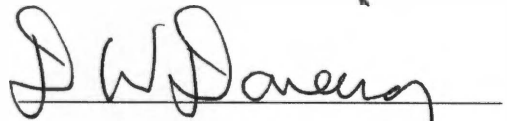


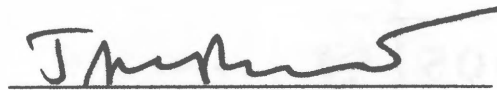
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Don Dareing, Major Professor

We have read this thesis and
recommend its acceptance:





Acceptance for the Council:


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Dean of Graduate Studies

**ANALYTICAL AND EXPERIMENTAL MODAL ANALYSIS OF A
MICRO CANTILEVER BEAM**

**A Thesis
Presented for the
Master of Science
Degree
The University of Tennessee, Knoxville**

**Aaron Lee Echols
May 2005**

TO THE HONORABLE MEMBERS OF THE HOUSE OF REPRESENTATIVES
AND THE SENATE OF THE UNITED STATES

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IN WITNESS WHEREOF, I have hereunto set my hand and the seal of the
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Abstract

Micro cantilever beams are used extensively in experimental nanomechanics. They were initially used as an integral component in atomic force microscopes (AFM) to map surfaces or objects with micro, nano, or even atomic precision. Recently, micro cantilever sensors are being studied for possible use in determining bio/chemical agents under a variety of military and homeland security conditions. The key here is to identify the absorbed agent in terms of cantilever deflections, whether static or dynamic. Micro cantilevers are also being studied for possible use in micro/nano engines. In this case, micro particles impact micro cantilevers as a result of Brownian motion causing the cantilevers to vibrate. Kinetic energy associated with these vibrations is converted into electrical energy. Typically, only the first mode of vibration has been considered. However, higher modes may convey additional information and energy. This study gives the dynamic response of micro cantilever in terms of the responses of each accumulate natural mode of vibration.

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List of Symbols

- a – length along the beam which the beam is hit
- A – cross-sectional area of the beam
- c – damping factor of the beam
- C – modal damping factor of the beam
- E – modulus of elasticity of the beam
- H – direct transfer function of the beam
- i – subscript used to represent mode number
- I – cross-sectional moment of inertia
- j – the imaginary number $\sqrt{-1}$
- K – modal stiffness of the beam
- L – length of the beam
- M – modal mass of the beam
- p – multiplication factor representing the integral of the
different modes evaluated along the beam's length
- q – variable used for simple substitution
- Q – scaling factor used with mode shape equation
- r – frequency ratio ω/ω_n
- Δr – change in frequency ratio
- t – time
- v – relative motion of the beam with respect to base motion
- w – distributed load on the beam (varies with time and distance along beam)
- x – distance along beam
- X – spatial function of the modes of the beam based on x only
- y – absolute motion of the beam with respect to a fixed frame
- Y – amplitude of mode of the beam with respect to base motion
- z – motion of the base the beam is attached to
- z_0 – amplitude of base motion

β – eigenvalue of the beam divided by the length of the beam

δ – unit impulse

ϕ – angle component of the frequency response of the beam

η – temporal function of the modes of the beam based on time only

φ – lag of the mode with respect to base motion

λ – eigenvalue of the beam

ρ – density of the beam

ω – frequency in radians per second

ω_d – damped natural frequency in radians per second

ω_n – undamped natural frequency in radians per second

$\Delta\omega$ – change in frequency in radians per second

ζ – modal damping coefficient

σ – coefficient used in mode shape equation

I. Introduction

Micro cantilevers are necessary structures that are vital to the function of atomic force microscopes (AFM). AFMs are used to provide high-resolution images of surfaces at the nano, and even, the atomic scale. One of the beneficial aspects of an AFM is that the scanning surface does not need to be electrically conductive. Electron microscopes can only scan on surface that can conduct electricity. An atomic force microscope works by using a micro cantilever with a special tip. This micro cantilever is dragged (or tapped) across the surface. An attached laser shoots a beam at the cantilever, which reflects at an oblique angle onto a segmented photodiode. This photodiode then amplifies the signals received from the laser into a topographical map of the surface being scanned. A simple schematic of the setup of an AFM is shown in Figure 1. Studies have been prompted into the fact that these same micro cantilevers used for scanning a surface could be used to detect certain physical properties of the surface or particulates on the surface. When the micro cantilever hits certain materials or particulates, statically or dynamically, the response of the modes could be unique based on the type of material or element the beam is in contact with. By analyzing the response of the different modes, it could be possible to develop a modal “signature.” Understanding and identifying these different signatures could lead to the identification of micro sized biological and chemical agents. The key to this success is translating a micro cantilever beam’s deformation into an identifying bio/chemical agent signature [1]. This type of use of a micro cantilever is best represented by base motion excitation, which will be considered in Chapter IV.

Another application that is possible with micro cantilever beams is the possibility to extract a small power unit while Brownian motion excites the beam. Brownian motion is defined as random movement of microscopic particles suspended in liquids or gases resulting from the impact of molecules of the fluid surrounding the particles. If these particles were to strike a micro cantilever beam into vibration, it could then be possible to obtain energy from the movement. As a result of the deformation, a piezoelectric material deforming with the beam would then generate a minute amount of electrical energy.

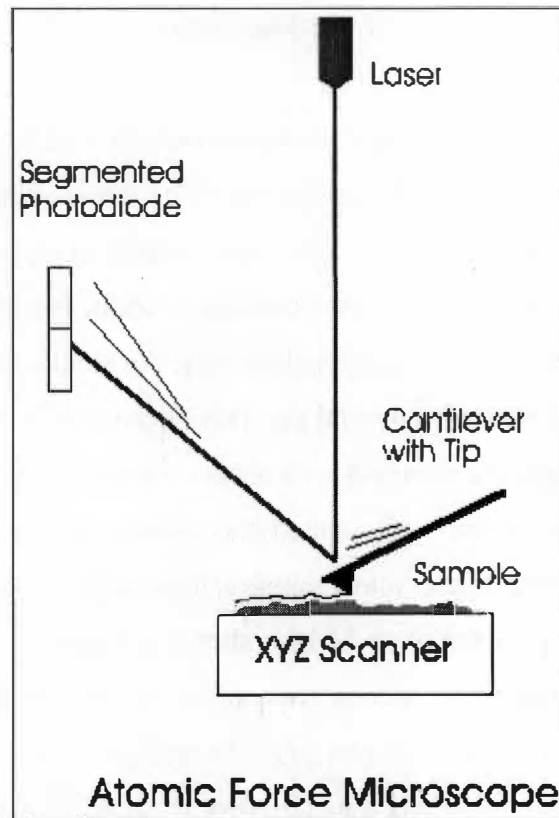


Figure 1: A simple schematic of an atomic force microscope [2].

With a large array of the micro cantilever energy receptors, it could be possible to extract a usable amount of energy from Brownian motion. This type of use of a micro cantilever is best represented by impulse excitation, which will also be considered in Chapter IV.

