P'_{n-1}(x)$ where n is the number of points in the Gauss-Lobatto quadrature (4). It can be shown (1) that the relationship between $Q_L(x)$ and $P'_L(x)$ is

$$Q_L(x) = (1 - x^2)/L(L + 1)\ P'_L(x)$$

Thus, x_i is the ith zero of $Q_{n-1}(x)$ as well as of $P'_{n-1}(x)$. Let us take a moment of the form $(\bar{n} - 1, \bar{n} - 1)$ where $\bar{n}$ is the number of points in the Gauss-Lobatto quadrature used. Then an equation of the form of Equation 9.2, for example, would be the following

$$\sum_{j=1}^{J} \sum_{i=1}^{I} w_i w_j\ U(x_i,y_j) P_{\bar{n}-1}(x_i) Q_{\bar{n}-1}(y_j)$$
$$- \sum_{j=1}^{J} \sum_{i=1}^{I} w_i w_j\ V(x_i,y_j) Q_{\bar{n}-1}(x_i) P_{\bar{n}-1}(y_j) = 0$$

In the equation the coefficients of both $V(x_i,y_j)$ and $U(x_i,y_j)$ will drop out leaving zeroes as the coefficients. This produces a row of zeroes in the system of equations

which is eliminated by the algorithm described in the Appendix.

Another problem which results from the forming of the proper number of equations is that of linear dependency. Some of the equations formed turn out to be linear combinations of the other equations. The reason for this linear dependency is still unclear, but this difficulty is overcome by the algorithm used.

III. ERROR ANALYSIS

Let us look at one of the equations in the Neumann problem

$$\int_{-1}^{1}\int_{-1}^{1} U(x,y)P_m(x)Q_n(y)dxdy + \int_{-1}^{1}\int_{-1}^{1} V(x,y)Q_m(x)P_n(y)dxdy$$

$$= \int_{-1}^{1}\int_{-1}^{1} F(x,y)Q_m(x)Q_n(y)dxdy$$

and calculate the order of accuracy of the difference equation obtained from it by Gauss-Lobatto quadrature. Let ϕ be a polynomial of degree (x^{α},y^{β}). Then, let the degree of $U(x,y)$ be represented by $\theta(x^{\alpha-1},y^{\beta})$ since $U = \phi_x$ and the degree of $V(x,y)$ be rèpresented by $\theta(x^{\alpha},y^{\beta-1})$ since $V = \phi_y$. Likewise, the degree of $F(x,y)$ would be $\theta(x^{\alpha-2},y^{\beta-2})$ since $F(x,y) = \phi_{xx} + \phi_{yy}$. From the properties of Legendre polynomials we know that the degree of $P_m(x)$ is m. We also know that the degree of $P'_m(x)$ is m - 1 and the degree of $Q_n(x)$ is

$n + 1$. We also know (4) that the order of approximation by Gauss-Lobatto quadrature is $2\bar{m} - 3$ where $\bar{m}$ is the number of quadrature points of the particular Gauss-Lobatto quadrature. Thus, for the first term on the left-hand side we have

$$\alpha - 1 + m = 2\bar{m} - 3$$

$$\alpha = 2\bar{m} - m - 2$$

and

$$\beta + n + 1 = 2\bar{n} - 3$$

$$\beta = 2\bar{n} - n - 4$$

Here,

$$\alpha = \text{Min } (2\bar{m} - m - 2)$$

and

$$\beta = \text{Min } (2\bar{n} - n - 4)$$

The second term is treated in a like manner and we obtain

$$\alpha = \text{Min } (2\bar{m} - m - 4)$$

and

$$\beta = \text{Min } (2\bar{n} - n - 2)$$

The order of accuracy of an equation such as this is the order of accuracy of the least accurate term. Thus the order of accuracy of the left-hand side is

$$\alpha = \text{Min } (2\bar{m} - m - 4)$$

$$\beta = \text{Min } (2\bar{n} - n - 4) \qquad (15)$$

By a similar analysis of the right-hand side we find the order of accuracy to be

$$\alpha = \text{Min } (2\bar{m} - m - 2)$$

$$\beta = \text{Min } (2\bar{n} - n - 2)$$

Therefore the order of accuracy for this equation is given by Equation 15. A similar analysis is done on the other equation in the Neumann problem

$$\int_{-1}^{1}\int_{-1}^{1} U(x,y)Q_m(x)P_n(y)\,dxdy - \int_{-1}^{1}\int_{-1}^{1} V(x,y)P_m(x)Q_n(y)\,dxdy = 0$$

and we find that the order of accuracy for it is also given by Equation 15.

Likewise, the procedure described above is applied to the equation in the Dirichlet problem

$$-\int_{-1}^{1}\int_{-1}^{1} \phi(x,y)P_m'(x)Q_n(y) - \int_{-1}^{1}\int_{-1}^{1} \phi(x,y)P_n'(y)Q_m(x)\,dxdy$$

$$= \int_{-1}^{1}\int_{-1}^{1} F(x,y)Q_m(x)Q_n(y)\,dxdy$$

$$+ \int_{-1}^{1} [\phi(-1,y)P_m(-1) - \phi(1,y)P_m(1)]Q_n(y)\,dy$$

$$+ \int_{-1}^{1} [\phi(x,-1)P_n(-1) - \phi(x,1)P_n(1)]Q_m(x)\,dx$$

Here, we find the order of accuracy of the left-hand side and the order of accuracy of the right-hand side to be

$$\alpha = \text{Min } (2\bar{m} - m - 4)$$

$$\beta = \text{Min } (2\bar{n} - n - 4)$$

which is the same as that given by Equation 15. Thus, the order of accuracy is the same for both the Dirichlet problem and the Neumann problem and is given by Equation 15. If $\bar{m} = \bar{n} = N$ and m and n range up to N, we find that the order of accuracy according to this analysis is at least N - 4 with N point Gauss-Lobatto formula. Numerical calculations reported later will show that we actually get slightly better accuracy, on the order of N - 2.

CHAPTER III

SPECIFIC EXAMPLES AND RESULTS

The present difference scheme has been evaluated and tested using three different types of forcing functions. These include: combinations of cosine functions, exponential functions, and polynomial functions. Both formulations of Poisson's equation, the direct and the Cauchy-Riemann system of equations, have been tested with the different forcing functions. These different functions are used with different numbers of Gauss-Lobatto points. While the two formulations in some cases are evaluated for a number of different points, the majority of the tests are done for two particular sets of points because of the necessity of having a number that would produce even grids for the five point finite difference scheme which has been used for comparison purposes. All the forcing functions have been used both with the present method and the five point finite difference scheme in order that accurate comparisons could be made.

The two grid sizes used for the five point finite difference scheme are $h = 0.5$ and $h = 0.2$. These grid sizes correspond to $N = 5$ and $N = 11$ points respectively for the present difference scheme. We are dealing with a rather coarse grid for the five point finite difference scheme so that the results from that scheme generally do not achieve even one decimal place of accuracy. However, this same

number of points with the uneven grid produce a reasonable and in a number of cases a very good accuracy for the present scheme. This occurrence, thereby, shows the feasibility of the scheme studied.

Polynomials of a number of different successively higher degrees ranging from 2 to 15 have been used. It is important that a number of different degrees be used in order that an estimate of the order of accuracy of the scheme could be obtained. For a number of degrees a general polynomial has also been utilized by the present scheme in order to substantiate the accuracy obtained. As has been noted in Chapter II, Section III, an estimate for the order of accuracy is evolved analytically also. This estimate is upheld and bettered by the actual numerical calculations.

