October, 1973

To the Graduate Council:

I am submitting herewith a thesis written by Michael Terry Doyle entitled "Stresses in the Intersection of Two Right Circular Cylinders." I recommend that it be accepted for nine quarter hours of credit in partial fulfillment of the requirements for the degree of Master of Science, with a major in Mechanical Engineering.

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STRESSES IN THE INTERSECTION OF TWO RIGHT CIRCULAR CYLINDERS

A Thesis

Presented to

the Graduate Council of

The University of Tennessee

In Partial Fulfillment

of the Requirements for the Degree

Master of Science

---

by

Michael Terry Doyle

December 1973

ABSTRACT

This paper represents the determination of the stresses near the normal intersection of two right circular cylinders whose radii are such that their ratio is less than one-third. The only loading considered is internal pressure. Both cylinders are assumed to be thin-walled. Donnell shell theory is used for the large shell, while the more exact Flugge shell theory is used for the smaller shell.

The area of the intersection is treated as being approximately plane, and a defining differential equation derived by previous researchers is used to describe the behavior of each shell in the neighborhood of the junction. Following this same method, a series solution to the differential equation for each shell is used to represent a stress function and a radial displacement.

Donnell's equations expressing stress resultants in terms of displacements are compared with Flugge's equations for stress resultants in terms of displacements. It is found that Flugge's stress resultants are equivalent to Donnell's stress resultants with small terms added. It is hypothesized that the terms in Flugge's theory common to both theories can be represented by the original series. The quantities added to these common terms convert the small shell analysis to Flugge theory. The radial displacement is taken in its original form. Since the other displacements are found by integration, they also have small added terms to account for the conversion to Flugge theory. Thus all quantities are expressed as functions of the original series solution.

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The series solutions have a total of eight unknown constants for each term in the series. A large number of points is taken on the intersection of the two cylinders, and eight junction compatibility equations used by previous researchers are applied to each chosen point. A least squares technique allows determinations of the unknown constants by computer. The stresses in the region of the intersection are then calculated using the series truncated at the desired number of terms.

Various methods are applied in an attempt to improve the accuracy of the numeric results, and final results are compared to the experimental data of other researchers. It is found that, while the series converge rapidly and the effects of the intersection are local in nature, the numeric results do not compare well with experimental data. A failure to totally debug the computer programs is the suspected cause of this error.

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LIST OF SYMBOLS

A_{n1}, A_{n2}, B_{n1}, B_{n2}	Series coefficients of the small shell
C_{2n-1}, C_{2n}, D_{2n-1}, D_{2n}	Expanded series coefficients of the large shell
D	Flexural rigidity of the small shell, $Et^3/12(1 - \nu^2)$
D'	Flexural rigidity of the large shell, $ET^3/12(1 - \nu^2)$
E	Modulus of elasticity in tension and compression
G	Modulus of elasticity in shear. Extension rigidity of the small shell, $Et/(1 - \nu^2)$
K	Constant in the definition of ψ; $[12(1 - \nu^2)]^{1/2}/ET^2\rho_o$
M_ρ, M_ϕ, $M_{\phi\rho}$	Large shell moment resultants in polar coordinates
M_X, M_θ, $M_{X\theta}$	Large shell moment resultants in cylindrical coordinates
N_ρ, N_ϕ, $N_{\rho\phi}$	Large shell stress resultants in polar coordinates
N_X, N_θ, $N_{X\theta}$	Large shell stress resultants in cylindrical coordinates
P	Internal pressure
Q_ρ, Q_ϕ	Large shell transverse shear stress resultants in polar coordinates

Q_X, Q_θ	Large shell transverse shear stress resultants in cylindrical coordinates
R	Radius of large shell
R_1, R_2	Constants of integration
T	Thickness of large shell
U_ρ, U_ϕ	Components of large shell displacements in polar coordinates
U, V	Components of large shell displacements in cylindrical coordinates
W	Radial displacement of large shell, positive inward
X, θ, Z	Cylindrical coordinates of the large shell
a_{2n-1}, a_{2n}	Unexpanded series coefficients of the large shell
k	Constant in the definition of $\tilde{\psi}$; $[12(1 - \nu^2]^{1/2}/Et^2\rho_o$
m_ϕ, m_y, $m_{\phi y}$, $m_{y\phi}$	Small shell moment resultants
m	Ratio of shell radii, ρ_o/R
n_ϕ, n_y, $n_{\phi y}$, $n_{y\phi}$	Small shell stress resultants
q_ϕ, q_y	Small shell transverse shear stress resultants
r	Radius of a shell under consideration
t	Thickness of the small shell. Tangential force when subscripted.
u, v, w	Displacements of the small shell
y, ϕ, z	Small shell cylindrical coordinates
$\tilde{y}$	Dimensionless axial coordinate of the small shell

Γ	Real part of the Hankel function
Δ	Imaginary part of the Hankel function
$\tilde{\Phi}$	Stress function for the small shell
Φ	Stress function for the large shell
$\tilde{\psi}$	Small shell series solution
ψ	Large shell series solution
$\Omega_1, \Omega_2, \Omega_3, \Omega_4$	Trignometric combinations of terms in the large shell series
$\gamma_{X\theta}$	Shear strain in the (X, θ,Z) coordinate system
$\varepsilon_X, \varepsilon_\theta$	Extension strains in the (X,θ,Z) coordinate system
ε	Constant in the differential equation defining $\tilde{\psi}$; $\{\frac{[3(1-\nu^2)]^{1/2}}{4\rho_o t}\}^{1/2}$
μ	Angle
ν	Poisson's ratio
ξ	Constant in the differential equation defining ψ; $\{\frac{[3(1-\nu^2)]^{1/2}}{4RT}\}^{1/2}$
π_{n1}	Numeric factor in the small shell series, $-(\frac{n^2}{2}+\sqrt{\frac{n^4}{4}+\frac{3}{16}(1-\nu^2)(\frac{\rho_o}{t})^2})^{1/2} - (\frac{\sqrt{3(1-\nu^2)}}{4}\frac{\rho_o}{t})^{1/2}$
π_{n2}	Numeric factor in the small shell series, $-(\frac{n^2}{2}+\sqrt{\frac{n^4}{4}+\frac{3}{16}(1-\nu^2)\frac{\rho_o^2}{t^2}})^{1/2} + (\frac{\sqrt{3(1-\nu^2)}}{4}\frac{\rho_o}{t})^{1/2}$
(ρ,ϕ)	Intersection polar coordinates
ρ_o	Small shell radius

$\tilde{\rho}$	Nondimensional intersection radial coordinate, ρ/ρ_o
σ_y, σ_ϕ	Normal stresses of small shell
σ_ρ, σ_ϕ	Normal stresses of large shell in intersection polar coordinates
$\tau_{y\phi}$, $\tau_{\phi y}$	Shear stresses of the small shell
ω_{n1}	Numeric factor in small shell series, $-\left(\frac{\sqrt{3(1-\nu^2)}}{4}\frac{\rho_o}{t}\right)^{1/2}-\left(\frac{-n^2}{2}+\sqrt{\frac{n^4}{4}+\frac{3}{16}(1-\nu^2)\left(\frac{\rho_o}{t}\right)^2}\right)^{1/2}$
ω_{n2}	Numeric factor in small shell series, $-\left(\frac{\sqrt{3(1-\nu^2)}}{4}\frac{\rho_o}{t}\right)^{1/2}+\left(\frac{-n^2}{2}+\sqrt{\frac{n^4}{4}+\frac{3}{16}(1-\nu^2)\left(\frac{\rho_o}{t}\right)^2}\right)^{1/2}$
A	Subscript for quantities added to Donnell theory due to Flugge theory
D	Subscript for quantities found by Donnell theory
S, μ	Intersection subscripts

CHAPTER I

INTRODUCTION

Historically, much work has been done on the elastic behavior of thin shells. In the 1930's, Donnell (1)[1] developed the assumptions leading to a set of simplified shell theory equations. Flugge (2) has derived a more rigorous shell theory involving less restrictive assumptions. The partial differential equations describing Flugge's theory are more accurate than the corresponding equations in Donnell's theory, but are also more complex and hence more difficult to solve, both theoretically and numerically.

The analysis of two right circular cylinders intersecting normally is a particularly difficult problem, yet of significant interest because of the many practical applications in which it occurs (e.g., pressure vessels, piping, reactor vessels). In recent years, several attempts have been made to determine the stresses in the neighborhood of the intersection itself. Reidelbach (3) formulated compatibility relations between the stress resultants and displacements of the two shells. These equations represent the boundary conditions imposed on each shell by the other shell, and have been part of most work on this problem since 1961.

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[1] Throughout this paper, numbers in parentheses will refer to numbered references listed in the list of references.

Naghdi and Eringen (4), working with a circular cutout in a cylindrical shell, proposed a series type solution. This series solution would then be satisfied at a finite number of points on the boundary, and the series coefficients determined by a least squares technique.

Edmondson (5) has done extensive analysis of the intersection of two cylinders using a series solution for each shell. The series for each shell is the solution to the governing differential equation for that shell. By use of Reidelbach's junction compatibility equations, the coefficients in the two series are determined at a chosen number of points on the intersection. The proper values of the coefficients are taken as those which satisfy Reidelbach's equations in a least squares sense. Edmondson's solution involves representing both shells by Donnell theory, and is restricted to ratios of shell diameters of such a magnitude that the intersection can be expressed in polar coordinates.

Previous work has produced a fairly good agreement with experimental data. However, an attempt to provide a method of analysis using the more exact theory, by formulating the analysis as Edmondson, but with the exception of the use of Flugge theory for the smaller of the two cylinders, will be useful. Numerical results will be obtained by use of the least squares colocation technique making use of Reidelbach's junction compatibility conditions. The case of internal pressure is considered exclusively.

CHAPTER II

THE PHYSICAL SYSTEM AND SERIES APPROACH

The Description of the Physical Case

The geometric configuration to be analyzed is that of two right circular cylinders intersecting normally. That is, the axes of the cylinders intersect and form a right angle at their intersection, as shown in Figure 1.

It is assumed that the thickness of each shell is small compared to its radius. In restricting the shells to thin shell theory, the following assumptions, as in Flugge (2, p. 210), are made:

1. Deformations due to transverse shears are negligible. This implies that any set of points which form a normal to the undeformed shell midsurface will form a normal to the midsurface after deformation.
2. Both stresses and strains in the thickness direction are negligible when compared with stresses and strains in the axial and circumferential directions.
3. All displacements are small when compared to the radius of the cylinder in which they occur. Further, the first derivatives of displacements are small compared to unity.

The previous three restrictions are assumed to apply to both shells. One further restriction is now applied to the shells in their

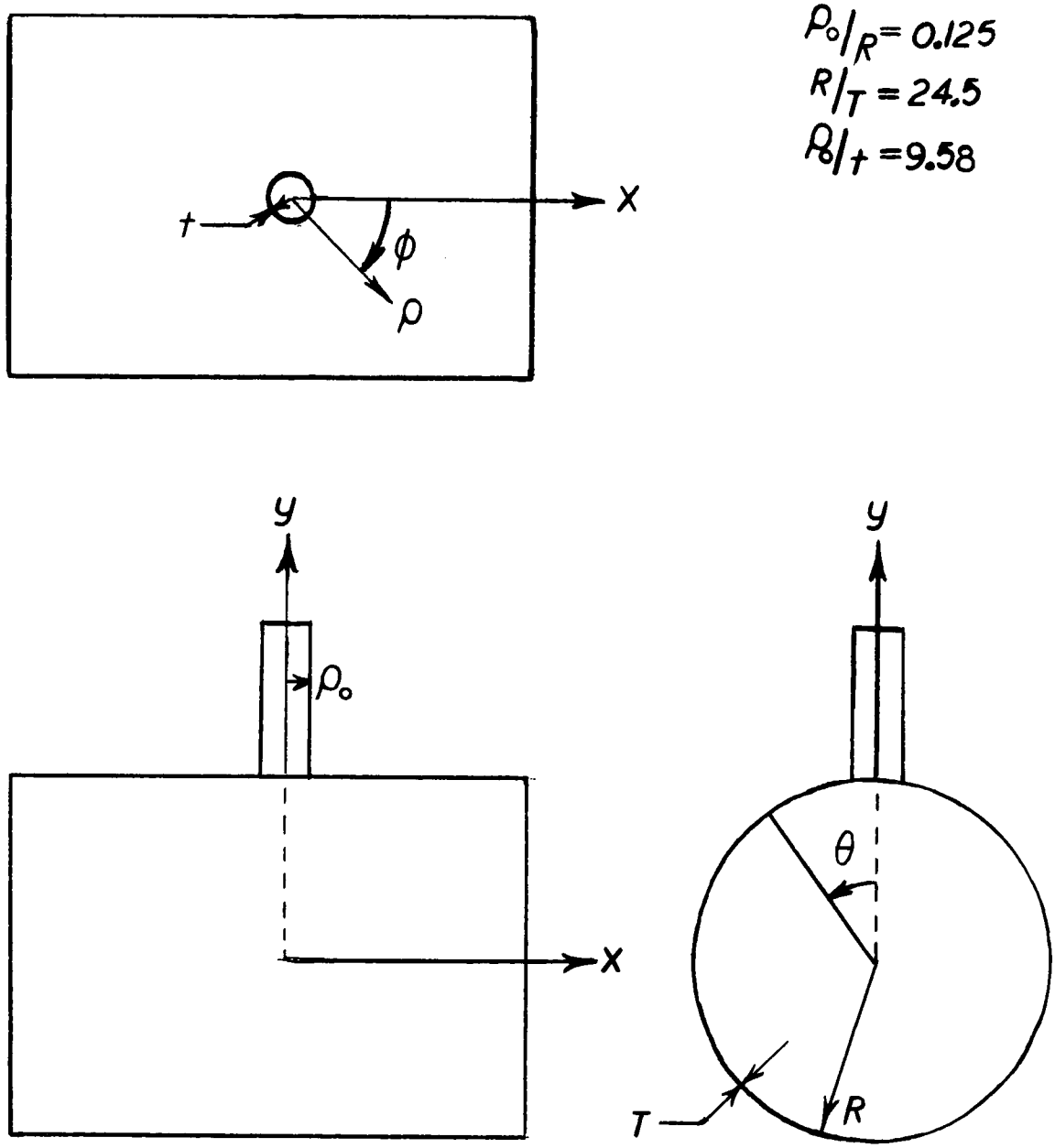

FIGURE 1

CONFIGURATION OF THE INTERSECTION REGION

comparison to each other. The ratio of the radius of the intersecting shell, ρ_o, to the radius of the intersected shell, R, is assumed to be less than one-third. As shown in Figure 2, page 6, as the ratio of the shell radii, m, becomes less than one-half, the intersection approaches a circle. Reidelbach (3, p. 14) shows that the deviation of the intersection from a circle is less than 5% when m is equal to one-half. By restricting m to be less than one-third, the deviation from a circle of intersection is certainly less than 5%. In fact, using the same method as Reidelbach, the deviation of the intersection from a circle:

$$\Delta m = \frac{1}{16} m^2 + \frac{3}{40} m^4 + \ldots$$

is found to be less than 0.8% for a value of m less than one-third. By use of the approximation of a circular intersection, and by treating the intersection as being plane, the Laplacian operator for the large, intersected shell in the region of the intersection can be converted to polar coordinates, (ρ,ϕ), located in the intersection plane. The origin of this polar coordinate system is on the axis of the small cylinder and coincides with the origin of the cylindrical coordinates describing the small shell. Thus, for the small shell:

$$\nabla^2 = \frac{\partial^2}{\partial y^2} + \frac{1}{\rho_o^2} \frac{\partial^2}{\partial \phi^2}$$

while, for the larger shell, the Laplacian operator is:

$$\nabla^2 = \frac{\partial^2}{\partial X^2} + \frac{\partial^2}{R^2 \partial \theta^2}$$

But, by use of the plane circle approximation for the intersection, i.e.,

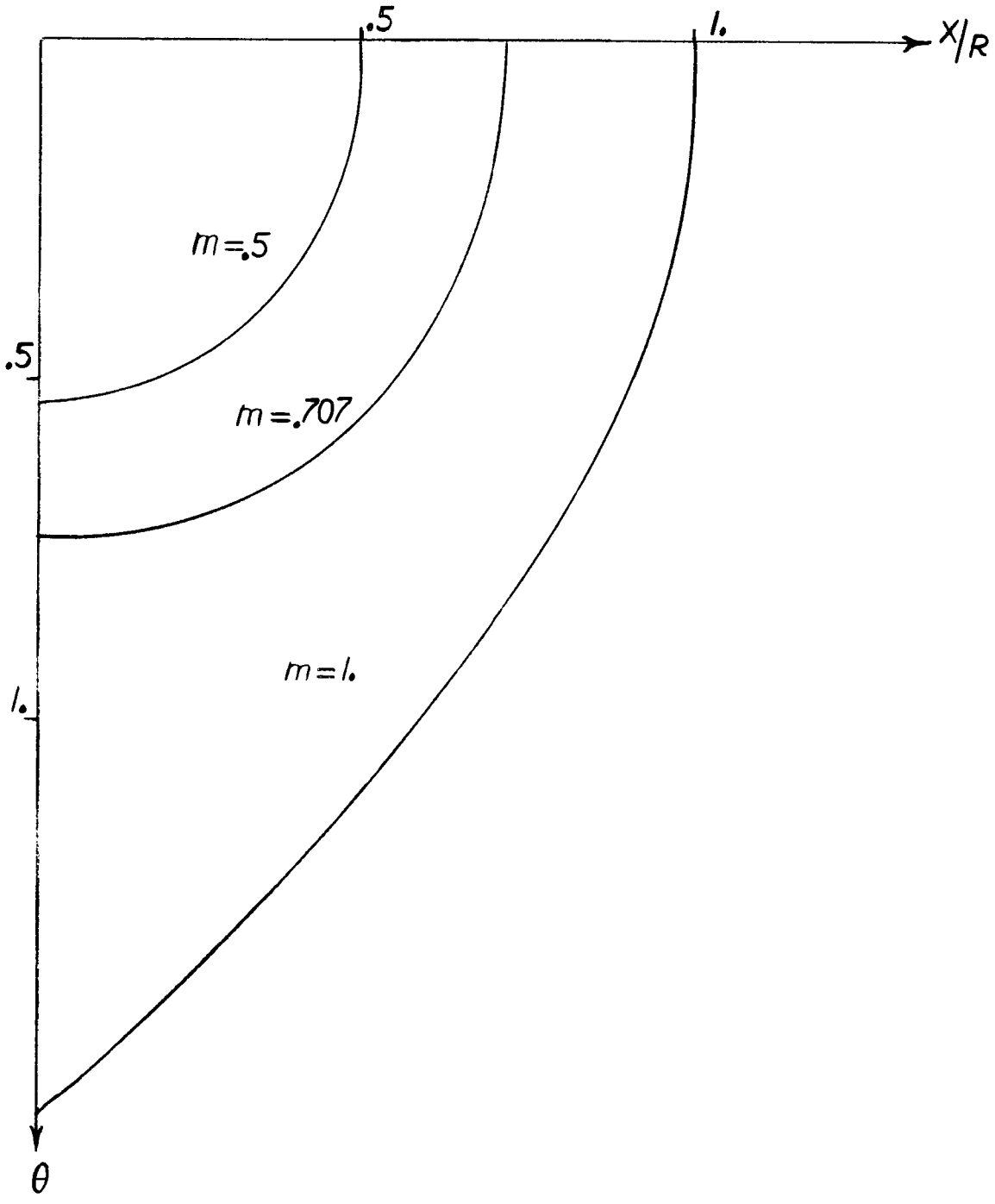

FIGURE 2

APPROXIMATE CIRCULAR INTERSECTION PLANE

$$X = \rho \cos \phi \qquad\qquad Y = R\theta \tilde{=} \rho \sin \phi$$

the following relation is arrived at:

$$\nabla^2 = \frac{\partial^2}{\partial X^2} + \frac{\partial^2}{R^2 \partial \theta^2} \tilde{=} \frac{\partial^2}{\partial \rho^2} + \frac{1}{\rho}\frac{\partial}{\partial \rho} + \frac{1}{\rho^2}\frac{\partial^2}{\partial \phi^2}$$

The large shell is now, in effect, a shallow shell in the region of the intersection when compared to the small shell.

The Series Approach to the Problem

The series type solution for each shell represents a solution to the defining differential equation for that shell. The derivation of each equation is described in more detail in Chapter III. The defining equation for the large shell may be expressed as:

$$\nabla^4 \psi + 8i\xi^2 \frac{\partial^2 \psi}{\partial X^2} = \frac{-12(1 - \nu^2)}{ET^3 \rho_o} P$$

In a like manner, the defining differential equation for the small shell is:

$$\nabla^4 \tilde{\psi} + 8i\varepsilon^2 \frac{\partial^2 \tilde{\psi}}{\partial \tilde{y}^2} = \frac{-12(1 - \nu^2)}{Et^3 \rho_o} P$$

As the series for each shell is also defined in terms of a stress function and the radial displacement for that shell, each stress resultant and displacement can be expressed in terms of the chosen series. The series for each shell involves four numeric constants for each shell for each term of the series for that shell. These constants are to be determined from a least squares collocation technique applied to the boundary between the two shells.

Following the method of previous researchers, such as Eringen (4) and Edmondson (5), the governing series for the large shell of:

$$\psi_{OPB} = \sum_{n=1}^{\infty} a_{2n}(1 + i) \sinh [(1-i)\xi\rho \cos \phi] H_{2n}^{(1)}[(2i)^{1/2}\xi\rho] \quad (2.1)$$

$$\sin 2n\phi + a_{2n-1} \cosh [(1-i)\xi\rho \cos \phi] H_{2n-1}^{(1)} [(2i)^{1/2}\xi\rho]$$

$$\sin (2n - 1)\phi$$

is obtained for the case of out of plane bending as the applied load.

As shown in Table I, the series will not have the same symmetry for loadings of out of plane bending and internal pressure. Modifications must be made on the above series, Equation 2.1, in order to obtain the proper symmetry for the case of internal pressure.

TABLE I

LARGE SHELL SYMMETRIES FOR OUT OF PLANE BENDING AND INTERNAL PRESSURE

Loading	Symmetry in X	Symmetry in ϕ
Out of plane bending	Even function	Odd function
Internal pressure	Even function	Even function

Out of plane bending symmetries arise from the fact that, for out of plane bending, both geometry and response of a shell element to external loading are identical for two shell elements located at the same ϕ, but having X coordinates equal in magnitude and opposite in sign. However, for two shell elements having the same value of X and values of ϕ equal in magnitude but opposite in

sign, the geometries will be identical and response to loading will be opposite in sign. For the case of internal pressure, external loading is everywhere the same. The geometry of the large shell is identical in each of four quadrants determined by two vertical planes, one containing the X axis and the other perpendicular to the first plane at X equal to zero.

The even and odd characteristics of the non-constant components of ψ are shown in Table II, together with the characteristics of $\cos(n\phi)$. By adjusting the circular trignometric functions contained in ψ, the proper symmetries for the large shell are obtained. The symmetry of ψ is independent of the Hankel function of the first kind of argument $[(2i)^{1/2}\xi\rho]$, since ρ can be specified independently of either X or ϕ. The correct form of ψ for the intersected shell for the case of internal pressure becomes:

$$\psi_{IP} = \sum_{n=1}^{\infty} a_{2n-1}(1+i)\sinh[(1-i)\xi\rho\cos\phi]H_{2n-1}^{(1)}[(2i)^{1/2}\xi\rho]$$
$$\cos[(2_{n-1})\phi] + a_{2n}\cosh[(1-i)\xi\rho\cos\phi]H_{2n}^{(1)}$$
$$[(2i)^{1/2}\xi\rho]\cos 2n\phi \qquad (2.2)$$

For the small shell, previous researchers have assumed a series solution of the following form for the case of out of plane bending:

$$\tilde{\psi}_{OPB} = \sum_{n=1,3,5}^{\infty}\sum_{j=1}^{2}[A_{nj}(\cos\omega_{nj}\tilde{y} + i\sin\omega_{nj}\tilde{y}) +$$
$$B_{nj}(-\sin\omega_{nj}\tilde{y} + i\cos\omega_{nj}\tilde{y}]e^{\pi_{nj}\tilde{y}}\sin n\phi \qquad (2.3)$$

TABLE II

SYMMETRIES OF TERMS IN THE LARGE SHELL SOLUTION TOGETHER WITH SYMMETRIES OF cos (Nϕ)

Term	Symmetry in X		Symmetry in ϕ	
	Even n	Odd n	Even n	Odd n
sinh [(1 - i)$\xi\rho\cos\phi$]	Odd	Odd	Even	Even
sin (2nϕ)	Odd	Odd	Odd	Odd
cosh [(1 - i)$\xi\rho\cos\phi$]	Even	Even	Even	Even
sin (2n - 1)ϕ	Even	Even	Odd	Odd
cos (nϕ)	Even	Odd	Even	Even

The even and odd symmetries in X and ϕ for both out of plane bending and internal pressures are exactly the same for the small shell as for the large shell. In contrast to the large shell, however, of all the components of $\tilde{\psi}_{OPB}$, only $\sin(n\phi)$ is a function of X or ϕ. Since $\sin(n\phi)$ is an odd function for all values of n, it must be replaced by $\cos(n\phi)$. If n is then restricted to even values, the proper symmetry for internal pressure is obtained. It must also be remembered that the proper symmetry is obtained when n is equal to zero. The correct series for the small shell is:

$$\tilde{\psi}_{IP} = \sum_{n=0,2,4}^{\infty} \sum_{j=1}^{2} [A_{nj}(\cos \omega_{nj} \tilde{y} + i \sin \omega_{nj} \tilde{y}) + B_{nj}(-\sin \omega_{nj} \tilde{y} + i \cos \omega_{nj} \tilde{y})]e^{\tau_{nj}\tilde{y}} \cos n\phi \qquad (2.4)$$

Use of Series Solutions

The governing series, ψ and $\tilde{\psi}$, are the homogeneous solutions to the defining differential equations, one of which can be derived for each shell. The forms of these two equations, shown previously, are similar for the two shells. Thus, in reality, only one distinct differential equation exists. This equation has been used by previous researchers and is based on the equilibrium equations written for a small element of a cylindrical shell, which are introduced in the next chapter. The different forms of the homogeneous solutions ψ and $\tilde{\psi}$ are due to the different coordinate systems previously chosen for the two shells. Each homogeneous solution will vary only to fit the symmetry of the problem under consideration. To each homogeneous solution must be added the particular solution of the differential

equation. The particular solution, of course, will be different for each loading condition.

Each series is composed of the non-dimensionalized radial displacement for that shell and a stress function multiplied by an appropriate constant. By use of the stress functions and the equilibrium equations, strain displacement equations, Appendix A, and Hooke's Law (Appendix B), the stress resultants and displacements can be expressed in terms of the proper series. Reidelbach's junction compatibility equations, together with a least squares fit allow a numeric determination of the constants in the series defining the two stress functions, and hence all stress resultants and displacements.

CHAPTER III

TRANSFORMATION OF THE SMALL SHELL ANALYSIS TO FLUGGE THEORY

Equilibrium Equations

Two major shell theories consistent with the present assumptions are Flugge theory and Donnell theory. Equilibrium equations for the element of the nozzle shown in Figures 3 and 4 can be written from element geometry and purely static considerations. Neglecting higher order terms (those on the order of a product of three differential quantities or more), the equilibrium equations for the case of internal pressure are:

$$\frac{\partial n_y}{\partial y} + \frac{1}{\rho_o}\frac{\partial n_{\phi y}}{\partial \phi} = 0$$

$$\frac{1}{\rho_o}\frac{\partial n_\phi}{\partial \phi} + \frac{\partial n_{y\phi}}{\partial y} - \frac{q_\phi}{\rho_o} = 0$$

$$\frac{1}{\rho_o}\frac{\partial q_\phi}{\partial \phi} + \frac{\partial q_y}{\partial y} + \frac{n_\phi}{\rho_o} = P$$

$$\frac{1}{\rho_o}\frac{\partial m_\phi}{\partial \phi} + \frac{\partial m_{y\phi}}{\partial y} - q_\phi = 0$$

$$\frac{\partial m_y}{\partial y} + \frac{1}{\rho_o}\frac{\partial m_{\phi y}}{\partial \phi} - q_y = 0$$

$$n_{y\phi} - n_{\phi y} + \frac{m_{\phi y}}{\rho_o} = 0 \qquad (3.1)$$

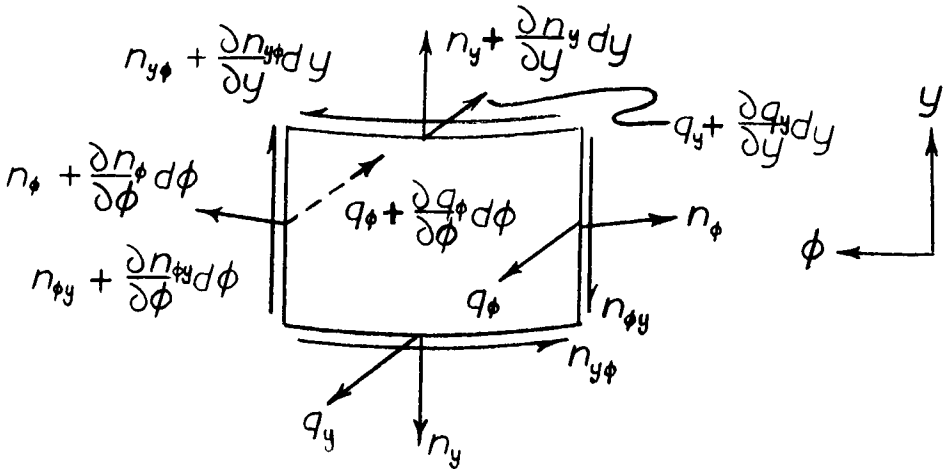

FIGURE 3

STRESS RESULTANTS

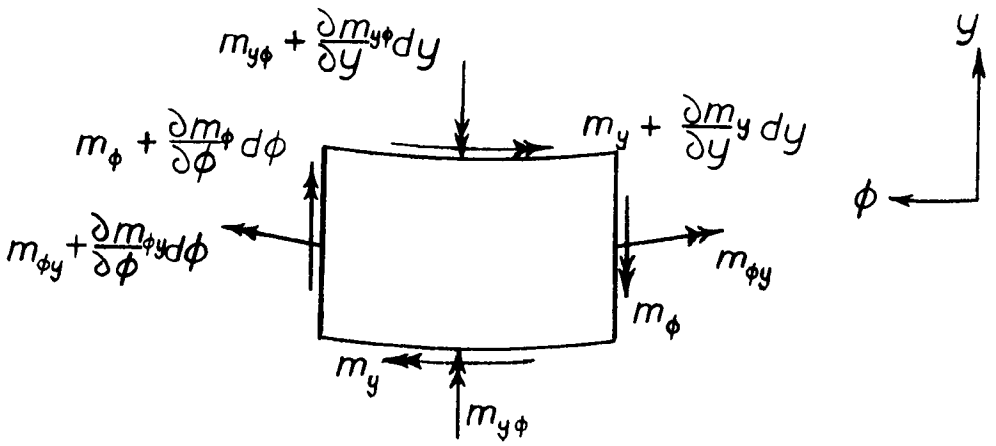

FIGURE 4

MOMENTS

The preceding six equilibrium equations form the basis of Flugge theory. It is important to note here that $n_{\phi y}$ does not necessarily equal $n_{y\phi}$ (note the sixth equation of Equations 3.1). Also note that $m_{\phi y}$ does not necessarily equal $m_{y\phi}$.

