

To the Graduate Council:

I am submitting herewith a dissertation written by Grady Dewayne Bullington entitled "On the expected number of generators of a submodule of a free module over a finite principal ideal ring." I have examined the final copy of this dissertation for form and content and recommend that it be accepted in partial fulfillment of the requirements for the degree Doctor of Philosophy, with a major in Mathematics.

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ON THE EXPECTED NUMBER OF GENERATORS OF A
SUBMODULE OF A FREE MODULE OVER A FINITE PRINCIPAL
IDEAL RING

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ABSTRACT

This work concerns the expected minimal cardinality of a generating set of an arbitrary submodule of a finitely generated free module over a finite (commutative) principal ideal ring (PIR). Our primary objective is to obtain ring- and module-theoretic generalizations of the work that had been done earlier in the finite field-vector space situation by Dobbs and Lancaster.

In Chapter I, we provide background material, conventions, and some of the definitions needed for the work in Chapters II-IV.

Chapter II addresses finding the expected minimal cardinality of a generating set of an arbitrary submodule of a finitely generated free module over a finite special principal ideal ring (SPIR). We generalize the various “weighted” limiting expected values found by Dobbs in the field-vector space situation to an SPIR-module setting which Dobbs had explored in a special case. As a byproduct, we find a formula for the number of submodules of a given finitely generated free module (over a finite SPIR) which require a prescribed number of generators.

In Chapter III, using a basic structure theorem of PIRs and results from Chapter II, we obtain limiting expected values for the PIR-module situation. Here we investigate the various “weighting” contexts for limiting expected values which were introduced by Dobbs and we also address some other “weighting” contexts that do not arise naturally for vector spaces.

In contrast to the PIR settings in Chapters II and III, Chapter IV investigates some ring-module situations which have limiting expected values that behave in qualitatively different ways from the answers found in the (S)PIR settings.

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CHAPTER I

INTRODUCTION

In this chapter, we provide some background material for the work in this dissertation. All rings are commutative with 1. Cardinality is denoted with absolute value notation; i.e., if S is a set, then $|S|$ denotes the cardinal number of S . Empty sums are 0, and empty products are 1. If E is an R -module, then $\nu_R(E)$ denotes the minimal cardinality of an R -generating set of E . Thus, by Nakayama's Lemma, if (R, M) is a local ring and E is a finitely generated R -module, then $\nu_R(E) = \dim_{R/M}(E/ME)$.

Most of Chapter 2 concerns special principal ideal rings (SPIRs). Following [7], a ring R is called a PIR (principal ideal ring) if each ideal of R is principal; and a PIR is called an SPIR if it has only one prime ideal M and M is nilpotent, that is, $M^n = 0$ for some $n > 0$. By the *index of nilpotency* of M , we mean the least positive integer e such that $M^e = 0$. Observe that fields are precisely the SPIR's for which $e = 1$. Two important facts about SPIR's are recalled next.

THEOREM 1.1 [7, p. 245] *Let (R, M) be an SPIR with e the index of nilpotency of M . The complete list of all ideals of R , with no repetitions, is $\{R = M^0, M = M^1, \dots, M^{e-1}, M^e = 0\}$.*

THEOREM 1.2 *Let (R, M) be an SPIR with e the index of nilpotency of M . Let π be a generator of the ideal M , i.e., $M = R\pi$. Then for all $0 \leq i \leq e$, $(0 :_R \pi^i) = M^{e-i} = R\pi^{e-i}$.*

Proof. As noted in [7, p. 245], each nonzero element $r \in R$ can be written uniquely as $r = u\pi^k$, where u is a unit of R and k is a non-negative integer. Then $r \in (0 :_R \pi^i) \Leftrightarrow u\pi^{k+i} = 0 \Leftrightarrow$ (by Theorem 1.1) $k + i \geq e \Leftrightarrow k \geq e - i \Leftrightarrow r \in M^{e-i}$. $\square$

There is a useful generalization of Theorem 1.2.

THEOREM 1.3 *Let V be a finitely generated free R -module (where, as above, (R, M) is an SPIR with e the index of nilpotency of M). Then, for $0 \leq i \leq e$, $(0 :_R M^i V) = M^{e-i}$ and $(0 :_V M^i) = M^{e-i} V$.*

Proof. Argue via coordinates and apply Theorem 1.2. $\square$

Chapter 3 is concerned with principal ideal rings (PIRs). A useful fact about PIRs is the following result.

THEOREM 1.4 [7, Lemma 1, p. 246] *Let R be a PIR. If W is a submodule of a finitely generated R -module V , then $\nu_R(W) \leq \nu_R(V)$.*

Of course, any SPIR is a PIR. Moreover, we have the following basic structure theorem for PIRs.

THEOREM 1.5 [7, Theorem 33, p. 245] *A ring R is a PIR if and only if R can be expressed as an internal direct product $\prod_{i=1}^n R_i$, where each R_i is either a principal ideal domain or an SPIR.*

Our concern in this dissertation is to generalize various results of D. E. Dobbs and M. J. Lancaster on the expected dimension (resp., minimal cardinality of a generating set) of a subspace (resp., submodule) of a given finite- n -dimensional (resp., finitely generated) vector space (resp., module) over a finite field (resp., finite SPIR with $e = n = 2$). The precise probabilistic formulations and contexts will be explained in Chapter 2. For the moment, we recall only the following motivating results and underlying formulas.

In Theorems 1.6-1.10, let F be a finite field of cardinality q , and let W be an n -dimensional F -vector space.

THEOREM 1.6 (Dobbs-Lancaster [4, Proposition 3.1 (b)]) *For each integer $i, 0 \leq i \leq n$, the cardinality of the set of i -dimensional F -subspaces of W is given by*

$$\nu_i := \frac{\prod_{\gamma=1}^i (q^n - q^{\gamma-1})}{\prod_{\gamma=1}^i (q^i - q^{\gamma-1})}.$$

THEOREM 1.7 (Dobbs-Lancaster [4, Corollary 3.3]) *If $0 \leq i \leq n$, then the formula in Theorem 1.6 gives a monic polynomial $f \in Z[X]$ of degree $(n-i)i$ such that $\nu_i = f(q)$.*

THEOREM 1.8 (Dobbs-Lancaster [4, Theorem 3.4]) *The expected dimension of an F -subspace of W is $\frac{n}{2}$.*

THEOREM 1.9 (Dobbs [1, Corollary 2.6]) *Let U vary over the F -subspaces of W , with weighting by dimension. If n is even, then $\lim_{q \rightarrow \infty} E(\dim_F(U)) = \frac{n}{2}$. If n is odd, then $\lim_{q \rightarrow \infty} E(\dim_F(U)) = \frac{n}{2} + \frac{1}{2n}$.*

THEOREM 1.10 (Dobbs [1, Corollary 2.5]) *Let U vary over the F -subspaces of W , with weighting by cardinality. Then, $\lim_{q \rightarrow \infty} E(\dim_F(U)) = \frac{n+1}{2}$.*

For the reader acquainted with [2], we offer the following orientation to the present work. In generalizing the methodology of [2] in terms of a linear transformation Δ , one works in Chapter 2 over a base SPIR (R, M) (rather than a base field). This is done with the aide of e suitable R -module homomorphisms and suitable new equivalence relations which are designed to identify minimal generating sets of submodules and thus determine the number of submodules whose minimal generating sets each have prescribed cardinality. One then proceeds to study limiting expected values so as to generalize Theorems 1.8-1.10. In Chapter 3, one studies analogous limits over PIRs by using the results of Chapter 2 in conjunction with Theorem 1.5 and module structure theory. Finally, in Chapter 4, qualitatively different behavior

of limiting expected values is obtained when the ambient module (resp., ring) is not free (resp., not a PIR), by analyzing cyclic modules and a generalization of the ring of dual numbers.

CHAPTER II

ON THE EXPECTED NUMBER OF GENERATORS OF A SUBMODULE OF A FREE MODULE OVER A FINITE SPECIAL PRINCIPAL IDEAL RING

We begin by recalling the following useful definition.

DEFINITION 2.1 Let X be a vector space over a field R . A finite nonempty subset $L = \{x_1, x_2, \dots, x_k\}$ of (k pairwise distinct) elements of X is said to be *linearly independent (over R)* if it has the following property:

If $r_1, r_2, \dots, r_k \in R \ni \sum_{j=1}^k r_j x_j = 0$, then $r_1 = r_2 = \dots = r_k = 0$.

Otherwise, L is said to be *linearly dependent (over R)*.

PROPOSITION 2.2 Let A be a module over a ring R which has a maximal ideal M . Let $S = \{a_1, a_2, \dots, a_k\}$, $1 \leq k < \infty$, be a subset of A . The following conditions are equivalent (notice that each of these conditions implies that $|S| = k$):

- (i) $L = \{a_1 + MA, a_2 + MA, \dots, a_k + MA\}$ is an R/M -linearly independent subset of A/MA with k pairwise distinct elements;
- (ii) $a_i \in A \setminus ((\sum_{j=1}^{i-1} Ra_j) + MA)$ for each $i \in \{1, \dots, k\}$.

Proof. Since each of (i) and (ii) implies that $|S| = k$, we may assume this condition henceforth.

$\sim (ii) \Rightarrow \sim (i)$: Suppose that there exists $i \in \{1, \dots, k\}$ such that $a_i \in (\sum_{j=1}^{i-1} Ra_j) + MA$. Without loss of generality, also assume that $|L| = k$. Thus, there exist $r_1, r_2, \dots, r_{i-1} \in R$ and $g \in MA$ such that $a_i = (\sum_{j=1}^{i-1} r_j a_j) + g$, giving that $(\sum_{j=1}^{i-1} (r_j + M)(a_j + MA)) + (-1 + M)(a_i + MA) = 0 + MA \in A/MA$. Therefore, L is R/M -linearly dependent.

$\sim (i) \Rightarrow \sim (ii)$: Suppose that (i) does not hold. There are two cases.

Case 1: $1 \leq |L| < k$. In other words, $a_i + MA = a_\tau + MA$ for some i and τ such that $1 \leq \tau < i \leq k$. In this case, (ii) does not hold since $a_i \in a_\tau + MA \subseteq (\sum_{j=1}^{i-1} Ra_j) + MA$.

Case 2: L is an R/M -linearly dependent subset of A/MA with k pairwise distinct elements. Thus, $\sum_{j=1}^k (r_j + M)(a_j + MA) = 0 \in A/MA$ for some $r_1 + M, r_2 + M, \dots, r_k + M$ so that not each $r_j + M = 0$ in R/M . Pick i maximal such that $r_i + M \neq 0$ in R/M . Since R/M is a field, it follows that

$$a_i + MA = -(\sum_{j=1}^{i-1} (r_j + M)^{-1} (r_j + M)(a_j + MA)).$$

Thus, $a_i \in (\sum_{j=1}^{i-1} Ra_j) + MA$. $\square$

We pause to establish **running hypotheses and notation for the rest of this chapter**. Let R be a special principal ideal ring (SPIR) with maximal ideal $M = R\pi$ and e the index of nilpotency of M . (Since the case in which R is a field has been discussed in [4] and [1], we often implicitly assume that $\pi \neq 0$ and $e \geq 2$, that is, R is not a field.) Let V be a nonzero finitely-generated R -module which is R -free on a basis of n elements ($1 \leq n < \infty$).

LEMMA 2.3 $\nu_R(M^i V) = n$ for each $i \in \{0, \dots, e-1\}$.

Proof. Observe via Nakayama's Lemma that $\nu_R(M^i V) = \dim_{R/M}(M^i V/M^{i+1} V)$. Thus, it suffices to show that $M^i V/M^{i+1} V \cong \oplus_{j=1}^n R/M$ as R/M -modules. As $V = Rf_1 \oplus Rf_2 \oplus \dots \oplus Rf_n \cong R^{(n)}$ as R -modules for some R -basis $\{f_1, \dots, f_n\}$ of V , it follows that

$$M^i V = M^i f_1 \oplus M^i f_2 \oplus \dots \oplus M^i f_n \text{ and } M^{i+1} V = M^{i+1} f_1 \oplus M^{i+1} f_2 \oplus \dots \oplus M^{i+1} f_n.$$

Using standard isomorphism theorems, observe that

$$\frac{M^i V}{M^{i+1} V} = \frac{(M^i f_1) \oplus (M^i f_2) \oplus \dots \oplus (M^i f_n)}{(M^{i+1} f_1) \oplus (M^{i+1} f_2) \oplus \dots \oplus (M^{i+1} f_n)} \cong \left(\frac{M^i f_1}{M^{i+1} f_1} \right) \oplus \dots \oplus \left(\frac{M^i f_n}{M^{i+1} f_n} \right) \cong \bigoplus_{j=1}^n \left(\frac{M^i}{M^{i+1}} \right).$$

Thus, the assertion follows since the above isomorphisms are, in fact, R/M -linear transformations and $\frac{M^i}{M^{i+1}} = (R/M)(\pi^i + M^{i+1})$ is a one-dimensional R/M -vector space. $\square$

DEFINITION 2.4 Let W be an R -submodule of V . For each $i \in \{1, \dots, e\}$, let $\Delta_W^{(i)}$ denote the R -homomorphism from $(W \cap M^{i-1}V)/M^i W$ to $M^{i-1}V/M^i V$ given by $w + M^i W \mapsto w + M^i V$. Notice that the image of $\Delta_W^{(i)}$ is the R/M -subspace $((W \cap M^{i-1}V) + M^i V)/M^i V$ of $M^{i-1}V/M^i V$. Let $p_{W,i}$ denote the R/M -dimension of the image of $\Delta_W^{(i)}$. When there is no ambiguity, we denote this dimension by p_i . Also, by convention, we let $p_{W,i} = p_i = 0$ for any integer $i \notin \{1, \dots, e\}$.

REMARK 2.5 (a) $0 \leq p_1 \leq p_2 \leq \dots \leq p_e \leq \nu_R(W)$ for any R -submodule W of V .

Proof. For each $i \in \{1, \dots, e\}$, observe via Nakayama's Lemma that

$$p_i = \dim_{R/M}(\text{im}(\Delta_W^{(i)})) = \nu_R(\text{im}(\Delta_W^{(i)})).$$

Moreover, $\nu_R(\text{im}(\Delta_W^{(i)})) \leq \nu_R((W \cap M^{i-1}V)/M^i W) \leq \nu_R(W \cap M^{i-1}V) \leq \nu_R(W)$,

the last inequality holding because R is a principal ideal ring [7, Lemma 1, p. 246].

Fix $i \in \{1, \dots, e-1\}$. To show that $p_i \leq p_{i+1}$, it suffices to verify that

$$\Gamma : ((W \cap M^{i-1}V) + M^i V)/M^i V \longrightarrow ((W \cap M^i V) + M^{i+1} V)/M^{i+1} V,$$

given by $z + M^i V \mapsto \pi z + M^{i+1} V$, is a one-to-one R/M -linear transformation. By the universal mapping property of factor modules, Γ is a well-defined R/M -linear transformation. To show that Γ is one-to-one, letting $z \in W \cap M^{i-1}V$ and

$\pi z \in M^{i+1}V$, it suffices to show that $z \in M^iV$. Since $\pi z \in M^{i+1}V$, there exists $g \in M^iV$ such that $\pi(z - g) = 0$. Thus $z - g \in (0 :_V \pi) = M^{e-1}V$. Since $i \leq e - 1$, it follows that $M^{e-1}V \subseteq M^iV$. Thus, $z \in M^iV$. $\square$

(b) $W = 0 \iff p_{W,i} = 0$ for each $i \in \{1, \dots, e\}$.

Proof. $\Rightarrow$: Suppose that $W = 0$. Fix $i \in \{1, \dots, e\}$. Since $\text{im}(\Delta_W^{(i)}) = ((W \cap M^{i-1}V) + M^iV)/M^iV = (0 + M^iV)/M^iV = 0$, it follows that $p_{W,i} = 0$.

$\Leftarrow$: Suppose that $p_{W,i} = 0$ for each $i \in \{1, \dots, e\}$. Equivalently, $\text{im}(\Delta_W^{(i)}) = ((W \cap M^{i-1}V) + M^iV)/M^iV = 0$ for each $i \in \{1, \dots, e\}$. Thus, $W \cap M^{i-1}V \subseteq M^iV$ for each $i \in \{1, \dots, e\}$, giving $W = W \cap M^0V \subseteq W \cap M^1V \subseteq \dots \subseteq W \cap M^{e-1}V \subseteq M^eV = 0$. $\square$

(c) $W = V \iff p_{W,i} = n$ for each $i \in \{1, \dots, e\}$.

Proof. $\Rightarrow$: Fix $i \in \{1, \dots, e\}$. By the definition of $\Delta_V^{(i)}$, we have that $\text{im}(\Delta_V^{(i)}) = \frac{(V \cap M^{i-1}V) + M^iV}{M^iV} = \frac{M^{i-1}V}{M^iV}$. Observe via Nakayama's Lemma that $\dim_{R/M}(\frac{M^{i-1}V}{M^iV}) = \nu_R(M^{i-1}V)$. Therefore, by Lemma 2.3, the assertion follows.

$\Leftarrow$: Using Nakayama's Lemma and Lemma 2.3, observe that $\dim_{R/M}(V/MV) = \nu_R(V) = n$. Since $\dim_{R/M}(\text{im}(\Delta_W^{(1)})) = p_{W,1} = n$ and $\text{im}(\Delta_W^{(1)}) = \frac{W+MV}{MV}$ is an R/M -vector subspace of the n -dimensional space V/MV , it follows that $\frac{W+MV}{MV} = \frac{V}{MV}$, and so $W + MV = V$. Therefore, Nakayama's Lemma gives that $W = V$. $\square$

(d) If $e = 1$ (i.e., R is a field), then $\Delta_W^{(1)}$ can be identified with the inclusion map from W to V .

(e) The paper [2] addresses the special case $e = 2 = n$ for the finite SPIR, R . In that paper, Dobbs partitions the set of submodules of V into six classes or types of submodules and counts the number of submodules of each type. (A main technical tool in [2] is a map denoted by Δ ; in the above notation, Δ is just $\Delta_W^{(1)}$.) In particular, the six identified types of submodules are: $p_1 = 0, \nu_R(W) = 0$;

$p_1 = 0, \nu_R(W) = 1; p_1 = 1, \nu_R(W) = 1; p_1 = 0, \nu_R(W) = 2; p_1 = 1, \nu_R(W) = 2; p_1 = 2, \nu_R(W) = 2.$

For general n and e , we shall see in Corollary 2.16 that $\nu_R(W) = p_{W,e}$. Thus, by part (a), one can say in effect that [2] counted submodules in terms of their values of p_1 and p_2 . More generally, for arbitrary n and e , we shall classify submodules in terms of their values of $p_1, p_2, \dots, p_e$.

The next two arguments identify some behavior of the $p_{W,i}$ notation that facilitates some induction and counting arguments later.

PROPOSITION 2.6 $p_{M^j W, i} = p_{W, i-j}$ for any $i \in \{1, \dots, e\}$ and nonnegative integer j .

Proof. When $j = 0$, equality holds since $M^0 = R$.

Suppose that $i \leq j$. Then $M^j V \subseteq M^i V$. It follows that $\text{im}(\Delta_{M^j W}^{(i)}) = ((M^j W \cap M^{i-1} V) + M^i V) / M^i V = 0$, giving $p_{M^j W, i} = 0 = p_{W, i-j}$.

Suppose that $0 < j < i$. To prove that $p_{M^j W, i} = p_{W, i-j}$, it suffices to show that $\Gamma : \text{im}(\Delta_W^{(i-j)}) = ((W \cap M^{(i-j)-1} V) + M^{i-j} V) / M^{i-j} V \longrightarrow \text{im}(\Delta_{M^j W}^{(i)}) = ((M^j W \cap M^{i-1} V) + M^i V) / M^i V$ given by $z + M^{i-j} V \longmapsto \pi^j z + M^i V$ is an R/M -isomorphism. Notice that Γ is a well-defined R/M -linear transformation by the universal mapping property of factor modules.

To show that Γ is one-to-one, it suffices to show that if $z \in W \cap M^{(i-j)-1} V$ and $\pi^j z \in M^i V$, then $z \in M^{i-j} V$. Since $\pi^j z \in M^i V$, it follows that $\pi^j(z - g) = 0$ for some $g \in M^{i-j} V$. Thus, $z - g \in (0 :_V \pi^j) = M^{e-j} V$. Since $i - j \leq e - j$, it follows that $M^{e-j} V \subseteq M^{i-j} V$, giving that $z = (z - g) + g \in M^{i-j} V$.

To show that Γ is onto, it suffices to show that if $w \in M^j W \cap M^{i-1} V$, then $w \in \pi^j(W \cap M^{(i-j)-1} V)$. Since $w \in M^j W$, there exists $w' \in W$ such that $w = \pi^j w'$. Since $\pi^j w' = w \in M^{i-1} V$, it follows that there exists $h \in M^{(i-j)-1} V$ such that

$\pi^j(w' - h) = 0$. Hence, $w' - h \in (0 :_V \pi^j) = M^{e-j}V$. Since $(i - j) - 1 \leq e - j$, it follows that $M^{e-j}V \subseteq M^{(i-j)-1}V$, yielding that $w' = (w' - h) + h \in W \cap M^{(i-j)-1}V$. Hence, $w = \pi^j w' \in \pi^j(W \cap M^{(i-j)-1}V)$. $\square$

PROPOSITION 2.7 *For any $i \in \{1, \dots, e\}$ and nonnegative integer j ,*

$$p_{W \cap M^j V, i} = \begin{cases} 0 & \text{if } i \leq j \\ p_{W, i} & \text{if } j < i. \end{cases}$$

Proof. Suppose that $i \leq j$. Then $M^j V \subseteq M^i V$. It follows that $\text{im}(\Delta_{W \cap M^j V}^{(i)}) = (((W \cap M^j V) \cap M^{i-1}V) + M^i V) / M^i V = 0$. Hence, $p_{W \cap M^j V, i} = 0$.

Suppose $j < i$ (i.e., $j \leq i - 1$). Then $M^{i-1}V \subseteq M^j V$, giving that $\text{im}(\Delta_{W \cap M^j V}^{(i)}) = (((W \cap M^j V) \cap M^{i-1}V) + M^i V) / M^i V = ((W \cap M^{i-1}V) + M^i V) / M^i V = \text{im}(\Delta_W^{(i)})$. Therefore, $p_{W \cap M^j V, i} = p_{W, i}$. $\square$

The next result will be used in the proof of Proposition 2.14 to construct minimal generating sets.

PROPOSITION 2.8 *Let $\{z_1, z_2, \dots, z_k\}$, $1 \leq k \leq n$, be a subset of $M^{i-1}V$, $1 \leq i \leq e - 1$. If $\alpha \in \{1, \dots, e - i\}$ and $\{z_j + M^i V : j \in \{1, \dots, k\}\}$ is an R/M -linearly independent subset of $M^{i-1}V / M^i V$ with k pairwise distinct elements, then $\{\pi^\alpha z_j + M^{\alpha+i} V : j \in \{1, \dots, k\}\}$ is an R/M -linearly independent subset of $M^{\alpha+i-1}V / M^{\alpha+i} V$ with k pairwise distinct elements.*

Proof. We proceed by contradiction. Fix $\alpha \in \{1, \dots, e - i\}$. Assume that $\{z_j + M^i V : j \in \{1, \dots, k\}\}$ is an R/M -linearly independent subset of $M^{i-1}V / M^i V$ with k pairwise distinct elements and $\{\pi^\alpha z_j + M^{\alpha+i} V : j \in \{1, \dots, k\}\}$ is not

an R/M -linearly independent subset of $M^{\alpha+i-1}V/M^{\alpha+i}V$ with k pairwise distinct elements. In view of Proposition 2.2, there exists $s \in \{1, \dots, k\}$ such that $\pi^\alpha z_s \in (\sum_{j=1}^{s-1} R\pi^\alpha z_j) + M^{\alpha+i}V = \pi^\alpha((\sum_{j=1}^{s-1} Rz_j) + M^iV)$, giving that $\pi^\alpha(z_s - g) = 0$ for some $g \in (\sum_{j=1}^{s-1} Rz_j) + M^iV$. Thus, $z_s - g \in (0 :_V \pi^\alpha)$. As R is an SPIR, $(0 :_R \pi^\alpha) = M^{e-\alpha}$. Hence, since V is R -free, $(0 :_V \pi^\alpha) = M^{e-\alpha}V$. Since $i \leq e - \alpha$, it follows that $M^{e-\alpha}V \subseteq M^iV$, giving that $z_s = (z_s - g) + g \in (\sum_{j=1}^{s-1} Rz_j) + M^iV$. This is a contradiction by Proposition 2.2. $\square$

DEFINITION 2.9 Call $(d_1, d_2, \dots, d_e)$ an *ascending e -tuple* if $d_1, d_2, \dots, d_e \in \{0, \dots, n\}$ and $d_1 \leq d_2 \leq \dots \leq d_e$. By convention, let $d_0 = 0$.

DEFINITION 2.10 Let $(d_1, d_2, \dots, d_e)$ be a nonzero ascending e -tuple.

- (1) Say that an R -submodule W of V *realizes* $(d_1, d_2, \dots, d_e)$ if $p_{W,i} = d_i$ for each $i \in \{1, \dots, e\}$.
- (2) Let $\lambda_0 = 0$, and let $\{\lambda_1, \lambda_2, \dots, \lambda_\beta\}$ be the set $\{i \in \{1, \dots, e\} : d_i - d_{i-1} \neq 0\}$ labeled such that $\lambda_1 < \lambda_2 < \dots < \lambda_\beta$. For each $k \in \{1, \dots, \beta\}$, let ψ_k denote $d_{\lambda_k} - d_{\lambda_{k-1}}$. (Notice $d_{\lambda_\beta} = d_e$ and the telescoping sum $\sum_{i=1}^\beta \psi_i = d_{\lambda_\beta}$.) After labeling the coordinates of a d_e -tuple w so that

$$w = (w_{1,1}, w_{1,2}, \dots, w_{1,\psi_1}, w_{2,1}, w_{2,2}, \dots, w_{2,\psi_2}, \dots, w_{\beta,1}, w_{\beta,2}, \dots, w_{\beta,\psi_\beta}),$$

say that the tuple w *realizes* $(d_1, d_2, \dots, d_e)$ if

$$w_{k,\gamma} \in M^{\lambda_k-1}V \setminus ((\sum_{i=1}^{k-1} \sum_{j=1}^{\psi_i} R\pi^{\lambda_k-\lambda_i} w_{i,j}) + (\sum_{j=1}^{\gamma-1} R w_{k,j}) + M^{\lambda_k}V)$$

for each $k \in \{1, \dots, \beta\}$ and $\gamma \in \{1, \dots, \psi_k\}$. (Notice, by an inductive argument, that $(\sum_{i=1}^{k-1} \sum_{j=1}^{\psi_i} R\pi^{\lambda_k-\lambda_i} w_{i,j}) + (\sum_{j=1}^{\gamma-1} R w_{k,j}) + M^{\lambda_k}V$ is an R -submodule of $M^{\lambda_k-1}V$ for all k, γ as above.)

REMARK 2.11 (a) There are a total of $\binom{n+e}{e}$ ascending e -tuples. To see this, recall (cf. [3, p. 74]) that $mHr = \binom{m+r-1}{r}$ counts the number of ways of selecting r objects from (at least r each of) m types of objects; apply this basic combinatorial fact, with $m = n + 1$ and $r = e$.

Using the above tools, with $m = k + 1$ and $r = e - 1$, we also obtain the following fact. For any $k \in \{0, \dots, n\}$, there are $\binom{k+e-1}{e-1} = \binom{k+e-1}{k}$ ascending e -tuples having $d_e = k$.

(b) The coordinates of any tuple which realizes a given nonzero ascending e -tuple $(d_1, d_2, \dots, d_e)$ are pairwise distinct.

Proof. Let w be a tuple which realizes $(d_1, d_2, \dots, d_e)$. Define $\beta, \lambda_0, \lambda_1, \dots, \lambda_\beta, \psi_1, \dots, \psi_\beta$ as in Definition 2.10 (2), label the coordinates of w so that

$$w = (w_{1,1}, w_{1,2}, \dots, w_{1,\psi_1}, w_{2,1}, w_{2,2}, \dots, w_{2,\psi_2}, \dots, w_{\beta,1}, w_{\beta,2}, \dots, w_{\beta,\psi_\beta}),$$

and let $B = \{w_{k,\gamma} : k \in \{1, \dots, \beta\}, \gamma \in \{1, \dots, \psi_k\}\}$. To prove Remark 2.11 (b), it suffices to show that there is a one-to-one correspondence between the elements of B and the index set $\{(k, \gamma) : k \in \{1, \dots, \beta\}, \gamma \in \{1, \dots, \psi_k\}\}$. Proceeding by contradiction, we have two cases.

Case 1: Suppose that $w_{k,\sigma} = w_{k,\gamma}$ for some $k \in \{1, \dots, \beta\}$ and $1 \leq \sigma < \gamma \leq \psi_k$. In this case, $w_{k,\gamma} \in Rw_{k,\sigma} + M^{\lambda_k}V \subseteq (\sum_{i=1}^{k-1} \sum_{j=1}^{\psi_i} R\pi^{\lambda_k - \lambda_i} w_{i,j}) + (\sum_{j=1}^{\gamma-1} Rw_{k,j}) + M^{\lambda_k}V$, contradicting the "realizing" assumption about w .

Case 2: Suppose that $w_{s,\sigma} = w_{k,\gamma}$ for some $1 \leq s < k \leq \beta$, $\sigma \in \{1, \dots, \psi_s\}$, and $\gamma \in \{1, \dots, \psi_k\}$. In this case, $w_{s,\sigma} \in M^{\lambda_s-1}V \setminus M^{\lambda_s}V$ and $w_{s,\sigma} = w_{k,\gamma} \in M^{\lambda_k-1}V \subseteq M^{\lambda_s}V$, a contradiction. $\square$

LEMMA 2.12 *Let w be a tuple which realizes a nonzero ascending e -tuple*

$(d_1, d_2, \dots, d_e)$. Define $\beta, \lambda_0, \lambda_1, \dots, \lambda_\beta, \psi_1, \dots, \psi_\beta$ as in Definition 2.10 (2), and label coordinates of w so that

$$w = (w_{1,1}, w_{1,2}, \dots, w_{1,\psi_1}, w_{2,1}, w_{2,2}, \dots, w_{2,\psi_2}, \dots, w_{\beta,1}, w_{\beta,2}, \dots, w_{\beta,\psi_\beta}).$$

For any $\alpha \in \{\lambda_1, \dots, e\}$ and $f := \max\{i \in \{1, \dots, \beta\} : \lambda_i \leq \alpha\}$, $L_\alpha := \{\pi^{\alpha-\lambda_i} w_{i,j} + M^\alpha V : i \in \{1, \dots, f\}, j \in \{1, \dots, \psi_i\}\}$ is an R/M -linearly independent subset of $M^{\alpha-1}V/M^\alpha V$ with d_{λ_f} pairwise distinct elements.

Proof. The number of indices (i, j) in the definition of L_α is the telescoping sum $\sum_{i=1}^f \psi_i = d_{\lambda_f}$. Now, assume that the assertion is false. Then, by Proposition 2.2, there exists $k \in \{1, \dots, f\}$ and $\gamma \in \{1, \dots, \psi_k\}$ such that

$$\pi^{\alpha-\lambda_k} w_{k,\gamma} \in (\sum_{i=1}^{k-1} \sum_{j=1}^{\psi_i} R\pi^{\alpha-\lambda_i} w_{i,j}) + (\sum_{j=1}^{\gamma-1} R\pi^{\alpha-\lambda_k} w_{k,j}) + M^\alpha V.$$

Thus, there exists $g \in (\sum_{i=1}^{k-1} \sum_{j=1}^{\psi_i} R\pi^{\lambda_k-\lambda_i} w_{i,j}) + (\sum_{j=1}^{\gamma-1} R w_{k,j}) + M^{\lambda_k} V$ such that $\pi^{\alpha-\lambda_k}(w_{k,\gamma} - g) = 0$. Hence, since V is R -free, $w_{k,\gamma} - g \in (0 :_V \pi^{\alpha-\lambda_k}) = M^{e-(\alpha-\lambda_k)} V$. As $\lambda_k \leq e - (\alpha - \lambda_k)$, we have that $M^{e-(\alpha-\lambda_k)} V \subseteq M^{\lambda_k} V$. Therefore,

$$w_{k,\gamma} = (w_{k,\gamma} - g) + g \in (\sum_{i=1}^{k-1} \sum_{j=1}^{\psi_i} R\pi^{\lambda_k-\lambda_i} w_{i,j}) + (\sum_{j=1}^{\gamma-1} R w_{k,j}) + M^{\lambda_k} V,$$

contradicting the “realizing” assumption about w . $\square$

PROPOSITION 2.13 *Let W be a nonzero R -submodule of V which realizes the e -tuple $(d_1, d_2, \dots, d_e)$. If W contains the set of coordinates of a tuple w which realizes $(d_1, d_2, \dots, d_e)$, then the set of coordinates of w is a minimal R -generating subset of W .*

Proof. Let w be a tuple which realizes $(d_1, d_2, \dots, d_e)$ where the set of coordinates of w are contained in W . Define $\beta, \lambda_0, \lambda_1, \dots, \lambda_\beta, \psi_1, \dots, \psi_\beta$ for $(d_1, d_2, \dots, d_e)$ as in Definition 2.10 (2), label the coordinates of w so that

$$w = (w_{1,1}, w_{1,2}, \dots, w_{1,\psi_1}, w_{2,1}, w_{2,2}, \dots, w_{2,\psi_2}, \dots, w_{\beta,1}, w_{\beta,2}, \dots, w_{\beta,\psi_\beta}),$$

and let $B = \{w_{k,\gamma} : k \in \{1, \dots, \beta\}, \gamma \in \{1, \dots, \psi_k\}\}$. By Remark 2.11 (b) and Remark 2.5 (a), observe that $|B| = \sum_{k=1}^{\beta} \psi_k = d_{\lambda_\beta} = d_e = p_{W,e} \leq \nu_R(W)$. Thus, to show that B is a minimal R -generating subset of W , it suffices to show that W is contained in the R -submodule W_0 of V generated by B . We claim that $W \cap M^{\alpha-1}V \subseteq W_0 + (W \cap M^\alpha V)$ for each $\alpha \in \{1, \dots, e\}$. Assuming this claim, the assertion follows since $W = W \cap M^0V \subseteq W_0 + (W \cap M^1V) \subseteq W_0 + (W \cap M^2V) \subseteq \dots \subseteq W_0 + (W \cap M^{e-1}V) \subseteq W_0 + (W \cap M^eV) = W_0 + 0 = W_0$.