This study will focus on the modal analysis of cantilevers and the contribution to the total response of the different modal components. In particular, three different cases of excitation will be considered. The first case is that of a cantilever beam under a distributed load varying with time. The second case (which is a special case of the first case) is a cantilever beam excited with a unit impulse at some distance along the beam's length. The third case is a cantilever beam excited by base motion. In any case of excitation, modal mass and modal stiffness can be calculated using the beam's physical properties. However, modal damping is best determined experimentally because of its dynamic nature. This study includes both the analysis of the previously mention cases, as

well as an experimental component to verify the modal mass and modal stiffness of a particular beam and acquire the modal damping factors.

Basic vibration theory will be used to determine the modal response of the beam under each case of excitation. Using the experimental beam's physical properties, the modal mass and modal stiffness will be found. Then, using experimental methods, the modal mass and modal stiffness will be verified. The modal damping factors will also be determined from the experimental data. The results of the data acquired from the experiment will be discussed. Using the experimental data and the analytical models, certain conclusions will be drawn as to the significance of the findings and any suggestions for any similar studies done in the future.

II. Objective of Study

The objective of this study is to define the dynamic response of cantilevers in terms of their modal components. Much attention is given in the literature to only the first mode of vibration response and is often only modeled as a single degree of freedom system. Under many types of forced vibration, this model becomes inadequate as the higher modes become more important. As mentioned earlier, these higher modes could have a direct significance in identifying specific biological and chemical agents. The objective here is driven by the need to understand the fundamental behavior of micro cantilevers in terms of all of its modal components.

III. Eigenvectors and Eigenvalues

For this study, the discussion of the response of a cantilever beam is based on the sum of all of its modal components. For completeness to this presentation, a short review of eigenvectors and eigenvalues, and their application to cantilever beams, will be shown. Eigenvectors (mode shapes of a beam) and eigenvalues (natural frequencies of a beam) are an essential part of the modal analysis method of vibration response. The eigenvalues, or natural frequencies, for a cantilever beam can be found in most textbooks [3-5] by using the following equation if the length of the beam is known.

$$\cos(\beta_i L) \cosh(\beta_i L) = -1 \quad (1)$$

The eigenvalues for the first four modes of vibrations for a cantilever beam are

$$\begin{aligned}\beta_1 L = \lambda_1 &= 1.87510407 \\ \beta_2 L = \lambda_2 &= 4.69409113 \\ \beta_3 L = \lambda_3 &= 7.85475744 \\ \beta_4 L = \lambda_4 &= 10.99554073\end{aligned}$$

Additionally, most textbooks [3-5] show that the mode shapes can be found using

$$X_i(x) = Q_i \{ \cosh(\beta_i x) - \cos(\beta_i x) - \sigma_i [\sinh(\beta_i x) - \sin(\beta_i x)] \} \quad (2)$$

where

$$\sigma_i = \frac{\sinh(\beta_i L) - \sin(\beta_i L)}{\cosh(\beta_i L) + \cos(\beta_i L)}$$

The mode shapes are included in Figure 2 for convenience.

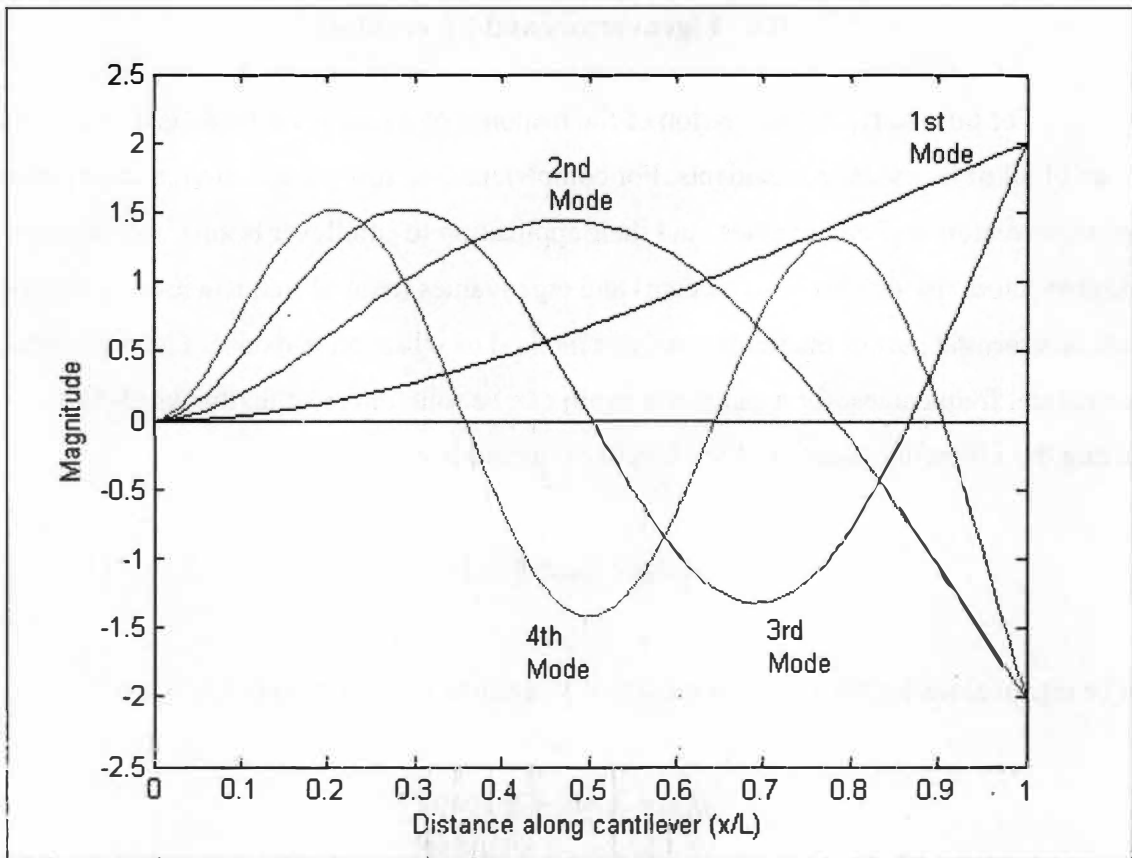


Figure 2: The first four mode shapes of a cantilever beam.

For this discussion, consider, specifically, a micro cantilever having the following dimensions:

Length: $120\ \mu\text{m}$

Width: $30\ \mu\text{m}$

Thickness: $2\ \mu\text{m}$

Density: $1000\ \text{kg/m}^3$

Modulus of Elasticity: $179\ \text{GPa}$

The corresponding values for β for the first four modes are

$$\beta_1 = 15626$$

$$\beta_2 = 39117$$

$$\beta_3 = 65456$$

$$\beta_4 = 91630$$

It will be shown later that

$$\omega_i = \beta_i^2 \sqrt{\frac{EI}{\rho A}}$$

Using the previous equation yields the first four natural frequencies of the micro cantilever beam to be

Frequency 1: 300 kHz

Frequency 2: 1881 kHz

Frequency 3: 5267 kHz

Frequency 4: 10322 kHz

IV. Three Types of Driving Forces Considered

This section develops the mathematical formulation, which predicts the vibration response of cantilevers in terms of its modal components. The three types of loading considered for this study are:

- Distributed load along the beam
- Impulse at some point along the beam
- Base motion exciting the beam

These three forcing types were chosen because they have a direct relevance to micro cantilever sensors.