The second equation of Equations 3.1 arises from setting the summation of forces in the ϕ direction equal to zero. In its unsimplified form this equation is:

$$\frac{1}{\rho_o}\frac{\partial n_\phi}{\partial \phi}\ \rho_o\ d\phi dy + \frac{\partial n_{y\phi}}{\partial y}\ dy\rho_o d\phi - q_\phi\ dy\ \frac{\rho_o d\phi}{\rho_o} = 0$$

Donnell (1) states that the quantity q_ϕ is an order of magnitude smaller than the other terms and can therefore be neglected compared to those terms. This assumption can be made due to the thinness of the shell. The second equilibrium equation in Donnell theory is then:

$$\frac{1}{\rho_o}\ \frac{\partial n_\phi}{\partial \phi} + \ \frac{\partial n_{y\phi}}{\partial y} = 0$$

Donnell also assumes that the two shear stress resultants, $n_{y\phi}$ and $n_{\phi y}$, are equal. The twisting moment resultants, $m_{\phi y}$ and $m_{y\phi}$ are likewise assumed to be equivalent. These two equalities are due to the fact that Donnell considers a shell to be a special case of a flat plate. Obviously, for equilibrium of a flat plate the two equalities must hold true. The sixth equation of Equations 3.1 in Donnell theory is trivial in light of the above assumptions:

$$n_{y\phi} - n_{\phi y} = 0.$$

The Stress Function Approach to the Small Shell

The series analysis referred to in Chapter II satisfies Donnell's equilibrium equations by defining a stress function $\tilde{\Phi}$ such that:

$$n_y = \frac{\partial^2 \tilde{\Phi}}{\rho_o^2 \partial \phi^2} \qquad n_\phi = \frac{\partial^2 \tilde{\Phi}}{\partial y^2} \qquad n_{y\phi} = -\frac{\partial^2 \tilde{\Phi}}{\rho_o \partial \phi \partial y}$$

The series analysis combines $\tilde{\Phi}$ with the radial displacement, w, to form:

$$\tilde{\psi} = \frac{w}{\rho_o} + ik\tilde{\Phi} \tag{3.2}$$

Where the constant k is defined by:

$$k = \frac{[12(1 - \nu^2)]^{1/2}}{Et^2\rho_o}$$

By use of the compatibility and equilibrium equations, together with the definitions of the stress resultants, the following differential equation in $\tilde{\psi}$ is found:

$$\nabla^4 \tilde{\psi} + 8i\varepsilon^2 \frac{\partial^2 \tilde{\psi}}{\partial y^2} = \frac{-12(1 - \nu^2)}{Et^3\rho_o} P \tag{3.3}$$

Here the constant ε is defined by:

$$\varepsilon^2 = \frac{[3(1 - \nu^2)]^{1/2}}{4\rho_o t}$$

Exactly the same procedure is used to find ψ for the large shell.

$$\nabla^4 \psi + 8i\xi^2 \frac{\partial^2 \psi}{\partial X^2} = \frac{-12(1 - \nu^2)}{ET^3\rho_o} P \tag{3.4}$$

Where the constant ξ is defined by:

$$\xi^2 = \frac{[3(1 - \nu)^2]^{1/2}}{4RT}$$

Modification of the Series Approach to Flugge Theory

By virtue of the fact that m has been previously restricted to be less than 1/3, the large shell is treated as a shallow shell in the neighborhood of the intersection. The portion of the large shell in the immediate area of the intersection can thus be thought of as a deformed, thin, flat plate. In this case, Donnell's theory should apply with satisfactory accuracy since, as previously stated, Donnell's theory is based on treating the shell element by certain plate theory methods. The series ψ for the large shell, with the changes in symmetry to convert to internal pressure loading as in Chapter II, should still be used. Indeed, without the assumed shallow shell behavior of the large cylinder the problem is extremely complicated and difficult to solve.

The small shell forms a complete circular cross-section in the area of the intersection. For this nozzle, Donnell's assumptions may over-simplify the problem to the point of an unacceptable error. The small shell is therefore treated by the more accurate Flugge theory.

Flugge defines the stress resultants by:

$$n_y = \int_{-t/2}^{t/2} \sigma_y (1 - \frac{z}{\rho_o}) dz \qquad n_\phi = \int_{-t/2}^{t/2} \sigma_\phi dz$$

$$n_{y\phi} = \int_{-t/2}^{t/2} \tau_{y\phi} (1 - \frac{z}{\rho_o}) dz \qquad n_{\phi y} = \int_{-t/2}^{t/2} \tau_{\phi y} \, dz$$

$$m_y = \int_{-t/2}^{t/2} \sigma_y (1 - \frac{z}{\rho_o}) z dz \qquad m_\phi = \int_{-t/2}^{t/2} \sigma_\phi \, z dz \tag{3.5}$$

$$m_{y\phi} = \int_{-t/2}^{t/2} \tau_{y\phi} (1 - \frac{z}{\rho_o}) z dz \qquad m_{\phi y} = \int_{-t/2}^{t/2} \tau_{\phi y} \, z dz$$

Donnell defines the stress resultants as Flugge does with the exception that Donnell neglects the term z/ρ_o when compared to one, such that:

$$1 - z/\rho_o \approx 1$$

By the use of Hooke's Law and the strain displacement relations, the stress resultants can be expressed in terms of the displacements and the derivatives of the displacements.

Flugge (2, pp. 213-214) carries out this derivation, retaining terms of order z/ρ_o, but dropping all higher order terms. The stress resultants in Flugge theory expressed in terms of the displacements are:

$$n_\phi = \frac{G}{\rho_o}(\frac{\partial v}{\partial \phi} - w + \rho_o \nu \frac{\partial u}{\partial y}) - \frac{D}{\rho_o^3}(w + \frac{\partial^2 w}{\partial \phi^2})$$

$$n_y = \frac{G}{\rho_o}(\rho_o \frac{\partial u}{\partial y} + \nu \frac{\partial v}{\partial \phi} - \nu w) + \frac{D}{\rho_o} \frac{\partial^2 w}{\partial y^2}$$

$$n_{\phi y} = \frac{G}{\rho_o} \frac{1 - \nu}{2}(\frac{\partial u}{\partial \phi} + \rho_o \frac{\partial v}{\partial y}) + \frac{D}{\rho_o^3} \frac{1 - \nu}{2}(\frac{\partial u}{\partial \phi} - \rho_o \frac{\partial^2 w}{\partial y \partial \phi})$$

$$n_{y\phi} = \frac{G}{\rho_o}(\frac{1 - \nu}{2})(\frac{\partial u}{\partial \phi} + \rho_o \frac{\partial v}{\partial y}) + \frac{D}{\rho_o^3}(\frac{1 - \nu}{2})(\rho_o \frac{\partial v}{\partial y} + \rho_o \frac{\partial^2 w}{\partial \phi \partial y})$$

$$m_\phi = (\frac{-D}{\rho_o^2})(w + \frac{\partial^2 w}{\partial \phi^2} + \nu \rho_o^2 \frac{\partial^2 w}{\partial y^2})$$

$$m_y = (\frac{-D}{\rho_o^2})(\rho_o^2 \frac{\partial^2 w}{\partial y^2} + \nu \frac{\partial^2 w}{\partial \phi^2} + \rho_o \frac{\partial u}{\partial y} + \nu \frac{\partial v}{\partial \phi})$$

$$m_{\phi y} = \left(\frac{-D}{\rho_o^2}\right)(1-\nu)\left(\rho_o \frac{\partial^2 w}{\partial y \partial \phi} - \frac{1}{2}\frac{\partial u}{\partial \phi} + \frac{\rho_o}{2}\frac{\partial v}{\partial y}\right)$$

$$m_{y\phi} = \left(\frac{-D}{\rho_o^2}\right)(1-\nu)\left(\rho_o \frac{\partial^2 w}{\partial y \partial \phi} + \rho_o \frac{\partial v}{\partial y}\right) \qquad (3.6)$$

where the factors G and D are defined by:

$$G = \frac{Et}{1-\nu^2} \qquad D = \frac{Et^3}{12(1-\nu^2)}$$

Since Donnell neglects terms of order z/ρ_o compared to one in the definitions of stress resultants, the same should be done in the expression of stress resultants in terms of displacements and derivatives of displacements. Flugge (2, pp. 216-217) has also derived expressions for the stress resultants wherein the factor z/ρ_o is neglected when compared to one. These simplified stress resultants are the ones due to Donnell theory and are given by:

$$n_\phi = \frac{G}{\rho_o}\left(\frac{\partial v}{\partial \phi} - w + \nu\rho_o \frac{\partial u}{\partial y}\right)$$

$$n_y = \frac{G}{\rho_o}\left(\rho_o \frac{\partial u}{\partial y} + \nu \frac{\partial v}{\partial \phi} - \nu w\right)$$

$$n_{y\phi} = n_{\phi y} = \frac{G(1-\nu)}{2\rho_o}\left(\frac{\partial u}{\partial \phi} + \rho_o \frac{\partial v}{\partial y}\right)$$

$$m_\phi = \frac{-D}{\rho_o^2}\left(\frac{\partial^2 w}{\partial \phi^2} + \nu\rho_o^2 \frac{\partial^2 w}{\partial y^2}\right)$$

$$m_y = \frac{-D}{\rho_o^2}\left(\rho_o^2 \frac{\partial^2 w}{\partial y^2} + \nu \frac{\partial^2 w}{\partial \phi^2}\right)$$

$$m_{\phi y} = m_{y\phi} = \frac{-D(1-\nu)}{\rho_o}\frac{\partial^2 w}{\partial y \partial \phi} \qquad (3.7)$$

When the stress resultants due to Flugge theory as stated in Equations (3.6) are compared with those in Donnell theory as stated in Equations (3.7), a striking similarity comes to light. Flugge's

stress resultants can be written in terms of Donnell's stress resultants and smaller, added terms. That these added terms are smaller than the terms common to the two theories can easily be shown. For the normal and shearing stress resultants, the added terms represent quantities of the same order as the common terms but multiplied by a small constant. This is shown by the relationship between G and D:

$$\frac{D}{\rho_o^3} = \frac{G}{\rho_o} \left(\frac{t^2}{12\rho_o^2}\right)$$

For the moment resultants, it has already been assumed that the displacements and the first derivatives of displacements are small when compared to the radius of the shell and to unity, respectively. This does not imply, however, that the second derivatives of displacements are small when compared to the shell radius or to unity. In fact, if appreciable bending moments exist, the second derivatives are not small, as may be seen from elementary beam or plate theory or from Equations (3.7). Thus the additional terms in the bending stress resultants in Flugge's theory, being all either displacements or the first derivatives of displacements, are smaller than the second derivative terms. All terms in the moment stress resultants common to both theories are second derivatives.

The stress function approach of Edmondson and others should remain valid for Flugge theory if the additional small terms are added to the representation of stress resultants and displacements, and all terms are expressed as functions of $\tilde{\psi}$. It will be recalled that the

defining differential equation for $\tilde{\psi}$, Equation (3.3), involves Donnell's simplified equilibrium equations in its derivation. On this basis, it can be argued that a new differential equation in $\tilde{\psi}$ should be formulated for use with Flugge theory. There are important reasons why this is not done.

If the second equilibrium equation is not simplified by Donnell's assumptions, no simple stress function can be found which will satisfy the first two equilibrium equations. The compatibility equation in terms of any stress function is unduly complicated as a result.

If, as in Flugge theory:

$$n_{y\phi} \neq n_{\phi y} \qquad\qquad m_{y\phi} \neq m_{\phi y}$$

the stress function problem is further complicated, and the effect noted above is worsened. The third, fourth and fifth equilibrium equations can still be combined to yield one differential equation in w and stress resultants, but, lack of a stress function in simple form makes this equation also very complicated.

Due to the combined effects of the above complications, no simple function such as $\tilde{\psi}$ can be found. Thus there is no series expression for stress resultants and displacements. What finally results is only a long and very complicated differential equation with no apparent closed form solution. A brief outline of such an attempt is given in Appendix C.

It has just been shown that the differential equation defining $\tilde{\psi}$ is left in its original form for reasons of difficulty in formulation. This, of course, forces the series expression for $\tilde{\psi}$ to be identical with

that used by Edmondson and other researchers, with the exception of modification of symmetry to account for internal pressure loading. That this form should be valid for Flugge theory follows from the following considerations:

1. All stress resultants and displacements are expressed as slightly different functions of $\tilde{\psi}$ than those of the original approach. The difference is due to the addition of small terms found in Flugge theory but not in Donnell theory. These added terms should improve accuracy and account for the terms in the equilibrium equations retained by Flugge but dropped by Donnell due to their small magnitude compared to other terms.
2. The added terms have been shown previously to be small compared to the terms common to both theories. This implies that the original defining differential equation for $\tilde{\psi}$ should be approximately satisfied.
3. All the numeric constants, A_{n1}, A_{n2}, B_{n1}, and B_{n2}, which appear in $\tilde{\psi}$ are determined by forcing the stress resultants and displacements, with added terms to satisfy prescribed boundary conditions as represented by Reidelbach's junction compatibility equations. The least squares approach used should eliminate errors caused solely by using the original form of $\tilde{\psi}$ with small added terms. This should be true within the degree of accuracy of Reidelbach's equations. In the next section Reidelbach's equations are modified so that the

small shell terms conform to Flugge theory as much as possible. The large shell terms pose no limit to the accuracy of the final results since the large shell is sufficiently described by Donnell theory in the region of the intersection.

The approach of using the original form of $\tilde{\psi}$, but with proper added terms, may not be as accurate as a rigorously derived solution following the method of Appendix C. The original series has the advantage of a relatively simple form, which is not attainable with the approach of Appendix C. The use of the original series with added terms to account for the differences due to the use of Flugge theory should be more accurate than a theory based on Donnell's assumptions alone.

Indicating the stress resultants found by Donnell theory by the subscript D, the terms in Flugge theory are:

$$n_\phi = (n_\phi)_D - (D/\rho_o^3)(w + \frac{\partial^2 w}{\partial \phi^2})$$

$$n_y = (n_y)_D + (D/\rho_o)(\partial^2 w/\partial y^2)$$

$$n_{\phi y} = (n_{\phi y})_D + (D/\rho_o^3)(1 - \nu)(\partial u/\partial \phi - \rho_o \partial^2 w/\partial y \partial \phi)/2$$

$$n_{y\phi} = (n_{y\phi})_D + (D/\rho_o^3)(1 - \nu)(\rho_o\ \partial v/\partial y + \rho_o \partial^2 w/\partial \phi \partial y)/2$$

$$m_\phi = (m_\phi)_D - Dw/\rho_o^2$$

$$m_y = (m_y)_D - (D/\rho_o^2)(\rho_o\ \partial u/\partial y + \nu \frac{\partial v}{\partial \phi})$$

$$m_{\phi y} = (m_{\phi y})_D + (D/\rho_o^2)(1 - \nu)(\partial u/\partial \phi - \rho_o\ \partial v/\partial y)/2$$

$$m_{y\phi} = (m_{y\phi})_D - (D/\rho_o^2)(\rho_o\ \partial v/\partial y)(1 - \nu) \qquad (3.8)$$

From Donnell theory, it follows that:

$$(n_{\phi y})_D = (n_{y\phi})_D \qquad (m_{\phi y})_D = (m_{y\phi})_D$$

All stress resultants and displacements found by Donnell theory are expressed in terms of $\tilde{\psi}$. Hence, stress resultants for Flugge theory are expressed in terms of the series $\tilde{\psi}$ by the use of Equations (3.8).

Modification of Reidelbach's Junction Compatibility Equations

The eight equations due to Reidelbach (3) which express compatibility between the two shells at their intersection allow a numeric solution of the unknown constants in each series. The final forms of Reidelbach's equations involve assumptions which are equivalent to the simplifications Donnell introduced. Strictly speaking, Reidelbach's equations apply only for Donnell theory. As with the differential equation for $\tilde{\psi}$, the simplification of the second equilibrium equation must be used in order to have a simple stress function which satisfies equilibrium. Reidelbach's derivation involves such a stress function. Using the same arguments which previously justified the acceptance of the unaltered form of $\tilde{\psi}$, this simplification is allowable. It should have even less significant consequences in the derivation of Reidelbach's equations because Donnell theory is still used for the large shell, and Flugge theory for only the small shell.

Reidelbach also assumes in his final equations that $n_{y\phi}$ equals $n_{\phi y}$, and that $m_{y\phi}$ is equal to $m_{\phi y}$. His equations are easily modified on this account to fit Flugge theory, as shown in Appendix D.

Edmondson (5) derives a revised form of Reidelbach's first and second equations. These revised forms have the virtue of eliminating all displacements of the large shell, except for the radial displacement, from the junction compatibility equations. The revised forms of the first two equilibrium equations are used in place of the original forms. With this substitution and the modifications in Appendix D, the final forms of Reidelbach's junction compatibility equations are:

$$\frac{\rho_o}{ET}(N_\phi - \nu N_\rho) = [-w + \frac{\partial v}{\partial \phi} - \frac{m}{2}\frac{\partial u}{\partial \phi}\sin 2\phi - \frac{mu}{2}(1 + \cos 2\phi) - mW\cos^2\phi]$$

$$\frac{\rho_o}{ET}[2(1+\nu)\frac{\partial N_{\rho\phi}}{\partial \phi} + (1+\nu)(N_\rho - N_\phi) - (\frac{\partial N_\phi}{\partial \rho} - \nu\frac{\partial N_\rho}{\partial \rho})]$$

$$= \{-\frac{\partial^2 w}{\partial \phi^2} - \frac{\partial v}{\partial \phi} - m[\frac{\partial^2 u}{\partial \phi^2}\sin^2\phi + \frac{3}{2}\frac{\partial u}{\partial \phi}\sin 2\phi + u(\cos 2\phi)]$$

$$- m[\frac{\partial W}{\partial \phi}\sin 2\phi - \frac{\partial W}{\partial \rho}\cos^2\phi + W\cos 2\phi]\}$$

$$W + v\frac{m}{2}\sin 2\phi + u - w\frac{m}{2}(1 - \cos 2\phi) = 0$$

$$N_\rho - n_y\frac{m}{2}(1 - \cos 2\phi) - q_\phi\frac{m}{2}\sin 2\phi - q_y = 0$$

$$N_{\rho\phi} + n_{y\phi} + (n_\phi - n_y)\frac{m}{2}\sin 2\phi = 0$$

$$Q_\rho + \frac{m}{2}N_\rho(1 - \cos 2\phi) - (n_y + m\frac{n_{y\phi} + n_{\phi y}}{2}\sin 2\phi) = 0$$

$$M_\rho - m_y - \frac{m_{y\phi} + m_{\phi y}}{2}m\sin 2\phi = 0$$

$$\frac{\partial W}{\partial \tilde{\rho}} + \frac{\partial w}{\partial \tilde{y}} + \frac{\partial w}{\partial \phi}\frac{m}{2}\sin 2\phi = 0 \qquad (3.9)$$

The small shell analysis is now in the form of Flugge theory.

CHAPTER IV

STRESS RESULTANTS AND DISPLACEMENTS EXPRESSED IN SERIES FORM

A numerical least squares technique is used to solve for the series coefficients, which are the constants of integration in the solutions to Equations (3.3) and (3.4). The junction compatibility equations are solved, for the chosen number of terms in each series, at a large number of points on the intersection of the shell and the nozzle. Due to the large number of calculations, a computer is used to find the numeric value of the multipliers of each series coefficient at every chosen point. Thus, the system of equations expressing the compatibility of the two shells in the intersection is highly over-determined. By a least squares technique, the large number of equations are reduced to the number of equations sufficient for determination of the series coefficients. The junction compatibility equations are satisfied at all chosen points in a least squares sense.

In order to use this approach, all needed stress resultants and their derivatives are first expressed in terms of the series ψ or $\tilde{\psi}$, whichever is appropriate.

Large Shell

The series describing the behavior of the large shell has been previously described as:

$$\psi = \sum_{n=1}^{\infty} a_{2n-1}(1+i)\sinh[(1-i)\xi\rho\cos\phi]H_{2n-1}^{(1)}$$

$$[(2i)^{1/2}\xi\rho]\cos(2n-1)\phi + a_{2n}\cosh[(1-i)\xi\rho\cos\phi]H_{2n}^{(1)}$$

$$[(2i)^{1/2}\xi\rho]\cos 2n\phi \qquad (4.1)$$

which is a solution to the differential equation defining ψ, Equation (3.4). This differential equation exists in its current form due to the fact that:

$$\psi = \frac{W}{\rho_o} + iK\Phi \qquad (4.2)$$

In view of Equation (4.2), W is easily expressed in terms of ψ as:

$$W = \rho_o \mathrm{Re}(\psi) \qquad (4.3)$$

Similarly, the stress function $\bar{\Phi}$ is given by:

$$\bar{\Phi} = K^{-1}\mathrm{Im}(\psi) \qquad (4.4)$$

It is easily shown that the real or imaginary part of a partial derivative is equal to the partial derivative of the real or imaginary part, respectively. By virtue of the above property and the definition of the stress function $\bar{\Phi}$, the stress resultant N_x is found to be:

$$N_x = \frac{\partial^2\bar{\Phi}}{R^2\partial\theta^2} = \frac{1}{KR^2}\mathrm{Im}\left(\frac{\partial^2\psi}{\partial\theta^2}\right) \qquad (4.5)$$

In view of the previous restriction that m must be less than 1/3 and the resulting simplification of the intersection to approximate plane, all quantities in the X,θ coordinate system are replaced by corresponding quantities in the ρ,ϕ coordinate system. The equations leading up to the defining differential equation in ψ, Equation (3.4), change in coordinates, as does Equation (3.4) itself. This

transformation has been carried out by previous researchers, and is the reason the quantities appearing in Equation (4.1) are functions of ρ and ϕ, instead of X and θ. Expressions in terms of X and θ can be directly converted or completely rederived in terms of ρ and ϕ. Reidelbach (3, p. 26) rederives the stress function in polar coordinates, and states normal and shearing stress resultants in this form. The results are then:

$$N_\rho = \frac{1}{\rho}\frac{\partial\Phi}{\partial\rho} + \frac{1}{\rho^2}\frac{\partial^2\Phi}{\partial\phi^2}$$

$$N_\phi = \frac{\partial^2\Phi}{\partial\rho^2}$$

$$N_{\rho\phi} = N_{\phi\rho} = \frac{1}{\rho^2}\frac{\partial\Phi}{\partial\phi} - \frac{1}{\rho}\frac{\partial^2\Phi}{\partial\rho\partial\phi} \tag{4.6}$$

By use of Equation (4.4), Equations (4.6) can be rewritten as:

$$N_\rho = \frac{1}{K}\,\mathrm{Im}\left(\frac{1}{\rho}\frac{\partial\psi}{\partial\rho} + \frac{1}{\rho^2}\frac{\partial^2\psi}{\partial\phi^2}\right)$$

$$N_\phi = \frac{1}{K}\,\mathrm{Im}\left(\frac{\partial^2\psi}{\partial\rho^2}\right)$$

$$N_{\rho\phi} = N_{\phi\rho} = \frac{1}{K}\,\mathrm{Im}\left(\frac{1}{\rho^2}\frac{\partial\psi}{\partial\phi} - \frac{1}{\rho}\frac{\partial^2\psi}{\partial\rho\partial\phi}\right) \tag{4.7}$$

Timoshenko (6, p. 282) has an excellent derivation of the moments in terms of the deflection W. Although Timoshenko's derivation is for a flat plate, physically it is a valid representation for the large shell in the region of intersection. The intersection, it will be recalled, is taken to be approximately plane. The intersection region of the large shell appears as deformed flat plate with a circular central hole. Due to these same assumptions, W is approximately normal to this plate. The moments in terms of W are given by:

$$M_\rho = -D'\left(\frac{\partial^2 W}{\partial X^2} + \nu \frac{\partial^2 W}{\partial Y^2}\right)$$

$$M_\phi = =D'\left(\frac{\partial^2 W}{\partial Y^2} + \nu \frac{\partial^2 W}{\partial X^2}\right)$$

$$M_{\rho\phi} = -M_{\phi\rho} = (1 - \nu)D'\left(\frac{\partial^2 W}{\partial Y \partial X}\right) \qquad (4.8)$$

These same equations for a shell element are found in Timoshenko (6, p. 523) and in Flugge (2, p. 217). Since Donnell's theory treats a shell element as a special case, it is not surprising that Timoshenko's plate derivation arrives at the same results. When Equations (4.8) are converted to polar coordinates, as in Timoshenko (6, p. 283), the results are:

$$M_\rho = -D'\left[\frac{\partial^2 W}{\partial \rho^2} + \nu \left(\frac{1}{\rho}\frac{\partial W}{\partial \rho} + \frac{1}{\rho^2}\frac{\partial^2 W}{\partial \phi^2}\right)\right]$$

$$M_\phi = -D'\left(\frac{1}{\rho}\frac{\partial W}{\partial \rho} + \frac{1}{\rho^2}\frac{\partial^2 W}{\partial \phi^2} + \nu \frac{\partial^2 W}{\partial \rho^2}\right)$$

$$M_{\rho\phi} = -M_{\phi\rho} = (1 - \nu)D'\left(\frac{1}{\rho}\frac{\partial^2 W}{\partial \rho \partial \phi} - \frac{1}{\rho^2}\frac{\partial W}{\partial \phi}\right) \qquad (4.9)$$

By use of Equation (4.3), Equation (4.9) can be rewritten as:

$$M_\rho = -D'\rho_o \mathrm{Re}\left[\frac{\partial^2 \psi}{\partial \rho^2} + \nu \left(\frac{1}{\rho}\frac{\partial \psi}{\partial \rho} + \frac{1}{\rho^2}\frac{\partial^2 \psi}{\partial \phi^2}\right)\right]$$

$$M_\phi = -D'\rho_o \mathrm{Re}\left(\frac{1}{\rho}\frac{\partial \psi}{\partial \rho} + \frac{1}{\rho^2}\frac{\partial^2 \psi}{\partial \phi^2} + \nu \frac{\partial^2 \psi}{\partial \rho^2}\right)$$

$$M_{\rho\phi} = -M_{\phi\rho} = (1 - \nu)\rho_o D' \mathrm{Re}\left(\frac{1}{\rho}\frac{\partial^2 \psi}{\partial \rho \partial \phi} - \frac{1}{\rho^2}\frac{\partial \psi}{\partial \phi}\right) \qquad (4.10)$$

The quantities U_ρ and U_ϕ never appear due to the revision of Reidelbach's first and second junction equations. These two quantities are not of primary interest. Since they would have meaning only very close to the intersection curve (where the plane approximation is good), they are not formulated.

The transverse shear, Q_ρ, is used in the junction equations, and hence must be expressed in terms of ψ. In many shell problems the transverse shear is replaced by Kirchhoff shear in order that four boundary conditions include the effects of five stress resultants on a free edge. Kirchhoff shear need not be used in Reidelbach's equations. No free edge is involved so no problem in including all known boundary conditions exists. When considered in the shell element equilibrium, effective forces added to shears cancel out. The final argument for the use of transverse shears in Reidelbach's equations is that Reidelbach himself takes this approach. The equations from which Reidelbach derives his junction equations involve transverse, not Kirchhoff forces.

A radial element of the large shell is shown in Figure 5, page 31. Only stress resultants which contribute to moment about the ϕ direction are shown. Figure 6, page 31, shows the contribution of M_ϕ. Equilibrium of moments about the ϕ direction yields:

$$(M_\rho + \frac{\partial M_\rho}{\partial \rho} d\rho)(\rho + d\rho)d\phi - M_\rho \rho d\phi - Q_\rho(\rho d\phi)\frac{d\rho}{2}$$
$$- (Q_\rho + \frac{\partial Q_\rho}{\partial \rho} d\rho)(\rho + d\rho)d\phi \frac{d\rho}{2} - (M\phi d\phi)d\rho$$
$$- M_{\phi\rho} d\rho + (M_{\phi\rho} + \frac{\partial M_{\phi\rho}}{\phi\rho} d\phi)d\rho = 0$$

Simplifying and neglecting higher order terms:

$$\frac{\partial M_\rho}{\partial \rho} \rho - \rho Q_\rho - M_\phi + M_\rho + \frac{\partial M_{\phi\rho}}{\partial \phi} = 0$$

So that:

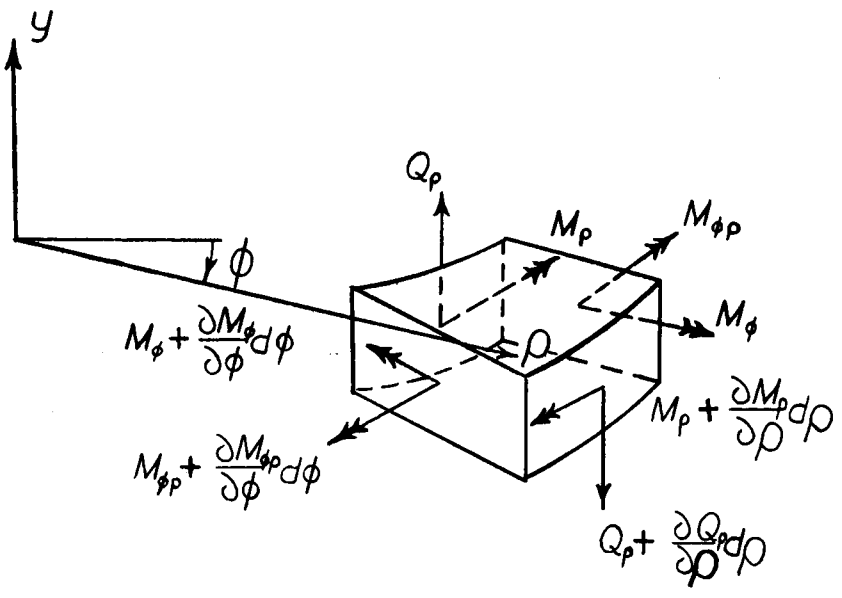

FIGURE 5

RADIAL ELEMENT OF LARGE SHELL

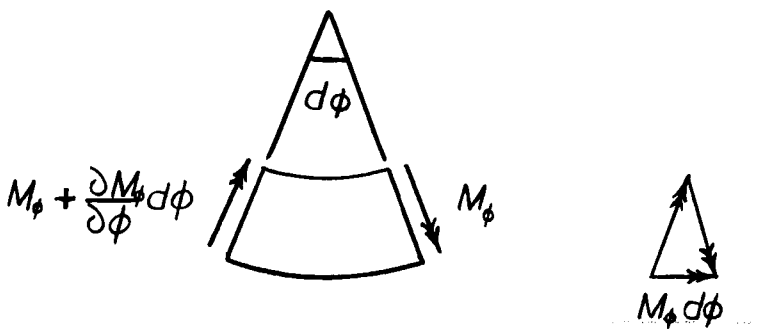

FIGURE 6

CONTRIBUTION OF M_ϕ

$$Q_\rho = \frac{1}{\rho}(M_\rho + \frac{\partial M_\rho}{\partial \rho}\rho - M_\phi + \frac{\partial M_{\phi\rho}}{\partial \phi}) \tag{4.11}$$

By use of Equations(4.9), Q_ρ may be expressed in terms of W:

$$Q_\rho = \frac{-D'}{\rho}[\frac{\partial^2 W}{\partial \rho^2} + \frac{\partial^3 W}{\partial \rho^3}\rho - \frac{1}{\rho}\frac{\partial W}{\partial \rho} + \frac{1}{\rho}\frac{\partial^3 W}{\partial \rho \partial \phi^2} - \frac{2}{\rho^2}\frac{\partial^2 W}{\partial \phi^2}]$$

Finally, by use of Equation (4.3), Q_ρ is expressed in terms of ψ:

$$Q_\rho = \frac{-D'\rho_o}{\rho}\,\mathrm{Re}(\frac{\partial^2 \psi}{\partial \rho^2} + \frac{\partial^3 \psi}{\partial \rho^3}\rho - \frac{1}{\rho}\frac{\partial \psi}{\partial \rho} + \frac{1}{\rho}\frac{\partial^3 \psi}{\partial \rho \partial \phi^2} - \frac{2}{\rho^2}\frac{\partial^2 \psi}{\partial \phi^2}) \tag{4.12}$$

All necessary quantities for the large shell are now in terms of ψ. These are Equations (4.3), (4.7), (4.10) and (4.12).