I. COSINE FUNCTIONS

One of the cases studied for the forcing function is

$$\phi = 1/\pi \cos \pi\mu x \cos \pi\nu y$$

where μ and ν are some real numbers. For this test we let $\mu = \nu = 0.5, 1.0, 2.0$. The errors obtained, found from the exact solution minus the numerical solution, can be seen in Table I for both formulations and the corresponding finite difference schemes. The direct formulation with one exception is better than the five point finite difference scheme for both grid sizes. In the case where $\mu = \nu = 2.0$ the finite difference absolute error value on the five point

TABLE I

CASES STUDIED AND MAXIMUM ABSOLUTE ERROR OBSERVED FOR THE PROBLEM $\phi = 1/\pi \cos \pi\mu x \cos \pi\nu y$

Number of Points	$\mu (=\nu)$	Maximum Absolute Error			
		Dirichlet Problem		Neumann Problem	
		PS	FD	PS	FD
5	0.5	4.73×10^{-2}	1.11×10^{-1}	2.14	3.63×10^{-1}
5	1.0	1.19×10^{-2}	1.69×10^{-2}	5.00×10^{-1}	1.29×10^{-2}
5	2.0	1.03	7.01×10^{-1}	1.09×10^{1}	4.67×10^{-1}
11	0.5	6.53×10^{-6}	1.48×10^{-2}	1.42×10^{-2}	6.13×10^{-2}
11	1.0	1.21×10^{-8}	2.63×10^{-3}	3.84×10^{-5}	3.91×10^{-3}
11	2.0	6.36×10^{-4}	4.75×10^{-2}	3.93	4.62×10^{-1}

PS = Present scheme.

FD = Five point finite difference scheme.

grid is 0.7 while the error value for the present scheme is 1.03. However, for this case where the cosine functions have two cycles, five points are too few for any method. In the other two cases on the five point grid the error is better, but on the same order (10^{-2}) in the $\mu = \nu = 1.0$ cases and better by an order of 10 in the $\mu = \nu = 0.5$ case. When the number of quadrature points is raised to 11, the error is better in the present scheme case by an order of 10^2 (in the $\mu = \nu = 2.0$ case) and an order of 10^4 (in the $\mu = \nu = 1.0$ case) and an order of 10^5 (in the $\mu = \nu = 0.5$ case).

In Table II we see the maximum relative percent error for this problem. Comparisons of the Dirichlet problem cannot easily be made because the maximum errors for the five point finite difference scheme occur at points where the solution is zero. As a result a percent error for the five point difference scheme cannot be calculated. However, we can note that the percent errors for the Dirichlet problem on the 11 point grid are quite good, the largest being 1.99×10^{-1}.

In Figure 1 we have a comparison of the exact solution for the Dirichlet problem on an 11 point grid with the solution from the present scheme and the solution from the five point difference scheme. In this figure we choose wave number, $\mu = \nu = 1$. We can see that the solution from the present scheme and the exact solution are identical for a plot. The five point difference scheme varies some from

TABLE II

CASES STUDIED AND MAXIMUM RELATIVE PERCENT ERROR OBSERVED FOR THE PROBLEM $\phi = 1/\pi \cos \pi\mu x \cos \pi\nu y$

Number of Points	$\mu (=\nu)$	Maximum Relative Percent Error			
		Dirichlet Problem		Neumann Problem	
		PS	FD	PS	FD
5	0.5	3.73	*	1.00×10^{2}	5.19
5	1.0	6.81×10^{1}	*	2.43×10^{2}	3.63×10^{1}
5	2.0	3.22×10^{2}	2.20×10^{2}	6.59×10^{2}	*
11	0.5	3.80×10^{-6}	*	2.34×10^{-3}	8.24×10^{-1}
11	1.0	2.05×10^{-3}	*	2.27×10^{-1}	6.45
11	2.0	1.99×10^{-1}	*	2.01×10^{2}	2.43×10^{1}

PS = Present scheme.

FD = Five point finite difference scheme.

* indicates exact solution is zero at the points of maximum error.

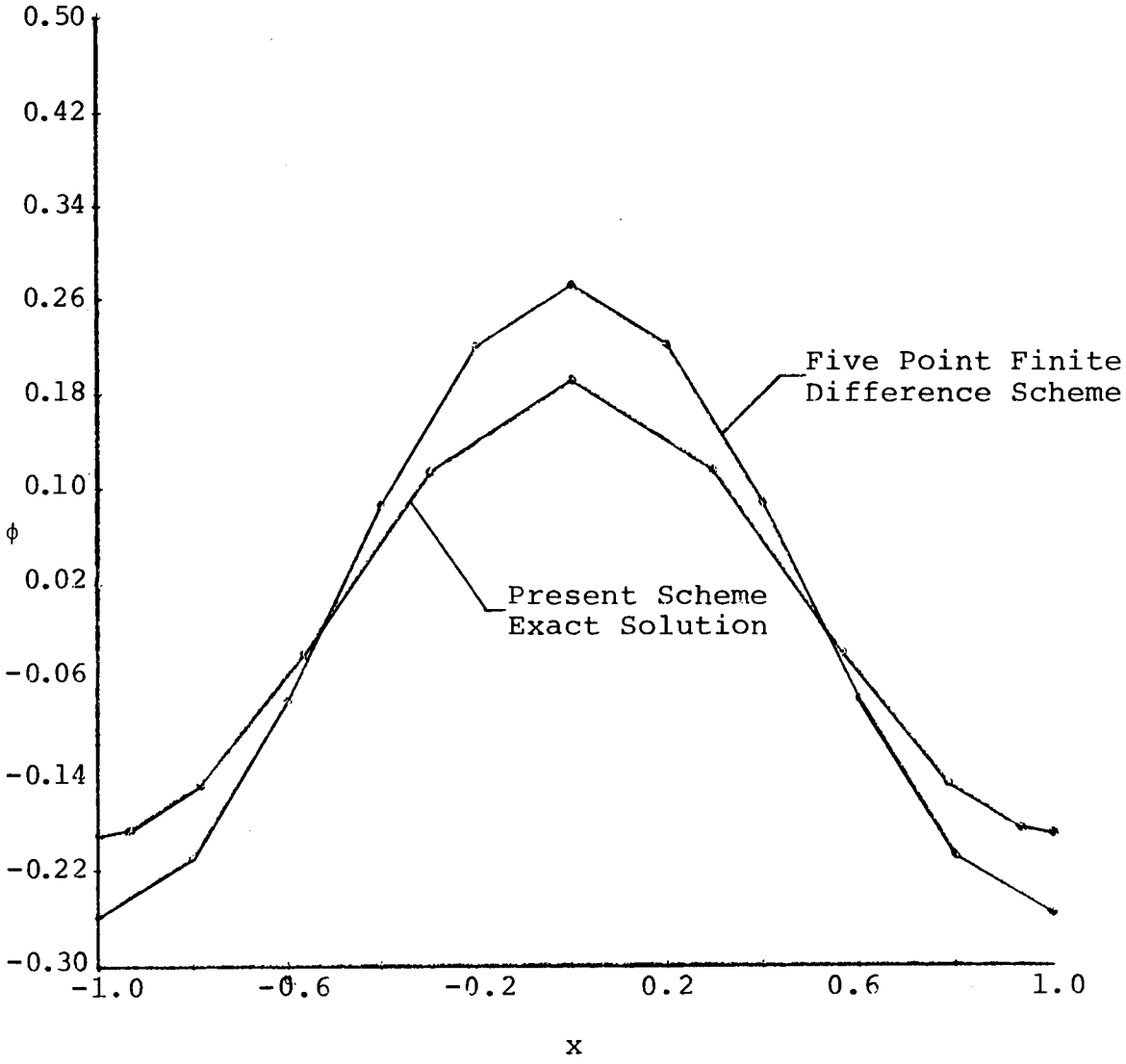


Figure 1. Solution of the Dirichlet problem corresponding to the exact solution, $1/\pi \cos \pi x \cos \pi y$, at $y = -0.2$ on an 11 point grid.

the exact solution. In Figure 2 we have a similar comparison of the Dirichlet problem on an 11 point grid for wave number $\mu = \nu = 0.5$. Again the exact solution and the solution from the present scheme are almost identical while the five point difference scheme has the most error. Figure 3 contains a comparison for a $\mu = \nu = 2$ Dirichlet problem with 11 points. Once again the present scheme solution and the exact solution are almost identical. In this figure the largest degree of variation is noted between the exact solution and the five point difference scheme solution of any of these three figures.