Case 1 – Distributed Load Excitation

We begin by considering a distributed load which changes with time, $w(x,t)$, as shown in Figure 3. This forcing function is typically defined as

$$w(x,t) = W(x)\cos(\omega t) \quad (3)$$

where $W(x)$ is a load varying along the length of the beam and ω is the driving frequency in radians/second. A model of the bending vibration of a cantilever beam can be derived from examining the force diagram of an infinitesimal element of the beam as shown in Figure 4.

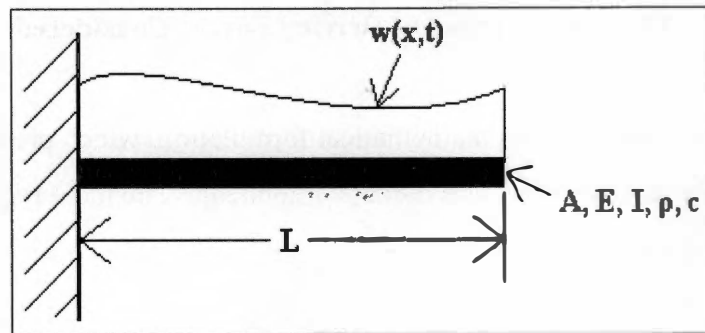


Figure 3: A cantilever beam with a distributed load that varies with time.

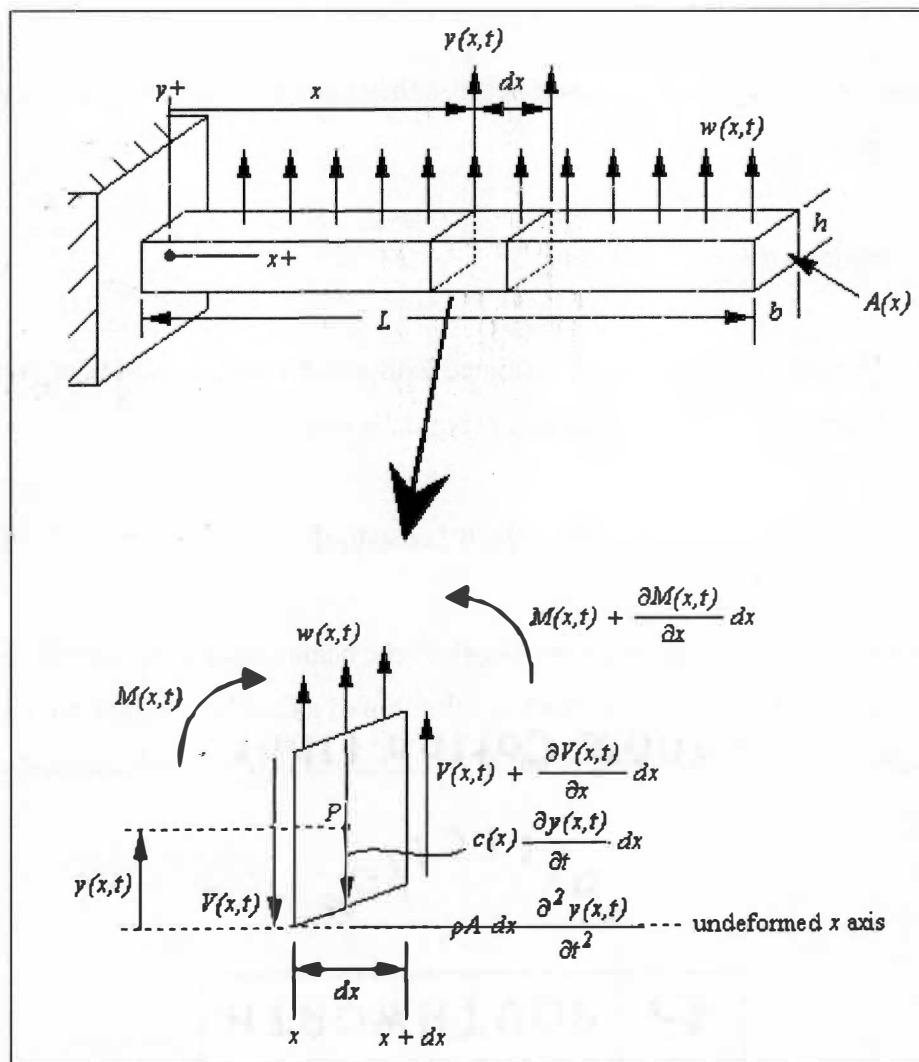


Figure 4: Simple beam in transverse vibration and a free-body diagram of a small element of the beam as it is deformed by a distributed load. [5]

In this figure, the deflection of the beam is $y(x,t)$, $M(x,t)$ is bending moment on the beam, $V(x,t)$ is the shear force on the beam. The beam's properties are length, L ; height, h ; thickness, b ; cross-sectional area, $A(x)$; density, ρ ; and damping per unit length, $c(x)$. Here the damping is modeled as environmental damping, not structural damping, which depends only on the first time derivative of the absolute motion of the beam. Assuming that the deformation of the element is small enough such that the shear deformation is much smaller than $y(x,t)$, a summation of forces in the y -direction yields

$$\left(V(x,t) + \frac{\partial V(x,t)}{\partial x} dx \right) - V(x,t) - c(x) \frac{\partial y(x,t)}{\partial t} dx + w(x,t) dx = \rho A(x) dx \frac{\partial^2 y(x,t)}{\partial t^2} \quad (4)$$

The assumption of small shear deformation used in the Equation (4) is true if $L/b \geq 10$ and $L/h \geq 10$ (i.e. long, slender beams). Next, summation of the moments about point P of the element results in

$$\left[M(x,t) + \frac{\partial M(x,t)}{\partial x} dx \right] - M(x,t) + \left[V(x,t) + \frac{\partial V(x,t)}{\partial x} dx \right] dx + [w(x,t) dx] \frac{dx}{2} = 0 \quad (5)$$

The left-hand side of the equation is zero since it is assumed that the rotary inertia of the element is negligible. Simplifying Eq. (5) yields

$$\left[\frac{\partial M(x,t)}{\partial x} + V(x,t) \right] dx + \left[\frac{\partial V(x,t)}{\partial x} + \frac{f(x,t)}{2} \right] (dx)^2 = 0 \quad (6)$$

Since dx is assumed to be very small (but not zero), then $(dx)^2$ can be assumed to be almost zero, so that Eq. (6) can be rewritten as

$$V(x,t) = -\frac{\partial M(x,t)}{\partial x} \quad (7)$$

Equation (7) states that the shear force is proportional to the spatial change in the bending moment. Substituting Eq. (7) into Eq. (4) yields

$$-\frac{\partial^2}{\partial x^2} [M(x,t)] - c(x) \frac{\partial y(x,t)}{\partial t} + w(x,t) = \rho A(x) \frac{\partial^2 y(x,t)}{\partial t^2} \quad (8)$$