Donnell Theory Terms for the Small Shell

The series describing the small shell has been previously written as:

$$\tilde{\psi} = \sum_{n=0}^{\infty}\sum_{j=1}^{2}[A_{2n,j}(\cos \omega_{2n,j}\tilde{y} + i \sin \omega_{2n,j}\tilde{y})$$

$$+ B_{2n,j}(-\sin \omega_{2n,j}\tilde{y} + i \cos \omega_{2n,j}\tilde{y})]e^{\pi_{2n,j}\tilde{y}}\cos 2n\phi \tag{4.13}$$

As with the large shell, the form of function $\tilde{\psi}$ is originally defined in terms of radial displacement and a stress function.

$$\tilde{\psi} = \frac{w}{\rho_o} + ik\tilde{\Phi} \tag{4.14}$$

Or, in terms of $\tilde{\psi}$:

$$w = \rho_o \mathrm{Re}(\tilde{\psi}) \tag{4.15}$$

$$\tilde{\Phi} = k^{-1}\mathrm{Im}(\tilde{\psi}) \tag{4.16}$$

The previously introduced notation is used, letting subscript D indicate that part of stress resultants and displacements common to

both Donnell and Flugge theory. Then, as for the large shell:

$$(n_y)_D = \frac{\partial^2 \overset{\sim}{\Phi}}{\rho_o^2 \, \partial\phi^2}$$

$$(n_\phi)_D = \frac{\partial^2 \overset{\sim}{\Phi}}{\partial y^2}$$

$$(n_{y\phi})_D = (n_{\phi y})_D = - \frac{\partial^2 \overset{\sim}{\Phi}}{\rho_o \partial y \partial \phi} \tag{4.17}$$

By virtue of Equation (4.16), Equations (4.17) may be rewritten as:

$$(n_y)_D = k^{-1} \mathrm{Im}\left(\frac{\partial^2 \overset{\sim}{\psi}}{\rho_o^2 \, \partial\phi^2}\right)$$

$$(n_\phi)_D = k^{-1} \mathrm{Im}\left(\frac{\partial^2 \overset{\sim}{\psi}}{\partial y^2}\right)$$

$$(n_{\phi y})_D = (n_{y\phi})_D = k^{-1} \mathrm{Im}\left(- \frac{\partial^2 \overset{\sim}{\psi}}{\rho_o \partial y \partial \phi}\right) \tag{4.18}$$

It is noted that although the stress resultants thus far expressed are given the subscript D, the radial displacement W is given no subscript. The omission of subscript indicates that W as expressed in Equation (4.14) is the correct W for Flugge theory. This is due to the determination of coefficients by forcing a fit to the junction compatibility equations, and the way $\overset{\sim}{\psi}$ is defined. The same holds true for $\overset{\sim}{\Phi}$. Its definition has not changed in terms of $\overset{\sim}{\psi}$. The addition of terms to the Donnell stress resultants recognizes the fact that, if these stress resultants were defined in terms of $\overset{\sim}{\Phi}$ alone, the Flugge equilibrium equations would not be satisfied. The small added terms then represent those quantities needed to account for terms which Donnell dropped. The heart of this

approach assumes that these added terms are small enough to allow the same differential equation defining $\tilde{\psi}$ to be used, but large enough that they are still of significance.

Mention should be made of the choice of coordinate system for the small shell. Certain previous assumptions involved neglecting effects of stress and strain in the thickness direction, thin shells, and small displacements compared to the radii of the shells. In view of these assumptions, it would be inappropriate to express the small shell quantities in terms of ρ, since ρ is the thickness direction of the small shell. The intersection is assumed to be approximately circular. The small cylinder is circular in cross-section; hence, it is sufficient to use the original cylindrical coordinates, neglecting changes in the radial direction. No transformation of coordinates need take place, as is used with the large shell.

The parts of the moment resultants common to both theories are easily formulated. As before, following Timoshenko (6, p. 523) or Flugge (2, p. 217), the results are given as:

$$(m_y)_D = -D\left(\frac{\partial^2 w}{\partial y^2} + \nu \frac{\partial^2 w}{\rho_o^2 \partial \phi^2}\right)$$

$$(m_\phi)_D = -D\left(\frac{\partial^2 w}{\rho_o^2 \partial \phi^2} + \nu \frac{\partial^2 w}{\partial y^2}\right)$$

$$(m_{y\phi})_D = (m_{\phi y})_D = -(1 - \nu)D\left(\frac{\partial^2 w}{\rho_o \partial y \partial \phi}\right) \qquad (4.19)$$

Here, the fact that $(m_{y\phi})_D$ and $(m_{\phi y})_D$ have the same sign, while for the large shell, opposite signs were evident, arises from the difference in positive directions for $M_{\rho\phi}$ and $m_{y\phi}$. By use of Equation

(4.15), Equations (4.19) may be rewritten as:

$$(m_y)_D - D\rho_o Re\left(\frac{\partial^2\tilde{\psi}}{\partial y^2} + \nu\,\frac{\partial^2\tilde{\psi}}{\rho_o^2\partial\phi^2}\right)$$

$$(m_\phi)_D = -D\rho_o Re\left(\frac{\partial^2\tilde{\psi}}{\rho_o^2\partial\phi^2} + \nu\,\frac{\partial^2\tilde{\psi}}{\partial y^2}\right)$$

$$(m_{y\phi})_D = (m_{\phi y})_D = -(1-\nu)D\rho_o Re\left(\frac{\partial^2\tilde{\psi}}{\rho_o^2\partial y\partial\phi}\right) \qquad (4.20)$$

As with the large shell, the transverse shears are used in the juncture compatibility equations. The argument for not using Kirchhoff shears is exactly the same as for the large shell.

Since no change in coordinates is involved in the small shell, the original equilibrium equations are used to express the transverse shears as:

$$(q_\phi)_D = \frac{1}{\rho_o}\,\frac{\partial(m_\phi)_D}{\partial\phi} + \frac{\partial(m_{y\phi})_D}{\partial y}$$

$$(q_y)_D = \frac{\partial(m_y)_D}{\partial y} + \frac{1}{\rho_o}\,\frac{\partial(m_{\phi y})_D}{\partial\phi}$$

By use of Equations (4.20), the transverse shears may be expressed in terms of $\tilde{\psi}$.

$$(q_\phi)_D = -D\rho_o Re\left(\frac{\partial^3\tilde{\psi}}{\rho_o^3\partial\phi^3} + \frac{\partial^3\tilde{\psi}}{\rho_o\partial\phi\partial y^2}\right)$$

$$(q_y)_D = -D\rho_o Re\left(\frac{\partial^3\tilde{\psi}}{\partial y^3} + \frac{\partial^3\tilde{\psi}}{\rho_o^2\partial y\partial\phi^2}\right) \qquad (4.21)$$

The displacements u and v must also be expressed in terms of $\tilde{\psi}$. The strain displacement relations, with both z and w taken positive inward, can be combined with Hooke's law to yield:

$$\frac{\partial u}{\partial y} - z\frac{\partial^2 w}{\partial y^2} = \frac{1}{E}(\sigma_y - \nu\sigma_\phi)$$

$$\frac{\partial v}{\rho_o \partial \phi} - \frac{w}{\rho_o - z} - \frac{z}{\rho_o(\rho_o - z)}\frac{\partial^2 w}{\partial \phi^2} = \frac{1}{E}(\sigma_\phi - \nu\sigma y) \qquad (4.22)$$

Normal stresses can also be expressed in terms of stress resultants.

$$\sigma_y = \frac{n_y}{t} + \frac{12\, m_y z}{t^3}$$

$$\sigma_\phi = \frac{n_\phi}{t} + \frac{12\, m_\phi z}{t^3} \qquad (4.23)$$

The positive signs in Equations (4.23) arise from the consideration that positive bending moments should give negative stresses at the outside surface.

Strictly speaking, the stress resultants n_y and n_ϕ are defined by:

$$n_\phi = \int_{-t/2}^{t/2} \sigma_\phi dz$$

$$n_y = \int_{-t/2}^{t/2} \sigma_y(1 - \frac{z}{\rho_o})dz$$

For a thin shell, the membrane stresses on the surfaces ϕ = constant can be assumed to have a uniform distribution with little or no loss of accuracy. The expression for n_y, however, indicates that due to the trapezoidal shape of the faces y = constant, a slightly non-uniform distribution of membrane stresses on these faces is possible. This implies that the value of σ_y at the mid-surface is not necessarily the average across the surface. Little error should be introduced by assuming a uniform distribution of membrane stresses on all faces of a shell element. In fact, it

appears that this assumption must be used in order to obtain a solution. In view of this assumption, the membrane stresses are represented as:

$$\sigma_\phi = \frac{n_\phi}{t}$$

$$\sigma_y = \frac{n_y}{t} \tag{4.24}$$

Since u and v are the displacements of the mid-surface of the shell, they can be thought of as displacements due to the membrane part of the problem. Thus, u and v can be expressed independently of bending terms. If z is set equal to zero in Equations (4.22) and Equations (4.24), the membrane solution results.

$$\frac{\partial u}{\partial y} = \frac{1}{E}(\sigma_y - \nu\sigma_\phi) = \frac{1}{Et}(n_y - \nu n_\phi)$$

$$\frac{\partial v}{\rho_o \partial \phi} = \frac{1}{E}(\sigma_\phi - \nu\sigma_y) + \frac{w}{\rho_o} = \frac{1}{Et}(n_\phi - \nu n_y) + \frac{w}{\rho_o} \tag{4.25}$$

The displacements u and v may be determined by integration of Equations (4.25).

$$u = \frac{1}{Et}\int (n_y - \nu n_\phi)\, dy + R_1$$

$$v = \rho_o \int [\frac{1}{Et}(n_\phi - \nu n_y) + \frac{w}{\rho_o}] d\phi + R_2 \tag{4.26}$$

Where terms R_1 and R_2 are used to express rigid body displacement of the small shell due to displacements of the large shell. Due to the consideration of the internal pressure case only, R_2 is set equal to zero. That there is no rigid body displacement in the v direction (i.e., no rotation) is seen from symmetry. Due to consideration of the large shell as a shallow shell in the region of

intersection, there should be no rigid body radial displacement, w. Setting of the rigid body part of w equal to zero should be no worse than the assumption of a circular intersection. The displacements of the small shell are always measured from the original location of the origin of u, v, and w. Hence, the rigid body displacement in the u direction, R_1, will be a large contribution to the total u. This is due to the radial displacement of the large shell, which is certainly not negligible.

By use of Equations (4.15, 4.18, 4.24, and 4.25) the displacements u and v are expressed in terms of $\tilde{\psi}$. Since Equations (4.18) deal only with the part of the stress resultants common to both Donnell and Flugge theory, it is evident that their use yields only that part of displacements common to both theories.

$$(u)_D = \frac{1}{Etk} \text{Im} \int \left(\frac{\partial^2 \tilde{\psi}}{\rho_o^2 \partial \phi^2} - \nu \frac{\partial^2 \tilde{\psi}}{\partial y^2}\right) dy + R_1$$

$$(v)_D = \frac{\rho_o}{Etk} \text{Im} \int \left(\frac{\partial^2 \tilde{\psi}}{\partial y^2} - \nu \frac{\partial^2 \tilde{\psi}}{\rho_o^2 \partial \phi^2}\right) d\phi + \rho_o \text{Re} \int \tilde{\psi} d\phi \qquad (4.27)$$

Added Quantities Due to Flugge Theory

Since the small shell is being treated by Flugge's theory, the added quantities on stress resultants and displacements must be formulated. The key to this approach is that w remains the same for both theories. The reasoning behind this approach was given in the last section. All added terms have previously been given in terms of displacements and their derivatives. The original displacements have already been formulated. It is possible to express the added

displacements in terms of w. Thus, since the complete w is already known, solution of all added quantities is possible.

The added quantities, both stress resultants and displacements, are denoted by a subscript A. The added stress resultants may be formulated as:

$$(n_\phi)_A = \frac{-D}{\rho_o^3}\left(w + \frac{\partial^2 w}{\partial \phi^2}\right)$$

$$(n_y)_A = \frac{D}{\rho_o^3}\left(\rho_o^2 \frac{\partial^2 w}{\partial y^2}\right)$$

$$(n_{\phi y})_A = \frac{D(1-\nu)}{2\rho_o^3}\left(\frac{\partial u}{\partial \phi} - \rho_o \frac{\partial^2 w}{\partial y \partial \phi}\right)$$

$$(n_{y\phi})_A = \frac{D(1-\nu)}{2\rho_o^3}\left(\rho_o \frac{\partial v}{\partial y} + \rho_o \frac{\partial^2 w}{\partial y \partial \phi}\right)$$

$$(m_\phi)_A = \frac{-D}{\rho_o^2} w \qquad (4.28)$$

$$(m_y)_A = \frac{-D}{\rho_o^2}\left(\rho_o \frac{\partial u}{\partial y} + \nu \frac{\partial v}{\partial \phi}\right)$$

$$(m_{\phi y})_A = \frac{D}{2\rho_o^2}(1-\nu)\left(\frac{\partial u}{\partial \phi} - \rho_o \frac{\partial v}{\partial y}\right)$$

$$(m_{y\phi})_A = \frac{-D}{\rho_o^2}(1-\nu)\rho_o \frac{\partial v}{\partial y}$$

$$(q_\phi)_A = \frac{-D}{\rho_o^3}\left[\frac{\partial w}{\partial \phi} + (1-\nu)\rho_o^2 \frac{\partial^2 v}{\partial y^2}\right]$$

$$(q_y)_A = \frac{-D}{\rho_o^3}\left[\rho_o^2 \frac{\partial^2 u}{\partial y^2} + \rho_o \nu \frac{\partial^2 v}{\partial y \partial \phi} + \frac{(1-\nu)}{2}\left(\rho_o \frac{\partial^2 v}{\partial y \partial \phi} - \frac{\partial^2 u}{\partial \phi^2}\right)\right]$$

Note particularly that the addition of these quantities to the original quantities common to both Donnell and Flugge theory has as one effect that $n_{\phi y}$ no longer equals $n_{y\phi}$, and $m_{\phi y}$ no longer equals

$m_{y\phi}$. It is also important to recognize that the displacements in Equations (4.28) have no subscripts. This indicates that total displacements are used in formulating the added quantities. Since most of the added quantities involve either u or v, it is necessary to first derive the complete displacements.

As previously stated, w is complete as originally defined by Equation (4.15). The other small shell displacements have additional parts due to the way they are found. Similar to the Equations (4.26) then, the added displacements may be expressed by:

$$(u)_A = \frac{1}{Et} \int [(n_y)_A - \nu\ (n_\phi)_A] dy$$

$$(v)_A = \frac{\rho_o}{ET} \int\ [(n_\phi)_A - \ \nu(n_y)_A] d\phi \qquad (4.29)$$

Only two major differences in form between Equations (4.26) and (4.29) are evident. The added displacements have no constants of integration (which are used to represent rigid body motion). Since the rigid body displacements are due to deformation of the large shell, and the large shell is represented by Donnell's theory, the Donnell terms should include all rigid body displacement. In any case, the constants which could result from Equations (4.29) can be combined with those due to integration of Equations (4.25). That the only non-zero rigid body displacement is in the u direction has already been shown. Further, this rigid body displacement cannot be a function of ϕ without disturbing the symmetry of the problem or violating the basic assumptions of intersection behavior.

The other difference between Equations (4.26) and (4.29) is that the radial displacement term in the integration yielding v is missing. This is due to the fact that w is complete as given, i.e.:

$$(w)_A = 0.$$

By use of Equations (4.28), the added displacements can be expressed in terms of w as:

$$(u)_A = \frac{D}{\rho_o^3 Et} \int [\rho_o^2 \frac{\partial^2 w}{\partial y^2} + \nu \ (w + \frac{\partial^2 w}{\partial \phi^2})] dy$$

$$(v)_A = \frac{-D}{\rho_o^2 Et} \int (w + \frac{\partial^2 w}{\partial \phi^2} + \nu \rho_o^2 \ \frac{\partial^2 w}{\partial y^2}) d\phi$$

And, by use of Equation (4.15), $(u)_A$ and $(v)_A$ are expressed in terms of the series $\tilde{\psi}$:

$$(u)_A = \frac{D}{\rho_o^2 Et} \text{ Re } \int [(\rho_o^2 \frac{\partial^2 \tilde{\psi}}{\partial y^2} + \nu(\tilde{\psi} + \frac{\partial^2 \tilde{\psi}}{\partial \phi^2})] dy \qquad (4.30)$$

$$(v)_A = \frac{-D}{\rho_o Et} \text{ Re } \int (\tilde{\psi} + \frac{\partial^2 \tilde{\psi}}{\partial \phi^2} + \nu \rho_o^2 \frac{\partial^2 \tilde{\psi}}{\partial y^2}) d\phi$$

With the formulation of $(u)_A$ and $(v)_A$ the formulation of the total displacements is now complete.

$$u = (u)_D + (u)_A$$

$$v = (v)_D + (v)_A$$

$$w_A = 0$$

$$(w)_D = w$$

All other added quantities can now be expressed in terms of $\tilde{\psi}$. Since all quantities common to both theories have already been expressed in terms of $\tilde{\psi}$, all total quantities are now completely described by $\tilde{\psi}$.

Particular Solutions for Both Shells

The quantities thus far given for each shell describe the modifications in behavior of the shells near the intersection. These quantities expressed in series form should become smaller as the distances from the intersection increase. To those series quantities must be added particular solutions which describe the behavior the shells would exhibit if no intersection were present.

In order to arrive at these particular solutions, some assumptions must be made as to the constraints on each shell. It is assumed here that each shell is "capped" far away from the intersection. The equilibrium equations are greatly simplified due to symmetry. The solution to this problem is well known and may be found in Timoshenko (7, p. 45) and Timoshenko (6, p. 482).

$$n_\phi = P\rho_o$$

$$n_y = \frac{P\rho_o}{2}$$

$$N_x = \frac{PR}{2}$$

$$N_\theta = PR$$

$$w = -\frac{P\rho_o^2}{Et}\left(1 - \frac{\nu}{2}\right)$$

$$W = \frac{-PR^2}{ET}\left(1 - \frac{\nu}{2}\right)$$

$$u = \frac{P\rho_o}{ET}\left(\frac{1}{2} - \nu\right)y$$

$$U = \frac{PR}{ET}\left(\frac{1}{2} - \nu\right)X$$

All other quantities in the particular solutions are equal to zero. The particular solutions for the large shell must be converted to the (ρ,ϕ) coordinate system. Edmondson (5) has made this conversion and arrived at:

$$N_\rho = \frac{PR}{4}(3 - \cos 2\phi)$$

$$N_\phi = \frac{PR}{4}(3 + \cos 2\phi)$$

$$N_{\rho\phi} = \frac{PR}{4}(\sin 2\phi)$$

$$U_\rho = \frac{PR}{4ET}[1 - 2\nu](1 + \cos 2\phi)\rho$$

$$U_\phi = \frac{-PR}{4ET}[1 - 2\nu](\sin 2\phi)\rho$$

The addition of the particular solutions to the series form of stress resultants and displacements completely specifies the behavior of the shells.

Dimensionless Quantities

In order to facilitate comparison of results, all quantities are made dimensionless. All displacements for both shells are divided by ρ_o. Normal and shear stress resultants have units of the type psi·inch. Accordingly, these terms are divided by $P\rho_o$. Moment stress resultants have units of the type psi·inch^2. Moment stress resultants are divided by $P\rho_o^2$.

Although stress resultants are made dimensionless by the above operations, they all involve the quantity E/P. The factor E/P arises from the use of D and D', and the use of K and k in the relations between the series solutions and the stress functions.

Recall that:

$$D = \frac{Et^3}{12(1 - \nu^2)} \qquad D' = \frac{ET^3}{12(1 - \nu^2)}$$

$$K^{-1} = \frac{ET^2\rho_o}{[12(1 - \nu^2)]^{1/2}} \qquad k^{-1} = \frac{Et^2\rho_o}{[12(1 - \nu^2)]^{1/2}}$$

Whenever any of these quantities is divided by $P\rho_o$ or $P\rho_o^2$, the result involves E/P. It should be possible to express the stress resultants independently of material, and hence independently of E. Since the constants in each series are solved numerically, E/P can be combined with the coefficients as yet to be determined. It should be noted that this combination is made only to allow solution independently of E, and is not in fact a change in the meaning of the series coefficients. The numeric answers generated by the computer will in reality be E/P times the appropriate coefficient, rather than the coefficient itself. There is no harm in this, since the constants remain dimensionless.

The combination of E/P with the series coefficients will, however, affect the series expressions for displacements. The displacements are the only quantities which did not originally contain a factor of E/P. All series expressions for displacements must be multiplied by a factor of P/E to counteract the effect of E/P in the numeric solutions for the coefficients.

Since dimensionless terms are used exclusively to allow numeric solution and comparison of results, it is convenient to replace derivatives with respect to ρ and y with new dimensionless derivatives. Defining new systems of coordinates as $(\tilde{y}, \phi)$ and $(\tilde{\rho}, \phi)$, this is easily accomplished. Use is made of the identity:

$$\frac{\partial ( )}{\partial r} = a \frac{\partial ( )}{\partial S}$$

where r is defined by:

$$r = S/a.$$

The series $\tilde{\psi}$ is already expressed in terms of $(\tilde{y}, \phi)$. The series ψ is easily converted to dimensionless coordinates by the use of the above identity.

All necessary stress resultants and displacements are expressed in terms of ψ and $\tilde{\psi}$ in Appendix E. All needed partial derivatives of ψ, and series expansions for the small shell terms are given in Appendix F.

Computer Solution of Coefficients

By use of the expressions given in Appendix E and Appendix F, the junction compatibility equations can be solved for the unknown coefficients at a chosen number of points. The small shell series contains four coefficients for each term of the series. Each of the two coefficients in the series ψ for the large shell for each series term is broken down into a real and an imaginary part. Thus a total of eight series coefficients exists for each term in the two series. Since Reidelbach has derived eight compatibility equations at the junction, a solution is possible if enough points are chosen.

One additional constant exists, however, in the form of the portion of u due to rigid body displacement. Fortunately, this constant appears only once in the expression for u and not once for each term in the series. In all small shell terms the coefficient $B_{o,2}$ is always multiplied by zero whenever it appears. $B_{o,2}$ is therefore

indeterminate, and any value for $B_{o,2}$ will satisfy the physical problem. A value of zero is chosen for $B_{o,2}$ for reasons of simplicity. Even with the addition of the rigid body constant, the first term of the small shell series has a total of only four unknown constants.

By virtue of the symmetry of the problem, only the quadrant of the intersection from $\phi = 0$ to $\phi = \pi/2$ is used. A large number of equally spaced points are chosen for each term of the series. The total number of unknown constants is eight times the number of terms used. The system of junction compatibility equations is thus highly over-determined, and a least squares approach is used to properly fit the constants.

All quantities describing the behavior of either shell can now be evaluated.

The final stresses are found by the formulas listed below:

$$\sigma_y = \frac{n_y}{t} + \frac{12m_y z}{t^3}$$

$$\sigma_\phi = \frac{n_\phi}{t} + \frac{12m_\phi z}{t^3} \qquad \text{Small shell}$$

$$\sigma_\rho = \frac{N_\rho}{T} + \frac{12M_\rho z}{T^3}$$

$$\sigma_\phi = \frac{N_\phi}{T} + \frac{12M_\phi z}{T^3} \qquad \text{Large shell} \qquad (4.31)$$

CHAPTER V

NUMERIC SOLUTION AND DISCUSSION

By use of the approach previously described, the series coefficients are calculated and stress resultants evaluated for each shell at varying distances from the intersection. While each series appears to converge fairly quickly, the results do not look promising when compared to experimental data.

Factors Investigated in Determination of Coefficients

The most obvious factor affecting the quality of the solution is the number of terms chosen for each series. Due to the large amount of computer time required for determination of series coefficients for a high number of series terms used, no investigation of more than seven terms is made. With only seven terms used, computer time for the determination of the series coefficients alone is on the order of 50 minutes. Comparison of coefficients for the cases of three, five, and seven terms shows that five terms is sufficient to determine the values of the stress resultants. Comparison of results for the cases of five and seven terms shows that the variation in the values of the coefficients themselves is generally much less than 1% in the coefficients of the dominant terms. Comparison of the magnitudes of the coefficients shows that the first three terms of the small shell series and the first two terms of the large shell series are the dominant terms.

The number of points chosen for each term of the series used does not appear to affect the results as long as a sufficient number of points for a least squares solution is used. Comparison of the coefficients for the choice of 200 points per term and the choice of 400 points per term used shows negligible variation in the coefficients for truncation of the series at five terms. The same number of terms is always chosen for each of the series.

Two possible methods of evaluating the rigid body component of u exist. The preferred method is to treat this rigid body part as an unknown constant to be found by the least squares solution to the junction compatibility equations. This is the method used in the determination of coefficients in the majority of cases. As a check, however, one run with the rigid body displacement taken as the average of the radial displacements of the large shell at $\phi = 0$, and $\phi = \pi/2$ shows very little change.

Only one factor is shown to be of great significance in the determination of the coefficients and evaluation of stresses from the coefficients. Many of the terms involved in the displacements u and v for the small shell are found by integration of terms containing the quantity $\cos(n\phi)$. These results cause no difficulty except when $n = 0$. In this case, the quantities π_{02} and ω_{02} both equal zero. Integration for the case $n = 0$ must be carried out separately because of the zero values of the two constants π_{02} and ω_{02}. It is also observed that the integral of $\cos(n\phi)$ with respect to ϕ is $\sin(n\phi)/n$. As $n \to 0$, the limit of $\sin(n\phi)/n$ is ϕ. Terms which involve some factor times ϕ are double valued functions of ϕ however.

For instance, their values at $\phi = 0$ and at $\phi = 2\pi$ are not identical. For the physical case, since the quantities involved are displacements, clearly any double valued function violates the symmetry of this problem. The existence of double valued terms causes little change in the values of the coefficients. It does give rise to large changes in the stresses in the intersection. Inparticular, the in-plane shear of the small shell is zero at $\phi = 0$. The use of double valued terms causes the evaluated in-plane shear of the small shell to be of significant size at $\phi = \pi$, and even larger at $\phi = 2\pi$. In order to satisfy the requirements of symmetry, all double valued terms are set equal to zero for the term n = 0 only. With this change, the values of the in-plane shear for the small shell is found to be approximately zero at the points $\phi = \pi$ and $\phi = \pi/2$.

Values for the coefficients are given in Appendix G.

Revised Displacement Conditions with Respect to Strain

As mentioned previously, Edmondson (5) has revised the first two of Reidelbach's junction compatibility equations. In an attempt to gain greater accuracy, the same two equations are further revised now. Edmondson uses the relation:

$$W = -u.$$

This relation comes from the third of the junction compatibility equations evaluated at $\phi = 0$ (Equations 3.9). Strictly speaking, this leads to a revision of Reidelbach's equations good only at $\phi = 0$. The more exact relations are derived using the unsimplified form of the third junction compatibility equation to modify the first two. The original forms of Reidelbach's first two junction

equations are:

$$U_{\phi} - v + \frac{um}{2} \sin 2\phi = 0$$

$$U_{\rho} + u\frac{m}{2}(1 - \cos 2\phi) + w = 0$$

By use of the strain-displacement equations, the strains for the large shell may be expressed in terms of the quantities U_{ρ}, U_{ϕ}, and their derivatives, as given in the above equations. The use of Hooke's law allows the replacement of strains by stress resultants. With this line of development, and using the complete, unsimplified form of the third of Reidelbach's equations, the following equations are used to replace the first two of Equations(3.9):

$$\frac{\rho_o}{Et}(N_{\phi} - \nu N_{\phi}) = -w + \frac{\partial v}{\partial \phi} - \frac{m}{2}\frac{\partial u}{\partial \phi} \sin 2\phi - m(u + W)\cos 2\phi$$

$$\frac{\rho_o}{ET}[(1 + \nu)(2\frac{\partial N_{\phi\rho}}{\partial \phi} + N_{\rho} - N_{\phi}) - \frac{\partial N_{\phi}}{\partial \rho} + \nu\frac{\partial N_{\rho}}{\partial \rho}] =$$

$$\{ - \frac{\partial^2 w}{\partial \phi^2} - \frac{\partial v}{\partial \phi} + m[\frac{\partial^2 W}{\partial \phi^2} \sin^2\phi + \sin 2\phi(\frac{\partial u}{2\partial \phi} - \frac{\partial W}{\partial \phi}) +$$

$$\cos 2\phi(u + W) + \cos^2\phi \frac{\partial W}{\partial \rho}] \qquad (5.1)$$

The use of these two equations brings no apparent increase in accuracy over those derived by Edmondson.

Effect of Sign Change on Results

In an attempt to achieve better agreement between experimental data and the theory presented here, the effects of change of sign of the series terms are examined. The signs of the particular solutions are, of course, not changed. Changing the sign of all series terms has the effect of reflecting all solutions about the axes represented

by the particular solutions. While this method appears to bring some quantities into better agreement with experimental data, the overall agreement is still not good. The changing of signs of series terms is purely analytical in nature. It has no justification in theory, but represents the possibility of the existence of a sign error in the derivation of the series solutions. If such were the case, it would not change the absolute value of any series coefficients, having the effect of multiplying the entire coefficient matrix by minus one. No such sign error could be found, but the better agreement with experimental data tends to support the change in sign on purely analytical grounds.

From the definition of the stress functions Φ and $\tilde{\Phi}$, it is observed that multiplying the stress functions by minus one still represents a solution to the simplified first and second equilibrium equations. With reasoning along the lines given above, the possibility of a sign error in the stress function portion of the series solution alone is examined. It should be noted that a change in sign of the terms arising from the stress functions alone will have the effect of altering the values of the series coefficients. Such a change has the effect of reversing the sign of only a portion of the terms in the coefficient matrix. The solution to this new matrix gives highly questionable results, such as a negative rigid body displacement of the small shell in its axial direction. Since the results of this line of analysis are not good, they are not presented here. Evidently no sign error is present in the stress function part of the problem alone.

The Final Results

The final results are presented in graphical form in Figures 7 through 14, together with a comparison with experimental results [as given in Gwaltney (8)]. The major weakness appears to be due to the fact that the transverse moments on both shells are too high. The final results are for the use of five terms in each series, with 2000 equally spaced points between $\phi = 0$, and $\phi = \pi/2$ used in the determination of the series coefficients. All doubly defined terms are set equal to zero in both the determination of coefficients and the evaluation of stresses. The sign of all series terms is reversed from the original derivation, as is discussed above.

The major strong points of the solution due to the theory presented here are:

1. Both series converge very rapidly with respect to the number of terms used. The solution for five terms is virtually the same as the solution for seven terms.
2. The effects of the intersection die out quickly as distances away from the intersection, as is expected from Saint-Venant's principle.
3. The solutions converge to the values given by the particular solutions at values far away from the intersection.
4. The magnitude of the stresses at small distances from the intersection are of the same order of magnitude as experimental data in several of the cases shown in Figures 7 through 14.