Proof of the claim. Suppose $\alpha \in \{1, \dots, \lambda_1 - 1\}$. Since $p_{W,\alpha} = 0$, we have that $0 = \text{im}(\Delta_W^{(\alpha)}) = ((W \cap M^{\alpha-1}V) + M^\alpha V) / M^\alpha V$, giving $(W \cap M^{\alpha-1}V) + M^\alpha V = M^\alpha V$. Therefore, $W \cap M^{\alpha-1}V \subseteq W_0 + (W \cap M^\alpha V)$.

Suppose $\alpha \in \{\lambda_1, \dots, e\}$. Let $f := \max\{i \in \{1, \dots, \beta\} : \lambda_i \leq \alpha\}$. Observe via Proposition 2.12 that $L_\alpha := \{\pi^{\alpha-\lambda_i} w_{i,j} + M^\alpha V : i \in \{1, \dots, f\}, j \in \{1, \dots, \psi_i\}\}$ is an R/M -linearly independent subset of $((W \cap M^{\alpha-1}V) + M^\alpha V) / M^\alpha V = \text{im}(\Delta_W^{(\alpha)})$ with d_{λ_f} pairwise distinct elements. Therefore, L_α is an R/M -basis of $\text{im}(\Delta_W^{(\alpha)})$ since $p_{W,\alpha} = d_\alpha = d_{\lambda_f}$. Consequently, $(W \cap M^{\alpha-1}V) + M^\alpha V = (\sum_{i=1}^f \sum_{j=1}^{\psi_i} R \pi^{\alpha-\lambda_i} w_{i,j}) + M^\alpha V$. Hence, $W \cap M^{\alpha-1}V \subseteq W \cap (W_0 + M^\alpha V) = W_0 + (W \cap M^\alpha V)$. $\square$

PROPOSITION 2.14 *Any nonzero R -submodule W of V which realizes the e -tuple $(d_1, d_2, \dots, d_e)$ contains the set of coordinates of some tuple which realizes $(d_1, d_2, \dots, d_e)$.*

Proof. For $(d_1, d_2, \dots, d_e)$, define $\beta, \lambda_0, \lambda_1, \dots, \lambda_\beta, \psi_1, \dots, \psi_\beta$ as in Definition 2.10 (2). To prove the assertion, we find (by weak induction on β) a tuple w which realizes $(d_1, d_2, \dots, d_e)$ with coordinates contained in W .

Suppose $\beta = 1$. Notice that $p_{W, \lambda_1} = d_{\lambda_1} \neq 0$ since $W \neq 0$. Thus, by the definition of $\Delta_W^{(\lambda_1)}$, there exists an R/M -basis $L_{\lambda_1} := \{w_{1, \gamma} + M^{\lambda_1} V : \gamma \in \{1, \dots, d_{\lambda_1}\}\}$ of $\text{im}(\Delta_W^{(\lambda_1)}) = ((W \cap M^{\lambda_1-1} V) + M^{\lambda_1} V) / M^{\lambda_1} V \subseteq M^{\lambda_1-1} V / M^{\lambda_1} V$. Without loss of generality, we may assume that $w_{1, \gamma} \in (W \cap M^{\lambda_1-1} V) \setminus M^{\lambda_1} V \subseteq W$ for each $\gamma \in \{1, \dots, d_{\lambda_1}\}$. Further, Proposition 2.2 gives that

$$w_{1, \gamma} \in M^{\lambda_1-1} V \setminus ((\sum_{j=1}^{\gamma-1} R w_{1, j}) + M^{\lambda_1} V)$$

for each $\gamma \in \{1, \dots, d_{\lambda_1} = \psi_1\}$. Therefore, it suffices to let w be the tuple $(w_{1,1}, w_{1,2}, \dots, w_{1, \psi_1})$.

Suppose $\beta > 1$. As an induction hypothesis, assume that the assertion holds for any nonzero ascending e -tuple with corresponding β -value less than β . By Proposition 2.6, $p_{M^{e-\lambda_{\beta-1}} W, e} = p_{W, \lambda_{\beta-1}} \neq 0$, giving that $M^{e-\lambda_{\beta-1}} W \neq 0$ by Remark 2.5 (b). In fact, Proposition 2.6 yields that $M^{e-\lambda_{\beta-1}} W$ realizes $(d'_1, d'_2, \dots, d'_e)$ where $d'_i = 0$ for all $i \in \{1, \dots, e-\lambda_{\beta-1}\}$ and $d'_i = d_{i-(e-\lambda_{\beta-1})}$ for all $i \in \{e-\lambda_{\beta-1}+1, \dots, e\}$. Defining $\beta', \lambda'_0, \lambda'_1, \dots, \lambda'_{\beta'}, \psi'_1, \dots, \psi'_{\beta'}$ for $(d'_1, d'_2, \dots, d'_e)$ as $\beta, \lambda_0, \lambda_1, \dots, \lambda_\beta, \psi_1, \dots, \psi_\beta$ were defined for $(d_1, d_2, \dots, d_e)$, observe that $\beta' = \beta - 1$, $\lambda'_0 := 0$, $\lambda'_i = (e - \lambda_{\beta-1}) + \lambda_i$, and $\psi'_i := \lambda'_i - \lambda'_{i-1} = \lambda_i - \lambda_{i-1} = \psi_i$ for each $i \in \{1, \dots, \beta'\}$. Since $\beta' = \beta - 1$, by the induction hypothesis, there exists a tuple

$$w' = (w'_{1,1}, w'_{1,2}, \dots, w'_{1, \psi'_1}, w'_{2,1}, w'_{2,2}, \dots, w'_{2, \psi'_2}, \dots, w'_{\beta',1}, w'_{\beta',2}, \dots, w'_{\beta', \psi'_{\beta'}})$$

which realizes $(d'_1, d'_2, \dots, d'_e)$ with coordinates in $M^{e-\lambda_{\beta-1}} W$. Therefore, for each $k \in \{1, \dots, \beta' = \beta - 1\}$ and $\gamma \in \{1, \dots, \psi'_k = \psi_k\}$, there exists $w_{k, \gamma} \in W$ such that $w'_{k, \gamma} = \pi^{e-\lambda_{\beta-1}} w_{k, \gamma}$. We claim that the tuple

$$(w_{1,1}, w_{1,2}, \dots, w_{1, \psi_1}, w_{2,1}, w_{2,2}, \dots, w_{2, \psi_2}, \dots, w_{\beta-1,1}, w_{\beta-1,2}, \dots, w_{\beta-1, \psi_{\beta-1}})$$

realizes $(d_1, d_2, \dots, d_{\lambda_{\beta-1}-1}, d_{\lambda_{\beta-1}}, \dots, d_{\lambda_{\beta-1}})$.

Assuming the above claim, it follows by Lemma 2.12 (letting $\alpha = \lambda_{\beta-1}$) that

$$L_{\lambda_{\beta-1}} := \{\pi^{\lambda_{\beta-1}-\lambda_i} w_{i,j} + M^{\lambda_{\beta-1}} V : i \in \{1, \dots, \beta-1\}, j \in \{1, \dots, \psi_i\}\}$$

is an R/M -linearly independent subset of $M^{\lambda_{\beta-1}-1}V/M^{\lambda_{\beta-1}}V$ with $d_{\lambda_{\beta-1}}$ pairwise distinct elements. Therefore, by Proposition 2.8,

$$S := \{\pi^{\lambda_{\beta}-\lambda_i} w_{i,j} + M^{\lambda_{\beta}} V : i \in \{1, \dots, \beta-1\}, j \in \{1, \dots, \psi_i\}\}$$

is an R/M -linearly independent subset of $M^{\lambda_{\beta}-1}V/M^{\lambda_{\beta}}V$ with $d_{\lambda_{\beta-1}}$ pairwise distinct elements. Thus, since $S \subseteq \text{im}(\Delta_W^{(\lambda_{\beta})})$ and $d_{\lambda_{\beta-1}} < d_{\lambda_{\beta}} = p_{W, \lambda_{\beta}}$, there exist $w_{\beta,1} + M^{\lambda_{\beta}} V, w_{\beta,2} + M^{\lambda_{\beta}} V, \dots, w_{\beta,\psi_{\beta}} + M^{\lambda_{\beta}} V \subseteq \text{im}(\Delta_W^{(\lambda_{\beta})})$ such that $L_{\lambda_{\beta}} := \{\pi^{\lambda_{\beta}-\lambda_i} w_{i,j} + M^{\lambda_{\beta}} V : i \in \{1, \dots, \beta\}, j \in \{1, \dots, \psi_i\}\}$ is an R/M -basis of $\text{im}(\Delta_W^{(\lambda_{\beta})})$. Without loss of generality, $w_{\beta,j} \in (W \cap M^{\lambda_{\beta}-1}V) \setminus M^{\lambda_{\beta}} V \subseteq W$ for each $j \in \{1, \dots, \psi_{\beta}\}$. Also, by the R/M -linear independence of $L_{\lambda_{\beta}}$ in $M^{\lambda_{\beta}-1}V/M^{\lambda_{\beta}}V$, Proposition 2.2 gives that

$$w_{\beta,\gamma} \in M^{\lambda_{\beta}-1}V \setminus ((\sum_{i=1}^{\beta-1} \sum_{j=1}^{\psi_i} R\pi^{\lambda_{\beta}-\lambda_i} w_{i,j}) + (\sum_{j=1}^{\gamma-1} R w_{\beta,j}) + M^{\lambda_{\beta}} V)$$

for each $\gamma \in \{1, \dots, \psi_{\beta}\}$. Therefore, it suffices to let

$$w = (w_{1,1}, w_{1,2}, \dots, w_{1,\psi_1}, w_{2,1}, w_{2,2}, \dots, w_{2,\psi_2}, \dots, w_{\beta,1}, w_{\beta,2}, \dots, w_{\beta,\psi_{\beta}}).$$

Proof of the claim. By Definition 2.10, it suffices to show that

$$w_{k,\gamma} \in M^{\lambda_k-1}V \setminus ((\sum_{i=1}^{k-1} \sum_{j=1}^{\psi_i} R\pi^{\lambda_k-\lambda_i} w_{i,j}) + (\sum_{j=1}^{\gamma-1} R w_{k,j}) + M^{\lambda_k} V)$$

for each $k \in \{1, \dots, \beta-1\}$ and $\gamma \in \{1, \dots, \psi_k\}$. Fix $k \in \{1, \dots, \beta-1\}$ and $\gamma \in \{1, \dots, \psi_k\}$. Since $w'_{k,\gamma} \in M^{\lambda'_k-1}V = M^{e-\lambda_{\beta-1}+\lambda_k-1}V$, there exists $g \in M^{\lambda_k-1}V$ such that $\pi^{e-\lambda_{\beta-1}}(w_{k,\gamma} - g) = 0$. Thus, $w_{k,\gamma} - g \in (0 :_V \pi^{e-\lambda_{\beta-1}}) = M^{\lambda_{\beta-1}}V$. As $\lambda_k - 1 \leq \lambda_{\beta-1}$, it follows that $M^{\lambda_{\beta-1}}V \subseteq M^{\lambda_k-1}V$, giving $w_{k,\gamma} = (w_{k,\gamma} - g) + g \in M^{\lambda_k-1}V$. Finally, since

$$\begin{aligned} \pi^{e-\lambda_{\beta-1}} w_{k,\gamma} &= w'_{k,\gamma} \notin (\sum_{i=1}^{k-1} \sum_{j=1}^{\psi_i} R\pi^{\lambda'_k-\lambda'_i} w'_{i,j}) + (\sum_{j=1}^{\gamma-1} R w'_{k,j}) + M^{\lambda'_k} V = \\ &= \pi^{e-\lambda_{\beta-1}} ((\sum_{i=1}^{k-1} \sum_{j=1}^{\psi_i} R\pi^{\lambda_k-\lambda_i} w_{i,j}) + (\sum_{j=1}^{\gamma-1} R w_{k,j}) + M^{\lambda_k} V), \end{aligned}$$

observe that $w_{k,\gamma} \notin (\sum_{i=1}^{k-1} \sum_{j=1}^{\psi_i} R\pi^{\lambda_k - \lambda_i} w_{i,j}) + (\sum_{j=1}^{\gamma-1} R w_{k,j}) + M^{\lambda_k} V$. $\square$

COROLLARY 2.15 *If W is a nonzero R -submodule of V which realizes the e -tuple $(d_1, d_2, \dots, d_e)$, then W has a minimal R -generating subset which is the set of coordinates of some tuple which realizes $(d_1, d_2, \dots, d_e)$.*

Proof. The assertion follows by Propositions 2.13 and 2.14. $\square$

COROLLARY 2.16 $p_{W,e} = \nu_R(W)$ for any R -submodule W of V .

Proof. Given $W = 0$, the assertion follows by Remark 2.5 (b).

Given $W \neq 0$, let $d_i := p_{W,i}$ for each $i \in \{1, \dots, e\}$. As W realizes the e -tuple $(d_1, d_2, \dots, d_e)$, the assertion follows by Corollary 2.15 and Remark 2.11 (b). $\square$

The next result is a natural companion for Proposition 2.14.

PROPOSITION 2.17 *If w is a tuple which realizes the nonzero e -tuple $(d_1, d_2, \dots, d_e)$, then the nonzero R -submodule W generated by the coordinates of w also realizes $(d_1, d_2, \dots, d_e)$. Moreover, the coordinates of w are a minimal generating set of W .*

Proof. Given that the tuple w realizes $(d_1, d_2, \dots, d_e)$, define $\beta, \lambda_0, \lambda_1, \dots, \lambda_\beta, \psi_1, \dots, \psi_\beta$ as in Definition 2.10 (2), and label the coordinates of w so that

$$w = (w_{1,1}, w_{1,2}, \dots, w_{1,\psi_1}, w_{2,1}, w_{2,2}, \dots, w_{2,\psi_2}, \dots, w_{\beta,1}, w_{\beta,2}, \dots, w_{\beta,\psi_\beta}).$$

Further, let W be the R -submodule of V generated by the coordinates of w .

Observe that $p_{W,i} = 0 = d_i$ for all $i \leq \lambda_1 - 1$ since $W \subseteq M^{\lambda_1 - 1} V$. Therefore, we need only show that $p_{W,i} = d_i$ for each $i \in \{\lambda_1, \dots, e\}$. For this purpose, let $\alpha \in \{1, \dots, e\}$ and let $f := \max\{i \in \{1, \dots, \beta\} : \lambda_i \leq \alpha\}$.

First, we show that $p_{W,\alpha} \leq d_\alpha$. Let $Y := \sum_{k=1}^f \sum_{\gamma=1}^{\psi_k} R w_{k,\gamma}$. Observe that if $f < k \leq \beta$, then $w_{k,\gamma} \in M^\alpha V$ for all $\gamma \in \{1, \dots, \psi_k\}$ since $\lambda_k - 1 \geq \alpha$. Hence,

$$((\sum_{k=1}^\beta \sum_{\gamma=1}^{\psi_k} R w_{k,\gamma}) \cap M^{\alpha-1} V) + M^\alpha V = ((\sum_{k=1}^f \sum_{\gamma=1}^{\psi_k} R w_{k,\gamma}) \cap M^{\alpha-1} V) + M^\alpha V.$$

It follows that $\text{im}(\Delta_W^{(\alpha)}) = \text{im}(\Delta_Y^{(\alpha)})$, giving $p_{W,\alpha} = p_{Y,\alpha} \leq \nu_R(Y)$ [by Remark 2.5(a)] $\leq \sum_{k=1}^f \psi_k$ [by the construction of Y] $= d_{\lambda_f} = d_\alpha$.

Since $p_{W,\alpha} \leq d_\alpha$, to prove $p_{W,\alpha} = d_\alpha$, it is enough to show that

$$L_\alpha := \{\pi^{\alpha-\lambda_i} w_{i,j} + M^\alpha V : i \in \{1, \dots, f\}, j \in \{1, \dots, \psi_i\}\}$$

is an R/M -linearly independent subset of $\text{im}(\Delta_W^{(\alpha)})$ with $d_{\lambda_f} = d_\alpha$ pairwise distinct elements. Since L_α is, in fact, a subset of $\text{im}(\Delta_W^{(\alpha)})$ ($\subseteq M^{\alpha-1} V / M^\alpha V$), by Proposition 2.2, it suffices to prove that

$$\begin{aligned} \pi^{\alpha-\lambda_k} w_{k,\gamma} \notin & (\sum_{i=1}^{k-1} \sum_{j=1}^{\psi_i} R \pi^{\alpha-\lambda_i} w_{i,j}) + (\sum_{j=1}^{\gamma-1} R \pi^{\alpha-\lambda_k} w_{k,j}) + M^\alpha V = \\ & \pi^{\alpha-\lambda_k} ((\sum_{i=1}^{k-1} \sum_{j=1}^{\psi_i} R \pi^{\lambda_k-\lambda_i} w_{i,j}) + (\sum_{j=1}^{\gamma-1} R w_{k,j}) + M^{\lambda_k} V) \end{aligned}$$

for each $k \in \{1, \dots, f\}$ and $\gamma \in \{1, \dots, \psi_k\}$. Proceeding by contradiction, assume that there exist $k \in \{1, \dots, f\}$ and $\gamma \in \{1, \dots, \psi_k\}$ such that

$$\pi^{\alpha-\lambda_k} w_{k,\gamma} \in \pi^{\alpha-\lambda_k} ((\sum_{i=1}^{k-1} \sum_{j=1}^{\psi_i} R \pi^{\lambda_k-\lambda_i} w_{i,j}) + (\sum_{j=1}^{\gamma-1} R w_{k,j}) + M^{\lambda_k} V).$$

Therefore, there exists $g \in (\sum_{i=1}^{k-1} \sum_{j=1}^{\psi_i} R \pi^{\lambda_k-\lambda_i} w_{i,j}) + (\sum_{j=1}^{\gamma-1} R w_{k,j}) + M^{\lambda_k} V$ such that $\pi^{\alpha-\lambda_k} (w_{k,\gamma} - g) = 0$. Thus, $w_{k,\gamma} - g \in (0 :_V \pi^{\alpha-\lambda_k}) = M^{e-(\alpha-\lambda_k)} V$. Since $\lambda_k \leq e - (\alpha - \lambda_k)$, it follows that $M^{e-(\alpha-\lambda_k)} V \subseteq M^{\lambda_k} V$. Hence, $w_{k,\gamma} = (w_{k,\gamma} - g) + g \in (\sum_{i=1}^{k-1} \sum_{j=1}^{\psi_i} R \pi^{\lambda_k-\lambda_i} w_{i,j}) + (\sum_{j=1}^{\gamma-1} R w_{k,j}) + M^{\lambda_k} V$, contradicting the "realizing" assumption about the tuple w .

Finally, the "moreover" assertion follows from Corollary 2.16 and the fact that the number of coordinates of w is $\sum_{i=1}^\beta \psi_i = d_{\lambda_\beta} = d_e = p_{W,e} = \nu_R(W)$. $\square$

THEOREM 2.18 *For any nonzero R -submodule W of V and nonzero e -tuple $(d_1, d_2, \dots, d_e)$, W realizes $(d_1, d_2, \dots, d_e)$ if and only if W is R -generated by the set of coordinates of some tuple which realizes $(d_1, d_2, \dots, d_e)$. Moreover, if these equivalent conditions hold, then each of the generating sets in question are minimal generating sets of W .*

Proof. The assertion follows by Corollary 2.15 and Proposition 2.17. $\square$

The next six results display some connections between R -freeness of submodules of V and the “realizing” concepts. These results will be especially useful in Chapter 3 in treating modules over an arbitrary principal ideal ring.

PROPOSITION 2.19 *Suppose that $k \in \{1, \dots, n\}$, and W is an R -submodule of V which realizes $(k, k, \dots, k)$. Then W is R -free on a basis of k elements.*

Proof. Since W realizes $(k, k, \dots, k)$, Theorem 2.18 gives that W has a minimal R -generating set which is the set of coordinates of a tuple $(w_{1,1}, w_{1,2}, \dots, w_{1,k})$ realizing $(k, k, \dots, k)$; i.e., $w_{1,\gamma} \in V \setminus ((\sum_{j=1}^{\gamma-1} R w_{1,j}) + MV)$ for each $\gamma \in \{1, \dots, k\}$. Thus, by Proposition 2.2, $L := \{w_{1,j} + MV : j \in \{1, \dots, k\}\}$ is an R/M -linearly independent subset of V/MV with k pairwise distinct elements. With this in hand, to prove that W is R -free on a basis of k elements, it suffices to show that if $r_1, r_2, \dots, r_k \in R$ such that $\sum_{j=1}^k r_j w_{1,j} = 0$, then $r_j = 0$ for each $j \in \{1, \dots, k\}$.

Suppose $r_1, r_2, \dots, r_k \in R$ such that $\sum_{j=1}^k r_j w_{1,j} = 0 \in MV$. Then, by the R/M -linear independence of L , $r_j \in M$ for each $j \in \{1, \dots, k\}$. Let ξ be the largest value in $\{1, \dots, e\}$ such that $r_1, r_2, \dots, r_k \in M^\xi$. If $e = \xi$, then the assertion follows since $M^e = 0$. Suppose that $\xi < e$. By the definition of ξ , for each $j \in \{1, \dots, k\}$, there exists $r'_j \in R$ such that $r_j = \pi^\xi r'_j$. Thus, since $\pi^\xi (\sum_{j=1}^k r'_j w_{1,j}) = \sum_{j=1}^k r_j w_{1,j} = 0$, we have that $\sum_{j=1}^k r'_j w_{1,j} \in (0 :_V \pi^\xi) = M^{e-\xi} V \subseteq MV$. Therefore, by the R/M -

linear independence of L , $r'_j \in M$ for each $j \in \{1, \dots, k\}$. Hence $r_1, \dots, r_k \in M^{\xi+1}$, contradicting the maximality of ξ . $\square$

The next result is essentially the converse of Proposition 2.19.

PROPOSITION 2.20 *Let $k \in \{1, \dots, n\}$. Given a (nonzero) submodule W of V which is R -free on a basis of k elements, any R -basis of W is the set of coordinates of some tuple which realizes $(k, k, \dots, k)$. (Thus, by Theorem 2.18, W realizes $(k, k, \dots, k)$.)*

Proof. Let $\{f_1, f_2, \dots, f_k\}$ be an R -basis of W . Since $(0 :_R f_j) = 0$ for each $j \in \{1, \dots, k\}$, observe that $f_j \in V \setminus MV$ for each $j \in \{1, \dots, k\}$. We claim that $\{f_j + MV : j \in \{1, \dots, k\}\}$ is an R/M -linearly independent subset of V/MV with k pairwise distinct elements. Assuming this claim, it follows, by Proposition 2.2 and Definition 2.10 (2), that the tuple $(f_1, f_2, \dots, f_k)$ realizes $(k, k, \dots, k)$.

Proof of the claim. Suppose that $r_1, r_2, \dots, r_k \in R$ such that $\sum_{i=1}^k (r_i + M)(f_i + MV) = 0 \in V/MV$; i.e., $\sum_{i=1}^k r_i f_i \in MV$. Then, $\pi^{e-1}(\sum_{i=1}^k r_i f_i) = \sum_{i=1}^k (\pi^{e-1} r_i) f_i \in \pi^{e-1} MV = M^e V = 0$. As $\{f_1, f_2, \dots, f_k\}$ is an R -basis of W , it follows that $\pi^{e-1} r_i = 0$ for each $i \in \{1, \dots, k\}$. Thus, $r_i \in (0 :_R \pi^{e-1}) = M$, and so $r_i + M = 0 \in R/M$ for each $i \in \{1, \dots, k\}$, proving the claim. $\square$

COROLLARY 2.21 *Let W be a nonzero R -submodule of V , and let $k = \nu_R(W)$. Then W is R -free on a basis of k elements if and only if W realizes $(k, k, \dots, k)$.*

Proof. Since $W \neq 0$ and R is a principal ideal ring, $k \in \{1, \dots, n\}$. The assertion follows by Proposition 2.19 and Proposition 2.20. $\square$

PROPOSITION 2.22 *Let W be an R -submodule of V . Then the following conditions are equivalent:*

- (i) V/W is R -flat;
- (ii) V/W is R -free;
- (iii) W is R -free;
- (iv) $MV \cap W = MW$.

Proof. (i) $\Rightarrow$ (ii) follows since any finitely generated flat module over a commutative local ring is free.

(ii) $\Rightarrow$ (iii): Given that V/W is R -free, the canonical short exact sequence $0 \rightarrow W \rightarrow V \rightarrow V/W \rightarrow 0$ is R -split. Thus W is R -projective. The assertion follows since every finitely generated projective module over a local ring is free (cf. [5, Theorem 2]).

(iii) $\Rightarrow$ (iv): Suppose W is R -free. If $W = 0$, then the assertion follows trivially. Thus, assume that $W \neq 0$. Let $\{w_1, w_2, \dots, w_k\}$ be an R -basis of W (with $k \in \{1, \dots, n\}$). By Proposition 2.20, $(w_1, w_2, \dots, w_k)$ is a tuple which realizes $(k, k, \dots, k)$; i.e., $w_\gamma \in V \setminus ((\sum_{j=1}^{\gamma-1} R w_j) + MV)$ for each $\gamma \in \{1, \dots, k\}$. Therefore, by Proposition 2.2, $L := \{w_j + MV : j \in \{1, \dots, k\}\}$ is an R/M -linearly independent subset of V/MV with k pairwise distinct elements. As a result, if $w = \sum_{j=1}^k r_j w_j \in MV \cap W$ for some $r_j \in R$, then, by the R/M -linear independence of L , $r_j \in M$ for all $j \in \{1, \dots, k\}$, giving that $w \in MW$. Therefore, $MV \cap W \subseteq MW$; the reverse inclusion is clear.

(iv) $\Rightarrow$ (i): Assume (iv). Since R is Noetherian local and V/W is a finitely generated R -module, it suffices to show that the canonical homomorphism $f : M \otimes_R (V/W) \rightarrow V/W$ is a monomorphism (cf. [6, Corollaire 1, p. IV-34]).

Let $\xi \in \ker(f)$. Then $\xi = \pi \otimes (v + W)$ for some $v \in V$ such that $\pi v + W = 0 \in V/W$. Hence $\pi v \in W \cap MV = MW$, and so $\pi v = \pi w$ for some $w \in W$. Then

$v - w \in (0 :_V \pi) = M^{e-1}V$, and so $v + W = \pi^{e-1}z + W \in V/W$ for some $z \in V$. Therefore $\xi = \pi \otimes (\pi^{e-1}z + W) = \pi^e \otimes (z + W) = 0 \otimes (z + W) = 0$. $\square$

COROLLARY 2.23 *Given R -submodules W, W' of V such that $W' \subseteq W$, if W' is R -free, then $\nu_R(W/W') = \nu_R(W) - \nu_R(W')$.*

Proof. Consider the canonical short exact sequence $0 \rightarrow W' \rightarrow W \rightarrow W/W' \rightarrow 0$. Tensoring by R/M gives that $R/M \otimes_R W' \rightarrow R/M \otimes_R W \rightarrow R/M \otimes_R W/W' \rightarrow 0$ is exact. Observe via Proposition 2.22 that $MV \cap W' = MW'$, and so the canonical homomorphism $W'/MW' \rightarrow W/MW$ is one-to-one. As a consequence, $0 \rightarrow R/M \otimes_R W' \rightarrow R/M \otimes_R W \rightarrow R/M \otimes_R W/W' \rightarrow 0$ is short exact. Thus, since this short exact sequence is R/M -split, it follows by Nakayama's Lemma that $\nu_R(W) = \dim_{R/M}(R/M \otimes_R W) = \dim_{R/M}(R/M \otimes_R W') + \dim_{R/M}(R/M \otimes_R W/W') = \nu_R(W') + \nu_R(W/W')$. $\square$

We pause to note that Corollary 2.23 fails if W' is not assumed to be R -free. For an example, it suffices to consider $W' = M$ and $W = R = V$ in case $e \geq 2$.

PROPOSITION 2.24 *Given an R -submodule W of V , the maximum rank of any R -free submodule of W is $p_{W,1} (= \dim_{R/M}(\text{im}(\Delta_W^{(1)})))$.*

Proof. Given an R -submodule W' of W , observe that $\text{im}(\Delta_{W'}^{(i)}) \subseteq \text{im}(\Delta_W^{(i)})$ for each $i \in \{1, \dots, e\}$. Thus, $p_{W',i} \leq p_{W,i}$ for each $i \in \{1, \dots, e\}$. Therefore, as a consequence of Corollary 2.21, the rank of any R -free submodule of W is less than or equal to $p_{W,1}$. Therefore, if $p_{W,1} = 0$, then the assertion follows from Remark 2.5 (b).

Suppose that $p_{W,1} \neq 0$. Define $d_i := p_{W,i}$ for each $i \in \{1, \dots, e\}$. By Theorem 2.18 and Remark 2.5 (b), W is R -generated by the coordinates of a tuple

$$(w_{1,1}, w_{1,2}, \dots, w_{1,\psi_1}, w_{2,1}, w_{2,2}, \dots, w_{2,\psi_2}, \dots, w_{\beta,1}, w_{\beta,2}, \dots, w_{\beta,\psi_\beta})$$

which realizes $(d_1, d_2, \dots, d_e)$, where $\beta, \lambda_0, \lambda_1, \dots, \lambda_\beta, \psi_1, \dots, \psi_\beta$ are as in Definition 2.10 (2). Let W' be the submodule of W which is R -generated by the set $\{w_{1,\gamma} : \gamma \in \{1, \dots, \psi_1 = p_{W,1}\}\}$. Observe via Theorem 2.18 that W' realizes $(p_{W,1}, p_{W,1}, \dots, p_{W,1})$. By Corollaries 2.21 and 2.16, the assertion follows. $\square$

Throughout the rest of this chapter, we **add to the running hypotheses** that the ring R is finite. Also, let q denote the cardinality of R/M .

PROPOSITION 2.25 $|W| = q^{\sum_{i=1}^e p_{W,i}}$ for any R -submodule W of V .

Proof. If $W = 0$, then the assertion follows by Remark 2.5 (b).

Suppose that $W \neq 0$. Then, by Lagrange's theorem, it follows that

$$|W| = |W/MW| |MW/M^2W| |M^2W/M^3W| \dots |M^{e-1}W/M^eW| |M^eW|.$$

Notice that $|M^eW| = 1$ since $M^e = 0$. Further, observe via Nakayama's Lemma and Corollary 2.16 that $\dim_{R/M}(M^jW/M^{j+1}W) = \nu_R(M^jW) = p_{M^jW,e}$ for each $j \in \{0, \dots, e-1\}$. Thus, since $p_{M^jW,e} = p_{W,e-j}$ by Proposition 2.6, we have that $|M^jW/M^{j+1}W| = q^{p_{W,e-j}}$ for each $j \in \{0, \dots, e-1\}$, giving that

$$|W| = q^{p_{W,e-0}} q^{p_{W,e-1}} \dots q^{p_{W,e-(e-1)}} 1 = q^{\sum_{i=1}^e p_{W,i}}. \quad \square$$

COROLLARY 2.26 Let W be an R -submodule of V and let j be a nonnegative integer. Then:

$$(1) \quad |M^jW| = q^{\sum_{i=1}^{e-j} p_{W,i}}.$$

$$(2) \quad |W \cap M^jV| = q^{\sum_{i=j+1}^e p_{W,i}}.$$

Proof. (1) Using Proposition 2.25 and Proposition 2.6, observe that $|M^j W| = q^{\sum_{i=1}^e p_{M^j W, i}} = q^{\sum_{i=1}^e p_{W, i-j}} = q^{\sum_{i=1}^{e-j} p_{W, i}}$.

(2) Using Proposition 2.25 and Proposition 2.7, observe that $|W \cap M^j V| = q^{\sum_{i=1}^e p_{W \cap M^j V, i}} = q^{\sum_{i=j+1}^e p_{W, i}}$. $\square$

REMARK 2.27 Let $(d_1, d_2, \dots, d_e)$ be a nonzero ascending e -tuple. Define an (equivalence) relation $\sim_{(d_1, d_2, \dots, d_e)}$ on the set $T_{(d_1, d_2, \dots, d_e)}$ of all tuples which realize $(d_1, d_2, \dots, d_e)$ as follows. For $x, y \in T_{(d_1, d_2, \dots, d_e)}$, let $x \sim_{(d_1, d_2, \dots, d_e)} y$ if the set of coordinates of x generates the same R -submodule of V as the set of coordinates of y .

By Proposition 2.25, any two R -submodules of V which both realize the same $(d_1, d_2, \dots, d_e)$ have the same cardinality. In view of Proposition 2.17, this observation yields the following useful fact. For $x, y \in T_{(d_1, d_2, \dots, d_e)}$, $x \sim_{(d_1, d_2, \dots, d_e)} y$ if and only if the coordinates of x are contained in the R -submodule of V generated by the coordinates of y .