From mechanics of materials, it is known that the beam sustains a bending moment, which is related to the beam deflection by

$$M(x,t) = EI(x) \frac{\partial^2 y(x,t)}{\partial x^2} \quad (9)$$

Then, substituting Eq. (9) into Eq. (8) and rewriting results in

$$\rho A(x) \frac{\partial^2 y(x,t)}{\partial t^2} + c(x) \frac{\partial y(x,t)}{\partial t} + \frac{\partial^2}{\partial x^2} \left[EI(x) \frac{\partial^2 y(x,t)}{\partial x^2} \right] = w(x,t) \quad (10)$$

For this study, only cantilever beams of uniform cross-sections are considered. Thus, Eq. (10) can be rewritten as the final differential equation used for cantilever beams

$$\rho A \frac{\partial^2 y(x,t)}{\partial t^2} + c \frac{\partial y(x,t)}{\partial t} + EI \frac{\partial^4 y(x,t)}{\partial x^4} = w(x,t) \quad (11)$$

An assumed solution can be used in order to separate the spatial and temporal variables. For this study, only the first four modes of the infinite solution are considered.

$$y(x,t) = \sum_{i=1}^4 X_i(x) \eta_i(t) \quad (12)$$

$$\frac{\partial^4 y}{\partial x^4} = y^{(4)}(x,t) = \sum_{i=1}^4 X_i^{(4)}(x) \eta_i(t)$$

$$\frac{\partial y}{\partial t} = \dot{y}(x,t) = \sum_{i=1}^4 X_i(x) \dot{\eta}_i(t)$$

$$\frac{\partial^2 y}{\partial t^2} = \ddot{y}(x,t) = \sum_{i=1}^4 X_i(x) \ddot{\eta}_i(t)$$

Substituting Eq. (12) into Eq. (11), multiplying by X_j , and integrating yields

$$\begin{aligned} EI \int_0^L \sum_{i=1}^4 X_i^{(4)}(x) X_j(x) \eta_i(t) dx + c \int_0^L \sum_{i=1}^4 X_i(x) X_j(x) \dot{\eta}_i(t) dx + \rho A \int_0^L \sum_{i=1}^4 X_i(x) X_j(x) \ddot{\eta}_i(t) dx \\ = \int_0^L X_j(x) w(x,t) dx \end{aligned} \quad (13)$$

Since the modes are orthogonal to each other

$$\int_0^L X_i(x) X_j(x) dx = 0 \quad \text{if } i \neq j$$

$$\int_0^L X_i^{(4)}(x) X_j(x) dx = 0 \quad \text{if } i \neq j$$

Also, note the following identities

$$\int_0^L X_i^2(x) dx = Q_i^2 L$$

$$X_i^{(4)} X_i = X_i^2 \beta_i^4$$

$$\int_0^L X_i^{(4)}(x) X_i(x) dx = \beta_i^4 Q_i^2 L$$

The previous equations are found by using basic mathematical principles on Eq. (2) and taking the integral across the full length of the beam. Substituting the previous equations into Eq. (13) results in

$$\rho ALQ_i^2 \ddot{\eta}_i(t) + cLQ_i^2 \dot{\eta}_i(t) + EIL\beta_i^4 Q_i^2 \eta_i(t) = \int_0^L X_i(x) w(x,t) dx \quad (14)$$

This can further be simplified by making the following substitutions

$$\begin{aligned} M_i &= \rho ALQ_i^2 \\ C_i &= cLQ_i^2 \\ K_i &= EIL\beta_i^4 Q_i^2 \\ F_i(t) &= \int_0^L X_i(x) w(x,t) dx \end{aligned}$$

which results in

$$M_i \ddot{\eta}_i(t) + C_i \dot{\eta}_i(t) + K_i \eta_i(t) = F_i(t) \quad (15)$$

Finally, Eq. (15) can be rewritten as

$$\ddot{\eta}_i(t) + 2\zeta_i \omega_{ni} \dot{\eta}_i(t) + \omega_{ni}^2 \eta_i(t) = \frac{F_i(t)}{\rho ALQ_i^2} \quad (16)$$

where

$$\begin{aligned} \omega_{ni} &= \sqrt{\frac{K_i}{M_i}} = \sqrt{\frac{EIL\beta_i^4 Q_i^2}{\rho ALQ_i^2}} = \beta^2 \sqrt{\frac{EI}{\rho A}} \\ \zeta_i &= \frac{C_i}{2M_i \omega_{ni}} = \frac{cLQ_i^2}{2\rho ALQ_i^2 \omega_{ni}} = \frac{c}{2\rho A \omega_{ni}} \end{aligned}$$

It can be noted that the term $F_i(t)$, in Eq (16), will have a Q_i that can be cancelled with one of the Q_i terms in the denominator of the left hand side of Eq. (16). Further noting that when the solution to Eq. (16) is substituted into Eq. (12), the remaining Q_i term will cancel out of the total response. Because of this fact, from this point forward, for convenience, the term Q_i will be arbitrarily set to a value of one. The solution to Eq. (16), $\eta_i(t)$, when combined with the mode shape represented by Eq. (2), $X_i(x)$, according to Eq. (12) gives the modal response.

If, for example, the driving force were represented by a uniformly distributed load driven so that all points along the beam are in phase, the modal force would be

$$\begin{aligned} F_i(t) &= \int_0^L X_i(x) w(x, t) dx = \int_0^L X_i(x) W_0 \cos(\omega t) dx \\ &= W_0 \cos(\omega t) \int_0^L X_i(x) dx = 2\sigma_i \frac{L}{\lambda_i} W_0 \cos(\omega t) \end{aligned}$$

and the resulting modal differential equation would be

$$\ddot{\eta}_i(t) + 2\zeta_i \omega_{ni} \dot{\eta}_i(t) + \omega_{ni}^2 \eta_i(t) = \frac{2W_0 \sigma_i}{\rho A \lambda_i} \cos(\omega t)$$

Case 2 – Unit Impulse Excitation

Case 2 is a special subset of case 1. This case represents the situation of a micro cantilever excited by particulates in a fluid or gas undergoing Brownian motion. This case is shown in Figure 5 as an impulse hitting the beam at some arbitrary point where $x=a$. If $w(x, t)$ is a unit impulse applied at some distance, a , along the beam,

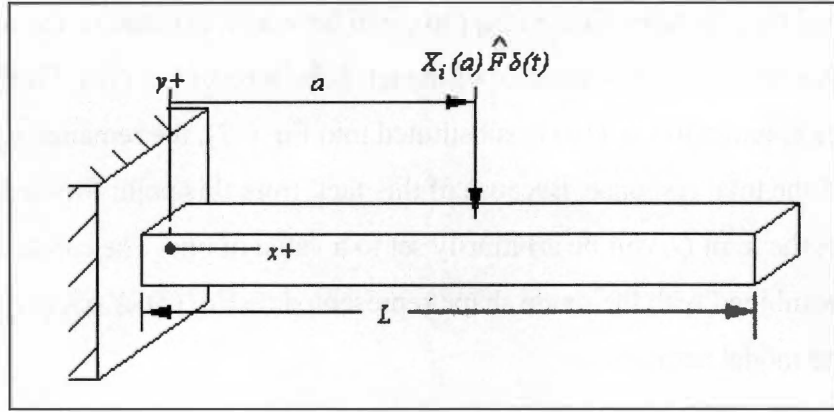


Figure 5: A unit impulse applied to a cantilever beam at some distance 'a' along the beam.

