

To the graduate Council: I am submitting herewith a thesis written by Despina Andrea Stavri entitled "Transformations of Differential Equations and Applications." I have examined the final electronic copy of this thesis for form and content and recommend that it be accepted in partial fulfillment of the requirements for the degree of Master of Science, with a major in Mathematics.

Don Hinton, Major Professor

We have read this dissertation
and recommend its acceptance:

Accepted for the Council:

Vice Chancellor and Dean of
Graduate Studies

TRANSFORMATIONS OF DIFFERENTIAL EQUATIONS AND APPLICATIONS

A Thesis
Presented for the
Master of Science
Degree
The University of Tennessee, Knoxville

Despina Andrea Stavri
August , 2007

Dedication

This thesis is dedicated to the loving memory
of my cousin:

ΧΡΙΣΤΟΦΟΡΟΣ ΓΕΩΡΓΙΟΥ ΜΑΤΣΙΟΥΚΚΟΥ

(Christophoros Georgiou Matsiukkou)

November 6, 1986 - May 10, 2007

Acknowledgments

I would like to take this opportunity to thank the following individuals for their invaluable input, without which none of this would be possible.

My profound gratitude and deep regards goes to my thesis advisor Dr. Don Hinton for his immense inspiration and guidance while preparing this thesis. Sincere appreciation is extended to Dr. Ohannes Karakashian and Dr. Suzanne Lenhart for taking the time to serve on my committee during this endeavor.

Sincere appreciation to all of my professors at the University of Tennessee. I also thank my colleagues and friends for their interest and encouragement throughout the past six years of my studies.

Finally, my deepest gratitude and love goes to my parents, Andreas and Anna, siblings Katerina, Panayiota and Constantina, nieces Anna and Constantina, nephew Perikli, and boyfriend George Panayiotou. Their undying love and faith in God, has allowed me to have enough courage to face each day with the best of my ability, and achieve all my goals.

To all of you, thank you.

Abstract

The Sturm-Liouville problems introduced by Jacques Charles Francois Sturm and Joseph Liouville in the 19th century are very important in applied mathematics. Singular Sturm-Liouville problems and two physical problems are discussed. The two physical problems are solved using two methods: Exact and asymptotic solutions. Also, the Sturm-Liouville problem's are classified using the Weyl-Kodaira Theorem.

A general transformation of a third order differential equation is introduced. Oscillation and non-oscillation theorems on an infinite interval for a third order differential equation are stated. New versions of these theorems are introduced for the special cases and examples are given to illustrate them. Finally, a canonical third order differential equation is discussed.

Contents

1	Introduction	1
2	2nd Order Differential Equations	3
3	Asymptotic Theorems	7
4	Physical Problems	10
4.1	Singular Sturm-Liouville Problems	11
4.2	Two Physical Problems	12
4.2.1	Vibrations of a Circular Membrane	12
4.2.2	Torsional Waves of an Elastic Rod	16
4.3	Asymptotic Solutions of the Physical Problems	21
4.3.1	Asymptotic Solutions of the Vibrations of a Circular Membrane	22
4.3.2	Asymptotic Solutions for the Torsional Waves of an Elastic Rod	34
5	3rd Order Differential Equations	51

5.1	General Transformation for a 3rd Order	
	Differential Equation	51
5.2	Theorems for 3rd Order Differential Equations	54
6	Canonical 3rd Order Differential Equations	76

Chapter 1

Introduction

The purpose of this thesis is to transform second and third order differential equations from a finite singularity into an infinite singularity.

In chapter 2, a second order differential equation, to which a general transformation is applied, is introduced. Three special cases with a finite singularity are considered and transformed into an infinite singularity by the variable change method.

In chapter 3, three asymptotic theorems on an infinite interval are introduced.

In chapter 4, two physical problems are considered: Vibrations of a Circular Membrane and Torsional Waves of an Elastic Rod. In the first section, the physical problems are solved by finding their exact solutions. The Weyl-Kodaira theorem is applied to classify the exact solutions by showing square integrability with respect to the weight function. In the second section, the two physical problems are transformed from a finite singularity to an infinite singularity before the three asymptotic theorems in chapter 3 can be used to find their asymptotic solutions.

In chapter 5, a third order differential equation is introduced to which a general transformation is applied. Two of the special cases in chapter 2 with finite singularity are considered and transformed into an infinite singularity by the variable change method. Oscillation and non-oscillation theorems for third order differential equations with a singularity at infinity are stated. By application of the transformation theory, new versions of these theorems are obtained for a singularity at zero and examples are given to illustrate them.

In chapter 6, a general form of third order differential equation is considered. A general change of independent and dependent variable is made. It is proved the equation is transformed into an equation of the same type and explicit formulas are given to the transformed coefficients.

Chapter 2

2nd Order Differential Equations

In this chapter we consider the second order differential equation,

$$\frac{d^2y(x)}{dx^2} + p_1(x)\frac{dy}{dx} + p_2(x)y(x) = 0, \quad x \in X \quad (2.0.1)$$

where X is an interval.

First we change the variables of (2.0.1) by introducing a general transformation. Second the middle term of the transformed equation is removed. Finally, three special cases with a finite singularity, say at 0, are considered and transformed into a singularity at infinity by the variable change method. One of the purposes of the transformation theory is to apply known results for a singularity at infinity to derive new results to a singularity at zero.

In order to change the variables of (2.0.1), we need to introduce the general transfor-

mation,

$$y(x) = \mu(x)z(t), \quad t = f(x) \quad (2.0.2)$$

with $f : X \rightarrow T$, monotone and $\mu(x) > 0$. Note that $\mu(x)$ and $f'(x)$ are never zero on X .

Denote $\gamma(x) \equiv f'(x) = \frac{dt}{dx}$.

Differentiating (2.0.2) with respect to x we get

$$y'(x) = \mu'(x)z(t) + \gamma(x)\mu(x)\dot{z}(t), \quad (2.0.3)$$

where $' = \frac{d}{dx}$ and $\dot{} = \frac{d}{dt}$.

Differentiating (2.0.3) with respect to x and simplifying we get

$$y''(x) = \mu''(x)z(t) + 2\gamma(x)\mu'(x)\dot{z}(t) + \gamma'(x)\mu(x)\dot{z}(t) + \gamma^2(x)\mu(x)\ddot{z}(t). \quad (2.0.4)$$

Substituting (2.0.3) and (2.0.4) into the second order differential equation (2.0.1) and simplifying we obtain

$$\begin{aligned} \ddot{z}(t) + \left(\frac{2\gamma(x)\mu'(x) + \gamma'(x)\mu(x) + p_1(x)\gamma(x)\mu(x)}{\gamma^2(x)\mu(x)} \right) \dot{z}(t) + \\ \left(\frac{\mu''(x) + p_1(x)\mu'(x) + p_2(x)\mu(x)}{\gamma^2(x)\mu(x)} \right) z(t) = 0. \end{aligned} \quad (2.0.5)$$

To remove the middle term of the above equation (2.0.5), let $f(x) = x$ and hence $\gamma(x) = f'(x) = 1$. Substituting $f(x)$ into (2.0.5) and simplifying we obtain

$$\ddot{z}(t) + \left(\frac{2\mu'(x) + p_1(x)\mu(x)}{\mu(x)} \right) \dot{z}(t) + \left(\frac{\mu''(x) + p_1(x)\mu'(x) + p_2(x)\mu(x)}{\mu(x)} \right) z(t) = 0. \quad (2.0.6)$$

Since we want to remove the middle term of (2.0.6) then we choose $\mu(x)$ to set the coefficient of the middle term of the above equation to be equal to zero, i.e,

$$\mu'(x) + \frac{p_1(x)}{2}\mu(x) = 0 \quad \text{where} \quad \mu(x) = \mu(0) \exp \left(- \int \frac{p_1(x)}{2} dx \right)$$

The middle term can be removed assume in (2.0.1) that $p_1(x) = 0$ and $p_2(x) = p(x)$ such that equation (2.0.1) becomes

$$y''(x) + p(x)y(x) = 0, \quad (2.0.7)$$

If $2\gamma(x)\mu'(x) + \gamma'(x)\mu(x) = 0$, and $p_1(x) = 0$ then there will be no middle term in the transformation of equation (2.0.7) which is

$$\ddot{z}(t) + \left(\frac{\mu''(x) + p(x)\mu(x)}{\gamma^2(x)\mu(x)} \right) z(t) = 0.$$

We now consider three special cases for equation (2.0.7) which will applied later.

CASE 1: $t = f(x) = \frac{1}{x}$, $\mu(x) = x$ where $x \in (0, a]$.

Using the general transformation (2.0.2) we have $y(x) = xz(t)$, and by (2.0.4) we

obtain $y''(x) = \frac{1}{x^3}\ddot{z}(t)$. Substituting $y(x) = xz(t)$ and $y''(x) = \frac{1}{x^3}\ddot{z}(t)$ into (2.0.7) and simplifying we get

$$\ddot{z}(t) + \frac{p(\frac{1}{t})}{t^4}z(t) = 0, \quad t \in [1/a, \infty). \quad (2.0.8)$$

CASE 2: $t = -\ln(x)$ $\mu(x) = x^{1/2}$ where $x \in (0, a]$.

Hence by the general transformation (2.0.2) we get $y(x) = x^{1/2}z(t)$ and by (2.0.4) we obtain $y''(x) = x^{-3/2}(\ddot{z}(t) - \frac{1}{4}z(t))$. Substituting $y(x)$ and $y''(x)$ into (2.0.7) and simplifying we get

$$\ddot{z}(t) + \left(\frac{p(e^{-t})}{e^{2t}} - \frac{1}{4} \right) z(t) = 0, \quad t \in [-\ln(a), \infty). \quad (2.0.9)$$

CASE 3: $t = \ln(x)$ $\mu(x) = x^{1/2}$ where $x \in [a, \infty]$.

The general transformation (2.0.2) gives us $y(x) = x^{1/2}z(t)$ and by (2.0.4) we have $y''(x) = x^{-3/2}(\ddot{z}(t) - \frac{1}{4}z(t))$. By substitution of $y(x)$ and $y''(x)$ into (2.0.7) and some simplification we obtain the same equation as above (2.0.9) but $t \in [\ln(a), \infty)$.

Chapter 3

Asymptotic Theorems

In this chapter we state three asymptotic theorems on $[a, \infty)$. These theorems can be used in Chapter 4 to find the asymptotic solutions of the two physical problems.

Asymptotic Theorems on $[a, \infty)$ for $\ddot{w}(t) + Q(t)w(t) = 0$

The first theorem is from Coddington and Levinson [3], page 92.

Theorem 3.0.1. *Let A be a constant matrix with characteristic roots μ_j , $j = 1, \dots, n$, all of which are distinct. Let the matrix V be differentiable and satisfy*

$$\int_0^\infty |V'(t)| dt < \infty,$$

and let $V(t) \rightarrow 0$ as $t \rightarrow \infty$. Let the matrix R be integrable and let

$$\int_0^\infty |R(t)| dt < \infty.$$

Let the roots of $\det(A + V(t) - \lambda I) = 0$ be denoted by $\lambda_j(t)$, $j = 1, \dots, n$. Clearly, by reordering the μ_j if necessary, $\lim_{t \rightarrow \infty} \lambda_j(t) = \mu_j$. For a given k , let

$$D_{kj}(t) = \operatorname{Re}(\lambda_k(t) - \lambda_j(t))$$

Suppose all j , $1 \leq j \leq n$, fall into one of two classes I_1 and I_2 , where $j \in I_1$ if

$$\int_0^t D_{kj}(\tau) d\tau \rightarrow \infty \text{ as } t \rightarrow \infty \text{ and } \int_{t_1}^{t_2} D_{kj}(\tau) d\tau > -K \quad (0 \leq t_1 \leq t_2)$$

$j \in I_2$ if $\int_{t_1}^{t_2} D_{kj}(\tau) d\tau < K \quad (0 \leq t_1 \leq t_2)$ where k is fixed and where K is a constant.

Let p_k be a characteristic vector of A associated with μ_k , so that

$$Ap_k = \mu_k p_k.$$

Then there is a solution $\phi_k(t)$ of $x' = (A + V(t) + R(t))x$ and a t_0 , $0 \leq t_0 < \infty$, such that

$$\lim_{t \rightarrow \infty} \phi_k(t) \exp \left[- \int_{t_0}^t \lambda_k(\tau) d\tau \right] = p_k.$$

Theorem 3.0.2. (Ghizzetti's Theorem) [4], page 92. Let

$$z''(t) + a(t)z'(t) + b(t)z(t) = 0, \quad c < t < \infty. \quad (3.0.1)$$

Suppose $\int_c^\infty |a(t)| dt < \infty$ and $\int_c^\infty t|b(t)| dt < \infty$. Then there are two independent solutions z_0 and z_1 of (3.0.1) such that

- $z_0(t) \rightarrow 1$, $tz_0'(t) \rightarrow 0$ as $t \rightarrow \infty$ and

- $\frac{z_1(t)}{t} \rightarrow 1, z_1'(t) \rightarrow 1$ as $t \rightarrow \infty$

For the following theorem we consider the second-order equation,

$$(p(t)w'(t))' \pm q(t)w(t) = 0, \text{ with } q(t) > 0 \text{ on } [a, \infty). \quad (3.0.2)$$

Theorem 3.0.3. [5] Suppose that $\left(\frac{q}{p}\right)^{1/2}$ does not belong to $\mathbf{L}(a, \infty)$ and that each of $\left[\left(\frac{q}{p}\right)^{-1/2}\frac{p'}{p}\right]', \left[\left(\frac{q}{p}\right)^{-1/2}\frac{q'}{q}\right]', \left[\left(\frac{q}{p}\right)^{-1/2}\left(\frac{p'}{p}\right)^2\right]$ and $\left[\left(\frac{q}{p}\right)^{-1/2}\left(\frac{q'}{q}\right)^2\right] \in \mathbf{L}(a, \infty)$.

Then there is a fundamental matrix W for $W' = AW$ such that as $t \rightarrow \infty, \hat{Q}(t)W(t)E(t) \rightarrow L$ where

$$A = \begin{bmatrix} 0 & 1/p \\ \mp q & 0 \end{bmatrix}, \hat{Q} = \begin{bmatrix} q^{1/4} & 0 \\ 0 & q^{-1/4} \end{bmatrix},$$

$$L = \begin{bmatrix} 1 & 1 \\ \mu_1 & \mu_2 \end{bmatrix}, E(t) = \begin{bmatrix} \exp[-\mu_1 \int_a^t (\frac{q}{p})^{1/2}] & 0 \\ 0 & \exp[-\mu_2 \int_a^t (\frac{q}{p})^{1/2}] \end{bmatrix}$$

and μ_1, μ_2 are the eigenvalues of $w''(t) \pm qw(t) = 0$.

Remark: The following corollary simplifies the hypothesis of Theorem 3 for the special case $p \equiv 1$.

Corollary: [5]

If $p \equiv 1$ and $[q^{-3/2}q''] \in \mathbf{L}(a, \infty)$, then the hypothesis of Theorem 3 is satisfied.

Chapter 4

Physical Problems

In this chapter, we start by defining singular Sturm-Liouville problems and then consider two physical problems: Vibrations of a circular membrane, and torsional waves of an elastic rod. In section 4.2, we solve the two physical problems by finding their exact solutions. Second, we use the Weyl Kodaira Theorem to classify their solutions by testing square integrability with respect to the weight function. In section 4.3, we find the asymptotic solutions of these problems by first transforming their equations from a finite singularity to an infinite singularity. Second, we find the asymptotic solutions by applying the asymptotic theorems in Chapter 5.

4.1 Singular Sturm-Liouville Problems

A Sturm-Liouville problem

$$-\frac{d}{dx} \left(p(x) \frac{du}{dx} \right) + q(x)u = \lambda w(x)u \quad (4.1.1)$$

is *singular* if at least one of the following occurs on the interval (a, b) ,

1. either a or b (or both) are infinite.
2. $p(a) = 0$ or $p(b) = 0$ (or both).
3. $\int_a^b \left(\frac{1}{p} + |q| + w \right) dx = +\infty$.

We assume that $p(x), p'(x), q(x)$, and $w(x)$ are continuous on the open interval $a < x < b$ with $p(x) > 0$ and $w(x) > 0$.

Example 4.1.1. Is the following problem singular?

$$-y'' = \lambda y, \quad 0 < x < \infty, \quad y(0) = 0 \quad (4.1.2)$$

The problem (4.1.2) is singular because of its interval $(0, \infty)$.

Example 4.1.2. Is the following problem singular?

$$y'' + \left(\frac{1 + \lambda x}{x} \right) y = 0, \quad 0 < x < 2, \quad y(0) = 0, \quad y(2) = 0 \quad (4.1.3)$$

Note that $p(x) = 1$, $\omega(x) = 1$ and $q(x) = \frac{1}{x}$, where $q(x)$ is not continuous at $x = 0$ and also $q(x)$ is undefined at $x = 0$. Thus problem (4.1.3) is singular because $q(x)$ violates the third condition.

Theorem 4.1.1. *(Weyl-Kodaira) [7] For either endpoint and for any nonreal λ there is at least one nonzero solution of (4.1.1) that is square-integrable at that endpoint.*

There are two cases:

1. *Limit Circle: All solutions of (4.1.1) are square-integrable at the endpoint for all λ .*
2. *Limit Point: Just one nonzero solution is square-integrable at the endpoint for all non-real λ .*

4.2 Two Physical Problems

In this section we derive two physical problems for which exact representations can be formed for their solutions. In the next section we show how to obtain asymptotic solutions of these two equations to get basic information on how the solutions behave near the singular point. The asymptotic methods are also applicable where exact solutions cannot be found.

4.2.1 Vibrations of a Circular Membrane

Consider the physical problem for the vibrations of a circular membrane. The partial differential equation for a circular membrane (a drumhead) is modelled with the two

dimensional wave equation in polar coordinates due to the circular geometry of the drumhead;

$$\frac{\partial^2 u}{\partial t^2} = c^2 \left(\frac{\partial^2 u}{\partial r^2} + \frac{1}{r} \frac{\partial u}{\partial r} + \frac{1}{r^2} \frac{\partial^2 u}{\partial \theta^2} \right) \quad (4.2.1)$$

where $c^2 = \frac{\tau}{\rho}$, τ = tension, ρ = density, and c = wave speed.