Solutions of the Neumann problem with Cauchy-Riemann formulation with these forcing functions are not quite as accurate as the corresponding Dirichlet problem solutions. The grid size with five quadrature points produces answers that are larger than the five point finite difference scheme results in some cases. The 11 point grid produces in two cases smaller errors in the present scheme than in the five point difference scheme. Thus, we can see that the present scheme is accurate for cosine function solution when there are enough number of points per wavelength.

Table II, which contains the relative percent error results for this problem, exaggerates the errors mainly because of near zero solution at points of maximum error. Otherwise, results indicated are similar to the ones mentioned earlier.

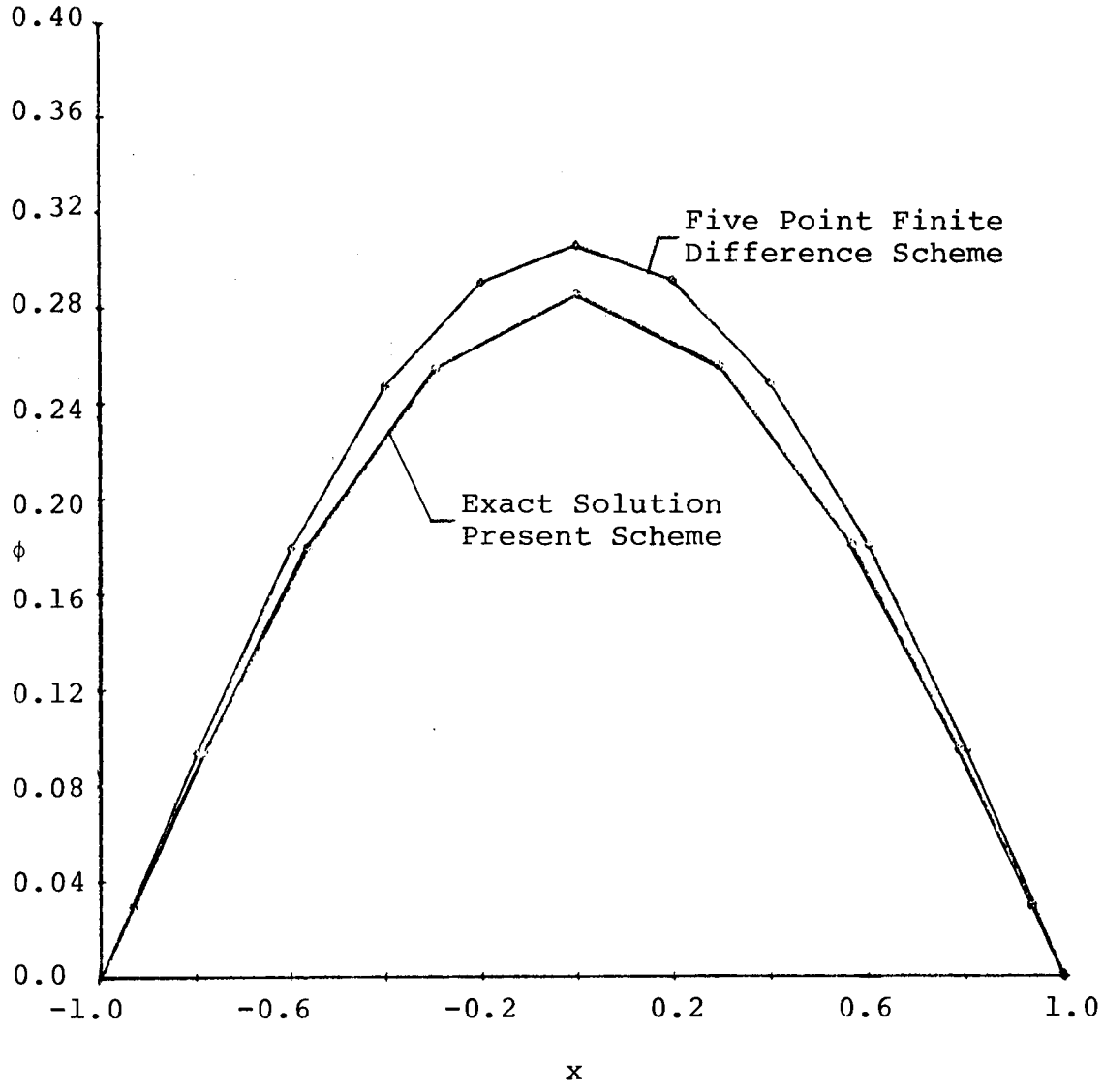


Figure 2. Solution of the Dirichlet problem at y = -0.2 corresponding to the exact solution, $1/\pi$ COS $\pi/2$ x COS $\pi/2$ y on an 11 point grid.