then the excitation force is zero everywhere along the beam except where $x = a$ which results in

$$F_i = \int_0^L X_i(x) w(x, t) dx = X_i(a) \delta(t)$$

So, then Eq. (16) can be rewritten as

$$\ddot{\eta}_i(t) + 2\zeta_i \omega_{ni} \dot{\eta}_i(t) + \omega_{ni}^2 \eta_i(t) = \frac{X_i(a)}{\rho A L} \delta(t) \quad (17)$$

where

$$\omega_{ni} = \sqrt{\frac{K_i}{M_i}} = \sqrt{\frac{EIL\beta_i^4}{\rho AL}} = \beta^2 \sqrt{\frac{EI}{\rho A}}$$

$$\zeta_i = \frac{C_i}{2M_i \omega_{ni}} = \frac{cL}{2\rho AL \omega_{ni}} = \frac{c}{2\rho A \omega_{ni}}$$

Each mode can now be considered independently as a simple single degree of freedom system. The solution to Eq. (17) can classically be found by using Duhamel's

integral. The solution to Eq. (17) can also be found in many textbooks [3-5] for an underdamped system as

$$\eta_i = \frac{X_i(a)}{\rho AL} \cdot \frac{1}{\omega_{di}} \exp(-\zeta_i \omega_{ni} t) \sin(\omega_{di} t) \quad (18)$$

where

$$\omega_{di} = \omega_{ni} \sqrt{1 - \zeta_i^2}$$

It can be shown that the frequency response of a system is equal to the Fourier transform of the response, Eq. (18), to an impulse [4] or

$$H_i(\omega) = \int_{-\infty}^{\infty} \eta_i(t) e^{-j\omega t} dt$$

Note that this is also how modern electronic equipment calculates the frequency response experimentally. Since no actions existing before the starting time will be considered, the integral can be rewritten as

$$H_i(\omega) = \frac{X_i(a)}{\rho AL \omega_{di}} \int_0^{\infty} e^{-(j\omega + \zeta_i \omega_{ni})t} \sin(\omega_{di} t) dt$$

$$H_i(\omega) = \frac{X_i(a)}{\rho AL \omega_{ni}^2} \left[\frac{1}{1 - r_i^2 + j2\zeta_i r_i} \right]$$

$$\text{where } r_i = \frac{\omega}{\omega_{ni}}$$

Given that

$$\omega_{ni}^2 = \frac{K_i}{M_i} = \frac{K_i}{\rho AL}$$

the final complex frequency response is

$$H_i(\omega) = \frac{X_i(a)}{K_i} \left[\frac{1}{1 - r_i^2 + j2\zeta_i r_i} \right] \quad (19)$$

Case 3 – Base Motion Excitation

Another case to be considered is a cantilever beam excited by base motion. This type of motion is typical for micro cantilever sensors and is shown in Figure 6. The absolute motion of any point on the beam is defined by

$$y(x, t) = z(t) + v(x, t) \quad (20)$$

The differential equation of motion for this case is

$$EI \frac{\partial^4 y}{\partial x^4} + c \frac{\partial y}{\partial t} + \rho A \frac{\partial^2 y}{\partial t^2} = 0 \quad (21)$$

Substitution of Eq. (20) into Eq. (21) gives

$$EI \frac{\partial^4 v}{\partial x^4} + c \frac{\partial v}{\partial t} + \rho A \frac{\partial^2 v}{\partial t^2} = -c\dot{z}(t) - \rho A\ddot{z}(t) \quad (22)$$

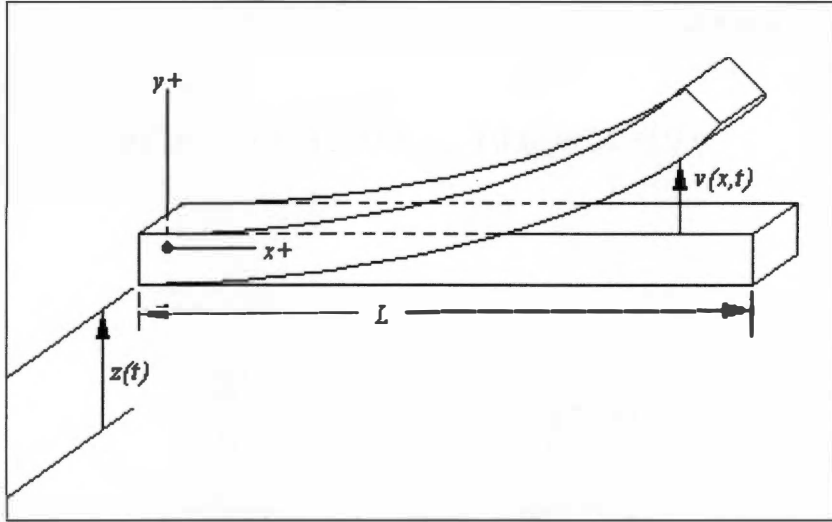


Figure 6: A cantilever beam excited by base motion.

Using the same procedure used in Case 1, we can derive an equation similar to Eq. (15).

$$M_i \ddot{\eta}_i(t) + C_i \dot{\eta}_i(t) + K_i \eta_i(t) = (-\rho A \ddot{z} - c \dot{z}) q_i \quad (23)$$

where

$$M_i = \rho A L$$

$$C_i = c L$$

$$K_i = E I \beta_i^4$$

$$q_i = \int_0^L X_i dx = L p_i$$

where

$$p_i = \frac{2 \sigma_i}{\lambda_i}$$

Eq. (23) can be rewritten as

$$\ddot{\eta}_i(t) + 2\zeta_i \omega_{ni} \dot{\eta}_i(t) + \omega_{ni}^2 \eta_i(t) = (-\ddot{z} - 2\zeta_i \omega_{ni} \dot{z}) p_i \quad (24)$$

where

$$\omega_{ni} = \sqrt{\frac{K_i}{M_i}} = \sqrt{\frac{EIL\beta_i^4}{\rho AL}} = \beta^2 \sqrt{\frac{EI}{\rho A}}$$

$$\zeta_i = \frac{C_i}{2M_i \omega_{ni}} = \frac{cL}{2\rho AL \omega_{ni}} = \frac{c}{2\rho A \omega_{ni}}$$

The equations for base motion can be converted to complex form

$$z(t) = z_0 e^{j\omega t}$$

$$\dot{z}(t) = z_0 j \omega e^{j\omega t}$$

$$\ddot{z}(t) = -z_0 \omega^2 e^{j\omega t}$$

Substituting into Eq. (24) gives

$$\ddot{\eta}_i(t) + 2\zeta_i \omega_{ni} \dot{\eta}_i(t) + \omega_{ni}^2 \eta_i(t) = z_0 e^{j\omega t} [\omega^2 - j2\zeta_i \omega \omega_{ni}] p_i \quad (25)$$