Consider a circular membrane with radius 1. Note that that our solutions depend only on radius r and time t . In other words, we assume that our solution does not depend on θ , and hence $u_\theta = 0$. Therefore, equation (4.2.1) becomes

$$\frac{\partial^2 u}{\partial t^2} = c^2 \left(\frac{\partial^2 u}{\partial r^2} + \frac{1}{r} \frac{\partial u}{\partial r} \right), \quad (4.2.2)$$

where $0 < r \leq 1$ and $t > 0$. Since the membrane is clamped at the circumference, we have the boundary condition

$$u(1, t) = 0, \quad t \geq 0. \quad (4.2.3)$$

Also, we have an implicit boundary condition where $r = 0$. Note that $r = 0$ is a singular point of equation (4.2.2) which gives us the implicit condition

$$R(r) \text{ is finite as } r \rightarrow 0^+ \quad (4.2.4)$$

for the radial component $R(r)$ of the wave equation in polar form.

We can solve the boundary value problem (4.2.2)- (4.2.4) using the separation of variables method. We summarize some of the calculations of [8] and hence by the method of separation of variables we arrive at the Sturm-Liouville equation for the

circular membrane:

$$-(rR'(r))' = \lambda\rho(r)rR(r), \quad 0 < r \leq 1 \quad (4.2.5)$$

Let the density be $\rho(r) = r^m$. Hence equation (4.2.5) becomes

$$-(rR'(r))' = \lambda r^{m+1}R(r), \quad 0 < r \leq 1 \quad (4.2.6)$$

In order to classify the Sturm-Liouville problem using the Weyl-Kodaira theorem we first need to find the general solution of equation (4.2.6). Since the Limit Circle and Limit Point classifications are independent of λ , we can choose different values of λ . Suppose that $\lambda = 0$; hence equation (4.2.6) becomes

$$-(rR'(r))' = 0. \quad (4.2.7)$$

Integrating both sides of equation (4.2.7) with respect to r , we get

$$R(r) = C \ln |r| + D. \quad (4.2.8)$$

We define

$$R_1(r) = \ln |r| \quad \text{and} \quad R_2(r) = 1.$$

The next step is to determine if $R_1(r)$ and $R_2(r)$ are square integrable with respect to the weight function $\omega(r)$, where $\omega(r) = r^{m+1}$.

First, we determine if $R_1(r)$ is square integrable with respect to the weight function.

$$\int_0^1 R_1^2(r)\omega(r) dr = \int_0^1 (\ln r)^2 r^{m+1} dr.$$

This integral can be evaluated exactly [8] yielding

$$\begin{aligned} \int_0^1 R_1^2(r)\omega(r) dr &= \int_0^1 (\ln r)^2 r^{m+1} dr \\ &= \begin{cases} \frac{2}{(m+2)^3} & \text{when } m > -2, \\ \infty & \text{when } m \leq -2. \end{cases} \end{aligned} \quad (4.2.9)$$

We conclude that R_1 is square integrable when $m > -2$, and it is not square integrable when $m \leq -2$.

Now we determine if $R_2(r)$ is square integrable.

$$\begin{aligned} \int_0^1 R_2^2(r)\omega(r) dr &= \int_0^1 r^{m+1} dr \\ &= -\lim_{r \rightarrow 0^+} \frac{r^{2+m} - 1}{2+m} \\ &= \begin{cases} \frac{1}{m+2} & \text{when } m > -2, \\ \infty & \text{when } m \leq -2. \end{cases} \end{aligned}$$

Hence, $R_2(r)$ is square integrable when $m > -2$, and it is not square integrable when $m \leq -2$. Applying the Weyl-Kodaira theorem we obtain a Limit Circle classification when $m > -2$.

4.2.2 Torsional Waves of an Elastic Rod

We now introduce the second physical problem: Torsional Waves of an Elastic Rod. Generally, torsion can be thought of as a twisting of an elastic body. In particular, torsion is the force that acts on the elastic cylinder when the two ends of the cylinder are twisted in opposite directions or if one end is turned while the other is held fixed. One can derive the partial differential equation for the torsion of a cylinder from the equations of elasticity in cylindrical polar coordinates; with the assumptions that there are no body forces and that the shear modulus of the circular rod depend only on the radius. One can use the separation of variables method to arrive at a singular Sturm-Liouville problem [8]. Finally, the singular Sturm-Liouville problem for the torsion of a cylinder is introduced and the Weyl Kodaira theorem is used to classify it.

Consider,

$$-\frac{d}{dr} \left(r^3 \mu(r) \frac{dR}{dr} \right) + \gamma^2 r^3 \mu(r) R(r) = \rho \omega^2 r^3 R(r) \quad (4.2.10)$$

with boundary conditions $R'(1) = 0$

In order to apply the Weyl-Kodaira theorem, we need to find the solution of the Sturm-Liouville equation (4.2.10). Assume that the shear modulus $\mu(r) = \alpha r^m$.

Hence the singular Sturm-Liouville equation (4.2.10) becomes

$$-\frac{d}{dr} \left(\alpha r^{3+m} \frac{dR}{dr} \right) + \gamma^2 \alpha r^{3+m} R(r) = \rho \omega^2 r^3 R(r). \quad (4.2.11)$$

Note that the Limit Point and Limit Circle classifications are independent of ω . To find the general solution of (4.2.11), we let $\omega = 0$ and $R(r) = fr^{-1-\frac{m}{2}}$.

Differentiating $R(r)$, we get

$$\frac{dR}{dr} = f'r^{-1-\frac{m}{2}} + \left(-1 - \frac{m}{2}\right)r^{-2-\frac{m}{2}}f. \quad (4.2.12)$$

Substitute $\omega = 0$, $R(r)$, and (4.2.12) into equation (4.2.11) to obtain

$$-\frac{d}{dr} \left[\alpha r^{3+m} \left(f'r^{-1-\frac{m}{2}} + \left(-1 - \frac{m}{2}\right)r^{-2-\frac{m}{2}} \right) \right] + \gamma^2 \alpha r^{3+m} f r^{-1-\frac{m}{2}} = 0. \quad (4.2.13)$$

Dividing by α and differentiating the first term with respect to r of equation (4.2.13) yields

$$\left(2 + \frac{m}{2}\right)r^{1+\frac{m}{2}}f' + r^{2+\frac{m}{2}}f'' + \left(-1 - \frac{m}{2}\right)r^{1+\frac{m}{2}}f' - \left(1 + \frac{m}{2}\right)^2 r^{\frac{m}{2}}f - \gamma^2 r^{2+\frac{m}{2}}f = 0.$$

After some algebraic manipulations we arrive at the Bessel equation

$$r^2 f'' + r f' + f \left(-\gamma^2 r^2 - \left(1 + \frac{m}{2}\right)^2 \right) = 0. \quad (4.2.14)$$

At this point, we will reduce equation (4.2.14) by change of variables. Let $z = \sqrt{-\gamma^2 r^2} = i\gamma r$, $dz = i\gamma dr$ and let $f(r) = f\left(\frac{z}{i\gamma}\right) = F(z)$. From chain rule, $f'(r) = F'(z)\frac{dz}{dr} = i\gamma F'(z)$; likewise, $f''(r) = (i\gamma)^2 F''(z)$. Substituting into equation (4.2.14)

we obtain a Bessel equation of order $(1 + \frac{m}{2})$:

$$z^2 F''(z) + zF'(z) + F(z) \left(z^2 - (1 + \frac{m}{2})^2 \right) = 0.$$

Therefore, the general solution of (4.2.11) is

$$R(r) = C_1 r^{-1-\frac{1}{2}m} J_{1+\frac{1}{2}m}(i\gamma r) + C_2 r^{-1-\frac{1}{2}m} Y_{1+\frac{1}{2}m}(i\gamma r), \quad (4.2.15)$$

where $J_p(r), Y_p(r)$ are Bessel functions of the first and second kind.

We define

$$R_1(r) = C_1 r^{-1-\frac{1}{2}m} J_{1+\frac{1}{2}m}(i\gamma r) \quad (4.2.16)$$

and

$$R_2(r) = C_2 r^{-1-\frac{1}{2}m} Y_{1+\frac{1}{2}m}(i\gamma r). \quad (4.2.17)$$

In order to check the classification of this problem we need to use the Weyl-Kodaira Theorem. We need to determine if $R_1(r)$ and $R_2(r)$ are square-integrable with respect to the weight function $\omega(r) = r^3$.

We begin by testing if $R_1(r)$ is square integrable.

$$\int_0^1 (R_1(r))^2 r^3 dr = \int_0^1 \left(r^{-1-\frac{1}{2}m} J_{1+\frac{1}{2}m}(i\gamma r) \right)^2 r^3 dr = \int_0^1 r^{1-m} J_{1+\frac{1}{2}m}^2(i\gamma r) dr$$

Note that $J_p(ir) = i^p I_p(r)$, where $I_p(r)$ is the modified Bessel function of the first kind. Hence,

$$\int_0^1 r^{1-m} \left(i^{1+\frac{1}{2}m} I_{1+\frac{1}{2}m}(\gamma r) \right)^2 dr = - \int_0^1 r^{1-m} i^m I_{1+\frac{1}{2}m}^2(\gamma r) dr.$$

Using the substitution $z = \gamma r$ and $dz = \gamma dr$, then

$$\begin{aligned} \int_0^1 (R_1(r))^2 r^3 dr &= -i^m \gamma^{m-2} \int_0^\gamma z^{1-m} I_{1+\frac{1}{2}m}^2(z) dz \\ &= -i^m \gamma^{m-2} \int_0^\gamma z^{1-m} \left[\sum_{k=0}^{\infty} \frac{\left(\frac{z}{2}\right)^{2k+\frac{1}{2}m}}{k! \Gamma(k+2+\frac{1}{2}m)} \right]^2 dz \\ &= -i^m \gamma^{m-2} \int_0^1 \left[\sum_{k=0}^{\infty} \frac{\left(\frac{z}{2}\right)^{2k+\frac{1}{2}}}{k! \Gamma(k+2+\frac{1}{2}m)} \right]^2 dz. \end{aligned} \quad (4.2.18)$$

It can be shown that the series in (4.2.18) is uniformly convergent [8]. Thus, we obtain

$$\int_0^1 (R_1(r))^2 r^3 dr = -i^m \gamma^{m-2} \int_0^\gamma \left[\sum_{k=0}^{\infty} \frac{\left(\frac{z}{2}\right)^{2k+\frac{1}{2}}}{k! \Gamma(k+2+\frac{1}{2}m)} \right]^2 dz < \infty$$

Therefore $R_1(r)$ is square-integrable, for all m .

Now we need to check if $R_2(r)$ is square-integrable with respect to the weight function

$$\omega(r) = r^3.$$

$$\int_0^1 (R_2(r))^2 r^3 dr = \int_0^1 \left(r^{-1-\frac{1}{2}m} Y_{1+\frac{1}{2}m}(i\gamma r) \right)^2 r^3 dr = \int_0^1 r^{1-m} Y_{1+\frac{1}{2}m}^2(i\gamma r) dr.$$

Note that $Y_p(ir) = i^p K_p(r)$, where $K_p(r)$ is the modified Bessel function of the second kind. Hence,

$$\int_0^1 r^{1-m} \left(i^{1+\frac{1}{2}m} K_{1+\frac{1}{2}m}(\gamma r) \right)^2 dr = - \int_0^1 r^{1-m} i^m K_{1+\frac{1}{2}m}^2(\gamma r) dr.$$

Using the substitution $z = \gamma r$, then $dz = \gamma dr$, and so

$$\int_0^1 (R_2(r))^2 r^3 dr = -i^m \gamma^{m-2} \int_0^\gamma z^{1-m} K_{1+\frac{1}{2}m}^2(z) dz. \quad (4.2.19)$$

Since $K_p(z) = \frac{\pi}{2 \sin p\pi} (I_{-p}(z) - I_p(z))$, and $p = 1 + \frac{1}{2}m$ then 4.2.19 becomes

$$-i^m \gamma^{m-2} \left(\frac{\pi}{2 \sin(1 + \frac{1}{2}m)\pi} \right)^2 \int_0^\gamma z^{1-m} \left(I_{-1-\frac{1}{2}m}(z) - I_{1+\frac{1}{2}m}(z) \right)^2 dz$$

The following results are worked out in [8] by means of examining the series expressions for $K_p(z)$.

CASE 1: Assume that $m \geq 0$.

In this case,

$$\int_0^1 (R_2(r))^2 r^3 dr = \infty$$

and hence $R_2(r)$ is not square integrable.

CASE 2: Assume that $m < 0$.

In this case,

$$\int_0^1 (R_2(r))^2 r^3 dr < \infty,$$

and therefore $R_2(r)$ is square integrable.

Thus by Weyl-Kodaira Theorem we obtain a Limit Circle classification for case 1, and a Limit Point classification for case 2.

In the next section we obtain asymptotic solutions of (4.2.6) and (4.2.10) for all values of m . The Limit Point - Limit Circle classifications are obtained.

4.3 Asymptotic Solutions of the Physical Problems

In the previous section two singular Sturm Liouville problems were classified by finding their general solutions and by using the Weyl-Kodaira Theorem. In this section the same singular problems are used to get their asymptotic solutions. In order to get their asymptotic solutions we first need to transform the two equations from a finite singularity to an infinite singularity. Second, we remove the second term of the transformed equation. Third, we transform the transformed equation without the middle term to a perturbation of a constant coefficient equation. Finally the asymptotic theorems in Chapter 3 can be used to find the asymptotic solutions. Note that the asymptotic solutions can provide us with the basic information on how the solutions behave near the singular point.

4.3.1 Asymptotic Solutions of the Vibrations of a Circular Membrane

Recall the singular Sturm-Liouville equation for the vibrations of a circular membrane,

$$\begin{aligned} -(rR'(r))' &= \lambda r^{m+1}R(r), \\ R''(r) + \frac{1}{r}R'(r) + \lambda r^m R(r) &= 0, \quad 0 < r \leq 1. \end{aligned} \quad (4.3.1)$$

We remove the middle term of (4.3.1), by letting $R(r) = r^{-1/2}z(r)$. The first derivative of $R(r)$ is

$$R'(r) = \frac{1}{r^{1/2}}z'(r) - \frac{1}{2r^{3/2}}z(r)$$

and the second derivative is

$$R''(r) = \frac{1}{r^{1/2}}z''(r) - \frac{1}{r^{3/2}}z'(r) + \frac{3}{4r^{5/2}}z(r).$$

Substituting $R(r)$ and its derivatives into (4.3.1) and simplifying we obtain

$$z''(r) + \left(\frac{1}{4r^2} + \lambda r^m \right) z(r) = 0, \quad 0 < r \leq 1 \quad (4.3.2)$$

Next we want to transform (4.3.2) to a perturbation of a constant coefficient equation.

This will enable us to apply the asymptotic theorems of chapter 2. One can do that in two different ways.

Method 1: Let $z(r) = \mu(r)w(t)$, where $t = -\ln r$, and $\mu(r) = r^{1/2}$.

We now differentiate $z(r)$ twice and we get

$$z''(r) = \frac{1}{r^{3/2}}\ddot{w}(t) - \frac{1}{4r^{3/2}}w(t), \quad \cdot = \frac{d}{dt}.$$

Substituting $z(r)$ and $z''(r)$ into (4.3.2) we get

$$\begin{aligned} \ddot{w}(t) + \lambda r^{m+2}w(t) &= 0 \\ \ddot{w}(t) + \lambda e^{-t(m+2)}w(t) &= 0, \quad 0 \leq t < \infty. \end{aligned} \quad (4.3.3)$$

Method 2:

Method 2 requires two steps in order to transform (4.3.2) to perturbation of constant coefficients.

First Step:

Transform (4.3.2) by letting $z(r) = \frac{1}{x}y(x)$, $x = \frac{1}{r}$. Note that $0 < r \leq 1$ and $1 \leq x < \infty$. Differentiating $z(r)$ twice we get

$$z''(r) = \frac{1}{r^3}y(x)$$

and substituting into (4.3.2) we obtain

$$y''(x) + \left(\frac{r^2}{4} + \lambda r^{m+4} \right) y(x) = 0, \quad (4.3.4)$$

Second Step:

Transform (4.3.4) to a perturbation of a constant coefficient equation. Let $y(x) = \mu(x)w(t)$, $t = \ln(x)$ and $\mu(x) = x^{1/2}$. We again differentiate $y(x)$ twice and we get

$$y''(x) = \frac{1}{x^{3/2}}\ddot{w}(t) - \frac{1}{4x^{3/2}}w(t).$$

Substituting into (4.3.4) and simplifying we obtain

$$\ddot{w}(t) + \lambda e^{-t(m+2)}w(t) = 0,$$

which is the same equation as (4.3.3). Note that in our problem we will use the first method.

Recall the transformed differential equations:

$$\begin{aligned}\ddot{w}(t) + \lambda r^{m+2}w(t) &= 0 \\ \ddot{w}(t) + \lambda e^{-t(m+2)}w(t) &= 0, \quad 0 \leq t < \infty.\end{aligned}$$

CASE 1A: $m = -2, \lambda = 0$

Since $m = -2$ and $\lambda = 0$ then equation (4.3.3) becomes $\ddot{w}(t) = 0$. In this case it is easy to find the general solution of $\ddot{w}(t) = 0$ which is $w(t) = At + B$. Note that A and B are constants. We now can classify the above problem by letting $w_1(t) = At$ and $w_2(t) = B$ and testing if the two solutions are square integrable with respect to

the weight function, $\bar{w}(t) = 1$.

Hence,

$$\int_0^\infty |w_1(t)|^2 1 \, dt = \int_0^\infty A^2 t^2 \, dt = \infty.$$

Note that

$$\int_0^\infty |w_1(t)|^2 1 \, dt = \int_0^\infty A^2 t^2 \, dt = \int_0^1 \frac{A^2 (\ln(r))^2}{r} \, dr = \int_0^1 \frac{(R_1(r))^2}{r} \, dr = \infty,$$

since $w(t) = r^{-1/2} z(r)$ and $t = -\ln(r)$.

Similarly for $w_2(t) = B$,

$$\int_0^\infty |w_2(t)|^2 1 \, dt = \int_0^\infty B^2 \, dt = \infty = \int_0^1 B^2 r^{-1} \, dr = \int_0^1 |R_2(r)|^2 r^{-1} \, dr.$$

Therefore in this case we have a Limit Point classification.