PROPOSITION 2.28 *Given a nonzero ascending e -tuple $(d_1, d_2, \dots, d_e)$, define $\beta, \lambda_0, \lambda_1, \dots, \lambda_\beta, \psi_1, \dots, \psi_\beta$ as in Definition 2.10 (2). Then:*

(a) *The cardinality of the set of all tuples which realize $(d_1, d_2, \dots, d_e)$ is*

$$q^{\epsilon_1} \left[\prod_{\gamma=1}^{d_{\lambda_\beta}} (q^\gamma - q^{\gamma-1}) \right], \text{ where}$$

$$\epsilon_1 = nd_{\lambda_\beta}(e - \lambda_\beta) + \left(\sum_{k=1}^{\beta-1} nd_{\lambda_k}(\lambda_{k+1} - \lambda_k) \right).$$

(b) *The cardinality of each $\sim_{(d_1, d_2, \dots, d_e)}$ -equivalence class is*

$$q^{\epsilon_2} \left[\prod_{k=1}^{\beta} \prod_{\gamma=1}^{\psi_k} (q^{d_{\lambda_k}} - q^{d_{\lambda_{k-1}} + \gamma - 1}) \right], \text{ where}$$

$$\epsilon_2 = d_{\lambda_\beta}^2 (e - \lambda_\beta) + \left(\sum_{k=1}^{\beta-1} d_{\lambda_k}^2 (\lambda_{k+1} - \lambda_k) \right) + \left(\sum_{k=1}^{\beta-1} d_{\lambda_k} (d_{\lambda_{k+1}} - d_{\lambda_k}) \right).$$

(c) *The number of submodules of V which realize $(d_1, d_2, \dots, d_e)$ is*

$$\Lambda(d_1, d_2, \dots, d_e) := q^\epsilon \left[\prod_{k=1}^{\beta-1} \frac{\prod_{\gamma=1}^{d_{\lambda_k}} (q^{d_{\lambda_k+1}-q^{\gamma-1}})}{\prod_{\gamma=1}^{d_{\lambda_k}} (q^{d_{\lambda_k}-q^{\gamma-1}})} \right] \left[\frac{\prod_{\gamma=1}^{d_{\lambda_\beta}} (q^n - q^{\gamma-1})}{\prod_{\gamma=1}^{d_{\lambda_\beta}} (q^{d_{\lambda_\beta}-q^{\gamma-1}})} \right], \text{ where}$$

$$\epsilon = d_{\lambda_\beta}(n - d_{\lambda_\beta})(e - \lambda_\beta) + \left(\sum_{k=1}^{\beta-1} d_{\lambda_k}(n - d_{\lambda_k})(\lambda_{k+1} - \lambda_k) \right) - \left(\sum_{k=1}^{\beta-1} d_{\lambda_k}(d_{\lambda_{k+1}} - d_{\lambda_k}) \right).$$

Further, $\Lambda(d_1, d_2, \dots, d_e)$ is the evaluation at q of a monic polynomial in $Z[X]$ of degree $\sum_{i=1}^e d_i(n - d_i)$; it can be arranged that this polynomial is independent of q .

Proof. (a) To find the cardinality of the set of all tuples which realize the given nonzero e -tuple $(d_1, d_2, \dots, d_e)$, we count the number of ways to successively choose the coordinates of a tuple

$$w = (w_{1,1}, w_{1,2}, \dots, w_{1,\psi_1}, w_{2,1}, w_{2,2}, \dots, w_{2,\psi_2}, \dots, w_{\beta,1}, w_{\beta,2}, \dots, w_{\beta,\psi_\beta})$$

with coordinates in V so that, after all its coordinates have been chosen, w realizes $(d_1, d_2, \dots, d_e)$. For this purpose, fix $k \in \{1, \dots, \beta\}$ and $\gamma \in \{1, \dots, \psi_k\}$. Also, assume that any coordinates preceding $w_{k,\gamma}$ in the tuple w have been chosen “properly”. In other words, assume that

$$w_{k',\gamma'} \in M^{\lambda_{k'}-1}V \setminus ((\sum_{i=1}^{k'-1} \sum_{j=1}^{\psi_i} R\pi^{\lambda_{k'}-\lambda_i} w_{i,j}) + (\sum_{j=1}^{\gamma'-1} R w_{k',j}) + M^{\lambda_{k'}}V)$$

for any $k' \in \{1, \dots, k-1\}$ and $\gamma' \in \{1, \dots, \psi_{k'}\}$, and that

$$w_{k,\gamma'} \in M^{\lambda_k-1}V \setminus ((\sum_{i=1}^{k-1} \sum_{j=1}^{\psi_i} R\pi^{\lambda_k-\lambda_i} w_{i,j}) + (\sum_{j=1}^{\gamma'-1} R w_{k,j}) + M^{\lambda_k}V)$$

for any $\gamma' \in \{1, \dots, \gamma-1\}$. As a consequence, observe that

$$(\sum_{i=1}^{k-1} \sum_{j=1}^{\psi_i} R\pi^{\lambda_k-\lambda_i} w_{i,j}) + (\sum_{j=1}^{\gamma-1} R w_{k,j}) + M^{\lambda_k}V \subseteq M^{\lambda_k-1}V.$$

Therefore, by Definition 2.10 (2), the number of ways to choose $w_{k,\gamma}$ from V is

$$\begin{aligned} & |M^{\lambda_k-1}V \setminus ((\sum_{i=1}^{k-1} \sum_{j=1}^{\psi_i} R\pi^{\lambda_k-\lambda_i} w_{i,j}) + (\sum_{j=1}^{\gamma-1} R w_{k,j}) + M^{\lambda_k}V)| = \\ & |M^{\lambda_k-1}V| - |(\sum_{i=1}^{k-1} \sum_{j=1}^{\psi_i} R\pi^{\lambda_k-\lambda_i} w_{i,j}) + (\sum_{j=1}^{\gamma-1} R w_{k,j}) + M^{\lambda_k}V| = \\ & |M^{\lambda_k-1}V| - \left| \frac{(\sum_{i=1}^{k-1} \sum_{j=1}^{\psi_i} R\pi^{\lambda_k-\lambda_i} w_{i,j}) + (\sum_{j=1}^{\gamma-1} R w_{k,j}) + M^{\lambda_k}V}{M^{\lambda_k}V} \right| |M^{\lambda_k}V|, \end{aligned}$$

with the last equality holding by Lagrange's Theorem.

By Corollary 2.26 (1) and Remark 2.5 (c), we have that

$$|M^{\lambda_k-1}V| = q^{\sum_{i=1}^{e-(\lambda_k-1)} n} = q^{n(e-\lambda_k+1)} \text{ and } |M^{\lambda_k}V| = q^{\sum_{i=1}^{e-\lambda_k} n} = q^{n(e-\lambda_k)}.$$

Next, we find the cardinality of $\frac{(\sum_{i=1}^{k-1} \sum_{j=1}^{\psi_i} R\pi^{\lambda_k-\lambda_i} w_{i,j}) + (\sum_{j=1}^{\gamma-1} R w_{k,j}) + M^{\lambda_k}V}{M^{\lambda_k}V}$.

There are two cases.

Case 1: *There are no coordinates preceding $w_{k,\gamma}$ in w (i.e., $k = 1$ and $\gamma = 1$).*

Therefore,

$$\begin{aligned} \frac{(\sum_{i=1}^{k-1} \sum_{j=1}^{\psi_i} R\pi^{\lambda_k-\lambda_i} w_{i,j}) + (\sum_{j=1}^{\gamma-1} R w_{k,j}) + M^{\lambda_k}V}{M^{\lambda_k}V} &= \frac{M^{\lambda_k}V}{M^{\lambda_k}V} = 0, \text{ giving that} \\ \left| \frac{(\sum_{i=1}^{k-1} \sum_{j=1}^{\psi_i} R\pi^{\lambda_k-\lambda_i} w_{i,j}) + (\sum_{j=1}^{\gamma-1} R w_{k,j}) + M^{\lambda_k}V}{M^{\lambda_k}V} \right| &= 1 = q^{d_{\lambda_{k-1}} + \gamma - 1}. \end{aligned}$$

Case 2: *There are coordinates preceding $w_{k,\gamma}$ in w (i.e., either $k \neq 1$ or $\gamma \neq 1$).*

Observe via the proof of Lemma 2.12 that

$$\begin{aligned} &\{\pi^{\lambda_k-\lambda_i} w_{i,j} + M^{\lambda_k}V : i \in \{1, \dots, k-1\}, j \in \{1, \dots, \psi_i\}\} \cup \\ &\{w_{k,j} + M^{\lambda_k}V : j \in \{1, \dots, \gamma-1\}\} \end{aligned}$$

is an R/M -linearly independent subset of $M^{\lambda_k-1}V/M^{\lambda_k}V$ with $d_{\lambda_{k-1}} + \gamma - 1$ pairwise distinct elements. Thus, $\frac{(\sum_{i=1}^{k-1} \sum_{j=1}^{\psi_i} R\pi^{\lambda_k-\lambda_i} w_{i,j}) + (\sum_{j=1}^{\gamma-1} R w_{k,j}) + M^{\lambda_k}V}{M^{\lambda_k}V}$ is a $d_{\lambda_{k-1}} + \gamma - 1$ dimensional R/M -vector space, giving that

$$\left| \frac{(\sum_{i=1}^{k-1} \sum_{j=1}^{\psi_i} R\pi^{\lambda_k-\lambda_i} w_{i,j}) + (\sum_{j=1}^{\gamma-1} R w_{k,j}) + M^{\lambda_k}V}{M^{\lambda_k}V} \right| = q^{d_{\lambda_{k-1}} + \gamma - 1}.$$

As a result of the above calculations, we see that in all cases, the number of ways to choose $w_{k,\gamma}$ (assuming the earlier coordinates have been chosen "properly") is

$$\left| M^{\lambda_k-1} V \right| - \left| \frac{(\sum_{i=1}^{k-1} \sum_{j=1}^{\psi_i} R \pi^{\lambda_k - \lambda_i} w_{i,j}) + (\sum_{j=1}^{\gamma-1} R w_{k,j}) + M^{\lambda_k} V}{M^{\lambda_k} V} \right| \left| M^{\lambda_k} V \right| =$$

$$q^{n(e-\lambda_k+1)} - q^{d_{\lambda_{k-1}}+\gamma-1} q^{n(e-\lambda_k)} = q^{n(e-\lambda_k)} (q^n - q^{d_{\lambda_{k-1}}+\gamma-1})$$

Therefore, the number of tuples which realize $(d_1, d_2, \dots, d_e)$ is

$$\prod_{k=1}^{\beta} \prod_{\gamma=1}^{\psi_k} [q^{n(e-\lambda_k)} (q^n - q^{d_{\lambda_{k-1}}+\gamma-1})] = \prod_{k=1}^{\beta} [q^{\psi_k n(e-\lambda_k)} \prod_{\gamma=1}^{\psi_k} (q^n - q^{d_{\lambda_{k-1}}+\gamma-1})] =$$

$$\left[q^{\sum_{k=1}^{\beta} \psi_k n(e-\lambda_k)} \right] \left[\prod_{k=1}^{\beta} \prod_{\gamma=1}^{\psi_k} (q^n - q^{d_{\lambda_{k-1}}+\gamma-1}) \right].$$

Notice that as k runs over $\{1, \dots, \beta\}$ and γ runs over $\{1, \dots, \psi_k\}$, then $d_{\lambda_{k-1}} + \gamma$ runs over $\{1, \dots, d_{\lambda_{\beta}}\}$. Therefore, the assertion follows since

$$n(\sum_{k=1}^{\beta} \psi_k (e - \lambda_k)) = n\left(\sum_{k=1}^{\beta} (d_{\lambda_k} - d_{\lambda_{k-1}})(e - \lambda_k)\right) =$$

$$n\left(d_{\lambda_{\beta}}(e - \lambda_{\beta}) - d_{\lambda_{\beta-1}}(e - \lambda_{\beta}) + \left(\sum_{k=1}^{\beta-1} (d_{\lambda_k} - d_{\lambda_{k-1}})(e - \lambda_k)\right)\right) =$$

$$n\left(d_{\lambda_{\beta}}(e - \lambda_{\beta}) - d_{\lambda_{\beta-1}}(e - \lambda_{\beta}) + e\left(\sum_{k=1}^{\beta-1} d_{\lambda_k}\right) - \left(\sum_{k=1}^{\beta-1} d_{\lambda_k} \lambda_k\right) -\right.$$

$$\left. e\left(\sum_{k=1}^{\beta-2} d_{\lambda_k}\right) + \left(\sum_{k=1}^{\beta-2} d_{\lambda_k} \lambda_{k+1}\right)\right) =$$

$$n\left(d_{\lambda_{\beta}}(e - \lambda_{\beta}) + \left(\sum_{k=1}^{\beta-1} d_{\lambda_k} (\lambda_{k+1} - \lambda_k)\right)\right).$$

(b) Let z be a tuple which realizes $(d_1, d_2, \dots, d_e)$, and let W be the R -submodule of V generated by the coordinates of z . To determine the cardinality of the $\sim_{(d_1, d_2, \dots, d_e)}$ -equivalence class of z , it suffices, by Remark 2.27, to find the number of tuples which realize $(d_1, d_2, \dots, d_e)$ and have coordinates which are all contained in W . Thus, we count the number of ways to successively choose the coordinates of a tuple

$$w = (w_{1,1}, w_{1,2}, \dots, w_{1,\psi_1}, w_{2,1}, w_{2,2}, \dots, w_{2,\psi_2}, \dots, w_{\beta,1}, w_{\beta,2}, \dots, w_{\beta,\psi_{\beta}})$$

with coordinates in W so that, after all the coordinates have been chosen, w realizes $(d_1, d_2, \dots, d_e)$. For this purpose, fix $k \in \{1, \dots, \beta\}$ and $\gamma \in \{1, \dots, \psi_k\}$.

Also, assume that any coordinates preceding $w_{k,\gamma}$ in the tuple w have been chosen “properly”. In other words, assume that

$$w_{k',\gamma'} \in (W \cap M^{\lambda_{k'}-1}V) \setminus ((\sum_{i=1}^{k'-1} \sum_{j=1}^{\psi_i} R\pi^{\lambda_{k'}-\lambda_i} w_{i,j}) + (\sum_{j=1}^{\gamma'-1} R w_{k',j}) + M^{\lambda_{k'}}V)$$

for any $k' \in \{1, \dots, k-1\}$ and $\gamma' \in \{1, \dots, \psi_{k'}\}$, and that

$$w_{k,\gamma'} \in (W \cap M^{\lambda_k-1}V) \setminus ((\sum_{i=1}^{k-1} \sum_{j=1}^{\psi_i} R\pi^{\lambda_k-\lambda_i} w_{i,j}) + (\sum_{j=1}^{\gamma'-1} R w_{k,j}) + M^{\lambda_k}V)$$

for any $\gamma' \in \{1, \dots, \gamma-1\}$. As a consequence, observe that

$$\begin{aligned} W \cap ((\sum_{i=1}^{k-1} \sum_{j=1}^{\psi_i} R\pi^{\lambda_k-\lambda_i} w_{i,j}) + (\sum_{j=1}^{\gamma-1} R w_{k,j}) + M^{\lambda_k}V) = \\ (\sum_{i=1}^{k-1} \sum_{j=1}^{\psi_i} R\pi^{\lambda_k-\lambda_i} w_{i,j}) + (\sum_{j=1}^{\gamma-1} R w_{k,j}) + (W \cap M^{\lambda_k}V) \subseteq W \cap M^{\lambda_k-1}V. \end{aligned}$$

Thus, by Definition 2.10 (2), the number of ways to choose $w_{k,\gamma}$ from W is

$$\begin{aligned} |(W \cap M^{\lambda_k-1}V) \setminus ((\sum_{i=1}^{k-1} \sum_{j=1}^{\psi_i} R\pi^{\lambda_k-\lambda_i} w_{i,j}) + (\sum_{j=1}^{\gamma-1} R w_{k,j}) + (W \cap M^{\lambda_k}V))| = \\ |W \cap M^{\lambda_k-1}V| - |(\sum_{i=1}^{k-1} \sum_{j=1}^{\psi_i} R\pi^{\lambda_k-\lambda_i} w_{i,j}) + (\sum_{j=1}^{\gamma-1} R w_{k,j}) + (W \cap M^{\lambda_k}V)| = \\ |W \cap M^{\lambda_k-1}V| - \left| \frac{(\sum_{i=1}^{k-1} \sum_{j=1}^{\psi_i} R\pi^{\lambda_k-\lambda_i} w_{i,j}) + (\sum_{j=1}^{\gamma-1} R w_{k,j}) + (W \cap M^{\lambda_k}V)}{W \cap M^{\lambda_k}V} \right| |W \cap M^{\lambda_k}V|, \end{aligned}$$

with the last equality holding by Lagrange’s Theorem.

By Corollary 2.26 (2) and Proposition 2.17, we have that $|W \cap M^{\lambda_k-1}V| = q^{\sum_{i=\lambda_k}^e p_{W,i}} = q^{\sum_{i=\lambda_k}^e d_i}$ and $|W \cap M^{\lambda_k}V| = q^{\sum_{i=\lambda_k+1}^e p_{W,i}} = q^{\sum_{i=\lambda_k+1}^e d_i}$.

Next, we find the cardinality of $\frac{(\sum_{i=1}^{k-1} \sum_{j=1}^{\psi_i} R\pi^{\lambda_k-\lambda_i} w_{i,j}) + (\sum_{j=1}^{\gamma-1} R w_{k,j}) + (W \cap M^{\lambda_k}V)}{W \cap M^{\lambda_k}V}$.

There are two cases.

Case 1: *There are no coordinates preceding $w_{k,\gamma}$ in w (i.e., $k = 1$ and $\gamma = 1$).*

Therefore,

$$\begin{aligned} \frac{(\sum_{i=1}^{k-1} \sum_{j=1}^{\psi_i} R\pi^{\lambda_k-\lambda_i} w_{i,j}) + (\sum_{j=1}^{\gamma-1} R w_{k,j}) + (W \cap M^{\lambda_k}V)}{(W \cap M^{\lambda_k}V)} = \frac{W \cap M^{\lambda_k}V}{W \cap M^{\lambda_k}V} = 0, \text{ giving that} \\ \left| \frac{(\sum_{i=1}^{k-1} \sum_{j=1}^{\psi_i} R\pi^{\lambda_k-\lambda_i} w_{i,j}) + (\sum_{j=1}^{\gamma-1} R w_{k,j}) + (W \cap M^{\lambda_k}V)}{(W \cap M^{\lambda_k}V)} \right| = 1 = q^{d_{\lambda_{k-1}} + \gamma - 1}. \end{aligned}$$

Case 2: There are coordinates preceding $w_{k,\gamma}$ in w (i.e., either $k \neq 1$ or $\gamma \neq 1$).

Observe via the proof of Lemma 2.12 that

$$\{\pi^{\lambda_k - \lambda_i} w_{i,j} + M^{\lambda_k} V : i \in \{1, \dots, k-1\}, j \in \{1, \dots, \psi_i\}\} \cup \\ \{w_{k,j} + M^{\lambda_k} V : j \in \{1, \dots, \gamma-1\}\}$$

is an R/M -linearly independent subset of $M^{\lambda_k - 1} V / M^{\lambda_k} V$ with $d_{\lambda_{k-1}} + \gamma - 1$ pairwise distinct elements. Therefore, using the definition of linear independence, notice that

$$\{\pi^{\lambda_k - \lambda_i} w_{i,j} + (W \cap M^{\lambda_k} V) : i \in \{1, \dots, k-1\}, j \in \{1, \dots, \psi_i\}\} \cup \\ \{w_{k,j} + (W \cap M^{\lambda_k} V) : j \in \{1, \dots, \gamma-1\}\}$$

is an R/M -linearly independent subset of $(W \cap M^{\lambda_k - 1} V) / (W \cap M^{\lambda_k} V)$ with $d_{\lambda_{k-1}} + \gamma - 1$ pairwise distinct elements. Thus, $\frac{(\sum_{i=1}^{k-1} \sum_{j=1}^{\psi_i} R\pi^{\lambda_k - \lambda_i} w_{i,j}) + (\sum_{j=1}^{\gamma-1} R w_{k,j}) + (W \cap M^{\lambda_k} V)}{W \cap M^{\lambda_k} V}$ is a $d_{\lambda_{k-1}} + \gamma - 1$ dimensional R/M -vector space, giving that

$$\left| \frac{(\sum_{i=1}^{k-1} \sum_{j=1}^{\psi_i} R\pi^{\lambda_k - \lambda_i} w_{i,j}) + (\sum_{j=1}^{\gamma-1} R w_{k,j}) + (W \cap M^{\lambda_k} V)}{W \cap M^{\lambda_k} V} \right| = q^{d_{\lambda_{k-1}} + \gamma - 1}.$$

As a result of the above calculations, we see that in all cases, the number of ways to choose $w_{k,\gamma}$ (assuming that the earlier coordinates have been chosen “properly”) is

$$\left| W \cap M^{\lambda_k - 1} V \right| - \left| \frac{(\sum_{i=1}^{k-1} \sum_{j=1}^{\psi_i} R\pi^{\lambda_k - \lambda_i} w_{i,j}) + (\sum_{j=1}^{\gamma-1} R w_{k,j}) + (W \cap M^{\lambda_k} V)}{W \cap M^{\lambda_k} V} \right| \left| W \cap M^{\lambda_k} V \right| = \\ (q^{\sum_{i=\lambda_k}^e d_i}) - (q^{d_{\lambda_{k-1}} + \gamma - 1}) (q^{\sum_{i=\lambda_k+1}^e d_i}) = q^{\sum_{i=\lambda_k+1}^e d_i} (q^{\lambda_k} - q^{d_{\lambda_{k-1}} + \gamma - 1}).$$

Therefore, the number of tuples realizing $(d_1, d_2, \dots, d_e)$ having all coordinates in W is

$$\prod_{k=1}^{\beta} \prod_{\gamma=1}^{\psi_k} \left[(q^{\sum_{i=\lambda_k+1}^e d_i}) (q^{\lambda_k} - q^{d_{\lambda_{k-1}} + \gamma - 1}) \right] =$$

$$\prod_{k=1}^{\beta} \left[q^{\psi_k(\sum_{i=\lambda_k+1}^e d_i)} \left(\prod_{\gamma=1}^{\psi_k} (q^{d_{\lambda_k}} - q^{d_{\lambda_{k-1}}+\gamma-1}) \right) \right] =$$

$$q^{\sum_{k=1}^{\beta} \psi_k(\sum_{i=\lambda_k+1}^e d_i)} \left[\prod_{k=1}^{\beta} \prod_{\gamma=1}^{\psi_k} (q^{d_{\lambda_k}} - q^{d_{\lambda_{k-1}}+\gamma-1}) \right].$$

The assertion now follows from Definition 2.10 since

$$\begin{aligned} \sum_{k=1}^{\beta} \psi_k \left(\sum_{i=\lambda_k+1}^e d_i \right) &= \sum_{k=1}^{\beta} (d_{\lambda_k} - d_{\lambda_{k-1}}) \left(\sum_{i=\lambda_k+1}^e d_i \right) = \\ &= \left(\sum_{k=1}^{\beta} d_{\lambda_k} \left(\sum_{i=\lambda_k+1}^e d_i \right) \right) - \left(\sum_{k=1}^{\beta} d_{\lambda_{k-1}} \left(\sum_{i=\lambda_k+1}^e d_i \right) \right) = \\ &= \left(d_{\lambda_1} (d_{\lambda_1+1} + \dots + d_e) + d_{\lambda_2} (d_{\lambda_2+1} + \dots + d_e) + \dots + d_{\lambda_{\beta}} (d_{\lambda_{\beta}+1} + \dots + d_e) \right) - \\ &= \left(d_{\lambda_1} (d_{\lambda_2+1} + \dots + d_e) + d_{\lambda_2} (d_{\lambda_3+1} + \dots + d_e) + \dots + d_{\lambda_{\beta-1}} (d_{\lambda_{\beta}+1} + \dots + d_e) \right) = \\ &= d_{\lambda_1} (d_{\lambda_1+1} + \dots + d_{\lambda_2}) + d_{\lambda_2} (d_{\lambda_2+1} + \dots + d_{\lambda_3}) + \dots + d_{\lambda_{\beta-1}} (d_{\lambda_{\beta-1}+1} + \dots + d_{\lambda_{\beta}}) + \\ &= d_{\lambda_{\beta}} (d_{\lambda_{\beta}+1} + \dots + d_e) = d_{\lambda_1} (d_{\lambda_1+1} + \dots + d_{\lambda_2-1}) + d_{\lambda_1} d_{\lambda_2} + d_{\lambda_2} (d_{\lambda_2+1} + \dots + d_{\lambda_3-1}) + \\ &= d_{\lambda_2} d_{\lambda_3} + \dots + d_{\lambda_{\beta-1}} (d_{\lambda_{\beta-1}+1} + \dots + d_{\lambda_{\beta}-1}) + d_{\lambda_{\beta-1}} d_{\lambda_{\beta}} + d_{\lambda_{\beta}} (d_{\lambda_{\beta}+1} + \dots + d_e) = \\ &= d_{\lambda_1}^2 (\lambda_2 - \lambda_1 - 1) + d_{\lambda_1} d_{\lambda_2} + d_{\lambda_2}^2 (\lambda_3 - \lambda_2 - 1) + d_{\lambda_2} d_{\lambda_3} + \dots + d_{\lambda_{\beta-1}}^2 (\lambda_{\beta} - \lambda_{\beta-1} - 1) + d_{\lambda_{\beta-1}} d_{\lambda_{\beta}} + \\ &= d_{\lambda_{\beta}}^2 (e - \lambda_{\beta}) = \left(\sum_{k=1}^{\beta-1} d_{\lambda_k}^2 (\lambda_{k+1} - \lambda_k) \right) - \left(\sum_{k=1}^{\beta-1} d_{\lambda_k}^2 \right) + \left(\sum_{k=1}^{\beta-1} d_{\lambda_k} d_{\lambda_{k+1}} \right) + d_{\lambda_{\beta}}^2 (e - \lambda_{\beta}) = \\ &= d_{\lambda_{\beta}}^2 (e - \lambda_{\beta}) + \left(\sum_{k=1}^{\beta-1} d_{\lambda_k}^2 (\lambda_{k+1} - \lambda_k) \right) + \left(\sum_{k=1}^{\beta-1} d_{\lambda_k} (d_{\lambda_{k+1}} - d_{\lambda_k}) \right). \end{aligned}$$

(c) By Theorem 2.18, the number of submodules of V which realize $(d_1, d_2, \dots, d_e)$ is the ratio of the value found in (a) to the value found in (b). Thus, by multiplying both the numerator and the denominator of this quotient by $\prod_{k=1}^{\beta-1} \prod_{\gamma=1}^{d_{\lambda_k}} (q^{d_{\lambda_{k+1}}} - q^{\gamma-1})$ (and recalling our convention that empty products are 1), we can write the quotient in the form

$$\Lambda(d_1, d_2, \dots, d_e) := q^{\epsilon} \left[\prod_{k=1}^{\beta-1} \frac{\prod_{\gamma=1}^{d_{\lambda_k}} (q^{d_{\lambda_{k+1}}} - q^{\gamma-1})}{\prod_{\gamma=1}^{d_{\lambda_k}} (q^{d_{\lambda_k}} - q^{\gamma-1})} \right] \left[\frac{\prod_{\gamma=1}^{d_{\lambda_{\beta}}} (q^n - q^{\gamma-1})}{\prod_{\gamma=1}^{d_{\lambda_{\beta}}} (q^{d_{\lambda_{\beta}}} - q^{\gamma-1})} \right]$$

where $\epsilon = \epsilon_1 - \epsilon_2 =$

$$d_{\lambda_{\beta}} (n - d_{\lambda_{\beta}}) (e - \lambda_{\beta}) + \left(\sum_{k=1}^{\beta-1} d_{\lambda_k} (n - d_{\lambda_k}) (\lambda_{k+1} - \lambda_k) \right) - \left(\sum_{k=1}^{\beta-1} d_{\lambda_k} (d_{\lambda_{k+1}} - d_{\lambda_k}) \right).$$

By applying [4, Corollary 3.3] β times, we see that $\Lambda(d_1, d_2, \dots, d_e)$ is the evaluation at q of a monic polynomial in $Z[X]$ of degree

$$\begin{aligned} d_{\lambda_\beta}(n - d_{\lambda_\beta})(e - \lambda_\beta) + \left(\sum_{k=1}^{\beta-1} d_{\lambda_k}(n - d_{\lambda_k})(\lambda_{k+1} - \lambda_k) \right) - \left(\sum_{k=1}^{\beta-1} d_{\lambda_k}(d_{\lambda_{k+1}} - d_{\lambda_k}) \right) + \\ \left(\sum_{k=1}^{\beta-1} d_{\lambda_k}(d_{\lambda_{k+1}} - d_{\lambda_k}) \right) + d_{\lambda_\beta}(n - d_{\lambda_\beta}) = \\ \left(\sum_{k=1}^{\beta-1} d_{\lambda_k}(n - d_{\lambda_k})(\lambda_{k+1} - \lambda_k) \right) + d_{\lambda_\beta}(n - d_{\lambda_\beta})(e - \lambda_\beta + 1). \end{aligned}$$

Therefore, the assertion follows since

$$\begin{aligned} \left(\sum_{k=1}^{\beta-1} d_{\lambda_k}(n - d_{\lambda_k})(\lambda_{k+1} - \lambda_k) \right) + d_{\lambda_\beta}(n - d_{\lambda_\beta})(e - \lambda_\beta + 1) = \\ 0 + (\lambda_2 - \lambda_1)d_{\lambda_1}(n - d_{\lambda_1}) + (\lambda_3 - \lambda_2)d_{\lambda_2}(n - d_{\lambda_2}) + \dots + \\ (\lambda_\beta - \lambda_{\beta-1})d_{\lambda_{\beta-1}}(n - d_{\lambda_{\beta-1}}) + (e - \lambda_\beta + 1)d_{\lambda_\beta}(n - d_{\lambda_\beta}) = \sum_{i=1}^{\lambda_1-1} d_{\lambda_0}(n - d_{\lambda_0}) + \\ \sum_{i=\lambda_1}^{\lambda_2-1} d_{\lambda_1}(n - d_{\lambda_1}) + \dots + \sum_{i=\lambda_{\beta-1}}^{\lambda_\beta-1} d_{\lambda_{\beta-1}}(n - d_{\lambda_{\beta-1}}) + \sum_{i=\lambda_\beta}^e d_{\lambda_\beta}(n - d_{\lambda_\beta}) = \\ \sum_{i=1}^e d_i(n - d_i). \quad \square \end{aligned}$$

In the next corollary, we establish that Proposition 2.28 (c) is indeed a generalization of [4, Proposition 3.1 (b)], a formula for the number of k -dimensional ($k \in \{0, \dots, n\}$) subspaces of an n -dimensional vector space over a finite field.

COROLLARY 2.29 *Let $k \in \{0, 1, \dots, n\}$. The number of submodules of V which are R -free on a basis of k elements is*

$$q^{k(n-k)(e-1)} \left[\frac{\prod_{\gamma=1}^k (q^n - q^{\gamma-1})}{\prod_{\gamma=1}^k (q^k - q^{\gamma-1})} \right].$$

As a consequence, if $e = 1$ (i.e., R is a finite field of cardinality q), then the number of k -dimensional subspaces of the n -dimensional R -vector space V is

$$\frac{\prod_{\gamma=1}^k (q^n - q^{\gamma-1})}{\prod_{\gamma=1}^k (q^k - q^{\gamma-1})}.$$

Proof. If $k = 0$, then the assertion follows since, by convention, an empty product is 1.

Suppose that $k \in \{1, \dots, n\}$. By Corollary 2.21 and Proposition 2.28 (c), the number of submodules of V which are R -free on a basis of k elements is $\Lambda(k, k, \dots, k)$. Defining $\beta, \lambda_0, \lambda_1, \dots, \lambda_\beta, \psi_1, \dots, \psi_\beta$ for $(d_1, d_2, \dots, d_e) := (k, k, \dots, k)$ as in Definition 2.10 (2), observe that $\beta = 1$, $\lambda_1 = 1$, and $\psi_1 = d_{\lambda_1} = k$. Thus, by simplifying the formula in Proposition 2.28 (c), we see that

$$\Lambda(k, k, \dots, k) = q^{k(n-k)(e-1)} \left[\frac{\prod_{\gamma=1}^k (q^n - q^{\gamma-1})}{\prod_{\gamma=1}^k (q^k - q^{\gamma-1})} \right]. \quad \square$$

In [4, Theorem 3.4], Dobbs and Lancaster showed that the expected value of the dimension of an arbitrary subspace of an n -dimensional vector space over a finite field is $\frac{n}{2}$. We generalize this result by finding the expected value of the cardinality of a minimal generating set (i.e., the ν_R -value) of an arbitrary R -submodule of the given R -free module V (where R is the given finite SPIR). As in [1], we will also determine the expected value of the ν_R -value of an arbitrary R -submodule W of V when W is weighted by either $\nu_R(W)$ or the cardinality of W . More generally, in Chapter 3, we will determine all the above types of expected values (and others) when R is a finite principal ideal ring (PIR).

Let η_k denote the number of R -submodules W of V such that $\nu_R(W) = k$. Hence, $\eta_0 = 1$ and, in view of Corollary 2.16, for each $k \in \{1, \dots, n\}$,

$$\eta_k = \sum_{0 \leq d_1 \leq d_2 \leq \dots \leq d_{e-1} \leq k} \Lambda(d_1, d_2, \dots, d_{e-1}, k)$$

where $\Lambda(d_1, d_2, \dots, d_{e-1}, k)$ is given in Proposition 2.28 (c). Moreover, the (unweighted) expected value of the cardinality of a minimal generating set of an arbitrary R -submodule W of V is given by

$$E(\nu_R(W)) = \frac{\sum_{k=0}^n k \eta_k}{\sum_{k=0}^n \eta_k}.$$

Similarly, the expected value of the cardinality of a minimal generating set of an arbitrary R -submodule W of V , when W is weighted by $\nu_R(W)$, is given by

$$E(\nu_R(W)) = \frac{\sum_{k=0}^n k^2 \eta_k}{\sum_{k=0}^n k \eta_k}.$$

REMARK 2.30 When $e = 1$ (i.e., R is a finite field) and $k \in \{1, \dots, n\}$, observe via Corollary 2.29 that $\eta_k = \Lambda(k)$ is the number of k -dimensional subspaces of an n -dimensional vector space over R . Thus, the above method for expressing (unweighted) $E(\nu_R(W))$ generalizes the method in [4, Theorem 3.4], although we shall see in Remark 2.36 that $E(\nu_R(W))$ may depend on q (as well as n) when $e > 1$.