This can be solved setting

$$\bar{\eta}_i = \bar{H}_i e^{j\omega t}$$

which yields

$$\bar{H}_i = z_0 p_i \frac{r_i^2 - j2\zeta_i r_i}{(1 - r_i^2) + j2\zeta_i r_i}$$

Also note that

$$\bar{\eta}_i = H_i e^{j\phi_i} e^{j\omega t}$$

where

$$\frac{H_i}{z_0} = \frac{\sqrt{r_i^4 + (2\zeta_i r_i)^2} p_i}{\sqrt{(1-r_i^2)^2 + (2\zeta_i r_i)^2}} \quad (26)$$

$$\tan \phi_i = \frac{-2\zeta_i r_i}{r_i^2(1-r_i^2)^2 - (2\zeta_i r_i)^2} \quad (27)$$

Since each modal response is small compared to the frequency spacing of each resonant frequency and frequency responses of interest are usually around modal resonance, we can focus on each modal response separately using

$$\begin{aligned} y(x, t) &= X_i(x) \eta_i(t) \\ y(x, t) &= H_i X_i(x) \cos(\omega t + \phi_i) \end{aligned}$$

Note that in this case, the motion of each point in the mode has the same phase lag, ϕ_i , with respect to the base motion.

It should be noted that in all of these cases the modal damping factor is dependent on the natural frequency as shown below.

$$\zeta_i = \frac{c}{2\rho A \omega_{ni}}$$

Since the density, ρ , the cross-sectional area, A , and the damping, c , are all constant, the damping factor decreases as the natural frequency increases. This will be discussed more in later sections.

V. Experimental Modal Analysis

In order to fully understand the response of a cantilever beam to excitation, the properties of the beam, modal mass, modal damping, and modal stiffness (M_i , C_i , K_i) must be known. Modal mass and modal stiffness can be found using the beam's physical properties, but modal damping is typically found experimentally. There is currently no experimental modal analysis equipment available for determining the modal damping on the micro scale. For this reason, an experiment was done on a larger scale with a steel beam with the following properties:

Length: 20 inches

Width: 1.5 inches

Thickness: 0.25 inches

Density: 0.00880 slugs/in³

Modulus of Elasticity: 30,000,000 lbf/in²

Mass of accelerometer: 0.00446 slugs

It can be seen that, although the scale of the larger scale model is not completely consistent with the dimensions of the micro cantilever beam mentioned previously, it is reasonably close. The beam was used in conjunction with a Brüel & Kjær Dual-Channel Signal Analyzer Type 2034. For specific details about the experimental setup please refer to Appendix A. A Brüel & Kjær Accelerometer Type 4339 was affixed to the underside of the beam at the free end. A Brüel & Kjær Impact Hammer Type 8202 was used to impact the beam on the topside at the free end. Both the accelerometer and hammer were connected to the signal analyzer. The signal analyzer then calculated the complex direct transfer function based on approximately ten averages.

The variable $X_i(a)$ in Eq. (19) can be given a value of two since the beam will be struck at the free end. Equation (19) can also be represented by a real part and an

imaginary by multiplying the numerator and denominator by the complex conjugate of the denominator. This results in

$$\text{Re}(H(\omega)) = \frac{2}{K_i} \left[\frac{1 - r_i^2}{(1 - r_i^2)^2 + (2\zeta_i r_i)^2} \right] \quad (28)$$

$$\text{Im}(H(\omega)) = \frac{2}{K_i} \left[\frac{-2\zeta_i r_i}{(1 - r_i^2)^2 + (2\zeta_i r_i)^2} \right] \quad (29)$$

The real part, Eq. (28), and the imaginary part, Eq. (29), can be plotted against frequency (in radians per second or Hertz) or frequency ratio so that all the beam's parameters can be defined. An example of such a plot can be seen in Figure 7. It can easily be shown that when the real part is equal to zero, r_i is equal to one, and ω_{ni} is equal to the frequency at that point. Taking the derivative of Eq. (28) and setting it equal to zero will determine the locations of the maxima and minima along the x-axis. Subtracting the location of the local maximum from the location of the associated minimum will result in a function for ζ_i dependent on the distance between the maximum and minimum.

$$\zeta_i = \frac{\Delta r}{4} \sqrt{4 - (\Delta r)^2} \quad \text{where} \quad \Delta r = \frac{\Delta \omega}{\omega_{ni}} \quad (30)$$

For extremely small values of Δr , the previous equation can be approximated as

$$\zeta_i = \frac{\Delta \omega}{2\omega_{ni}} \quad (31)$$

The largest magnitude of the imaginary part occurs at the natural frequency.