CASE 1B: $m = -2, \lambda \neq 0$

Substituting $m = -2$ and $\lambda \neq 0$ into (4.3.3) we obtain

$$\ddot{w}(t) + \lambda w(t) = 0, \quad 0 \leq t < \infty. \quad (4.3.5)$$

The general solution of (4.3.5) is

$$w(t) = Ae^{t\sqrt{-\lambda}} + Be^{-t\sqrt{-\lambda}}, \quad \text{if } \lambda < 0 \quad (4.3.6)$$

$$w(t) = A \cos \sqrt{\lambda} t + B \sin \sqrt{\lambda} t, \quad \text{if } \lambda > 0. \quad (4.3.7)$$

One can show that one solution of (4.3.5) for $\lambda < 0$ is square integrable with respect to the weight function, $\bar{\omega}(t) = 1$. Let $w_1(t) = Ae^{t\sqrt{-\lambda}}$ and $w_2(t) = Be^{-t\sqrt{-\lambda}}$.

Thus, we obtain

$$\int_0^\infty |w_1(t)|^2 1 dt = \int_0^\infty A^2 e^{2t\sqrt{-\lambda}} dt = \infty \quad \text{because of exponential growth.}$$

One can transform $w_1(t)$ back to $R_1(r)$ by letting $w_1(t) = z_1(r)r^{-1/2}$ and $t = -\ln r \Leftrightarrow r = e^{-t}$.

Therefore,

$$\int_0^\infty |w_1(t)|^2 1 dt = \int_0^\infty A^2 e^{2t\sqrt{-\lambda}} dt = \infty = \int_0^1 A^2 r^{-2\sqrt{-\lambda}-1} dr = \int_0^1 |R_1(r)|^2 r^{-1} dr = \infty$$

Similarly for $w_2(t)$:

$$\int_0^\infty |w_2(t)|^2 1 dt = \int_0^\infty B^2 e^{-2t\sqrt{-\lambda}} dt = \int_0^1 B^2 r^{2\sqrt{-\lambda}-1} dr = \int_0^1 |R_2(r)|^2 r^{-1} dr < \infty,$$

because of exponential decay.

In a similar way one can show (4.3.7) is not square integrable with respect to the weight function.

We can conclude that for case 1B with $\lambda \neq 0$ we have a Limit Point classification.

CASE 2: $m > -2$

Recall the second order differential equation (4.3.3), $\ddot{w}(t) + \lambda e^{-t(m+2)}w(t) = 0$. In this case we cannot find the general solution of (4.3.3), so we will apply the Ghizzetti's

Theorem to get the asymptotic solutions of (4.3.3).

Recall the Ghizzetti's Theorem:

Theorem 4.3.1. *Consider the second order differential equation*

$$z''(t) + a(t)z'(t) + b(t)z(t) = 0, \quad c < t < \infty \quad (4.3.8)$$

Suppose $\int_c^\infty |a(t)| dt < \infty$ and $\int_c^\infty t|b(t)| dt < \infty$. Then there are two independent solutions z_0 and z_1 of (4.3.8) such that

- $z_0(t) \rightarrow 1, tz'_0(t) \rightarrow 0$ as $t \rightarrow \infty$ and
- $\frac{z_1(t)}{t} \rightarrow 1, z'_1(t) \rightarrow 1$ as $t \rightarrow \infty$.

We now apply Ghizzetti's Theorem on (4.3.3). First we check the assumptions of theorem (4.3.1)

Since $a(t) = 0$ then

$$\int_0^\infty |a(t)| dt = \int_0^\infty 0 dt < \infty.$$

Since $b(t) = \lambda e^{-t(m+2)}$ then

$$\int_0^\infty t|b(t)| dt = \int_0^\infty t\lambda e^{-t(m+2)} dt < \infty \text{ if } m+2 > 0.$$

Hence the assumptions of theorem (4.3.1) are satisfied and then

$$w_0(t) = r^{-1/2}z_0(r) = r^{-1/2}r^{1/2}R_0(r) = R_0(r) \rightarrow 1, \quad \text{as } r \rightarrow 0,$$

and

$$\frac{w_1(t)}{t} = \frac{r^{-1/2}z_1(r)}{-\ln r} = -\frac{R_1(r)}{\ln r} \rightarrow 1 \quad \text{as} \quad r \rightarrow 0.$$

Therefore,

$$R_0(r) = (1 + o(1))$$

$$R_1(r) = -\ln r(1 + o(1)).$$

We now need to determine square integrability of $R_0(r)$ and $R_1(r)$ with respect to the weight function $\bar{\omega}(t) = e^{-t(m+2)}$. Hence,

$$\int_0^\infty |w_0(t)|^2 e^{-t(m+2)} dt = \int_0^\infty (1 + o(1))^2 e^{-t(m+2)} < \infty$$

because of exponential decay.

We can transform $w_0(t)$ back to $R_0(r)$ with $w_0(t) = r^{-1/2}z_0(r)$, $z_0(r) = r^{1/2}R_0(r)$ and $t = -\ln r$.

Thus,

$$\int_0^\infty |w_0(t)|^2 e^{-t(m+2)} dt = \int_0^1 |R_0(r)|^2 r^{m+1} dr < \infty.$$

Similarly for $R_1(r)$:

$$\int_0^\infty |w_1(t)|^2 e^{-t(m+2)} dt = \int_0^\infty t^2 e^{-t(m+2)} dt = \int_0^1 |R_1(r)|^2 r^{m+1} dr < \infty.$$

Therefore for case 2 with $m > -2$ we obtain a Limit Circle classification.

CASE 3: $m < -2$

Recall Theorem 3.0.3: Consider the second-order equation,

$$(pw'(t))' \pm qw(t) = 0, \quad \text{with } q > 0 \text{ on } [a, \infty). \quad (4.3.9)$$

Theorem 4.3.2. *Suppose that $\left(\frac{q}{p}\right)^{1/2}$ does not belong to $\mathbf{L}(a, \infty)$ and that each of $\left[\left(\frac{q}{p}\right)^{-1/2}\frac{p'}{p}\right]', \left[\left(\frac{q}{p}\right)^{-1/2}\frac{q'}{q}\right]', \left[\left(\frac{q}{p}\right)^{-1/2}\left(\frac{p'}{p}\right)^2\right]$ and $\left[\left(\frac{q}{p}\right)^{-1/2}\left(\frac{q'}{q}\right)^2\right] \in \mathbf{L}(a, \infty)$. Then there is a fundamental matrix W for $W' = AW$ such that as $t \rightarrow \infty$, $\hat{Q}(t)W(t)E(t) \rightarrow L$ where*

$$A = \begin{bmatrix} 0 & 1/p \\ \mp q & 0 \end{bmatrix}, \quad \hat{Q} = \begin{bmatrix} q^{1/4} & 0 \\ 0 & q^{-1/4} \end{bmatrix},$$

$$L = \begin{bmatrix} 1 & 1 \\ \mu_1 & \mu_2 \end{bmatrix}, \quad E(t) = \begin{bmatrix} \exp[-\mu_1 \int_a^t (\frac{q}{p})^{1/2}] & 0 \\ 0 & \exp[-\mu_2 \int_a^t (\frac{q}{p})^{1/2}] \end{bmatrix}$$

and μ_1, μ_2 are the eigenvalues of $\mu^2 \pm 1 = 0$.

Remark: The following corollary simplifies the hypothesis of Theorem 4.3.2 for the special case $p \equiv 1$.

Corollary:

If $p \equiv 1$ and $[q^{-3/2}q''] \in \mathbf{L}(a, \infty)$, then the hypothesis of Theorem 4.3.2 is satisfied.

Note that in our problem $p \equiv 1$, $q(t) = \lambda e^{-t(m+2)}$ and hence we can use the above corollary.

Hence,

$$\begin{aligned}
\int_0^\infty (q^{-3/2}q'')^{1/2} dt &= \int_0^\infty (\lambda e^{-t(m+2)}(\lambda e^{-t(m+2)})'')^{1/2} dt \\
&= \int_0^\infty \lambda^{-1/4}(m+2)e^{t(m+2)/4} dt \\
&< \infty \quad \text{since } m < -2 \text{ (exponential decay)}
\end{aligned}$$

Therefore the hypothesis of Theorem 4.3.2 is satisfied by corollary and hence we can apply Theorem 4.3.2 on (4.3.3) with $m < -2$. Thus $\hat{Q}(t)W(t)E(t) \rightarrow L$ as $t \rightarrow \infty$.

Hence,

$$\begin{aligned}
\hat{Q}(t)W(t)E(t) &= \begin{bmatrix} q^{1/4} & 0 \\ 0 & q^{-1/4} \end{bmatrix} \cdot \begin{bmatrix} w_1 & w_2 \\ w'_1 & w'_2 \end{bmatrix} \cdot \begin{bmatrix} e^{-\mu_1 Q(t)} & 0 \\ 0 & e^{-\mu_2 Q(t)} \end{bmatrix} \\
&= \begin{bmatrix} (1 + o_1(t)) & (1 + o_2(t)) \\ (\mu_1 + o_3(t)) & (\mu_2 + o_4(t)) \end{bmatrix}
\end{aligned}$$

Since $\hat{Q}(t)$ and $E(t)$ are nonsingular matrices then we can solve for $W(t)$. Thus,

$$W = \begin{bmatrix} w_1 & w_2 \\ w'_1 & w'_2 \end{bmatrix} = \begin{bmatrix} e^{\mu_1 Q(t)} q^{-1/4} (1 + o_1(t)) & e^{\mu_2 Q(t)} q^{-1/4} (1 + o_2(t)) \\ e^{\mu_1 Q(t)} q^{1/4} (\mu_1 + o_3(t)) & e^{\mu_2 Q(t)} q^{1/4} (\mu_2 + o_4(t)) \end{bmatrix}$$

where $q(t) = \lambda e^{-t(m+2)}$, $Q(t) = \int_a^t (q(s))^{1/2} ds$, and μ_1, μ_2 are the eigenvalues of $\mu^2 \pm 1 = 0$.

Subcase 3.1: $m < -2$ and $\lambda > 0$

Note that $\mu_1 = i$ and $\mu_2 = -i$. For our problem we just consider $w_1(t)$ and $w_2(t)$:

$$w_1(t) = e^{iQ(t)} q^{-1/4} (1 + o_1(t)) = (1 + o_1(t)) (\lambda e^{-t(m+2)})^{-1/4} e^{i \int_a^t (\lambda e^{-s(m+2)})^{1/2} ds}$$

$$w_2(t) = e^{-iQ(t)} q^{-1/4} (1 + o_2(t)) = (1 + o_1(t)) (\lambda e^{-t(m+2)})^{-1/4} e^{-i \int_a^t (\lambda e^{-s(m+2)})^{1/2} ds}.$$

We let,

$$\hat{w}_1(t) = \frac{w_1(t) + w_2(t)}{2} = q^{-1/4} (\cos Q(t) + o(t))$$

because we just want to work with the real solutions of our problem.

$$\hat{w}_1(t) = \lambda^{-1/4} e^{t(m+2)/4} \left(\cos \left[-\frac{2\sqrt{\lambda}}{m+2} (e^{-t(m+2)/2} - e^{-a(m+2)/2}) \right] + o(t) \right)$$

Note that $Q(t) = \int_a^t (q(s))^{1/2} ds = \int_a^t (\lambda e^{-s(m+2)})^{1/2} ds = \infty$ as $t \rightarrow \infty$, because of exponential growth ($m+2 < 0$).

Since $\cos Q(t)$ is bounded as $t \rightarrow \infty$ and $e^{t(m+2)/4} \rightarrow 0$ as $t \rightarrow \infty$ then

$$\hat{w}_1(t) \rightarrow 0 \text{ as } t \rightarrow \infty$$

We now can transform $\hat{w}_1(t)$ back to $\hat{R}_1(r)$ by $z(r) = r^{1/2} w(t)$, $R(r) = r^{-1/2} z(r)$ and $r = e^{-t}$. Thus,

$$\hat{R}_1(r) = \lambda^{-1/4} r^{-(m+2)/4} \left(\cos \left[-\frac{2\sqrt{\lambda}}{m+2} (r^{(m+2)/2} - e^{-a(m+2)/2}) \right] + o(r) \right),$$

and

$$\hat{R}_1(r) \rightarrow 0 \quad \text{as } r \rightarrow 0.$$

One can show the lack of square integrability for $w_1(t)$ with respect to the weight function $\bar{\omega}(t) = e^{-t(m+2)}$:

$$\begin{aligned} \int_0^\infty |w_1(t)|^2 \bar{\omega}(t) dt &= \int_0^\infty |e^{iQ(t)}|^2 |\lambda^{-1/4} e^{t(m+2)/4} (1 + o(t))|^2 e^{-t(m+2)} dt \\ &= \lambda^{-1/2} \int_0^\infty e^{-t(m+2)/2} (1 + o(t))^2 dt \\ &= \infty \quad \text{since } m+2 < 0. \end{aligned}$$

Transforming back to $R_1(r)$ we have

$$\int_0^\infty |w_1(t)|^2 \bar{\omega}(t) dt = \int_0^1 |R_1(r)|^2 r^{m+1} dr = \infty.$$

Similarly for

$$\hat{w}_2(t) = \frac{w_1(t) - w_2(t)}{2i} = q^{-1/4} (\sin Q(t) + o(t))$$

Therefore, $\hat{w}_2(t) \rightarrow 0$ as $t \rightarrow \infty$ and

$$\hat{R}_2(r) = \lambda^{-1/4} r^{-(m+2)/4} \left(\sin \left[\frac{-2\sqrt{\lambda}}{m+2} (r^{(m+2)/2} - e^{-a(m+2)/2}) \right] + o(r) \right),$$

and $\hat{R}_2(r) \rightarrow 0$, as $r \rightarrow 0$.

Note that since we show that one of the solutions is infinite then we do not need to test square integrability for $w_2(t)$ to determine the classification. Thus for subcase

3.1 with $m < -2$ and $\lambda > 0$ we have a Limit Point classification.

Subcase 3.2: $m < -2$ and $\lambda < 0$

In this subcase we again use theorem 4.3.2. We already show that the assumptions of Theorem 4.3.2 are satisfied and hence $\hat{Q}(t)W(t)E(t) \rightarrow L$. So,

$$W = \begin{bmatrix} w_1 & w_2 \\ w'_1 & w'_2 \end{bmatrix} = \begin{bmatrix} e^{\mu_1 Q(t)} q^{-1/4} (1 + o_1(t)) & e^{\mu_2 Q(t)} q^{-1/4} (1 + o_2(t)) \\ e^{\mu_1 Q(t)} q^{1/4} (\mu_1 + o_3(t)) & e^{\mu_2 Q(t)} q^{1/4} (\mu_2 + o_4(t)) \end{bmatrix}$$

where $q(t) = -\lambda e^{-t(m+2)}$, $Q(t) = \int_a^t (q(s))^{1/2} ds$, and μ_1, μ_2 are the eigenvalues of $\mu^2 - 1 = 0$. So $\mu_1 = 1$ and $\mu_2 = -1$.

Consider $w_1(t)$ and $w_2(t)$,

$$w_1(t) = (1 + o_1(t)) q^{-1/4} e^{Q(t)} = (1 + o_1(t)) (|\lambda| e^{-t(m+2)})^{-1/4} e^{\int_a^t |\lambda|^{1/2} e^{-s(m+2)/2} ds}$$

and

$$w_2(t) = (1 + o_2(t)) q^{-1/4} e^{-Q(t)} = (1 + o_2(t)) (|\lambda| e^{-t(m+2)})^{-1/4} e^{-\int_a^t |\lambda|^{1/2} e^{-s(m+2)/2} ds}.$$

Now as $t \rightarrow \infty$ we have

$$w_1(t) \rightarrow \infty \quad \text{and} \quad w_2(t) \rightarrow 0,$$

and as $r \rightarrow 0$ we get

$$R_1(r) \rightarrow \infty \quad \text{and} \quad R_2(r) \rightarrow 0.$$

One can show square integrability as in subcase 3.1. Therefore for subcase 3.2 we have a Limit Point classification.

4.3.2 Asymptotic Solutions for the Torsional Waves of an Elastic Rod

We now find the asymptotic solutions for the second physical problem: Torsional waves of an elastic rod. Consider the second order differential equation

$$R''(r) + \frac{3+m}{r}R'(r) + \left(\frac{\lambda}{r^m} - c\right)R(r) = 0, \quad 0 < r \leq 1. \quad (4.3.10)$$

One can remove the middle term of (4.3.10) by letting $R(r) = r^{-(3+m)/2}z(r)$. The first derivative of $R(r)$ is

$$R'(r) = r^{-(3+m)/2}z'(r) - \frac{(3+m)}{2}r^{-(5+m)/2}z(r)$$

and the second derivative is

$$R''(r) = r^{-(3+m)/2}z''(r) - (3+m)r^{-(5+m)/2}z'(r) + \frac{3+m}{2}\frac{5+m}{2}r^{-(7+m)/2}z(r).$$

Substituting $R(r)$ and its derivatives into (4.3.10) and simplifying we obtain

$$z''(r) + \left[\left(\frac{\lambda}{r^m} - c \right) - \frac{(m+3)(m+1)}{4r^2} \right] z(r) = 0, \quad 0 < r \leq 1. \quad (4.3.11)$$

We continue with the transformation of (4.3.11) to a perturbation of constant coefficients equation. So we let $z(r) = r^{1/2}w(t)$, $t = -\ln r$. Differentiating $z(r)$ twice we get

$$z''(r) = r^{-3/2}\ddot{w}(t) - \frac{r^{-3/2}}{4}w(t), \quad \cdot = \frac{d}{dt},$$

substituting into (4.3.11) and simplifying we obtain

$$\ddot{w}(t) + \left(-\frac{m^2 + 4m + 4}{4} + \lambda e^{t(m-2)} - ce^{-2t} \right) w(t) = 0, \quad 0 \leq t < \infty. \quad (4.3.12)$$

We now want to find the asymptotic solutions of (4.3.12) for all m . In order to find the asymptotic solutions we need to consider three cases.

CASE 1: $m < 2$

For this case we need to apply Theorem 3.0.1. First we show that the assumptions of Theorem 3.0.1 are satisfied, however in order to do that we consider two subcases.