Since the cardinalities of R -submodules of V with the same ν_R -value may differ, one needs different notation when weighting by the cardinality of the R -submodule. For each $k \in \{0, \dots, n\}$, let $\eta'_k = \sum_{\nu_R(W)=k} |W|$. Hence $\eta'_0 = 1$ and, for each $k \in \{1, \dots, n\}$,

$$\eta'_k = \sum_{0 \leq d_1 \leq d_2 \leq \dots \leq d_{e-1} \leq k} \Lambda'(d_1, d_2, \dots, d_{e-1}, k),$$

with $\Lambda'(d_1, d_2, \dots, d_{e-1}, k) := q^{(\sum_{i=1}^{e-1} d_i) + k} \Lambda(d_1, d_2, \dots, d_{e-1}, k)$. (We have used Proposition 2.25 to note that the cardinality of any submodule which realizes $(d_1, d_2, \dots, d_{e-1}, k)$ is $q^{(\sum_{i=1}^{e-1} d_i) + k}$.) With this in hand, the expected value of the cardinality of a minimal generating set of an arbitrary R -submodule W of V , when W is weighted by cardinality, is given by

$$E(\nu_R(W)) = \frac{\sum_{k=0}^n k \eta'_k}{\sum_{k=0}^n \eta'_k}.$$

The next four results will help in the calculation of limiting values, as $q \rightarrow \infty$, of the various types of expected values which were introduced above. For this purpose, it is convenient, given $k \in \{1, \dots, n\}$, to define

$$\Gamma(k) := \max\{\deg(\Lambda(d_1, \dots, d_{e-1}, k)) : (d_1, \dots, d_{e-1}, k) \text{ ascending } e\text{-tuple}\}.$$

For sufficiently large q , the next proposition essentially determines which ascending e -tuples of the form $(d_1, d_2, \dots, d_{e-1}, k)$ have maximum Λ -value.

PROPOSITION 2.31 (a) Suppose that n is even and $k \in \{1, \dots, \frac{n}{2}\}$.

Then $\Gamma(k) = ek(n - k)$. Further, $\deg(\Lambda(d_1, d_2, \dots, d_{e-1}, k)) = \Gamma(k)$ if and only if $(d_1, d_2, \dots, d_{e-1}, k) = (k, k, \dots, k)$.

(b) Suppose that n is even and $k \in \{\frac{n}{2}, \dots, n\}$. Then

$\Gamma(k) = (e - 1)(\frac{n}{2})^2 + k(n - k)$. Further, $\deg(\Lambda(d_1, d_2, \dots, d_{e-1}, k)) = \Gamma(k)$ if and only if $(d_1, d_2, \dots, d_{e-1}, k) = (\frac{n}{2}, \frac{n}{2}, \dots, \frac{n}{2}, k)$.

(c) Suppose n is odd and $k \in \{1, \dots, \frac{n-1}{2}\}$. Then $\Gamma(k) = ek(n - k)$. Further, $\deg(\Lambda(d_1, d_2, \dots, d_{e-1}, k)) = \Gamma(k)$ if and only if $(d_1, d_2, \dots, d_{e-1}, k) = (k, k, \dots, k)$.

(d) Suppose n is odd and $k \in \{\frac{n+1}{2}, \dots, n\}$. Then $\Gamma(k) = (e - 1)(\frac{n^2-1}{4}) + k(n - k)$. Further, $\deg(\Lambda(d_1, d_2, \dots, d_{e-1}, k)) = \Gamma(k)$ if and only if $d_i \in \{\frac{n-1}{2}, \frac{n+1}{2}\}$ for each $i \in \{1, \dots, e - 1\}$ and $d_1 \leq d_2 \leq \dots \leq d_{e-1}$.

Proof. Given $k \in \{1, \dots, n\}$, by Proposition 2.28 (c), the degree of

$\Lambda(d_1, d_2, \dots, d_{e-1}, k)$ in q is $\sum_{i=1}^{e-1} d_i(n - d_i) + k(n - k)$ for any ascending e -tuple of the form $(d_1, d_2, \dots, d_{e-1}, k)$. In each case, we use this fact to find the ascending e -tuples where $\Gamma(k)$ is attained.

(a) Suppose that n is even and $k \in \{1, \dots, \frac{n}{2}\}$. Since

$$\max\{t(n - t) : t \in \{0, \dots, k\}\} = k(n - k),$$

we have that $\sum_{i=1}^{e-1} d_i(n - d_i) + k(n - k) \leq (e - 1)k(n - k) + k(n - k) = ek(n - k)$ for any ascending e -tuple of the form $(d_1, d_2, \dots, d_{e-1}, k)$, and equality holds if and only if $d_i = k$ for each $i \in \{1, \dots, e - 1\}$. Hence $\Gamma(k) = ek(n - k)$; and $\deg(\Lambda(d_1, \dots, d_{e-1}, k)) = \Gamma(k)$ if and only if $(d_1, \dots, d_{e-1}, k) = (k, k, \dots, k)$.

(b) Suppose that n is even and $k \in \{\frac{n}{2}, \dots, n\}$. Since

$$\max\{t(n - t) : t \in \{0, \dots, k\}\} = \frac{n}{2}(n - \frac{n}{2}) = (\frac{n}{2})^2,$$

observe that $\sum_{i=1}^{e-1} d_i(n - d_i) + k(n - k) \leq (e - 1)(\frac{n}{2})^2 + k(n - k)$ for any ascending e -tuple of the form $(d_1, \dots, d_{e-1}, k)$, and equality holds if and only if $d_i = \frac{n}{2}$ for each $i \in \{1, \dots, e - 1\}$. Hence $\Gamma(k) = (e - 1)(\frac{n}{2})^2 + k(n - k)$; and $\deg(\Lambda(d_1, \dots, d_{e-1}, k)) = \Gamma(k)$ if and only if $(d_1, \dots, d_{e-1}, k) = (\frac{n}{2}, \dots, \frac{n}{2}, k)$.

(c) Suppose that n is odd and $k \in \{1, \dots, \frac{n-1}{2}\}$. Since

$$\max\{t(n - t) : t \in \{0, \dots, k\}\} = k(n - k),$$

it follows that $\sum_{i=1}^{e-1} d_i(n - d_i) + k(n - k) \leq (e - 1)k(n - k) + k(n - k) = ek(n - k)$ for any ascending e -tuple of the form $(d_1, \dots, d_{e-1}, k)$, and equality holds if and only if $d_i = k$ for each $i \in \{1, \dots, e - 1\}$. Hence $\Gamma(k) = ek(n - k)$; and $\deg(\Lambda(d_1, \dots, d_{e-1}, k)) = \Gamma(k)$ if and only if $(d_1, \dots, d_{e-1}, k) = (k, k, \dots, k)$.

(d) Suppose n is odd and $k \in \{\frac{n+1}{2}, \dots, n\}$. Since

$\max\{t(n - t) : t \in \{0, \dots, k\}\} = \frac{n^2 - 1}{4}$ is achieved precisely when $t \in \{\frac{n-1}{2}, \frac{n+1}{2}\}$, observe that $\sum_{i=1}^{e-1} d_i(n - d_i) + k(n - k) \leq (e - 1)(\frac{n^2 - 1}{4}) + k(n - k)$ for any ascending e -tuple $(d_1, \dots, d_{e-1}, k)$, and equality holds if and only if $d_i \in \{\frac{n-1}{2}, \frac{n+1}{2}\}$ for each $i \in \{1, \dots, e - 1\}$. Hence $\Gamma(k) = (e - 1)(\frac{n^2 - 1}{4}) + k(n - k)$; and $\deg(\Lambda(d_1, \dots, d_{e-1}, k)) = \Gamma(k)$ if and only if $d_i \in \{\frac{n-1}{2}, \frac{n+1}{2}\}$ for each $i \in \{1, \dots, e - 1\}$ and $d_1 \leq d_2 \leq \dots \leq d_{e-1}$. $\square$

For sufficiently large q , the following corollary essentially determines which types of submodules are the most numerous in V . For convenience, we define

$\Gamma := \max\{\Gamma(k) : k \in \{1, \dots, n\}\}$. Notice that Γ is the maximum value of $\deg(\Lambda(d_1, d_2, \dots, d_e))$, as $(d_1, d_2, \dots, d_e)$ ranges over the set of all nonzero ascending e -tuples.

COROLLARY 2.32 (a) Suppose that n is even. Then $\Gamma = e \left(\frac{n}{2}\right)^2$; and $\deg(\Lambda(d_1, d_2, \dots, d_e)) = \Gamma$ if and only if $(d_1, d_2, \dots, d_e) = (\frac{n}{2}, \frac{n}{2}, \dots, \frac{n}{2})$.

(b) Suppose $n \geq 3$ is odd. Then $\Gamma = e \left(\frac{n^2-1}{4}\right)$; and $\deg(\Lambda(d_1, d_2, \dots, d_e)) = \Gamma$ if and only if $d_i \in \{\frac{n-1}{2}, \frac{n+1}{2}\}$ for each $i \in \{1, \dots, e\}$ and $d_1 \leq d_2 \leq \dots \leq d_e$.

Proof. The assertion of (a) follows from Proposition 2.31 (a) and (b). Indeed, one checks that in part (a) (resp., (b)) of Proposition 2.31, $\Gamma(k)$ is maximized only at $k = \frac{n}{2}$. Similarly, the assertion of (b) follows from the analysis supporting Proposition 2.31 (c) and (d). $\square$

The next result is the analogue of Proposition 2.31 for the case where R -submodules are weighted by their cardinalities. For convenience, given $k \in \{1, \dots, n\}$, define

$$\Gamma'(k) := \max\{\deg(\Lambda'(d_1, \dots, d_{e-1}, k)) : (d_1, \dots, d_{e-1}, k) \text{ ascending } e\text{-tuple}\}.$$

PROPOSITION 2.33 (a) Suppose that n is even and $k \in \{1, \dots, \frac{n}{2}\}$. Then $\Gamma'(k) = ek((n+1) - k)$. Further, $\deg(\Lambda'(d_1, d_2, \dots, d_{e-1}, k)) = \Gamma'(k)$ if and only if $(d_1, d_2, \dots, d_{e-1}, k) = (k, k, \dots, k)$.

(b) Suppose that n is even and $k \in \{\frac{n}{2} + 1, \dots, n\}$. Then $\Gamma'(k) = (e-1)\left(\frac{(n+1)^2-1}{4}\right) + k((n+1) - k)$. Further, $\deg(\Lambda'(d_1, d_2, \dots, d_{e-1}, k)) = \Gamma'(k)$ if and only if $d_i \in \{\frac{n}{2}, \frac{n}{2} + 1\}$ for each $i \in \{1, \dots, e-1\}$ and $d_1 \leq d_2 \leq \dots \leq d_{e-1}$.

(c) Suppose that n is odd and $k \in \{1, \dots, \frac{n+1}{2}\}$. Then $\Gamma'(k) = ek((n+1) - k)$. Further, $\deg(\Lambda'(d_1, d_2, \dots, d_{e-1}, k)) = \Gamma'(k)$ if and only if $(d_1, d_2, \dots, d_{e-1}, k) = (k, k, \dots, k)$.

(d) Suppose that n is odd and $k \in \{\frac{n+1}{2}, \dots, n\}$. Then $\Gamma'(k) = (e-1)(\frac{n+1}{2})^2 + k((n+1) - k)$. Further, $\deg(\Lambda'(d_1, d_2, \dots, d_{e-1}, k)) = \Gamma'(k)$ if and only if $(d_1, d_2, \dots, d_{e-1}, k) = (\frac{n+1}{2}, \frac{n+1}{2}, \dots, \frac{n+1}{2}, k)$.

Proof. Given $k \in \{1, \dots, n\}$, by Proposition 2.28 (c), the degree of $\Lambda'(d_1, d_2, \dots, d_{e-1}, k)$ in q is $\sum_{i=1}^{e-1} d_i((n+1) - d_i) + k((n+1) - k)$ for any ascending e -tuple $(d_1, d_2, \dots, d_{e-1}, k)$. In each case, we use this fact to find the ascending e -tuples where $\Gamma'(k)$ is attained.

(a) Suppose that n is even and $k \in \{1, \dots, \frac{n}{2}\}$. Since

$$\max\{t((n+1) - t) : t \in \{0, \dots, k\}\} = k((n+1) - k),$$

it follows that

$$\begin{aligned} \sum_{i=1}^{e-1} d_i((n+1) - d_i) + k((n+1) - k) &\leq \\ (e-1)(k((n+1) - k)) + k((n+1) - k) &= ek((n+1) - k) \end{aligned}$$

for any ascending e -tuple of the form $(d_1, \dots, d_{e-1}, k)$, with equality holding if and only if $d_i = k$ for each $i \in \{1, \dots, e-1\}$. Hence $\Gamma'(k) = ek((n+1) - k)$; and $\deg(\Lambda'(d_1, \dots, d_{e-1}, k)) = \Gamma'(k)$ if and only if $(d_1, \dots, d_{e-1}, k) = (k, k, \dots, k)$.

(b) Suppose that n is even and $k \in \{\frac{n}{2} + 1, \dots, n\}$. Since

$\max\{t((n+1) - t) : t \in \{0, \dots, k\}\} = \frac{(n+1)^2 - 1}{4}$ is achieved only when $t \in \{\frac{n}{2}, \frac{n}{2} + 1\}$, observe that $\sum_{i=1}^{e-1} d_i((n+1) - d_i) + k((n+1) - k) \leq (e-1)(\frac{(n+1)^2 - 1}{4}) + k((n+1) - k)$ for any ascending e -tuple $(d_1, \dots, d_{e-1}, k)$, with equality holding if and only if $d_i \in \{\frac{n}{2}, \frac{n}{2} + 1\}$ for each $i \in \{1, \dots, e-1\}$. Hence $\Gamma'(k) = (e-1)(\frac{(n+1)^2 - 1}{4}) + k((n+1) - k)$;

and $\deg(\Lambda'(d_1, \dots, d_{e-1}, k)) = \Gamma'(k)$ if and only if $d_i \in \{\frac{n}{2}, \frac{n}{2} + 1\}$ for each $i \in \{1, \dots, e-1\}$ and $d_1 \leq d_2 \leq \dots \leq d_{e-1}$.

(c) Suppose that n is odd and $k \in \{1, \dots, \frac{n+1}{2}\}$. Since

$$\max\{t((n+1)-t) : t \in \{0, \dots, k\}\} = k((n+1)-k),$$

we have that

$$\begin{aligned} \sum_{i=1}^{e-1} d_i((n+1)-d_i) + k((n+1)-k) &\leq \\ (e-1)k((n+1)-k) + k((n+1)-k) &= ek((n+1)-k) \end{aligned}$$

for any ascending e -tuple of the form $(d_1, d_2, \dots, d_{e-1}, k)$, with equality holding if and only if $d_i = k$ for each $i \in \{1, \dots, e-1\}$. Hence $\Gamma'(k) = ek((n+1)-k)$; and $\deg(\Lambda'(d_1, \dots, d_{e-1}, k)) = \Gamma'(k)$ if and only if $(d_1, \dots, d_{e-1}, k) = (k, k, \dots, k)$.

(d) Suppose that n is odd and $k \in \{\frac{n+1}{2}, \dots, n\}$. Since

$$\max\{t((n+1)-t) : t \in \{0, \dots, k\}\} = (\frac{n+1}{2})^2,$$

observe that $\sum_{i=1}^{e-1} d_i((n+1)-d_i) + k((n+1)-k) \leq (e-1)(\frac{n+1}{2})^2 + k((n+1)-k)$ for any ascending e -tuple of the form $(d_1, \dots, d_{e-1}, k)$, with equality holding if and only if $d_i = \frac{n+1}{2}$ for each $i \in \{1, \dots, e-1\}$. Hence $\Gamma'(k) = (e-1)(\frac{n+1}{2})^2 + k((n+1)-k)$; and $\deg(\Lambda'(d_1, \dots, d_{e-1}, k)) = \Gamma'(k)$ if and only if $(d_1, \dots, d_{e-1}, k) = (\frac{n+1}{2}, \dots, \frac{n+1}{2}, k)$.

□

For sufficiently large q , the following corollary essentially determines which types of submodules are most numerous in V when weighted by cardinality. For convenience, define $\Gamma' := \max\{\Gamma'(k) : k \in \{1, \dots, n\}\}$. Notice that Γ' is the maximum value of $\deg(\Lambda'(d_1, d_2, \dots, d_e))$, as $(d_1, d_2, \dots, d_e)$ ranges over the set of all nonzero ascending e -tuples.

COROLLARY 2.34 (a) Suppose n is even. Then $\Gamma' = e \left(\frac{(n+1)^2 - 1}{4} \right)$; and $\deg(\Lambda'(d_1, d_2, \dots, d_e)) = \Gamma'$ if and only if $d_i \in \{\frac{n}{2}, \frac{n}{2} + 1\}$ for each $i \in \{1, \dots, e\}$ and $d_1 \leq d_2 \leq \dots \leq d_e$.

(b) Suppose n is odd. Then $\Gamma' = e \left(\frac{n+1}{2} \right)^2$; and $\deg(\Lambda'(d_1, d_2, \dots, d_e)) = \Gamma'$ if and only if $(d_1, d_2, \dots, d_e) = (\frac{n+1}{2}, \frac{n+1}{2}, \dots, \frac{n+1}{2})$.

Proof. The assertion of (a) follows from Proposition 2.33 (a) and (b). Indeed, one checks that $\Gamma'(k)$ is maximized only at $k = \frac{n}{2}$ and $k = \frac{n}{2} + 1$. Similarly, the assertion of (b) follows from the analysis supporting Proposition 2.33 (c) and (d). $\square$

In [1], Dobbs found the limiting expected value of the dimension of an arbitrary subspace, weighted by dimension (resp., weighted by cardinality), of a finite-dimensional vector space over a finite field, as the cardinality of the finite field goes to infinity. Further, in [2], Dobbs found the limiting expected value of the ν_S -value of an arbitrary (unweighted) S -submodule of a module which is S -free on a basis of two elements, where S is a finite special principal ideal ring having maximal ideal with nilpotency index 2, the limit being taken as the cardinality of the residue field of S goes to infinity. The next three theorems are generalizations of these results since our ambient V is an R -free module of arbitrary finite rank over the given arbitrary finite special principal ideal ring R .

THEOREM 2.35 (unweighted) As W varies over the set of R -submodules of V ,

$$\lim_{q \rightarrow \infty} E(\nu_R(W)) = \begin{cases} \frac{n}{2} & \text{if } n \text{ is even} \\ \frac{n}{2} + \frac{1}{2} - \frac{1}{e+1} & \text{if } n \text{ is odd.} \end{cases}$$

Proof. In both cases, it will be helpful to recall that $E(\nu_R(W)) = \frac{\sum_{k=0}^n k \eta_k}{\sum_{k=0}^n \eta_k}$.

Case 1: Suppose n is odd. When $n = 1$, it will follow from Proposition 4.1 (a) that $\lim_{q \rightarrow \infty} E(\nu_R(W)) = \frac{e}{e+1} = \frac{1}{2} + \frac{1}{2} - \frac{1}{e+1}$. Therefore, assume that $n \geq 3$. With this in hand, observe via Corollary 2.32 (b) that $\sum_{k=0}^n \eta_k$ is a polynomial in q of degree $e \left(\frac{n^2-1}{4} \right)$ with leading coefficient $e + 1$. (The number of ascending e -tuples with entries in $\{\frac{n-1}{2}, \frac{n+1}{2}\}$ is $e + 1$. Of these, e end with $d_e = \frac{n+1}{2}$, and only one ends with $d_e = \frac{n-1}{2}$.) Similarly, $\sum_{k=0}^n k\eta_k$ is a polynomial in q of degree $e \left(\frac{n^2-1}{4} \right)$ with leading coefficient $\left(\frac{n-1}{2} \right) + e \left(\frac{n+1}{2} \right)$. Thus, $\lim_{q \rightarrow \infty} E(\nu_R(W)) = \frac{(\frac{n-1}{2}) + e(\frac{n+1}{2})}{e+1} = \frac{n(e+1) + (e-1)}{2(e+1)} = \frac{n}{2} + \frac{e-1}{2(e+1)} = \frac{n}{2} + \frac{1}{2} - \frac{1}{e+1}$.

Case 2: Suppose n is even. Observe via Corollary 2.32 (a) that $\sum_{k=0}^n k\eta_k$ is a polynomial in q of degree $e \left(\frac{n}{2} \right)^2$ with leading coefficient $\frac{n}{2}$. Similarly, $\sum_{k=0}^n \eta_k$ is a monic polynomial in q of degree $e \left(\frac{n}{2} \right)^2$. Hence, $\lim_{q \rightarrow \infty} E(\nu_R(W)) = \frac{n}{2}$. $\square$

REMARK 2.36 When $e = 1$ (i.e., R is a finite field), it was shown in [1, Example 2.1] that the expected dimension of an arbitrary subspace of an n -dimensional vector space over R may depend on q (as well as n) when weighting by cardinality or dimension. Further, in contrast to the situation when R is a field [4, Theorem 3.4], $E(\nu_R(W))$ in the “unweighted” context of Theorem 2.35 may also depend on both q and n when $e > 1$. For instance, one can use Proposition 2.28 (c) to check that if $n = 3$ and $e = 2$, $E(\nu_R(W))$ is given by

$$E(\nu_R(W)) = \frac{5q^4 + 7q^3 + 16q^2 + 11q + 15}{3q^4 + 4q^3 + 8q^2 + 5q + 7}.$$

For these reasons, as in [2], we are subjecting the various expected values to the limiting process $\lim_{q \rightarrow \infty}$.

THEOREM 2.37 (*weighting by ν_R -value*) Let W vary over the set of R -submodules of V . When W is weighted $\nu_R(W)$,

$$\lim_{q \rightarrow \infty} E(\nu_R(W)) = \begin{cases} \frac{n}{2} & \text{if } n \text{ is even} \\ \frac{n}{2} + \frac{1}{2} - \frac{1}{e+1} + \frac{2e}{(e+1)((e+1)(n+1)-2)} & \text{if } n \text{ is odd.} \end{cases}$$

Proof. Recall that $E(\nu_R(W)) = \frac{\sum_{k=0}^n k^2 \eta_k}{\sum_{k=0}^n k \eta_k}$ when the R -submodule W is weighted by $\nu_R(W)$.

Case 1: Suppose n is odd. When $n = 1$, it will follow from Proposition 4.1 (b) that $\lim_{q \rightarrow \infty} E(\nu_R(W)) = 1 = \frac{1}{2} + \frac{1}{2} - \frac{1}{e+1} + \frac{2e}{(e+1)((e+1)(1+1)-2)}$. Therefore, assume that $n \geq 3$. With this in hand, observe via Corollary 2.32 (b) that $\sum_{k=0}^n k \eta_k$ is a polynomial in q of degree $e \left(\frac{n^2-1}{4} \right)$ with leading coefficient $\left(\frac{n-1}{2} \right) + e \left(\frac{n+1}{2} \right)$. (The number of ascending e -tuples with entries in $\{ \frac{n-1}{2}, \frac{n+1}{2} \}$ is $e+1$. Of these, e end with $d_e = \frac{n+1}{2}$, and only one ends with $d_e = \frac{n-1}{2}$.) Similarly, $\sum_{k=0}^n k^2 \eta_k$ is a polynomial in q of degree $e \left(\frac{n^2-1}{4} \right)$ with leading coefficient $\left(\frac{n-1}{2} \right)^2 + e \left(\frac{n+1}{2} \right)^2$. Thus,

$$\begin{aligned} \lim_{q \rightarrow \infty} E(\nu_R(W)) &= \frac{\left(\frac{n-1}{2} \right)^2 + e \left(\frac{n+1}{2} \right)^2}{\left(\frac{n-1}{2} \right) + e \left(\frac{n+1}{2} \right)} = \frac{(n-1)^2 + e(n+1)^2}{2((n-1) + e(n+1))} = \frac{(e+1)n^2 + 2(e-1)n + (e+1)}{2(e+1)n + 2(e-1)} = \\ &= \frac{n}{2} + \frac{e-1}{2(e+1)} + \frac{(e+1) - \left(\frac{e-1}{e+1} \right)}{2(e+1)n + 2(e-1)} = \frac{n}{2} + \frac{1}{2} - \frac{1}{e+1} + \frac{2e}{(e+1)((e+1)(n+1)-2)}. \end{aligned}$$

Case 2: Suppose n is even. Observe via Corollary 2.32 (a) that $\sum_{k=0}^n k \eta_k$ is a polynomial in q of degree $e \left(\frac{n}{2} \right)^2$ with leading coefficient $\frac{n}{2}$. Similarly, $\sum_{k=0}^n k^2 \eta_k$ is a polynomial in q of degree $e \left(\frac{n}{2} \right)^2$ with leading coefficient $\left(\frac{n}{2} \right)^2$. Hence,

$$\lim_{q \rightarrow \infty} E(\nu_R(W)) = \frac{\left(\frac{n}{2} \right)^2}{\frac{n}{2}} = \frac{n}{2}. \quad \square$$

THEOREM 2.38 (*weighting by cardinality*) *Let W vary over the set of R -submodules of V . When W is weighted by cardinality,*

$$\lim_{q \rightarrow \infty} E(\nu_R(W)) = \begin{cases} \frac{n+1}{2} + \frac{1}{2} - \frac{1}{e+1} & \text{if } n \text{ is even} \\ \frac{n+1}{2} & \text{if } n \text{ is odd.} \end{cases}$$

Proof. In both cases, it will be helpful to recall that $E(\nu_R(W)) = \frac{\sum_{k=0}^n k\eta'_k}{\sum_{k=0}^n \eta'_k}$ when weighting by the cardinality of the R -submodule W .

Case 1: Suppose n is even. Observe via Corollary 2.34 (a) that $\sum_{k=0}^n \eta'_k$ is a polynomial in q of degree $e \left(\frac{(n+1)^2-1}{4} \right)$ with leading coefficient $e+1$. (The number of ascending e -tuples with entries in $\{\frac{n}{2}, \frac{n}{2}+1\}$ is $e+1$. Of these, e end with $d_e = \frac{n}{2}+1$, and only one ends with $d_e = \frac{n}{2}$.) Similarly, $\sum_{k=0}^n k\eta'_k$ is a polynomial in q of degree $e \left(\frac{(n+1)^2-1}{4} \right)$ with leading coefficient $\left(\frac{n}{2} \right) + e \left(\frac{n}{2} + 1 \right)$. Thus,

$$\begin{aligned} \lim_{q \rightarrow \infty} E(\nu_R(W)) &= \frac{\left(\frac{n}{2} \right) + e \left(\frac{n}{2} + 1 \right)}{e+1} = \frac{\left(\frac{n}{2} \right) + e \left(\frac{n+2}{2} \right)}{e+1} = \frac{n+e(n+2)}{2(e+1)} = \frac{n(e+1)+2e}{2(e+1)} = \frac{n}{2} + \frac{e}{e+1} = \\ &= \frac{n+1}{2} - \frac{1}{2} + \frac{e}{e+1} = \frac{n+1}{2} + \frac{e-1}{2(e+1)} = \frac{n+1}{2} + \frac{1}{2} - \frac{1}{e+1}. \end{aligned}$$

Case 2: Suppose n is odd. Observe via Corollary 2.34 (b) that $\sum_{k=0}^n k\eta'_k$ is a polynomial in q of degree $e \left(\frac{n+1}{2} \right)^2$ with leading coefficient $\frac{n+1}{2}$, and $\sum_{k=0}^n \eta'_k$ is a monic polynomial in q of degree $e \left(\frac{n+1}{2} \right)^2$. Hence, $\lim_{q \rightarrow \infty} E(\nu_R(W)) = \frac{n+1}{2}$. $\square$

CHAPTER III

ON THE EXPECTED NUMBER OF GENERATORS OF A SUBMODULE OF A FREE MODULE OVER A FINITE PRINCIPAL IDEAL RING

We begin with a result which will allow us to use results from Chapter 2 to calculate limiting expected values when the ambient ring is a finite PIR.

PROPOSITION 3.1 *Let S and T be rings. Let R be the ring $S \times T$. If A is an S -module, B is an T -module, and M is the R -module $A \oplus B$, then $\nu_R(M) = \max\{\nu_S(A), \nu_T(B)\}$.*

Proof. We first show that $\nu_R(M) \leq \max\{\nu_S(A), \nu_T(B)\}$. Let k_1 and k_2 denote $\nu_S(A)$ and $\nu_T(B)$ respectively. Without loss of generality, $k_1 \leq k_2 < \infty$. Thus, there exist an S -generating subset $\{a_i : i \in \{1, \dots, k_1\}\}$ of A and T -generating subset $\{b_i : i \in \{1, \dots, k_2\}\}$ of B . A straightforward calculation shows that $\{(a_i, b_i) : i \in \{1, \dots, k_1\}\} \cup \{(0, b_i) : i \in \{k_1 + 1, \dots, k_2\}\}$ is an R -generating subset of M , and so $\nu_R(M) \leq \max\{\nu_S(A), \nu_T(B)\}$.

For the reverse inequality, we may suppose, without loss of generality, that $k := \nu_R(M) < \infty$. Thus, there exists a R -generating subset $\{(a_i, b_i) \in M : i \in \{1, \dots, k\}\}$. Observe that $\{a_i : i \in \{1, \dots, k\}\}$ is an S -generating subset of A and $\{b_i : i \in \{1, \dots, k\}\}$ is a T -generating subset of B , giving that $\nu_R(M) \geq \max\{\nu_S(A), \nu_T(B)\}$. $\square$

We pause to establish **running hypotheses and notation for the rest of this chapter**. Let $R = R_1 \times \dots \times R_m$, where, for each $i \in \{1, \dots, m\}$, R_i is an SPIR (possibly a field). Let $M_i = R\pi_i$ denote the maximal ideal of R_i , and let e_i denote the index of nilpotency of M_i . Let V be a module which is R -free on a basis

of n elements ($1 \leq n < \infty$). Since the category of R -modules is equivalent to the product of the categories of R_i -modules ($1 \leq i \leq m$), V is uniquely expressible as an internal direct sum $V = \bigoplus_{i=1}^m V_i$, for suitable R_i -modules V_i . In view of the canonical R -module isomorphism $R^{(n)} \cong R_1^{(n)} \times \dots \times R_m^{(n)}$, it follows that $V_i \cong R_i^{(n)}$; that is, V_i is a free R_i -module of rank n , for each i . Observe also that every R -submodule of V is an internal direct sum of the form $W_1 \oplus \dots \oplus W_m$, where W_i is an R_i -submodule of V_i for each i .

The next two propositions will be used later in this chapter to find various types of expected values over finite PIRs.

PROPOSITION 3.2 *Let W be an R -submodule of V ; put $k = \nu_R(W)$. As above, express $W = W_1 \oplus W_2 \oplus \dots \oplus W_m$, where W_i is an R_i -submodule of V_i for each i . Then W is R -free on a basis of k elements if and only if each W_i is R_i -free on a basis of k elements.*

Proof. Since R is a PIR, $k \leq \nu_R(V) = n$, and so $k < \infty$. As the assertion follows trivially when $W = 0$, we assume that $W \neq 0$. For the “if” assertion, observe that $W_i \cong R_i^{(k)}$ for each i , and so $W \cong R_1^{(k)} \times \dots \times R_m^{(k)} \cong R^{(k)}$ is, indeed, R -free on a basis of k elements. The “only if” assertion follows since $W_i \cong R_i \otimes_R W$ for each i .
□

PROPOSITION 3.3 *Let W be an R -submodule of V , expressed as usual as $W = W_1 \oplus W_2 \oplus \dots \oplus W_m$, where W_i is an R_i -module. For each i , let $p_{W_i,1}$ denote the R_i/M_i -dimension of the image of $\Delta_{W_i}^{(1)}$ in Definition 2.4 using the base ring R_i and ambient R_i -free module V_i . Then the highest rank attained by an R -free submodule of W is $\min\{p_{W_i,1} : i \in \{1, \dots, m\}\}$.*

Proof. Using Propositions 3.2 and 2.24, observe that the rank of any R -free submodule of W is less than or equal to $z := \min\{p_{W_i,1} : i \in \{1, \dots, m\}\}$. It remains only to find an R -submodule of W which is R -free on a basis of z elements.

For each i , Proposition 2.24 shows that W_i contains an R_i -submodule F_i which is R_i -free on a basis of $p_{W_i,1}$ elements. Choose F'_i to be an R_i -submodule of F_i which is R_i -free on a basis of z elements. Thus, by Proposition 3.2, $F'_1 \oplus \dots \oplus F'_m$ is a submodule of W which is R -free on a basis of z elements. $\square$

For the rest of this chapter, we **add to the running hypotheses** that R is finite, with $q_i := |R_i/M_i|$ for each $i \in \{1, \dots, m\}$. Observe via [7, Theorem 33, p. 245] that any finite PIR can be expressed as an internal direct product of finitely many finite SPIRs. In particular, any finite PIR can serve as the ambient ring R in the following discussion.

In order to calculate various expected values over the finite PIR R , we introduce the following notation. For each $i \in \{1, \dots, m\}$ and $j \in \{0, \dots, n\}$, let $\eta_{i,j}$ denote the number of R_i -submodules of V_i with ν_{R_i} -value j . Thus, for each $i \in \{1, \dots, m\}$, $\eta_{i,0} = 1$, and, for each $j \in \{1, \dots, n\}$, we have that $\eta_{i,j} = \sum_{0 \leq d_1 \leq \dots \leq d_{e_i-1} \leq j} \Lambda_i(d_1, d_2, \dots, d_{e_i-1}, j)$, where Λ_i denotes Λ in Proposition 2.28 (c) using R_i as the base ring and V_i as the ambient R_i -free module. For each $k \in \{0, \dots, n\}$, let η_k denote the number of R -submodules of V with ν_R -value k . Thus, for each $k \in \{0, \dots, n\}$, Proposition 3.1 yields that

$$\eta_k = (\sum_{j=0}^k \eta_{1,j})(\sum_{j=0}^k \eta_{2,j}) \dots (\sum_{j=0}^k \eta_{m,j}) - (\sum_{j=0}^{k-1} \eta_{1,j})(\sum_{j=0}^{k-1} \eta_{2,j}) \dots (\sum_{j=0}^{k-1} \eta_{m,j}).$$

In the next two lemmas, we find expected values stated in terms of the above notation. Readers interested in practical computations of expected values should keep in mind that each $\eta_{i,j}$ can be found in terms of q_i and e_i (and the ascending

e_i -tuples) using Proposition 2.28 (c).