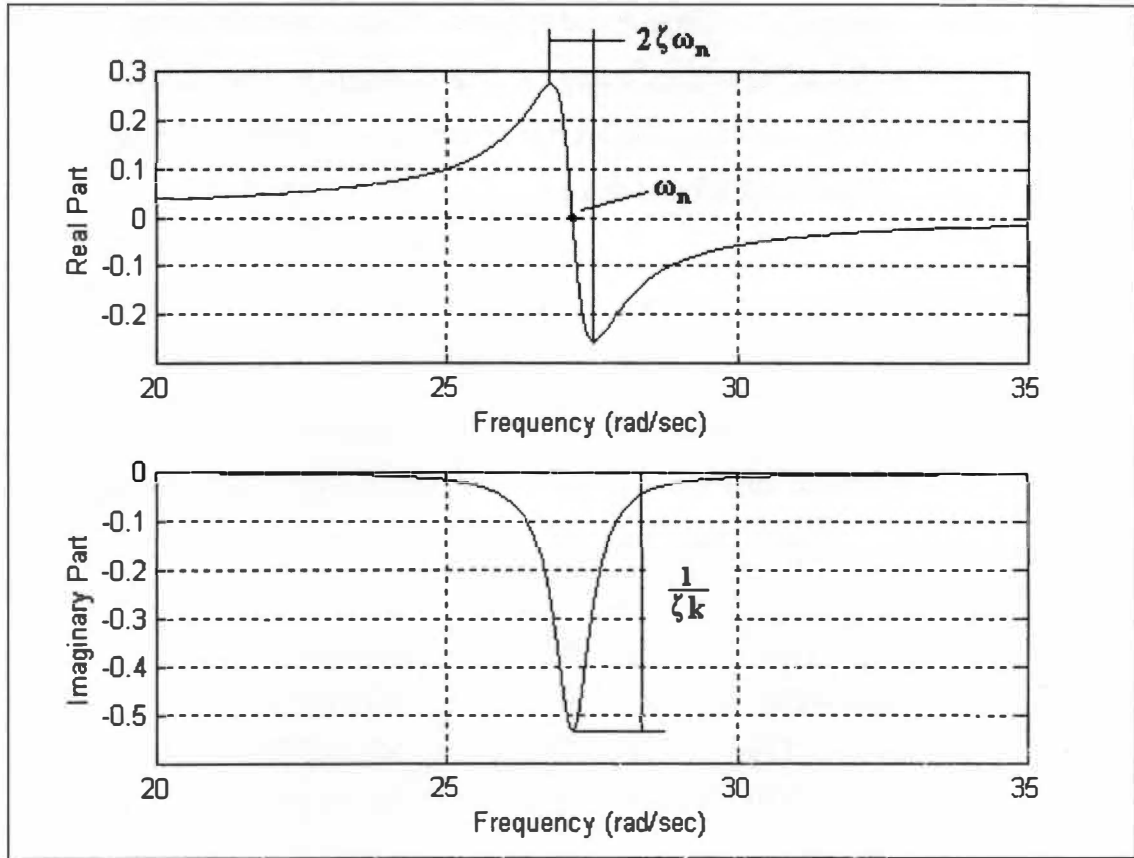


Figure 7; Example of a Single Degree of Freedom Complex Frequency Response plot

So, setting r_i equal to one in Eq. (29) leads to

$$\text{Im}(H(\omega)) = -\frac{1}{\zeta_i K_i} \quad (32)$$

After completing the previous step, all necessary beam parameters are defined.

Once the frequency response for each mode is known, we can verify the values for the modal mass and the modal stiffness. The modal stiffness is found directly from the data using Eq. (32). The modal mass can be found using

$$M_i = \omega_{ni}^2 K_i$$

If the modal mass and modal stiffness found analytically agree with the values for the modal mass and modal stiffness found using the dual-channel spectrum analyzer, then it can be assumed that the values for modal damping factors are accurate. The modal damping coefficient can then be found using

$$C_i = 2M_i\zeta_i\omega_{ni}$$

The measured data for modes 1-4 are included in Appendix A. The beam's parameters, along with the equations for modal mass, modal damping, and modal stiffness, can be used to find, theoretically, that

$\omega_{n1} = 124$	$M_1 = 0.0705$	$K_1 = 1087$
$\omega_{n2} = 778$	$M_2 = 0.0705$	$K_2 = 42673$
$\omega_{n3} = 2179$	$M_3 = 0.0705$	$K_3 = 334560$
$\omega_{n4} = 4270$	$M_4 = 0.0705$	$K_4 = 1284721$

where natural frequency is in units of radians/sec, mass is in units of slugs, and stiffness is in units of lbf/ft. It should be noted, as stated earlier, that the damping factor is inversely proportional to the natural frequency. This is shown below.

$$\zeta_i\omega_{ni} = \zeta_1\omega_{n1}$$

$$\zeta_i = \zeta_1\left(\frac{\omega_{n1}}{\omega_{ni}}\right)$$

So, given that the first damping factor is approximately 0.001691, then the other damping factors, and damping coefficients can be determined theoretically as

$$\begin{array}{ll}
\zeta_1 = 0.00169127 & C_1 = 0.02959 \\
\zeta_2 = 0.00026987 & C_2 = 0.02959 \\
\zeta_3 = 0.00009638 & C_3 = 0.02959 \\
\zeta_4 = 0.00004918 & C_4 = 0.02959
\end{array}$$

where the damping factor is unitless and the damping coefficient is in slugs/sec (lbf•sec/ft). Using the data acquired from the frequency response results in

$$\begin{array}{lll}
\omega_{n1} = 117 & M_1 = 0.0741 & K_1 = 1015 \\
\omega_{n2} = 737 & M_2 = 0.0760 & K_2 = 41244 \\
\omega_{n3} = 2130 & M_3 = 0.0788 & K_3 = 357483 \\
\omega_{n4} = 4166 & M_4 = 0.0915 & K_4 = 1588478
\end{array}$$

$$\begin{array}{ll}
\zeta_1 = 0.001691 & C_1 = 0.02935 \\
\zeta_2 = 0.002401 & C_2 = 0.26882 \\
\zeta_3 = 0.037584 & C_3 = 12.61563 \\
\zeta_4 = 0.004902 & C_4 = 3.73837
\end{array}$$

where natural frequency is in units of radians/sec, mass is in units of slugs, stiffness is in units of lbf/ft, the damping factor is unitless and the damping coefficient is in slugs/sec (lbf•sec/ft).

As it can be seen from the results comparing natural frequency, mass, and stiffness, the data is fairly accurate. The damping factor for the third mode seems to be off somewhat compared to the damping factors for the other modes. This is most likely due to noise around the third natural frequency while testing. This can also be seen by the “messiness” of the third mode plot in Figure A.4 compared to the other modal frequency response plots in Figures A.2, A.3, and A.5. Note, however, that theoretically it was found that the damping factors would decrease with higher modes, yet the opposite is what was found experimentally. For this study, damping was assumed to be environmental, which does decrease as the natural frequency. The opposite is true for

structural damping [6]. Obviously, for our experimental setup, the damping is not entirely environmental and, therefore, linear damping may not be accurate. The relative errors for natural frequency, mass, and stiffness are shown below.

<i>frequency</i>	<i>mass</i>	<i>stiffness</i>
$err_1 = 5.77\%$	$err_1 = 5.24\%$	$err_1 = 6.55\%$
$err_2 = 5.34\%$	$err_2 = 7.87\%$	$err_2 = 3.35\%$
$err_3 = 2.26\%$	$err_3 = 11.84\%$	$err_3 = 6.85\%$
$err_4 = 2.45\%$	$err_4 = 29.92\%$	$err_4 = 23.64\%$

The errors for all modes seem to be increasing as frequency increases with higher modes. Mode 4 has a higher error than the other modes. Also, mode 3 seems to have a high amount of noise, although, the error associated with mode 3 is comparable to the errors for modes 1-2 for frequency, mass, and stiffness.

Often a continuous system can be modeled as a multiple degree of freedom by discretizing it. An example of this cantilever beam discretized into four equally spaced lumped points is shown in Appendix B.

VI. Discussion of Results

Since all the necessary data has been acquired from the large-scale model, response predictions can be made for the three different excitation cases mentioned previously. Using the data acquired from the experimental setup, we can assume the same damping factors for the micro cantilever mentioned previously. Recall that the properties of the micro cantilever were given as

Length: 120 μm

Width: 30 μm

Thickness: 2 μm

Density: 1000 kg/m^3

Modulus of Elasticity: 179 GPa

This would result in modal masses, modal stiffnesses, and natural frequencies as follows

$M_1 = 7.2 \times 10^{-12}$	$K_1 = 25.612$	$\omega_{n1} = 300.17$
$M_2 = 7.2 \times 10^{-12}$	$K_2 = 1005.878$	$\omega_{n2} = 1881.16$
$M_3 = 7.2 \times 10^{-12}$	$K_3 = 7886.247$	$\omega_{n3} = 5267.31$
$M_4 = 7.2 \times 10^{-12}$	$K_4 = 30283.471$	$\omega_{n4} = 10321.83$

where modal mass is in kilograms, modal stiffness is in N/m, and natural frequency is in kHz. Shown below are the damping factors used and the corresponding modal damping coefficients.