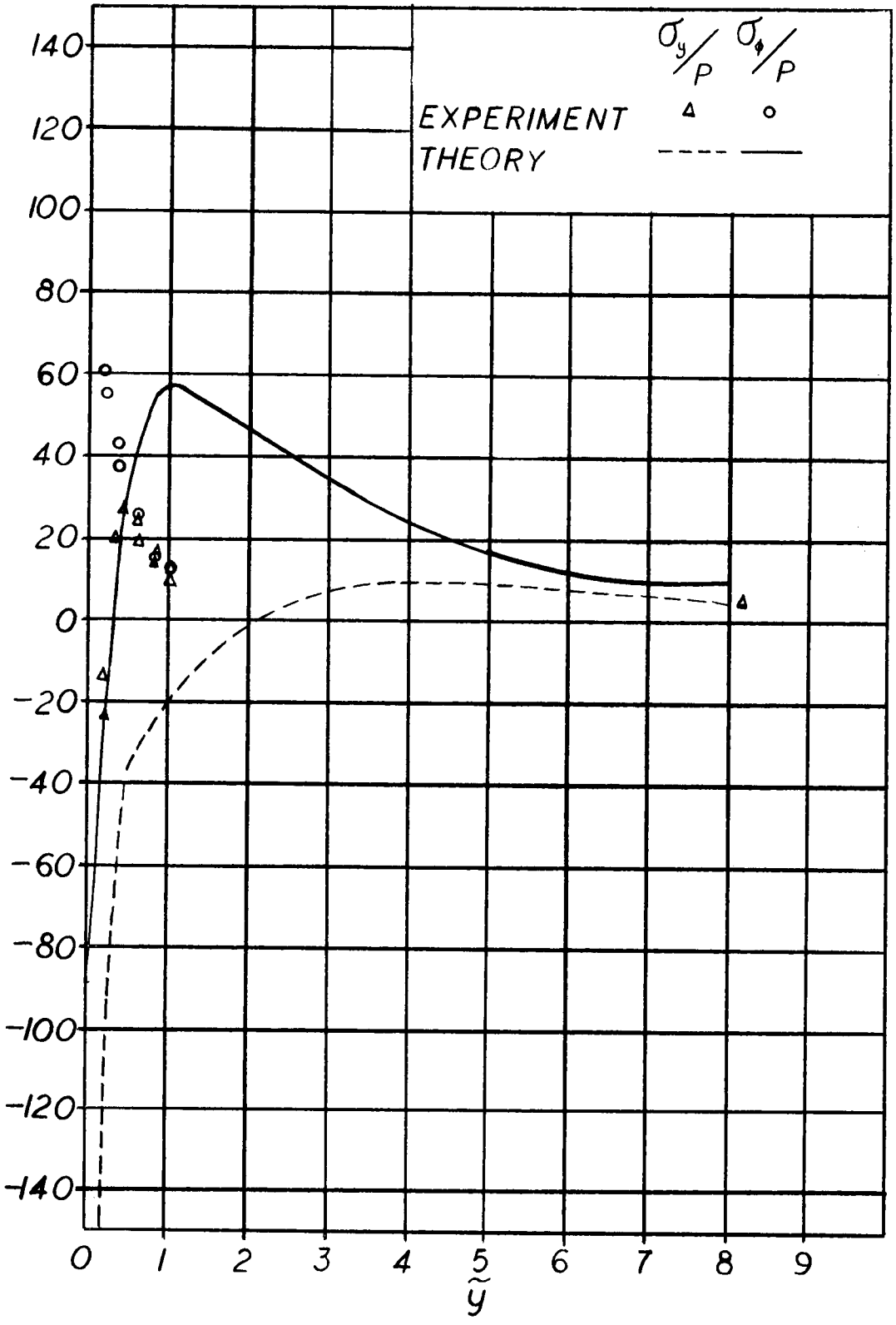

FIGURE 7

SMALL SHELL INSIDE AT $\phi = 0$

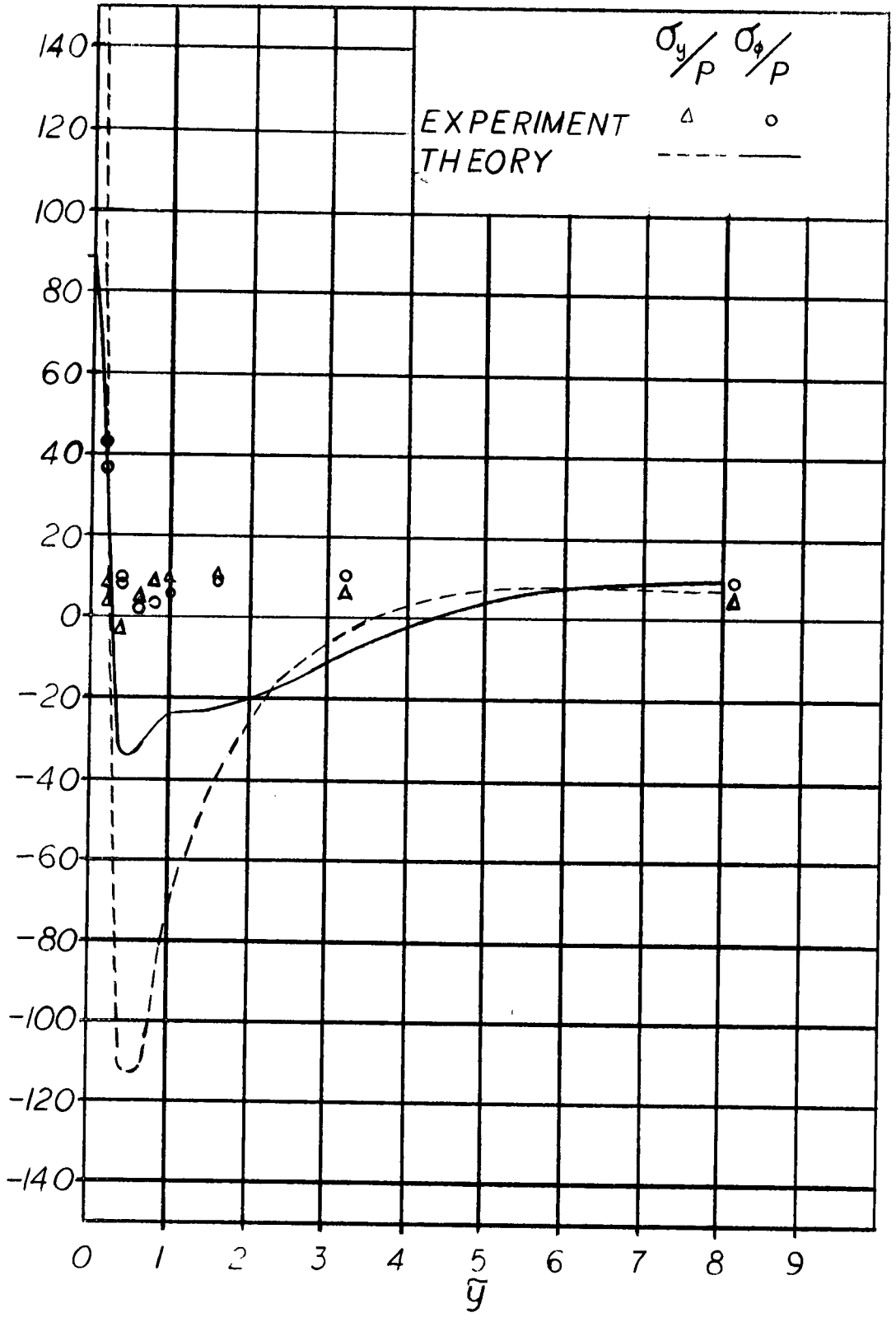

FIGURE 8

SMALL SHELL OUTSIDE AT $\phi=0$

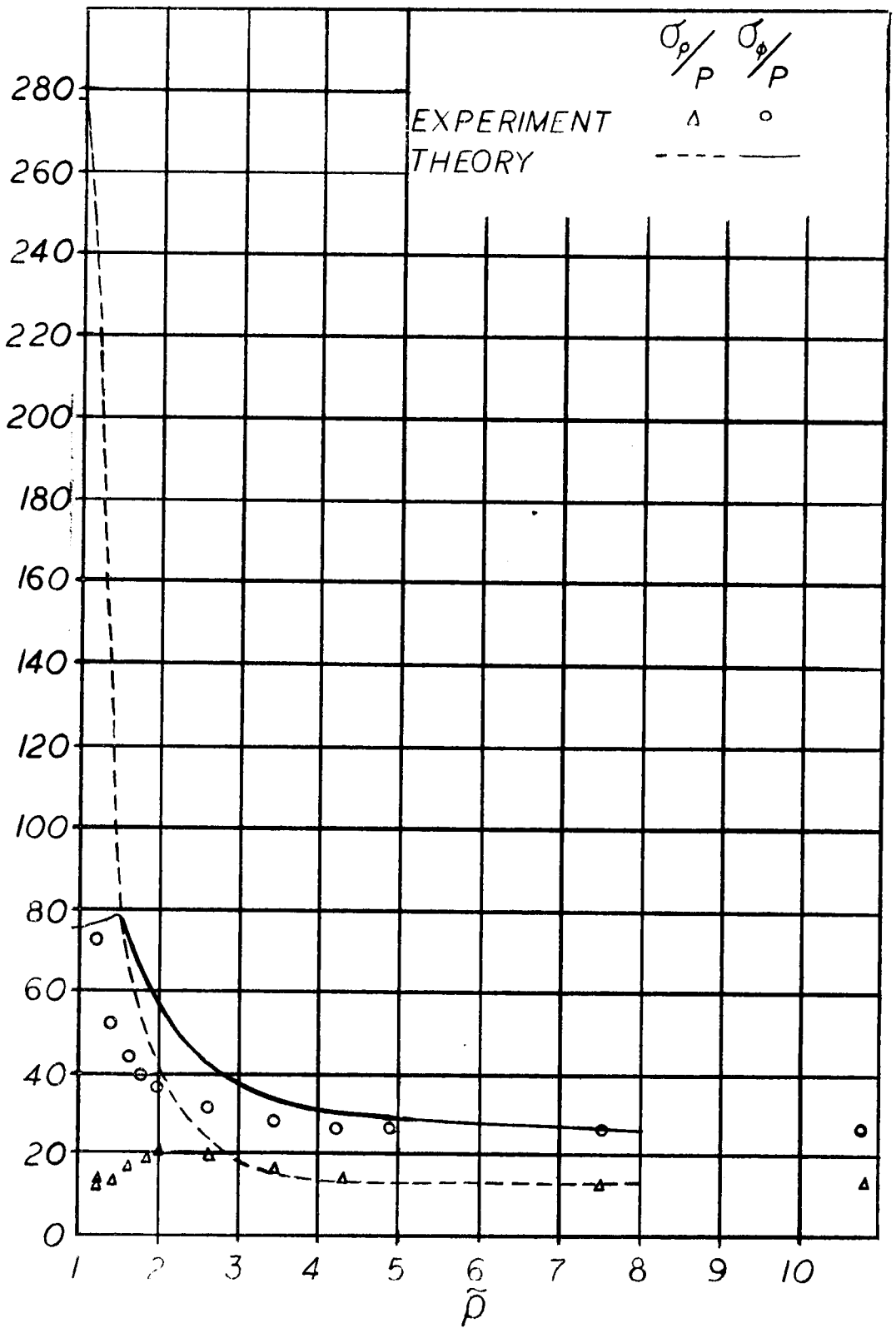

FIGURE 9

LARGE SHELL INSIDE AT $\phi = 0$

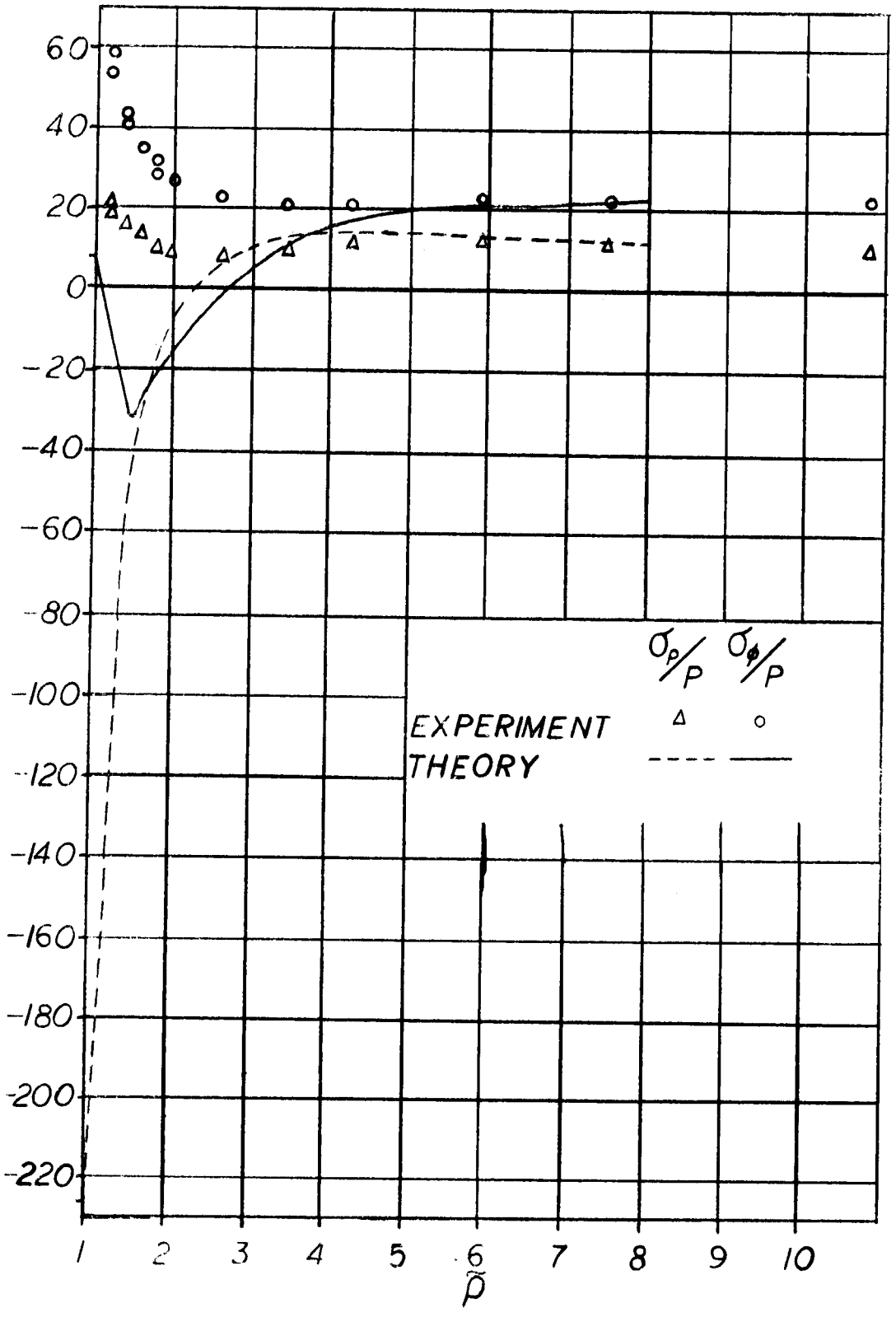

FIGURE 10

LARGE SHELL OUTSIDE AT $\phi = 0$

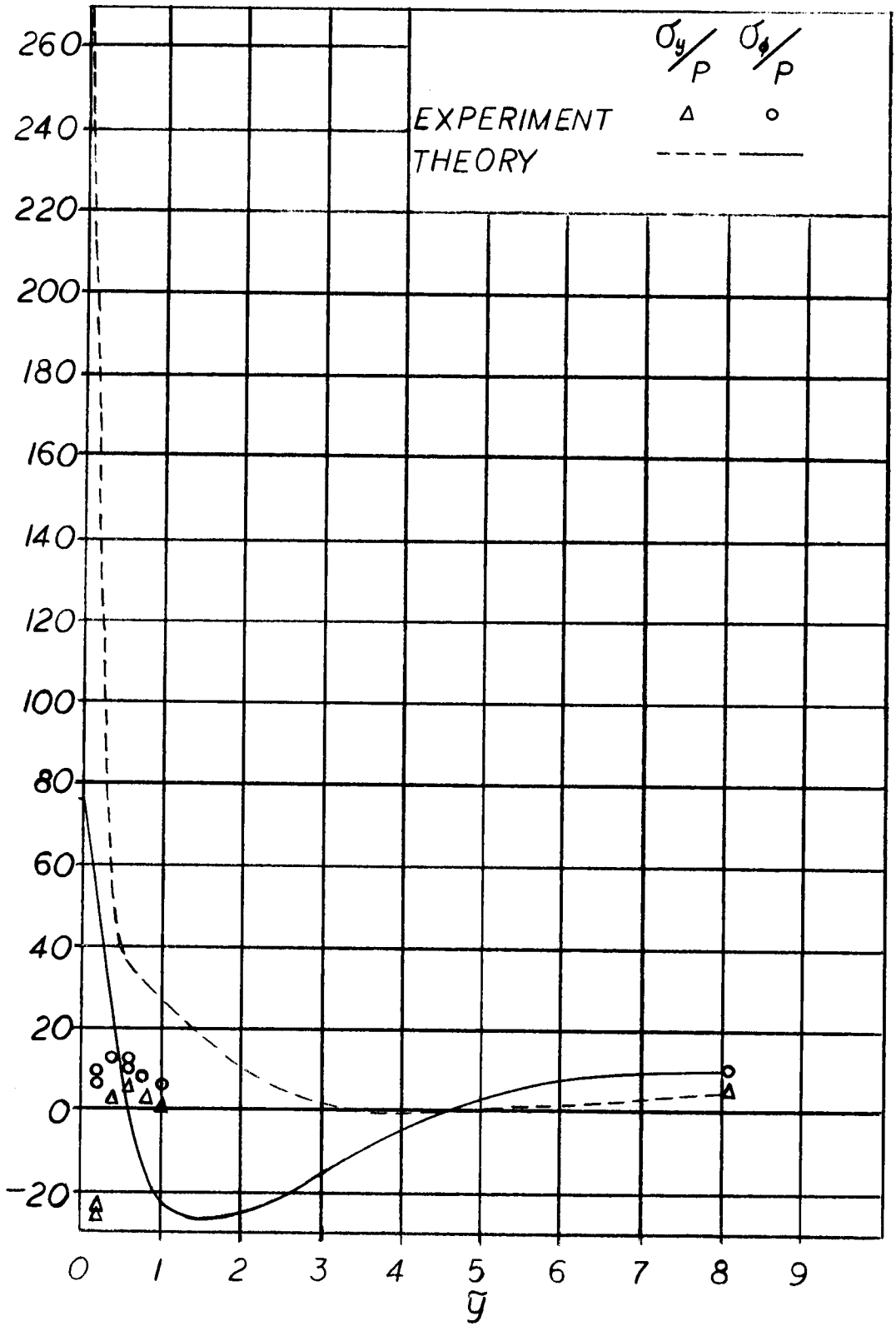

FIGURE 11

SMALL SHELL INSIDE AT $\phi = \pi/2$

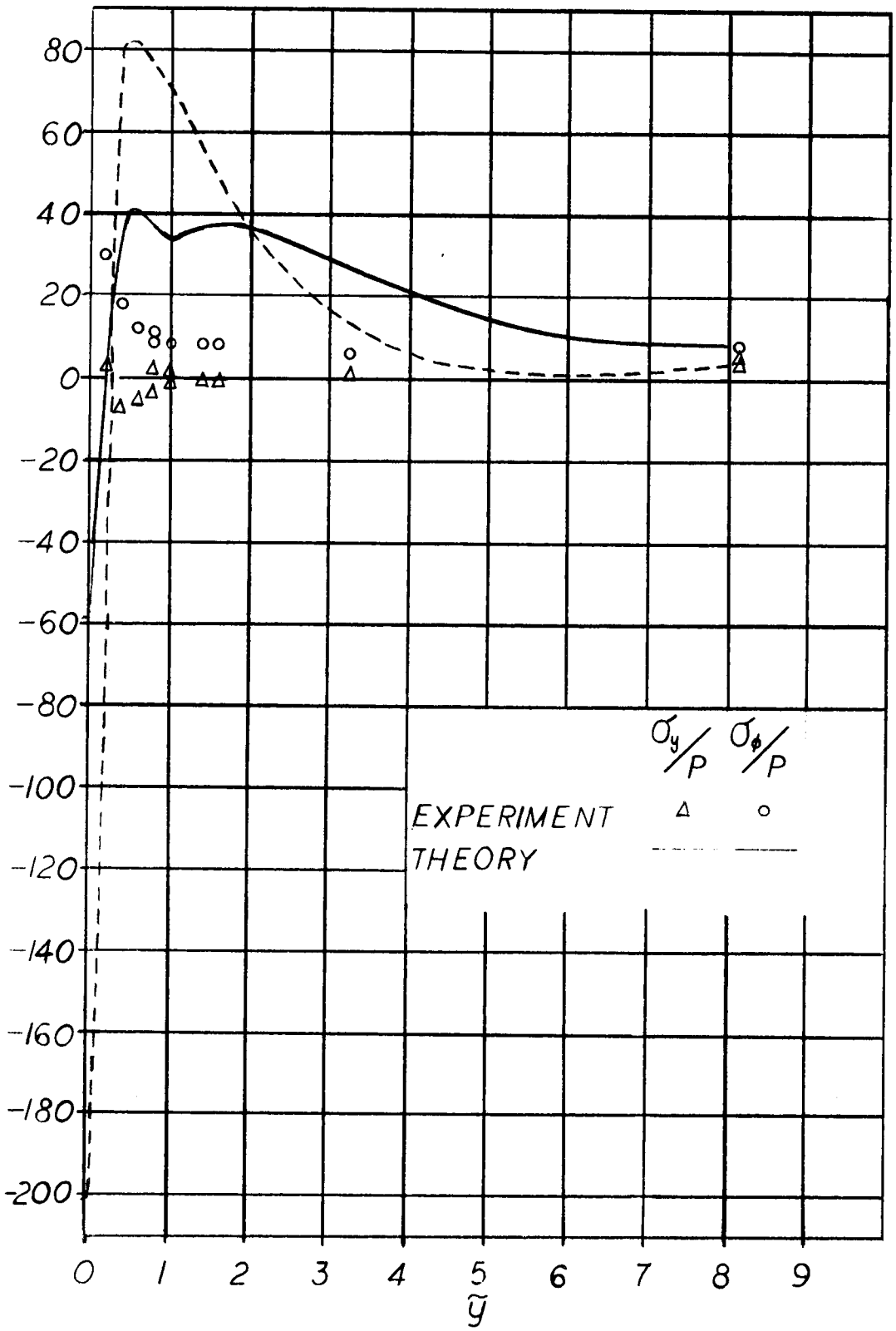

FIGURE 12

SMALL SHELL OUTSIDE AT $\phi = \pi/2$

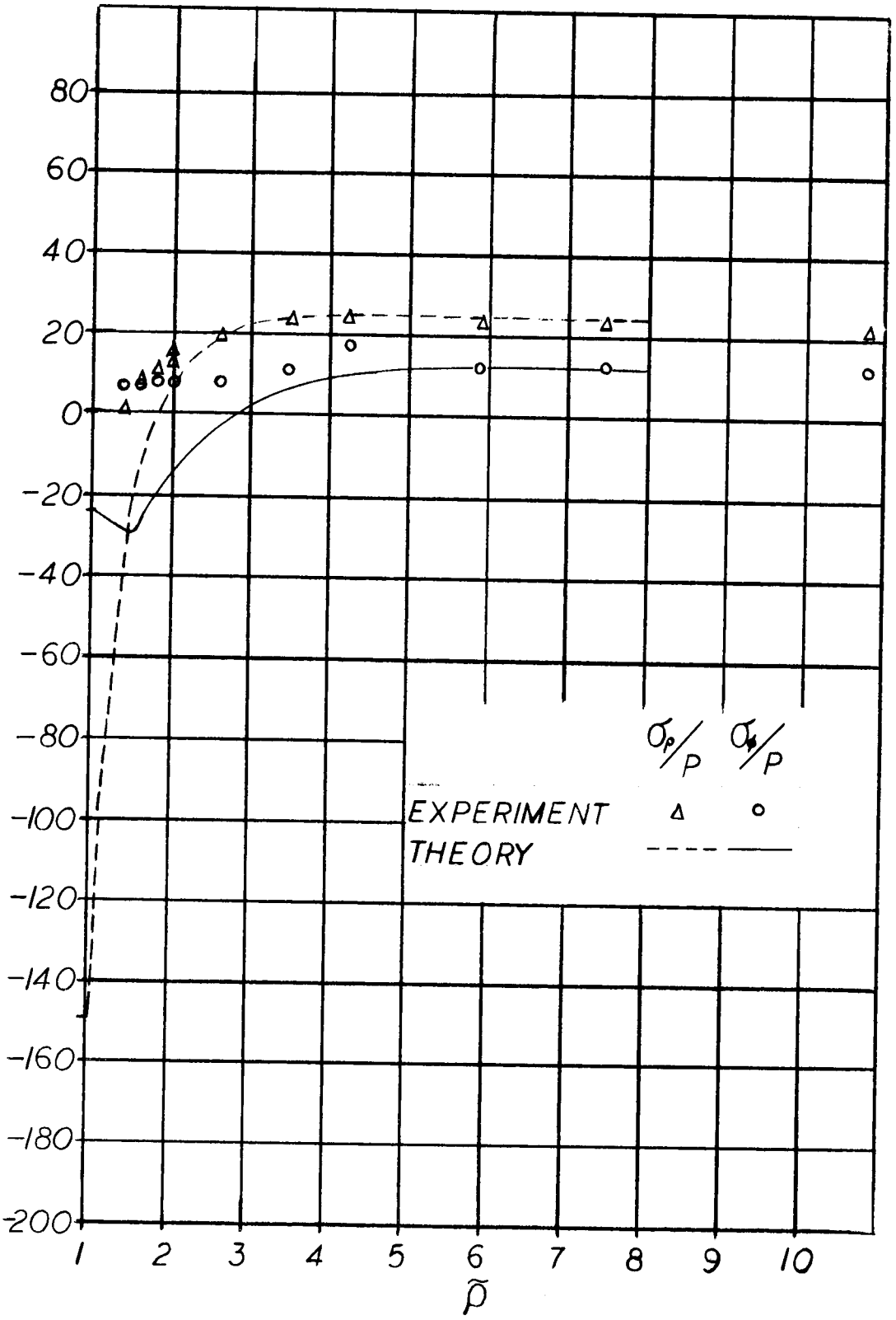

FIGURE 13

LARGE SHELL INSIDE AT $\phi = \pi/2$

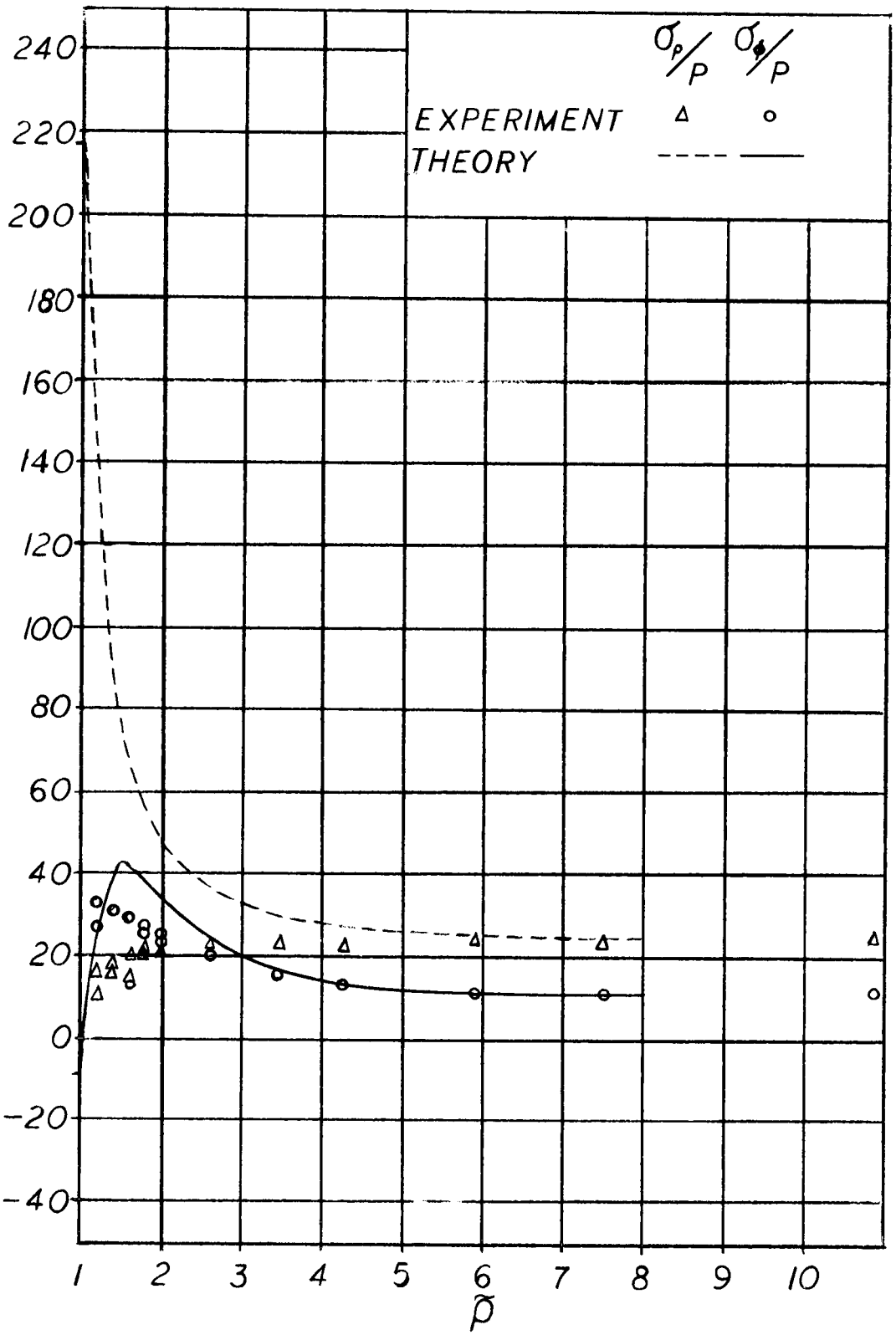

FIGURE 14

LARGE SHELL OUTSIDE AT $\phi = \pi/2$

In the actual intersection (at $\tilde{\rho} = 1$, $\tilde{y} = 0$), the physical region is actually contained within both shells. Thus in this region, results may not be good for the small shell because it is not really a thin shell here. Until the point where y is greater than T/2, the small shell and large shell regions overlap. This may be the reason for the less than good agreement between theory and experiment. It may be that the thicknesses of the two shells must be close in value to allow the use of Flugge theory for even one of the shells. A future investigation of this effect might prove useful.

There may be some undetermined fault in the approach used in this paper to allow modification of the small shell to Flugge theory. Where results are not as good as desired, this possibility is always present.

Due to the extremely large computer programs involved, the possibility of a logic or key-punch error is likely. A large amount of time and effort in this direction has failed to improve the results to the degree desired. The experience of corrected errors in these two programs shows that the solution is very sensitive to small changes in expressions. Hence, due to the computer method of solution, small key punch errors in one or two terms have a large effect on the magnitude of the final stresses. For this reason, it is suspected that the two computer programs (involving a total of almost 3000 cards) have not entirely been debugged.

LIST OF REFERENCES

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APPENDICES

APPENDIX A

STRAIN DISPLACEMENT EQUATIONS

Two forms of strain displacement equations exist for thin shells restricted to small displacements. The differences in these two forms are due to whether terms on the order z/r are retained or neglected, and whether z is considered negligible when compared to r.

Flugge (2) considers these small terms involving z as being non-negligible in the derivation of his shell theory. The strain displacement equations due to Flugge (2, p. 212) are as follows:

$$\varepsilon_x = \frac{\partial u}{\partial x} - z\frac{\partial^2 w}{\partial x^2}$$

$$\varepsilon_\theta = \frac{1}{r}\frac{\partial v}{\partial \theta} - \frac{z}{r}\frac{1}{r-z}\frac{\partial^2 w}{\partial \theta^2} - \frac{w}{r-z}$$

$$\gamma_{x\theta} = \frac{1}{r-z}\frac{\partial u}{\partial \theta} + \frac{r-z}{r}\frac{\partial v}{\partial x} - \frac{\partial^2 w}{\partial x \partial \theta}\left(\frac{z}{r} + \frac{z}{r-z}\right) \qquad \text{(A-1)}$$

Flugge's shell theory and stress resultant definitions involve the same assumptions as used in the derivation of Equation (A-1). The use of Flugge shell theory implies the use of the previous form of strain displacement relations.