LEMMA 3.4 (*unweighted*) *As W varies over the set of R -submodules of V ,*

$$E(\nu_R(W)) = \frac{n(\sum_{j=0}^n \eta_{1,j}) \cdots (\sum_{j=0}^n \eta_{m,j}) - (\sum_{k=0}^{n-1} (\sum_{j=0}^k \eta_{1,j}) \cdots (\sum_{j=0}^k \eta_{m,j}))}{(\sum_{j=0}^n \eta_{1,j}) \cdots (\sum_{j=0}^n \eta_{m,j})}.$$

Proof. For each $k \in \{0, \dots, n\}$, let a_k denote $(\sum_{j=0}^k \eta_{1,j})(\sum_{j=0}^k \eta_{2,j}) \cdots (\sum_{j=0}^k \eta_{m,j})$; put $a_{-1} = 0$. By the above remarks, $\eta_k = a_k - a_{k-1}$ for each k . By the discussion preceding Remark 2.30, it follows that

$$\begin{aligned} E(\nu_R(W)) &= \frac{\sum_{k=0}^n k \eta_k}{\sum_{k=0}^n \eta_k} = \frac{\sum_{k=1}^n k(a_k - a_{k-1})}{a_0 + \sum_{k=1}^n (a_k - a_{k-1})} = \frac{(\sum_{k=1}^n k a_k) - (\sum_{k=1}^n k a_{k-1})}{(\sum_{k=0}^n a_k) - (\sum_{k=1}^n a_{k-1})} = \\ &= \frac{(\sum_{k=1}^n k a_k) - (\sum_{k=0}^{n-1} (k+1) a_k)}{(\sum_{k=0}^n a_k) - (\sum_{k=0}^{n-1} a_k)} = \frac{n a_n - (\sum_{k=0}^{n-1} a_k)}{a_n} = \\ &= \frac{n(\sum_{j=0}^n \eta_{1,j}) \cdots (\sum_{j=0}^n \eta_{m,j}) - (\sum_{k=0}^{n-1} (\sum_{j=0}^k \eta_{1,j}) \cdots (\sum_{j=0}^k \eta_{m,j}))}{(\sum_{j=0}^n \eta_{1,j}) \cdots (\sum_{j=0}^n \eta_{m,j})}. \quad \square \end{aligned}$$

LEMMA 3.5 (*weighting by ν_R -value*) *Let W vary over the set of R -submodules of V . When W is weighted by $\nu_R(W)$,*

$$E(\nu_R(W)) = \frac{n^2(\sum_{j=0}^n \eta_{1,j}) \cdots (\sum_{j=0}^n \eta_{m,j}) - (\sum_{k=0}^{n-1} (2k+1)(\sum_{j=0}^k \eta_{1,j}) \cdots (\sum_{j=0}^k \eta_{m,j}))}{n(\sum_{j=0}^n \eta_{1,j}) \cdots (\sum_{j=0}^n \eta_{m,j}) - (\sum_{k=0}^{n-1} (\sum_{j=0}^k \eta_{1,j}) \cdots (\sum_{j=0}^k \eta_{m,j}))}.$$

Proof. As above, for each $k \in \{0, \dots, n\}$, let a_k denote $(\sum_{j=0}^k \eta_{1,j}) \cdots (\sum_{j=0}^k \eta_{m,j})$; and put $a_{-1} = 0$. When W is weighted by $\nu_R(W)$, by the discussion preceding Remark 2.30, we have that

$$\begin{aligned} E(\nu_R(W)) &= \frac{\sum_{k=0}^n k^2 \eta_k}{\sum_{k=0}^n k \eta_k} = \frac{\sum_{k=1}^n k^2 (a_k - a_{k-1})}{\sum_{k=1}^n k (a_k - a_{k-1})} = \frac{(\sum_{k=1}^n k^2 a_k) - (\sum_{k=1}^n k^2 a_{k-1})}{(\sum_{k=1}^n k a_k) - (\sum_{k=1}^n k a_{k-1})} = \\ &= \frac{(\sum_{k=1}^n k^2 a_k) - (\sum_{k=0}^{n-1} (k+1)^2 a_k)}{(\sum_{k=1}^n k a_k) - (\sum_{k=0}^{n-1} (k+1) a_k)} = \frac{(\sum_{k=1}^n k^2 a_k) - ((\sum_{k=1}^{n-1} k^2 a_k) + 2(\sum_{k=0}^{n-1} k a_k) + (\sum_{k=0}^{n-1} a_k))}{(\sum_{k=1}^n k a_k) - ((\sum_{k=1}^{n-1} k a_k) + (\sum_{k=0}^{n-1} a_k))} = \\ &= \frac{n^2 a_n - (\sum_{k=0}^{n-1} (2k+1) a_k)}{n a_n - (\sum_{k=0}^{n-1} a_k)} = \frac{n^2(\sum_{j=0}^n \eta_{1,j}) \cdots (\sum_{j=0}^n \eta_{m,j}) - (\sum_{k=0}^{n-1} (2k+1)(\sum_{j=0}^k \eta_{1,j}) \cdots (\sum_{j=0}^k \eta_{m,j}))}{n(\sum_{j=0}^n \eta_{1,j}) \cdots (\sum_{j=0}^n \eta_{m,j}) - (\sum_{k=0}^{n-1} (\sum_{j=0}^k \eta_{1,j}) \cdots (\sum_{j=0}^k \eta_{m,j}))}. \quad \square \end{aligned}$$

In order to calculate $E(\nu_R(W))$ when W is weighted by cardinality, we introduce the following notation. For each $i \in \{1, \dots, m\}$ and $j \in \{0, \dots, n\}$, let $\eta'_{i,j}$ denote the sum of cardinalities of all R_i -submodules of V_i with ν_{R_i} -value j . Thus, for each $i \in \{1, \dots, m\}$, $\eta'_{i,0} = 1$, and, for each $j \in \{1, \dots, n\}$, by Propositions 2.25 and 2.28 (c), we have that $\eta'_{i,j} = \sum_{0 \leq d_1 \leq \dots \leq d_{e_i-1} \leq j} \Lambda'_i(d_1, d_2, \dots, d_{e_i-1}, j) = \sum_{0 \leq d_1 \leq \dots \leq d_{e_i-1} \leq j} q^{(\sum_{s=1}^{e_i-1} d_s) + j} \Lambda_i(d_1, d_2, \dots, d_{e_i-1}, j)$, where Λ_i denotes Λ using R_i as the base ring and V_i as the ambient R_i -free module. For $k \in \{0, \dots, n\}$, let η'_k denote the sum of cardinalities of all R -submodules of V with ν_R -value k . Hence, for each $k \in \{0, \dots, n\}$, observe via Proposition 3.1 (and the distributive property) that

$$\eta'_k = (\sum_{j=0}^k \eta'_{1,j})(\sum_{j=0}^k \eta'_{2,j}) \cdots (\sum_{j=0}^k \eta'_{m,j}) - (\sum_{j=0}^{k-1} \eta'_{1,j})(\sum_{j=0}^{k-1} \eta'_{2,j}) \cdots (\sum_{j=0}^{k-1} \eta'_{m,j}).$$

LEMMA 3.6 (*weighting by cardinality*) *Let W vary over the set of R -submodules of V . When W is weighted by the cardinality,*

$$E(\nu_R(W)) = \frac{n(\sum_{j=0}^n \eta'_{1,j}) \cdots (\sum_{j=0}^n \eta'_{m,j}) - (\sum_{k=0}^{n-1} (\sum_{j=0}^k \eta'_{1,j}) \cdots (\sum_{j=0}^k \eta'_{m,j}))}{(\sum_{j=0}^n \eta'_{1,j}) \cdots (\sum_{j=0}^n \eta'_{m,j})}.$$

Proof. In view of the comments following Remark 2.30, one may adapt the proof of Lemma 3.4, replacing each $\eta_{i,j}$ by $\eta'_{i,j}$. $\square$

In the next three theorems, letting $r \in \{1, \dots, m\}$, we take the limit of the expected values found in Lemmas 3.4, 3.5, and 3.6 respectively as $q_1, q_2, \dots, q_r$ all approach infinity. (More precisely, for each $\epsilon > 0$, we require that there exist an $N > 0$ such that the expected value is within ϵ of the limit value whenever $q_1, q_2, \dots, q_r \geq N$.)

THEOREM 3.7 (unweighted) Let $r \in \{1, \dots, m\}$. For each $k \in \{0, \dots, n\}$, let

$$\sigma_k := \begin{cases} (\sum_{j=0}^k \eta_{r+1,j})(\sum_{j=0}^k \eta_{r+2,j}) \dots (\sum_{j=0}^k \eta_{m,j}) & \text{if } r < m \\ 1 & \text{if } r = m. \end{cases}$$

Further, let W vary over the set of R -submodules of V .

(a) If n is even, then

$$\lim_{q_1, q_2, \dots, q_r \rightarrow \infty} E(\nu_R(W)) = \frac{n\sigma_n - (\sum_{k=\frac{n}{2}}^{n-1} \sigma_k)}{\sigma_n}.$$

In particular, when $r = m$, we have that

$$\lim_{q_1, q_2, \dots, q_m \rightarrow \infty} E(\nu_R(W)) = \frac{n}{2}.$$

(b) If n is odd, then

$$\lim_{q_1, q_2, \dots, q_r \rightarrow \infty} E(\nu_R(W)) = \frac{n(e_1+1) \dots (e_r+1)\sigma_n - (\sigma_{\frac{n-1}{2}} + (e_1+1) \dots (e_r+1)(\sum_{k=\frac{n+1}{2}}^{n-1} \sigma_k))}{(e_1+1) \dots (e_r+1)\sigma_n}.$$

In particular, when $r = m$, we have that

$$\lim_{q_1, q_2, \dots, q_m \rightarrow \infty} E(\nu_R(W)) = \frac{n}{2} + \frac{1}{2} - \frac{1}{(e_1+1) \dots (e_m+1)}.$$

Proof. Referring to the numerator and denominator appearing in Lemma 3.4, let N denote

$$n(\sum_{j=0}^n \eta_{1,j}) \dots (\sum_{j=0}^n \eta_{r,j})\sigma_n - (\sum_{k=0}^{n-1} (\sum_{j=0}^k \eta_{1,j}) \dots (\sum_{j=0}^k \eta_{r,j})\sigma_k),$$

and let D denote $(\sum_{j=0}^n \eta_{1,j}) \dots (\sum_{j=0}^n \eta_{r,j})\sigma_n$.

(a) Suppose that n is even. To prove the assertion, after rewriting both N and D as polynomials in $q_1, q_2, \dots, q_r$, with coefficients viewed as integral polynomials in $q_{r+1}, \dots, q_m$, observe that it suffices to prove the following three statements. For each $i \in \{1, \dots, r\}$, the highest power of q_i in both N and D is at most $e_i \left(\frac{n}{2}\right)^2$; the

coefficient of the $q_1^{e_1(\frac{n}{2})^2} \dots q_r^{e_r(\frac{n}{2})^2}$ term of N is $n\sigma_n - (\sum_{k=\frac{n}{2}}^{n-1} \sigma_k)$; and the coefficient of the $q_1^{e_1(\frac{n}{2})^2} \dots q_r^{e_r(\frac{n}{2})^2}$ term of D is σ_n .

Let $i \in \{1, \dots, r\}$. For each $k \in \{0, \dots, n\}$, using Proposition 2.28 (c) and the discussion preceding Lemma 3.4, observe that $\sum_{j=0}^k \eta_{i,j}$ is an integral polynomial in q_i . Using Corollary 2.32 (a), one checks that the degree of $\sum_{j=0}^k \eta_{i,j}$ in q_i is at most $e_i(\frac{n}{2})^2$, with equality if and only if $k \in \{\frac{n}{2}, \dots, n\}$. In particular, for each $k \in \{\frac{n}{2}, \dots, n\}$, observe via Corollary 2.32 (a) and Proposition 2.28 (c) that $\sum_{j=0}^k \eta_{i,j}$ is a monic polynomial in q_i .

Thus, after rewriting

$$N = n(\sum_{j=0}^n \eta_{1,j}) \dots (\sum_{j=0}^n \eta_{r,j}) \sigma_n - (\sum_{k=0}^{n-1} (\sum_{j=0}^k \eta_{1,j}) \dots (\sum_{j=0}^k \eta_{r,j}) \sigma_k)$$

as a polynomial in $q_1, \dots, q_r$, with coefficients viewed as integral polynomials in $q_{r+1}, \dots, q_m$, the coefficient of the $q_1^{e_1(\frac{n}{2})^2} \dots q_r^{e_r(\frac{n}{2})^2}$ term of N is $n\sigma_n - (\sum_{k=\frac{n}{2}}^{n-1} \sigma_k)$.

Similarly, after rewriting $D = (\sum_{j=0}^n \eta_{1,j}) \dots (\sum_{j=0}^n \eta_{r,j}) \sigma_n$ as a polynomial in $q_1, \dots, q_r$, with coefficients viewed as integral polynomials in $q_{r+1}, \dots, q_m$, the coefficient of the $q_1^{e_1(\frac{n}{2})^2} \dots q_r^{e_r(\frac{n}{2})^2}$ term of D is σ_n . Therefore, the assertion follows.

When $r = m$, since $\sigma_k = 1$ for each $k \in \{0, \dots, n\}$, it follows that

$$\lim_{q_1, q_2, \dots, q_m \rightarrow \infty} E(\nu_R(W)) = \frac{n\sigma_n - (\sum_{k=\frac{n}{2}}^{n-1} \sigma_k)}{\sigma_n} = \frac{n - (\sum_{k=\frac{n}{2}}^{n-1} 1)}{1} = \frac{n}{2}.$$

(b) Suppose that n is odd. When $n = 1$, Proposition 4.1 (a) will establish that $E(\nu_R(W)) = 1 - \frac{1}{\prod_{i=1}^m (e_i + 1)}$. Then, after some calculation, the assertion follows for $n = 1$ (even before the limit is taken) since, using Proposition 2.28 (c) or the fact that R_i is an SPIR, one checks that $\sum_{j=0}^1 \eta_{i,j} = e_i + 1$ for every $i \in \{1, \dots, m\}$. Therefore, without loss of generality, suppose $n \geq 3$. To prove the assertion, after rewriting both N and D as polynomials in $q_1, \dots, q_r$, with coefficients viewed as integral

polynomials in $q_{r+1}, \dots, q_m$, it suffices to prove the following three statements. For each $i \in \{1, \dots, r\}$, the highest power of q_i in both N and D is at most $e_i \left(\frac{n^2-1}{4} \right)$;

the coefficient of the $q_1^{e_1 \left(\frac{n^2-1}{4} \right)} \dots q_r^{e_r \left(\frac{n^2-1}{4} \right)}$ term of N is

$$n(e_1 + 1) \dots (e_r + 1) \sigma_n - (\sigma_{\frac{n-1}{2}} + (e_1 + 1) \dots (e_r + 1) (\sum_{k=\frac{n+1}{2}}^{n-1} \sigma_k));$$

and the coefficient of the $q_1^{e_1 \left(\frac{n^2-1}{4} \right)} \dots q_r^{e_r \left(\frac{n^2-1}{4} \right)}$ term of D is $(e_1 + 1) \dots (e_r + 1) \sigma_n$.

Let $i \in \{1, \dots, r\}$. For each $k \in \{0, \dots, n\}$, using Proposition 2.28 (c) and the discussion preceding Lemma 3.4, observe that $\sum_{j=0}^k \eta_{i,j}$ is an integral polynomial in q_i . Using Corollary 2.32 (b), one checks that the degree of $\sum_{j=0}^k \eta_{i,j}$ in q_i is at most $e_i \left(\frac{n^2-1}{4} \right)$, with equality if and only if $k \in \{\frac{n-1}{2}, \dots, n\}$. In particular, observe via Corollary 2.32 (b) and Proposition 2.28 (c) that $\sum_{j=0}^{\frac{n-1}{2}} \eta_{i,j}$ is a monic polynomial in q_i . Similarly, using Corollary 2.32 (b) and Proposition 2.28 (c), one checks, for each $k \in \{\frac{n+1}{2}, \dots, n\}$, that the leading coefficient of $\sum_{j=0}^k \eta_{i,j}$ as a polynomial in q_i is $e_i + 1$.

Thus, after rewriting

$$N = n(\sum_{j=0}^n \eta_{1,j}) \dots (\sum_{j=0}^n \eta_{r,j}) \sigma_n - (\sum_{k=0}^{n-1} (\sum_{j=0}^k \eta_{1,j}) \dots (\sum_{j=0}^k \eta_{r,j}) \sigma_k)$$

as a polynomial in $q_1, \dots, q_r$, with coefficients viewed as integral polynomials in

$q_{r+1}, \dots, q_m$, the coefficient of the $q_1^{e_1 \left(\frac{n^2-1}{4} \right)} \dots q_r^{e_r \left(\frac{n^2-1}{4} \right)}$ term of N is

$$n(e_1 + 1) \dots (e_r + 1) \sigma_n - (\sigma_{\frac{n-1}{2}} + (e_1 + 1) \dots (e_r + 1) (\sum_{k=\frac{n+1}{2}}^{n-1} \sigma_k)).$$

Similarly, after rewriting $D = (\sum_{j=0}^n \eta_{1,j}) \dots (\sum_{j=0}^n \eta_{r,j}) \sigma_n$ as a polynomial in $q_1, \dots, q_r$, with coefficients viewed as integral polynomials in $q_{r+1}, \dots, q_m$, the coefficient of the $q_1^{e_1 \left(\frac{n^2-1}{4} \right)} \dots q_r^{e_r \left(\frac{n^2-1}{4} \right)}$ term of D is $(e_1 + 1) \dots (e_r + 1) \sigma_n$. Therefore, the assertion follows.

When $r = m$, since $\sigma_k = 1$ for each $k \in \{0, \dots, n\}$, it follows that

$$\begin{aligned} \lim_{q_1, q_2, \dots, q_m \rightarrow \infty} E(\nu_R(W)) &= \frac{n(e_1+1)\dots(e_r+1)\sigma_n - (\sigma_{\frac{n-1}{2}} + (e_1+1)\dots(e_r+1)(\sum_{k=\frac{n+1}{2}}^{n-1} \sigma_k))}{(e_1+1)\dots(e_r+1)\sigma_n} = \\ &= \frac{n(e_1+1)\dots(e_m+1) - (1 + (e_1+1)\dots(e_m+1)(\sum_{k=\frac{n+1}{2}}^{n-1} 1))}{(e_1+1)\dots(e_m+1)} = \frac{(\frac{n+1}{2})(e_1+1)\dots(e_m+1) - 1}{(e_1+1)\dots(e_m+1)} = \\ &= \frac{n}{2} + \frac{1}{2} - \frac{1}{(e_1+1)\dots(e_m+1)}. \quad \square \end{aligned}$$

THEOREM 3.8 (*weighting by ν_R -value*) Let $r \in \{1, \dots, m\}$. For each $k \in \{0, \dots, n\}$, let

$$\sigma_k := \begin{cases} (\sum_{j=0}^k \eta_{r+1,j})(\sum_{j=0}^k \eta_{r+2,j}) \dots (\sum_{j=0}^k \eta_{m,j}) & \text{if } r < m \\ 1 & \text{if } r = m. \end{cases}$$

Let W vary over the set of R -submodules of V . Also, suppose that W is weighted by $\nu_R(W)$. Then:

(a) If n is even, then

$$\lim_{q_1, q_2, \dots, q_r \rightarrow \infty} E(\nu_R(W)) = \frac{n^2 \sigma_n - (\sum_{k=\frac{n}{2}}^{n-1} (2k+1) \sigma_k)}{n \sigma_n - (\sum_{k=\frac{n}{2}}^{n-1} \sigma_k)}.$$

In particular, when $r = m$, we have that $\lim_{q_1, q_2, \dots, q_m \rightarrow \infty} E(\nu_R(W)) = \frac{n}{2}$.

(b) If n is odd, then

$$\lim_{q_1, q_2, \dots, q_r \rightarrow \infty} E(\nu_R(W)) = \frac{n^2(e_1+1)\dots(e_r+1)\sigma_n - (n\sigma_{\frac{n-1}{2}} + (e_1+1)\dots(e_r+1)(\sum_{k=\frac{n+1}{2}}^{n-1} (2k+1)\sigma_k))}{n(e_1+1)\dots(e_r+1)\sigma_n - (\sigma_{\frac{n-1}{2}} + (e_1+1)\dots(e_r+1)(\sum_{k=\frac{n+1}{2}}^{n-1} \sigma_k))}.$$

In particular, when $r = m$, we have that

$$\lim_{q_1, q_2, \dots, q_m \rightarrow \infty} E(\nu_R(W)) = \frac{n}{2} + \frac{1}{2} - \frac{1}{(e_1+1)\dots(e_m+1)} + \frac{2((e_1+1)\dots(e_m+1)-1)}{(e_1+1)\dots(e_m+1)((e_1+1)\dots(e_m+1)(n+1)-2)}.$$

Proof. Using the fact that $\sigma_k \leq \sigma_n$ for each $k \in \{0, \dots, n\}$, one checks that the denominators found in the assertions of both (a) and (b) are indeed nonzero. Referring to Lemma 3.5, let N denote

$$n^2(\sum_{j=0}^n \eta_{1,j}) \cdots (\sum_{j=0}^n \eta_{r,j}) \sigma_n - (\sum_{k=0}^{n-1} (2k+1) (\sum_{j=0}^k \eta_{1,j}) \cdots (\sum_{j=0}^k \eta_{r,j}) \sigma_k),$$

and let D denote

$$n(\sum_{j=0}^n \eta_{1,j}) \cdots (\sum_{j=0}^n \eta_{r,j}) \sigma_n - (\sum_{k=0}^{n-1} (\sum_{j=0}^k \eta_{1,j}) \cdots (\sum_{j=0}^k \eta_{r,j}) \sigma_k).$$

(a) Suppose that n is even. To prove the assertion, after rewriting both N and D as polynomials in $q_1, \dots, q_r$, with coefficients viewed as integral polynomials in $q_{r+1}, \dots, q_m$, observe that it suffices to prove the following three statements. For each $i \in \{1, \dots, r\}$, the highest power of q_i in both N and D is at most $e_i \left(\frac{n}{2}\right)^2$; the coefficient of the $q_1^{e_1 \left(\frac{n}{2}\right)^2} \cdots q_r^{e_r \left(\frac{n}{2}\right)^2}$ term of N is $n^2 \sigma_n - (\sum_{k=\frac{n}{2}}^{n-1} (2k+1) \sigma_k)$; and the coefficient of the $q_1^{e_1 \left(\frac{n}{2}\right)^2} \cdots q_r^{e_r \left(\frac{n}{2}\right)^2}$ term of D is $n \sigma_n - (\sum_{k=\frac{n}{2}}^{n-1} \sigma_k)$.

Let $i \in \{1, \dots, r\}$. For each $k \in \{0, \dots, n\}$, using Proposition 2.28 (c) and the discussion preceding Lemma 3.4, observe that $\sum_{j=0}^k \eta_{i,j}$ is an integral polynomial in q_i . Using Corollary 2.32 (a), one checks that the degree of $\sum_{j=0}^k \eta_{i,j}$ in q_i is at most $e_i \left(\frac{n}{2}\right)^2$, with equality if and only if $k \in \{\frac{n}{2}, \dots, n\}$. In particular, for each $k \in \{\frac{n}{2}, \dots, n\}$, observe via Corollary 2.32 (a) and Proposition 2.28 (c) that $\sum_{j=0}^k \eta_{i,j}$ is a monic polynomial in q_i .

Thus, after rewriting

$$N = n^2(\sum_{j=0}^n \eta_{1,j}) \cdots (\sum_{j=0}^n \eta_{r,j}) \sigma_n - (\sum_{k=0}^{n-1} (2k+1) (\sum_{j=0}^k \eta_{1,j}) \cdots (\sum_{j=0}^k \eta_{r,j}) \sigma_k)$$

as a polynomial in $q_1, \dots, q_r$, with coefficients viewed as integral polynomials in $q_{r+1}, \dots, q_m$, the coefficient of the $q_1^{e_1 \left(\frac{n}{2}\right)^2} \cdots q_r^{e_r \left(\frac{n}{2}\right)^2}$ term of N is $n^2 \sigma_n - (\sum_{k=\frac{n}{2}}^{n-1} (2k+1) \sigma_k)$. Similarly, after rewriting

$$D = n(\sum_{j=0}^n \eta_{1,j}) \cdots (\sum_{j=0}^n \eta_{r,j}) \sigma_n - (\sum_{k=0}^{n-1} (\sum_{j=0}^k \eta_{1,j}) \cdots (\sum_{j=0}^k \eta_{r,j}) \sigma_k)$$

as a polynomial in $q_1, \dots, q_r$, with coefficients viewed as integral polynomials in $q_{r+1}, \dots, q_m$, the coefficient of the $q_1^{e_1(\frac{n}{2})^2} \dots q_r^{e_r(\frac{n}{2})^2}$ term of D is $n\sigma_n - (\sum_{k=\frac{n}{2}}^{n-1} \sigma_k)$.

Therefore, the assertion follows.

When $r = m$, since $\sigma_k = 1$ for each $k \in \{0, \dots, n\}$, it follows that

$$\begin{aligned} \lim_{q_1, q_2, \dots, q_m \rightarrow \infty} E(\nu_R(W)) &= \frac{n^2 \sigma_n - (\sum_{k=\frac{n}{2}}^{n-1} (2k+1) \sigma_k)}{n \sigma_n - (\sum_{k=\frac{n}{2}}^{n-1} \sigma_k)} = \frac{n^2 - (\sum_{k=\frac{n}{2}}^{n-1} (2k+1))}{n - (\sum_{k=\frac{n}{2}}^{n-1} 1)} = \\ &= \frac{n^2 - 2(\sum_{k=\frac{n}{2}}^{n-1} k) - (\sum_{k=\frac{n}{2}}^{n-1} 1)}{n - \frac{n}{2}} = \frac{n^2 - (\frac{3n^2 - 2n}{2}) - \frac{n}{2}}{\frac{n}{2}} = \frac{n}{2}. \end{aligned}$$

(b) Suppose that n is odd. When $n = 1$, Proposition 4.1 (b) will establish that $E(\nu_R(W)) = 1$. Then, the assertion follows for $n = 1$ (even before the limit is taken). Therefore, without loss of generality, suppose $n \geq 3$. To prove the assertion, after rewriting both N and D as polynomials in $q_1, \dots, q_r$, with coefficients viewed as integral polynomials in $q_{r+1}, \dots, q_m$, observe that it suffices to prove the following three statements. For each $i \in \{1, \dots, r\}$, the highest power of q_i in both N and D is at most $e_i \left(\frac{n^2-1}{4} \right)$; the coefficient of the $q_1^{e_1 \left(\frac{n^2-1}{4} \right)} \dots q_r^{e_r \left(\frac{n^2-1}{4} \right)}$ term of N is

$$n^2(e_1 + 1) \dots (e_r + 1) \sigma_n - (n \sigma_{\frac{n-1}{2}} + (e_1 + 1) \dots (e_r + 1) (\sum_{k=\frac{n+1}{2}}^{n-1} (2k+1) \sigma_k));$$

and the coefficient of the $q_1^{e_1 \left(\frac{n^2-1}{4} \right)} \dots q_r^{e_r \left(\frac{n^2-1}{4} \right)}$ term of D is

$$n(e_1 + 1) \dots (e_r + 1) \sigma_n - (\sigma_{\frac{n-1}{2}} + (e_1 + 1) \dots (e_r + 1) (\sum_{k=\frac{n+1}{2}}^{n-1} \sigma_k)).$$

Let $i \in \{1, \dots, r\}$. For each $k \in \{0, \dots, n\}$, using Proposition 2.28 (c) and the discussion preceding Lemma 3.4, observe that $\sum_{j=0}^k \eta_{i,j}$ is an integral polynomial in q_i . Using Corollary 2.32 (b), one checks that the degree of $\sum_{j=0}^k \eta_{i,j}$ in q_i is at most $e_i \left(\frac{n^2-1}{4} \right)$, with equality if and only if $k \in \{\frac{n-1}{2}, \dots, n\}$. In particular, observe via Corollary 2.32 (b) and Proposition 2.28 (c) that $\sum_{j=0}^{\frac{n-1}{2}} \eta_{i,j}$ is a monic polynomial

in q_i . Similarly, using Corollary 2.32 (b) and Proposition 2.28 (c), one checks, for each $k \in \{\frac{n+1}{2}, \dots, n\}$, that the leading coefficient of $\sum_{j=0}^k \eta_{i,j}$ as a polynomial in q_i is $e_i + 1$.

Thus, after rewriting

$$N = n^2(\sum_{j=0}^n \eta_{1,j}) \dots (\sum_{j=0}^n \eta_{r,j}) \sigma_n - (\sum_{k=0}^{n-1} (2k+1) (\sum_{j=0}^k \eta_{1,j}) \dots (\sum_{j=0}^k \eta_{r,j}) \sigma_k)$$

as a polynomial in $q_1, \dots, q_r$, with coefficients viewed as integral polynomials in

$q_{r+1}, \dots, q_m$, the coefficient of the $q_1^{e_1 \left(\frac{n^2-1}{4}\right)} \dots q_r^{e_r \left(\frac{n^2-1}{4}\right)}$ term of N is

$$n^2(e_1 + 1) \dots (e_r + 1) \sigma_n - (n \sigma_{\frac{n-1}{2}} + (e_1 + 1) \dots (e_r + 1) (\sum_{k=\frac{n+1}{2}}^{n-1} (2k+1) \sigma_k)).$$

Similarly, after rewriting

$$D = n(\sum_{j=0}^n \eta_{1,j}) \dots (\sum_{j=0}^n \eta_{r,j}) \sigma_n - (\sum_{k=0}^{n-1} (\sum_{j=0}^k \eta_{1,j}) \dots (\sum_{j=0}^k \eta_{r,j}) \sigma_k)$$

as a polynomial in $q_1, \dots, q_r$, with coefficients viewed as integral polynomials in

$q_{r+1}, \dots, q_m$, the coefficient of the $q_1^{e_1 \left(\frac{n^2-1}{4}\right)} \dots q_r^{e_r \left(\frac{n^2-1}{4}\right)}$ term of D is

$$n(e_1 + 1) \dots (e_r + 1) \sigma_n - (\sigma_{\frac{n-1}{2}} + (e_1 + 1) \dots (e_r + 1) (\sum_{k=\frac{n+1}{2}}^{n-1} \sigma_k)).$$

Therefore, the assertion follows.

When $r = m$, since $\sigma_k = 1$ for each $k \in \{0, \dots, n\}$, it follows that

$$\begin{aligned} \lim_{q_1, q_2, \dots, q_m \rightarrow \infty} E(\nu_R(W)) &= \\ \frac{n^2(e_1+1)\dots(e_r+1)\sigma_n - (n\sigma_{\frac{n-1}{2}} + (e_1+1)\dots(e_r+1)(\sum_{k=\frac{n+1}{2}}^{n-1} (2k+1)\sigma_k))}{n(e_1+1)\dots(e_r+1)\sigma_n - (\sigma_{\frac{n-1}{2}} + (e_1+1)\dots(e_r+1)(\sum_{k=\frac{n+1}{2}}^{n-1} \sigma_k))} &= \\ \frac{n^2(e_1+1)\dots(e_m+1) - (n + (e_1+1)\dots(e_m+1)(\sum_{k=\frac{n+1}{2}}^{n-1} (2k+1)))}{n(e_1+1)\dots(e_m+1) - (1 + (e_1+1)\dots(e_m+1)(\sum_{k=\frac{n+1}{2}}^{n-1} 1))} &= \\ \frac{n^2(e_1+1)\dots(e_m+1) - (n + (e_1+1)\dots(e_m+1)(\frac{3n^2-4n+1}{4} + \frac{n-1}{2}))}{n(e_1+1)\dots(e_m+1) - (1 + (e_1+1)\dots(e_m+1)(\frac{n-1}{2}))} &= \end{aligned}$$

$$\begin{aligned}
& \frac{n^2(e_1+1)\dots(e_m+1)-(n+(e_1+1)\dots(e_m+1)(\frac{3n^2-2n-1}{4}))}{(\frac{n+1}{2})(e_1+1)\dots(e_m+1)-1} = \\
& \frac{4n^2(e_1+1)\dots(e_m+1)-(4n+(e_1+1)\dots(e_m+1)(3n^2-2n-1))}{2(n+1)(e_1+1)\dots(e_m+1)-4} = \\
& \frac{(e_1+1)\dots(e_m+1)n^2+(2(e_1+1)\dots(e_m+1)-4)n+(e_1+1)\dots(e_m+1)}{2(e_1+1)\dots(e_m+1)n+2(e_1+1)\dots(e_m+1)-4} = \\
& \frac{n}{2} + \frac{(e_1+1)\dots(e_m+1)-2}{2(e_1+1)\dots(e_m+1)} + \left(\frac{(e_1+1)\dots(e_m+1)-(\frac{(e_1+1)\dots(e_m+1)-2}{2(e_1+1)\dots(e_m+1)})(2(e_1+1)\dots(e_m+1)-4)}{2(e_1+1)\dots(e_m+1)n+2(e_1+1)\dots(e_m+1)-4} \right) = \\
& \frac{n}{2} + \frac{1}{2} - \frac{1}{(e_1+1)\dots(e_m+1)} + \frac{2((e_1+1)\dots(e_m+1))^2 - ((e_1+1)\dots(e_m+1)-2)(2(e_1+1)\dots(e_m+1)-4)}{2(e_1+1)\dots(e_m+1)(2(e_1+1)\dots(e_m+1)n+2(e_1+1)\dots(e_m+1)-4)} = \\
& \frac{n}{2} + \frac{1}{2} - \frac{1}{(e_1+1)\dots(e_m+1)} + \frac{2((e_1+1)\dots(e_m+1)-1)}{(e_1+1)\dots(e_m+1)((e_1+1)\dots(e_m+1)(n+1)-2)}. \quad \square
\end{aligned}$$

THEOREM 3.9 (*weighting by the cardinality*) Let $r \in \{1, \dots, m\}$. For each $k \in \{0, \dots, n\}$, let

$$\sigma'_k := \begin{cases} (\sum_{j=0}^k \eta'_{r+1,j})(\sum_{j=0}^k \eta'_{r+2,j}) \dots (\sum_{j=0}^k \eta'_{m,j}) & \text{if } r < m \\ 1 & \text{if } r = m. \end{cases}$$

Let W vary over the set of R -submodules of V . Also, suppose that W is weighted by cardinality. Then:

(a) If n is even, then

$$\lim_{q_1, q_2, \dots, q_r \rightarrow \infty} E(\nu_R(W)) = \frac{n(e_1+1)\dots(e_r+1)\sigma'_n - (\sigma'_{\frac{n}{2}} + (e_1+1)\dots(e_r+1)(\sum_{k=\frac{n}{2}+1}^{n-1} \sigma'_k))}{(e_1+1)\dots(e_r+1)\sigma'_n}.$$

In particular, when $r = m$, we have that

$$\lim_{q_1, q_2, \dots, q_m \rightarrow \infty} E(\nu_R(W)) = \frac{n+1}{2} + \frac{1}{2} - \frac{1}{(e_1+1)\dots(e_m+1)}.$$

(b) If n is odd, then

$$\lim_{q_1, q_2, \dots, q_r \rightarrow \infty} E(\nu_R(W)) = \frac{n\sigma'_n - (\sum_{k=\frac{n+1}{2}}^{n-1} \sigma'_k)}{\sigma'_n}.$$

In particular, when $r = m$, we have that

$$\lim_{q_1, q_2, \dots, q_m \rightarrow \infty} E(\nu_R(W)) = \frac{n+1}{2}.$$

Proof. Referring to the numerator and denominator appearing in Lemma 3.6, let N denote

$$n(\sum_{j=0}^n \eta'_{1,j}) \dots (\sum_{j=0}^n \eta'_{r,j}) \sigma'_n - (\sum_{k=0}^{n-1} (\sum_{j=0}^k \eta'_{1,j}) \dots (\sum_{j=0}^k \eta'_{r,j}) \sigma'_k),$$

and let D denote $(\sum_{j=0}^n \eta'_{1,j}) \dots (\sum_{j=0}^n \eta'_{r,j}) \sigma'_n$.