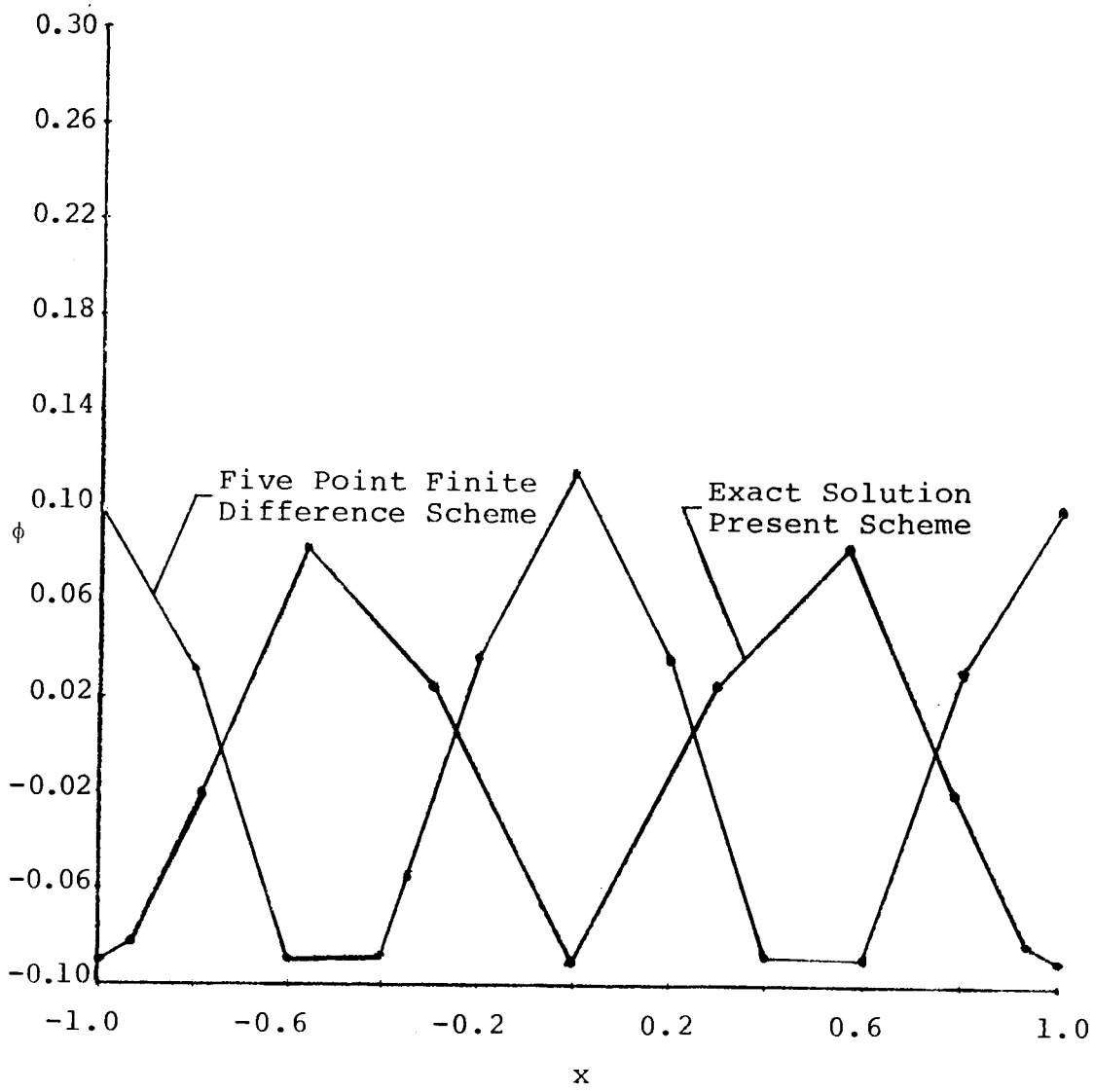


Figure 3. Solution of the Dirichlet problem corresponding to the exact solution $1/\pi$ COS $2\pi x$ COS $2\pi y$ at $y = -0.2$ on an 11 point grid.

II. EXPONENTIAL FUNCTIONS

A second case considered for the forcing function is the exponential case, in the form

$$\phi = e^{x/2} + e^{y/2}$$

Again, solutions of the Dirichlet problem for this case are excellent. In fact, on the 11 point grid the problem is solved almost exactly by the present scheme as can be seen in Table III. Even on the five point grid the present scheme solves the case better than the five point finite difference scheme though both are of the same order of accuracy for such a coarse grid. In fact, the five point grid with the present scheme solves this exponential forcing function problem to the same order of accuracy as does the 11 point grid with the five point finite difference scheme. This shows the feasibility of the present scheme in reducing the number of points required for the same accuracy. Here, the present scheme allows us to use five points instead of 11 to achieve the same order of accuracy, 10^{-4}.

In Table IV, the relative percent error table for the exponential case, we again have lower errors for the present scheme than for the five point finite difference scheme. In fact, in this table all the percent errors are of a desirable accuracy including the five point difference scheme errors.

TABLE III

MAXIMUM ABSOLUTE ERROR OBSERVED FOR THE PROBLEM $\phi = e^{x/2} + e^{y/2}$

Grid Size	Dirichlet Problem	Neumann Problem
Present Scheme:		
5 point	5.84×10^{-4}	6.43×10^{-4}
11 point	4.99×10^{-13}	2.31×10^{-9}
Five Point Finite Difference Scheme:		
5 point	7.44×10^{-4}	6.71×10^{-3}
11 point	1.24×10^{-4}	1.24×10^{-3}

TABLE IV

MAXIMUM RELATIVE PERCENT ERROR OBSERVED FOR THE PROBLEM $\phi = e^{x/2} + e^{y/2}$

Grid Size	Dirichlet Problem	Neumann Problem
Present Scheme:		
5 point	2.92×10^{-2}	1.78×10^{-1}
11 point	2.49×10^{-11}	7.38×10^{-7}
Five Point Finite Difference Scheme:		
5 point	3.72×10^{-2}	1.04×10^{-1}
11 point	6.20×10^{-3}	1.67×10^{-1}

In Figure 4 we have a comparison for the exponential case between the exact solution, the present scheme solution, and the five point difference scheme solution. This comparison is for the Dirichlet problem on an 11 point grid. We see again that the exact solution and the solution obtained by the present technique are almost identical, with the five point difference scheme solution deviating some from the exact solution though it is still of a desirable accuracy.

The Neumann problem, or the Cauchy-Riemann type system of equations, is also solved well by the present scheme. For the five point grid the error is again on the order of 10^{-4} which was one order better than the finite difference scheme which is only on an order of 10^{-3}. The 11 point grid produces an error for the present scheme on the order of 10^{-9} as opposed to an error of 10^{-3} for the five point finite difference scheme. Here, for both cases we can see a vast improvement in the error for the present scheme over the five point finite difference scheme. Again the present scheme has a lower error on the five point grid than the five point finite difference scheme on either the 5 point grid or the 11 point grid.

In Table IV on the five point grid, the relative percent error for the present scheme and the five point difference scheme are on the same order. However, the present scheme has a slightly higher percent error than the five point difference scheme. On the 11 point grid, it can

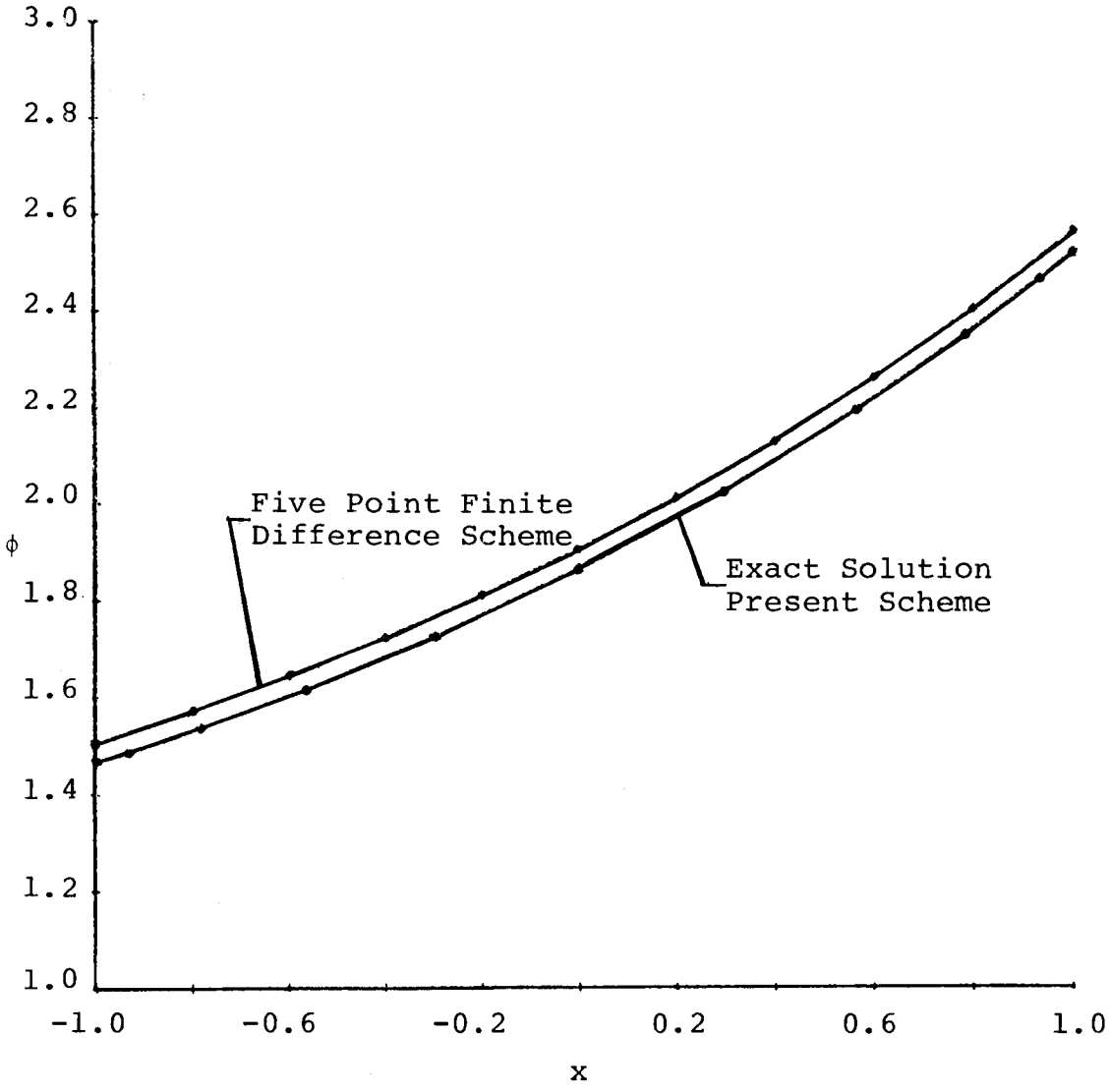


Figure 4. Solution of the Dirichlet problem corresponding to the exact solution, $e^{x/2} + e^{y/2}$, at y = -0.2 on an 11 point grid.

be seen that the relative percent error is much lower for the present scheme than for the five point difference scheme. We see again the feasibility of the scheme proposed, particularly in using less points to achieve the same or better accuracy than the standard scheme.