The assumptions due to Donnell (1) lead to a more simplified form of Equations (A-1). By neglecting z when compared to r, the following form of the strain displacement equations are arrived at:

$$\varepsilon_x = \frac{\partial u}{\partial x} - z\frac{\partial^2 w}{\partial x^2}$$

$$\varepsilon_\theta = \frac{1}{r}\frac{\partial v}{\partial \theta} - \frac{z}{r^2}\frac{\partial^2 w}{\partial \theta^2} - \frac{w}{r}$$

$$\gamma_{x\theta} = \frac{1}{r}\frac{\partial u}{\partial \theta} + \frac{\partial v}{\partial x} - \frac{2z}{r}\frac{\partial^2 w}{\partial x \partial \theta} \qquad \text{(A-2)}$$

Reidelbach (3) uses the simplified form expressed by Equations (A-2). In both sets of equations, the strains at any point are defined in terms of the displacements of the midsurface of the shell wall. It is to be noted, however, that for strains at the midsurface itself, both the simplified and unsimplified forms of the strain displacement relations reduce to exactly the same quantities. Setting z equal to zero in either set, the strain displacement relations at the midsurface become:

$$\varepsilon_x = \frac{\partial u}{\partial x}$$

$$\varepsilon_\theta = \frac{1}{r}\frac{\partial v}{\partial \theta} - \frac{w}{r}$$

$$\gamma_{x\theta} = \frac{1}{r}\frac{\partial u}{\partial \theta} + \frac{\partial v}{\partial x} \qquad \text{(A-3)}$$

Equations (A-3) are the equations of membrane strains in the shell element. Thus both sets of equations assume that strain due to bending is zero at the midsurface.

APPENDIX B

HOOKE'S LAW

Hooke's law must be used in order to relate stresses to strains. In keeping with the thin shell assumptions, the effects of stress and strain in the radial direction are neglected. With this restriction, Hooke's law becomes:

$$\varepsilon_x = \frac{1}{E}(\sigma_x - \nu\sigma_\theta)$$

$$\varepsilon_\theta = \frac{1}{E}(\sigma_\theta - \nu\sigma_x)$$

$$\gamma_{x\theta} = \frac{\tau_{x\theta}}{G}$$

From the viewpoint of the theory of elasticity, these equations are not exact, due to neglecting radial stress and radial strain. In particular, the equation for ε_z in terms of stresses is not satisfied. This is true for both Flugge and Donnell theory.

APPENDIX C

A DEFINING DIFFERENTIAL EQUATION BASED ON FLUGGE THEORY

Following previous researchers' work in the derivation of a defining differential equation, combine the third, fourth, and fifth equilibrium equations.

$$\frac{\partial^2 m_y}{\partial y^2} + \frac{\partial^2 m_{\phi y}}{\rho_o \partial\phi \partial y} + \frac{\partial^2 m_\phi}{\rho_o^2 \partial\phi^2} + \frac{\partial^2 m_{y\phi}}{\rho_o \partial y \partial\phi} = \frac{\partial q_\phi}{\rho_o \partial\phi} + \frac{\partial q_y}{\partial y}$$

$$\frac{\partial^2 m_y}{\partial y^2} + \frac{\partial^2 m_{\phi y}}{\rho_o \partial\phi \partial y} + \frac{\partial^2 m_\phi}{\rho_o^2 \partial\phi^2} + \frac{\partial^2 m_{y\phi}}{\rho_o \partial y \partial\phi} = P - \frac{n_\phi}{\rho_o}$$

Since $m_{\phi y}$ does not equal $m_{y\phi}$, the partial derivatives of these two terms cannot be combined. In keeping with previous work, a stress function of the following form is assumed:

$$n_y = \frac{\partial^2 \tilde{\Phi}}{\rho_o^2 \partial\phi^2} \qquad n = \frac{\partial^2 \tilde{\Phi}}{\partial y^2} \qquad n_{\phi y} = -\frac{\partial^2 \tilde{\Phi}}{\rho_o \partial\phi \partial y}$$

The first equilibrium equation is satisfied, but the second equilibrium equation now yields:

$$\frac{\partial n_{y\phi}}{\partial y} = -\frac{1}{\rho_o}\frac{\partial^3 \tilde{\Phi}}{\partial y^2 \partial\phi} + \frac{1}{\rho_o^2}\frac{\partial m_\phi}{\partial\phi} + \frac{\partial m_{y\phi}}{\rho_o \partial y}$$

When substitution is made in terms of displacements, one additional unknown remains: $n_{y\phi}$. Also, instead of having an equation involving $\nabla^4 w$ and $\tilde{\Phi}$, u and v and many of the derivatives of u and v appear. Substitution can be made for $n_{y\phi}$ by use of the second

equilibrium equation. Use of the second equilibrium equation also forces any solution to satisfy that equation, which has been a problem previously. The final equilibrium equation is:

$$\frac{Et^3}{\rho_o^2 12(1-\nu^2)}\left\{\left[\rho_o^4\frac{\partial^4 w}{\partial y^4}+2\rho_o^2\frac{\partial^4 w}{\partial y^2\partial\phi^2}+\frac{\partial^4 w}{\partial\phi^4}+\frac{\partial^2 w}{\partial\phi^2}-\rho_o^3\frac{\partial^3 u}{\partial y^3}\right.\right.$$

$$\left.+\frac{\rho_o}{2}\frac{\partial^3 u}{\partial y\partial\phi^2}-\frac{3\rho_o^2}{2}\frac{\partial^3 v}{\partial y^2\partial\phi}\right]+\nu\left[\frac{\rho_o^2}{2}\frac{\partial^2 v}{\partial y^2}-\frac{\rho_o}{2}\frac{\partial^3 u}{\partial y\partial\phi^2}\right]\Bigg\}$$

$$+\frac{Et^3}{\rho_o 12(1-\nu^2)}\left\{\frac{\partial^3 w}{\partial\phi^3}+\frac{\partial w}{\partial\phi}+\frac{3\rho_o^2}{2}\frac{\partial^3 w}{\partial y^2\partial\phi}-\frac{3\rho_o^2}{2}\frac{\partial^2 v}{\partial y^2}+\nu\left[\frac{3\rho_o^2}{2}\frac{\partial^2 v}{\partial y^2}\right.\right.$$

$$\left.\left.+\frac{\rho_o^2}{2}\frac{\partial^3 w}{\partial y^2\partial\phi}\right]\right\}=P\rho_o^2+\frac{\partial^3\tilde{\Phi}}{\partial y^2\partial\phi}-\frac{\partial^2\tilde{\Phi}}{\rho_o\partial y^2}+\frac{\rho_o Et}{2(1+\nu)}$$

$$\left(\rho_o\frac{\partial^2 u}{\partial y\partial\phi}+\rho_o^2\frac{\partial^2 v}{\partial y^2}\right)$$

A simple transformation which will convert this extremely long and tedious equation to one with a possible solution cannot be found. It has now been shown that Flugge's assumption (or rather the absence of Donnell's simplifications) vastly complicate the formulation of a quantity corresponding to $\tilde{\psi}$. The impossibility of deriving a usable series solution based on Flugge's equations leaves no choice other than to modify the original series, which was based on Donnell theory, in order to fit Flugge theory more closely. This modification is carried out in Chapter III.

APPENDIX D

TRANSFORMATION OF REIDELBACH'S EQUATIONS FOR NONEQUIVALENT STRESS RESULTANTS

In order to arrive at the junction compatibility equations, Reidelbach (3) sums the forces due to stress resultants acting on the boundary between the two shells. Reidelbach, however, sets $n_{y\phi}$ equal to $n_{\phi y}$, and $m_{y\phi}$ equal to $m_{\phi y}$ in this summation. Following Reidelbach, an element of the small shell at the boundary is shown in Figure 15. Since Donnell theory is used for the large shell, no changes are made in stress resultants of the large shell. Note that the areas of the sides of the small shell element are related by:

$$A_1 = A \sin \mu \qquad A_2 = A \cos \mu$$

Setting the summation of forces acting on the small shell element in the s direction equal to zero yields:

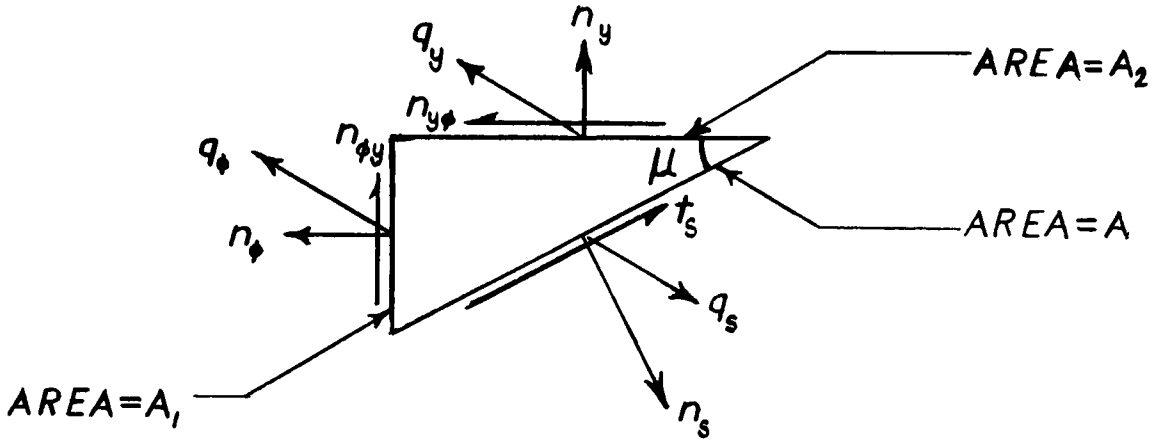
$$An_s = n_y(\cos \mu)A_2 + n_\phi(\sin \mu)A_1 + n_{y\phi}(\sin \mu)A_2$$

$$+ n_{\phi y}(\cos \mu)A_1$$

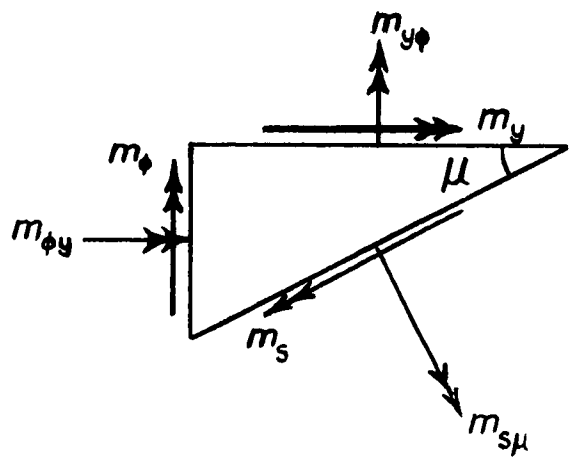
On simplification of the previous equation, the following expression for n_s is obtained:

$$n_s = \frac{1}{2}(n_y + n_\phi) + \frac{1}{2}(n_y - n_\phi)\cos 2\mu + (n_{y\phi} + n_{\phi y})\frac{\sin 2\mu}{2}$$

Similarly, after simplification, the following expressions for m_s, $m_{s\mu}$, and t_s are obtained:



(A) STRESS RESULTANTS



(B) MOMENTS

FIGURE 15
SMALL SHELL BOUNDARY ELEMENT

$$m_s = \frac{1}{2}(m_y + m_\phi) + \frac{1}{2}(m_y - m_\phi)\cos 2\mu + (m_{y\phi} + m_{\phi y})\frac{\sin 2\mu}{2}$$

$$m_{s\mu} = \frac{1}{2}(m_\phi - m_y)\sin 2\mu + m_{y\phi}\cos^2\mu - m_{\phi y}\sin^2\mu$$

$$t_s = \frac{1}{2}(n_\phi - n_y)\sin 2\mu + n_{y\phi}\cos^2\mu - n_{\phi y}\sin^2\mu$$

Since m is restricted to be less than one-third, terms on the order of $m^2/2$ and higher powers of m are neglected. For the case of small m, Reidelbach uses the approximations:

$$\cos\mu \approx 1 \qquad\qquad \sin 2\mu \approx m \sin 2\phi$$

With these assumptions in mind, only the fifth, sixth, and eighth of Reidelbach's equations need be modified. These three equations can be expressed as:

$$0 = m_s - M_\rho$$

$$0 = t_s + N_{\rho\phi}$$

$$0 = n_s - Q_\rho - N_\rho \frac{m}{2}(1 - \cos 2\phi)$$

On substitution of the new equations for t_s, n_s, and m_s, the final forms of the affected equations are:

$$0 = m_y + \frac{(m_{\phi y} + m_{y\phi})}{2} m \sin 2\phi - M_\rho$$

$$0 = N_{\rho\phi} + (n_\phi - n_y)\frac{m}{2}\sin 2\phi + n_{y\phi}$$

$$0 = n_y + \frac{n_{y\phi} + n_{\phi y}}{2} m \sin 2\phi - Q_\rho - N_\rho \frac{m}{2}(1 - \cos 2\phi)$$

APPENDIX E

FINAL FORM OF STRESS RESULTANTS AND DISPLACEMENTS

For the large shell, the stress resultants and displacements are:

$$N_\rho = \frac{1}{K\rho_o^2}\,\mathrm{Im}\left(-\frac{1}{\tilde{\rho}^2}\frac{\partial^2\psi}{\partial\phi^2} + \frac{1}{\tilde{\rho}}\frac{\partial\psi}{\partial\tilde{\rho}}\right) + \frac{PR}{4}(3-\cos 2\phi)$$

$$N_\phi = \frac{1}{K\rho_o^2}\,\mathrm{Im}\left(\frac{\partial^2\psi}{\partial\tilde{\rho}^2}\right) + \frac{PR}{4}(3+\cos 2\phi)$$

$$N_{\rho\phi} = \frac{1}{K\rho_o^2}\,\mathrm{Im}\left(\frac{1}{\tilde{\rho}^2}\frac{\partial\psi}{\partial\phi} - \frac{1}{\tilde{\rho}}\frac{\partial^2\psi}{\partial\tilde{\rho}\partial\phi}\right) + \frac{PR}{4}\sin 2\phi$$

$$M_\phi = -\frac{D'}{\rho_o}\,\mathrm{Re}\left(\frac{1}{\tilde{\rho}}\frac{\partial\psi}{\partial\tilde{\rho}} + \frac{1}{\tilde{\rho}^2}\frac{\partial^2\psi}{\partial\phi^2} + \nu\frac{\partial^2\psi}{\partial\tilde{\rho}^2}\right)$$

$$M_\rho = -\frac{D'}{\rho_o}\,\mathrm{Re}\left[\frac{\partial^2\psi}{\partial\tilde{\rho}^2} + \nu\left(\frac{1}{\tilde{\rho}^2}\frac{\partial^2\psi}{\partial\phi^2} + \frac{1}{\tilde{\rho}}\frac{\partial\psi}{\partial\tilde{\rho}}\right)\right]$$

$$M_{\rho\phi} = -M_{\phi\rho} = \frac{(1-\nu)D'}{\rho_o}\,\mathrm{Re}\left(\frac{1}{\tilde{\rho}}\frac{\partial^2\psi}{\partial\tilde{\rho}\partial\phi} - \frac{1}{\tilde{\rho}^2}\frac{\partial\psi}{\partial\phi}\right)$$

$$Q_\rho = -\frac{D'}{\tilde{\rho}\,\rho_o^2}\,\mathrm{Re}\left(\frac{\partial^2\psi}{\partial\tilde{\rho}^2} + \frac{\partial^3\psi}{\partial\tilde{\rho}^3}\tilde{\rho} - \frac{1}{\tilde{\rho}}\frac{\partial\psi}{\partial\tilde{\rho}} + \frac{\partial^3\psi}{\tilde{\rho}\partial\tilde{\rho}\partial\phi^2} - \frac{2}{\tilde{\rho}^2}\frac{\partial^2\psi}{\partial\phi^2}\right)$$

$$W = \rho_o\,\mathrm{Re}(\psi) - \frac{PR^2}{ET}\left(1-\frac{\nu}{2}\right)$$

For the small shell, stress resultants and displacements are:

$$n_\phi = \frac{1}{k\rho_o^2}\,\mathrm{Im}\left(\frac{\partial^2\tilde{\psi}}{\partial\tilde{y}^2}\right) - \frac{D}{\rho_o^2}\,\mathrm{Re}\left(\tilde{\psi} + \frac{\partial^2\tilde{\psi}}{\partial\phi^2}\right) + P\rho_o$$

$$n_y = \frac{1}{k\rho_o^2}\,\mathrm{Im}\left(\frac{\partial^2\tilde{\psi}}{\partial\phi^2}\right) + \frac{D}{\rho_o^2}\,\mathrm{Re}\left(\frac{\partial^2\tilde{\psi}}{\partial\tilde{y}^2}\right) + \frac{P\rho_o}{2}$$

$$n_{\phi y} = \frac{-1}{k\rho_o^2}\,\mathrm{Im}\left(\frac{\partial^2\tilde{\psi}}{\partial\tilde{y}\partial\phi}\right) + \frac{D(1-\nu)}{2\rho_o^3}\left\{\frac{\rho_o}{Etk}\,\mathrm{Im}\int\left(\frac{\partial^3\tilde{\psi}}{\rho_o^2\partial\phi^3} - \frac{\nu}{\rho_o^2}\frac{\partial^3\tilde{\psi}}{\partial\phi\partial\tilde{y}^2}\right)d\tilde{y}\right.$$

$$+\frac{D}{\rho_o Et}\ \mathrm{Re}\int\left[\frac{\partial^3\tilde{\psi}}{\partial\phi\partial\tilde{y}^2}+\nu\left(\frac{\partial\tilde{\psi}}{\partial\phi}+\frac{\partial^3\tilde{\psi}}{\partial\phi^3}\right)\right]d\tilde{y}-\rho_o\mathrm{Re}\ \frac{\partial^2\tilde{\psi}}{\partial\tilde{y}\partial\phi}\Big\}$$

$$n_{y\phi}=-\frac{1}{k\rho_o^2}\ \mathrm{Im}\left(\frac{\partial^2\tilde{\psi}}{\partial\tilde{y}\partial\phi}\right)+\frac{D(1-\nu)}{2\rho_o^3}\left\{\frac{\rho_o}{Etk}\ \mathrm{Im}\int\left(\frac{1}{\rho_o^2}\ \frac{\partial^3\tilde{\psi}}{\partial\tilde{y}^3}\right.\right.$$

$$\left.-\frac{\nu}{\rho_o^2}\ \frac{\partial^2\tilde{\psi}}{\partial\phi^2\partial\tilde{y}}\right)d\phi+\rho_o\mathrm{Re}\int\frac{\partial\tilde{\psi}}{\partial\tilde{y}}\ d\phi-\frac{D}{\rho_o Et}\ \mathrm{Re}\int\left(\frac{\partial\tilde{\psi}}{\partial\tilde{y}}+\frac{\partial^3\tilde{\psi}}{\partial\tilde{y}\partial\phi^2}+\right.$$

$$\left.\nu\ \frac{\partial^3\tilde{\psi}}{\partial\tilde{y}^3}\right)\ d\phi+\rho_o\mathrm{Re}\ \frac{\partial^2\tilde{\psi}}{\partial\tilde{y}\partial\phi}\Big\}$$

$$m_\phi=-D\rho_o\mathrm{Re}\left(\frac{\partial^2\tilde{\psi}}{\rho_o^2\partial\phi^2}+\frac{\nu}{\rho_o^2}\ \frac{\partial^2\tilde{\psi}}{\partial\tilde{y}^2}\right)-\frac{D}{\rho_o}\ \mathrm{Re}\tilde{\psi}$$

$$m_y=-D\rho_o\mathrm{Re}\left(\frac{\partial^2\tilde{\psi}}{\rho_o^2\partial\tilde{y}^2}+\frac{\nu}{\rho_o^2}\ \frac{\partial^2\tilde{\psi}}{\partial\phi^2}\right)-\frac{D}{\rho_o^2}\left\{\frac{\rho_o}{Etk}\ \mathrm{Im}\left(\frac{\partial^2\tilde{\psi}}{\rho_o^2\partial\phi^2}-\frac{\nu}{\rho_o^2}\ \frac{\partial^2\tilde{\psi}}{\partial\tilde{y}^2}\right)\right.$$

$$+\frac{D}{\rho_o Et}\ \mathrm{Re}\left[\frac{\partial^2\tilde{\psi}}{\partial\tilde{y}^2}+\nu\left(\tilde{\psi}+\frac{\partial^2\tilde{\psi}}{\partial\phi^2}\right)\right]+\nu\left[\frac{\rho_o}{Etk}\ \mathrm{Im}\left(\frac{1}{\rho_o^2}\ \frac{\partial^2\tilde{\psi}}{\partial\tilde{y}^2}\right.\right.$$

$$\left.\left.-\frac{\nu}{\rho_o^2}\ \frac{\partial^2\tilde{\psi}}{\partial\phi^2}\right)+\rho_o\mathrm{Re}\tilde{\psi}-\frac{D}{\rho_o Et}\ \mathrm{Re}\left(\tilde{\psi}+\frac{\partial^2\tilde{\psi}}{\partial\phi^2}+\nu\ \frac{\partial^2\tilde{\psi}}{\partial\tilde{y}^2}\right)\right]\Big\}$$

$$m_{\phi y}=-(1-\nu)D\rho_o\mathrm{Re}\left(\frac{\partial^2\tilde{\psi}}{\rho_o^2\partial\tilde{y}\partial\phi}\right)+\frac{D(1-\nu)}{2\rho_o^2}\left\{\frac{\rho_o}{Etk}\ \mathrm{Im}\int\left(\frac{\partial^3\tilde{\psi}}{\rho_o^2\partial\phi^3}\right.\right.$$

$$\left.-\frac{\nu}{\rho_o^2}\ \frac{\partial^3\tilde{\psi}}{\partial\phi\partial\tilde{y}^2}\right)\ d\tilde{y}+\frac{D}{\rho_o Et}\ \mathrm{Re}\int\left[\frac{\partial^3\tilde{\psi}}{\partial\phi\partial\tilde{y}^2}+\nu\left(\frac{\partial\tilde{\psi}}{\partial\phi}+\frac{\partial^3\tilde{\psi}}{\partial\phi^3}\right)\right]\ d\tilde{y}$$

$$-\frac{\rho_o}{Etk}\ \mathrm{Im}\int\left(\frac{1}{\rho_o^2}\ \frac{\partial^3\tilde{\psi}}{\partial\tilde{y}^3}-\frac{\nu}{\rho_o^2}\ \frac{\partial^3\tilde{\psi}}{\partial\tilde{y}\partial\phi^2}\right)d\phi-\rho_o\mathrm{Re}\int\frac{\partial\tilde{\psi}}{\partial\tilde{y}}\ d\phi$$

$$+\frac{D}{\rho_o Et}\ \mathrm{Re}\int\left(\frac{\partial\tilde{\psi}}{\partial\tilde{y}}+\frac{\partial^3\tilde{\psi}}{\partial\tilde{y}\partial\phi^2}+\nu\ \frac{\partial^3\tilde{\psi}}{\partial\tilde{y}^3}\right)d\phi\Big\}$$

$$m_{y\phi}=-(1-\nu)D\rho_o\mathrm{Re}\left(\frac{\partial^2\tilde{\psi}}{\rho_o^2\partial\tilde{y}\partial\phi}\right)-\frac{D}{\rho_o^2}(1-\nu)\left\{\frac{\rho_o}{Etk}\ \mathrm{Im}\int\left(\frac{1}{\rho_o^2}\ \frac{\partial^3\tilde{\psi}}{\partial\tilde{y}^3}\right.\right.$$

$$\left.-\nu\ \frac{\partial^3\tilde{\psi}}{\rho_o^2\partial\tilde{y}\partial\phi^2}\right)d\phi+\rho_o\mathrm{Re}\int\frac{\partial\tilde{\psi}}{\partial\tilde{y}}\ d\phi-\frac{D}{\rho_o Et}\ \mathrm{Re}\int\left(\frac{\partial\tilde{\psi}}{\partial\tilde{y}}\right.$$

$$\left.+\frac{\partial^3\tilde{\psi}}{\partial\tilde{y}\partial\phi^2}+\nu\frac{\partial^3\tilde{\psi}}{\partial\tilde{y}^3}\right)d\phi\Big\}$$

$$q_y = -D\rho_o \mathrm{Re}\left(\frac{1}{\rho_o^3}\frac{\partial^3\tilde{\psi}}{\partial\tilde{y}^3} + \frac{\partial^3\tilde{\psi}}{\rho_o^3\partial\tilde{y}\partial\phi^2}\right) - \frac{D}{\rho_o^3}\left\{\frac{\rho_o}{Etk}\ \mathrm{Im}\left(\frac{\partial^3\tilde{\psi}}{\rho_o^2\partial\phi^2\partial\tilde{y}}\right.\right.$$

$$\left.-\nu\frac{\partial^3\tilde{\psi}}{\rho_o^2\partial\tilde{y}^3}\right) + \frac{D}{\rho_o Et}\ \mathrm{Re}\left[\frac{\partial^3\tilde{\psi}}{\partial\tilde{y}^3} + \nu\left(\frac{\partial\tilde{\psi}}{\partial\tilde{y}} + \frac{\partial^3\tilde{\psi}}{\partial\tilde{y}\partial\phi^2}\right)\right] +$$

$$\frac{(1+\nu)}{2}\left[\frac{\rho_o}{Etk}\ \mathrm{Im}\left(\frac{\partial^3\tilde{\psi}}{\rho_o^2\partial\tilde{y}^3} - \nu\frac{\partial^3\tilde{\psi}}{\rho_o^2\partial\tilde{y}\partial\phi^2}\right) + \rho_o \mathrm{Re}\ \frac{\partial\tilde{\psi}}{\partial\tilde{y}}\right.$$

$$\left.- \frac{D}{\rho_o Et}\ \mathrm{Re}\left(\frac{\partial\tilde{\psi}}{\partial\tilde{y}} + \frac{\partial^3\tilde{\psi}}{\partial\tilde{y}\partial\phi^2} + \nu\frac{\partial^3\tilde{\psi}}{\partial\tilde{y}^3}\right)\right] + \frac{(\nu-1)}{2}\left[\frac{\rho_o}{EtK}\ \mathrm{Im}\int\left(\frac{1}{\rho_o^2}\frac{\partial^4\tilde{\psi}}{\partial\phi^4}\right.\right.$$

$$\left.\left.- \frac{\nu}{\rho_o^2}\frac{\partial^4\tilde{\psi}}{\partial\tilde{y}^2\partial\phi^2}\right)d\tilde{y} + \frac{D}{\rho_o Et}\ \mathrm{Re}\int\left[\frac{\partial^4\tilde{\psi}}{\partial\tilde{y}^2\partial\phi^2} + \nu\left(\frac{\partial^2\tilde{\psi}}{\partial\phi^2} + \frac{\partial^4\tilde{\psi}}{\partial\phi^4}\right)\right]d\tilde{y}\right\}$$

$$u = \frac{\rho_o}{Etk}\ \mathrm{Im}\int\left(\frac{\partial^2\tilde{\psi}}{\rho_o^2\partial\phi^2} - \frac{\nu}{\rho_o^2}\frac{\partial^2\tilde{\psi}}{\partial\tilde{y}^2}\right)d\tilde{y} + \frac{D}{\rho_o Et}\ \mathrm{Re}\int\left[\frac{\partial^2\tilde{\psi}}{\partial\tilde{y}^2} + \nu\left(\tilde{\psi} + \frac{\partial^2\tilde{\psi}}{\partial\phi^2}\right)\right]$$

$$d\tilde{y} + R_1 + P\rho_o/Et\ \left[\frac{1}{2} - \nu\right]\tilde{y}$$

$$v = \frac{\rho_o}{Etk}\ \mathrm{Im}\int\left(\frac{\partial^2\tilde{\psi}}{\rho_o^2\partial\tilde{y}^2} - \frac{\nu}{\rho_o^2}\frac{\partial^2\tilde{\psi}}{\partial\phi^2}\right)d\phi + \rho_o \mathrm{Re}\int\tilde{\psi}d\phi\ - \frac{D}{\rho_o Et}\ \mathrm{Re}$$

$$\int\left(\tilde{\psi} + \frac{\partial^2\tilde{\psi}}{\partial\phi^2} + \nu\frac{\partial^2\tilde{\psi}}{\partial\tilde{y}^2}\right)d\phi$$

$$w = \rho_o \mathrm{Re}(\tilde{\psi}) - \frac{P\rho_o^2}{Et}\left(1 - \frac{\nu}{2}\right)$$

$$q_\phi = -D\rho_o \mathrm{Re}\left(\frac{\partial^3\tilde{\psi}}{\rho_o^3\partial\phi^3} + \frac{\partial^3\tilde{\psi}}{\rho_o^3\partial\phi\partial\tilde{y}^2}\right) - \frac{D}{\rho_o^3}\left\{\rho_o \mathrm{Re}\ \frac{\partial\tilde{\psi}}{\partial\phi} + (1-\nu)\right.$$

$$\left[\frac{\rho_o}{Etk}\ \mathrm{Im}\int\left(\frac{1}{\rho_o^2}\frac{\partial^4\tilde{\psi}}{\partial\tilde{y}^4} - \frac{\nu}{\rho_o^2}\frac{\partial^4\tilde{\psi}}{\partial\tilde{y}^2\partial\phi^2}\right)d\phi + \rho_o \mathrm{Re}\int\frac{\partial^2\tilde{\psi}}{\partial\tilde{y}^2}\,d\phi\right.$$

$$\left.\left.- \frac{D}{\rho_o Et}\ \mathrm{Re}\int\left(\frac{\partial^2\tilde{\psi}}{\partial\tilde{y}^2} + \frac{\partial^4\tilde{\psi}}{\partial\tilde{y}^2\partial\phi^2} + \nu\frac{\partial^4\tilde{\psi}}{\partial\tilde{y}^4}\right)d\phi\right\}$$