(a) Suppose that n is even. To prove the assertion, after rewriting both N and D as polynomials in $q_1, \dots, q_r$, with coefficients viewed as integral polynomials in $q_{r+1}, \dots, q_m$, observe that it suffices to prove the following three statements. For each $i \in \{1, \dots, r\}$, the highest power of q_i in both N and D is at most $e_i \left(\frac{(n+1)^2-1}{4} \right)$;

the coefficient of the $q_1^{e_1 \left(\frac{(n+1)^2-1}{4} \right)} \dots q_r^{e_r \left(\frac{(n+1)^2-1}{4} \right)}$ term of N is

$$n(e_1 + 1) \dots (e_r + 1) \sigma'_n - (\sigma'_{\frac{n}{2}} + (e_1 + 1) \dots (e_r + 1) (\sum_{k=\frac{n}{2}+1}^{n-1} \sigma'_k));$$

and the coefficient of the $q_1^{e_1 \left(\frac{(n+1)^2-1}{4} \right)} \dots q_r^{e_r \left(\frac{(n+1)^2-1}{4} \right)}$ term of D is $(e_1 + 1) \dots (e_r + 1) \sigma'_n$.

Let $i \in \{1, \dots, r\}$. For each $k \in \{0, \dots, n\}$, using Proposition 2.28 (c) and the discussion preceding Lemma 3.6, observe that $\sum_{j=0}^k \eta'_{i,j}$ is an integral polynomial in q_i . Using Corollary 2.34 (a), one checks that the degree of $\sum_{j=0}^k \eta'_{i,j}$ in q_i is at most $e_i \left(\frac{(n+1)^2-1}{4} \right)$, with equality if and only if $k \in \{\frac{n}{2}, \dots, n\}$. In particular, observe via Corollary 2.34 (a) and Proposition 2.28 (c) that $\sum_{j=0}^{\frac{n}{2}} \eta'_{i,j}$ is a monic polynomial in q_i . Similarly, using Corollary 2.34 (a) and Proposition 2.28 (c), one checks, for each $k \in \{\frac{n}{2} + 1, \dots, n\}$, that the leading coefficient of $\sum_{j=0}^k \eta'_{i,j}$ as a polynomial in q_i is $e_i + 1$.

Thus, after rewriting

$$N = n(\sum_{j=0}^n \eta'_{1,j}) \cdots (\sum_{j=0}^n \eta'_{r,j}) \sigma'_n - (\sum_{k=0}^{n-1} (\sum_{j=0}^k \eta'_{1,j}) \cdots (\sum_{j=0}^k \eta'_{r,j}) \sigma'_k)$$

as a polynomial in $q_1, \dots, q_r$, with coefficients viewed as integral polynomials in

$q_{r+1}, \dots, q_m$, the coefficient of the $q_1^{e_1 \left(\frac{(n+1)^2-1}{4} \right)} \cdots q_r^{e_r \left(\frac{(n+1)^2-1}{4} \right)}$ term of N is

$$n(e_1 + 1) \cdots (e_r + 1) \sigma'_n - (\sigma'_{\frac{n}{2}} + (e_1 + 1) \cdots (e_r + 1) (\sum_{k=\frac{n}{2}+1}^{n-1} \sigma'_k)).$$

Similarly, after rewriting $D = (\sum_{j=0}^n \eta'_{1,j}) \cdots (\sum_{j=0}^n \eta'_{r,j}) \sigma'_n$ as a polynomial in

$q_1, \dots, q_r$, with coefficients viewed as integral polynomials in $q_{r+1}, \dots, q_m$, the coefficient of the $q_1^{e_1 \left(\frac{(n+1)^2-1}{4} \right)} \cdots q_r^{e_r \left(\frac{(n+1)^2-1}{4} \right)}$ term of D is $(e_1 + 1) \cdots (e_r + 1) \sigma'_n$. Therefore,

the assertion follows.

When $r = m$, as $\sigma'_k = 1$ for each $k \in \{0, \dots, n\}$, it follows that

$$\begin{aligned} \lim_{q_1, q_2, \dots, q_m \rightarrow \infty} E(\nu_R(W)) &= \frac{n(e_1+1) \cdots (e_r+1) \sigma'_n - (\sigma'_{\frac{n}{2}} + (e_1+1) \cdots (e_r+1) (\sum_{k=\frac{n}{2}+1}^{n-1} \sigma'_k))}{(e_1+1) \cdots (e_r+1) \sigma'_n} = \\ &= \frac{n(e_1+1) \cdots (e_m+1) - (1 + (e_1+1) \cdots (e_m+1) (\sum_{k=\frac{n}{2}+1}^{n-1} 1))}{(e_1+1) \cdots (e_m+1)} = \frac{n(e_1+1) \cdots (e_m+1) - (1 + (e_1+1) \cdots (e_m+1) (\frac{n}{2} - 1))}{(e_1+1) \cdots (e_m+1)} = \\ &= \frac{(\frac{n}{2}+1)(e_1+1) \cdots (e_m+1) - 1}{(e_1+1) \cdots (e_m+1)} = \frac{n+1}{2} + \frac{1}{2} - \frac{1}{(e_1+1) \cdots (e_m+1)}. \end{aligned}$$

(b) Suppose that n is odd. To prove the assertion, after rewriting both N and D as polynomials in $q_1, \dots, q_r$, with coefficients viewed as integral polynomials in $q_{r+1}, \dots, q_m$, observe that it suffices to prove the following three statements. For each $i \in \{1, \dots, r\}$, the highest power of q_i in both N and D is at most $e_i \left(\frac{n+1}{2} \right)^2$; the coefficient of the $q_1^{e_1 \left(\frac{n+1}{2} \right)^2} \cdots q_r^{e_r \left(\frac{n+1}{2} \right)^2}$ term of N is $n\sigma'_n - (\sum_{k=\frac{n+1}{2}}^{n-1} \sigma'_k)$; and the coefficient of the $q_1^{e_1 \left(\frac{n+1}{2} \right)^2} \cdots q_r^{e_r \left(\frac{n+1}{2} \right)^2}$ term of D is σ'_n .

Let $i \in \{1, \dots, r\}$. For each $k \in \{0, \dots, n\}$, using Proposition 2.28 (c) and the discussion preceding Lemma 3.6, observe that $\sum_{j=0}^k \eta'_{i,j}$ is an integral polynomial in q_i . Using Corollary 2.34 (b), one checks that the degree of $\sum_{j=0}^k \eta'_{i,j}$ in q_i is at

most $e_i \left(\frac{n+1}{2} \right)^2$, with equality if and only if $k \in \{ \frac{n+1}{2}, \dots, n \}$. In particular, for each $k \in \{ \frac{n+1}{2}, \dots, n \}$, observe via Corollary 2.34 (b) and Proposition 2.28 (c) that $\sum_{j=0}^k \eta'_{i,j}$ is a monic polynomial in q_i .

Thus, after rewriting

$$N = n(\sum_{j=0}^n \eta'_{1,j}) \dots (\sum_{j=0}^n \eta'_{r,j}) \sigma'_n - (\sum_{k=0}^{n-1} (\sum_{j=0}^k \eta'_{1,j}) \dots (\sum_{j=0}^k \eta'_{r,j}) \sigma'_k)$$

as a polynomial in $q_1, \dots, q_r$, with coefficients viewed as integral polynomials in $q_{r+1}, \dots, q_m$, the coefficient of the $q_1^{e_1 \left(\frac{n+1}{2} \right)^2} \dots q_r^{e_r \left(\frac{n+1}{2} \right)^2}$ term of N is $n\sigma'_n - (\sum_{k=\frac{n+1}{2}}^{n-1} \sigma'_k)$. Similarly, after rewriting $D = (\sum_{j=0}^n \eta'_{1,j}) \dots (\sum_{j=0}^n \eta'_{r,j}) \sigma'_n$ as a polynomial in $q_1, \dots, q_r$, with coefficients viewed as integral polynomials in $q_{r+1}, \dots, q_m$, the coefficient of the $q_1^{e_1 \left(\frac{n+1}{2} \right)^2} \dots q_r^{e_r \left(\frac{n+1}{2} \right)^2}$ term of D is σ'_n . Therefore, the assertion follows.

When $r = m$, since $\sigma'_k = 1$ for each $k \in \{0, \dots, n\}$, it follows that

$$\lim_{q_1, q_2, \dots, q_m \rightarrow \infty} E(\nu_R(W)) = \frac{n\sigma'_n - (\sum_{k=\frac{n+1}{2}}^{n-1} \sigma'_k)}{\sigma'_n} = \frac{n - (\sum_{k=\frac{n+1}{2}}^{n-1} 1)}{1} = n - \left(\frac{n}{2} - \frac{1}{2} \right) = \frac{n+1}{2}. \quad \square$$

In order to calculate the expected rank of an arbitrary R -free submodule of V , we introduce the following notation. For each $i \in \{1, \dots, m\}$ and $j \in \{0, \dots, n\}$, let $\tau_{i,j}$ denote the number of R_i -free submodules of V_i with rank j . Thus, for each $i \in \{1, \dots, m\}$, $\tau_{i,0} = 1$, and, for each $j \in \{1, \dots, n\}$, observe via Corollary 2.21 that $\tau_{i,j} = \Lambda_i(j, j, \dots, j)$, where Λ_i denotes Λ in Proposition 2.28 (c) using R_i as the base ring and V_i as the ambient R_i -free module. For each $k \in \{0, \dots, n\}$, let τ_k denote the number of R -free submodules of V with rank k . Thus, for each $k \in \{0, \dots, n\}$, Proposition 3.2 yields that $\tau_k = \tau_{1,k} \tau_{2,k} \dots \tau_{m,k}$.

With the above notation in hand, the unweighted expected rank of an arbitrary R -free submodule F of V is given by $E(\nu_R(F)) = \frac{\sum_{k=0}^n k\tau_k}{\sum_{k=0}^n \tau_k}$. Similarly, the expected rank of an arbitrary R -free submodule F of V , when F is weighted by $\nu_R(F)$, is given by $E(\nu_R(F)) = \frac{\sum_{k=0}^n k^2\tau_k}{\sum_{k=0}^n k\tau_k}$.

For each $i \in \{1, \dots, m\}$ and $j \in \{0, \dots, n\}$, let $\tau'_{i,j}$ denote the sum of cardinalities of all R_i -free submodules of V_i with rank j . Thus, for each $i \in \{1, \dots, m\}$, $\tau'_{i,0} = 1$, and, for each $j \in \{1, \dots, n\}$, observe via Corollary 2.21, Proposition 2.25, and Proposition 2.28 (c) that $\tau'_{i,j} = \Lambda'_i(j, j, \dots, j) = q_i^{e_{ij}} \Lambda_i(j, j, \dots, j)$ where Λ_i denotes Λ using R_i as the base ring and V_i as the ambient R_i -free module. For each $k \in \{0, \dots, n\}$, let τ'_k denote the sum of cardinalities of all R -free submodules of V with rank k . Thus, for each $k \in \{0, \dots, n\}$, Proposition 3.2 yields that $\tau'_k = \tau'_{1,k} \tau'_{2,k} \dots \tau'_{m,k}$. Hence, the expected rank of an arbitrary R -free submodule F of V , when F is weighted by cardinality, is given by $E(\nu_R(F)) = \frac{\sum_{k=0}^n k\tau'_k}{\sum_{k=0}^n \tau'_k}$.

In the next three theorems, letting $r \in \{1, \dots, m\}$, we take the limit of the various expected ranks discussed above as $q_1, q_2, \dots, q_r$ all approach infinity. (More precisely, for each $\epsilon > 0$, we require that there exist an $N > 0$ such that the expected value is within ϵ of the limit value whenever $q_1, q_2, \dots, q_r \geq N$.)

THEOREM 3.10 (*unweighted*) *Let $r \in \{1, \dots, m\}$. For each $k \in \{0, \dots, n\}$, let*

$$\sigma_k := \begin{cases} \tau_{r+1,k} \tau_{r+2,k} \dots \tau_{m,k} & \text{if } r < m \\ 1 & \text{if } r = m. \end{cases}$$

Further, let F vary over the set of R -free submodules of V .

(a) *If n is even, then*

$$\lim_{q_1, \dots, q_r \rightarrow \infty} E(\nu_R(F)) = \frac{n}{2}.$$

(b) If n is odd, then

$$\lim_{q_1, \dots, q_r \rightarrow \infty} E(\nu_R(F)) = \frac{\left(\frac{n-1}{2}\right)\sigma_{\frac{n-1}{2}} + \left(\frac{n+1}{2}\right)\sigma_{\frac{n+1}{2}}}{\sigma_{\frac{n-1}{2}} + \sigma_{\frac{n+1}{2}}}.$$

In particular, when $r = m$, we have that $\lim_{q_1, \dots, q_m \rightarrow \infty} E(\nu_R(F)) = \frac{n}{2}$.

Proof. Let N denote $\sum_{k=0}^n k\tau_k = \sum_{k=0}^n k\tau_{1,k}\tau_{2,k} \dots \tau_{r,k}\sigma_k$, and let D denote

$$\sum_{k=0}^n \tau_k = \sum_{k=0}^n \tau_{1,k}\tau_{2,k} \dots \tau_{r,k}\sigma_k.$$

(a) Suppose that n is even. To prove the assertion, after rewriting both N and D as polynomials in $q_1, q_2, \dots, q_r$, with coefficients viewed as integral polynomials in $q_{r+1}, \dots, q_m$, observe that it suffices to prove the following three statements. For each $i \in \{1, \dots, r\}$, the highest power of q_i in both N and D is at most $e_i \left(\frac{n}{2}\right)^2$; the coefficient of the $q_1^{e_1(\frac{n}{2})^2} \dots q_r^{e_r(\frac{n}{2})^2}$ term of N is $\left(\frac{n}{2}\right)\sigma_{\frac{n}{2}}$; and the coefficient of the $q_1^{e_1(\frac{n}{2})^2} \dots q_r^{e_r(\frac{n}{2})^2}$ term of D is $\sigma_{\frac{n}{2}}$.

Let $i \in \{1, \dots, r\}$. For each $k \in \{0, \dots, n\}$, using Proposition 2.28 (c) and the above discussion, observe that $\tau_{i,k}$ is a monic polynomial in q_i . By Corollary 2.32 (a), the degree of $\tau_{i,k}$ in q_i is at most $e_i \left(\frac{n}{2}\right)^2$, with equality if and only if $k = \frac{n}{2}$.

Thus, after rewriting $N = \sum_{k=0}^n k\tau_{1,k}\tau_{2,k} \dots \tau_{r,k}\sigma_k$ as a polynomial in $q_1, q_2, \dots, q_r$, with coefficients viewed as integral polynomials in $q_{r+1}, \dots, q_m$, the coefficient of the $q_1^{e_1(\frac{n}{2})^2} \dots q_r^{e_r(\frac{n}{2})^2}$ term of N is $\left(\frac{n}{2}\right)\sigma_{\frac{n}{2}}$. Similarly, after rewriting $D = \sum_{k=0}^n \tau_{1,k}\tau_{2,k} \dots \tau_{r,k}\sigma_k$ as a polynomial in $q_1, q_2, \dots, q_r$, with coefficients viewed as integral polynomials in $q_{r+1}, \dots, q_m$, the coefficient of the $q_1^{e_1(\frac{n}{2})^2} \dots q_r^{e_r(\frac{n}{2})^2}$ term of D is $\sigma_{\frac{n}{2}}$. Therefore, the assertion follows.

(b) Suppose that n is odd. When $n = 1$, V is the only R -free submodule of V with ν_R -value 1. Thus, the assertion follows for $n = 1$ since $N = 0\tau_0 + 1\tau_1 = 1$ and $D = \tau_0 + \tau_1 = 2$. Therefore, without loss of generality, suppose

$n \geq 3$. To prove the assertion, after rewriting both N and D as polynomials in $q_1, q_2, \dots, q_r$, with coefficients viewed as integral polynomials in $q_{r+1}, \dots, q_m$, observe that it suffices to prove the following three statements. For each $i \in \{1, \dots, r\}$, the highest power of q_i in both N and D is at most $e_i \left(\frac{n^2-1}{4} \right)$; the coefficient of the $q_1^{e_1 \left(\frac{n^2-1}{4} \right)} \dots q_r^{e_r \left(\frac{n^2-1}{4} \right)}$ term of N is $\left(\frac{n-1}{2} \right) \sigma_{\frac{n-1}{2}} + \left(\frac{n+1}{2} \right) \sigma_{\frac{n+1}{2}}$; and the coefficient of the $q_1^{e_1 \left(\frac{n^2-1}{4} \right)} \dots q_r^{e_r \left(\frac{n^2-1}{4} \right)}$ term of D is $\sigma_{\frac{n-1}{2}} + \sigma_{\frac{n+1}{2}}$.

Let $i \in \{1, \dots, r\}$. For each $k \in \{0, \dots, n\}$, using Proposition 2.28 (c) and the above discussion, observe that $\tau_{i,k}$ is a monic polynomial in q_i . By Corollary 2.32 (b), the degree of $\tau_{i,k}$ in q_i is at most $e_i \left(\frac{n^2-1}{4} \right)$, with equality if and only if either $k = \frac{n-1}{2}$ or $k = \frac{n+1}{2}$.

Thus, after rewriting $N = \sum_{k=0}^n k \tau_{1,k} \tau_{2,k} \dots \tau_{r,k} \sigma_k$ as a polynomial in $q_1, q_2, \dots, q_r$, with coefficients viewed as integral polynomials in $q_{r+1}, \dots, q_m$, the coefficient of the $q_1^{e_1 \left(\frac{n^2-1}{4} \right)} \dots q_r^{e_r \left(\frac{n^2-1}{4} \right)}$ term of N is $\left(\frac{n-1}{2} \right) \sigma_{\frac{n-1}{2}} + \left(\frac{n+1}{2} \right) \sigma_{\frac{n+1}{2}}$. Similarly, after rewriting $D = \sum_{k=0}^n \tau_{1,k} \tau_{2,k} \dots \tau_{r,k} \sigma_k$ as a polynomial in $q_1, q_2, \dots, q_r$, with coefficients viewed as integral polynomials in $q_{r+1}, \dots, q_m$, the coefficient of the $q_1^{e_1 \left(\frac{n^2-1}{4} \right)} \dots q_r^{e_r \left(\frac{n^2-1}{4} \right)}$ term of D is $\sigma_{\frac{n-1}{2}} + \sigma_{\frac{n+1}{2}}$. Therefore, the assertion follows.

When $r = m$, since $\sigma_k = 1$ for each $k \in \{0, \dots, n\}$, it follows that

$$\lim_{q_1, \dots, q_m \rightarrow \infty} E(\nu_R(F)) = \frac{\left(\frac{n-1}{2} \right) + \left(\frac{n+1}{2} \right)}{2} = \frac{n}{2}. \quad \square$$

THEOREM 3.11 (*weighting by ν_R -value*) Let $r \in \{1, \dots, m\}$. For each $k \in \{0, \dots, n\}$, let

$$\sigma_k := \begin{cases} \tau_{r+1,k} \tau_{r+2,k} \dots \tau_{m,k} & \text{if } r < m \\ 1 & \text{if } r = m. \end{cases}$$

Let F vary over the set of R -free submodules of V . Also, suppose that F is weighted by $\nu_R(F)$. Then:

(a) If n is even, then

$$\lim_{q_1, \dots, q_r \rightarrow \infty} E(\nu_R(F)) = \frac{n}{2}.$$

(b) If n is odd, then

$$\lim_{q_1, \dots, q_r \rightarrow \infty} E(\nu_R(F)) = \frac{\left(\frac{n-1}{2}\right)^2 \sigma_{\frac{n-1}{2}} + \left(\frac{n+1}{2}\right)^2 \sigma_{\frac{n+1}{2}}}{\left(\frac{n-1}{2}\right) \sigma_{\frac{n-1}{2}} + \left(\frac{n+1}{2}\right) \sigma_{\frac{n+1}{2}}}.$$

In particular, when $r = m$, we have that $\lim_{q_1, \dots, q_m \rightarrow \infty} E(\nu_R(F)) = \frac{n}{2} + \frac{1}{2n}$.

Proof. Let N denote $\sum_{k=0}^n k^2 \tau_k = \sum_{k=0}^n k^2 \tau_{1,k} \tau_{2,k} \dots \tau_{r,k} \sigma_k$, and let D denote $\sum_{k=0}^n k \tau_k = \sum_{k=0}^n k \tau_{1,k} \tau_{2,k} \dots \tau_{r,k} \sigma_k$.

(a) Suppose that n is even. To prove the assertion, after rewriting both N and D as polynomials in $q_1, q_2, \dots, q_r$, with coefficients viewed as integral polynomials in $q_{r+1}, \dots, q_m$, observe that it suffices to prove the following three statements. For each $i \in \{1, \dots, r\}$, the highest power of q_i in both N and D is at most $e_i \left(\frac{n}{2}\right)^2$; the coefficient of the $q_1^{e_1 \left(\frac{n}{2}\right)^2} \dots q_r^{e_r \left(\frac{n}{2}\right)^2}$ term of N is $\left(\frac{n}{2}\right)^2 \sigma_{\frac{n}{2}}$; and the coefficient of the $q_1^{e_1 \left(\frac{n}{2}\right)^2} \dots q_r^{e_r \left(\frac{n}{2}\right)^2}$ term of D is $\left(\frac{n}{2}\right) \sigma_{\frac{n}{2}}$.

Let $i \in \{1, \dots, r\}$. For each $k \in \{0, \dots, n\}$, using Proposition 2.28 (c) and the discussion preceding Theorem 3.10, observe that $\tau_{i,k}$ is a monic polynomial in q_i . By Corollary 2.32 (a), the degree of $\tau_{i,k}$ in q_i is at most $e_i \left(\frac{n}{2}\right)^2$, with equality if and only if $k = \frac{n}{2}$.

Thus, after rewriting $N = \sum_{k=0}^n k^2 \tau_{1,k} \tau_{2,k} \dots \tau_{r,k} \sigma_k$ as a polynomial in $q_1, q_2, \dots, q_r$, with coefficients viewed as integral polynomials in $q_{r+1}, \dots, q_m$, the coefficient of the $q_1^{e_1 \left(\frac{n}{2}\right)^2} \dots q_r^{e_r \left(\frac{n}{2}\right)^2}$ term of N is $\left(\frac{n}{2}\right)^2 \sigma_{\frac{n}{2}}$. Similarly, after rewriting $D = \sum_{k=0}^n k \tau_{1,k} \tau_{2,k} \dots \tau_{r,k} \sigma_k$ as a polynomial in $q_1, q_2, \dots, q_r$, with coefficients viewed

as integral polynomials in $q_{r+1}, \dots, q_m$, the coefficient of the $q_1^{e_1(\frac{n}{2})^2} \dots q_r^{e_r(\frac{n}{2})^2}$ term of D is $\left(\frac{n}{2}\right) \sigma_{\frac{n}{2}}$. Therefore, the assertion follows.

(b) Suppose that n is odd. When $n = 1$, V is the only R -free submodule of V with ν_R -value 1. Thus, the assertion follows for $n = 1$ since $N = 0^2\tau_0 + 1^2\tau_1 = 1$ and $D = 0\tau_0 + 1\tau_1 = 1$. Therefore, without loss of generality, suppose $n \geq 3$. To prove the assertion, after rewriting both N and D as polynomials in $q_1, q_2, \dots, q_r$, with coefficients viewed as integral polynomials in $q_{r+1}, \dots, q_m$, observe that it suffices to prove the following three statements. For each $i \in \{1, \dots, r\}$, the highest power of q_i in both N and D is at most $e_i\left(\frac{n^2-1}{4}\right)$; the coefficient of the $q_1^{e_1\left(\frac{n^2-1}{4}\right)} \dots q_r^{e_r\left(\frac{n^2-1}{4}\right)}$ term of N is $\left(\frac{n-1}{2}\right)^2 \sigma_{\frac{n-1}{2}} + \left(\frac{n+1}{2}\right)^2 \sigma_{\frac{n+1}{2}}$; and the coefficient of the $q_1^{e_1\left(\frac{n^2-1}{4}\right)} \dots q_r^{e_r\left(\frac{n^2-1}{4}\right)}$ term of D is $\left(\frac{n-1}{2}\right) \sigma_{\frac{n-1}{2}} + \left(\frac{n+1}{2}\right) \sigma_{\frac{n+1}{2}}$.

Let $i \in \{1, \dots, r\}$. For each $k \in \{0, \dots, n\}$, using Proposition 2.28 (c) and the above discussion, observe that $\tau_{i,k}$ is a monic polynomial in q_i . By Corollary 2.32 (b), the degree of $\tau_{i,k}$ in q_i is at most $e_i\left(\frac{n^2-1}{4}\right)$, with equality if and only if either $k = \frac{n-1}{2}$ or $k = \frac{n+1}{2}$.

Thus, after rewriting $N = \sum_{k=0}^n k^2 \tau_{1,k} \tau_{2,k} \dots \tau_{r,k} \sigma_k$ as a polynomial in $q_1, q_2, \dots, q_r$, with coefficients viewed as integral polynomials in $q_{r+1}, \dots, q_m$, the coefficient of the $q_1^{e_1\left(\frac{n^2-1}{4}\right)} \dots q_r^{e_r\left(\frac{n^2-1}{4}\right)}$ term of N is $\left(\frac{n-1}{2}\right)^2 \sigma_{\frac{n-1}{2}} + \left(\frac{n+1}{2}\right)^2 \sigma_{\frac{n+1}{2}}$. Similarly, after rewriting $D = \sum_{k=0}^n k \tau_{1,k} \tau_{2,k} \dots \tau_{r,k} \sigma_k$ as a polynomial in $q_1, q_2, \dots, q_r$, with coefficients viewed as integral polynomials in $q_{r+1}, \dots, q_m$, the coefficient of the $q_1^{e_1\left(\frac{n^2-1}{4}\right)} \dots q_r^{e_r\left(\frac{n^2-1}{4}\right)}$ term of D is $\left(\frac{n-1}{2}\right) \sigma_{\frac{n-1}{2}} + \left(\frac{n+1}{2}\right) \sigma_{\frac{n+1}{2}}$. Therefore, the assertion follows.

When $r = m$, since $\sigma_k = 1$ for each $k \in \{0, \dots, n\}$, it follows that

$$\lim_{q_1, \dots, q_m \rightarrow \infty} E(\nu_R(F)) = \frac{\left(\frac{n-1}{2}\right)^2 + \left(\frac{n+1}{2}\right)^2}{\left(\frac{n-1}{2}\right) + \left(\frac{n+1}{2}\right)} = \frac{n}{2} + \frac{1}{2n}. \quad \square$$

THEOREM 3.12 (*weighting by cardinality*) Let $r \in \{1, \dots, m\}$. For each $k \in \{0, \dots, n\}$, let

$$\sigma'_k := \begin{cases} \tau'_{r+1,k} \tau'_{r+2,k} \dots \tau'_{m,k} & \text{if } r < m \\ 1 & \text{if } r = m. \end{cases}$$

Let F vary over the set of R -free submodules of V . Also, suppose that F is weighted by cardinality. Then:

(a) If n is even, then

$$\lim_{q_1, \dots, q_r \rightarrow \infty} E(\nu_R(F)) = \frac{\left(\frac{n}{2}\right) \sigma'_{\frac{n}{2}} + \left(\frac{n}{2} + 1\right) \sigma'_{\frac{n}{2}+1}}{\sigma'_{\frac{n}{2}} + \sigma'_{\frac{n}{2}+1}}.$$

In particular, when $r = m$, we have that $\lim_{q_1, \dots, q_m \rightarrow \infty} E(\nu_R(F)) = \frac{n+1}{2}$.

(b) If n is odd, then

$$\lim_{q_1, \dots, q_r \rightarrow \infty} E(\nu_R(F)) = \frac{n+1}{2}.$$

Proof. Let N denote $\sum_{k=0}^n k \tau'_k = \sum_{k=0}^n k \tau'_{1,k} \tau'_{2,k} \dots \tau'_{r,k} \sigma'_k$, and let D denote $\sum_{k=0}^n \tau'_k = \sum_{k=0}^n \tau'_{1,k} \tau'_{2,k} \dots \tau'_{r,k} \sigma'_k$.

(a) Suppose that n is even. To prove the assertion, after rewriting both N and D as polynomials in $q_1, q_2, \dots, q_r$, with coefficients viewed as integral polynomials in $q_{r+1}, \dots, q_m$, observe that it suffices to prove the following three statements. For each $i \in \{1, \dots, r\}$, the highest power of q_i in both N and D is at most $e_i \left(\frac{(n+1)^2 - 1}{4} \right)$; the coefficient of the $q_1^{e_1 \left(\frac{(n+1)^2 - 1}{4} \right)} \dots q_r^{e_r \left(\frac{(n+1)^2 - 1}{4} \right)}$ term of N is $\left(\frac{n}{2}\right) \sigma'_{\frac{n}{2}} + \left(\frac{n}{2} + 1\right) \sigma'_{\frac{n}{2}+1}$; and the coefficient of the $q_1^{e_1 \left(\frac{(n+1)^2 - 1}{4} \right)} \dots q_r^{e_r \left(\frac{(n+1)^2 - 1}{4} \right)}$ term of D is $\sigma'_{\frac{n}{2}} + \sigma'_{\frac{n}{2}+1}$.

Let $i \in \{1, \dots, r\}$. For each $k \in \{0, \dots, n\}$, using Proposition 2.28 (c) and the discussion preceding Theorem 3.10, observe that $\tau'_{i,k}$ is a monic polynomial in q_i . By Corollary 2.34 (a), the degree of $\tau'_{i,k}$ in q_i is at most $e_i \left(\frac{(n+1)^2-1}{4} \right)$, with equality if and only if either $k = \frac{n}{2}$ or $k = \frac{n}{2} + 1$.

Thus, after rewriting $N = \sum_{k=0}^n k \tau'_{1,k} \tau'_{2,k} \dots \tau'_{r,k} \sigma'_k$ as a polynomial in $q_1, q_2, \dots, q_r$, with coefficients viewed as integral polynomials in $q_{r+1}, \dots, q_m$, the coefficient of the $q_1^{e_1 \left(\frac{(n+1)^2-1}{4} \right)} \dots q_r^{e_r \left(\frac{(n+1)^2-1}{4} \right)}$ term of N is $\left(\frac{n}{2} \right) \sigma'_{\frac{n}{2}} + \left(\frac{n}{2} + 1 \right) \sigma'_{\frac{n}{2}+1}$. Similarly, after rewriting $D = \sum_{k=0}^n \tau'_{1,k} \tau'_{2,k} \dots \tau'_{r,k} \sigma'_k$ as a polynomial in $q_1, q_2, \dots, q_r$, with coefficients viewed as integral polynomials in $q_{r+1}, \dots, q_m$, the coefficient of the $q_1^{e_1 \left(\frac{(n+1)^2-1}{4} \right)} \dots q_r^{e_r \left(\frac{(n+1)^2-1}{4} \right)}$ term of D is $\sigma'_{\frac{n}{2}} + \sigma'_{\frac{n}{2}+1}$. Therefore, the assertion follows.

When $r = m$, since $\sigma'_k = 1$ for each $k \in \{0, \dots, n\}$, it follows that

$$\lim_{q_1, \dots, q_m \rightarrow \infty} E(\nu_R(F)) = \frac{\left(\frac{n}{2} \right) + \left(\frac{n}{2} + 1 \right)}{2} = \frac{n+1}{2}.$$

(b) Suppose that n is odd. To prove the assertion, after rewriting both N and D as polynomials in $q_1, q_2, \dots, q_r$, with coefficients viewed as integral polynomials in $q_{r+1}, \dots, q_m$, observe that it suffices to prove the following three statements. For each $i \in \{1, \dots, r\}$, the highest power of q_i in both N and D is at most $e_i \left(\frac{n+1}{2} \right)^2$; the coefficient of the $q_1^{e_1 \left(\frac{n+1}{2} \right)^2} \dots q_r^{e_r \left(\frac{n+1}{2} \right)^2}$ term of N is $\left(\frac{n+1}{2} \right) \sigma'_{\frac{n+1}{2}}$; and the coefficient of the $q_1^{e_1 \left(\frac{n+1}{2} \right)^2} \dots q_r^{e_r \left(\frac{n+1}{2} \right)^2}$ term of D is $\sigma'_{\frac{n+1}{2}}$.