III. POLYNOMIAL FUNCTIONS

The third case of forcing functions are the polynomial forms. Polynomials of the form

$$\phi = x^n + y^n$$

have been considered for both grid sizes with n varying from 2 to 15. The number of different powers used is in order to ensure that a proper estimate of the accuracy could be made. In a number of cases more general polynomial equations have also been used in order to ensure that the order of accuracy is not just a result of symmetry. This solution of the general degree polynomials supports our statements on the accuracy of the polynomials.

For the Dirichlet problem, with the direct formulation, we have solved polynomials up to the third degree exactly on the five point grid. As can be seen in Table V, for polynomials of higher order the answers are generally better than in the five point finite difference scheme, but neither are very good.

However, the 11 point grid shows some remarkable results. On this grid, polynomials of up to ninth degree

TABLE V

MAXIMUM ABSOLUTE ERROR OBSERVED FOR THE DIRICHLET PROBLEM $\phi = x^n + y^n$

Degree of Polynomial n	Present Scheme		Five Point Finite Difference Scheme	
	5 pt grid	11 pt grid	5 pt grid	11 pt grid
2	5.13×10^{-16}	4.34×10^{-16}	5.55×10^{-17}	1.46×10^{-15}
3	5.13×10^{-16}	3.04×10^{-14}	3.57×10^{-1}	2.63×10^{-16}
4	2.21×10^{-1}	3.31×10^{-14}	9.37×10^{-1}	4.68×10^{-2}
5	4.72×10^{-16}	4.17×10^{-14}	2.23×10^{-1}	8.08×10^{-1}
6	3.69×10^{-1}	2.15×10^{-14}	6.17×10^{-1}	6.73×10^{-1}
7	1.62×10^{-1}	2.15×10^{-14}	5.36×10^{-1}	3.03×10^{-1}
8	4.32×10^{-1}	7.01×10^{-14}	8.65×10^{-1}	1.40×10^{-1}
9	3.69×10^{-1}	4.73×10^{-14}	6.55×10^{-1}	3.52×10^{-2}
10	6.20×10^{-1}	1.71×10^{-3}	9.56×10^{-1}	1.98×10^{-1}
11	5.48×10^{-1}	3.97×10^{-14}	6.75×10^{-1}	2.63×10^{-1}
12	7.65×10^{-1}	5.75×10^{-3}	9.68×10^{-1}	4.24×10^{-1}
13	6.75×10^{-1}	6.92×10^{-4}	7.00×10^{-1}	4.36×10^{-1}
14	8.63×10^{-1}	1.06×10^{-2}	9.89×10^{-1}	5.86×10^{-1}
15	7.57×10^{-1}	2.54×10^{-3}	7.09×10^{-1}	5.79×10^{-1}

have been solved almost exactly by the present method while the finite difference scheme is still not getting even one decimal place of accuracy. For polynomials above ninth degree the errors are on the order of 10^{-3} and 10^{-4} which is much better than the 10^{-1} errors the finite difference scheme has been producing on the 11 point grid.

As can be seen in Table VI the relative percent errors for the present scheme on the five point grid are high, but are lower than the percent errors for the five point difference scheme on the five point grid. The present scheme solves polynomials up to third degree exactly while the five point difference scheme solves exactly polynomials only up to second degree. The 11 point grid with two exceptions has very good results. The percent errors on the 11 point grid are lower in all cases than the five point difference scheme.

Figure 5 shows us a comparison for a third degree polynomial of the exact solution, the present scheme solution, and the five point difference scheme solution for the Dirichlet problem on an 11 point grid. The five point difference scheme solves this problem exactly as does the present scheme. So we find all three solutions to be almost identical. Figure 6 gives a similar comparison for a fourth degree polynomial in a Dirichlet problem on an 11 point grid. Here, the five point difference scheme does not solve the problem exactly so we see variation between the five point difference scheme solution and the exact and present

TABLE VI

MAXIMUM RELATIVE PERCENT ERROR OBSERVED FOR THE DIRICHLET PROBLEM $\phi = x^n + y^n$

Degree of Polynomial n	Present Scheme		Five Point Finite Difference Scheme	
	5 pt grid	11 pt grid	5 pt grid	11 pt grid
2	5.99×10^{-14}	2.49×10^{-14}	1.11×10^{-14}	4.55×10^{-13}
3	9.15×10^{-14}	1.87×10^{-12}	1.43×10^{2}	4.58×10^{-14}
4	*	2.17×10^{-12}	*	*
5	1.96×10^{-13}	2.94×10^{-12}	3.57×10^{2}	1.36×10^{2}
6	*	1.62×10^{-12}	*	8.22×10^{3}
7	1.57×10^{2}	*	3.43×10^{3}	8.28×10^{1}
8	6.40×10^{2}	5.66×10^{-12}	*	9.99×10^{1}
9	8.36×10^{2}	4.08×10^{-12}	1.68×10^{4}	1.56×10^{1}
10	2.14×10^{3}	1.67×10^{4}	*	9.48×10^{4}
11	2.89×10^{3}	4.21×10^{-12}	6.91×10^{4}	1.90×10^{2}
12	6.17×10^{3}	*	*	1.26×10^{6}
13	8.32×10^{3}	5.76×10^{-2}	2.87×10^{5}	3.96×10^{2}
14	1.62×10^{4}	*	*	3.74×10^{4}
15	2.18×10^{4}	4.84	2.32×10^{6}	8.23×10^{2}

* indicates exact solution is zero at the points of maximum error.