APPENDIX F

SERIES EXPANSIONS

It is necessary to express the derivatives of ψ and $\tilde{\psi}$ in the final expressions for stress resultants and displacements in series form. By a suitable combination of terms, the derivatives of ψ can be taken with respect to $\tilde{\rho}$. For convenience, define the following quantities:

$$b_1 = \xi\rho_o \qquad b_2 = (b_1)^2 \qquad b_3 = b_1 b_2$$

$$\Omega_1 = \cosh(b_1\tilde{\rho}\cos\phi)\cos(b_1\tilde{\rho}\cos\phi)$$

$$\Omega_2 = \sinh(b_1\tilde{\rho}\cos\phi)\cos(b_1\tilde{\rho}\cos\phi)$$

$$\Omega_3 = \sinh(b_1\tilde{\rho}\cos\phi)\sin(b_1\tilde{\rho}\cos\phi)$$

$$\Omega_4 = \cosh(b_1\tilde{\rho}\cos\phi)\sin(b_1\tilde{\rho}\cos\phi)$$

$$\Gamma_n = \mathrm{Re}H_n^{(1)}[(2i)^{1/2}b_1\tilde{\rho}]$$

$$\Delta_n = \mathrm{Im}H_n^{(1)}[(2i)^{1/2}b_1\tilde{\rho}]$$

$$C_n = \mathrm{Re}\ a_n$$

$$D_n = \mathrm{Im}\ a_n$$

The series ψ can now be expressed as:

$$\psi = \sum_{n=1}^{\infty} (C_{2n-1} + iD_{2n-1})(1+i)\sinh[(1-i)b_1\tilde{\rho}\cos\phi]$$

$$[\Gamma_{2n-1} + i\Delta_{2n-1}]\cos(2n-1)\phi + (C_{2n} + iD_{2n})\cosh[(1-i)b_1$$

$$\tilde{\rho}\cos\phi][\Gamma_{2n} + i\Delta_{2n}]\cos 2n\phi$$

$$\text{Re}\,\frac{\partial\psi}{\partial\phi} = \sum_{n=1}^{\infty} -\{(2n-1)[(\Omega_2 + \Omega_4)\Gamma_{2n-1} - (\Omega_2 - \Omega_4)\Delta_{2n-1}]\sin(2n-1)\phi$$

$$+ 2b_1\tilde{\rho}\sin\phi[\Omega_1\Gamma_{2n-1} + \Omega_3\Delta_{2n-1}]\cos(2n-1)\phi\}C_{2n-1}$$

$$- \{2n[\Omega_1\Gamma_{2n} + \Omega_3\Delta_{2n}]\sin 2n\phi + b_1\tilde{\rho}\sin\phi[(\Omega_2 - \Omega_4)\Gamma_{2n}$$

$$+ (\Omega_2 + \Omega_4)\Delta_{2n}]\cos 2n\phi\}C_{2n} + \{(2n-1)[(\Omega_2 + \Omega_4)\Delta_{2n-1}$$

$$+ (\Omega_2 - \Omega_4)\Gamma_{2n-1}]\sin(2n-1)\phi + 2b_1\tilde{\rho}\sin\phi[\Omega_1\Delta_{2n-1}$$

$$- \Omega_3\Gamma_{2n-1}]\cos(2n-1)\phi\}D_{2n-1} +$$

$$\{2n[\Omega_1\Delta_{2n} - \Omega_3\Gamma_{2n}]\sin 2n\phi + b_1\tilde{\rho}\sin\phi$$

$$[(\Omega_2 - \Omega_4)\Delta_{2n} - (\Omega_2 + \Omega_4)\Gamma_{2n}]\cos 2n\phi\}D_{2n}$$

$$\text{Im}\,\frac{\partial\psi}{\partial\phi} = \sum_{n=1}^{\infty} -\{(2n-1)[(\Omega_2 + \Omega_4)\Delta_{2n-1} + (\Omega_2 - \Omega_4)\Gamma_{2n-1}]$$

$$\sin(2n-1)\phi + 2b_1\tilde{\rho}\sin\phi[\Omega_1\Delta_{2n-1} - \Omega_3\Gamma_{2n-1}]$$

$$\cos(2n-1)\phi\}C_{2n-1} - \{2n[\Omega_1\Delta_{2n} - \Omega_3\Gamma_{2n}]\sin 2n\phi$$

$$+ b_1\tilde{\rho}\sin\phi[(\Omega_2 - \Omega_4)\Delta_{2n} - (\Omega_2 + \Omega_4)\Gamma_{2n}]\cos 2n\phi\}C_{2n} -$$

$$\{(2n-1)[(\Omega_2 + \Omega_4)\Gamma_{2n-1} - (\Omega_2 - \Omega_4)\Delta_{2n-1}]\sin(2n-1)\phi +$$

$$2b_1\tilde{\rho}\sin\phi[\Omega_1\Gamma_{2n-1} + \Omega_3\Delta_{2n-1}]\cos(2n-1)\phi\}D_{2n-1}$$

$$- \{2n[\Omega_1\Gamma_{2n} + \Omega_3\Delta_{2n}]\sin 2n\phi + b_1\tilde{\rho}\sin\phi[(\Omega_2 - \Omega_4)\Gamma_{2n}$$

$$+ (\Omega_2 + \Omega_4)\Delta_{2n}]\cos 2n\phi\}D_{2n}$$

$$\text{Re}\,\frac{\partial\psi}{\partial\phi^2} = \sum_{n=1}^{\infty} \{4(2n-1)b_1\tilde{\rho}\sin\phi[\Omega_1\Gamma_{2n-1} + \Omega_3\Delta_{2n-1}]$$

$$\sin(2n-1)\phi - [(2n-1)^2(\Omega_2 + \Omega_4) + 2b_1\tilde{\rho}\cos\phi\Omega_1$$

$$-2(b_1\tilde{\rho})^2\sin^2\phi\,(\Omega_2 - \Omega_4)]\Gamma_{2n-1}\cos(2n-1)\phi$$

$$+ [(2n-1)^2(\Omega_2 - \Omega_4) - 2b_1\tilde{\rho}\cos\phi\Omega_3 + 2(b_1\tilde{\rho})^2\sin^2\phi(\Omega_2 + \Omega_4)]$$

$$\Delta_{2n-1}\cos(2n-1)\phi\}C_{2n-1} + \{4nb_1\tilde{\rho}\sin\phi[(\Omega_2 - \Omega_4)\Gamma_{2n}$$

$$+ (\Omega_2 + \Omega_4)\Delta_{2n}]\sin 2n\phi - [(2n)^2\Omega_1 + b_1\tilde{\rho}$$

$$\cos\phi(\Omega_2 - \Omega_4) - 2(b_1\tilde{\rho})^2\sin^2\phi\Omega_3]\Gamma_{2n}\cos 2n\phi - [(2n)^2\,\Omega_3$$

$$+ b_1\tilde{\rho}\cos\phi(\Omega_2 + \Omega_4) + 2(b_1\tilde{\rho})^2\sin^2\phi\Omega_1]\Delta_{2n}\cos 2n\phi\}C_{2n}$$

$$-\{4(2n-1)b_1\tilde{\rho}\sin\phi[\Omega_1\Delta_{2n-1} - \Omega_3\Gamma_{2n-1}]\sin(2n-1)\phi$$

$$- [(2n-1)^2(\Omega_2 - \Omega_4) - 2b_1\tilde{\rho}\cos\phi\Omega_3 + 2(b_1\tilde{\rho})^2\sin^2\phi(\Omega_2 + \Omega_4)]$$

$$\Gamma_{2n-1}\cos(2n-1)\phi - [(2n-1)^2(\Omega_2 + \Omega_4) + 2b_1\tilde{\rho}\cos\phi\Omega_1$$

$$-2(b_1\tilde{\rho})^2\sin^2\phi(\Omega_2 - \Omega_4)]\Delta_{2n-1}\cos(2n-1)\phi\}\,D_{2n-1}$$

$$-\{4nb_1\tilde{\rho}\sin\phi[(\Omega_2 - \Omega_4)\Delta_{2n} - (\Omega_2 + \Omega_4)\Gamma_{2n}]\sin 2n\phi$$

$$+[(2n)^2\Omega_3 + b_1\tilde{\rho}\cos\phi(\Omega_2 + \Omega_4) + 2(b_1\tilde{\rho})^2\sin^2\phi\Omega_1]\Gamma_{2n}$$

$$\cos 2n\phi - [(2n)^2\Omega_1 + b_1\tilde{\rho}\cos\phi(\Omega_2 - \Omega_4) - 2(b_1\tilde{\rho})^2\sin^2\phi\Omega_3]$$

$$\Delta_{2n}\cos 2n\phi\}D_{2n}$$

$$\text{Im}\frac{\partial^2\psi}{\partial\phi^2} = \sum_{n=1}^{\infty} \{4(2n-1)b_1\tilde{\rho}\sin\phi[\Omega_1\Delta_{2n-1} - \Omega_3\Gamma_{2n-1}]\sin(2n-1)\phi$$

$$-[(2n-1)^2(\Omega_2 - \Omega_4) - 2b_1\tilde{\rho}\cos\phi\Omega_3 + 2(b_1\tilde{\rho})^2\sin^2\phi(\Omega_2 + \Omega_4)]$$

$$\Gamma_{2n-1}\cos(2n-1)\phi - [(2n-1)^2(\Omega_2 + \Omega_4) + 2b_1\tilde{\rho}\cos\phi\Omega_1$$

$$-2(b_1\tilde{\rho})^2\sin^2\phi\ (\Omega_2 - \Omega_4)]\Delta_{2n-1}\cos(2n-1)\phi\}C_{2n-1}$$

$$+ \{4nb_1\tilde{\rho}\sin\phi[(\Omega_2 - \Omega_4)\Delta_{2n} - (\Omega_2 + \Omega_4)\Gamma_{2n}]\sin 2n\phi$$

$$+ [(2n)^2\Omega_3 + b_1\tilde{\rho}\cos\phi(\Omega_2 + \Omega_4) + 2(b_1\tilde{\rho})^2\sin^2\phi\Omega_1]\Gamma_{2n}\cos 2n\phi$$

$$- [(2n)^2\Omega_1 + b_1\tilde{\rho}\cos\phi(\Omega_2 - \Omega_4) - 2(b_1\tilde{\rho})^2\sin^2\phi\Omega_3]\Delta_{2n}\cos 2n\phi\}C_{2n}$$

$$+ \{4(2n-1)b_1\tilde{\rho}\sin\phi[\Omega_1\Gamma_{2n-1} + \Omega_3\Delta_{2n-1}]\sin(2n-1)\phi$$

$$- [(2n-1)^2(\Omega_2 + \Omega_4) + 2b_1\tilde{\rho}\cos\phi\Omega_1 - 2(b_1\tilde{\rho})^2\sin^2\phi(\Omega_2 - \Omega_4)]$$

$$\Gamma_{2n-1}\cos(2n-1)\phi + [(2n-1)^2(\Omega_2 - \Omega_4) - 2b_1\tilde{\rho}\cos\phi\Omega_3$$

$$+ 2(b_1\tilde{\rho})^2\sin^2\phi(\Omega_2 + \Omega_4)]\Delta_{2n-1}\cos(2n-1)\phi\}\ D_{2n-1}$$

$$+\{4n\ b_1\tilde{\rho}\sin\phi[(\Omega_2 - \Omega_4)\Gamma_{2n} + (\Omega_2 + \Omega_4)\Delta_{2n}]\sin 2n\phi$$

$$-[(2n)^2\Omega_1 + b_1\tilde{\rho}\cos\phi(\Omega_2 - \Omega_4) - 2(b_1\tilde{\rho})^2\sin^2\phi\Omega_3]\Gamma_{2n}$$

$$\cos 2n\phi - [(2n)^2\Omega_3 + b_1\tilde{\rho}\cos\phi(\Omega_2 + \Omega_4) + 2(b_1\tilde{\rho})^2\sin^2\phi\Omega_1]$$

$$\Delta_{2n}\cos 2n\phi\}\ D_{2n}$$

$$\mathrm{Re}\,\frac{\partial\psi}{\partial\tilde{\rho}} = \sum_{n=1}^{\infty} \{2b_1 \cos\phi(\Omega_1\Gamma_{2n-1} + \Omega_3\Delta_{2n-1}) - b_1[\Omega_2(\Delta_{2n-2} - \Delta_{2n})$$

$$- \Omega_4[\Gamma_{2n-2} - \Gamma_{2n})]\}\cos\phi(2n-1)\phi\ C_{2n-1} - \{2b_1 \cos\phi(\Omega_1\Delta_{2n-1}$$

$$- \Omega_3\Gamma_{2n-1}) + b_1[\Omega_2(\Gamma_{2n-2} - \Gamma_{2n}) + \Omega_4(\Delta_{2n-2} - \Delta_{2n})]\}$$

$$\cos\ (2n-1)\phi\ D_{2n-1} + \{b_1 \cos\phi[\Omega_2(\Gamma_{2n} + \Delta_{2n}) - \Omega_4(\Gamma_{2n} - \Delta_{2n})$$

$$+ \frac{b_1}{2}[\Omega_1(\Gamma_{2n-1} - \Gamma_{2n+1} - \Delta_{2n-1} + \Delta_{2n+1}) + \Omega_3(\Delta_{2n-1} - \Delta_{2n+1}$$

$$+ \Gamma_{2n-1} - \Gamma_{2n+1})]\}\cos\ 2n\phi\ C_{2n} + \{b_1\ \cos\phi[\Omega_2(\Gamma_{2n} - \Delta_{2n})$$

$$+ \Omega_4(\Gamma_{2n} + \Delta_{2n})] - \frac{b_1}{2}\ [\Omega_1(\Gamma_{2n-1} - \Gamma_{2n+1} + \Delta_{2n-1} - \Delta_{2n+1})$$

$$+ \Omega_3(\Delta_{2n-1} - \Delta_{2n+1} - \Gamma_{2n-1} + \Gamma_{2n+1})]\}\ \cos\ 2n\phi\ D_{2n}$$

$$\mathrm{Im}\,\frac{\partial\psi}{\partial\tilde{\rho}} = \sum_{n=1}^{\infty} \{2b_1 \cos\phi(\Omega_1\Delta_{2n-1} - \Omega_3\Gamma_{2n-1}) + b_1[\Omega_2(\Gamma_{2n-2} - \Gamma_{2n})$$

$$+ \Omega_4(\Delta_{2n-2} - \Delta_{2n})]\}\cos(2n-1)\phi C_{2n-1} + \{2b_1 \cos\phi(\Omega_1\Gamma_{2n-1}$$

$$+ \Omega_3\Delta_{2n-1}) - b_1[\Omega_2(\Delta_{2n-2} - \Delta_{2n}) - \Omega_4(\Gamma_{2n-2} - \Gamma_{2n})]\}$$

$$\cos(2n-1)\phi\ D_{2n-1} - \{b_1 \cos\phi[\Omega_2(\Gamma_{2n} - \Delta_{2n}) + \Omega_4(\Gamma_{2n} + \Delta_{2n})]$$

$$- \frac{b_1}{2}[\Omega_1(\Gamma_{2n-1} - \Gamma_{2n+1} + \Delta_{2n-1} - \Delta_{2n+1}) + \Omega_3(\Delta_{2n-1} - \Delta_{2n+1}$$

$$- \Gamma_{2n-1} + \Gamma_{2n+1})]\}\ \cos\ 2n\phi\ C_{2n} + \{b_1\ \cos\phi[\Omega_2(\Gamma_{2n} + \Delta_{2n})$$

$$- \Omega_4(\Gamma_{2n} - \Delta_{2n})] + \frac{b_1}{2}[\Omega_1(\Gamma_{2n-1} - \Gamma_{2n+1} - \Delta_{2n-1} + \Delta_{2n+1})$$

$$+ \Omega_3(\Delta_{2n-1} - \Delta_{2n+1} + \Gamma_{2n-1} - \Gamma_{2n+1})]\}\ \cos\ 2n\phi\ D_{2n}$$

$$\mathrm{Re}\,\frac{\partial^2\psi}{\partial\rho^{\sim 2}} = \sum_{n=1}^{\infty} \{-\frac{b_2}{2}[(\Omega_2 - \Omega_4)\Gamma_{2n-3} + (\Omega_2 + \Omega_4)\Delta_{2n-3}]$$

$$+ 2b_2 \cos\phi[(\Omega_1 + \Omega_3)\Gamma_{2n-2} - (\Omega_1 - \Omega_3)\Delta_{2n-2}]$$

$$+ b_2(1 + 2\cos^2\phi)[(\Omega_2 - \Omega_4)\Gamma_{2n-1} + (\Omega_2 + \Omega_4)\Delta_{2n-1}]$$

$$- 2b_2 \cos\phi[(\Omega_1 + \Omega_3)\Gamma_{2n} - (\Omega_1 - \Omega_3)\Delta_{2n}] - \frac{b_2}{2}[(\Omega_2 - \Omega_4)$$

$$\Gamma_{2n+1} + (\Omega_2 + \Omega_4)\Delta_{2n+1}]\} C_{2n-1} \cos(2n-1)\phi - \{-\frac{b_2}{2}[(\Omega_2 - \Omega_4)$$

$$\Delta_{2n-3} - (\Omega_2 + \Omega_4)\Gamma_{2n-3}] + 2b_2 \cos\phi[(\Omega_1 - \Omega_3)\Gamma_{2n-2}$$

$$+ (\Omega_1 + \Omega_3)\Delta_{2n-2}] + b_2(1 + 2\cos^2\phi)[(\Omega_2 - \Omega_4)\Delta_{2n-1}$$

$$- (\Omega_2 + \Omega_4)\Gamma_{2n-1}] - 2b_2 \cos\phi[(\Omega_1 - \Omega_3)\Gamma_{2n} + (\Omega_1 + \Omega_3)\Delta_{2n}]$$

$$- \frac{b_2}{2}[(\Omega_2 - \Omega_4)\Delta_{2n+1} - (\Omega_2 + \Omega_4)\Gamma_{2n+1}]\} D_{2n-1} \cos(2n-1)\phi$$

$$+\{\frac{b_2}{2}[(\Omega_3\Gamma_{2n-2} - \Omega_1\Delta_{2n-2})] + 2b_2 \cos\phi[\Omega_2\Gamma_{2n-1} + \Omega_4\Delta_{2n-1}]$$

$$- b_2(1 + 2\cos^2\phi)[\Omega_3\Gamma_{2n} - \Omega_1\Delta_{2n}] - 2b_2 \cos\phi[\Omega_2\Gamma_{2n+1} + \Omega_4\Delta_{2n+1}]$$

$$+ \frac{b_2}{2}[\Omega_3\Gamma_{2n+2} - \Omega_1\Delta_{2n+2}]\} C_{2n} \cos 2n\phi - \{\frac{b_2}{2}[\Omega_1\Gamma_{2n-2} + \Omega_3\Delta_{2n-2}]$$

$$+ 2b_2 \cos\phi[\Omega_2\Delta_{2n-1} - \Omega_4\Gamma_{2n-1}] - b_2(1 + 2\cos^2\phi)$$

$$[\Omega_1\Gamma_{2n} + \Omega_3\Delta_{2n}] - 2b_2 \cos\phi[\Omega_2\Delta_{2n+1} - \Omega_4\Gamma_{2n+1}]$$

$$+ \frac{b_2}{2}[\Omega_1\Gamma_{2n+2} + \Omega_3\Delta_{2n+2}]\} D_{2n} \cos 2n\phi$$

$$\text{Im}\,\frac{\partial^2\psi}{\partial\tilde{\rho}^2} = \sum_{n=1}^{\infty} \{-\frac{b_2}{2}[(\Omega_2 - \Omega_4)\Delta_{2n-3} - (\Omega_2 + \Omega_4)\Gamma_{2n-3}]$$

$$+ 2b_2\cos\phi[(\Omega_1 - \Omega_3)\Gamma_{2n-2} + (\Omega_1 + \Omega_3)\Delta_{2n-2}] + b_2(1 + 2\cos^2\phi)$$

$$[(\Omega_2 - \Omega_4)\Delta_{2n-1} - (\Omega_2 + \Omega_4)\Gamma_{2n-1}] - 2b_2\cos\phi[(\Omega_1 - \Omega_3)\Gamma_{2n}$$

$$+ (\Omega_1 + \Omega_3)\Delta_{2n}] - \frac{b_2}{2}[(\Omega_2 - \Omega_4)\Delta_{2n+1} - (\Omega_2 + \Omega_4)\Gamma_{2n+1}]\}$$

$$C_{2n-1}\cos(2n-1)\phi + \{\frac{-b_2}{2}[(\Omega_2 - \Omega_4)\Gamma_{2n-3} + (\Omega_2 + \Omega_4)\Delta_{2n-3}]$$

$$+ 2b_2\cos\phi[(\Omega_1 + \Omega_3)\Gamma_{2n-2} - (\Omega_1 - \Omega_3)\Delta_{2n-2}]$$

$$+ b_2(1 + 2\cos^2\phi)[(\Omega_2 - \Omega_4)\Gamma_{2n-1} + (\Omega_2 + \Omega_4)\Delta_{2n-1}]$$

$$- 2b_2\cos\phi[(\Omega_1 + \Omega_3)\Gamma_{2n} - (\Omega_1 - \Omega_3)\Delta_{2n}] - \frac{b_2}{2}[(\Omega_2 - \Omega_4)\Gamma_{2n+1} +$$

$$(\Omega_2 + \Omega_4)\Delta_{2n+1}]\}D_{2n-1}\cos(2n-1)\phi + \{\frac{b_2}{2}[\Omega_1\Gamma_{2n-2} + \Omega_3\Delta_{2n-2}]$$

$$+ 2b_2\cos\phi[\Omega_2\Delta_{2n-1} - \Omega_4\Gamma_{2n-1}] - b_2(1 + 2\cos^2\phi)$$

$$[\Omega_1\Gamma_{2n} + \Omega_3\Delta_{2n}] - 2b_2\cos\phi[\Omega_2\Delta_{2n+1} - \Omega_4\Gamma_{2n+1}] + \frac{b_2}{2}$$

$$[\Omega_1\Gamma_{2n+2} + \Omega_3\Delta_{2n+2}]\}C_{2n}\cos 2n\phi + \{\frac{b_2}{2}[\Omega_3\Gamma_{2n-2} - \Omega_1\Delta_{2n-2}]$$

$$+ 2b_2\cos\phi[\Omega_2\Gamma_{2n-1} + \Omega_4\Delta_{2n-1}] - b_2(1 + 2\cos^2\phi)$$

$$[\Omega_3\Gamma_{2n} - \Omega_1\Delta_{2n}] - 2b_2\cos\phi[\Omega_2\Gamma_{2n+1} + \Omega_4\Delta_{2n+1}]$$

$$+ \frac{b_2}{2}[\Omega_3\Gamma_{2n+2} - \Omega_1\Delta_{2n+2}]\}D_{2n}\cos 2n\phi$$

$$\mathrm{Re}\,\frac{\partial^2\psi}{\partial\tilde{\rho}\partial\phi} = \sum_{n=1}^{\infty} -\Big\{\{2b_1(\Omega_1\Gamma_{2n-1} + \Omega_3\Delta_{2n-1}) + 2b_2\tilde{\rho}\cos\phi$$

$$[\Omega_2(\Gamma_{2n-1} + \Delta_{2n-1}) - \Omega_4(\Gamma_{2n-1} - \Delta_{2n-1})] + b_2\tilde{\rho}$$

$$[\Omega_1(\Gamma_{2n-2} - \Delta_{2n-2} - \Gamma_{2n} + \Delta_{2n}) + \Omega_3(\Gamma_{2n-2} + \Delta_{2n-2} - \Gamma_{2n} - \Delta_{2n})]\}$$

$$\sin\phi\cos(2n-1)\phi + (2n-1)\{2b_1\cos\phi(\Omega_1\Gamma_{2n-1} + \Omega_3\Delta_{2n-1})$$

$$- b_1[\Omega_2(\Delta_{2n-2} - \Delta_{2n}) - \Omega_4(\Gamma_{2n-2} - \Gamma_{2n})]\}\sin(2n-1)\phi\Big\}\, C_{2n-1} +$$

$$\Big\{\{2b_1(\Omega_1\Delta_{2n-1} - \Omega_3\Gamma_{2n-1}) - 2b_2\tilde{\rho}\cos\phi[\Omega_2(\Gamma_{2n-1} - \Delta_{2n-1})$$

$$+ \Omega_4(\Gamma_{2n-1} + \Delta_{2n-1})] + b_2\tilde{\rho}[\Omega_1(\Gamma_{2n-2} + \Delta_{2n-2} - \Gamma_{2n} - \Delta_{2n})$$

$$- \Omega_3(\Gamma_{2n-2} - \Delta_{2n-2} - \Gamma_{2n} + \Delta_{2n})]\}\sin\phi\cos(2n-1)\phi + (2n-1)$$

$$\{2b_1\cos\phi(\Omega_1\Delta_{2n-1} - \Omega_3\Gamma_{2n-1}) + b_1[\Omega_2(\Gamma_{2n-2} - \Gamma_{2n}) + \Omega_4(\Delta_{2n-2}$$

$$- \Delta_{2n})]\}\sin(2n-1)\phi\Big\}\, D_{2n-1} - \Big\{b_1[\Omega_2(\Gamma_{2n} + \Delta_{2n}) - \Omega_4(\Gamma_{2n} - \Delta_{2n})]$$

$$+ 2b_2\tilde{\rho}\cos\phi(\Omega_1\Delta_{2n} - \Omega_3\Gamma_{2n}) + b_2\tilde{\rho}[\Omega_2(\Gamma_{2n-1} - \Gamma_{2n+1})$$

$$+ \Omega_4(\Delta_{2n-1} - \Delta_{2n+1})]\}\sin\phi\cos 2n\phi + 2n\{b_1\cos\phi$$

$$[\Omega_2(\Gamma_{2n} + \Delta_{2n}) - \Omega_4(\Gamma_{2n} - \Delta_{2n})] + \frac{b_1}{2}[\Omega_1(\Gamma_{2n-1} - \Delta_{2n-1}$$

$$-\Gamma_{2n+1} + \Delta_{2n+1}) + \Omega_3(\Gamma_{2n-1} + \Delta_{2n-1} - \Gamma_{2n+1} - \Delta_{2n+1})]\}\sin 2n\phi\Big\}$$

$$C_{2n} - \Big\{\{b_1[\Omega_2(\Gamma_{2n} - \Delta_{2n}) + \Omega_4(\Gamma_{2n} + \Delta_{2n})] + 2b_2\tilde{\rho}\cos\phi$$

$$(\Omega_1\Gamma_{2n} + \Omega_3\Delta_{2n}) - b_2\tilde{\rho}\,[\Omega_2(\Delta_{2n-1} - \Delta_{2n+1}) - \Omega_4(\Gamma_{2n-1} - \Gamma_{2n+1})]\}$$

$$\sin\phi\cos 2n\phi + 2n\{b_1\cos\phi[\Omega_2(\Gamma_{2n} - \Delta_{2n}) + \Omega_4(\Gamma_{2n} + \Delta_{2n})$$

$$- \frac{b_1}{2}[\Omega_1(\Gamma_{2n-1} + \Delta_{2n-1} - \Gamma_{2n+1} - \Delta_{2n+1}) - \Omega_3(\Gamma_{2n-1} - \Delta_{2n-1}$$

$$- \Gamma_{2n-1} - \Delta_{2n+1})]\} \sin 2n\phi\Big] D_{2n}$$

$$\text{Im}\frac{\partial^2\psi}{\partial\tilde{\rho}\partial\phi} = \sum_{n=1}^{\infty}\Big\{-\{2b_1(\Omega_1\Delta_{2n-1} - \Omega_3\Gamma_{2n-1}) - 2b_2\tilde{\rho}\cos\phi$$

$$[\Omega_2(\Gamma_{2n-1} - \Delta_{2n-1}) + \Omega_4(\Gamma_{2n-1} + \Delta_{2n-1})] + b_2\tilde{\rho}$$

$$[\Omega_1(\Gamma_{2n-2} + \Delta_{2n-2} - \Gamma_{2n} - \Delta_{2n}) - \Omega_3(\Gamma_{2n-2} - \Delta_{2n-2}$$

$$-\Gamma_{2n} + \Delta_{2n})]\} \sin\phi\cos(2n-1)\phi + (2n-1)\{2b_1\cos\phi$$

$$(\Omega_3\Gamma_{2n-1} - \Omega_1\Delta_{2n-1}) - b_1[\Omega_2(\Gamma_{2n-2} - \Gamma_{2n}) + \Omega_4(\Delta_{2n-2} - \Delta_{2n})]\}$$

$$\sin(2n-1)\phi\Big] C_{2n-1} + \Big\{-\{2b_1\Omega_1\Gamma_{2n-1} + \Omega_3\Delta_{2n-1})$$

$$+ 2b_2\tilde{\rho}\cos\phi[\Omega_2(\Gamma_{2n-1} + \Delta_{2n-1}) - \Omega_4(\Gamma_{2n-1} - \Delta_{2n-1})]$$

$$+ b_2\tilde{\rho}[\Omega_1(\Gamma_{2n-2} - \Delta_{2n-2} - \Gamma_{2n} + \Delta_{2n}) + \Omega_3(\Gamma_{2n-2} + \Delta_{2n-2}$$

$$- \Gamma_{2n} - \Delta_{2n})]\} \sin\phi\cos(2n-1)\phi - (2n-1)\{2b_1\cos\phi(\Omega_1\Gamma_{2n-1}$$

$$+ \Omega_3\Delta_{2n-1}) + b_1[\Omega_4(\Gamma_{2n-2} - \Gamma_{2n}) - \Omega_2(\Delta_{2n-2} - \Delta_{2n})]\}\sin$$

$$(2n-1)\phi\Big] D_{2n-1} + \Big\{\{b_1[\Omega_2(\Gamma_{2n} - \Delta_{2n}) + \Omega_4(\Gamma_{2n} + \Delta_{2n})]$$

$$+ 2b_2\tilde{\rho}\cos\phi(\Omega_1\Gamma_{2n} + \Omega_3\Delta_{2n}) + b_2\tilde{\rho}[\Omega_4(\Gamma_{2n-1} - \Gamma_{2n+1})$$

$$- \Omega_2(\Delta_{2n-1} - \Delta_{2n+1})]\} \sin\phi\cos 2n\phi + 2n\{b_1\cos\phi[\Omega_2(\Gamma_{2n} - \Delta_{2n})$$

$$+ \Omega_4(\Gamma_{2n} + \Delta_{2n}) - \frac{b_1}{2}[\Omega_1(\Gamma_{2n-1} + \Delta_{2n-1} - \Gamma_{2n+1} - \Delta_{2n+1})$$

$$- \Omega_3(\Gamma_{2n-1} - \Delta_{2n-1} - \Gamma_{2n+1} + \Delta_{2n+1})]\}\sin 2n\phi\Big\} C_{2n}$$

$$-\Big\{\{b_1[\Omega_2(\Gamma_{2n}+\Delta_{2n})-\Omega_4(\Gamma_{2n}-\Delta_{2n})]+2b_2\tilde{\rho}\cos\phi$$

$$(\Omega_1\Delta_{2n}-\Omega_3\Gamma_{2n})+b_2\tilde{\rho}\,[\Omega_2(\Gamma_{2n-1}-\Gamma_{2n+1})+\Omega_4$$

$$(\Delta_{2n-1}-\Delta_{2n+1})]\}\sin\phi\cos 2n\phi+2n\,\{b_1\cos\phi$$

$$[\Omega_2(\Gamma_{2n}+\Delta_{2n})-\Omega_4(\Gamma_{2n}-\Delta_{2n})]+\frac{b_1}{2}\,[\Omega_1(\Gamma_{2n-1}-\Delta_{2n-1}$$

$$-\Gamma_{2n+1}+\Delta_{2n+1})+\Omega_3(\Gamma_{2n-1}+\Delta_{2n-1}-\Gamma_{2n+1}-\Delta_{2n+1})]\}$$

$$\sin 2n\phi\Big\}D_{2n}$$

$$\mathrm{Re}\,\frac{\partial^3\psi}{\partial\tilde{\rho}\partial\phi^2}=\sum_{n=1}^{\infty}\{4(2n-1)b_1\sin\phi[(\Omega_1\Gamma_{2n-1}+\Omega_3\Delta_{2n-1})$$

$$+b_1\tilde{\rho}\cos\phi(\Omega_2-\Omega_4)\Gamma_{2n-1}+\frac{b_1\tilde{\rho}}{2}\,\Omega_1(\Gamma_{2n-2}-\Gamma_{2n}-\Delta_{2n-2}+\Delta_{2n})$$

$$b_1\tilde{\rho}\cos\phi(\Omega_2+\Omega_4)\Delta_{2n-1}+\frac{b_1\tilde{\rho}}{2}\,\Omega_3(\Gamma_{2n-2}-\Gamma_{2n}+\Delta_{2n-2}-\Delta_{2n})]$$

$$\sin(2n-1)\phi-[2(2n-1)^2b_1\cos\phi\,\Omega_1+2b_1\cos\phi\Omega_1+2b_2\tilde{\rho}\cos^2\phi$$

$$(\Omega_2-\Omega_4)-4b_2\tilde{\rho}\sin^2\phi(\Omega_2-\Omega_4)+4b_1b_2\tilde{\rho}^2\sin^2\phi\cos\phi\,\Omega_3]$$

$$\Gamma_{2n-1}\cos(2n-1)\phi-[(2n-1)^2(\Omega_2+\Omega_4)+2b_1\tilde{\rho}\cos\phi\,\Omega_1$$

$$-2b_2\tilde{\rho}^2\sin^2\phi\,(\Omega_2-\Omega_4)]\,\frac{b_1}{2}(\Gamma_{2n-2}-\Gamma_{2n}-\Delta_{2n-2}+\Delta_{2n})$$

$$\cos(2n-1)\phi+[-2(2n-1)^2\,b_1\tilde{\rho}\cos\phi\,\Omega_3-2b_1\tilde{\rho}\cos\phi\,\Omega_3-2b_2\tilde{\rho}$$

$$\cos^2\phi(\Omega_2+\Omega_4)+4b_2\tilde{\rho}\sin^2\phi(\Omega_2+\Omega_4)+4b_1b_2\tilde{\rho}^2\sin^2\phi\ \cos\phi\,\Omega_1]$$

$$\Delta_{2n-1}\cos(2n-1)\phi+[(2n-1)^2(\Omega_2-\Omega_4)-2b_1\tilde{\rho}\cos\phi\,\Omega_3+2b_2$$

$$\tilde{\rho}^2\sin^2\phi(\Omega_2+\Omega_4)]\,\frac{b_1}{2}\,(\Gamma_{2n-2}-\Gamma_{2n}+\Delta_{2n-2}-\Delta_{2n})\cos(2n-1)\phi\}$$