Subcase 1.1: $m < 2$ and $C = \frac{m^2 + 4m + 4}{4} < 0$

Assumptions of Theorem 3.0.1:

- Let A be a constant matrix with characteristic distinct roots μ_1, μ_2 , and $A = \begin{bmatrix} 0 & 1 \\ C & 0 \end{bmatrix}$. The eigenvalues of A are $\mu = \pm\sqrt{C}$. Since $C < 0$ then $\mu_1 = i\sqrt{-C}$

and $\mu_2 = -i\sqrt{-C}$.

- Let the matrix V be differentiable and satisfy $\int_0^\infty |V'(t)| dt < \infty$ and let $V(t) \rightarrow 0$ as $t \rightarrow \infty$.

Since $V(t) = 0$ then the above assumptions for the matrix V are satisfied.

- Let the matrix S be integrable and let $\int_0^\infty |S(t)| dt < \infty$.

$$\begin{aligned} \int_0^\infty |S(t)| dt &= \int_0^\infty |\lambda e^{t(m-2)} - ce^{-2t}| dt \\ &< \infty \end{aligned}$$

since both exponentials are decaying.

- Let the roots of $\det(A + V(t) - \tilde{\lambda}E) = 0$ be denoted by $\tilde{\lambda}_j(t), j = 1, 2$. Clearly, by reordering the μ_j if necessary, $\lim_{t \rightarrow \infty} \tilde{\lambda}_j(t) = \mu_j$. So,

$$\det(A + V(t) - \tilde{\lambda}E) = \det \begin{bmatrix} -\tilde{\lambda} & 1 \\ C & -\tilde{\lambda} \end{bmatrix} = \tilde{\lambda}^2 - C = 0 \Leftrightarrow \tilde{\lambda} = \pm i\sqrt{-C}.$$

Thus the eigenvalues of $A + V(t)$ are $\tilde{\lambda}_1 = i\sqrt{-C}$ and $\tilde{\lambda}_2 = -i\sqrt{-C}$. Note that $\lim_{t \rightarrow \infty} \tilde{\lambda}_1(t) = \lim_{t \rightarrow \infty} i\sqrt{-C} = i\sqrt{-C}$ and $\lim_{t \rightarrow \infty} \tilde{\lambda}_2(t) = \lim_{t \rightarrow \infty} -i\sqrt{-C} = -i\sqrt{-C}$

- For a given k , let $D_{kj}(t) = \operatorname{Re}(\tilde{\lambda}_k(t) - \tilde{\lambda}_j(t))$. Suppose all $j, 1 \leq j \leq n$ fall into

one of two classes I_1 and I_2 , where

$$j \in I_1 \text{ if } \int_0^t D_{kj}(\tau) d\tau \rightarrow \infty \text{ as } t \rightarrow \infty,$$

$$\int_{t_1}^{t_2} D_{kj}(\tau) d\tau > -K (0 \leq t_1 \leq t_2),$$

and $j \in I_2 \text{ if } \int_{t_1}^{t_2} D_{kj}(\tau) d\tau < K, (0 \leq t_1 \leq t_2)$

where k is fixed and K is a constant. Since the eigenvalues $\tilde{\lambda}_1, \tilde{\lambda}_2$ are complex then $D_{kj}(t) = 0$ and hence $\int_{t_1}^{t_2} 0 d\tau < K$.

- Let p_k be a characteristic vector of A associated with μ_k , so that

$$Ap_k = \mu_k p_k.$$

The eigenvectors of A are $p_1 = [1, i\sqrt{-C}]^T$ and $p_2 = [1, -i\sqrt{-C}]^T$. with $\mu_1 = i\sqrt{-C}$ and $\mu_2 = -i\sqrt{-C}$ respectively.

Since all the assumptions of Theorem 3.0.1 are satisfied then there is a solution of (4.3.12) and a t_0 , such that $(0 \leq t_0 < \infty)$ and

$$\lim_{t \rightarrow \infty} \phi_k(t) \exp \left[- \int_{t_0}^t \tilde{\lambda}_k(\tau) d\tau \right] = p_k,$$

which implies

$$\begin{bmatrix} w_1(t) \\ \dot{w}_1(t) \end{bmatrix} \exp \left(- \int_{t_0}^t \tilde{\lambda}_1(\tau) d\tau \right) \rightarrow p_1 = \begin{bmatrix} 1 \\ i\sqrt{-C} \end{bmatrix}$$

and

$$\begin{bmatrix} w_2(t) \\ \dot{w}_2(t) \end{bmatrix} \exp \left(- \int_{t_0}^t \tilde{\lambda}_2(\tau) d\tau \right) \rightarrow p_2 = \begin{bmatrix} 1 \\ -i\sqrt{-C} \end{bmatrix}.$$

In our problem we consider $w_1(t)$ and $w_2(t)$. Rewriting $w_1(t)$ and $w_2(t)$ we get

$$w_1(t) = (1 + o(1)) \exp \left(i \int_{t_0}^t \sqrt{-C} d\tau \right) = (1 + o(1)) \exp \left(i\sqrt{-C}(t - t_0) \right)$$

and

$$w_2(t) = (1 + o(1)) \exp \left(-i \int_{t_0}^t \sqrt{-C} d\tau \right) = (1 + o(1)) \exp \left(-i\sqrt{-C}(t - t_0) \right).$$

We now consider only the real solutions of our problem. So we let

$$\tilde{w}_1(t) = \frac{w_1(t) + w_2(t)}{2} = \cos(\sqrt{-C}(t - t_0)) + o(1).$$

Therefore $\tilde{w}_1(t)$ is oscillating as $t \rightarrow \infty$. One can transform $\tilde{w}_1(t)$ back to $R_1(r)$ by

letting $\tilde{w}_1(t) = r^{-1/2} z_1(r)$ and $z_1(r) = r^{(3+m)/2} R_1(r)$ where $t = -\ln r$. Thus

$$R_1(r) = r^{-(m+2)/2} \left[\cos(\sqrt{-C}(-\ln r - t_0)) + o(1) \right] \rightarrow 0, \text{ as } r \rightarrow 0.$$

One can show that $w_1(t)$ and $w_2(t)$ are square integrable with respect to the weight function and also use the same transformation as before to show that $R_1(r)$ and $R_2(r)$ are square integrable:

$$\begin{aligned} \int_0^\infty |w_1(t)|^2 e^{t(m-2)} dt &= \int_0^\infty |(1 + o(1))e^{i\sqrt{-C}(t-t_0)}|^2 e^{t(m-2)} dt \\ &= \int_0^\infty (1 + o(1))e^{t(m-2)} dt \\ &< \infty \quad \text{because of exponential decay.} \end{aligned}$$

Transforming back to $R_1(r)$ we have

$$\int_0^\infty |w_1(t)|^2 e^{t(m-2)} dt = \int_0^1 r^3 |R_1(r)|^2 dr < \infty.$$

The same procedure applies for $w_2(t)$:

$$\int_0^\infty |w_2(t)|^2 e^{t(m-2)} dt = \int_0^1 r^3 |R_2(r)|^2 dr < \infty.$$

Therefore for subcase 1.1 with $m < 2$ and $C < 0$ we can conclude a Limit Circle classification.

Subcase 1.2: $m < 2$ and $C = \frac{m^2+4m+4}{4} > 0$

We again need to show that the assumptions of Theorem 3.0.1 are satisfied.

- Let A be a constant matrix with characteristic distinct roots μ_1, μ_2 , and $A = \begin{bmatrix} 0 & 1 \\ C & 0 \end{bmatrix}$. The eigenvalues of A are $\mu = \pm\sqrt{C}$. Since $C > 0$ then $\mu_1 = \sqrt{C}$

and $\mu_2 = -\sqrt{C}$.

- Let the matrix V be differentiable and satisfy $\int_0^\infty |V'(t)| dt < \infty$ and let $V(t) \rightarrow 0$ as $t \rightarrow \infty$.

$V(t) = 0$ and hence the above assumptions for the matrix V are satisfied since V is just the zero matrix.

- Let the matrix S be integrable and let $\int_0^\infty |S(t)| dt < \infty$.

$$\begin{aligned} \int_0^\infty |S(t)| dt &= \int_0^\infty |\lambda e^{t(m-2)} - ce^{-2t}| dt \\ &< \infty, \quad \text{since both exponentials are decaying} \end{aligned}$$

- Let the roots of $\det(A + V(t) - \tilde{\lambda}E) = 0$ be denoted by $\tilde{\lambda}_j(t), j = 1, 2$. Clearly, by reordering the μ_j if necessary, $\lim_{t \rightarrow \infty} \tilde{\lambda}_j(t) = \mu_j$. So,

$$\det(A + V(t) - \tilde{\lambda}E) = \det \begin{bmatrix} -\tilde{\lambda} & 1 \\ C & -\tilde{\lambda} \end{bmatrix} = \tilde{\lambda}^2 - C = 0 \Leftrightarrow \tilde{\lambda} = \pm\sqrt{C}.$$

Thus the eigenvalues of $A + V(t)$ are $\tilde{\lambda}_1 = \sqrt{C}$ and $\tilde{\lambda}_2 = -\sqrt{-C}$. Note that $\lim_{t \rightarrow \infty} \tilde{\lambda}_1(t) = \lim_{t \rightarrow \infty} \sqrt{C} = \sqrt{C}$ and $\lim_{t \rightarrow \infty} \tilde{\lambda}_2(t) = \lim_{t \rightarrow \infty} -\sqrt{C} = -\sqrt{C}$

- For a given k , let $D_{kj}(t) = \operatorname{Re}(\tilde{\lambda}_k(t) - \tilde{\lambda}_j(t))$. Suppose all $j, 1 \leq j \leq n$ fall into

one of two classes I_1 and I_2 , where

$$j \in I_1 \text{ if } \int_0^t D_{kj}(\tau) d\tau \rightarrow \infty \text{ as } t \rightarrow \infty,$$

$$\int_{t_1}^{t_2} D_{kj}(\tau) d\tau > -K (0 \leq t_1 \leq t_2),$$

and $j \in I_2 \text{ if } \int_{t_1}^{t_2} D_{kj}(\tau) d\tau < K, (0 \leq t_1 \leq t_2)$

where k is fixed and K is a constant. So for $k = 1$ and $j = 1$ we have $D_{11}(t) = \mathbb{R}(\tilde{\lambda}_1(\tau) - \tilde{\lambda}_1(\tau)) = \mathbb{R}(\sqrt{C} - \sqrt{C}) = 0$ and hence $j \in I_2$ since $\int_{t_1}^{t_2} 0 d\tau < K$.

Note that the same condition is true for $j = k = 2$.

For $k = 1$ and $j = 2$ we have $D_{12}(t) = \mathbb{R}(\sqrt{C} + \sqrt{C}) = 2\sqrt{C}$. Thus $j \in I_1$ since $\int_0^t D_{12}(\tau) d\tau = 2\sqrt{C}t \rightarrow \infty$ as $t \rightarrow \infty$. Also note that for $k = 2$ and $j = 1$ we have $j \in I_1$.

- Let p_k be a characteristic vector of A associated with μ_k , so that

$$Ap_k = \mu_k p_k.$$

The eigenvectors of A are $p_1 = [1, \sqrt{C}]^T$ and $p_2 = [1, -\sqrt{C}]^T$. with $\mu_1 = \sqrt{C}$ and $\mu_2 = -\sqrt{C}$, respectively.

Since all the assumptions of Theorem 3.0.1 are satisfied then there is a solution of (4.3.12) and a t_0 , such that $(0 \leq t_0 < \infty)$ and

$$\lim_{t \rightarrow \infty} \phi_k(t) \exp \left[- \int_{t_0}^t \tilde{\lambda}_k(\tau) d\tau \right] = p_k,$$

which implies

$$w_1(t) = (1 + o(1)) \exp \left(\int_{t_0}^t \sqrt{C} \, d\tau \right) = (1 + o(1)) \exp (\sqrt{C}(t - t_0)) \quad \text{as } t \rightarrow \infty$$

and

$$w_2(t) = (1 + o(1)) \exp \left(- \int_{t_0}^t \sqrt{C} \, d\tau \right) = (1 + o(1)) \exp (-\sqrt{C}(t - t_0)) \quad \text{as } t \rightarrow \infty.$$

One can transform $w_1(t)$ and $w_2(t)$ back to $R_1(r)$ and $R_2(r)$ using the same transformation as in special case 1.1. We now show square integrability for $w_1(t)$ and $w_2(t)$ with respect to the weight function. Thus for $w_1(t)$ we have

$$\begin{aligned} \int_0^\infty |w_1(t)|^2 e^{t(m-2)} \, dt &= e^{2\sqrt{C}t_0} \int_0^\infty (1 + o(1)) e^{-2\sqrt{C}t} e^{t(m-2)} \, dt \\ &< \infty, \quad \text{because of exponential decay.} \end{aligned}$$

and for $w_2(t)$ we find

$$\begin{aligned} \int_0^\infty |w_2(t)|^2 e^{t(m-2)} \, dt &= e^{-2\sqrt{C}t_0} \int_0^\infty (1 + o(1)) e^{2\sqrt{C}t} e^{t(m-2)} \, dt \\ &= \begin{cases} < \infty & \text{if } m < 0, \quad \text{because of exponential decay} \\ = \infty & \text{if } m > 0, \quad \text{because of exponential growth} \\ = \infty & \text{if } m = 0, \quad \text{asymptotically constant.} \end{cases} \end{aligned}$$

Note that $m < 0$ since $m - 2 + 2\sqrt{C} = m - 2 + 2(\sqrt{\frac{(m+2)^2}{4}}) < 0 \Leftrightarrow m < 0$.

The same procedure applies for square integrability of $R_1(r)$ and $R_2(r)$. The classification of this case is Limit Circle if $m < 0$ and Limit Point if $m \geq 0$.

CASE 2: $m = 2$

Subcase 2.1: $m = 2$ and $\lambda - \frac{7}{2} = 0$

Since $m = 2$ and $\lambda - \frac{7}{2} = 0$ then equation (4.3.12) becomes

$$\ddot{w}(t) - ce^{-2t}w(t) = 0. \quad (4.3.13)$$

In order to find the asymptotic solutions of (4.3.13) we apply the Ghizzetti's Theorem 3.0.2. First, we show that the assumptions of Theorem 3.0.2 are satisfied. Since $a(t) = 0$ then $\int_0^\infty |a(t)| dt < \infty$, and since $b(t) = -ce^{-2t}$ then

$$\int_0^\infty t|b(t)| dt = |c| \int_0^\infty te^{-2t} dt < \infty,$$

by integration by parts and the exponential is decaying. Hence, all the assumptions are satisfied and then there are two independent solutions $w_0(t)$ and $w_1(t)$ of (4.3.13) such that

- $w_0(t) = r^{-1/2}z_0(r) = r^{-1/2}r^{5/2}R_0(r) = r^2R_0(r) \rightarrow 1$ as $r \rightarrow 0$, and
- $\frac{w_1(t)}{t} = \frac{r^2R_1(r)}{-\ln r} \rightarrow 1$ as $r \rightarrow 0$.

Clearly, $\int_0^\infty |w_0(t)|^2 dt = \infty$ so the Limit Point classification holds.

Subcase 2.2 $m = 2$ and $\lambda - \frac{7}{2} \neq 0$

Substituting $m = 2$ and $\lambda - \frac{7}{2} \neq 0$ into equation (4.3.12) we obtain

$$\ddot{w}(t) + \left(\lambda - \frac{7}{2} - ce^{-2t} \right) w(t) = 0. \quad (4.3.14)$$

We find the asymptotic solutions of (4.3.14) by applying Theorem 3.0.1, but first we need to show that the assumptions of Theorem 3.0.1 hold. In order to do that we consider two subsubcases.

Subsubcase 2.2.1: $\lambda < \frac{7}{2}$

Consider the following assumptions of Theorem 3.0.1

1. Let the matrix $A = \begin{bmatrix} 0 & 1 \\ -\lambda + \frac{7}{2} & 0 \end{bmatrix}$ with distinct characteristic roots $\mu_1 = \sqrt{\frac{7}{2} - \lambda}$ and $\mu_2 = -\sqrt{\frac{7}{2} - \lambda}$.

2. Let the matrix $V(t) = 0$ which is differentiable and clearly satisfies $\int_0^\infty |V'(t)| dt < \infty$. Also, $V(t) \rightarrow 0$ as $t \rightarrow \infty$.

3. Let the matrix $S = -ce^{-2t} \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}$ be integrable.

Obviously, $\int_0^\infty |S(t)| dt = -c \int_0^\infty e^{-2t} dt < \infty$ since the exponential is decaying.

4. Let $A + V(t) = \begin{bmatrix} 0 & 1 \\ \frac{7}{2} - \lambda & 0 \end{bmatrix}$ with characteristic roots $\tilde{\lambda}_1 = \sqrt{\frac{7}{2} - \lambda}$ and $\tilde{\lambda}_2 = -\sqrt{\frac{7}{2} - \lambda}$

This case is like subcase 1.2 and similar calculations yield solutions $w_1(t), w_2(t)$ such that

$$\lim_{t \rightarrow \infty} w_1(t) \exp \left(- \int_{t_0}^t \tilde{\lambda}_1(\tau) d\tau \right) = \lim_{t \rightarrow \infty} w_1(t) \exp \left(-(t - t_0) \sqrt{\frac{7}{2} - \lambda} \right) = 1$$

and

$$\lim_{t \rightarrow \infty} w_2(t) \exp \left(- \int_{t_0}^t \tilde{\lambda}_2(\tau) d\tau \right) = \lim_{t \rightarrow \infty} w_2(t) \exp \left((t - t_0) \sqrt{\frac{7}{2} - \lambda} \right) = 1.$$

One can transform $w_1(t)$ and $w_2(t)$ back to $R_1(r)$ and $R_2(r)$ respectively. Thus,

$$\lim_{t \rightarrow \infty} w_1(t) \exp \left(-(t - t_0) \sqrt{\frac{7}{2} - \lambda} \right) = \exp \left(t_0 \sqrt{\frac{7}{2} - \lambda} \right) \lim_{r \rightarrow 0} R_1(r) r^{3 + \sqrt{\frac{7}{2} - \lambda}} = 1$$

and

$$\lim_{t \rightarrow \infty} w_2(t) \exp \left((t - t_0) \sqrt{\frac{7}{2} - \lambda} \right) = \exp \left(-t_0 \sqrt{\frac{7}{2} - \lambda} \right) \lim_{r \rightarrow 0} R_2(r) r^{3 - \sqrt{\frac{7}{2} - \lambda}} = 1.$$

In the end, we can show square integrability for $w_1(t)$ and $w_2(t)$ with respect to the weight function. So,

$$\int_0^\infty |w_1(t)|^2 \cdot 1 dt = \int_0^\infty (1 + o(1)) \exp \left(2(t - t_0) \sqrt{\frac{7}{2} - \lambda} \right) dt = \infty$$

and

$$\int_0^\infty |w_2(t)|^2 \cdot 1 \, dt = \int_0^\infty (1 + o(1)) \exp \left(-2(t - t_0) \sqrt{\frac{7}{2} - \lambda} \right) dt < \infty.$$

We conclude with a Limit Point classification for $m = 2$ and $\lambda < \frac{7}{2}$.