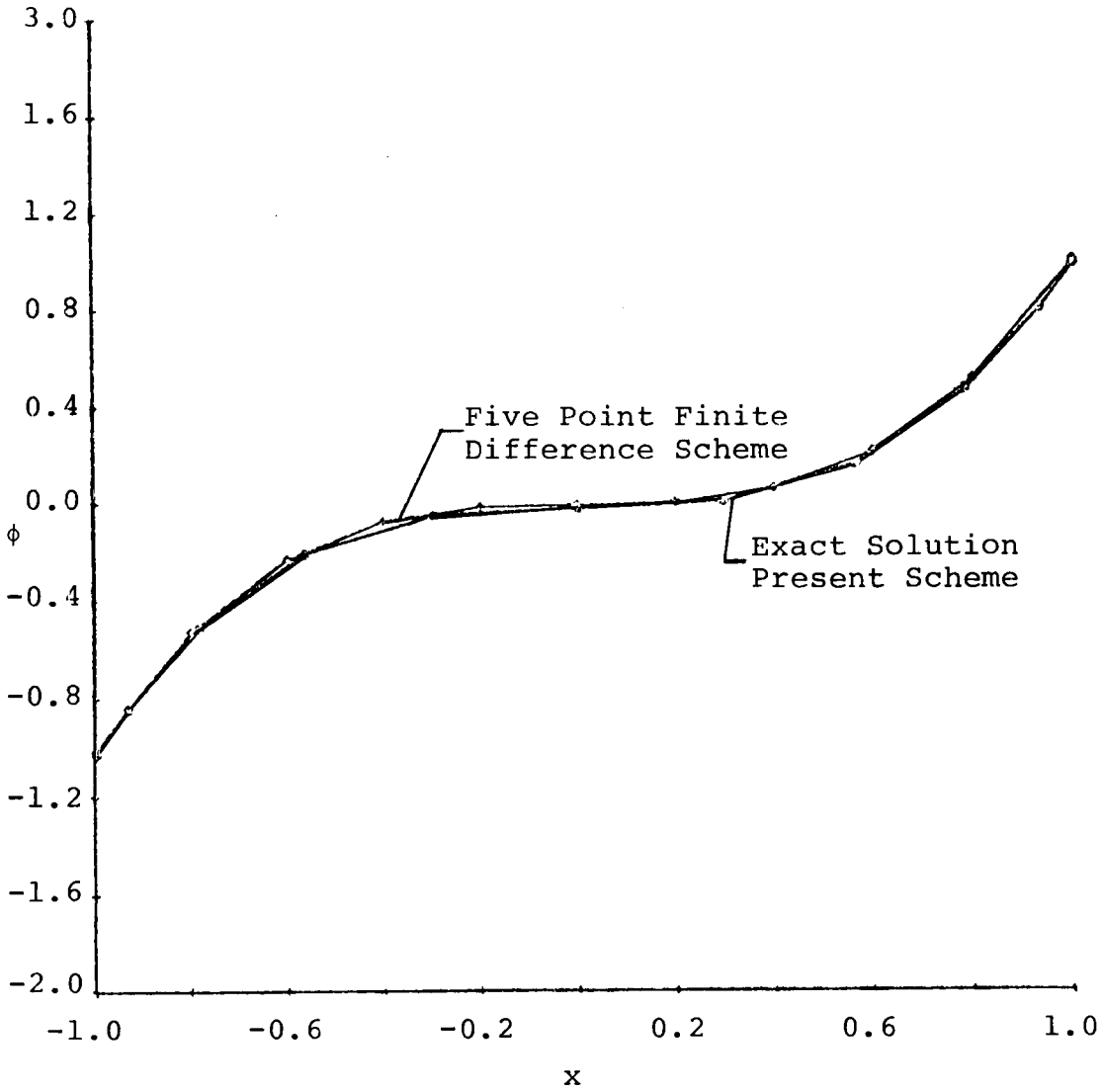


Figure 5. Solution of the Dirichlet problem on an 11 point grid at $y = -0.2$ corresponding to the exact solution $\phi = x^3 + y^3$.

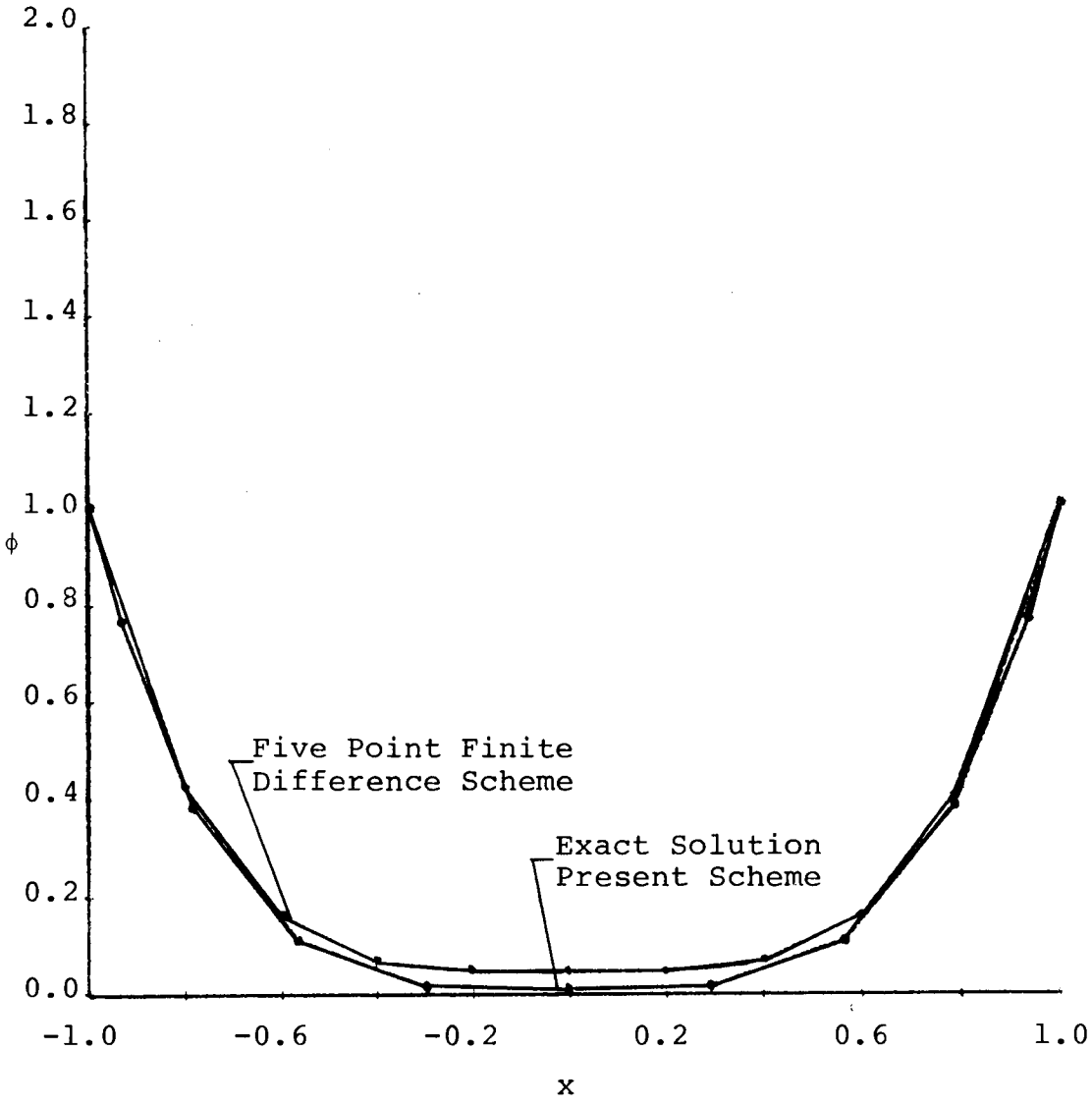


Figure 6. Solution of the Dirichlet problem corresponding to the exact solution $\phi = x^4 + y^4$ on an 11 point grid at $y = -0.2$.

scheme solution. Likewise, Figure 7 contains a comparison for an eighth degree polynomial. The present scheme solves it exactly so the present scheme solution and the exact solution are again almost identical. The five point difference scheme solution varies slightly from these two. In Figure 8 we have a comparison for a ninth degree polynomial which is the highest successive degree polynomial which the present scheme solves exactly. Again the present scheme solution and the exact solution are almost identical with the five point difference scheme varying slightly.

For the Neumann problem, the present scheme on a five point grid solves polynomials up to the degree of three exactly. The errors on polynomials above the fourth degree, however, are generally higher than the errors obtained as a result of the finite difference scheme. The exact results can be seen in Table VII. On the 11 point grid the polynomials again are solved exactly up to the ninth order. However, above ninth order for half the cases the errors are high, higher than the corresponding errors from the finite difference scheme. In the other half of the cases though, the errors are smaller than the finite difference errors though neither one achieves one decimal place of accuracy. It seems that in the case of the Neumann problem, the higher polynomials are not well integrated by this procedure.