Let $i \in \{1, \dots, r\}$. For each $k \in \{0, \dots, n\}$, using Proposition 2.28 (c) and the discussion preceding Theorem 3.10, observe that $\tau'_{i,k}$ is a monic polynomial in q_i . By Corollary 2.34 (b), the degree of $\tau'_{i,k}$ in q_i is at most $e_i \left(\frac{n+1}{2} \right)^2$, with equality if and only if $k = \frac{n+1}{2}$.

Thus, after rewriting $N = \sum_{k=0}^n k \tau'_{1,k} \tau'_{2,k} \dots \tau'_{r,k} \sigma'_k$ as a polynomial in

$q_1, q_2, \dots, q_r$, with coefficients viewed as integral polynomials in $q_{r+1}, \dots, q_m$, the coefficient of the $q_1^{e_1(\frac{n+1}{2})^2} \dots q_r^{e_r(\frac{n+1}{2})^2}$ term of N is $(\frac{n+1}{2})\sigma'_{\frac{n+1}{2}}$. Similarly, after rewriting $D = \sum_{k=0}^n \tau'_{1,k} \tau'_{2,k} \dots \tau'_{r,k} \sigma'_k$ as a polynomial in $q_1, q_2, \dots, q_r$, with coefficients viewed as integral polynomials in $q_{r+1}, \dots, q_m$, the coefficient of the $q_1^{e_1(\frac{n+1}{2})^2} \dots q_r^{e_r(\frac{n+1}{2})^2}$ term of D is $\sigma'_{\frac{n+1}{2}}$. Therefore, the assertion follows. $\square$

For the rest of this chapter, our focus changes. Instead of asking for the expected value of $\nu_R(W)$, for suitably restricted types of submodules W of V , we ask for the expected value of a quantity which measures the largest free submodule inside a suitably random submodule W of V .

For an arbitrary submodule W of V , let

$$\theta_R(W) := \max\{\nu_R(F) : F \text{ is an } R\text{-free submodule of } W\}.$$

In other words, $\theta_R(W)$ is the highest rank attained by an R -free submodule of W .

In order to calculate the expected θ_R -value of an arbitrary R -submodule of V , we introduce the following notation. For each $i \in \{1, \dots, m\}$ and $j \in \{0, \dots, n\}$, let $\mu_{i,j}$ denote the number of R_i -submodules of V_i with θ_{R_i} -value j . Thus, for each $i \in \{1, \dots, m\}$, Proposition 2.24 yields that $\mu_{i,0} = 1 + \sum_{0 \leq d_2 \leq \dots \leq d_{e_i} \leq n} \Lambda_i(0, d_2, \dots, d_{e_i})$ and, for each $j \in \{1, \dots, n\}$, $\mu_{i,j} = \sum_{j \leq d_2 \leq \dots \leq d_{e_i} \leq n} \Lambda_i(j, d_2, \dots, d_{e_i})$, where Λ_i denotes Λ in Proposition 2.28 (c) using R_i as the base ring and V_i as the ambient R_i -free module. For each $k \in \{0, \dots, n\}$, let μ_k denote the number of R -submodules of V with θ_R -value k . Thus, for each $k \in \{0, \dots, n\}$, using Proposition 3.3 and the discussion preceding Proposition 3.2, observe that

$$\mu_k = (\sum_{j=k}^n \mu_{1,j}) \dots (\sum_{j=k}^n \mu_{m,j}) - (\sum_{j=k+1}^n \mu_{1,j}) \dots (\sum_{j=k+1}^n \mu_{m,j}).$$

LEMMA 3.13 (unweighted) *As W varies over the set of R -submodules of V ,*

$$E(\theta_R(W)) = \frac{\sum_{k=1}^n (\sum_{j=k}^n \mu_{1,j}) (\sum_{j=k}^n \mu_{2,j}) \cdots (\sum_{j=k}^n \mu_{m,j})}{(\sum_{j=0}^n \mu_{1,j}) (\sum_{j=0}^n \mu_{2,j}) \cdots (\sum_{j=0}^n \mu_{m,j})}.$$

Proof. For each $k \in \{0, \dots, n\}$, let b_k denote $(\sum_{j=k}^n \mu_{1,j}) (\sum_{j=k}^n \mu_{2,j}) \cdots (\sum_{j=k}^n \mu_{m,j})$; put $b_{n+1} = 0$. By the above remarks, $\mu_k = b_k - b_{k+1}$ for each k . Hence,

$$\begin{aligned} E(\theta_R(W)) &= \frac{\sum_{k=0}^n k \mu_k}{\sum_{k=0}^n \mu_k} = \frac{(\sum_{k=0}^{n-1} k(b_k - b_{k+1})) + nb_n}{\sum_{k=0}^{n-1} (b_k - b_{k+1}) + b_n} = \frac{(\sum_{k=0}^{n-1} k b_k) - (\sum_{k=0}^{n-1} k b_{k+1}) + nb_n}{(\sum_{k=0}^{n-1} b_k) - (\sum_{k=0}^{n-1} b_{k+1}) + b_n} = \\ &= \frac{(\sum_{k=0}^{n-2} (k+1)b_{k+1}) - (\sum_{k=0}^{n-1} k b_{k+1}) + nb_n}{(\sum_{k=0}^{n-1} b_k) - (\sum_{k=1}^n b_k) + b_n} = \frac{(\sum_{k=0}^{n-2} b_{k+1}) - (n-1)b_n + nb_n}{b_0 - b_n + b_n} = \frac{\sum_{k=1}^n b_k}{b_0} = \\ &= \frac{\sum_{k=1}^n (\sum_{j=k}^n \mu_{1,j}) (\sum_{j=k}^n \mu_{2,j}) \cdots (\sum_{j=k}^n \mu_{m,j})}{(\sum_{j=0}^n \mu_{1,j}) (\sum_{j=0}^n \mu_{2,j}) \cdots (\sum_{j=0}^n \mu_{m,j})}. \quad \square \end{aligned}$$

LEMMA 3.14 (weighting by θ_R -value) *Let W vary over the set of R -submodules of V . When W is weighted by $\theta_R(W)$,*

$$E(\theta_R(W)) = \frac{\sum_{k=1}^n (2k-1) (\sum_{j=k}^n \mu_{1,j}) (\sum_{j=k}^n \mu_{2,j}) \cdots (\sum_{j=k}^n \mu_{m,j})}{\sum_{k=1}^n (\sum_{j=k}^n \mu_{1,j}) (\sum_{j=k}^n \mu_{2,j}) \cdots (\sum_{j=k}^n \mu_{m,j})}.$$

Proof. For each $k \in \{0, \dots, n\}$, let b_k denote $(\sum_{j=k}^n \mu_{1,j}) (\sum_{j=k}^n \mu_{2,j}) \cdots (\sum_{j=k}^n \mu_{m,j})$; put $b_{n+1} = 0$. When W is weighted by $\theta_R(W)$, we have that

$$\begin{aligned} E(\theta_R(W)) &= \frac{\sum_{k=0}^n k^2 \mu_k}{\sum_{k=0}^n k \mu_k} = \frac{(\sum_{k=0}^{n-1} k^2 (b_k - b_{k+1})) + n^2 b_n}{(\sum_{k=0}^{n-1} k (b_k - b_{k+1})) + nb_n} = \frac{(\sum_{k=0}^{n-1} k^2 b_k) - (\sum_{k=0}^{n-1} k^2 b_{k+1}) + n^2 b_n}{(\sum_{k=0}^{n-1} k b_k) - (\sum_{k=0}^{n-1} k b_{k+1}) + nb_n} = \\ &= \frac{(\sum_{k=0}^{n-2} (k+1)^2 b_{k+1}) - (\sum_{k=0}^{n-1} k^2 b_{k+1}) + n^2 b_n}{(\sum_{k=0}^{n-2} (k+1) b_{k+1}) - (\sum_{k=0}^{n-1} k b_{k+1}) + nb_n} = \\ &= \frac{(\sum_{k=0}^{n-2} k^2 b_{k+1}) + 2(\sum_{k=0}^{n-2} k b_{k+1}) + (\sum_{k=0}^{n-2} b_{k+1}) - (\sum_{k=0}^{n-1} k^2 b_{k+1}) + n^2 b_n}{(\sum_{k=0}^{n-2} k b_{k+1}) + (\sum_{k=0}^{n-2} b_{k+1}) - (\sum_{k=0}^{n-1} k b_{k+1}) + nb_n} = \\ &= \frac{(\sum_{k=0}^{n-2} (2k+1) b_{k+1}) - (n-1)^2 b_n + n^2 b_n}{(\sum_{k=0}^{n-2} b_{k+1}) - (n-1)b_n + nb_n} = \frac{\sum_{k=0}^{n-1} (2k+1) b_{k+1}}{\sum_{k=0}^{n-1} b_{k+1}} = \frac{\sum_{k=1}^n (2k-1) b_k}{\sum_{k=1}^n b_k} = \\ &= \frac{\sum_{k=1}^n (2k-1) (\sum_{j=k}^n \mu_{1,j}) (\sum_{j=k}^n \mu_{2,j}) \cdots (\sum_{j=k}^n \mu_{m,j})}{\sum_{k=1}^n (\sum_{j=k}^n \mu_{1,j}) (\sum_{j=k}^n \mu_{2,j}) \cdots (\sum_{j=k}^n \mu_{m,j})}. \quad \square \end{aligned}$$

In order to calculate $E(\nu_R(W))$ when W is weighted by cardinality, we introduce the following notation. For each $i \in \{1, \dots, m\}$ and $j \in \{0, \dots, n\}$, let $\mu'_{i,j}$ denote the sum of cardinalities of all of R_i -submodules of V_i with θ_{R_i} -value j . Thus, for each $i \in \{1, \dots, m\}$, Propositions 2.24 and 2.25 yield that

$$\begin{aligned}\mu'_{i,0} &= 1 + \sum_{0 \leq d_2 \leq \dots \leq d_{e_i} \leq n} \Lambda'_i(0, d_2, \dots, d_{e_i}) = \\ &= 1 + \sum_{0 \leq d_2 \leq \dots \leq d_{e_i} \leq n} (q^{\sum_{s=2}^{e_i} d_s}) \Lambda_i(0, d_2, \dots, d_{e_i})\end{aligned}$$

and, for each $j \in \{1, \dots, n\}$,

$$\begin{aligned}\mu'_{i,j} &= \sum_{j \leq d_2 \leq \dots \leq d_{e_i} \leq n} \Lambda'_i(j, d_2, \dots, d_{e_i}) = \\ &= \sum_{j \leq d_2 \leq \dots \leq d_{e_i} \leq n} (q^{\sum_{s=2}^{e_i} d_s + j}) \Lambda_i(j, d_2, \dots, d_{e_i}),\end{aligned}$$

where Λ_i denotes Λ in Proposition 2.28 (c) using R_i as the base ring and V_i as the ambient R_i -free module. For each $k \in \{0, \dots, n\}$, let μ'_k denote the sum of cardinalities of all R -submodules of V with θ_R -value k . Thus, for each $k \in \{0, \dots, n\}$, observe via Proposition 3.3 (using the distributive property) that

$$\mu'_k = (\sum_{j=k}^n \mu'_{1,j}) \cdots (\sum_{j=k}^n \mu'_{m,j}) - (\sum_{j=k+1}^n \mu'_{1,j}) \cdots (\sum_{j=k+1}^n \mu'_{m,j}).$$

LEMMA 3.15 (*weighting by cardinality*) *Let W vary over the set of R -submodules of V . When W is weighted by cardinality,*

$$E(\theta_R(W)) = \frac{\sum_{k=1}^n (\sum_{j=k}^n \mu'_{1,j}) (\sum_{j=k}^n \mu'_{2,j}) \cdots (\sum_{j=k}^n \mu'_{m,j})}{(\sum_{j=0}^n \mu'_{1,j}) (\sum_{j=0}^n \mu'_{2,j}) \cdots (\sum_{j=0}^n \mu'_{m,j})}.$$

Proof. When W is weighted by cardinality, $E(\theta_R(W)) = \frac{\sum_{k=0}^n k \mu'_k}{\sum_{k=0}^n \mu'_k}$. Thus, one may adapt the proof of Lemma 3.13, replacing $\mu_{i,j}$ by $\mu'_{i,j}$. $\square$

In the next three theorems, letting $r \in \{1, \dots, m\}$, we take the limit of the expected values found in Lemmas 3.13, 3.14, and 3.15 respectively as $q_1, q_2, \dots, q_r$

all approach infinity. (More precisely, for each $\epsilon > 0$, we require that there exist an $N > 0$ such that the expected value is within ϵ of the limit value whenever $q_1, q_2, \dots, q_r \geq N$.)

THEOREM 3.16 (*unweighted*) Let $r \in \{1, \dots, m\}$. For each $k \in \{0, \dots, n\}$, let

$$\sigma_k := \begin{cases} (\sum_{j=k}^n \mu_{r+1,j})(\sum_{j=k}^n \mu_{r+2,j}) \cdots (\sum_{j=k}^n \mu_{m,j}) & \text{if } r < m \\ 1 & \text{if } r = m. \end{cases}$$

Further, let W vary over the set of R -submodules of V .

(a) If n is even, then

$$\lim_{q_1, q_2, \dots, q_r \rightarrow \infty} E(\theta_R(W)) = \frac{\sum_{k=1}^{\frac{n}{2}} \sigma_k}{\sigma_0}.$$

In particular, when $r = m$, we have that

$$\lim_{q_1, q_2, \dots, q_m \rightarrow \infty} E(\theta_R(W)) = \frac{n}{2}.$$

(b) If n is odd, then

$$\lim_{q_1, q_2, \dots, q_r \rightarrow \infty} E(\theta_R(W)) = \frac{(e_1+1) \cdots (e_r+1) (\sum_{k=1}^{\frac{n-1}{2}} \sigma_k) + \sigma_{\frac{n+1}{2}}}{(e_1+1) \cdots (e_r+1) \sigma_0}.$$

In particular, when $r = m$, we have that

$$\lim_{q_1, q_2, \dots, q_m \rightarrow \infty} E(\theta_R(W)) = \frac{n}{2} - \frac{1}{2} + \frac{1}{(e_1+1) \cdots (e_m+1)}.$$

Proof. Referring to the numerator and denominator appearing in Lemma 3.13,

let N denote $\sum_{k=1}^n (\sum_{j=k}^n \mu_{1,j})(\sum_{j=k}^n \mu_{2,j}) \cdots (\sum_{j=k}^n \mu_{r,j}) \sigma_k$, and let D denote $(\sum_{j=0}^n \mu_{1,j})(\sum_{j=0}^n \mu_{2,j}) \cdots (\sum_{j=0}^n \mu_{r,j}) \sigma_0$.

(a) Suppose that n is even. To prove the assertion, after rewriting both N and D as polynomials in $q_1, q_2, \dots, q_r$, with coefficients viewed as integral polynomials in

$q_{r+1}, \dots, q_m$, observe that it suffices to prove the following three statements. For each $i \in \{1, \dots, r\}$, the highest power of q_i in both N and D is at most $e_i \left(\frac{n}{2}\right)^2$; the coefficient of the $q_1^{e_1 \left(\frac{n}{2}\right)^2} \dots q_r^{e_r \left(\frac{n}{2}\right)^2}$ term of N is $\sum_{k=1}^{\frac{n}{2}} \sigma_k$; and the coefficient of the $q_1^{e_1 \left(\frac{n}{2}\right)^2} \dots q_r^{e_r \left(\frac{n}{2}\right)^2}$ term of D is σ_0 .

Let $i \in \{1, \dots, r\}$. For each $k \in \{0, \dots, n\}$, using Proposition 2.28 (c) and the discussion preceding Lemma 3.13, observe that $\sum_{j=k}^n \mu_{i,j}$ is an integral polynomial in q_i . Using Corollary 2.32 (a), one checks that the degree of $\sum_{j=k}^n \mu_{i,j}$ in q_i is at most $e_i \left(\frac{n}{2}\right)^2$, with equality if and only if $k \in \{0, \dots, \frac{n}{2}\}$. In particular, for each $k \in \{0, \dots, \frac{n}{2}\}$, observe via Corollary 2.32 (a) and Proposition 2.28 (c) that $\sum_{j=k}^n \mu_{i,j}$ is a monic polynomial in q_i .

Thus, after rewriting $N = \sum_{k=1}^n (\sum_{j=k}^n \mu_{1,j}) (\sum_{j=k}^n \mu_{2,j}) \dots (\sum_{j=k}^n \mu_{r,j}) \sigma_k$ as a polynomial in $q_1, \dots, q_r$, with coefficients viewed as integral polynomials in $q_{r+1}, \dots, q_m$, the coefficient of the $q_1^{e_1 \left(\frac{n}{2}\right)^2} \dots q_r^{e_r \left(\frac{n}{2}\right)^2}$ term of N is $\sum_{k=1}^{\frac{n}{2}} \sigma_k$. Similarly, after rewriting $D = (\sum_{j=0}^n \mu_{1,j}) (\sum_{j=0}^n \mu_{2,j}) \dots (\sum_{j=0}^n \mu_{r,j}) \sigma_0$ as a polynomial in $q_1, \dots, q_r$, with coefficients viewed as integral polynomials in $q_{r+1}, \dots, q_m$, the coefficient of the $q_1^{e_1 \left(\frac{n}{2}\right)^2} \dots q_r^{e_r \left(\frac{n}{2}\right)^2}$ term of D is σ_0 . Therefore, the assertion follows.

When $r = m$, since $\sigma_k = 1$ for each $k \in \{0, \dots, n\}$, it follows that

$$\lim_{q_1, q_2, \dots, q_m \rightarrow \infty} E(\theta_R(W)) = \frac{\sum_{k=1}^{\frac{n}{2}} \sigma_k}{\sigma_0} = \frac{\sum_{k=1}^{\frac{n}{2}} 1}{1} = \frac{n}{2}.$$

(b) Suppose that n is odd. When $n = 1$, after some calculation, the assertion follows for $n = 1$ (even before the limit is taken) since, using Lemma 3.13 and the fact that R_i is an SPIR, one checks that $\mu_{i,0} = e_i$ and $\mu_{i,1} = 1$ for every $i \in \{1, \dots, m\}$. Therefore, without loss of generality, suppose $n \geq 3$. To prove the assertion, after rewriting both N and D as polynomials in $q_1, \dots, q_r$, with coefficients viewed as integral polynomials in $q_{r+1}, \dots, q_m$, it suffices to prove the following

three statements. For each $i \in \{1, \dots, r\}$, the highest power of q_i in both N and D is at most $e_i \left(\frac{n^2-1}{4} \right)$; the coefficient of the $q_1^{e_1 \left(\frac{n^2-1}{4} \right)} \dots q_r^{e_r \left(\frac{n^2-1}{4} \right)}$ term of N is $(e_1 + 1) \dots (e_r + 1) \left(\sum_{k=0}^{\frac{n-1}{2}} \sigma_k \right) + \sigma_{\frac{n+1}{2}}$; and the coefficient of the $q_1^{e_1 \left(\frac{n^2-1}{4} \right)} \dots q_r^{e_r \left(\frac{n^2-1}{4} \right)}$ term of D is $(e_1 + 1) \dots (e_r + 1) \sigma_0$.

Let $i \in \{1, \dots, r\}$. For each $k \in \{0, \dots, n\}$, using Proposition 2.28 (c) and the discussion preceding Lemma 3.13, observe that $\sum_{j=k}^n \mu_{i,j}$ is an integral polynomial in q_i . Using Corollary 2.32 (b), one checks that the degree of $\sum_{j=k}^n \mu_{i,j}$ in q_i is at most $e_i \left(\frac{n^2-1}{4} \right)$, with equality if and only if $k \in \{0, \dots, \frac{n+1}{2}\}$. In particular, observe via Corollary 2.32 (b) and Proposition 2.28 (c) that $\sum_{j=\frac{n+1}{2}}^n \mu_{i,j}$ is a monic polynomial in q_i . Similarly, using Corollary 2.32 (b) and Proposition 2.28 (c), one checks, for each $k \in \{0, \dots, \frac{n-1}{2}\}$, that the leading coefficient of $\sum_{j=k}^n \mu_{i,j}$ as a polynomial in q_i is $e_i + 1$.

Thus, after rewriting $N = \sum_{k=1}^n \left(\sum_{j=k}^n \mu_{1,j} \right) \left(\sum_{j=k}^n \mu_{2,j} \right) \dots \left(\sum_{j=k}^n \mu_{r,j} \right) \sigma_k$ as a polynomial in $q_1, \dots, q_r$, with coefficients viewed as integral polynomials in

$q_{r+1}, \dots, q_m$, the coefficient of the $q_1^{e_1 \left(\frac{n^2-1}{4} \right)} \dots q_r^{e_r \left(\frac{n^2-1}{4} \right)}$ term of N is

$$(e_1 + 1) \dots (e_r + 1) \left(\sum_{k=1}^{\frac{n-1}{2}} \sigma_k \right) + \sigma_{\frac{n+1}{2}}.$$

Similarly, after rewriting $D = \left(\sum_{j=0}^n \mu_{1,j} \right) \left(\sum_{j=0}^n \mu_{2,j} \right) \dots \left(\sum_{j=0}^n \mu_{r,j} \right) \sigma_0$ as a polynomial in $q_1, \dots, q_r$, with coefficients viewed as integral polynomials in $q_{r+1}, \dots, q_m$,

the coefficient of the $q_1^{e_1 \left(\frac{n^2-1}{4} \right)} \dots q_r^{e_r \left(\frac{n^2-1}{4} \right)}$ term of D is $(e_1 + 1) \dots (e_r + 1) \sigma_0$.

Therefore, the assertion follows.

When $r = m$, since $\sigma_k = 1$ for each $k \in \{0, \dots, n\}$, it follows that

$$\lim_{q_1, q_2, \dots, q_m \rightarrow \infty} E(\theta_R(W)) = \frac{(e_1+1) \dots (e_m+1) \left(\sum_{k=1}^{\frac{n-1}{2}} \sigma_k \right) + \sigma_{\frac{n+1}{2}}}{(e_1+1) \dots (e_m+1) \sigma_0} = \frac{(e_1+1) \dots (e_m+1) \left(\sum_{k=1}^{\frac{n-1}{2}} 1 \right) + 1}{(e_1+1) \dots (e_m+1)} =$$

$$\frac{(e_1+1)\dots(e_m+1)(\frac{n-1}{2}+1)}{(e_1+1)\dots(e_m+1)} = \frac{n}{2} - \frac{1}{2} + \frac{1}{(e_1+1)\dots(e_m+1)}. \quad \square$$

THEOREM 3.17 (*weighting by θ_R -value*) Let $r \in \{1, \dots, m\}$. For each $k \in \{0, \dots, n\}$, let

$$\sigma_k := \begin{cases} (\sum_{j=k}^n \mu_{r+1,j})(\sum_{j=k}^n \mu_{r+2,j}) \dots (\sum_{j=k}^n \mu_{m,j}) & \text{if } r < m \\ 1 & \text{if } r = m. \end{cases}$$

Let W vary over the set of R -submodules of V . Also, suppose that W is weighted by $\theta_R(W)$. Then:

(a) If n is even, then

$$\lim_{q_1, q_2, \dots, q_r \rightarrow \infty} E(\theta_R(W)) = \frac{\sum_{k=1}^{\frac{n}{2}} (2k-1)\sigma_k}{\sum_{k=1}^{\frac{n}{2}} \sigma_k}.$$

In particular, when $r = m$, we have that

$$\lim_{q_1, q_2, \dots, q_m \rightarrow \infty} E(\theta_R(W)) = \frac{n}{2}.$$

(b) If n is odd, then

$$\lim_{q_1, q_2, \dots, q_r \rightarrow \infty} E(\theta_R(W)) = \frac{(e_1+1)\dots(e_r+1)(\sum_{k=1}^{\frac{n-1}{2}} (2k-1)\sigma_k) + n\sigma_{\frac{n+1}{2}}}{(e_1+1)\dots(e_r+1)(\sum_{k=1}^{\frac{n-1}{2}} \sigma_k) + \sigma_{\frac{n+1}{2}}}.$$

In particular, when $r = m$, we have that

$$\lim_{q_1, q_2, \dots, q_m \rightarrow \infty} E(\theta_R(W)) = \frac{n}{2} - \frac{1}{2} + \frac{1}{(e_1+1)\dots(e_m+1)} + \frac{2((e_1+1)\dots(e_m+1)-1)}{(e_1+1)\dots(e_m+1)((e_1+1)\dots(e_m+1)(n-1)+2)}.$$

Proof. Referring to the numerator and denominator appearing in Lemma 3.14, let N denote $\sum_{k=1}^n (2k-1)(\sum_{j=k}^n \mu_{1,j})(\sum_{j=k}^n \mu_{2,j}) \dots (\sum_{j=k}^n \mu_{r,j})\sigma_k$, and let D denote $\sum_{k=1}^n (\sum_{j=k}^n \mu_{1,j})(\sum_{j=k}^n \mu_{2,j}) \dots (\sum_{j=k}^n \mu_{r,j})\sigma_k$.

(a) Suppose that n is even. To prove the assertion, after rewriting both N and D as polynomials in $q_1, q_2, \dots, q_r$, with coefficients viewed as integral polynomials in $q_{r+1}, \dots, q_m$, observe that it suffices to prove the following three statements. For each $i \in \{1, \dots, r\}$, the highest power of q_i in both N and D is at most $e_i \left(\frac{n}{2}\right)^2$; the coefficient of the $q_1^{e_1 \left(\frac{n}{2}\right)^2} \dots q_r^{e_r \left(\frac{n}{2}\right)^2}$ term of N is $\sum_{k=1}^{\frac{n}{2}} (2k-1) \sigma_k$; and the coefficient of the $q_1^{e_1 \left(\frac{n}{2}\right)^2} \dots q_r^{e_r \left(\frac{n}{2}\right)^2}$ term of D is $\sum_{k=1}^{\frac{n}{2}} \sigma_k$.

Let $i \in \{1, \dots, r\}$. For each $k \in \{0, \dots, n\}$, using Proposition 2.28 (c) and the discussion preceding Lemma 3.13, observe that $\sum_{j=k}^n \mu_{i,j}$ is an integral polynomial in q_i . Using Corollary 2.32 (a), one checks that the degree of $\sum_{j=k}^n \mu_{i,j}$ in q_i is at most $e_i \left(\frac{n}{2}\right)^2$, with equality if and only if $k \in \{0, \dots, \frac{n}{2}\}$. In particular, for each $k \in \{0, \dots, \frac{n}{2}\}$, observe via Corollary 2.32 (a) and Proposition 2.28 (c) that $\sum_{j=k}^n \mu_{i,j}$ is a monic polynomial in q_i .

Thus, after rewriting $N = \sum_{k=1}^n (2k-1) (\sum_{j=k}^n \mu_{1,j}) (\sum_{j=k}^n \mu_{2,j}) \dots (\sum_{j=k}^n \mu_{r,j}) \sigma_k$ as a polynomial in $q_1, \dots, q_r$, with coefficients viewed as integral polynomials in $q_{r+1}, \dots, q_m$, the coefficient of the $q_1^{e_1 \left(\frac{n}{2}\right)^2} \dots q_r^{e_r \left(\frac{n}{2}\right)^2}$ term of N is $\sum_{k=1}^{\frac{n}{2}} (2k-1) \sigma_k$. Similarly, after rewriting $D = \sum_{k=1}^n (\sum_{j=k}^n \mu_{1,j}) (\sum_{j=k}^n \mu_{2,j}) \dots (\sum_{j=k}^n \mu_{r,j}) \sigma_k$ as a polynomial in $q_1, \dots, q_r$, with coefficients viewed as integral polynomials in $q_{r+1}, \dots, q_m$, the coefficient of the $q_1^{e_1 \left(\frac{n}{2}\right)^2} \dots q_r^{e_r \left(\frac{n}{2}\right)^2}$ term of D is $\sum_{k=1}^{\frac{n}{2}} \sigma_k$. Therefore, the assertion follows.

When $r = m$, since $\sigma_k = 1$ for each $k \in \{0, \dots, n\}$, it follows that

$$\begin{aligned} \lim_{q_1, q_2, \dots, q_m \rightarrow \infty} E(\theta_R(W)) &= \frac{\sum_{k=1}^{\frac{n}{2}} (2k-1) \sigma_k}{\sum_{k=1}^{\frac{n}{2}} \sigma_k} = \frac{\sum_{k=1}^{\frac{n}{2}} (2k-1)}{\frac{n}{2}} = \frac{2(\sum_{k=1}^{\frac{n}{2}} k) - \frac{n}{2}}{\frac{n}{2}} = \\ &= \frac{2\left(\frac{\frac{n}{2}(\frac{n}{2}+1)}{2}\right) - \frac{n}{2}}{\frac{n}{2}} = \frac{n}{2}. \end{aligned}$$

(b) Suppose that n is odd. When $n = 1$, using Lemma 3.14, observe that $E(\theta_R(W)) = 1$. Thus, the assertion follows for $n = 1$ (even before the limit is taken). Therefore, without loss of generality, suppose $n \geq 3$. To prove the assertion, after rewriting both N and D as polynomials in $q_1, \dots, q_r$, with coefficients viewed as integral polynomials in $q_{r+1}, \dots, q_m$, it suffices to prove the following three statements. For each $i \in \{1, \dots, r\}$, the highest power of q_i in both N and D is at most $e_i \left(\frac{n^2-1}{4} \right)$; the coefficient of the $q_1^{e_1 \left(\frac{n^2-1}{4} \right)} \dots q_r^{e_r \left(\frac{n^2-1}{4} \right)}$ term of N is $(e_1 + 1) \dots (e_r + 1) \left(\sum_{k=1}^{\frac{n-1}{2}} (2k-1) \sigma_k \right) + n \sigma_{\frac{n+1}{2}}$; and the coefficient of the $q_1^{e_1 \left(\frac{n^2-1}{4} \right)} \dots q_r^{e_r \left(\frac{n^2-1}{4} \right)}$ term of D is $(e_1 + 1) \dots (e_r + 1) \left(\sum_{k=1}^{\frac{n-1}{2}} \sigma_k \right) + \sigma_{\frac{n+1}{2}}$.

Let $i \in \{1, \dots, r\}$. For each $k \in \{0, \dots, n\}$, using Proposition 2.28 (c) and the discussion preceding Lemma 3.13, observe that $\sum_{j=k}^n \mu_{i,j}$ is an integral polynomial in q_i . Using Corollary 2.32 (b), one checks that the degree of $\sum_{j=k}^n \mu_{i,j}$ in q_i is at most $e_i \left(\frac{n^2-1}{4} \right)$, with equality if and only if $k \in \{0, \dots, \frac{n+1}{2}\}$. In particular, observe via Corollary 2.32 (b) and Proposition 2.28 (c) that $\sum_{j=\frac{n+1}{2}}^n \mu_{i,j}$ is a monic polynomial in q_i . Similarly, using Corollary 2.32 (b) and Proposition 2.28 (c), one checks, for each $k \in \{0, \dots, \frac{n-1}{2}\}$, that the leading coefficient of $\sum_{j=k}^n \mu_{i,j}$ as a polynomial in q_i is $e_i + 1$.

Thus, after rewriting

$$N = \sum_{k=1}^n (2k-1) \left(\sum_{j=k}^n \mu_{1,j} \right) \left(\sum_{j=k}^n \mu_{2,j} \right) \dots \left(\sum_{j=k}^n \mu_{r,j} \right) \sigma_k$$

as a polynomial in $q_1, \dots, q_r$, with coefficients viewed as integral polynomials in

$q_{r+1}, \dots, q_m$, the coefficient of the $q_1^{e_1 \left(\frac{n^2-1}{4} \right)} \dots q_r^{e_r \left(\frac{n^2-1}{4} \right)}$ term of N is

$$(e_1 + 1) \dots (e_r + 1) \left(\sum_{k=1}^{\frac{n-1}{2}} (2k-1) \sigma_k \right) + n \sigma_{\frac{n+1}{2}}.$$

Similarly, after rewriting

$$D = \sum_{k=1}^n (\sum_{j=k}^n \mu_{1,j}) (\sum_{j=k}^n \mu_{2,j}) \cdots (\sum_{j=k}^n \mu_{r,j}) \sigma_k$$

as a polynomial in $q_1, \dots, q_r$, with coefficients viewed as integral polynomials in

$q_{r+1}, \dots, q_m$, the coefficient of the $q_1^{e_1 \left(\frac{n^2-1}{4} \right)} \cdots q_r^{e_r \left(\frac{n^2-1}{4} \right)}$ term of D is

$$(e_1 + 1) \cdots (e_r + 1) \left(\sum_{k=1}^{\frac{n-1}{2}} \sigma_k \right) + \sigma_{\frac{n+1}{2}}.$$

Therefore, the assertion follows.