Subsubcase 2.2.2: $\lambda > \frac{7}{2}$

In this case we use Theorem 3.0.1 again in order to find the asymptotic solutions of (4.3.14). Following the same procedure as in subsubcase 2.2.1 one can show that the same assumptions of Theorem 3.0.1 hold. Note that the eigenvalues of A are $\mu_1 = i\sqrt{\lambda - \frac{7}{2}}$, $\mu_2 = -i\sqrt{\lambda - \frac{7}{2}}$ and the associated eigenvectors are $p_1 = [1, i\sqrt{-C}]^T$, $p_2 = [1, -i\sqrt{-C}]^T$ respectively, where $C = \frac{7}{2} - \lambda$. Also the eigenvalues of $A + V(t)$ are $\tilde{\lambda} = \pm i\sqrt{\lambda - \frac{7}{2}}$. Therefore by Theorem 3.0.1 we get

$$w_1(t) = (1 + o(1)) \exp \left(-i\sqrt{-C}(t - t_0) \right)$$

and

$$w_2(t) = (1 + o(1)) \exp \left(i\sqrt{-C}(t - t_0) \right).$$

We now consider

$$\tilde{w}_1(t) = \frac{w_1(t) + w_2(t)}{2} = \cos(\sqrt{-C}(t - t_0)) + o(1)$$

and

$$\tilde{w}_2(t) = \frac{w_1(t) + w_2(t)}{2i} = \sin(\sqrt{-C}(t - t_0)) + o(1),$$

because we only want real solutions. One can show that square integrability fails for $w_1(t)$ and $w_2(t)$ as before so a Limit Point classification is obtained. Therefore for $m = 2$, $\lambda > \frac{7}{2}$ or $\lambda < \frac{7}{2}$ we obtain a Limit Point classification. Finally, we consider the third case with $m > 2$.

CASE 3: $m > 2$

In order to find the asymptotic solutions of (4.3.12) for $m > 2$ we need to apply Theorem 3.0.3. Since $p = 1$ and $q(t) = -\frac{1}{4}(m^2 + 4m + 2) - ce^{-2t} + \lambda e^{t(m-2)}$ then we can use the corollary of Theorem 3.0.3 to show that the assumptions of Theorem 3.0.3 are satisfied. Note that the sign of $q(t)$ for large t is determined by λ . So we need to show $\int_0^\infty |q^{-3/2}q''| dt < \infty$. Note that

$$q''(t) = e^{t(m-2)}[\lambda(m-2)^2 + o(1)]$$

and

$$q^{-3/2}(t)q''(t) = (o(1) + \lambda)^{-3/2}(o(1) + \lambda(m-2)^2)e^{-t(m-2)/2} \rightarrow 0, \quad \text{as } t \rightarrow \infty.$$

Therefore, $\int_0^\infty |q^{-3/2}q''| dt < \infty$ since $e^{-t(m-2)/2}$ is a decaying exponential, and the hypothesis of Theorem 3.0.3 holds. Then $\hat{Q}(t)W(t)E(t) \rightarrow L$ as $t \rightarrow \infty$, where

$$\begin{aligned}\hat{Q}(t)W(t)E(t) &= \begin{bmatrix} |q|^{1/4} & 0 \\ 0 & |q|^{-1/4} \end{bmatrix} \cdot \begin{bmatrix} w_1 & w_2 \\ w'_1 & w'_2 \end{bmatrix} \cdot \begin{bmatrix} e^{-\mu_1 Q(t)} & 0 \\ 0 & e^{-\mu_2 Q(t)} \end{bmatrix} \\ &= \begin{bmatrix} (1 + o_1(t)) & (1 + o_2(t)) \\ (\mu_1 + o_3(t)) & (\mu_2 + o_4(t)) \end{bmatrix}.\end{aligned}$$

Since $\hat{Q}(t)$ and $E(t)$ are nonsingular matrices then we solve for $W(t)$, where

$$W = \begin{bmatrix} w_1 & w_2 \\ w'_1 & w'_2 \end{bmatrix} = \begin{bmatrix} e^{\mu_1 Q(t)} |q|^{-1/4} (1 + o_1(t)) & e^{\mu_2 Q(t)} |q|^{-1/4} (1 + o_2(t)) \\ e^{\mu_1 Q(t)} |q|^{1/4} (\mu_1 + o_3(t)) & e^{\mu_2 Q(t)} |q|^{1/4} (\mu_2 + o_4(t)) \end{bmatrix}$$

and $q(t) = -\frac{1}{4}(m^2 + 4m + 2) - ce^{-2t} + \lambda e^{t(m-2)}$, $Q(t) = \int_0^t |q(s)|^{1/2} ds$, and μ_1, μ_2 are the eigenvalues of $\mu^2 \pm 1 = 0$. Note that " + " is for $\lambda > 0$ and " - " is for $\lambda < 0$.

Subcase 3.1: $m > 2$ and $\lambda < 0$

Consider

$$w_1(t) = e^{\mu_1 Q(t)} |q|^{-1/4} (1 + o_1(t))$$

and

$$w_2(t) = e^{\mu_2 Q(t)} |q|^{-1/4} (1 + o_2(t)),$$

where $\mu_1 = 1, \mu_2 = -1$. Note that

$$\begin{aligned}
Q(t) &= \int_0^t |q(s)|^{1/2} ds \\
&= \int_0^t \left(-\frac{1}{4}(m^2 + 4m + 2) - ce^{-2s} + \lambda e^{s(m-2)} \right)^{1/2} ds \\
&= \int_0^t \left((o(1) + \lambda)e^{s(m-2)} \right)^{1/2} ds \\
&= \infty \text{ because of exponential growth,}
\end{aligned}$$

and

$$\begin{aligned}
(q(t))^{-1/4} &= \left(-\frac{1}{4}(m^2 + 4m + 2) - ce^{-2t} + \lambda e^{t(m-2)} \right)^{-1/4} \\
&= \left((o(1) + \lambda)e^{t(m-2)} \right)^{-1/4} \\
&\rightarrow 0 \text{ as } t \rightarrow \infty \text{ since the exponential is decaying.}
\end{aligned}$$

Thus,

$$w_1(t) = e^{\mu_1 Q(t)} q^{-1/4} (1 + o_1(t)) \rightarrow \infty \text{ as } t \rightarrow \infty.$$

Similarly,

$$w_2(t) = e^{\mu_2 Q(t)} |q|^{-1/4} (1 + o_2(t)) \rightarrow 0 \text{ as } t \rightarrow \infty.$$

One can show that square integrability fails for $w_1(t)$ and holds for $w_2(t)$ as in previous cases. So for subcase 3.2 we obtain a Limit Point classification.

Subcase 3.2: $m > 2$ and $\lambda > 0$

Consider $w_1(t)$ and $w_2(t)$ with $\mu_1 = i, \mu_2 = -i$ respectively. Since we want to work

only with real solutions we consider

$$\tilde{w}_1(t) = \frac{w_1(t) + w_2(t)}{2} = \left((o(1) + \lambda)e^{t(m-2)}\right)^{-1/4} [\cos(Q(t)) + o(1)],$$

and

$$\tilde{w}_2(t) = \frac{w_1(t) - w_2(t)}{2i} = \left((o(1) + \lambda)e^{t(m-2)}\right)^{-1/4} [\sin(Q(t)) + o(t)].$$

Hence

$$\tilde{w}_1(t) \rightarrow 0 \text{ as } t \rightarrow \infty$$

and

$$\tilde{w}_2(t) \rightarrow 0 \text{ as } t \rightarrow \infty.$$

Testing square integrability with respect to the weight function $e^{t(m-2)}$ gives us a Limit Point classification since neither w_1 nor w_2 is square integrable.

Chapter 5

3rd Order Differential Equations

5.1 General Transformation for a 3rd Order Differential Equation

In this section, we begin with the general transformation of the third order differential equation,

$$y'''(x) + p_1(x)y''(x) + p_2(x)y'(x) + p_3(x)y(x) = 0. \quad (5.1.1)$$

Continuing with the removal of the second term of the transformed differential equation, we consider two special cases with a finite singularity at 0, and transform 0 into an infinite singularity.

Consider the general transformation,

$$y(x) = \mu(x)z(t), \quad t = f(x) \quad (5.1.2)$$

with $f : X \rightarrow T$, monotone and $\mu(x) > 0$. Note that $\mu(x)$ and $f'(x)$ are never zero on X .

Denote $\gamma(x) \equiv f'(x) = \frac{dt}{dx}$.

The first, second and third derivatives of (5.1.2) with respect to x are

$$y'(x) = \mu'(x)z(t) + \gamma(x)\mu(x)\dot{z}(t), \quad (5.1.3)$$

where $' = \frac{d}{dx}$ and $\dot{} = \frac{d}{dt}$,

$$y''(x) = \gamma^2(x)\mu(x)\ddot{z}(t) + 2\gamma(x)\mu'(x)\dot{z}(t) + \gamma'(x)\mu(x)\dot{z}(t) + \mu''(x)z(t), \quad (5.1.4)$$

and

$$y'''(x) = \mu\gamma^3\ddot{\dot{z}} + 3\mu\gamma\gamma'\ddot{z} + 3\mu'\gamma^2\ddot{z} + 3\mu''\gamma\dot{z} + 3\mu'\gamma'\dot{z} + \mu\gamma''\dot{z} + \mu'''z, \quad (5.1.5)$$

Substituting (5.1.3) - (5.1.5) into (5.1.2) and simplifying we obtain

$$\begin{aligned} & \ddot{\dot{z}} + (3\mu\gamma\gamma' + 3\mu'\gamma^2 + p_1\mu\gamma^2) \frac{\ddot{\dot{z}}}{\mu\gamma^3} + (3\mu''\gamma + 3\mu'\gamma') \frac{\dot{z}}{\mu\gamma^3} + \\ & (\mu\gamma'' + 2p_1\mu'\gamma + p_1\mu\gamma' + p_2\mu\gamma) \frac{\dot{z}}{\mu\gamma^3} + (\mu''' + p_1\mu'' + p_2\mu' + p_3\mu) \frac{z}{\mu\gamma^3} = 0, \end{aligned} \quad (5.1.6)$$

The second term of (5.1.6) can be removed if $p_1 = 0$ and $\mu\gamma = \text{constant}$. For example, we assume that $p_1 = 0$ and hereafter we require $3\mu\gamma\gamma' + 3\mu'\gamma^2 = 0 \Rightarrow 3\gamma(\mu\gamma)' = 0 \Rightarrow$

$(\mu\gamma)' = 0 \Rightarrow \mu\gamma = \text{constant}$. Therefore equation (5.1.6) with $p_1 = 0$ becomes

$$\ddot{z} + \frac{1}{\mu\gamma^3} (3\mu''\gamma + 3\mu'\gamma' + \mu\gamma'' + p_2\mu\gamma) \dot{z} + \frac{1}{\mu\gamma^3} (\mu''' + p_2\mu' + p_3\mu) z = 0, \quad (5.1.7)$$

We next consider two special cases for the transformation of the third order differential equation.

Special Case 1: Let $t = \frac{1}{x}$, $\mu(x) = x^2$, and $\gamma(x) = -\frac{1}{x^2}$.

Differentiating $\mu(x)$, $\gamma(x)$ and substituting into (5.1.7) we get

$$\ddot{z}(t) + b(t)\dot{z}(t) + c(t)z(t) = 0, \quad 1 < t < \infty, \quad (5.1.8)$$

where $b(t) = \frac{p_2(1/t)}{t^4}$ and $c(t) = -\frac{2p_2(1/t)}{t^5} - \frac{p_3(1/t)}{t^6}$.

Hence, letting $B(x) = p_2(x)$, $C(x) = p_3(x)$,

$$b(t) = \frac{B(1/t)}{t^4} = x^4 B(x)$$

and

$$c(t) = -\frac{2B(1/t)}{t^5} - \frac{C(1/t)}{t^6} = -2x^5 B(x) - x^6 C(x).$$

Special Case 2: $t = -\ln x$, $\mu(x) = x$, $\gamma(x) = -\frac{1}{x}$

Again we differentiate $\mu(x)$ and $\gamma(x)$, we substitute into (5.1.7) and simplify we obtain

$$\ddot{z}(t) + b(t)\dot{z}(t) + c(t)z(t) = 0, \quad 1 < t < \infty, \quad (5.1.9)$$

where $b(t) = p_2(e^{-t})e^{-2t} - 1$ and $c(t) = -p_2(e^{-t})e^{-2t} - p_3(e^{-t})e^{-3t}$. Note that we let $p_2(x) = B(x)$ and $p_3(x) = C(x)$

Furthermore,

$$b(t) = e^{-2t}B(e^{-t}) - 1 = x^2B(x) - 1$$

and

$$c(t) = -e^{-2t}B(e^{-t}) - e^{-3t}C(e^{-t}) = -x^2B(x) - x^3C(x).$$

5.2 Theorems for 3rd Order Differential Equations

In this section the definitions for oscillatory and non-oscillatory solutions and equations are introduced. Further, some known theorems of oscillation and nonoscillation criteria for third order differential equations are stated which have a singularity at infinity. By applying transformation theory we derive new theorems of oscillation and nonoscillation criteria for a singularity at zero. The transformations of special case 1 and special case 2 are used to derive the new theorems. Also, examples are given to illustrate the new theorems.

Consider the third order differential equation

$$y'''(x) + A(x)y''(x) + B(x)y'(x) + C(x)y(x) = 0, \quad 0 < x \leq 1 \quad (5.2.1)$$

where A , B , and C are real-valued continuous functions in $(0, 1]$. If $A(x) = 0$ then the differential equation (5.2.1) becomes

$$y'''(x) + B(x)y'(x) + C(x)y(x) = 0 \quad (5.2.2)$$

Remark: The second derivative term in (5.2.1) can always be removed by the transformation $y(x) = u(x) \exp\left(-\frac{1}{3} \int_0^x A(t) dt\right)$.

Definitions:

1. A solution of (5.2.1) is called oscillatory if and only if it has an infinite number of zeros in $(0, 1]$ and non-oscillatory otherwise.
2. Equation (5.2.1) is said to be oscillatory if and only if it has at least one (non-trivial) oscillatory solution.
3. Equation (5.2.1) is said to be non-oscillatory if and only if all of its solutions are non-oscillatory. Also, (5.2.1) is called disconjugate if and only if it has no (nontrivial) solution with more than two zeros in $(0, 1]$.

Remark: There always exists a solution with zeros at two arbitrary points.

We now state the oscillation and non-oscillation theorems for the third order differential equations

$$\ddot{z}(t) + b(t)\dot{z}(t) + c(t)z(t) = 0, \quad 1 < t < \infty, \quad (5.2.3)$$

and

$$y'''(x) + B(x)y'(x) + C(x)y(x) = 0, \quad 0 < x < 1. \quad (5.2.4)$$

The first oscillation theorem with a singularity at infinity is

Theorem 5.2.1. (Hanan) [9] Equation (5.2.3) is oscillatory if $b(t) \in \mathcal{C}^1$, $2c(t) > \dot{b}(t)$

$(1 < t < \infty)$, and

$$\liminf_{t \rightarrow \infty} t^2 b(t) > p, \quad \liminf_{t \rightarrow \infty} t^3 c(t) > -p$$

for some $p > 1$.

Using the transformation of special case 1 we can show that a version of Theorem 5.2.1

holds with singularity at zero. In this case we consider (5.2.4) which is oscillatory if

$$b(t) = \frac{B(1/t)}{t^4} = x^4 B(x) \Rightarrow B(x) \in \mathcal{C}^1,$$

$$2c(t) > \dot{b}(t) \text{ where } c(t) = -2t^{-5}B(1/t) - t^{-6}C(1/t) = -2x^5 B(x) - x^6 C(x),$$

$$\text{and } \dot{b}(t) = -t^{-6}B'(1/t) - 4t^{-5}B(1/t) = -x^6 B'(x) - 4x^5 B(x).$$

Thus, $2c(t) > \dot{b}(t) \Leftrightarrow$

$$2(-2x^5 B(x) - x^6 C(x)) > -x^6 B'(x) - 4x^5 B(x) \Leftrightarrow 2C(x) < B'(x).$$

Also, note that

$$\begin{aligned}
\liminf_{t \rightarrow \infty} t^2 b(t) &= \liminf_{x \rightarrow 0} x^2 B(x) > p, \quad \text{and} \\
\liminf_{t \rightarrow \infty} t^3 c(t) &= - \limsup_{t \rightarrow \infty} -t^3 c(t) > -p \\
&\Leftrightarrow \limsup_{x \rightarrow 0} (2x^2 B(x) + x^3 C(x)) < p.
\end{aligned}$$

Hence we obtain a new oscillation theorem with singularity at zero.

Theorem 5.2.2. *Equation (5.2.4) is oscillatory if $B(x) \in \mathcal{C}^1$, $2C(x) < B'(x)$,*

($0 < x < 1$), and

$$\liminf_{x \rightarrow 0} x^2 B(x) > p, \quad \limsup_{x \rightarrow 0} (2x^2 B(x) + x^3 C(x)) < p$$

for some number $p > 1$.