$$C_{2n-1}+\{4nb_1\sin\phi\,[(\Omega_2-\Omega_4)\Gamma_{2n}+(\Omega_2+\Omega_4)\Delta_{2n}]\sin 2n\phi$$

$$+ 4nb_1\tilde{\rho} \sin\phi[-2b_1 \cos\phi\Omega_3\Gamma_{2n} + \frac{b_1}{2}(\Omega_2 - \Omega_4)(\Gamma_{2n-1} - \Gamma_{2n+1}$$

$$- \Delta_{2n-1} + \Delta_{2n+1}) + 2b_1 \cos\phi\Omega_1\Delta_{2n} + \frac{b_1}{2}(\Omega_2 + \Omega_4)(\Gamma_{2n-1} - \Gamma_{2n+1}$$

$$+ \Delta_{2n-1} - \Delta_{2n+1})] \sin 2n\phi - [(2n)^2 b_1 \cos\phi(\Omega_2 - \Omega_4) + b_1 \cos\phi$$

$$(\Omega_2 - \Omega_4) - 2b_2\tilde{\rho}\cos^2\phi\,\Omega_3 + 4b_2\tilde{\rho}\sin^2\phi\Omega_3 - 2b_1b_2\tilde{\rho}^2\sin^2\phi \cos\phi$$

$$(\Omega_2 + \Omega_4)]\Gamma_{2n} \cos 2n\phi - [4n^2\Omega_1 + b_1\tilde{\rho} \cos\phi(\Omega_2 - \Omega_4) - 2b_2$$

$$\tilde{\rho}^2\sin^2\phi\Omega_3]\frac{b_1}{2}(\Gamma_{2n-1} - \Gamma_{2n+1} - \Delta_{2n-1} + \Delta_{2n+1}) \cos 2n\phi$$

$$-[4n^2b_1 \cos\phi(\Omega_2 + \Omega_4) + b_1 \cos\phi(\Omega_2 + \Omega_4) + 2b_2\tilde{\rho} \cos^2\phi\Omega_1$$

$$+ 4b_2\tilde{\rho} \sin^2\phi\Omega_1 + 2b_1b_2\tilde{\rho}^2\sin^2\phi \cos\phi(\Omega_2 - \Omega_4)]\Delta_{2n} \cos 2n\phi$$

$$- [4n^2\Omega_3 + b_1\tilde{\rho} \cos\phi(\Omega_2 + \Omega_4) + 2b_2\tilde{\rho}^2\sin^2\phi\Omega_1]\frac{b_1}{2}$$

$$(\Gamma_{2n-1} - \Gamma_{2n+1} + \Delta_{2n-1} - \Delta_{2n+1}) \cos 2n\phi\} C_{2n}$$

$$-\{4(2n-1)b_1 \sin\phi[(\Omega_1\Delta_{2n-1} - \Omega_3\Gamma_{2n-1}) + b_1\tilde{\rho} \cos\phi(\Omega_2 - \Omega_4)\Delta_{2n-1}$$

$$-b_1\tilde{\rho} \cos\phi(\Omega_2 + \Omega_4)\Gamma_{2n-1} + \frac{b_1\tilde{\rho}}{2}\Omega_1(\Gamma_{2n-2} - \Gamma_{2n} + \Delta_{2n-2} - \Delta_{2n})$$

$$- \frac{b_1\tilde{\rho}}{2}\Omega_3(\Gamma_{2n-2} - \Gamma_{2n} - \Delta_{2n-2} + \Delta_{2n})]\sin(2n-1)\phi - [-2(2n-1)^2$$

$$b_1 \cos\phi\Omega_3 - 2b_1 \cos\phi\Omega_3 - 2b_2\tilde{\rho}\cos^2\phi(\Omega_2 + \Omega_4) + 4b_2\tilde{\rho} \sin^2\phi$$

$$(\Omega_2 + \Omega_4) + 4b_1b_2\tilde{\rho}^2\sin^2\phi \cos\phi\Omega_1]\Gamma_{2n-1} \cos(2n-1)\phi - [(2n-1)^2$$

$$(\Omega_2 - \Omega_4) - 2b_1\tilde{\rho} \cos\phi\Omega_3 + 2b_2\tilde{\rho}^2\sin^2\phi(\Omega_2 + \Omega_4)]\frac{b_1}{2}(\Gamma_{2n-2} - \Gamma_{2n}$$

$$-\Delta_{2n-2} + \Delta_{2n}) \cos(2n-1)\phi - [2(2n-1)^2b_1 \cos\phi\Omega_1 +$$

$$+ 2b_1 \cos \phi\Omega_1 + 2b_2\tilde{\rho} \cos^2\phi(\Omega_2 - \Omega_4) - 4b_2\tilde{\rho} \sin^2 \phi$$

$$(\Omega_2 - \Omega_4) + 4b_1b_2\tilde{\rho}^2 \sin^2\phi \cos \phi\Omega_3]\Delta_{2n-1} \cos (2n-1)\phi$$

$$-[(2n-1)^2(\Omega_2 + \Omega_4) + 2b_1\tilde{\rho} \cos \phi\Omega_1 - 2b_2\tilde{\rho}^2\sin^2\phi(\Omega_2 - \Omega_4)]$$

$$\frac{b_1}{2}(\Gamma_{2n-2} - \Gamma_{2n} + \Delta_{2n-2} - \Delta_{2n}) \cos(2n-1)\phi\} D_{2n-1} -\{4nb_1 \sin \phi$$

$$[(\Omega_2 - \Omega_4)\Delta_{2n} - (\Omega_2 + \Omega_4)\Gamma_{2n}] \sin 2n\phi + 4nb_1\tilde{\rho} \sin \phi$$

$$[-2b_1 \cos \phi\Omega_3\Delta_{2n} + \frac{b_1}{2}(\Omega_2 - \Omega_4)(\Gamma_{2n-1} - \Gamma_{2n+1} + \Delta_{2n-1} - \Delta_{2n+1})$$

$$- 2b_1 \cos \phi\Omega_1\Gamma_{2n} - \frac{b_1}{2}(\Omega_2 + \Omega_4)(\Gamma_{2n-1} - \Gamma_{2n+1} - \Delta_{2n-1} + \Delta_{2n+1})]$$

$$\sin 2n\phi + [4n^2b_1 \cos \phi(\Omega_2 + \Omega_4) + b_1 \cos \phi(\Omega_2 + \Omega_4) + 2b_2$$

$$\tilde{\rho} \cos^2\phi\Omega_1 + 4b_2\tilde{\rho} \sin^2\phi\Omega_1 + 2b_1b_2\tilde{\rho}^2\sin^2\phi \cos \phi(\Omega_2 - \Omega_4)]$$

$$\Gamma_{2n} \cos 2n\phi + [4n^2\Omega_3 + b_1\tilde{\rho} \cos \phi(\Omega_2 + \Omega_4) + 2b_2\tilde{\rho}^2\sin^2\phi\Omega_1]$$

$$\frac{b_1}{2}(\Gamma_{2n-1} - \Gamma_{2n+1} - \Delta_{2n-1} + \Delta_{2n+1})\cos 2n\phi - [4n^2b_1 \cos \phi$$

$$(\Omega_2 - \Omega_4) + b_1 \cos \phi(\Omega_2 - \Omega_4) - 2b_2\tilde{\rho} \cos^2\phi\Omega_3 - 4b_2\tilde{\rho}\sin^2\phi\Omega_3$$

$$- 2b_1b_2\tilde{\rho}^2\sin^2\phi \cos \phi(\Omega_2 + \Omega_4)]\Delta_{2n} \cos 2n\phi - [4n^2\Omega_1 + b_1\tilde{\rho}$$

$$\cos\phi(\Omega_2 - \Omega_4) - 2b_2\tilde{\rho}^2\sin^2\phi\Omega_3] \frac{b_1}{2}(\Gamma_{2n-1} - \Gamma_{2n+1} + \Delta_{2n-1}$$

$$- \Delta_{2n+1})] \cos 2n\phi\} D_{2n}$$

$$\text{Im}\frac{\partial^3\psi}{\partial\tilde{\rho}\partial\phi^2} = \sum_{n=1}^{\infty}\Big\{4(2n-1)b_1\sin\phi\{(\Omega_1\Delta_{2n-1} - \Omega_3\Gamma_{2n-1}) + b_1\tilde{\rho}$$

$$\cos\phi[(\Omega_2 - \Omega_4)\Delta_{2n-1} - (\Omega_2 + \Omega_4)\Gamma_{2n-1}] + \frac{b_1\tilde{\rho}}{2}[\Omega_1(\Gamma_{2n-2}$$

$$- \Gamma_{2n} + \Delta_{2n-2} - \Delta_{2n}) - \Omega_3(\Gamma_{2n-2} - \Gamma_{2n} - \Delta_{2n-2} + \Delta_{2n})]\}$$

$$\sin(2n-1)\phi + \{(-2(2n-1)^2b_1\cos\phi(\Omega_1\Delta_{2n-1} - \Omega_3\Gamma_{2n-1}) - 2b_1$$

$$\cos\phi(\Omega_1\Delta_{2n-1} - \Omega_3\Gamma_{2n-1}) - 2b_2\tilde{\rho}\cos^2\phi[(\Omega_2 - \Omega_4)\Delta_{2n-1} -$$

$$(\Omega_2 + \Omega_4)\Gamma_{2n-1}] + 4b_2\tilde{\rho}\sin^2\phi[(\Omega_2 - \Omega_4)\Delta_{2n-1} - (\Omega_2 + \Omega_4)\Gamma_{2n-1}]$$

$$- 4b_1b_2\tilde{\rho}^2\sin^2\phi\cos\phi(\Omega_3\Delta_{2n-1} + \Omega_1\Gamma_{2n-1}) - (2n-1)^2b_1[\Omega_2$$

$$(\Gamma_{2n-2} - \Gamma_{2n}) + \Omega_4(\Delta_{2n-2} - \Delta_{2n})] + b_2\tilde{\rho}\cos\phi[\Omega_3(\Gamma_{2n-2} - \Gamma_{2n}$$

$$- \Delta_{2n-2} + \Delta_{2n}) - \Omega_1(\Gamma_{2n-2} - \Gamma_{2n} + \Delta_{2n-2} - \Delta_{2n})] + 2b_1b_2\tilde{\rho}^2\sin^2\phi$$

$$[\Omega_2(\Delta_{2n-2} - \Delta_{2n}) - \Omega_4(\Gamma_{2n-2} - \Gamma_{2n})]\}\cos(2n-1)\phi\Big\}C_{2n-1}$$

$$+\Big\{4(2n-1)b_1\sin\phi\{(\Omega_1\Gamma_{2n-1} + \Omega_3\Delta_{2n-1}) + b_1\tilde{\rho}\cos\phi[(\Omega_2 - \Omega_4)$$

$$\Gamma_{2n-1} + (\Omega_2 + \Omega_4)\Delta_{2n-1}] + \frac{b_1\tilde{\rho}}{2}[\Omega_1(\Gamma_{2n-2} - \Gamma_{2n} - \Delta_{2n-2} + \Delta_{2n})$$

$$+ \Omega_3(\Gamma_{2n-2} - \Gamma_{2n} + \Delta_{2n-2} - \Delta_{2n})]\}\sin(2n-1)\phi + \{2(2n-1)^2$$

$$b_1\cos\phi(\Omega_1\Gamma_{2n-1} + \Omega_3\Delta_{2n-1}) - 2b_1\cos\phi(\Omega_1\Gamma_{2n-1} + \Omega_3\Delta_{2n-1})$$

$$- 4b_1b_2\tilde{\rho}^2\sin^2\phi\cos\phi(\Omega_3\Gamma_{2n-1} - \Omega_1\Delta_{2n-1}) - 2b_2\tilde{\rho}\cos^2\phi$$

$$[(\Omega_2 - \Omega_4)\Gamma_{2n-1} + (\Omega_2 + \Omega_4)\Delta_{2n-1}] + 4b_2\tilde{\rho}\sin^2\phi[(\Omega_2 - \Omega_4)\Gamma_{2n+1}$$

$$(\Omega_2 + \Omega_4)\Delta_{2n-1}] - (2n-1)^2b_1[\Omega_4(\Gamma_{2n-2} - \Gamma_{2n}) - \Omega_2(\Delta_{2n-2} - \Delta_{2n})] - b_2\tilde{\rho}\cos\phi$$

$$[\Omega_1(\Gamma_{2n-2} - \Gamma_{2n} - \Delta_{2n-2} + \Delta_{2n}) + \Omega_3(\Gamma_{2n-2} - \Gamma_{2n} + \Delta_{2n} - \Delta_{2n-2})]$$

$$+ 2b_1b_2\tilde{\rho}^2\sin^2\phi[\Omega_2(\Gamma_{2n-2} - \Gamma_{2n}) + \Omega_4(\Delta_{2n-2} - \Delta_{2n})]\}$$

$$\cos(2n-1)\phi\Big]\Big\} D_{2n-1} + \Big\{\Big[4nb_1 \sin\phi\{[(\Omega_2 - \Omega_4)\Delta_{2n} - (\Omega_2 + \Omega_4)\Gamma_{2n}]$$

$$-2b_1\tilde{\rho}\cos\phi(\Omega_3\Delta_{2n} + \Omega_1\Gamma_{2n}) + \frac{b_1\tilde{\rho}}{2}[(\Omega_2 - \Omega_4)(\Gamma_{2n-1} - \Gamma_{2n+1}$$

$$+ \Delta_{2n-1} - \Delta_{2n+1}) - (\Omega_2 + \Omega_4)(\Gamma_{2n-1} - \Gamma_{2n+1} - \Delta_{2n-1} + \Delta_{2n+1})]\}$$

$$\sin 2n\phi + \{4n^2b_1\cos\phi[(\Omega_2 + \Omega_4)\Gamma_{2n} - (\Omega_2 - \Omega_4)\Delta_{2n}] + b_1\cos\phi$$

$$[(\Omega_2 + \Omega_4)\Gamma_{2n} - (\Omega_2 - \Omega_4)\Delta_{2n}] + b_2\tilde{\rho}\cos^2\phi[\Omega_1\Gamma_{2n} + \Omega_3\Delta_{2n}]$$

$$+ 4b_2\tilde{\rho}\sin^2\phi[\Omega_1\Gamma_{2n} + \Omega_3\Delta_{2n}] + 2b_1b_2\tilde{\rho}^2\sin^2\phi\cos\phi[(\Omega_2 - \Omega_4)\Gamma_{2n}$$

$$(\Omega_2 + \Omega_4)\Delta_{2n}] + 4n^2b_1[\frac{\Omega_3}{2}(\Gamma_{2n-1} - \Gamma_{2n+1} - \Delta_{2n-1} + \Delta_{2n+1})$$

$$- \frac{\Omega_1}{2}(\Gamma_{2n-1} - \Gamma_{2n+1} + \Delta_{2n-1} - \Delta_{2n+1})] + b_2\tilde{\rho}\cos\phi[\Omega_4(\Gamma_{2n-1}$$

$$- \Gamma_{2n+1}) - \Omega_2(\Delta_{2n-1} - \Delta_{2n+1})] + 2b_1b_2\tilde{\rho}^2\sin^2\phi[\frac{\Omega_1}{2}(\Gamma_{2n-1}$$

$$- \Gamma_{2n+1} - \Delta_{2n-1} + \Delta_{2n+1}) + \frac{\Omega_3}{2}(\Gamma_{2n-1} - \Gamma_{2n+1} + \Delta_{2n-1} - \Delta_{2n+1})]\}$$

$$\cos 2n\phi\Big]C_{2n} + \Big\{\Big[4nb_1 \sin\phi\{[(\Omega_2 - \Omega_4)\Gamma_{2n} + (\Omega_2 + \Omega_4)\Delta_{2n}]$$

$$-2b_1\tilde{\rho}\cos\phi[\Omega_3\Gamma_{2n} - \Omega_1\Delta_{2n}] + \frac{b_1\tilde{\rho}}{2}[(\Omega_2 - \Omega_4)(\Gamma_{2n-1} - \Gamma_{2n+1}$$

$$- \Delta_{2n-1} + \Delta_{2n+1}) + (\Omega_2 + \Omega_4)(\Gamma_{2n-1} - \Gamma_{2n+1} + \Delta_{2n-1} - \Delta_{2n+1})]\}$$

$$\sin 2n\phi - \{4n^2b_1\cos\phi[(\Omega_2 - \Omega_4)\Gamma_{2n} + (\Omega_2 + \Omega_4)\Delta_{2n}] + b_1\cos\phi$$

$$[(\Omega_2 - \Omega_4)\Gamma_{2n} + (\Omega_2 + \Omega_4)\Delta_{2n}] + 2b_2\tilde{\rho}\cos^2\phi[\Omega_1\Delta_{2n} - \Omega_3\Gamma_{2n}] +$$

$$4b_2\tilde{\rho}\sin^2\phi[-\Omega_3\Gamma_{2n}+\Omega_1\Delta_{2n}]-2b_1b_2\tilde{\rho}^2\sin^2\phi\cos\phi[(\Omega_2+\Omega_4)\Gamma_{2n}$$

$$-(\Omega_2-\Omega_4)\Delta_{2n}]+4n^2b_1[\frac{\Omega_1}{2}(\Gamma_{2n-1}-\Gamma_{2n+1}-\Delta_{2n-1}+\Delta_{2n+1})$$

$$+\frac{\Omega_3}{2}(\Gamma_{2n-1}-\Gamma_{2n+1}+\Delta_{2n+1}-\Delta_{2n+1})]+b_2\tilde{\rho}\cos\phi[\Omega_2(\Gamma_{2n-1}$$

$$-\Gamma_{2n+1})+\Omega_4(\Delta_{2n-1}-\Delta_{2n+1})]-2b_1b_2\tilde{\rho}^2\sin^2\phi[\frac{\Omega_3}{2}(\Gamma_{2n-1}-\Gamma_{2n+1}$$

$$-\Delta_{2n-1}+\Delta_{2n+1})-\frac{\dot{\Omega}_1}{2}(\Gamma_{2n-1}-\Gamma_{2n+1}+\Delta_{2n-1}-\Delta_{2n+1})]\}$$

$$\cos 2n\phi\Big\}D_{2n}$$

$$\mathrm{Re}\frac{\partial^3\psi}{\partial\tilde{\rho}^3}=\sum_{n=1}^{\infty}\{\frac{b_3}{2}\cos\phi[(\Omega_2+\Omega_4)(\Gamma_{2n-2}+\Gamma_{2n+2})-(\Omega_2-\Omega_4)$$

$$(\Delta_{2n-2}+\Delta_{2n+2})]\frac{b_3}{4}[\Omega_3(\Gamma_{2n-3}-\Gamma_{2n-1}+\Gamma_{2n+1}-\Gamma_{2n+3}$$

$$-\Delta_{2n-3}+\Delta_{2n-1}-\Delta_{2n+1}+\Delta_{2n+3})-\Omega_1(\Gamma_{2n-3}-\Gamma_{2n-1}+\Gamma_{2n+1}$$

$$-\Gamma_{2n+3}+\Delta_{2n-3}-\Delta_{2n-1}+\Delta_{2n+1}-\Delta_{2n+3})]+2b_3\cos\phi$$

$$[\cos\phi[(\Omega_1-\Omega_3)(\Gamma_{2n-1}-\Gamma_{2n+1})+(\Omega_1+\Omega_3)(\Delta_{2n-1}-\Delta_{2n+1})]$$

$$+\frac{\Omega_2}{2}(\Gamma_{2n-2}-2\Gamma_{2n}+\Gamma_{2n+2}-\Delta_{2n-2}+2\Delta_{2n}-\Delta_{2n+2})+\frac{\Omega_4}{2}$$

$$(\Gamma_{2n-2}-2\Gamma_{2n}+\Gamma_{2n+2}+\Delta_{2n-2}-2\Delta_{2n}+\Delta_{2n+2})]-b_3(1+2\cos^2\phi)$$

$$[\cos\phi((\Omega_2+\Omega_4)\Gamma_{2n}-(\Omega_2-\Omega_4)\Delta_{2n})+\frac{\Omega_3}{2}(\Gamma_{2n-1}-\Gamma_{2n+1}$$

$$-\Delta_{2n-1}+\Delta_{2n+1})-\frac{\Omega_1}{2}(\Gamma_{2n-1}-\Gamma_{2n+1}+\Delta_{2n-1}-\Delta_{2n+1})]\}$$

$$C_{2n}\cos 2n\phi-\{\frac{b_3}{2}\cos\phi[(\Omega_2-\Omega_4)(\Gamma_{2n-2}+\Gamma_{2n+2})+(\Omega_2+\Omega_4)$$

$$(\Delta_{2n-2}+\Delta_{2n+2})+\frac{b_3}{4}[\Omega_1(\Gamma_{2n-3}-\Gamma_{2n-1}+\Gamma_{2n+1}-\Gamma_{2n+3}+\Delta_{2n-1}$$

$$-\Delta_{2n-3}-\Delta_{2n+1}+\Delta_{2n+3})+\Omega_3(\Gamma_{2n-3}-\Gamma_{2n-1}+\Gamma_{2n+1}-\Gamma_{2n+3}+$$

$$\Delta_{2n-3} - \Delta_{2n-1} + \Delta_{2n+1} - \Delta_{2n+3})] + 2b_3 \cos\phi[\cos\phi$$

$$((\Omega_1 - \Omega_3)(\Delta_{2n-1} - \Delta_{2n+1}) - (\Omega_1 + \Omega_3)(\Gamma_{2n-1} - \Gamma_{2n+1}))$$

$$+ \frac{\Omega_2}{2}(\Gamma_{2n-2} - 2\Gamma_{2n} + \Gamma_{2n+2} + \Delta_{2n-2} - 2\Delta_{2n} + \Delta_{2n+2}) - \frac{\Omega_4}{2}$$

$$(\Gamma_{2n-2} - 2\Gamma_{2n} + \Gamma_{2n+2} - \Delta_{2n-2} + 2\Delta_{2n} - \Delta_{2n+2})] - b_3(1 + 2\cos^2\phi)$$

$$[\cos\phi((\Omega_2 - \Omega_4)\Gamma_{2n} + (\Omega_2 + \Omega_4)\Delta_{2n}) + \frac{\Omega_1}{2}(\Gamma_{2n-1} - \Gamma_{2n+1}$$

$$- \Delta_{2n-1} + \Delta_{2n+1}) + \frac{\Omega_3}{2}(\Gamma_{2n-1} - \Gamma_{2n+1} + \Delta_{2n-1} - \Delta_{2n+1})]\} D_{2n}$$

$$\cos 2n\phi + \frac{b_3}{2}[2\cos\phi(\Omega_3(\Gamma_{2n-3} + \Gamma_{2n+1}) - \Omega_1(\Delta_{2n-3} + \Delta_{2n+1}))$$

$$-(\frac{\Omega_3 - \Omega_4}{2})(\Gamma_{2n-4} - \Gamma_{2n-2} + \Gamma_{2n} - \Gamma_{2n+2} - \Delta_{2n-4} + \Delta_{2n-2} - \Delta_{2n}$$

$$+ \Delta_{2n+2}) - (\frac{\Omega_2 + \Omega_4}{2})(\Gamma_{2n-4} - \Gamma_{2n-2} + \Gamma_{2n} - \Gamma_{2n+2} + \Delta_{2n-4}$$

$$- \Delta_{2n-2} + \Delta_{2n} - \Delta_{2n+2})] + 2b_3\cos\phi[2\cos\phi(\Omega_2(\Gamma_{2n-2} - \Gamma_{2n})$$

$$+ \Omega_4(\Delta_{2n-2} - \Delta_{2n})) + (\frac{\Omega_1 + \Omega_3}{2})(\Gamma_{2n-3} - 2\Gamma_{2n-1} + \Gamma_{2n+1}$$

$$- \Delta_{2n-3} + 2\Delta_{2n-1} - \Delta_{2n+1}) - (\frac{\Omega_1 - \Omega_3}{2})(\Gamma_{2n-3} - 2\Gamma_{2n-1} + \Gamma_{2n+1} + \Delta_{2n-3}$$

$$- 2\Delta_{2n-1} + \Delta_{2n+1})] + b_3(1 + 2\cos^2\phi)[2\cos\phi(\Omega_1\Delta_{2n-1}$$

$$- \Omega_3\Gamma_{2n-1}) + (\frac{\Omega_2 - \Omega_4}{2})(\Gamma_{2n-2} - \Gamma_{2n} - \Delta_{2n-2} + \Delta_{2n}) + (\frac{\Omega_2 + \Omega_4}{2})$$

$$(\Gamma_{2n-2} - \Gamma_{2n} + \Delta_{2n-2} - \Delta_{2n})]\} C_{2n-1}\cos(2n-1)\phi - \{\frac{b_3}{2}[2\cos\phi$$

$$(\Omega_3(\Delta_{2n-3} + \Delta_{2n+1}) + \Omega_1(\Gamma_{2n-3} + \Gamma_{2n+1})) - (\frac{\Omega_2 - \Omega_4}{2})$$

$$(\Gamma_{2n-4} - \Gamma_{2n-2} + \Gamma_{2n} - \Gamma_{2n+2} + \Delta_{2n-4} - \Delta_{2n-2} + \Delta_{2n} - \Delta_{2n+2})$$

$$+(\frac{\Omega_2 + \Omega_4}{2})(\Gamma_{2n-4} - \Gamma_{2n-2} + \Gamma_{2n} - \Gamma_{2n+2} - \Delta_{2n-4} + \Delta_{2n-2} - \Delta_{2n}$$

$$+ \Delta_{2n+2})] + 2b_3 \cos\phi[2\cos\phi(\Omega_2(\Delta_{2n-2} - \Delta_{2n})$$

$$- \Omega_4(\Gamma_{2n-2} - \Gamma_{2n})) + (\frac{\Omega_1 - \Omega_3}{2})(\Gamma_{2n-3} - 2\Gamma_{2n-1} + \Gamma_{2n+1} - \Delta_{2n-3}$$

$$+2\Delta_{2n-1} - \Delta_{2n+1}) + (\frac{\Omega_1 + \Omega_3}{2})(\Gamma_{2n-3} - 2\Gamma_{2n-1} + \Gamma_{2n+1} + \Delta_{2n-3}$$

$$-2\Delta_{2n-1} + \Delta_{2n+1})] + b_3(1 + 2\cos^2\phi)[-2\cos\phi(\Omega_1\Gamma_{2n-1}$$

$$+ \Omega_3\Delta_{2n-1}) + (\frac{\Omega_2 - \Omega_4}{2})(\Gamma_{2n-2} - \Gamma_{2n} + \Delta_{2n-2} - \Delta_{2n}) - (\frac{\Omega_2 + \Omega_4}{2})$$

$$(\Gamma_{2n-2} - \Gamma_{2n} - \Delta_{2n-2} + \Delta_{2n})]\} D_{2n-1}\cos(2n-1)\phi$$

$$\text{Im}\frac{\partial^3\psi}{\partial\rho^3} = \sum_{n=1}^{\infty}\{\frac{b_3}{2}\cos\phi[(\Omega_2 - \Omega_4)(\Gamma_{2n-2} + \Gamma_{2n+2}) + (\Omega_2 + \Omega_4)$$

$$(\Delta_{2n-2} + \Delta_{2n+2})] + \frac{b_3}{4}[\Omega_1(\Gamma_{2n-3} - \Gamma_{2n-1} - \Delta_{2n-3} + \Delta_{2n-1}$$

$$+ \Gamma_{2n+1} - \Gamma_{2n+3} - \Delta_{2n+1} + \Delta_{2n+3}) + \Omega_3(\Gamma_{2n-3} - \Gamma_{2n-1} + \Delta_{2n-3}$$

$$- \Delta_{2n-1} + \Gamma_{2n+1} - \Gamma_{2n+3} + \Delta_{2n+1} - \Delta_{2n+3})] + 2b_3\cos\phi[\cos\phi$$

$$((\Omega_1 - \Omega_3)(\Delta_{2n-1} - \Delta_{2n+1}) - (\Omega_1 + \Omega_3)(\Gamma_{2n-1} - \Gamma_{2n+1})) + \frac{\Omega_2}{2}$$

$$(\Gamma_{2n-2} - 2\Gamma_{2n} + \Gamma_{2n+2} + \Delta_{2n-2} - 2\Delta_{2n} + \Delta_{2n+2}) - \frac{\Omega_4}{2}$$

$$(\Gamma_{2n-2} - 2\Gamma_{2n} + \Gamma_{2n+2} - \Delta_{2n-2} + 2\Delta_{2n} - \Delta_{2n+2})] - b_3$$

$$(1 + 2\cos^2\phi)[\cos\phi((\Omega_2 - \Omega_4)\Gamma_{2n} + (\Omega_2 + \Omega_4)\Delta_{2n}) + \frac{\Omega_1}{2}$$

$$(\Gamma_{2n-1} - \Gamma_{2n+1} - \Delta_{2n-1} + \Delta_{2n+1}) + \frac{\Omega_3}{2}(\Gamma_{2n-1} - \Gamma_{2n+1} + \Delta_{2n-1}$$

$$- \Delta_{2n+1})]\} C_{2n}\cos 2n\phi + \{\frac{b_3}{2}\cos\phi[(\Omega_2 + \Omega_4)(\Gamma_{2n-2} + \Gamma_{2n+2})$$

$$- (\Omega_2 - \Omega_4)(\Delta_{2n-2} + \Delta_{2n+2})] + \frac{b_3}{4}[\Omega_3(\Gamma_{2n-3} - \Gamma_{2n-1} - \Delta_{2n-3}$$

$$+ \Delta_{2n-1} + \Gamma_{2n+1} - \Gamma_{2n+3} - \Delta_{2n+1} + \Delta_{2n+3}) - \Omega_1(\Gamma_{2n-3} - \Gamma_{2n-1} +$$

$$\Delta_{2n-3} - \Delta_{2n-1} + \Gamma_{2n+1} - \Gamma_{2n+3} + \Delta_{2n+1} - \Delta_{2n+3})] + 2b_3 \cos\phi$$

$$[\cos\phi((\Omega_1 - \Omega_3)(\Gamma_{2n-1} - \Gamma_{2n+1}) + (\Omega_1 + \Omega_3)(\Delta_{2n-1} - \Delta_{2n+1}))$$

$$+ \frac{\Omega_2}{2}(\Gamma_{2n-2} - 2\Gamma_{2n} + \Gamma_{2n+2} - \Delta_{2n-2} + 2\Delta_{2n} - \Delta_{2n+2}) + \frac{\Omega_4}{2}$$

$$(\Gamma_{2n-2} - 2\Gamma_{2n} + \Gamma_{2n+2} + \Delta_{2n-2} - 2\Delta_{2n} + \Delta_{2n+2})] - b_3$$

$$(1 + 2\cos^2\phi)[\cos\phi((\Omega_2 + \Omega_4)\Gamma_{2n} - (\Omega_2 - \Omega_4)\Delta_{2n}) + \frac{\Omega_3}{2}$$

$$(\Gamma_{2n-1} - \Gamma_{2n+1} - \Delta_{2n-1} + \Delta_{2n+1}) - \frac{\Omega_1}{2}(\Gamma_{2n-1} - \Gamma_{2n+1}$$

$$+ \Delta_{2n-1} - \Delta_{2n+1})]\} D_{2n} \cos 2n\phi + \{\frac{b_3}{2}[2\cos\phi(\Omega_3(\Delta_{2n-3}$$

$$+ \Delta_{2n+1}) + \Omega_1(\Gamma_{2n-3} + \Gamma_{2n+1})) - (\frac{\Omega_2 - \Omega_4}{2})(\Gamma_{2n-4} - \Gamma_{2n-2}$$

$$+ \Gamma_{2n} - \Gamma_{2n+2} + \Delta_{2n-4} - \Delta_{2n-2} + \Delta_{2n} - \Delta_{2n+2}) + (\frac{\Omega_2 + \Omega_4}{2})$$