Table VIII justifies the statement that higher degree polynomials in the Neumann problem are not well solved by the present scheme. The percent errors on the

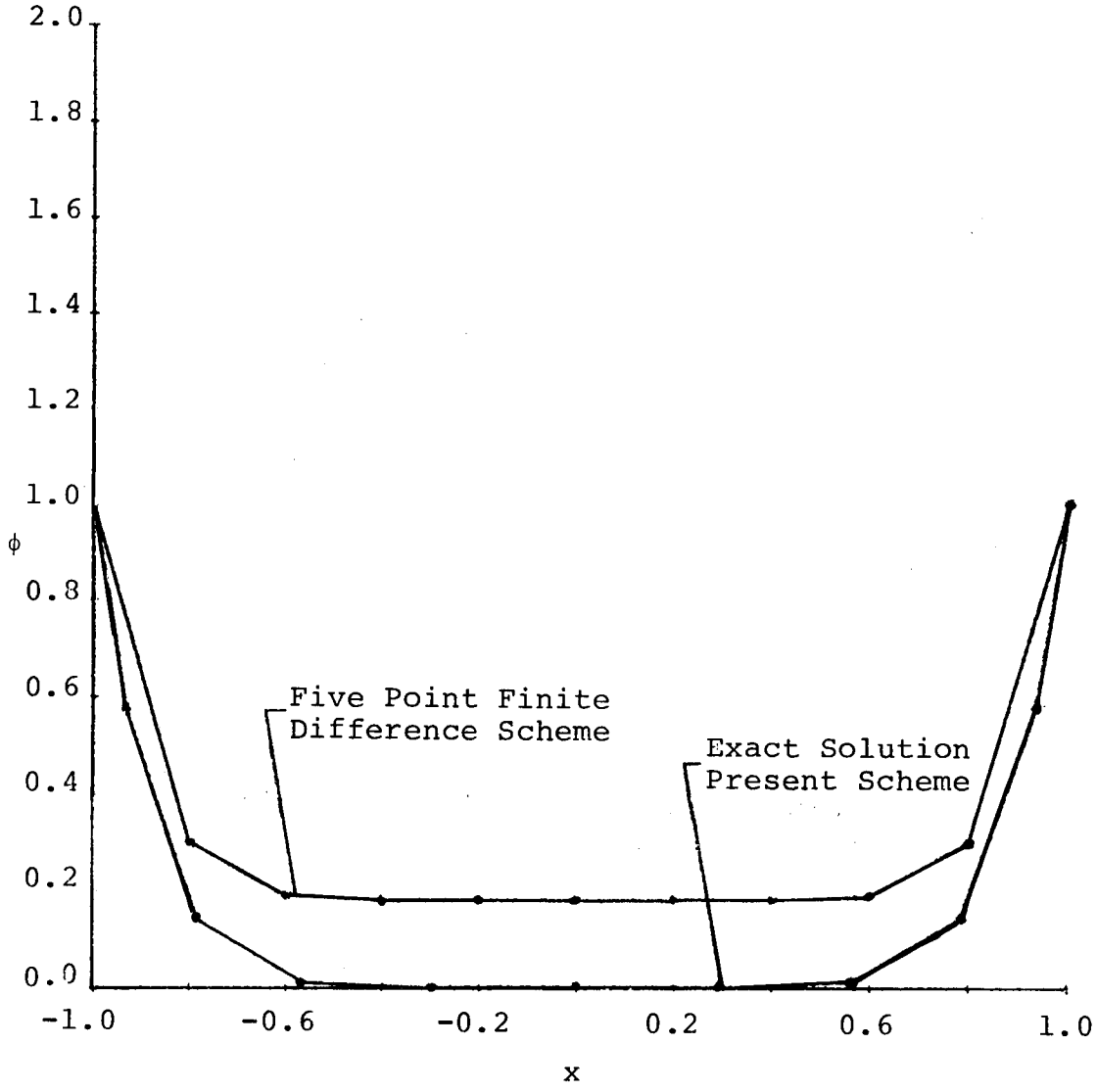


Figure 7. Solution of the Dirichlet problem at $y = -0.2$ corresponding to the exact solution, $\phi = x^8 + y^8$, on an 11 point grid.

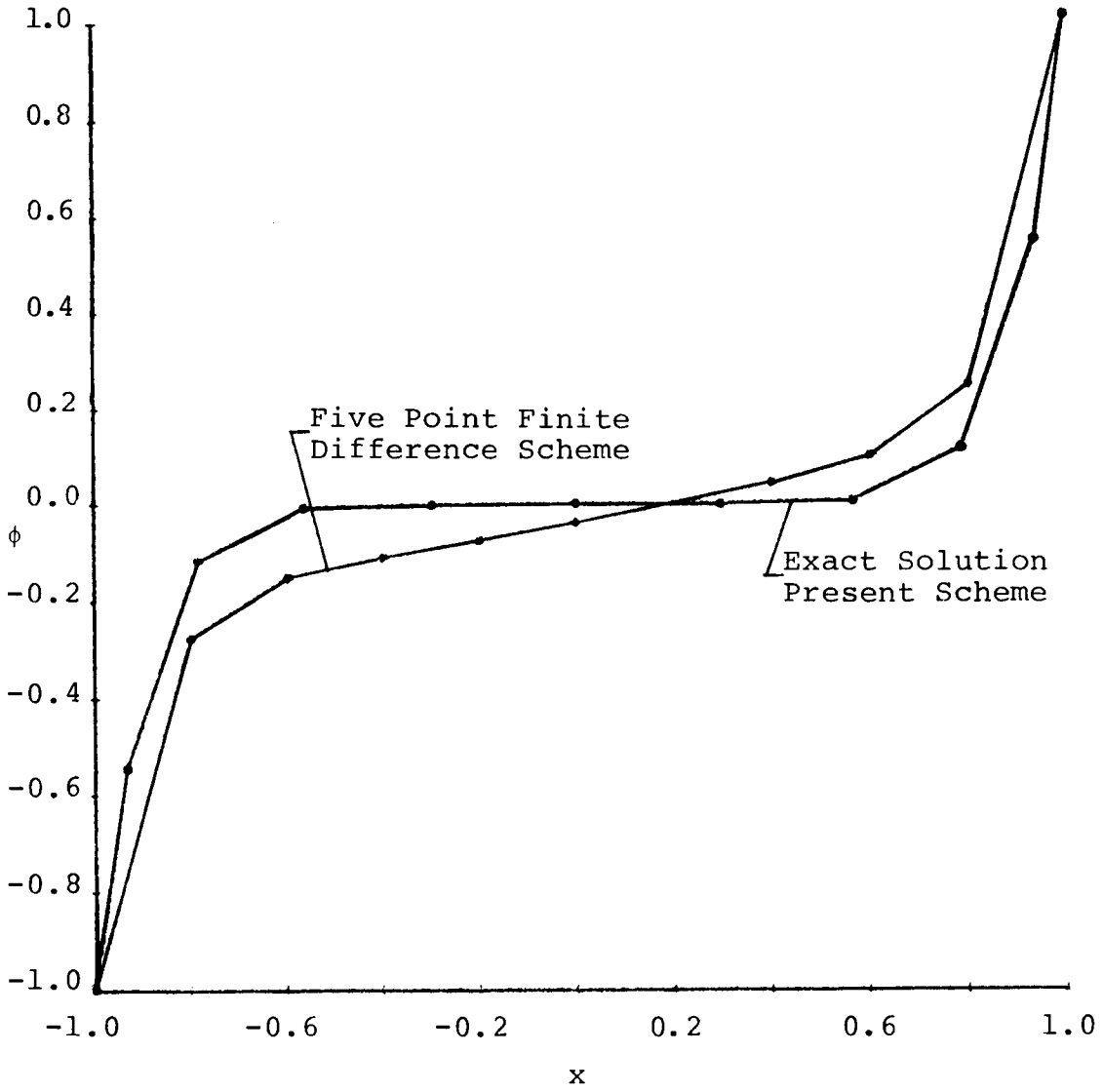


Figure 8. Solution of the Dirichlet problem on an 11 point grid corresponding to the exact solution, $\phi = x^9 + y^9$, at $y = -0.2$.