Example 5.2.1. This example illustrates Theorem 5.2.2. Let $B(x) = \alpha x^\beta$ and $C(x) = \gamma x^\delta$. We now show that the assumptions of Theorem 5.2.2 are satisfied. First note that $B(x) = \alpha x^\beta \in \mathcal{C}^1$, since it is just a power function. Consider the following conditions

1. $2C(x) < B'(x) \Leftrightarrow 2\gamma x^\delta < \alpha\beta x^{\beta-1}$,
2. $\lim_{x \rightarrow 0} \inf x^2 B(x) > p \Leftrightarrow \lim_{x \rightarrow 0} \alpha x^{\beta+2} > p$,
3. $\lim_{x \rightarrow 0} \sup (2x^2 B(x) + x^3 C(x)) < p \Leftrightarrow \lim_{x \rightarrow 0} 2\alpha x^{\beta+2} + \gamma x^{\delta+3} < p$

Condition 2 holds if

$$\begin{cases} \beta = -2 & \text{and } \alpha > p, \text{ or} \\ \beta < -2 & \text{and } \alpha > 0. \end{cases}$$

Note that if $\beta > -2$ then $\lim_{x \rightarrow 0} \alpha x^{\beta+2} = 0 < p$. This implies that condition 2 does not hold for $\beta > -2$ because $p > 1$.

Conditions 1 and 2 hold if

$$\begin{cases} \beta = -2, & \delta = -3, & \alpha > p, & \text{and } \gamma < -\alpha, & \text{or} \\ \beta < -2, & \delta = \beta - 1, & \alpha > 0, & \text{and } 2\gamma < \alpha\beta, & \text{or} \\ \beta < -2, & \delta < \beta - 1, & \alpha > 0, & \text{and } \gamma < 0. \end{cases}$$

Conditions 1,2, 3 hold if

$$\begin{cases} \beta = -2, & \delta = -3, & \alpha > p, & \text{and } \gamma < -\alpha, & \text{or} \\ \beta = -2, & \delta < -3, & \alpha > p & \text{and } \gamma < 0, & \text{or} \\ \beta < -2, & \delta = \beta - 1, & \alpha > 0, & 2\gamma < \alpha\beta, & \text{and } 2\alpha + \gamma < 0, & \text{or} \\ \beta < -2, & \delta < \beta - 1, & \delta < -3 & \alpha > 0, & \text{and } \gamma < 0. \end{cases}$$

Thus Theorem 5.2.2 holds if

$$\begin{cases} \beta = -2, & \alpha > p, & \delta = -3, & \text{and } \gamma < -\alpha, & \text{or} \\ \beta = -2, & \alpha > p, & \delta < -3, & \text{and } \gamma < 0, \end{cases}$$

or if

$$\left\{ \begin{array}{l} \beta < -2, \quad \alpha > 0, \quad \delta = \beta - 1, \quad 2\gamma < \alpha\beta, \quad \text{and} \quad 2\alpha + \gamma < 0, \quad \text{or} \\ \beta < -2, \quad \alpha > 0, \quad \delta < \beta - 1, \quad \gamma < 0, \quad \text{and} \quad \delta < -3. \end{array} \right.$$

Following the same procedure as in Theorem 5.2.2 we can obtain a new oscillation theorem for special case 2. So,

$$\begin{aligned} b(t) &= e^{-2t}B(e^{-t}) \\ &= x^2B(x) - 1, \\ c(t) &= -e^{-2t}B(e^{-t}) - e^{-3t}C(e^{-t}), \\ &= -x^2B(x) - x^3C(x), \\ \dot{b}(t) &= e^{-3t}B'(e^{-t}) - 2e^{-2t}B(e^{-t}), \\ &= -x^3B'(x) - 2x^2B(x), \\ t^2b(t) &= (\ln x)^2(x^2B(x) - 1), \\ t^3c(t) &= (\ln x)^3(-x^2B(x) - x^3C(x)). \end{aligned}$$

Theorem 5.2.3. *Equation (5.2.4) is oscillatory if $B(x) \in \mathcal{C}^1$, $2C(x) < B'(x)$,*

($0 < x < 1$), and

$$\liminf_{x \rightarrow 0} (\ln x)^2(x^2B(x) - 1) > p, \quad \limsup_{x \rightarrow 0} x^2(\ln x)^3(B(x) + xC(x)) < p$$

for some number $p > 1$.

Theorem 5.2.4. *(Hanan) [9] Suppose $b(t) \in \mathcal{C}^1$ and $2c(t) > \dot{b}(t)$, ($1 < t < \infty$).*

Then equation (5.2.3) is oscillatory if there exists a number p satisfying $0 < p < 1$ such that

$$\liminf_{t \rightarrow \infty} t^2 b(t) > p, \quad \liminf_{t \rightarrow \infty} t^3 c(t) > 2 \left(\frac{1-p}{3} \right)^{3/2} - p$$

and nonoscillatory if there exists a number p , ($0 < p < 1$), such that

$$\limsup_{t \rightarrow \infty} t^2 b(t) < p, \quad \limsup_{t \rightarrow \infty} t^3 c(t) < 2 \left(\frac{1-p}{3} \right)^{3/2} - p.$$

Using the same calculations as in Theorem 5.2.1, we can show that a version of Theorem 5.2.4 holds with singularity at zero.

Theorem 5.2.5. Suppose $B(x) \in \mathcal{C}^1$ and $2C(x) < B'(x)$ ($0 < x < 1$). Then equation (5.2.4) is oscillatory if there exists a p satisfying $0 < p < 1$ such that

$$\liminf_{x \rightarrow 0} x^2 B(x) > p, \quad \limsup_{x \rightarrow 0} 2x^2 B(x) + x^3 C(x) < p - \left(\frac{1-p}{3} \right)^{3/2},$$

and nonoscillatory if there exists a number p , ($0 < p < 1$) such that

$$\limsup_{x \rightarrow 0} x^2 B(x) < p, \quad \liminf_{x \rightarrow 0} 2x^2 B(x) + x^3 C(x) > p - \left(\frac{1-p}{3} \right)^{3/2}.$$

Also theorem 5.2.4 holds with singularity at 0 for special case 2.

Theorem 5.2.6. Suppose $(x^2 B(x) - 1) \in \mathcal{C}^1$ and $2C(x) < B'(x)$ ($0 < x < 1$). Then

equation (5.2.4) is oscillatory if there exists a p satisfying $0 < p < 1$ such that

$$\liminf_{x \rightarrow 0} (\ln x)^2 (x^2 B(x) - 1) > p, \quad \limsup_{x \rightarrow 0} (-x^2 (\ln x)^3 (B(x) + xC(x))) < p - \left(\frac{1-p}{3} \right)^{3/2},$$

and nonoscillatory if there exists a number p , ($0 < p < 1$) such that

$$\limsup_{x \rightarrow 0} (\ln x)^2 (x^2 B(x) - 1) < p, \quad \liminf_{x \rightarrow 0} (-x^2 (\ln x)^3 (B(x) + xC(x))) > p - \left(\frac{1-p}{3} \right)^{3/2}.$$

Theorem 5.2.7. (Hanan) [9] If $c(t)$ has constant sign, then $\ddot{z}(t) + c(t)z(t) = 0$, is oscillatory if

$$\liminf_{t \rightarrow \infty} t^3 |c(t)| > \frac{2}{3\sqrt{3}}$$

and nonoscillatory if

$$\limsup_{t \rightarrow \infty} t^3 |c(t)| < \frac{2}{3\sqrt{3}}.$$

Theorem 5.2.7 holds with singularity at zero only for special case 1. So by the transformation of special case 1 we have

$$t^3 |c(t)| = x^{-3} | -x^6 C(x) | = x^3 |C(x)|.$$

Theorem 5.2.8. If $C(x)$ has constant sign, then $y'''(x) + C(x)y(x) = 0$ is oscillatory if

$$\liminf_{x \rightarrow 0} x^3 |C(x)| > \frac{2}{3\sqrt{3}}$$

and nonoscillatory if

$$\limsup_{x \rightarrow 0} x^3 |C(x)| < \frac{2}{3\sqrt{3}}.$$

We proceed with an example that illustrates Theorem 5.2.8.

Example 5.2.2. Let $y'''(x) + C(x)y(x) = 0$ where $C(x) = \gamma x^\delta$. Note that $-x^6 C(x) = -\gamma x^{\delta+6}$ has a constant sign. Then $y'''(x) + \gamma x^\delta y(x) = 0$ is oscillatory if

$$\begin{aligned} \liminf_{x \rightarrow 0} x^3 |C(x)| &= \lim_{x \rightarrow 0} |x^{\delta+3} C(x)| \\ &> \frac{2}{3\sqrt{3}}, \quad \text{if } \delta + 3 < 0, \quad \gamma \neq 0 \quad \text{or} \\ \liminf_{x \rightarrow 0} x^3 |C(x)| &= \lim_{x \rightarrow 0} |x^{\delta+3} C(x)| \\ &= \liminf_{x \rightarrow 0} |\gamma| \\ &> \frac{2}{3\sqrt{3}}, \quad \text{if } \delta + 3 = 0, \quad |\gamma| > \frac{2}{3\sqrt{3}} \end{aligned}$$

and nonoscillatory if

$$\begin{aligned} \limsup_{x \rightarrow 0} x^3 |C(x)| &= \limsup_{x \rightarrow 0} |\gamma x^{\delta+3}| \\ &< \frac{2}{3\sqrt{3}} \quad \text{if } \delta + 3 > 0, \quad \text{or} \\ \limsup_{x \rightarrow 0} |\gamma x^{\delta+3}| &= \limsup_{x \rightarrow 0} |\gamma| \\ &< \frac{2}{3\sqrt{3}} \quad \text{if } \delta + 3 = 0, \quad |\gamma| < \frac{2}{3\sqrt{3}}. \end{aligned}$$

We now state the next theorem with singularity at infinity.

Theorem 5.2.9. (Kondrat'ev - Lazer) [9] Equation (5.2.3) is oscillatory if $b(t) \leq 0$,

$c(t) > 0$ ($0 < t < \infty$) and if there exists a number $a > 0$ such that

$$\int_a^\infty \left[c(t) - \frac{2}{3\sqrt{3}}(-b(t))^{3/2} \right] dt = \infty.$$

We show that a version of Theorem 5.2.9 holds with singularity at zero using the transformation of special case 1. So,

$$b(t) = x^4 B(x),$$

$$c(t) = -2x^5 B(x) - x^6 C(x).$$

We take $c(t) > 0$; Hence $-2x^5 B(x) - x^6 C(x) > 0 \Leftrightarrow 2B(x) + xC(x) < 0$. Also,

$$\begin{aligned} \int_a^\infty \left[c(t) - \frac{2}{3\sqrt{3}}(-b(t))^{3/2} \right] dt &= \infty \\ \int_{1/a}^0 \left[-2x^5 B(x) - x^6 C(x) - \frac{2}{3\sqrt{3}}(-x^4 B(x))^{3/2} \right] (-x^{-2}) dx &= \infty \\ \int_0^{1/a} \left[-x^3(2B(x) + xC(x)) - \frac{2x^4}{3\sqrt{3}}(-B(x))^{3/2} \right] dx &= \infty \end{aligned}$$

Theorem 5.2.10. Equation (5.2.4) is oscillatory if $B(x) \leq 0$, $2B(x) + xC(x) < 0$

($0 < x < 1$) and if there exists a number $a > 0$ such that

$$\int_0^{1/a} \left[-x^3(2B(x) + xC(x)) - \frac{2x^4}{3\sqrt{3}}(-B(x))^{3/2} \right] dx = \infty.$$

The following example illustrates Theorem 5.2.10

Example 5.2.3. Let $B(x) = \alpha x^\beta$ and $C(x) = \gamma x^\delta$. One can show that the assump-

tions of Theorem 5.2.10 are satisfied. Consider the following conditions:

1. $B(x) = \alpha x^\beta \leq 0$
2. $2B(x) + xC(x) = 2\alpha x^\beta + \gamma x^{\delta+1} < 0$
3. $\int_0^{1/a} \left[x^3(2\alpha x^\beta + \gamma x^{\delta+1}) + \frac{2x^4}{3\sqrt{3}}(-\alpha x^\beta)^{3/2} \right] dx$
 $= \int_0^{1/a} \left[2\alpha x^{\beta+3} + \gamma x^{\delta+4} + \frac{2(-\alpha)^{3/2}}{3\sqrt{3}} x^{(3\beta+8)/2} \right] dx$
 $= -\infty$

Condition 1 holds if $\alpha \leq 0$.

Conditions 1 and 2 hold if

$$\begin{cases} \gamma < 0 & \text{or} \\ \gamma = 0, \quad \alpha < 0 \end{cases}$$

or if

$$\gamma > 0, \quad \alpha < 0 \quad \begin{cases} \beta < \delta + 1 & \text{or} \\ \beta + \delta + 1 & \alpha < \gamma. \end{cases}$$

Conditions 1, 2 and 3 hold if

$$\alpha = 0, \quad \frac{\gamma}{\delta + 5} < 0, \quad \text{or}$$

$$\alpha < 0 \quad \begin{cases} \beta = \delta + 1, \quad \frac{2\alpha}{\beta+4} + \frac{\gamma}{\delta+5} < 0, \quad \beta + 4 < \frac{3}{2}\beta + 5 \Leftrightarrow -2 < \beta, \quad \text{or} \\ \beta < \delta + 1, \quad \frac{2\alpha}{\beta+4} < 0, \quad \beta + 4 < \frac{3}{2}\beta + 5 \Leftrightarrow -2 < \beta. \end{cases}$$

The next theorem is for special case 2 with singularity at zero. One can follow the same procedure as in Theorem 5.2.10 to show that Theorem 5.2.9 holds for special

case 2. Note that

$$\begin{aligned} b(t) &= x^2 B(x) - 1, \\ c(t) &= -x^2(B(x) + xC(x)) \end{aligned}$$

Also,

$$\begin{aligned} \int_a^\infty \left[c(t) - \frac{2}{3\sqrt{3}}(-b(t))^{3/2} \right] dt &= \infty \\ \int_{e^{-a}}^0 \left[-x^2 B(x) - x^3 C(x) - \frac{2}{3\sqrt{3}}(-x^2 B(x) + 1)^{3/2} \right] (-x^{-1}) dx &= \infty \\ \int_0^{e^{-a}} \left[-xB(x) - x^2 C(x) - \frac{2}{3x\sqrt{3}}(1 - x^2 B(x))^{3/2} \right] dx &= \infty. \end{aligned}$$

Theorem 5.2.11. *Equation (5.2.4) is oscillatory if $(x^2 B(x) - 1) \leq 0$, $B(x) + xC(x) < 0$ ($0 < x < 1$) and if there exists a number a such that*

$$\int_0^{e^{-a}} \left[-xB(x) - x^2 C(x) - \frac{2}{3x\sqrt{3}}(1 - x^2 B(x))^{3/2} \right] dx = \infty.$$

The next theorem with singularity at infinity is

Theorem 5.2.12. *(Greguš) [9] Suppose $b(t) \in \mathcal{C}^1$, $b(t) \leq 0$, $c(t) > 0$ and $2c(t) \geq \dot{b}(t)$ with equality occurring only at isolated points ($1 < t < \infty$). If equation (5.2.3) has one oscillatory solution, then all solutions of (5.2.3) are oscillatory except constant multiples of one nonvanishing solution $z_1(t)$; furthermore $z_1(t)$ tends to a finite limit as $t \rightarrow \infty$.*

Applying special case 1, with

$$\begin{aligned} b(t) &= x^4 B(x), \\ c(t) &= -2x^5 B(x) - x^6 C(x), \\ \dot{b}(t) &= -x^6 B'(x) - 4x^5 B(x), \\ z_1(t) &= \frac{y_1(x)}{x^2}. \end{aligned}$$

yields the following theorem.

Theorem 5.2.13. *Suppose $B(x) \in \mathcal{C}^1$, $B(x) \leq 0$, $2B(x) + xC(x) < 0$, and $2C(x) \leq B'(x)$ with equality occurring only at isolated points ($0 < x < 1$). If equation (5.2.4) has one oscillatory solution, then all solutions of (5.2.4) are oscillatory except constant multiples of one nonvanishing solution $y_1(x)$; furthermore $\frac{y_1(x)}{x^2}$ tends to a finite limit as $x \rightarrow 0$.*

This next example considers Theorem 5.2.13 with power coefficients.

Example 5.2.4. Let $B(x) = \alpha x^\beta$ and $C(x) = \gamma x^\delta$. Note that $B(x) \in \mathcal{C}^1$. We continue by showing that the assumptions of Theorem 5.2.13 are satisfied. Consider the following conditions:

1. $\alpha x^\beta \leq 0$,
2. $2\alpha x^\beta + \gamma x^{\delta+1} < 0$,
3. $2\gamma x^\delta \leq \alpha \beta x^{\beta-1} \Leftrightarrow 2\gamma \leq \alpha \beta x^{\beta-1-\delta}$

Conditions 1, 2 and 3 hold if $\alpha = 0$, $\gamma < 0$ and also if

$$\alpha < 0 \quad \left\{ \begin{array}{ll} \beta - 1 - \delta < 0, & \beta < 0, \quad \text{or} \quad (\beta = 0, \gamma \leq 0) \\ \beta - 1 - \delta > 0, & \gamma < 0, \quad \text{or} \quad (\beta \leq 0, \gamma = 0) \\ \beta - 1 - \delta = 0, & \gamma < -2\alpha, \quad 2\gamma \leq \alpha\beta. \end{array} \right.$$

Theorem 5.2.12 holds for special case 2 as well.

Theorem 5.2.14. *Suppose $x^2B(x) - 1 \in \mathcal{C}^1$, $x^2B(x) \leq 1$, $B(x) + xC(x) < 0$, and $2C(x) \leq B'(x)$ with equality occurring only at isolated points ($0 < x < 1$). If equation (5.2.4) has one oscillatory solution, then all solutions of (5.2.4) are oscillatory except constant multiples of one nonvanishing solution $y_1(x)$; furthermore $\frac{y_1(x)}{x^2}$ tends to a finite limit as $x \rightarrow 0$.*

Example 5.2.5. This example is an illustration of Theorem 5.2.14. Let $B(x) = \alpha x^\beta$ and $C(x) = \gamma x^\delta$. We now want to show that the assumptions of Theorem 5.2.14 are satisfied. Clearly $(x^2B(x) - 1) = (\alpha x^{\beta+2} - 1) \in \mathcal{C}^1$. We consider the following three conditions,

1. $\alpha x^{\beta+2} \leq 1$
2. $\alpha x^\beta + \gamma x^{\delta+1} < 0 \Leftrightarrow \alpha < -\gamma x^{\delta+1-\beta}$
3. $2\gamma x^\delta \leq \alpha \beta x^{\beta-1} \Leftrightarrow 2\gamma \leq \alpha \beta x^{\beta-1-\delta}$.