$$(\Gamma_{2n-4} - \Gamma_{2n-2} + \Gamma_{2n} - \Gamma_{2n+2} - \Delta_{2n-4} + \Delta_{2n-2} - \Delta_{2n} + \Delta_{2n+2})]$$

$$+ 2b_3 \cos\phi[2\cos\phi(\Omega_2(\Delta_{2n-2} - \Delta_{2n}) - \Omega_4(\Gamma_{2n-2} - \Gamma_{2n}))$$

$$+(\frac{\Omega_1 - \Omega_3}{2})(\Gamma_{2n-3} - 2\Gamma_{2n-1} + \Gamma_{2n+1} - \Delta_{2n-3} + 2\Delta_{2n-1}$$

$$- \Delta_{2n+1}) + (\frac{\Omega_1 + \Omega_3}{2})(\Gamma_{2n-3} - 2\Gamma_{2n-1} + \Gamma_{2n+1} + \Delta_{2n-3}$$

$$-2\Delta_{2n-1} + \Delta_{2n+1})] - b_3(1 + 2\cos^2\phi)[2\cos\phi(\Omega_3\Delta_{2n-1}$$

$$+ \Omega_1\Gamma_{2n-1}) - (\frac{\Omega_2 - \Omega_4}{2})(\Gamma_{2n-2} - \Gamma_{2n} + \Delta_{2n-2} - \Delta_{2n}) + (\frac{\Omega_2 + \Omega_4}{2})$$

$$(\Gamma_{2n-2} - \Gamma_{2n} - \Delta_{2n-2} + \Delta_{2n})]\} C_{2n-1} \cos(2n-1)\phi + \{\frac{b_3}{2}[2\cos\phi$$

$$(\Omega_3(\Gamma_{2n-3} + \Gamma_{2n+1}) - \Omega_1(\Delta_{2n-3} + \Delta_{2n+1})) - (\frac{\Omega_2 - \Omega_4}{2})$$

$$(\Gamma_{2n-4} - \Gamma_{2n-2} + \Gamma_{2n} - \Gamma_{2n+2} - \Delta_{2n-4} + \Delta_{2n-2} - \Delta_{2n} + \Delta_{2n+2})$$

$$-(\frac{\Omega_2 + \Omega_4}{2})(\Gamma_{2n-4} - \Gamma_{2n-2} + \Gamma_{2n} - \Gamma_{2n+2} + \Delta_{2n-4} - \Delta_{2n-2}$$

$$+ \Delta_{2n} - \Delta_{2n+2}) + 2b_3 \cos\phi[(\Omega_2(\Gamma_{2n-2} - \Gamma_{2n}) + \Omega_4(\Delta_{2n-2}$$

$$- \Delta_{2n}))2\cos\phi + (\frac{\Omega_1 + \Omega_3}{2})(\Gamma_{2n-3} - 2\Gamma_{2n-1} + \Gamma_{2n+1} - \Delta_{2n-3} + 2\Delta_{2n-1}$$

$$- \Delta_{2n+1}) - (\frac{\Omega_1 - \Omega_3}{2})(\Gamma_{2n-3} - 2\Gamma_{2n-1} + \Gamma_{2n+1} + \Delta_{2n-3} - 2\Delta_{2n-1}$$

$$+ \Delta_{2n+1})] + b_3(1 + 2\cos^2\phi)[2\cos\phi(\Omega_1\Delta_{2n-1} - \Omega_3\Gamma_{2n-1})$$

$$+ (\frac{\Omega_2 - \Omega_4}{2})(\Gamma_{2n-2} - \Gamma_{2n} - \Delta_{2n-2} + \Delta_{2n}) + (\frac{\Omega_2 + \Omega_4}{2})(\Gamma_{2n-2} - \Gamma_{2n}$$

$$+ \Delta_{2n-2} - \Delta_{2n})]\} D_{2n-1} \cos(2n-1)\phi$$

The series expansions of the needed displacements and stress resultants for the small shell are given below. The constants in the series $\mathring{\psi}$ are defined as:

$$\omega_{n1} = -(\frac{[3(1-\nu^2)]^{1/2}}{4}\frac{\rho_o}{t})^{1/2} - (\frac{-n^2}{2} + [\frac{n^4}{4} + \frac{3}{16}(1-\nu^2)(\frac{\rho_o}{t})^2]^{1/2})^{1/2}$$

$$\omega_{n2} = -(\frac{[3(1-\nu^2)]^{1/2}}{4}\frac{\rho_o}{t})^{1/2} + (\frac{-n^2}{4} + [\frac{n^4}{4} + \frac{3}{16}(1-\nu^2)(\frac{\rho_o}{t})^2]^{1/2})^{1/2}$$

$$\pi_{n1} = -(\frac{n^2}{2} + [\frac{n^4}{4} + \frac{3}{16}(1-\nu^2)(\frac{\rho_o}{t})^2]^{1/2})^{1/2} - (\frac{[3(1-\nu^2]1/2}{4}\frac{\rho_o}{t})^{1/2}$$

$$\pi_{n2} = -(\frac{n^2}{2} + [\frac{n^4}{4} + \frac{3}{16}(1-\nu^2)(\frac{\rho_o}{t})^2]^{1/2})^{1/2} + (\frac{[3(1-\nu^2)]^{1/2}}{4}\frac{\rho_o}{t})^{1/2}$$

$$(n_\phi)_D = \frac{1}{k\rho_o^2} \sum_{n=0,2,..}^{\infty} \sum_{j=1}^{2} \{A_{nj}[2\omega_{nj}\pi_{nj} \cos \omega_{nj}\tilde{y} + (\pi_{nj}^2 - \omega_{nj}^2)$$

$$\sin \omega_{nj}\tilde{y}] + B_{nj}[(\pi_{nj}^2 - \omega_{nj}^2)\cos \omega_{nj}\tilde{y} - 2\pi_{nj}\omega_{nj} \sin \omega_{nj}\tilde{y}]\}$$

$$e^{\pi_{nj}\tilde{y}} \cos n\phi$$

$$(n_y)_D = \frac{1}{k\rho_o^2} \sum_{n=0,2..}^{\infty} \sum_{j=1}^{2} (A_{nj} \sin \omega_{nj}\tilde{y} + B_{nj} \cos \omega_{nj}\tilde{y})$$

$$(-n^2 e^{\pi_{nj}\tilde{y}}) \cos n\phi$$

$$(n_{y\phi})_D = \frac{1}{k\rho_o^2} \sum_{n=o,2..}^{\infty} \sum_{j=1}^{2} [A_{nj}(\omega_{nj} \cos \omega_{nj}\tilde{y} + \pi_{nj} \sin \omega_{nj}\tilde{y})$$

$$+ B_{nj}(\pi_{nj} \cos \omega_{nj}\tilde{y} - \omega_{nj} \sin\omega_{nj}\tilde{y})(ne^{\pi_{nj}\tilde{y}})\sin n\phi$$

$$(m_\phi)_D = \frac{-D}{\rho_o} \sum_{n=0,2,..}^{\infty} \sum_{j=1}^{2} \Big\{ A_{nj}\{[\nu(\pi_{nj}^2 - \omega_{nj}^2) - n^2] \cos \omega_{nj}\tilde{y}$$

$$- 2\nu\pi_{nj}\omega_{nj} \sin \omega_{nj}\tilde{y}\} + B_{nj}\{[n^2 - \nu(\pi_{nj}^2 - \omega_{nj}^2)] \sin \omega_{nj}\tilde{y}$$

$$- 2\nu\pi_{nj}\omega_{nj} \cos \omega_{nj}\tilde{y}\} \Big] e^{\pi_{nj}\tilde{y}} \cos n\phi$$

$$(m_{y\phi})_D = \frac{-(1 - \nu)D}{\rho_o} \sum_{n=0,2..}^{\infty} \sum_{j=1}^{2} [A_{nj}(\omega_{nj} \sin \omega_{nj}\tilde{y} - \pi_{nj} \cos \omega_{nj}\tilde{y})$$

$$+ B_{nj}(\pi_{nj} \sin \omega_{nj}\tilde{y} + \omega_{nj} \cos \omega_{nj}\tilde{y})]ne^{\pi_{nj}\tilde{y}} \sin n\phi$$

$$(m_y)_D = \frac{-D}{\rho_o} \sum_{n=0,2..}^{\infty} \sum_{j=1}^{2} \{A_{nj}[(\pi_{nj}^2 - \omega_{nj} - \nu n^2) \cos \omega_{nj}\tilde{y}$$

$$- 2\pi_{nj}\omega_{nj} \sin \omega_{nj}\tilde{y}] + B_{nj}[(\nu n^2 + \omega_{nj}^2 - \pi_{nj}^2) \sin \omega_{nj}\tilde{y}$$

$$- 2\pi_{nj}\omega_{nj} \cos \omega_{nj}\tilde{y}]\}e^{\pi_{nj}\tilde{y}} \cos n\phi$$

$$(w)_D = \rho_o \sum_{n-0,2..}^{\infty} \sum_{j=1}^{2} (A_{nj} \cos \omega_{nj}\tilde{y} - B_{nj} \sin\omega_{nj}\tilde{y})e^{\pi_{nj}\tilde{y}} \cos n\phi$$

$$(u)_D = \frac{-1}{Etk\rho_o} \sum_{n=0,2..}^{\infty} \sum_{j=1}^{2} \{A_{nj}[\omega_{nj}(\nu - \frac{n^2}{\pi_{nj}^2 + \omega_{nj}^2}) \cos \omega_{nj}\tilde{y}$$

$$+ \pi_{nj}(\nu + \frac{n^2}{\pi_{nj}^2 + \omega_{nj}^2}) \sin \omega_{nj}\tilde{y}] + B_{nj}[\pi_{nj}(\nu + \frac{n^2}{\pi_{nj}^2 + \omega_{nj}^2})$$

$$\cos \omega_{nj}\tilde{y} - \omega_{nj}(\nu - \frac{n^2}{\pi_{nj}^2 + \omega_{nj}^2}) \sin \omega_{nj}\tilde{y}]\} e^{\pi_{nj}\tilde{y}} \cos n\phi$$

$$(v)_D = \frac{1}{Etk\rho_o} \sum_{n=0,2..}^{\infty} \sum_{j=1}^{2} \{A_{nj}[(2\pi_{nj}\omega_{nj} + Etk\rho_o^2) \cos \omega_{nj}\tilde{y}$$

$$+ (\pi_{nj}^2 - \omega_{nj}^2 + \nu n^2) \sin \omega_{nj}\tilde{y}] + B_{nj}[(\pi_{nj}^2 - \omega_{nj}^2 + \nu n^2)$$

$$\cos \omega_{nj}\tilde{y} - (2\pi_{nj}\omega_{nj} + Etk\rho_o^2) \sin \omega_{nj}\tilde{y}]\} e^{\pi_{nj}\tilde{y}} \frac{\sin n\phi}{n}$$

$$(\frac{\partial w}{\partial \tilde{y}})_D = \rho_o \sum_{n=0,2..}^{\infty} \sum_{j=1}^{2} [A_{nj}(\pi_{nj} \cos \omega_{nj}\tilde{y} - \omega_{nj} \sin \omega_{nj}\tilde{y})$$

$$-B_{nj}(\pi_{nj} \sin \omega_{nj}\tilde{y} + \omega_{nj} \cos \omega_{nj}\tilde{y})]e^{\pi_{nj}\tilde{y}} \cos n\phi$$

$$(\frac{\partial w}{\partial \phi})_D = \rho_o \sum_{n=0,2..}^{\infty} \sum_{j=1}^{2} n(-A_{nj} \cos \omega_{nj}\tilde{y} + B_{nj} \sin \omega_{nj}\tilde{y})e^{\pi_{nj}\tilde{y}}$$

$$\sin n\phi$$

$$(\frac{\partial v}{\partial \phi})_D = \frac{1}{Etk\rho_o} \sum_{n=0,2..}^{\infty} \sum_{j=1}^{2} \{A_{nj}[(2\pi_{nj}\omega_{nj} + Etk\rho_o^2) \cos \omega_{nj}\tilde{y}$$

$$+ (\pi_{nj}^2 - \omega_{nj}^2 + \nu n^2) \sin \omega_{nj}\tilde{y}] + B_{nj}[(\pi_{nj}^2 - \omega_{nj}^2 + \nu n^2)$$

$$\cos \omega_{nj}\tilde{y} - (2\pi_{nj}\omega_{nj} + Etk\rho_o^2) \sin \omega_{nj}\tilde{y}]\} e^{\pi_{nj}\tilde{y}} \cos n\phi$$

$$(\frac{\partial u}{\partial \phi})_D = \frac{1}{Etk\rho_o} \sum_{n=0,2..}^{\infty} \sum_{j=1}^{2} \{A_{nj}(\omega_{nj}(\nu - \frac{n^2}{\pi_{nj}^2 + \omega_{nj}^2}) \cos \omega_{nj}\tilde{y}$$

$$+ \pi_{nj}(\nu + \frac{n^2}{\pi_{nj}^2 + \omega_{nj}^2}) \sin \omega_{nj}\tilde{y}] + B_{nj}[\pi_{nj}(\nu + \frac{n^2}{\pi_{nj}^2 + \omega_{nj}^2})$$

$$\cos \omega_{nj}\tilde{y} - \omega_{nj}(\nu - \frac{n^2}{\pi_{nj}^2 + \omega_{nj}^2}) \sin \omega_{nj}\tilde{y}]]ne^{\pi_{nj}\tilde{y}} \sin n\phi$$

$$\left(\frac{\partial^2 w}{\partial \phi^2}\right)_D = -n^2 (w)_D$$

$$\left(\frac{\partial^2 u}{\partial \phi^2}\right)_D = -n^2 (u)_D$$

$$(q_y)_D = \frac{-D}{\rho_o^2} \sum_{n=0,2..}^{\infty} \sum_{j=1}^{2} \{A_{nj}[\cos \omega_{nj}\tilde{y}(\pi_{nj}^3 - 3\pi_{nj}\omega_{nj}^2 - \pi_{nj}n^2)$$

$$+ \sin \omega_{nj}\tilde{y}(\omega_{nj}^3 - 3\omega_{nj}\pi_{nj}^2 + \omega_{nj}n^2)] + B_{nj}[\cos \omega_{nj}\tilde{y}(\omega_{nj}^3 -$$

$$-3\omega_{nj}\pi_{nj}^2 + \omega_{nj}n^2) + \sin \omega_{nj}\tilde{y}(-\pi_{nj}^3 + 3\pi_{nj}\omega_{nj}^2 + \pi_{nj}n^2)]\}$$

$$e^{\pi_{nj}\tilde{y}} \cos n\phi$$

$$(q_\phi)_D = \frac{D}{\rho_o} \sum_{n=0,2..}^{\infty} \sum_{j=1}^{2} \{A_{nj}[(\pi_{nj}^2 - \omega_{nj}^2 - n^2) \cos \omega_{nj}\tilde{y}$$

$$-2\omega_{nj}\pi_{nj} \sin \omega_{nj}\tilde{y}] + B_{nj}[-2\pi_{nj}\omega_{nj} \cos \omega_{nj}\tilde{y}$$

$$+ (\omega_{nj}^2 - \pi_{nj}^2 + n^2) \sin \omega_{nj}\tilde{y}]\} ne^{\pi_{nj}\tilde{y}} \sin n\phi$$

$$\left(\frac{\partial^2 w}{\partial \tilde{y}^2}\right)_D = \rho_o \sum_{n=0,2..}^{\infty} \sum_{j=1}^{2} \{A_{nj}[\pi_{nj} \cos \omega_{nj}\tilde{y} - \omega_{nj} \sin \omega_{nj}\tilde{y})$$

$$- \omega_{nj}(\pi_{nj} \sin \omega_{nj}\tilde{y} + \omega_{nj} \cos \omega_{nj}\tilde{y})] - B_{nj}[\pi_{nj}(\pi_{nj} \sin \omega_{nj}\tilde{y}$$

$$+ \omega_{nj} \cos \omega_{nj}\tilde{y}) + \omega_{nj}(\pi_{nj} \cos \omega_{nj}\tilde{y} - \omega_{nj} \sin \omega_{nj}\tilde{y})]\} e^{\pi_{nj}\tilde{y}} \cos n\phi$$

$$\left(\frac{\partial^2 u}{\partial \tilde{y}^2}\right)_D = \frac{-1}{Etk\rho_o} \sum_{n=0,2..}^{\infty} \sum_{j=1}^{2} \Bigg\{ A_{nj}\{(\pi_{nj}^2 - \omega_{nj}^2)[\omega_{nj}(\nu - \frac{n^2}{\pi_{nj}^2 + \omega_{nj}^2})$$

$$\cos \omega_{nj}\tilde{y} + \pi_{nj}(\nu + \frac{n^2}{\pi_{nj}^2 + \omega_{nj}^2}) \sin \omega_{nj}\tilde{y}] + 2\omega_{nj}[\pi_{nj}(\nu +$$

$$\frac{n^2}{\pi_{nj}^2 + \omega_{nj}^2})\cos \omega_{nj}\tilde{y} - \omega_{nj}(\nu - \frac{n^2}{\pi_{nj}^2 + \omega_{nj}^2}) \sin \omega_{nj}\tilde{y}]\}$$

$$+ B_{nj}\{(\pi_{nj}^2 - \omega_{nj}^2)[\pi_{nj}(\nu + \frac{n^2}{\pi_{nj}^2 + \omega_{nj}^2}) \cos \omega_{nj}\tilde{y}$$

$$- \omega_{nj}(\nu - \frac{n^2}{\pi_{nj}^2 + \omega_{nj}^2}) \sin \omega_{nj}\tilde{y}] - 2\omega_{nj}[\omega_{nj}(\nu - \frac{n^2}{\pi_{nj}^2 + \omega_{nj}^2})$$

$$\cos \omega_{nj}\tilde{y} + \pi_{nj}(\nu + \frac{n^2}{\pi_{nj}^2 + \omega_{nj}^2}) \sin \omega_{nj}\tilde{y}]\}\Big] e^{\pi_{nj}\tilde{y}} \cos n\phi$$

$$(\frac{\partial^2 w}{\partial \tilde{y} \partial \phi})_D = -n(\frac{\partial w}{\partial \tilde{y}})_D \tan n\phi$$

$$(\frac{\partial u}{\partial \tilde{y}})_D = \frac{1}{Etk\rho_o}[(n_y)_D - \nu(n_\phi)_D]$$

$$(\frac{\partial v}{\partial \tilde{y}})_D = \frac{1}{Etk\rho_o} \sum_{n=0,2..}^{\infty} \sum_{j=1}^{2} \Big\{A_{nj}\{[\pi_{nj}(2\pi_{nj}\omega_{nj} + Etk\rho_o^2)$$

$$+ \omega_{nj}(\pi_{nj}^2 - \omega_{nj}^2 + \nu n^2)] \cos \omega_{nj}\tilde{y} + [\pi_{nj}(\pi_{nj}^2 - \omega_{nj}^2 + \nu n^2)$$

$$- \omega_{nj}(2\pi_{nj}\omega_{nj} + Etk\rho_o^2)] \sin \omega_{nj}\tilde{y}\} + B_{nj}\{[\pi_{nj}(\pi_{nj}^2 - \omega_{nj}^2 + \nu n^2)$$

$$- \omega_{nj}(2\pi_{nj}\omega_{nj} + Etk\rho_o^2)] \cos \omega_{nj}\tilde{y} - [\pi_{nj}(2\pi_{nj}\omega_{nj} + Etk\rho_o^2)$$

$$+ \omega_{nj}(\pi_{nj}^2 - \omega_{nj}^2 + \nu n^2) \sin \omega_{nj}\tilde{y}]\}\Big] e^{\pi_{nj}\tilde{y}} \frac{\sin n\phi}{n}$$

$$(\frac{\partial^2 v}{\partial \tilde{y}^2})_D = \frac{1}{Etk\rho_o} \sum_{n=0,2..}^{\infty} \sum_{j=1}^{2} \Big\{A_{nj}\{[(\pi_{nj}^2 - \omega_{nj}^2)(2\pi_{nj}\omega_{nj} + Etk\rho_o^2)$$

$$+ 2\pi_{nj}\omega_{nj}(\pi_{nj}^2 - \omega_{nj}^2 + \nu n^2)] \cos \omega_{nj}\tilde{y} + [(\pi_{nj}^2 - \omega_{nj}^2)$$

$$(\pi_{nj}^2 - \omega_{nj}^2 + \nu n^2) - 2\omega_{nj}\pi_{nj}(2\pi_{nj}\omega_{nj} + Etk\rho_o^2)] \sin \omega_{nj}\tilde{y}\}$$

$$+ B_{nj}\{[(\pi_{nj}^2 - \omega_{nj}^2)(\pi_{nj}^2 - \omega_{nj}^2 + \nu n^2) - 2\pi_{nj}\omega_{nj}(2\pi_{nj}\omega_{nj}$$

$$+ Etk\rho_o^2)] \cos \omega_{nj}\tilde{y} + [(-\pi_{nj}^2 + \omega_{nj}^2)(2\pi_{nj}\omega_{nj} + Etk\rho_o^2)$$

$$- 2\pi_{nj}\omega_{nj}(\pi_{nj}^2 - \omega_{nj}^2 + \nu n^2)] \sin \omega_{nj}\tilde{y}\}\Big] e^{\pi_{nj}\tilde{y}} \frac{\sin n\phi}{n}$$

$$\left(\frac{\partial^2 v}{\partial \tilde{y} \partial \phi}\right)_D = n\left(\frac{\partial v}{\partial \tilde{y}}\right)_D \cot n\phi$$

$$(u)_A = \frac{D}{Et\rho_o} \sum_{n=0,2..}^{\infty} \sum_{j=1}^{2} \{A_{nj}[\pi_{nj} \cos \omega_{nj}\tilde{y} - \omega_{nj} \sin \omega_{nj}\tilde{y}$$

$$+ \nu(1 - n^2)\frac{\pi_{nj} \cos \omega_{nj}\tilde{y} + \omega_{nj} \sin \omega_{nj}\tilde{y}}{\pi_{nj}^2 + \omega_{nj}^2}]$$

$$+ B_{nj}[-\pi_{nj} \sin \omega_{nj}\tilde{y} - \omega_{nj} \cos \omega_{nj}\tilde{y}$$

$$+\nu(1 - n^2)\frac{-\pi_{nj} \sin \omega_{nj}\tilde{y} + \omega_{nj} \cos \omega_{nj}\tilde{y}}{\pi_{nj}^2 + \omega_{nj}^2}]\}e^{\pi_{nj}\tilde{y}} \cos n\phi$$

$$(v)_A = \frac{D}{Et\ \rho_o} \sum_{n=0,2..}^{\infty} \sum_{j=1}^{2} \Big\{A_{nj}\{(1 - n^2) \cos \omega_{nj}\tilde{y} + \nu[(\pi_{nj}^2 - \omega_{nj}^2)$$

$$\cos \pi_{nj}\tilde{y} - 2\pi_{nj}\omega_{nj} \sin \omega_{nj}\tilde{y}]\} + B_{nj}\{(n^2 - 1) \sin \omega_{nj}\tilde{y}$$

$$+ \nu[(\omega_{nj}^2 - \pi_{nj}^2) \sin \omega_{nj}\tilde{y} - 2\pi_{nj}\omega_{nj} \cos \omega_{nj}\tilde{y}]\Big\} e^{\pi_{nj}\tilde{y}} \frac{\sin n\phi}{n}$$

$$\left(\frac{\partial v}{\partial \phi}\right)_A = n(v_A) \cot n\phi$$

$$\left(\frac{\partial u}{\partial \phi}\right)_A = -n(u)_A \tan n\phi$$

$$\left(\frac{\partial^2 u}{\partial \phi^2}\right)_A = -n^2(u)_A$$

$$\left(\frac{\partial u}{\partial \tilde{y}}\right)_A = \frac{D}{Et\ \rho_o} \sum_{n=0,2..}^{\infty} \sum_{j=1}^{2} \{A_{nj}[(\pi_{nj}^2 - \omega_{nj}^2) \cos \omega_{nj}\tilde{y}$$

$$- 2\pi_{nj}\omega_{nj} \sin \omega_{nj}\tilde{y} + \nu(1 - n^2) \cos \omega_{nj}\tilde{y}]$$

$$+ B_{nj}[(\omega_{nj}^2 - \pi_{nj}^2) \sin \omega_{nj}\tilde{y} - 2\pi_{nj}\omega_{nj} \cos \omega_{nj}\tilde{y}$$

$$- \nu(1 - n^2) \sin \omega_{nj}\tilde{y}]\} e^{\pi_{nj}\tilde{y}} \cos n\phi$$

$$\left(\frac{\partial^2 u}{\partial \tilde{y}^2}\right)_A = \frac{D}{Et\rho_o} \sum_{n=0,2..}^{\infty} \sum_{j=1}^{2} \{A_{nj}[(\pi_{nj}^3 - 3\omega_{nj}^2 \pi_{nj}) \cos \omega_{nj}\tilde{y}$$

$$(\omega_{nj}^3 - 3\pi_{nj}^2\omega_{nj}) \sin \omega_{nj}\tilde{y} + \nu(1 - n^2)(\pi_{nj} \cos \omega_{nj}\tilde{y}$$

$$- \omega_{nj} \sin \omega_{nj}\tilde{y})] + B_{nj}[(\omega_{nj}^3 - 3\pi_{nj}^2\omega_{nj}) \cos \omega_{nj}\tilde{y}$$

$$+ (3\pi_{nj}\omega_{nj}^2 - \pi_{nj}^3) \sin \omega_{nj}\tilde{y} - \nu(1 - n^2)(\omega_{nj} \cos \omega_{nj}\tilde{y}$$

$$+ \pi_{nj} \sin \omega_{nj}\tilde{y})]\} e^{\pi_{nj}\tilde{y}} \cos n\phi$$

$$\left(\frac{\partial v}{\partial \tilde{y}}\right)_A = \frac{D}{Et\rho_o} \sum_{n=0,2..}^{\infty} \sum_{j=1}^{2} \Big\{A_{nj}\{[1 - n^2 + \nu(\pi_{nj}^2 - \omega_{nj}^2)]$$

$$[\pi_{nj} \cos \omega_{nj}\tilde{y} - \omega_{nj} \sin \omega_{nj}\tilde{y}] - \nu(2\omega_{nj}\pi_{nj})$$

$$[\pi_{nj} \sin \omega_{nj}\tilde{y} + \omega_{nj} \cos \omega_{nj}\tilde{y}]\} + B_{nj}\{-2\nu\omega_{nj}\pi_{nj}$$

$$[\pi_{nj} \cos \omega_{nj}\tilde{y} - \omega_{nj} \sin \omega_{nj}\tilde{y}] - [1 - n^2 + \nu$$

$$(\pi_{nj}^2 - \omega_{nj}^2)][\pi_{nj} \sin \omega_{nj}\tilde{y} + \omega_{nj} \cos \omega_{nj}\tilde{y}]\Big\} e^{\pi_{nj}\tilde{y}} \frac{\sin n\phi}{n}$$

$$\left(\frac{\partial^2 v}{\partial \tilde{y}^2}\right)_A = \frac{D}{Et\rho_o} \sum_{n=0,2..}^{\infty} \sum_{j=1}^{2} \Big\{A_{nj}\{[1 - n^2 + \nu(\pi_{nj}^2 - \omega_{nj}^2)][(\pi_{nj}^2 - \omega_{nj}^2)$$

$$\cos \omega_{nj}\tilde{y} - 2\pi_{nj}\omega_{nj} \sin \omega_{nj}\tilde{y}] + 2\nu\pi_{nj}\omega_{nj}[(\omega_{nj}^2 - \pi_{nj}^2)$$

$$\sin \omega_{nj}\tilde{y} - 2\pi_{nj}\omega_{nj} \cos \omega_{nj}\tilde{y}]\} + B_{nj}\{-2\nu\omega_{nj}\pi_{nj}$$

$$[(\pi_{nj}^2 - \omega_{nj}^2) \cos \omega_{nj}\tilde{y} - 2\pi_{nj}\omega_{nj} \sin \omega_{nj}\tilde{y}] + [1 - n^2$$

$$+ \nu(\pi_{nj}^2 - \omega_{nj}^2)][(\omega_{nj}^2 - \pi_{nj}^2) \sin \omega_{nj}\tilde{y} - 2\pi_{nj}\omega_{nj}$$

$$\cos \omega_{nj}\tilde{y})]\Big\} e^{\pi_{nj}\tilde{y}} \frac{\sin n\phi}{n}$$

$$\left(\frac{\partial^2 v}{\partial \tilde{y} \partial \phi}\right)_A = n\left(\frac{\partial v}{\partial \tilde{y}}\right)_A \cot n\phi$$

APPENDIX G

COEFFICIENTS

The final coefficients are calculated by truncation of each series at five terms. A total of 2000 points evenly distributed between $\phi = 0$ and $\phi = \pi/2$ are used in the least squares collocation technique. The coefficients are:

$A_{01} = 0.66655 \times 10$

$A_{21} = -0.29575 \times 10^{2}$

$A_{41} = -0.10350 \times 10^{2}$

$A_{61} = -0.37537$

$A_{81} = -0.67718 \times 10^{-2}$

$A_{02} = 0.15010 \times 10$

$A_{22} = -0.13406 \times 10^{2}$

$A_{42} = 0.11313 \times 10^{2}$

$A_{62} = 0.34115$

$A_{82} = 0.61379 \times 10^{-2}$

$B_{01} = -0.37597$

$B_{21} = 0.15909 \times 10^{3}$

$B_{41} = 0.18898 \times 10^{2}$

$B_{61} = -0.11481 \times 10$

$B_{81} = 0.15023 \times 10^{-1}$

$B_{02} = 0.0$

$B_{22} = -0.71516 \times 10^{3}$

$B_{42} = -0.70792 \times 10^{2}$

$B_{62} = 0.77632$

$B_{82} = -0.47214 \times 10^{-1}$

$C_1 = 0.20231 \times 10^{3}$

$C_3 = 0.10046 \times 10$

$C_5 = 0.74767 \times 10^{-4}$

$C_7 = -0.76719 \times 10^{-8}$

$C_9 = 0.55726 \times 10^{-13}$

$D_1 = -0.56960 \times 10^{3}$

$D_3 = 0.49391$

$D_5 = 0.98227 \times 10^{-4}$

$D_7 = 0.20601 \times 10^{-8}$

$D_9 = 0.65891 \times 10^{-12}$

$C_2 = -0.55473 \times 10^{2}$

$C_4 = 0.25992 \times 10^{-2}$

$C_6 = -0.59010 \times 10^{-6}$

$C_8 = 0.15293 \times 10^{-9}$

$C_{10} = 0.13290 \times 10^{-14}$

$D_2 = 0.11039 \times 10^{2}$

$D_4 = -0.36962 \times 10^{-1}$

$D_6 = 0.54447 \times 10^{-6}$

$D_8 = -0.11767 \times 10^{-10}$

$D_{10} = -0.10118 \times 10^{-13}$

$R_1 = 0.41269 \times 10^{3}$

$R_2 = 0.0$

Note that all of these coefficients contain a factor of E/P, which was combined with the A's, B's, C's, and D's in order to allow a solution for stresses which is independent of material constants. As explained previously, the displacements must then be multiplied by P/E, since they are the only quantities that do not contain a factor of E/P when made dimensionless.

VITA

Michael Terry Doyle was born in Waverly, Tennessee on September 1, 1949. He attended elementary and high schools in McEwen, Tennessee, and graduated from McEwen High School in 1967. The following September he entered the Georgia Institute of Technology, and in June, 1971, he received a Bachelor of Aerospace Engineering degree, with honor. At that time he was commissioned an officer in the United States Army, and received an active duty delay in order to pursue a graduate degree.

He accepted a teaching assistantship at the University of Tennessee in September, 1971, and began study toward a Master's degree. He received a Master of Science degree with a major in Mechanical Engineering in December, 1973. In November, 1973, he began employment with the E. I. DuPont deNemours Company. He is a member of Phi Kappa Phi, Sigma Gamma Tau, and Tau Beta Pi.

He is married to the former Barbara Ellen Rister of Newberry, South Carolina.