TABLE VII

MAXIMUM ABSOLUTE ERROR FOR THE NEUMANN PROBLEM $\phi = x^n + y^n$

Degree of Polynomial n	Present Scheme		Five Point Finite Difference Scheme	
	5 pt grid	11 pt grid	5 pt grid	11 pt grid
2	1.26×10^{-14}	1.23×10^{-16}	4.16×10^{-17}	1.77×10^{-15}
3	1.13×10^{-14}	1.75×10^{-12}	2.50×10^{-1}	4.00×10^{-2}
4	2.06×10^{-14}	2.15×10^{-11}	5.00×10^{-1}	1.28×10^{-1}
5	1.63	4.04×10^{-12}	6.87×10^{-1}	2.28
6	9.62	2.74×10^{-11}	8.12×10^{-1}	2.06
7	4.24	2.63×10^{-12}	8.91×10^{-1}	1.35
8	2.66×10^{1}	3.61×10^{-11}	9.37×10^{-1}	1.13
9	7.07	1.19×10^{-12}	1.36	9.65×10^{-1}
10	4.45×10^{1}	8.55	9.80×10^{-1}	1.14
11	9.78	8.58×10^{-2}	1.39	1.53
12	6.16×10^{1}	2.06×10^{1}	9.94×10^{-1}	1.46
13	1.23×10^{1}	3.47×10^{-1}	1.42	1.60
14	7.71×10^{1}	3.15×10^{1}	9.98×10^{-1}	1.73
15	1.46×10^{1}	8.47×10^{-1}	1.42	1.84

TABLE VIII

MAXIMUM RELATIVE PERCENT ERROR FOR THE NEUMANN PROBLEM $\phi = x^n + y^n$

Degree of Polynomial n	Present Scheme		Five Point Finite Difference Scheme	
	5 pt grid	11 pt grid	5 pt grid	11 pt grid
2	9.67×10^{-13}	6.59×10^{-15}	*	1.11×10^{-13}
3	*	6.68×10^{-11}	3.33×10^{1}	2.08
4	1.69×10^{-12}	6.59×10^{-10}	1.00×10^{2}	6.25
5	1.78×10^{2}	1.06×10^{-10}	2.20×10^{2}	1.11×10^{2}
6	1.33×10^{2}	1.53×10^{-9}	4.33×10^{2}	1.05×10^{2}
7	7.70×10^{2}	1.61×10^{-10}	8.14×10^{2}	7.37×10^{1}
8	6.46×10^{2}	2.47×10^{-9}	1.50×10^{3}	6.74×10^{1}
9	2.33×10^{3}	9.28×10^{-11}	3.86×10^{3}	6.39×10^{1}
10	2.02×10^{4}	1.58×10^{2}	5.02×10^{3}	8.51×10^{1}
11	6.15×10^{3}	1.54	1.30×10^{4}	1.29×10^{2}
12	5.43×10^{4}	3.64×10^{2}	1.69×10^{4}	1.42×10^{2}
13	1.52×10^{4}	6.07	6.52×10^{4}	1.79×10^{2}
14	1.36×10^{5}	5.28×10^{2}	5.84×10^{4}	2.24×10^{2}
15	3.66×10^{4}	1.47×10^{1}	1.55×10^{5}	2.79×10^{2}

* indicates exact solution is zero at the point of maximum error.

five point grid for the present scheme are generally lower than those for the five point difference scheme, but were still not of a desirable accuracy. Likewise the percent errors on the 11 point grid are very good for the present scheme up to the tenth degree polynomial whereupon they rapidly deteriorate. The percent errors at this point are generally lower than the percent errors for the five point difference scheme solution but the order is still not desirable.

In conclusion, we can see that for exponential cases both the direct formulation and the Cauchy-Riemann system formulation are well integrated by the present scheme. This occurs as a result of the fact that exponentials are well approximated by a third degree polynomial and both formulations integrate a third degree polynomial exactly on either grid size. It can also be seen that for the cosine case on the 11 point grid, the integration is done well by either formulation. In addition, the direct formulation integrates cosines well on a five point grid. Finally, for polynomials of any degree up to 15, the direct formulation on an 11 point grid integrates them well at least to an order of 10^2 better than the five point finite difference scheme. The Cauchy-Riemann system up to ninth degree on an 11 point grid integrates the polynomials almost exactly. For both formulations on a five point grid a polynomial of up to third order is integrated accurately.

It has been shown that for almost all problems the present scheme, the combination of the weighted residual method and Gauss-Lobatto quadrature, integrates the problem better than the five point finite difference scheme. In fact, the integration is often of quite a better accuracy. Thus, this shows the feasibility of the present scheme for getting better accuracy with a given number of points than the standard difference schemes. In fact, in a number of cases we can see a better accuracy with fewer points.

CHAPTER IV

CONCLUSIONS

A feasible technique for solving Poisson's equation has been developed. The present scheme is based on the method of weighted residuals and Gauss-Lobatto quadrature. Difference equations are formed which relate solutions at all the points in the grid. Enough weighting functions are taken so that the proper number of algebraic equations is formed. Two formulations of Poisson's equation are studied. Their results are compared against the results obtained from the traditional five point finite difference scheme.

This procedure has been applied to Poisson's equation with three different classes of forcing functions. For almost all of the cases studied, the results obtained from the combination of weighted residuals and Gauss-Lobatto quadrature are more accurate, for the same number of points, than the results obtained from the five point finite difference scheme. Numerical results confirm the high order of accuracy predicted by the truncation error analysis for the present scheme.

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APPENDIX

APPENDIX

ALGORITHM FOR CHOOSING LINEARLY INDEPENDENT EQUATIONS

In forming the equations for this scheme two problems result as was noted in Chapter III. These two problems are the linear dependency problem and the problem of a row of zeroes. Because of these two problems and because of the fact that they do not occur for the same moments every time, it is not always easy to choose the moments which will give a linearly independent system of equations. Therefore, we allow the computer to eliminate the linearly dependent equations.

Once the difference equations have been formed, we evaluate them from one to N where N is the number of points in the Gauss-Lobatto quadrature. Thus, we have x equations in x unknowns. We then apply the boundary conditions and arrive at x equations in $\bar{x}$ unknowns. We now proceed to put the largest elements in each column on the diagonal by the following procedure. We search the first column for the largest value in absolute value. The row in which it is found we interchange with the first row. By row operations we zero out the first column from the second row down. We then proceed to the second column and search it from the second row down for the largest value. The row with the largest value is interchanged with the second row. The

second column is then zeroed out by row operations from the third row down. We proceed with this algorithm until we have an upper triangular matrix of $\bar{x}$ equations in $\bar{x}$ unknowns. The remaining $x - \bar{x}$ equations are all zero. Thus, we have $\bar{x}$ equations in $\bar{x}$ unknowns. We then proceed to solve this system by back substitution.

The question arises as to what happens if there are not $\bar{x}$ linearly independent equations. This is a valid question. However, because of the way the problem is stated and because of the method of forming equations, this problem does not arise. With the boundary conditions prescribed on all four sides, the number of variables, once the boundary conditions have been applied, is such that varying the difference equations from one to N, where N is the number of points in the Gauss-Lobatto quadrature, forms enough equations to compensate for the linearly dependent ones. As a result, the algorithm always produces a linearly independent solution.

VITA

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