Condition 1 holds if

$$\left\{ \begin{array}{ll} \alpha \leq 0, & \text{or} \\ \alpha > 0, & \beta + 2 > 0 \text{ or} \\ \beta + 2 = 0, & 0 < \alpha \leq 1. \end{array} \right.$$

Condition 2 holds if

$$\left\{ \begin{array}{lll} \delta + 1 - \beta > 0, & \alpha < 0, & \text{or} \\ \delta + 1 - \beta > 0, & \alpha = 0, & \gamma \leq 0 \\ \delta + 1 - \beta < 0, & \gamma < 0, & \text{or} \\ \delta + 1 - \beta < 0, & \alpha < 0, & \gamma = 0 \\ \delta + 1 - \beta = 0, & \alpha < -\gamma, & \text{or} \\ \delta + 1 - \beta = 0, & 2\gamma \leq \alpha\beta. \end{array} \right.$$

Condition 3 holds if

$$\left\{ \begin{array}{lll} \delta + 1 - \beta > 0, & \gamma < 0, & \text{or} \\ \delta + 1 - \beta > 0, & \gamma = 0, & \alpha = 0 \\ \delta + 1 - \beta < 0, & \alpha\beta > 0, & \text{or} \\ \delta + 1 - \beta < 0, & \alpha\beta = 0, & \gamma \leq 0 \\ \delta + 1 - \beta = 0, & 2\gamma \leq \alpha\beta. \end{array} \right.$$

Since all the assumptions of Theorem 5.2.14 are satisfied then the conclusion of the theorem holds.

Theorem 5.2.15. (Lazer) [9] Suppose $c(t) \in \mathcal{C}^2$, $b(t) \leq 0$, $c(t) > 0$ and

$$\frac{2b(t)}{c(t)} + \left[\frac{1}{c(t)} \right]'' \leq 0, \quad 0 < t < \infty.$$

Then the conclusion of theorem (5.2.12) is valid.

We again use the transformations for special case 1 to show that a version of Theorem 5.2.15 holds with singularity at zero.

Note that for special case 1

$$\frac{2b(t)}{c(t)} + \left[\frac{1}{c(t)} \right]'' \leq 0 \Leftrightarrow \frac{-2B(x)}{2xB(x) + x^2C(x)} + \left[-\frac{1}{2x^5B(x) + x^6C(x)} \right]'' \leq 0,$$

where $\left[\frac{1}{c(t)} \right]'' = \left[\frac{1}{c(t)} \right]'' (-x^2)^{-2} - 2x \left[\frac{1}{c(t)} \right]'$.

Theorem 5.2.16. Suppose $-x^5(2B(x) + xC(x)) \in \mathcal{C}^2$, $B(x) \leq 0$, $2B(x) + xC(x) < 0 \Leftrightarrow$, and

$$\frac{-2B(x)}{2xB(x) + x^2C(x)} + \left[-\frac{1}{2x^5B(x) + x^6C(x)} \right]'' \leq 0, \quad (0 < x < 1).$$

Then the conclusion of Theorem 5.2.13 is valid.

Theorem 5.2.17. (Švec - Villari) [9] If $b(t) \equiv 0$, $c(t) > 0$, and (5.2.3) has one oscillatory solution, then every nonoscillatory solution tends to zero as $t \rightarrow \infty$.

Theorem 5.2.18. If $B(x) \equiv 0$, $C(x) < 0$, and $y'''(x) + C(x)y(x) = 0$ has one oscillatory solution, then every nonoscillatory solution tends to zero as $x \rightarrow 0$.

Example 5.2.6. Let $C(x) = \gamma x^\delta$. Note that $\gamma x^\delta < 0$ if $\gamma < 0$. If the assumptions of Theorem 5.2.18 are satisfied then every nonoscillatory solution tends to zero as $x \rightarrow 0$, that is $\frac{y(x)}{x^2} \rightarrow 0$ or $y(x) = o(1)x^2$, if $y'''(x) + C(x)y(x) = 0$ has an oscillatory solution. Let $C(x) = -\frac{6}{x^3}$. Substituting $C(x)$ into $y'''(x) + C(x)y(x) = 0$ we have

$$y'''(x) - \frac{6}{x^3}y(x) = 0. \quad (5.2.5)$$

We can solve (5.2.5) by letting $y(x) = x^\sigma$. Differentiating $y(x)$ three times we get $y'''(x) = \sigma(\sigma-1)(\sigma-2)x^{\sigma-3}$. Substituting $y(x)$ and $y'''(x)$ into (5.2.5) and simplifying we obtain $\sigma^3 - 3\sigma^2 + 2\sigma - 6 = 0$. Solving the third degree polynomial we get $\sigma = 3, \sigma = \pm i\sqrt{2}$. Hence, the three linearly independent real valued solutions of (5.2.5) are

$$y_1(x) = x^3, \quad y_2(x) = \cos(\sqrt{2} \ln x) \quad \text{and} \quad y_3(x) = \sin(\sqrt{2} \ln x).$$

Therefore, (5.2.5) has two oscillatory solutions, i.e. $y_2(x), y_3(x)$, and one nonoscillatory solution $y_1(x)$ so that $\frac{y_1(x)}{x^2}$ tends to zero as $x \rightarrow 0$.

Theorem 5.2.19. *If $x^2B(x)-1 \equiv 0$, $C(x) < -\frac{1}{x^3}$ and $y'''(x)+B(x)y'(x)+C(x)y(x) = 0$ has one oscillatory solution, then every nonoscillatory solution tends to zero as $x \rightarrow 0$.*

Example 5.2.7. This example illustrates Theorem 5.2.19. Let $B(x) = 1/x^2$ and $C(x) = \gamma x^\delta$. We now need to show that the assumptions of Theorem 5.2.19 hold.

Consider the condition

$$\gamma x^\delta < -x^{-3} \Leftrightarrow \gamma < -x^{-3-\delta}.$$

The above condition holds if

$$\begin{cases} \delta = -3, & \gamma < -1, \text{ or} \\ \delta < -3, & \gamma < 0. \end{cases}$$

Note that if $\delta > -3$ the condition does not hold because it is impossible to find a condition on γ .

Theorem 5.2.20. *(Lazer) [9] Suppose $b(t) \leq 0, c(t) > 0 (0 < t < \infty)$ and $\int_a^\infty t^2 c(t) dt = \infty$ for some $a > 0$. Then if (5.2.3) has one oscillatory solution, every nonoscillatory solution tends to zero as $t \rightarrow \infty$.*

Theorem 5.2.20 leads to the following two theorems.

Theorem 5.2.21. *Suppose $B(x) \leq 0, 2B(x) + xC(x) < 0 (0 < x < 1)$ and*

$$\int_0^{1/a} (2xB(x) + x^2C(x)) dx = -\infty$$

for some $a > 0$. Then if (5.2.4) has one oscillatory solution, every nonoscillatory solution tends to zero as $x \rightarrow 0$.

The following example illustrates Theorem 5.2.21.

Example 5.2.8. Let $B(x) = \frac{\alpha}{x^2}$ and $C(x) = \frac{\beta}{x^3}$. Clearly, $B(x) = \frac{\alpha}{x^2} \leq 0$ if $\alpha \leq 0$,

and $2B(x) + xC(x) < 0$ if $2\alpha < -\beta$. Since $2\alpha + \beta < 0$ then

$$\begin{aligned} \int_0^{1/a} (2xB(x) + x^2C(x)) \, dx &= (2\alpha + \beta) \int_0^{1/a} \frac{1}{x} \, dx \\ &= (2\alpha + \beta) \ln x \Big|_0^{1/a} = -\infty. \end{aligned}$$

Theorem 5.2.22. *Suppose $x^2B(x) \leq 1$, $B(x) + xC(x) < 0$ ($0 < x < 1$) and*

$$\int_0^{e^{-a}} ((\ln x)^2(xB(x) + x^2C(x))) \, dx = -\infty$$

for some $a > 0$. Then if (5.2.4) has one oscillatory solution, every nonoscillatory solution tends to zero as $x \rightarrow 0$.

Theorem 5.2.23. *(Lazer) [9] If $b(t) \in \mathcal{C}^1$, $b(t) \leq 0$, $c(t) < \dot{b}(t)$ and*

$$\int_a^\infty \dot{b}(t) - c(t) - \frac{2}{3\sqrt{3}}(-b(t))^{3/2} \, dt = 0$$

has two linearly independent oscillatory solutions.

A version of Theorem 5.2.23 holds with singularity at 0 for special case 1 and special case 2.

Theorem 5.2.24. *If $B(x) \in \mathcal{C}^1$, $B(x) \leq 0$, $xC(x) > 2B(x) + xB'(x)$ and*

$$\int_0^{1/a} \left[x^3(xC(x) - 2B(x) - xB'(x)) + \frac{2x^4}{3\sqrt{3}}(-B(x))^{3/2} \right] \, dx = \infty$$

for some $a > 0$ then equation (5.2.4) has two linearly independent oscillatory solu-

tions.

Theorem 5.2.25. *If $(x^2B(x) - 1) \in \mathcal{C}^1$, $x^2B(x) \leq 1$, $xC(x) > B(x) + xB'(x)$ and*

$$\int_0^{e^{-a}} \left[x(xC(x) - B(x) - xB'(x)) + \frac{2x^{-1}}{3\sqrt{3}}(1 - x^2B(x))^{3/2} \right] dx = \infty$$

for some $a > 0$ then equation (5.2.4) has two linearly independent oscillatory solutions.

Theorem 5.2.26. *(Jones, G.) [6] If $2c(t) - \dot{b}(t)$ is of constant sign having only isolated zeros, then equation (5.2.3) is oscillatory whenever*

$$\ddot{z}(t) + \frac{b(t)}{4}z(t) = 0$$

is oscillatory.

Note that for special case 1, we have

$$\begin{aligned} 2c(t) - \dot{b}(t) &= 2(-2x^5B(x) - x^6C(x)) - (-x^6B'(x) - 4x^5B(x)) \\ &= x^6(B'(x) - 2C(x)), \end{aligned}$$

so that a version of Theorem 5.2.26 with singularity at 0 is:

Theorem 5.2.27. *Consider equation (5.2.4). If $B'(x) - 2C(x)$ is of constant sign having only isolated zeros, then equation (5.2.4) is oscillatory whenever $y''(x) + \frac{1}{4}x^4B(x)y(x) = 0$ is oscillatory.*

Example 5.2.9. Let $B(x) = \frac{4c}{x^6}$ with $c > \frac{1}{4}$ and $C(x)$ be any function such that $C(x) \geq 0$. Then

$$x^6(B'(x) - 2C(x)) = x^6(-24cx^{-7} - 2C(x)) = -24cx^{-1} - 2x^6C(x)$$

is of constant sign. Substituting $B(x) = \frac{4c}{x^6}$ into $y''(x) + \frac{1}{4}x^4B(x)y(x) = 0$ and simplifying we get an oscillatory Euler differential equation

$$y''(x) + \frac{c}{x^2}(x) = 0. \quad (5.2.6)$$

One can solve (5.2.6) by letting $y(x) = x^\sigma$. Differentiating $y(x)$ twice we have $y''(x) = \sigma(\sigma-1)x^{\sigma-2}$. Substituting $y(x)$ and its second derivative into the differential equation (5.2.6) and simplifying we obtain $\sigma^2 - \sigma + c = 0$. Solving the quadratic equation we get $\sigma = \frac{1 \pm \sqrt{1-4c}}{2}$. Since $c > \frac{1}{4}$ then $\sigma = \frac{1 \pm i\sqrt{4c-1}}{2}$. Therefore, the two linearly independent real valued solutions of (5.2.6) are

$$y_1(x) = x^{1/2} \cos \beta \ln x, \quad \text{and} \quad y_2(x) = x^{1/2} \sin \beta \ln x$$

where $\beta = \sqrt{4c-1}$. Note that $y_1(x)$ and $y_2(x)$ oscillate "infinitely rapidly" near the origin as $x \rightarrow 0$.

For special case 2, we have

$$\begin{aligned} 2c(t) - \dot{b}(t) &= 2(-x^2B(x) - x^3C(x)) - (-x^3B'(x) - 2x^2B(x)) \\ &= x^3(B'(x) - 2C(x)), \end{aligned}$$

so that a version of Theorem 5.2.26 with singularity at 0 is:

Theorem 5.2.28. *If $B'(x) - 2C(x)$ is of constant sign having only isolated zeros, then equation (5.2.4) is oscillatory whenever $y''(x) + \frac{1}{4}(x^2B(x) - 1)y(x) = 0$ is oscillatory.*

Chapter 6

Canonical 3rd Order Differential Equations

In this chapter we introduce a general third order differential equation considered by Barrett [1]. We show that the general transformation of Chapter 5 can be used to transform Barrett's equation into one of the same form. The third order canonical form of Barrett is

$$L[y] = \{r_2[(r_1y'(x))' + q_1y(x)]\}' + q_2(r_1y'(x)) = 0 \quad (6.0.1)$$

where $r_1, r_2 > 0$, $q_1, q_2 \in C(I)$, $I = (0, 1)$.

Remark: The differential equation (6.0.1) is more general than (5.1.1). This can be shown by the following Lemma from Barrett [1].

Lemma 5.1 [1]

Any classical third order equation

$$y'''(x) + p_1(x)y''(x) + p_2(x)y'(x) + p_3(x)y(x) = 0; \quad p_i \in \mathcal{C}(I),$$

can be expressed in the canonical form (6.0.1) since it is equivalent to

$$(s(x)y''(x))' + s(x)p_2(x)y'(x) + s(x)p_3(x)y(x) = 0; \quad s = e^{\int p_1},$$

and

$$\left[s(x)y''(x) + \left(\int s(x)p_3(x) \right) y(x) \right]' + \left[s(x)p_2(x) - \int s(x)p_3(x) \right] y'(x) = 0,$$

where $r_1(x) = 1, r_2(x) = s(x), q_1(x) = \int s(x)p_3(x) dx$, and $q_2(x) = s(x)p_2(x) - \int s(x)p_3(x) dx$.

Let $y^{[1]} = r_1 y'$ and $y^{[2]} = r_2[(r_1 y')' + q_1 y] = r_2[(y^{[1]})' + q_1 y]$.

Hence the equation (6.0.1) is equivalent to a special case of the first order vector-matrix equation:

$$Y' = A(x)Y, \quad A(x) = (a_{ij}(x)), \quad a_{ij} \in C(I).$$

Thus,

$$\begin{bmatrix} y \\ y^{[1]} \\ y^{[2]} \end{bmatrix}' = \begin{bmatrix} 0 & 1/r_1 & 0 \\ -q_1 & 0 & 1/r_2 \\ 0 & -q_2 & 0 \end{bmatrix} \begin{bmatrix} y \\ y^{[1]} \\ y^{[2]} \end{bmatrix}$$

We now want to transform equation (6.0.1) by using the general transformation of chapter 5 .

Let $y(x) = \mu(x)z(t)$, $t = f(x)$, $f'(x) = \gamma(x)$. We seek $\tilde{r}_1, \tilde{r}_2, \tilde{q}_1, \tilde{q}_2$ so that $L[y] = 0$ if and only if

$$\tilde{L}[z] = \{\tilde{r}_2[(\tilde{r}_1\dot{z})' + \tilde{q}_1z]\}' + \tilde{q}_2\tilde{r}_1\dot{z} = 0. \quad (6.0.2)$$

Equation (6.0.2) will have the same matrix form as Y with $z^{[1]} = \tilde{r}_1\dot{z}$, $z^{[2]} = \tilde{r}_2[(\tilde{r}_1\dot{z})' + \tilde{q}_1z]$. We expect a linear relation

$$Y = FZ, \quad Z = [z, z^{[1]}, z^{[2]}]^T, \quad F(x) = (f_{ij}(x)), \quad f_{ij} \in C(I).$$

It is clear that $F_{11} = \mu$, $F_{12} = F_{13} = F_{23} = 0$. We differentiate $y(x) = \mu(x)z(t)$ and we get $y'(x) = \mu'(x)z(t) + \mu(x)\gamma(x)\dot{z}(t)$. Multiplying $y'(x)$ by r_1 we obtain

$$r_1y' = r_1\mu'(x)z(t) + r_1\mu(x)\gamma(x)\dot{z}(t).$$

We should take $F_{21} = r_1\mu'$ since $y^{[1]} = r_1y' = F_{21}z + F_{22}z^{[1]}$. We will try to construct F with $F_{22} = 1$ and determine F_{31}, F_{32}, F_{33} labelled a, b, c. Thus,

$$\begin{bmatrix} y \\ y^{[1]} \\ y^{[2]} \end{bmatrix} = \begin{bmatrix} \mu & 0 & 0 \\ r_1\mu' & 1 & 0 \\ a & b & c \end{bmatrix} \begin{bmatrix} z \\ z^{[1]} \\ z^{[2]} \end{bmatrix}.$$

Since $Y = FZ$ then $Y' = F'Z + F\dot{Z}\gamma = AY = AFZ$.