When $r = m$, since $\sigma_k = 1$ for each $k \in \{0, \dots, n\}$, it follows that

$$\begin{aligned} \lim_{q_1, q_2, \dots, q_m \rightarrow \infty} E(\theta_R(W)) &= \frac{(e_1+1) \cdots (e_m+1) \left(\sum_{k=1}^{\frac{n-1}{2}} (2k-1) \sigma_k \right) + n \sigma_{\frac{n+1}{2}}}{(e_1+1) \cdots (e_m+1) \left(\sum_{k=1}^{\frac{n-1}{2}} \sigma_k \right) + \sigma_{\frac{n+1}{2}}} = \\ &= \frac{(e_1+1) \cdots (e_m+1) \left(\sum_{k=1}^{\frac{n-1}{2}} (2k-1) \right) + n}{(e_1+1) \cdots (e_m+1) \left(\sum_{k=1}^{\frac{n-1}{2}} 1 \right) + 1} = \frac{(e_1+1) \cdots (e_m+1) \left(2 \left(\sum_{k=1}^{\frac{n-1}{2}} k \right) - \left(\frac{n-1}{2} \right) \right) + n}{(e_1+1) \cdots (e_m+1) \left(\frac{n-1}{2} \right) + 1} = \\ &= \frac{(e_1+1) \cdots (e_m+1) \left(2 \left(\frac{(\frac{n-1}{2})(\frac{n+1}{2})}{2} \right) - \left(\frac{n-1}{2} \right) \right) + n}{(e_1+1) \cdots (e_m+1) \left(\frac{n-1}{2} \right) + 1} = \frac{(e_1+1) \cdots (e_m+1) \left(\frac{n-1}{2} \right)^2 + n}{(e_1+1) \cdots (e_m+1) \left(\frac{n-1}{2} \right) + 1} = \\ &= \frac{(e_1+1) \cdots (e_m+1) (n-1)^2 + 4n}{2(e_1+1) \cdots (e_m+1) (n-1) + 4} = \frac{(e_1+1) \cdots (e_m+1) n^2 + (4-2(e_1+1) \cdots (e_m+1))n + (e_1+1) \cdots (e_m+1)}{2(e_1+1) \cdots (e_m+1) n + 4 - 2(e_1+1) \cdots (e_m+1)} = \\ &= \frac{n}{2} + \frac{2 - (e_1+1) \cdots (e_m+1)}{2(e_1+1) \cdots (e_m+1)} + \frac{4(e_1+1) \cdots (e_m+1) - 4}{(e_1+1) \cdots (e_m+1) (2(e_1+1) \cdots (e_m+1) n + 4 - 2(e_1+1) \cdots (e_m+1))} = \\ &= \frac{n}{2} - \frac{1}{2} + \frac{1}{(e_1+1) \cdots (e_m+1)} + \frac{2((e_1+1) \cdots (e_m+1) - 1)}{(e_1+1) \cdots (e_m+1) ((e_1+1) \cdots (e_m+1) (n-1) + 2)}. \quad \square \end{aligned}$$

THEOREM 3.18 (*weighting by cardinality*) Let $r \in \{1, \dots, m\}$. For each $k \in \{0, \dots, n\}$, let

$$\sigma'_k := \begin{cases} (\sum_{j=k}^n \mu'_{r+1,j}) (\sum_{j=k}^n \mu'_{r+2,j}) \cdots (\sum_{j=k}^n \mu'_{m,j}) & \text{if } r < m \\ 1 & \text{if } r = m. \end{cases}$$

Let W vary over the set of R -submodules of V . Also, suppose W is weighted by cardinality. Then:

(a) If n is even, then

$$\lim_{q_1, q_2, \dots, q_r \rightarrow \infty} E(\theta_R(W)) = \frac{(e_1+1) \dots (e_r+1) (\sum_{k=1}^{\frac{n}{2}} \sigma'_k) + \sigma'_{\frac{n}{2}+1}}{(e_1+1) \dots (e_r+1) \sigma'_0}.$$

In particular, when $r = m$, we have that

$$\lim_{q_1, q_2, \dots, q_m \rightarrow \infty} E(\theta_R(W)) = \frac{n}{2} + \frac{1}{(e_1+1) \dots (e_m+1)}.$$

(b) If n is odd, then

$$\lim_{q_1, q_2, \dots, q_r \rightarrow \infty} E(\theta_R(W)) = \sum_{\sigma'_0}^{\frac{n+1}{2}} \sigma'_k.$$

In particular, when $r = m$, we have that

$$\lim_{q_1, q_2, \dots, q_m \rightarrow \infty} E(\theta_R(W)) = \frac{n+1}{2}.$$

Proof. Referring to the numerator and denominator appearing in Lemma 3.15, let N denote $\sum_{k=1}^n (\sum_{j=k}^n \mu'_{1,j}) (\sum_{j=k}^n \mu'_{2,j}) \dots (\sum_{j=k}^n \mu'_{r,j}) \sigma'_k$, and let D denote $(\sum_{j=0}^n \mu'_{1,j}) (\sum_{j=0}^n \mu'_{2,j}) \dots (\sum_{j=0}^n \mu'_{r,j}) \sigma'_0$.

(a) Suppose that n is even. To prove the assertion, after rewriting both N and D as polynomials in $q_1, \dots, q_r$, with coefficients viewed as integral polynomials in $q_{r+1}, \dots, q_m$, observe that it suffices to prove the following three statements. For each $i \in \{1, \dots, r\}$, the highest power of q_i in both N and D is at most $e_i \left(\frac{(n+1)^2-1}{4} \right)$;

the coefficient of the $q_1^{e_1 \left(\frac{(n+1)^2-1}{4} \right)} \dots q_r^{e_r \left(\frac{(n+1)^2-1}{4} \right)}$ term of N is

$$(e_1 + 1) \dots (e_r + 1) (\sum_{k=1}^{\frac{n}{2}} \sigma'_k) + \sigma'_{\frac{n}{2}+1};$$

and the coefficient of the $q_1^{e_1 \left(\frac{(n+1)^2-1}{4} \right)} \dots q_r^{e_r \left(\frac{(n+1)^2-1}{4} \right)}$ term of D is

$$(e_1 + 1) \dots (e_r + 1) \sigma'_0.$$

Let $i \in \{1, \dots, r\}$. For each $k \in \{0, \dots, n\}$, using Proposition 2.28 (c) and the discussion preceding Lemma 3.15, observe that $\sum_{j=k}^n \mu'_{i,j}$ is an integral polynomial in q_i . Using Corollary 2.34 (a), one checks that the degree of $\sum_{j=k}^n \mu'_{i,j}$ in q_i is at most $e_i \left(\frac{(n+1)^2-1}{4} \right)$, with equality if and only if $k \in \{0, \dots, \frac{n}{2} + 1\}$. In particular, observe via Corollary 2.34 (a) and Proposition 2.28 (c) that $\sum_{j=\frac{n}{2}+1}^n \mu'_{i,j}$ is a monic polynomial in q_i . Similarly, using Corollary 2.34 (a) and Proposition 2.28 (c), one checks, for each $k \in \{0, \dots, \frac{n}{2}\}$, that the leading coefficient of $\sum_{j=k}^n \mu'_{i,j}$ as a polynomial in q_i is $e_i + 1$.

Thus, after rewriting

$$N = \sum_{k=1}^n (\sum_{j=k}^n \mu'_{1,j}) (\sum_{j=k}^n \mu'_{2,j}) \cdots (\sum_{j=k}^n \mu'_{r,j}) \sigma'_k$$

as a polynomial in $q_1, \dots, q_r$, with coefficients viewed as integral polynomials in $q_{r+1}, \dots, q_m$, the coefficient of the $q_1^{e_1 \left(\frac{(n+1)^2-1}{4} \right)} \cdots q_r^{e_r \left(\frac{(n+1)^2-1}{4} \right)}$ term of N is

$$(e_1 + 1) \cdots (e_r + 1) (\sum_{k=1}^{\frac{n}{2}} \sigma'_k) + \sigma'_{\frac{n}{2}+1}.$$

Similarly, after rewriting $D = (\sum_{j=0}^n \mu'_{1,j}) (\sum_{j=0}^n \mu'_{2,j}) \cdots (\sum_{j=0}^n \mu'_{r,j}) \sigma'_0$ as a polynomial in $q_1, \dots, q_r$, with coefficients viewed as integral polynomials in $q_{r+1}, \dots, q_m$, the coefficient of the $q_1^{e_1 \left(\frac{(n+1)^2-1}{4} \right)} \cdots q_r^{e_r \left(\frac{(n+1)^2-1}{4} \right)}$ term of D is $(e_1 + 1) \cdots (e_r + 1) \sigma'_0$.

Therefore, the assertion follows.

When $r = m$, as $\sigma'_k = 1$ for each $k \in \{0, \dots, n\}$, it follows that

$$\lim_{q_1, q_2, \dots, q_m \rightarrow \infty} E(\theta_R(W)) = \frac{(e_1+1) \cdots (e_m+1) (\sum_{k=1}^{\frac{n}{2}} \sigma'_k) + \sigma'_{\frac{n}{2}+1}}{(e_1+1) \cdots (e_m+1) \sigma'_0} = \frac{(e_1+1) \cdots (e_m+1) (\frac{n}{2}) + 1}{(e_1+1) \cdots (e_m+1)} = \frac{\frac{n}{2} + \frac{1}{(e_1+1) \cdots (e_m+1)}}{1}.$$

(b) Suppose that n is odd. To prove the assertion, after rewriting both N and D as polynomials in $q_1, \dots, q_r$, with coefficients viewed as integral polynomials

in $q_{r+1}, \dots, q_m$, observe that it suffices to prove the following three statements. For each $i \in \{1, \dots, r\}$, the highest power of q_i in both N and D is at most $e_i \left(\frac{n+1}{2}\right)^2$; the coefficient of the $q_1^{e_1 \left(\frac{n+1}{2}\right)^2} \dots q_r^{e_r \left(\frac{n+1}{2}\right)^2}$ term of N is $\sum_{k=1}^{\frac{n+1}{2}} \sigma'_k$; and the coefficient of the $q_1^{e_1 \left(\frac{n+1}{2}\right)^2} \dots q_r^{e_r \left(\frac{n+1}{2}\right)^2}$ term of D is σ'_0 .

Let $i \in \{1, \dots, r\}$. For each $k \in \{0, \dots, n\}$, using Proposition 2.28 (c) and the discussion preceding Lemma 3.15, observe that $\sum_{j=k}^n \mu'_{i,j}$ is an integral polynomial in q_i . Using Corollary 2.34 (b), one checks that the degree of $\sum_{j=k}^n \mu'_{i,j}$ in q_i is at most $e_i \left(\frac{n+1}{2}\right)^2$, with equality if and only if $k \in \{0, \dots, \frac{n+1}{2}\}$. In particular, for each $k \in \{0, \dots, \frac{n+1}{2}\}$, observe via Corollary 2.34 (b) and Proposition 2.28 (c) that $\sum_{j=k}^n \mu'_{i,j}$ is a monic polynomial in q_i .

Thus, after rewriting

$$N = \sum_{k=1}^n (\sum_{j=k}^n \mu'_{1,j}) (\sum_{j=k}^n \mu'_{2,j}) \dots (\sum_{j=k}^n \mu'_{r,j}) \sigma'_k$$

as a polynomial in $q_1, \dots, q_r$, with coefficients viewed as integral polynomials in $q_{r+1}, \dots, q_m$, the coefficient of the $q_1^{e_1 \left(\frac{n+1}{2}\right)^2} \dots q_r^{e_r \left(\frac{n+1}{2}\right)^2}$ term of N is $\sum_{k=1}^{\frac{n+1}{2}} \sigma'_k$. Similarly, after rewriting $D = (\sum_{j=0}^n \mu'_{1,j}) (\sum_{j=0}^n \mu'_{2,j}) \dots (\sum_{j=0}^n \mu'_{r,j}) \sigma'_0$ as a polynomial in $q_1, \dots, q_r$, with coefficients viewed as integral polynomials in $q_{r+1}, \dots, q_m$, the coefficient of the $q_1^{e_1 \left(\frac{n+1}{2}\right)^2} \dots q_r^{e_r \left(\frac{n+1}{2}\right)^2}$ term of D is σ'_0 . Therefore, the assertion follows.

When $r = m$, since $\sigma'_k = 1$ for each $k \in \{0, \dots, n\}$, it follows that

$$\lim_{q_1, q_2, \dots, q_m \rightarrow \infty} E(\theta_R(W)) = \frac{\sum_{k=1}^{\frac{n+1}{2}} \sigma'_k}{\sigma'_0} = \sum_{k=1}^{\frac{n+1}{2}} 1 = \frac{n+1}{2}. \quad \square$$

In closing the chapter, we observe that Theorems 3.16-3.18 give another way, distinct from the SPIR studies in Theorems 2.35, 2.37 and 2.38, of generalizing the

motivating results on vector spaces in Theorems 1.8-1.10. To see the “generalizing” assertion, note first that if R is a field, then $\theta_R(W) = \dim_R(W)$; and then simply take $m = 1$ and $e_1 = 1$ in the above results. As for the “distinct” assertion, it suffices to observe that the formula in Theorem 3.16 (resp., 3.17; resp., 3.18) disagrees with that in Theorem 2.35 (resp., 2.37; resp., 2.38) in case n is odd, $m = 1$ and $e \geq 2$ (resp., $n = 3$, $m = 1$, $e = 2$; resp., n is even, $m = 1$ and $e \geq 2$). For an alternate approach to the “distinct” assertion, observe that $\theta \neq \nu$ as follows. Take an SPIR, (R, M) , with $e \geq 2$, ambient R -module $V := R$, and R -submodule $W := M$. Here, $\nu_R(W) = 1$, but it follows from Theorems 1.1 and 1.2 that $\theta_R(W) = 0$ since 0 is the only proper ideal of R which is R -free.

CHAPTER IV

WHEN THE UNIVERSE IS NOT A FINITELY GENERATED FREE MODULE OR THE BASE RING IS NOT A PRINCIPAL IDEAL RING

This short chapter illustrates the behavior of $E(\nu_R(W))$ in case the ambient module V is not finitely generated free or the ambient ring R is not a PIR. The former situation is addressed in Proposition 4.1, which also settles the case $n = 1$ which was announced in Theorem 2.35 and Theorem 2.37, and in Remark 4.2. The latter situation is dealt with in Example 4.3.

For Proposition 4.1 and Remark 4.2, [7, Theorem 33, p. 245] permits the ambient finite PIR, R , to be expressed as an internal direct product of finitely many finite special principal ideal rings (possibly fields), say $R = R_1 \times \dots \times R_m$, where, for each $i \in \{1, \dots, m\}$, R_i is a finite SPIR. Let $M_i = R\pi_i$ denote the maximal ideal of R_i , let e_i denote the index of nilpotency of M_i , and let $q_i := |R_i/M_i|$.

PROPOSITION 4.1 *Let R be as above. Let V be a nonzero cyclic R -module; that is, $V \cong R/I$ as R -modules, with $I = I_1 \times I_2 \times \dots \times I_m$ where, for each $i \in \{1, \dots, m\}$, $I_i = R_i\pi_i^{k_i}$ for some $k_i \in \{0, \dots, e_i\}$, and $k_j \neq 0$ for some j . Let W vary over the set of all R -submodules of V . Then:*

- (a) *For the unweighted context, $E(\nu_R(W)) = 1 - \frac{1}{\prod_{i=1}^m (k_i+1)}$. As a consequence, if $I = 0$, then $E(\nu_R(W)) = 1 - \frac{1}{\prod_{i=1}^m (e_i+1)}$.*
- (b) *If W is weighted by $\nu_R(W)$, then $E(\nu_R(W)) = 1$.*
- (c) *If W is weighted by cardinality, then*

$$E(\nu_R(W)) = 1 - \frac{1}{(\sum_{j=0}^{k_1} q_1^j)(\sum_{j=0}^{k_2} q_2^j) \dots (\sum_{j=0}^{k_m} q_m^j)}.$$

Proof. Since R is a PIR, $\nu_R(W) \leq 1$ for each R -submodule W of V . We use this fact throughout the proof. For $k \in \{0, 1\}$, let η_k denote the number of R -submodules of V with ν_R -value k . Since $V \neq 0$, it follows that $\eta_0 = 1$ and $\eta_1 \neq 0$.

(a) For $i \in \{1, \dots, m\}$, by [7, Example, p. 245], R_i has precisely $e_i + 1$ ideals (namely, $R_i, R_i\pi_i, R_i\pi_i^2, \dots, R_i\pi_i^{e_i} = 0$). Hence, for $i \in \{1, \dots, m\}$, R_i/I_i has $k_i + 1$ R_i -submodules. By the discussion preceding Proposition 3.2, there are $\prod_{i=1}^m (k_i + 1)$ R -submodules of $V \cong R/I \cong (R_1/I_1) \times \dots \times (R_m/I_m)$. Of these, only one (namely, 0) has ν_R -value 0. Therefore, by the reasoning for the “unweighted” case discussed prior to Remark 2.30,

$$E(\nu_R(W)) = \frac{\sum_{k=0}^1 k\eta_k}{\sum_{k=0}^1 \eta_k} = \frac{(\prod_{i=1}^m (k_i+1)) - 1}{\prod_{i=1}^m (k_i+1)} = 1 - \frac{1}{\prod_{i=1}^m (k_i+1)}.$$

Observe that if $I = 0$, then $k_i = e_i$ for each $i \in \{1, \dots, m\}$, and so $E(\nu_R(W)) = 1 - \frac{1}{\prod_{i=1}^m (e_i+1)}$.

(b) If W is weighted by $\nu_R(W)$, then it follows from the discussion prior to Remark 2.30 that $E(\nu_R(W)) = \frac{\sum_{k=0}^1 k^2\eta_k}{\sum_{k=0}^1 k\eta_k} = \frac{\eta_1}{\eta_1} = 1$.

(c) For $i \in \{1, \dots, m\}$ and $j \in \{0, \dots, k_i\}$, let $\tau_i(j) = \left| \frac{R_i\pi_i^j}{R_i\pi_i^{k_i}} \right|$. Combining Lagrange’s Theorem with two applications apiece of Corollary 2.26 (1) and Remark 2.5 (c), we see that $\tau_i(j) = \frac{|R_i\pi_i^j|}{|R_i\pi_i^{k_i}|} = \frac{q_i^{e_i-j}}{q_i^{e_i-k_i}} = q_i^{k_i-j}$ for each i and j . Further, the sum of cardinalities of all R -submodules of V is

$$\begin{aligned} \sum_{j_i \in \{0, \dots, k_i\}} \tau_1(j_1)\tau_2(j_2) \dots \tau_m(j_m) &= (\sum_{j=0}^{k_1} \tau_1(j))(\sum_{j=0}^{k_2} \tau_2(j)) \dots (\sum_{j=0}^{k_m} \tau_m(j)) = \\ &= (\sum_{j=0}^{k_1} q_1^{k_1-j})(\sum_{j=0}^{k_2} q_2^{k_2-j}) \dots (\sum_{j=0}^{k_m} q_m^{k_m-j}) = (\sum_{j=0}^{k_1} q_1^j)(\sum_{j=0}^{k_2} q_2^j) \dots (\sum_{j=0}^{k_m} q_m^j). \end{aligned}$$

Hence, the sum of cardinalities of all nonzero R -submodules of V is

$$(\sum_{j=0}^{k_1} q_1^j)(\sum_{j=0}^{k_2} q_2^j) \dots (\sum_{j=0}^{k_m} q_m^j) - 1.$$

Therefore, if W is weighted by cardinality,

$$E(\nu_R(W)) = \frac{(\sum_{j=0}^{k_1} q_1^j)(\sum_{j=0}^{k_2} q_2^j) \dots (\sum_{j=0}^{k_m} q_m^j) - 1}{(\sum_{j=0}^{k_1} q_1^j)(\sum_{j=0}^{k_2} q_2^j) \dots (\sum_{j=0}^{k_m} q_m^j)} = 1 - \frac{1}{(\sum_{j=0}^{k_1} q_1^j)(\sum_{j=0}^{k_2} q_2^j) \dots (\sum_{j=0}^{k_m} q_m^j)}. \quad \square$$

REMARK 4.2 (a) When the ambient R -module $V \cong R/I$ is not free, $I \neq 0$, and so $k_i < e_i$ for at least one i . Thus, in this case, Proposition 4.1 (a) shows that the (limiting) behavior of $E(\nu_R(W))$ depends on new parameters $(k_1, \dots, k_m)$ which played no role in the calculations of the expected values found in Chapters 2 and 3.

(b) Let V be as in the context of Proposition 4.1. Notice that in part (c), when W is weighted by cardinality, $E(\nu_R(W))$ depends on at least some of the q_i values. However, in parts (a) and (b), when submodules are unweighted or weighted by $\nu_R(W)$, $E(\nu_R(W))$ is independent of the q_i values. Nevertheless, in each part of Proposition 4.1, $E(\nu_R(W)) \leq 1$; and, in fact, by choosing R appropriately, one can arrange that $1 - E(\nu_R(W))$ be arbitrarily small.

(c) Let R be a finite SPIR with maximal ideal $M = R\pi$ and e the index of nilpotency of M . Let $V \cong R$ (i.e., $I = 0$ in Proposition 4.1). As anticipated by Theorem 2.35, Proposition 4.1 (a) gives that $\lim_{q \rightarrow \infty} E(\nu_R(W)) = 1 - \frac{1}{e+1} = \frac{e}{e+1}$. Similarly, when W is weighted by $\nu_R(W)$, Proposition 4.1 (b) gives that $\lim_{q \rightarrow \infty} E(\nu_R(W)) = 1$, a generalization of the $n = 1$ case of [1, Corollary 2.6]. When W is weighted by cardinality, Proposition 4.1 (c) gives that $\lim_{q \rightarrow \infty} E(\nu_R(W)) = \lim_{e \geq 1} (1 - \frac{1}{\sum_{j=0}^e q^j}) = 1$, a generalization of the $n = 1$ case of [1, Corollary 2.5].

Finally, we show that if a finite local ring R is not a PIR, then the (limiting) behavior of $E(\nu_R(W))$ is qualitatively different from that recorded in Chapters 2 and 3. We document the point for three contexts: when W is unweighted, when W

is weighted by $\nu_R(W)$, and when W is weighted by cardinality. In Remark 4.4 (b), we broach the problem of analyzing related examples arising from other finite local rings R which are not PIRs.

EXAMPLE 4.3 Let K be a finite field of cardinality q and let $d \geq 2$.

Let $R := K[X_1, \dots, X_d]/(\{X_i X_j : i, j \in \{1, \dots, d\}\}) = K[\bar{X}_1, \dots, \bar{X}_d] = K + \sum_{i=1}^d K\bar{X}_i$, where $\bar{X}_i$ denotes the coset represented by X_i . Observe that R is a finite local ring with maximal ideal $M := (X_1, \dots, X_d)/(\{X_i X_j : i, j \in \{1, \dots, d\}\}) = (\bar{X}_1, \dots, \bar{X}_d)$ with index of nilpotency 2, but R is not a PIR since $\nu_R(M) = d$.

(a) The maximal ideal M is a d -dimensional K -vector space with K -basis $\{\bar{X}_1, \dots, \bar{X}_d\}$. Further, the set of all proper ideals of R is precisely the set of all K -vector subspaces of M .

(b) For each proper ideal I of R , $\nu_R(I) = \dim_K(I)$.

In (c)-(e), let I vary over the set of ideals of R .

(c) In the unweighted context, $\lim_{q \rightarrow \infty} E(\nu_R(I)) = \frac{d}{2}$.

(d) When I is weighted by $\nu_R(I)$,

$$\lim_{q \rightarrow \infty} E(\nu_R(I)) = \begin{cases} \frac{d}{2} & \text{if } d \text{ is even} \\ \frac{d}{2} + \frac{1}{2d} & \text{if } d \text{ is odd.} \end{cases}$$

(e) When I is weighted by cardinality,

$$\lim_{q \rightarrow \infty} E(\nu_R(I)) = \begin{cases} 1 & \text{if } d = 2 \\ \frac{3}{2} & \text{if } d = 3 \\ \frac{d+1}{2} & \text{if } d \geq 4. \end{cases}$$

Proof. (a) If $k_1, k_2, \dots, k_d \in K$ and $k_1 \bar{X}_1 + k_2 \bar{X}_2 + \dots + k_d \bar{X}_d = \bar{0}$, then $k_1 = k_2 = \dots = k_d = 0$. Thus, M is a d -dimensional K -vector space with basis $\{\bar{X}_1, \dots, \bar{X}_d\}$.

Further, any K -vector subspace of M is an ideal of R since $M^2 = 0$ leads to

$$(k_0\bar{1} + k_1\bar{X}_1 + \dots + k_d\bar{X}_d)(k'_1\bar{X}_1 + k'_2\bar{X}_2 + \dots + k'_d\bar{X}_d) = k_0(k'_1\bar{X}_1 + k'_2\bar{X}_2 + \dots + k'_d\bar{X}_d)$$

for all $k_1, \dots, k_d, k'_1, \dots, k'_d \in K$. Of course, any proper ideal of R is a K -subspace of M .

(b) Since $M^2 = 0$ and $R/M \cong K$, it follows from Nakayama's Lemma that $\nu_R(I) = \dim_{R/M}(I/MI) = \dim_{R/M}(I/0) = \dim_{R/M}(I) = \dim_K(I)$.

(c) By (a), R is the only ideal of R which is not a K -subspace of M ; and, of course, $\nu_R(R) = 1$. Therefore, with ω_j denoting the number of j -dimensional K -subspaces of a d -dimensional K -vector space, we see via (b) and the discussion prior to Remark 2.30 that

$$E(\nu_R(I)) = \frac{0\omega_0+1(\omega_1+1)+\sum_{j=2}^d j\omega_j}{\omega_0+(\omega_1+1)+\sum_{j=2}^d \omega_j} = \frac{(\sum_{j=0}^d j\omega_j)+1}{(\sum_{j=0}^d \omega_j)+1}.$$

Since $d \geq 2$, [4, Corollary 3.3] gives that $\sum_{j=0}^d j\omega_j$ and $\sum_{j=0}^d \omega_j$ are both integral polynomials in q of degree ≥ 1 . Therefore, $\lim_{q \rightarrow \infty} E(\nu_R(I)) = \lim_{q \rightarrow \infty} \frac{(\sum_{j=0}^d j\omega_j)+1}{(\sum_{j=0}^d \omega_j)+1} = \lim_{q \rightarrow \infty} \frac{\sum_{j=0}^d j\omega_j}{\sum_{j=0}^d \omega_j} = \frac{d}{2}$, with the last equality holding (even before taking the limit) by the result on vector spaces in [4, Theorem 3.4].

(d) As in (c), let ω_j denote the number of j -dimensional K -subspaces of a d -dimensional K -vector space. When I is weighted by $\nu_R(I)$, by arguing as in (c), it follows that

$$E(\nu_R(I)) = \frac{0^2\omega_0+1^2(\omega_1+1)+\sum_{j=2}^d j^2\omega_j}{0\omega_0+1(\omega_1+1)+\sum_{j=2}^d j\omega_j} = \frac{(\sum_{j=0}^d j^2\omega_j)+1}{(\sum_{j=0}^d j\omega_j)+1}.$$

Since $d \geq 2$, [4, Corollary 3.3] gives that both $\sum_{j=0}^d j^2\omega_j$ and $\sum_{j=0}^d j\omega_j$ are polynomials in q of degree ≥ 1 . Therefore, $\lim_{q \rightarrow \infty} E(\nu_R(I)) = \lim_{q \rightarrow \infty} \frac{(\sum_{j=0}^d j^2\omega_j)+1}{(\sum_{j=0}^d j\omega_j)+1} = \lim_{q \rightarrow \infty} \frac{\sum_{j=0}^d j^2\omega_j}{\sum_{j=0}^d j\omega_j}$. Hence, the assertion follows by the result on vector spaces in [1, Corollary 2.6].

(e) Let ω'_j denote the sum of cardinalities of all j -dimensional K -vector subspaces of a d -dimensional K -vector space. Thus, since the cardinality of any j -dimensional K -vector space is q^j , [4, Proposition 3.1] gives that $\omega'_j = q^j \frac{\prod_{i=1}^j (q^d - q^{i-1})}{\prod_{i=1}^j (q^j - q^{i-1})}$ for each $j \in \{1, \dots, d\}$; the same is true for $j = 0$, by our convention on empty products. Further, by [4, Corollary 3.3], each ω'_j is a monic integral polynomial in q of degree $j + j(d - j) = j((d + 1) - j)$.

By (a), R is the only ideal of R which is not a K -subspace of M . Further, since M is a d -dimensional K -vector space and $R/M \cong K$, it follows, by Lagrange's Theorem, that $|R| = |R/M||M| = qq^d = q^{d+1}$. Thus, when W is weighted by cardinality, it follows from the discussion after Remark 2.30 that

$$E(\nu_R(I)) = \frac{0\omega'_0 + 1(\omega'_1 + |R|) + \sum_{j=2}^d j\omega'_j}{\omega'_0 + (\omega'_1 + |R|) + \sum_{j=2}^d \omega'_j} = \frac{(\sum_{j=0}^d j\omega'_j) + |R|}{(\sum_{j=0}^d \omega'_j) + |R|} = \frac{(\sum_{j=0}^d j\omega'_j) + q^{d+1}}{(\sum_{j=0}^d \omega'_j) + q^{d+1}}.$$

By Corollary 2.34 (a), using $n = d$, $e = 1$ with $\omega'_j = \Lambda'(j)$ if $j > 0$, observe that if d is even and $d \geq 4$, then the degree of both $\sum_{j=0}^d j\omega'_j$ and $\sum_{j=0}^d \omega'_j$ (as polynomials) in q , $\frac{(d+1)^2-1}{4}$, is greater than $d + 1$. Similarly, by Corollary 2.34 (b), observe that if d is odd and $d \geq 5$, then the degree of both $\sum_{j=0}^d \omega'_j$ and $\sum_{j=0}^d \omega'_j$ in q , $\left(\frac{d+1}{2}\right)^2$, is greater than $d + 1$. Therefore, when $d \geq 4$, it follows that

$$\lim_{q \rightarrow \infty} E(\nu_R(I)) = \lim_{q \rightarrow \infty} \frac{(\sum_{j=0}^d j\omega'_j) + q^{d+1}}{(\sum_{j=0}^d \omega'_j) + q^{d+1}} = \lim_{q \rightarrow \infty} \frac{\sum_{j=0}^d j\omega'_j}{\sum_{j=0}^d \omega'_j} = \frac{d+1}{2},$$

with the last equality holding by the result on vector spaces in [1, Corollary 2.5].

On the other hand, when $d = 2$, using [4, Proposition 3.1], observe that $\omega'_0 = 1$, $\omega'_1 = q^2 + q$, and $\omega'_2 = q^2$. Therefore, the above formula simplifies to

$$E(\nu_R(I)) = \frac{q^3 + 3q^2 + q}{q^3 + 2q^2 + q + 1},$$

giving that $\lim_{q \rightarrow \infty} E(\nu_R(I)) = 1$.

Similarly, when $d = 3$, one checks that $\omega'_0 = 1$, $\omega'_1 = q^3 + q^2 + q$, $\omega'_2 = q^4 + q^3 + q^2$, and $\omega'_3 = q^3$, giving

$$E(\nu_R(I)) = \frac{3q^4 + 6q^3 + 3q^2 + q}{2q^4 + 3q^3 + 2q^2 + q + 1},$$

and so $\lim_{q \rightarrow \infty} E(\nu_R(I)) = \frac{3}{2}$. $\square$

REMARK 4.4 (a) In contrast to the calculation of (limiting) expected values over finite PIRs in Chapters 2 and 3, all limiting expected values found in Example 4.3 (c),(d), and (e) depend on a parameter, d , of the ambient ring. Further, in contrast to the behavior over finite PIRs given for $I = 0$ and $V \cong R$ in Proposition 4.1 (a),(b),(c) and Remark 4.2 (b), each of the limits $\lim_{q \rightarrow \infty} E(\nu_R(I))$ is unbounded as d varies.

(b) Let R be the ring in Example 4.3. Since $\nu_R(M) = d \geq 2$, R is not an SPIR. However, if $d \geq 4$, then [1, Corollaries 2.7, 2.6, 2.5] shows that the limiting answers in Example 4.3 (c),(d),(e) are the same as the corresponding values of $\lim_{q \rightarrow \infty} E(\nu_F(W))$, where F is the finite field with q elements and W ranges over the subspaces of a d -dimensional F -vector space V . In particular, if $d \geq 4$, there is an SPIR-module context (indeed, a vector space context) which mimics the behavior of the R -module R in Example 4.3 (c),(d),(e). On the other hand, if $d = 2$ or $d = 3$, then we can use Theorems 2.35, 2.37, and 2.38 to show that there is no SPIR-module context (with fixed parameters e and n) for which $\lim_{q \rightarrow \infty} E(\nu_R(W))$ replicates the behavior of the R -module R in Example 4.3 (c),(d),(e). We thus have the following curiosity: for large d (i.e., $d \geq 4$), the ring R in Example 4.3 behaves more like an SPIR than is the case for $d = 2, 3$. Moreover, since the construction in Example 4.3 formally degenerates to the SPIR of dual numbers $K[X]/(X^2)$ in case $d = 1$, we see that this construction fails to exhibit SPIR-type behavior for $d = 2$ and $d = 3$, while exhibiting SPIR-like behavior for $d = 1, 4, 5, 6, 7, \dots$: a most remarkable pattern!

(c) For sufficiently large d , the answers in Example 4.3 (c),(d),(e) are formally the same as the corresponding answers given in [1, Corollaries 2.7, 2.6, 2.5] for the limiting values of $E(\dim_F(W))$ where W varies over the subspaces of a d -dimensional vector space over a finite field F with q elements. This raises the question of whether known results on vector spaces (or from Chapters 2 and 3) can be used to study limits of expected values over non-PIRs substantially different from the ring R in Example 4.3.

BIBLIOGRAPHY

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- [1] David E. Dobbs, *The expected weighted dimension of a sum of vector subspaces*, Lecture Notes in Pure and Appl. Math **185** (1997), 251–267, Dekker, New York.
- [2] ———, *On minimal generating sets of modules over a special principal ideal ring*, Lecture Notes in Pure and Appl. Math **185** (1997), 241–250, Dekker, New York.
- [3] David E. Dobbs and Robert Hanks, *A Modern Course on the Theory of Equations*, second ed., Polygonal Publishing House, Washington, New Jersey, 1992.
- [4] David E. Dobbs and Mark J. Lancaster, *The expected dimension of a sum of vector subspaces*, Bull. Austral. Math. Soc. **45** (1992), no. 3, 467–477.
- [5] Irving Kaplansky, *Projective Modules*, Ann. Math. **68** (1958), 372–377.
- [6] J-P Serre, *Algèbre Locale-Multiplicités*, Lecture Notes in Math., vol. 11, Springer-Verlag, Heidelberg, 1965.
- [7] Oscar Zariski and Pierre Samuel, *Commutative Algebra*, vol. I, Van Nostrand, Princeton/Toronto/London, 1958.

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