Therefore, $\dot{Z} = \frac{1}{\gamma}F^{-1}[AF - F']Z = \tilde{A}Z$,

where we have calculated

$$F^{-1} = \begin{bmatrix} \frac{1}{\mu} & 0 & 0 \\ \frac{-r_1\mu'}{\mu} & 1 & 0 \\ \frac{r_1\mu'b-a}{\mu c} & -\frac{b}{c} & \frac{1}{c} \end{bmatrix}.$$

From

$$A = \begin{bmatrix} 0 & \frac{1}{r_1} & 0 \\ -q_1 & 0 & \frac{1}{r_2} \\ 0 & -q_2 & 0 \end{bmatrix}, \quad F = \begin{bmatrix} \mu & 0 & 0 \\ r_1\mu' & 1 & 0 \\ a & b & c \end{bmatrix},$$

we have

$$\tilde{A} = \begin{bmatrix} 0 & 1/\tilde{r}_1 & 0 \\ -\tilde{q}_1 & 0 & 1/\tilde{r}_2 \\ 0 & -\tilde{q}_2 & 0 \end{bmatrix}, \quad \text{and} \quad F' = \begin{bmatrix} \mu' & 0 & 0 \\ (r_1\mu')' & 0 & 0 \\ a' & b' & c' \end{bmatrix}.$$

Note that $\dot{Z} = [\dot{z}, \dot{z}^{[1]}, \dot{z}^{[2]}]^T$, $Z = [z, z^{[1]}, z^{[2]}]^T$ and $\dot{Z} = \tilde{A}Z$. We now compute $\tilde{A} = \frac{1}{\gamma}F^{-1}[AF - F']$ to be

$$\begin{bmatrix} 0 & \frac{1}{\gamma r_1 \mu} & 0 \\ \frac{1}{\gamma} \left(-q_1 \mu - (r_1 \mu')' + \frac{a}{r_2} \right) & \frac{b}{\gamma r_2} - \frac{\mu'}{\gamma \mu} & \frac{c}{\gamma r_2} \\ \frac{1}{\gamma c} \left[b \left(q_1 \mu - \frac{a}{r_2} + (r_1 \mu')' \right) - q_2 r_1 \mu' - a' \right] & \frac{1}{\gamma c} \left(\frac{b \mu'}{\mu} - \frac{a}{r_1 \mu} - \frac{b^2}{r_2} - q_2 - b' \right) & -\frac{b}{\gamma r_2} - \frac{c'}{\gamma c} \end{bmatrix}.$$

GENERAL CASE:

In this case we want to find a, b, c so $\tilde{A}$ has the required structure and compute $\tilde{r}_1, \tilde{r}_2, \tilde{q}_1, \tilde{q}_2$. We begin by renaming the above matrix; $B =$

$$\begin{bmatrix} 0 & \frac{1}{\gamma r_1 \mu} & 0 \\ \frac{1}{\gamma} \left(-q_1 \mu - (r_1 \mu')' + \frac{a}{r_2} \right) & \frac{b}{\gamma r_2} - \frac{\mu'}{\gamma \mu} & \frac{c}{\gamma r_2} \\ \frac{1}{\gamma c} \left[b \left(q_1 \mu - \frac{a}{r_2} + (r_1 \mu')' \right) - q_2 r_1 \mu' - a' \right] & \frac{1}{\gamma c} \left(\frac{b \mu'}{\mu} - \frac{a}{r_1 \mu} - \frac{b^2}{r_2} - q_2 - b' \right) & -\frac{b}{\gamma r_2} - \frac{c'}{\gamma c} \end{bmatrix}.$$

Recall we seek $\tilde{A} = \begin{bmatrix} 0 & 1/\tilde{r}_1 & 0 \\ -\tilde{q}_1 & 0 & 1/\tilde{r}_2 \\ 0 & -\tilde{q}_2 & 0 \end{bmatrix}.$

In order to find a, b, c and compute $\tilde{r}_1, \tilde{r}_2, \tilde{q}_1, \tilde{q}_2$ we need to show that matrix B has the same structure with matrix $\tilde{A}$.

The matrix B has the same structure with matrix $\tilde{A}$ if

$$\tilde{a}_{11} = b_{11} \longleftrightarrow 0 = 0$$

$$\tilde{a}_{12} = b_{12} \longleftrightarrow \frac{1}{\tilde{r}_1} = \frac{1}{\gamma\mu r_1} \longleftrightarrow \tilde{r}_1 = \gamma\mu r_1$$

$$\tilde{a}_{13} = b_{13} \longleftrightarrow 0 = 0$$

$$\tilde{a}_{22} = b_{22} \longleftrightarrow -\frac{\mu'}{\gamma\mu} + \frac{b}{\gamma r_2} = 0 \longleftrightarrow b = \frac{r_2\mu'}{\mu}$$

$$\tilde{a}_{23} = b_{23} \longleftrightarrow \frac{1}{\tilde{r}_2} = \frac{c}{\gamma r_2} \longleftrightarrow \tilde{r}_2 = \frac{\gamma r_2}{c}.$$

One can find c by considering

$$\tilde{a}_{33} = b_{33} \longleftrightarrow \frac{b}{r_2} = -\frac{c'}{c}.$$

Thus $c = \frac{1}{\mu}$ and $\tilde{r}_2 = \gamma\mu r_2$.

From $\tilde{a}_{31} = b_{31}$ we have

$$\frac{1}{\gamma c} \left[b \left(q_1\mu - \frac{a}{r_2} + (r_1\mu')' \right) - q_2 r_1 \mu' - a' \right] = 0.$$

Multiplying both sides by γc , substituting $b = \frac{r_2\mu'}{\mu}$ into the above equation and simplifying we get

$$(a\mu)' = r_2 q_1 \mu' \mu - r_1 q_2 \mu' \mu + r_2 \mu' (r_1 \mu')'.$$

Integrating both sides of the above equation we can solve for a . Hence

$$a = \frac{1}{\mu} \int (\mu' \mu (r_2 q_1 - r_1 q_2) + r_2 \mu' (r_1 \mu')') dx.$$

From $\tilde{a}_{21} = b_{21}$ we have

$$-\tilde{q}_1 = \frac{1}{\gamma} \left[-q_1\mu + \frac{a}{r_2} - (r_1\mu')' \right].$$

Substituting a into the above equation and simplifying we get

$$-\tilde{q}_1 = \frac{1}{\gamma} \left[-q_1\mu - (r_1\mu')' + \frac{1}{\mu r_2} \int (\mu'\mu(r_2q_1 - r_1q_2) + r_2\mu'(r_1\mu')') dx \right].$$

Finally, from the last entries $\tilde{a}_{32} = b_{32}$ we have

$$-\tilde{q}_2 = \frac{1}{\gamma c} \left(\frac{\mu'b}{\mu} - \frac{a}{r_1\mu} - \frac{b^2}{r_2} - q_2 - b' \right).$$

We now substitute a, b, b' into the above equation and after long calculations we get

$$-\tilde{q}_2 = \frac{1}{\gamma} \left[\frac{r_2(\mu')^2}{\mu} - (r_2\mu')' - q_2\mu - \frac{1}{r_1\mu} \int (\mu'\mu(r_2q_1 - r_1q_2) + r_2\mu'(r_1\mu')') dx \right].$$

Summarizing,

$$\begin{aligned} a &= \frac{1}{\mu} \int (\mu'\mu(r_2q_1 - r_1q_2) + r_2\mu'(r_1\mu')') dx \\ b &= \frac{r_2\mu'}{\mu} \\ c &= \frac{1}{\mu} \end{aligned} \tag{6.0.3}$$

$$\begin{aligned}
\tilde{r}_1 &= \gamma\mu r_1 \\
-\tilde{q}_1 &= \frac{1}{\gamma} \left[-q_1\mu - (r_1\mu')' + \frac{1}{\mu r_2} \int (\mu'\mu(r_2q_1 - r_1q_2) + r_2\mu'(r_1\mu')') dx \right] \\
-\tilde{q}_2 &= \frac{1}{\gamma} \left[\frac{r_2(\mu')^2}{\mu} - (r_2\mu')' - q_2\mu - \frac{1}{r_1\mu} \int (\mu'\mu(r_2q_1 - r_1q_2) + r_2\mu'(r_1\mu')') dx \right] \\
\tilde{r}_2 &= \gamma\mu r_2
\end{aligned} \tag{6.0.4}$$

SPECIAL CASE: Self-Adjoint

If the equation (6.0.1) is self-adjoint, that is $r_1 = r_2$ and $q_1 = q_2$, then (6.0.1) becomes

$$L[y] = \{r_1 [(r_1 y')' + q_1 y]\}' + q_1(r_1 y') = 0. \tag{6.0.5}$$

Following the same procedure as in general case one can find a, b, c and compute

$\tilde{r}_1, \tilde{r}_2, \tilde{q}_1, \tilde{q}_2$ for the special case. From the general case we have

$$\begin{aligned}
a &= \frac{1}{\mu} \int (\mu'\mu(r_2q_1 - r_1q_2) + r_2\mu'(r_1\mu')') dx \\
b &= \frac{r_2\mu'}{\mu} \\
c &= \frac{1}{\mu} \\
\tilde{r}_1 &= \gamma\mu r_1 \\
\tilde{r}_2 &= \gamma\mu r_2 \\
-\tilde{q}_1 &= \frac{1}{\gamma} \left[-q_1\mu - (r_1\mu')' + \frac{1}{\mu r_2} \int (\mu'\mu(r_2q_1 - r_1q_2) + r_2\mu'(r_1\mu')') dx \right] \\
-\tilde{q}_2 &= \frac{1}{\gamma} \left[\frac{r_2(\mu')^2}{\mu} - (r_2\mu')' - q_2\mu - \frac{1}{r_1\mu} \int (\mu'\mu(r_2q_1 - r_1q_2) + r_2\mu'(r_1\mu')') dx \right].
\end{aligned}$$

Since $r_1 = r_2$ and $q_1 = q_2$ then the above equations become

$$\begin{aligned}
a &= \frac{1}{\mu} \int r_1 \mu' (r_1 \mu')' dx = \frac{(r_1 \mu')^2}{2\mu} + \frac{k}{\mu} \\
b &= \frac{r_1 \mu'}{\mu} \\
c &= \frac{1}{\mu} \\
\tilde{r}_1 &= \gamma \mu r_1 \\
\tilde{r}_2 &= \gamma \mu r_2 \\
-\tilde{q}_1 &= \frac{1}{\gamma} \left[-q_1 \mu - (r_1 \mu')' + \frac{1}{\mu r_2} \int r_1 \mu' (r_1 \mu')' dx \right] \\
&= \frac{1}{\gamma} \left[-q_1 \mu - (r_1 \mu')' + \frac{1}{\mu r_2} \left(\frac{(r_1 \mu')^2}{2} + k \right) \right] \\
-\tilde{q}_2 &= \frac{1}{\gamma} \left[\frac{r_2 (\mu')^2}{\mu} - (r_2 \mu')' - q_2 \mu - \frac{1}{r_1 \mu} \int r_2 \mu' (r_2 \mu')' dx \right] \\
&= \frac{1}{\gamma} \left[\frac{r_2 (\mu')^2}{\mu} - (r_2 \mu')' - q_2 \mu - \frac{1}{r_1 \mu} \left(\frac{(r_2 \mu')^2}{2} + k \right) \right],
\end{aligned}$$

where k is a constant. Therefore the transformed system $\dot{Z} = \tilde{A}Z$ with $\dot{Z} = [\dot{z}, \dot{z}^{[1]}, \dot{z}^{[2]}]^T$ and $Z = [z, z^{[1]}, z^{[2]}]^T$ is

$$\begin{bmatrix}
0 & \frac{1}{\gamma \mu r_1} & 0 \\
\frac{1}{\gamma} \left(\frac{r_1 (\mu')^2}{2\mu} - (r_1 \mu')' - q_1 \mu + \frac{k}{r_2 \mu} \right) & 0 & \frac{1}{\gamma \mu r_2} \\
0 & \frac{1}{\gamma} \left(\frac{r_2 (\mu')^2}{2\mu} - (r_2 \mu')' - q_1 \mu - \frac{k}{r_1 \mu} \right) & 0
\end{bmatrix}.$$

Note that the transformed system is also self-adjoint if we choose $k = 0$.

General Subcase 1:

In this case we use the general transformation of chapter 5 but let $\mu(x) = x^2$ and

$t = 1/x$. Note that $0 < x < 1$ and $1 < t < \infty$. We again want to find a, b, c and compute $\tilde{r}_1, \tilde{r}_2, \tilde{q}_1, \tilde{q}_2$. Since a, b, c are known from the general case then we just substitute $\mu(x) = x^2$ into (6.0.3) and we simplify,

$$\begin{aligned} a &= \frac{1}{x^2} \int (2x^3(r_2 q_1 - r_1 q_2) + 2xr_2(2xr_1)') dx \\ b &= \frac{2r_2}{x} \\ c &= \frac{1}{x^2}. \end{aligned}$$

Similarly for $\tilde{r}_1, \tilde{r}_2, \tilde{q}_1, \tilde{q}_2$, we have

$$\begin{aligned} \tilde{r}_1(t) &= -r_1(x) \\ \tilde{r}_2(t) &= -r_2(x) \\ -\tilde{q}_1(t) &= q_1(x)x^4 + x^2(2xr_1(x))' \\ &\quad - \frac{1}{r_2(x)} \int (2x^3(r_2(x)q_1(x) - r_1(x)q_2(x)) + 2xr_2(x)(2xr_1(x))') dx \\ -\tilde{q}_2(t) &= q_2(x)x^4 + x^2(2xr_2(x))' - 4x^2r_2(x) \\ &\quad + \frac{1}{r_1(x)} \int_x^1 (2x^3(r_2(x)q_1(x) - r_1(x)q_2(x)) + 2xr_2(x)(2xr_1(x))') dx. \end{aligned}$$

Special Subcase 1: Self-Adjoint

In the previous case we found a, b, c and $\tilde{r}_1, \tilde{r}_2, \tilde{q}_1, \tilde{q}_2$ when $y(x) = x^2 z(1/x)$. In this case we consider the same transformation as in the previous case but let $r_1 = r_2$ and

$$q_1 = q_2 :$$

$$a = 2r_1^2 + \frac{k}{x^2}$$

$$b = \frac{2r_1}{x}$$

$$c = \frac{1}{x^2}$$

$$\tilde{r}_1(t) = -r_1(x)$$

$$\tilde{r}_2(t) = -r_2(x)$$

$$-\tilde{q}_1(t) = q_1(x)x^4 + x^2(2xr_1(x))' - \frac{1}{r_2(x)} [2(xr_1(x))^2 + k]$$

$$-\tilde{q}_2(t) = q_2(x)x^4 + x^2(2xr_2(x))' - 4x^2r_2(x) + \frac{1}{r_1(x)} [2(xr_2(x))^2 + k] .$$

Therefore, the transformed system is also self adjoint if we choose $k = 0$. **General**

Subcase 2:

Let $y(x) = \mu(x)z(t)$ where $\mu(x) = x$ and $t = -\ln(x)$. Note that $0 < x < 1$ and $1 < t < \infty$. Following the same procedure as in general subcase 1 we can get a, b, c and $\tilde{r}_1, \tilde{r}_2, \tilde{q}_1, \tilde{q}_2$. Hence,

$$a = \frac{1}{x} \int [x(r_2q_1 - r_1q_2) + r_2r_1'] dx$$

$$b = \frac{r_2}{x}$$

$$c = \frac{1}{x}.$$

Recall that $\dot{Z} = \tilde{A}Z$ with $\dot{Z} = [\dot{z}, \dot{z}^{[1]}, \dot{z}^{[2]}]^T$, $Z = [z, z^{[1]}, z^{[2]}]^T$ and

$$A = \begin{bmatrix} 0 & 1/\tilde{r}_1 & 0 \\ -\tilde{q}_1 & 0 & 1/\tilde{r}_2 \\ 0 & -\tilde{q}_2 & 0 \end{bmatrix}.$$

Thus,

$$\tilde{r}_1(t) = -r_1(x)$$

$$\tilde{r}_2(t) = -r_2(x)$$

$$-\tilde{q}_1(t) = x^2 q_1(x) + x r_1'(x) - \frac{1}{r_2(x)} \int [x(r_2(x)q_1(x) - r_1(x)q_2(x)) + r_2(x)r_1'(x)] dx$$

$$-\tilde{q}_2(t) = q_2(x)x^2 - r_2(x) + x r_2'(x) + \frac{1}{r_1(x)} \int [x(r_2(x)q_1(x) - r_1(x)q_2(x)) + r_2(x)r_1'(x)] dx.$$

Special Subcase 2: Self-Adjoint

In this case we consider a, b, c and $\tilde{r}_1, \tilde{r}_2, \tilde{q}_1, \tilde{q}_2$ from the general subcase 2. Let $r_1 = r_2$

and $q_1 = q_2$. Applying this condition into a, b, c of the previous case we get

$$a = \frac{1}{2x} (r_1^2 + 2k)$$

$$b = \frac{r_1}{x}$$

$$c = \frac{1}{x}.$$

We also apply the same condition into $\tilde{r}_1, \tilde{r}_2, \tilde{q}_1, \tilde{q}_2$:

$$\tilde{r}_1(t) = -r_1(x)$$

$$\tilde{r}_2(t) = -r_2(x)$$

$$-\tilde{q}_1(t) = x^2 q_1(x) + x r_1'(x) - \frac{1}{2r_2(x)} (r_1^2(x) + 2k)$$

$$-\tilde{q}_2(t) = x^2 q_2(x) - r_2(x) + x r_2'(x) + \frac{1}{2r_1(x)} (r_2^2(x) + 2k) .$$

Therefore, the transformed system is also self adjoint if we choose $k = 0$.

Bibliography

Bibliography

- [1] J. H. Barrett. *Oscillation Theory of Ordinary Linear Differential Equations*. Advances in Mathematics 3, 1969, 415 - 509.
- [2] R. G. Bartle, and D. R. Sherbert. *Introduction to Real Analysis*. New York: Wiley, 2000.
- [3] E. A. Coddington, and N. Levinson. *Theory of Ordinary Differential Equation*. New York: Mc Graw Hill, 1955.
- [4] W. A. Coppel. *Stability and Asymptotic Behavior of Differential Equations*. Boston: D.C. Heath and Company, 1965.
- [5] D. Hinton. *Asymptotic Behavior of Solutions of $(ry^{(m)})^{(k)} \pm qy = 0$* . J. Differential Equations 4, 1968, 590 - 596.
- [6] G. Jones *Oscillation Criteria for Third Order Differential Equations*. J. Math. Anal. 7 (1976), no.1, 13-15.
- [7] J. D. Pryce. *Numerical Solution of Sturm-Liouville Problems* New York: Oxford, 1993.

- [8] D. A. Stavri. *Applications of Singular Sturm-Liouville Problems: Torsion of an Elastic Cylinder & Vibrations of a Circular Membrane*. MS Thesis, Murray State University, KY, 2005.
- [9] C. A. Swanson. *Comparison and Oscillation Theory of Linear Differential Equations*. New York: Academic P., 1968.

Vita

Despina Andrea Stavri was born in Sotera Famagusta, Cyprus on February 2, 1980. She lived in Sotera Famagusta, Cyprus until the age of 17. After graduating from Paralimni High School in 1997, she went to the Technological Educational Institutions at Thessaloniki, Greece to study computer science. In 2000, she transferred to Murray State University, Kentucky and graduated with a B.S. in applied mathematics with emphasis on computer science in 2003. After graduation, Despina started a Master of Science in Applied Mathematics at MSU and graduated in August 2005. From there she went to the University of Tennessee, Knoxville where she pursued a second M.S. in Mathematics. She graduated in August 2007.

Despina intends on pursuing a PhD in Mathematics.