$\zeta_1 = 0.001691$	$C_1 = 45.934 \times 10^{-9}$
$\zeta_2 = 0.002401$	$C_2 = 408.633 \times 10^{-9}$
$\zeta_3 = 0.037584$	$C_3 = 17911.607 \times 10^{-9}$
$\zeta_4 = 0.004902$	$C_4 = 4577.872 \times 10^{-9}$

where the damping factor is unitless and the damping coefficient is in units of kg/sec (N•sec/m). Obviously, the damping coefficient for the third mode is high. This could be because of frequency noise during experimentation occurring close to the third natural frequency.

Case 1 – Distributed Load Excitation

For the first type of excitation considered, a distributed load along the length of the beam, recall that the basic differential equation of motion shown as Eq. (15) is

$$M_i \ddot{\eta}_i(t) + C_i \dot{\eta}_i(t) + K_i \eta_i(t) = F_i(t)$$

Consider a uniformly distributed load. For this case, since the distributed load varies only with time, not along the length of the beam then the modal force is simply the integral of the mode shape multiplied by the load. This is represented by

$$w(x, t) = W_0 \cos(\omega t)$$

$$F_i = \int_0^L X_i(x) w(x, t) dx = W_0 \cos(\omega t) \int_0^L X_i(x) dx = 2\sigma_i \frac{L}{\lambda_i} W_0 \cos(\omega t)$$

The resulting equation of motion is

$$\ddot{\eta}_i(t) + 2\zeta_i \omega_{ni} \dot{\eta}_i(t) + \omega_{ni}^2 \eta_i(t) = \frac{2W_0 \sigma_i}{\rho A \lambda_i} \cos(\omega t)$$

and the total response of the beam is given by Eq. (12)

$$y_i(x, t) = \sum_{i=1}^{\infty} X_i(x) \eta_i(t)$$

The maximum amplitudes for underdamped, forced motion for a given input frequency (as shown above) can be represented by [5]

$$\eta_{i,\max}(\omega) = \left(\frac{2W_0\sigma_i}{\rho A \lambda_i} \right) * \frac{1}{\sqrt{(\omega_{ni}^2 - \omega^2)^2 - (2\zeta_i \omega_{ni} \omega)^2}} \quad (33)$$

To find the maximum amplitude along the length of the beam for a given input frequency, simply substitute Eq. (33) into Eq. (12), which results in

$$y_i(x, \omega) = X_i(x) \eta_{i,\max}(\omega)$$

Of course, when the driving frequency is equal to one of the natural frequencies, that particular mode will be excited a great deal more than it would be for other driving frequencies. The total maximum amplitudes for each mode driven at the four different natural frequencies are shown in Figures 8-11. Noting the different scales of the y-axes in the following figures reveals that the mode whose natural frequency coincides with the driving frequency is greatly amplified compared to the other modes.

Case 2 – Unit Impulse Excitation

For the second type of excitation, the main difference will be on the right hand side of Eq. (15). For this case, the force applied is considered a unit impulse. What becomes very important for this case is the point along the beam's length at which the impulse is applied. At the free end, all modes are excited and theoretically have the same maximum relative amplitude at the free end as shown in Figure 2. However, if the impulse is at a point along the beam's length that coincides with a node or anti-node of a mode shape, then the relative maximum amplitude of that mode could vary greatly.

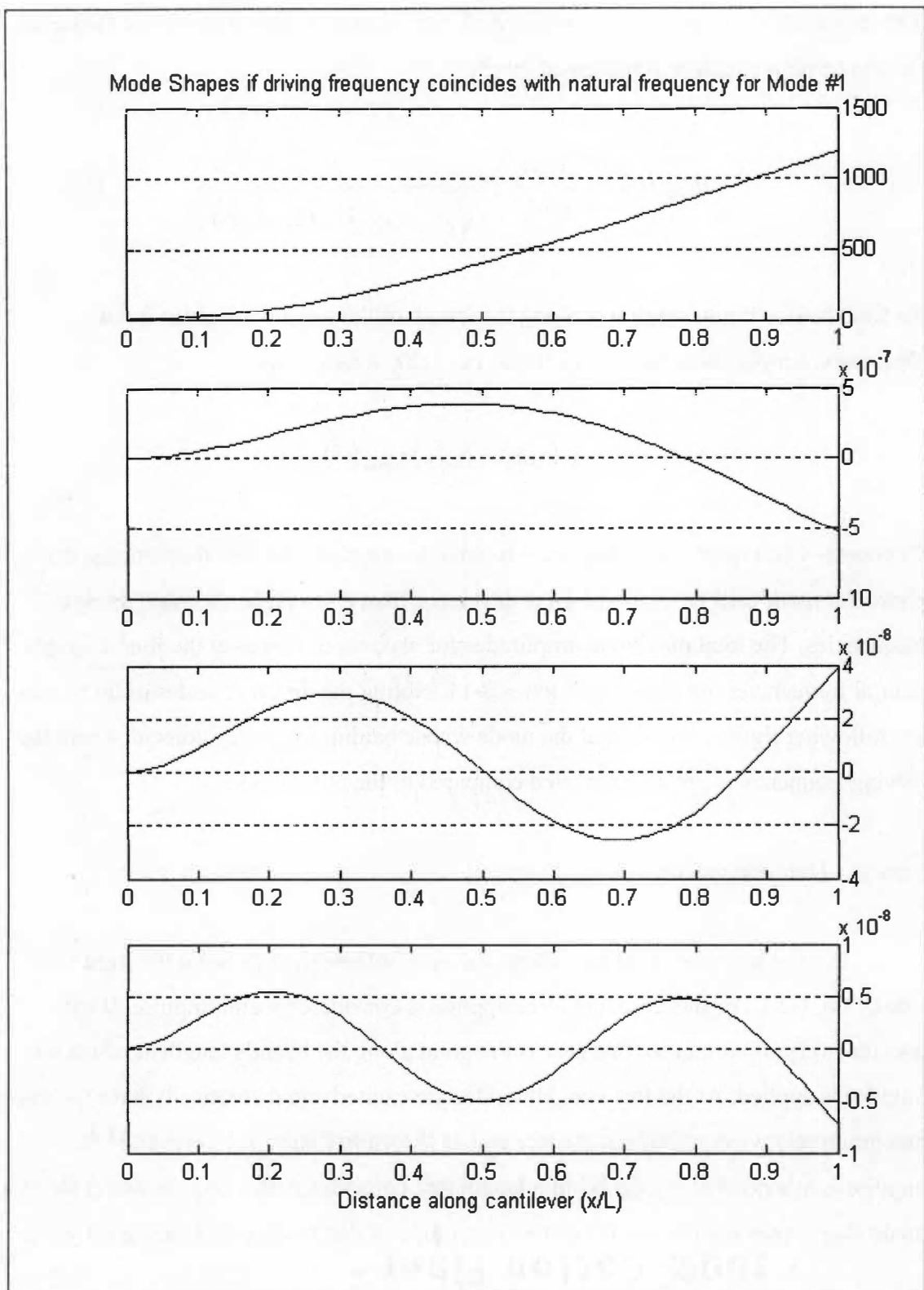


Figure 8: Relative magnitudes of mode shapes if the driving frequency is equal to the first natural frequency.

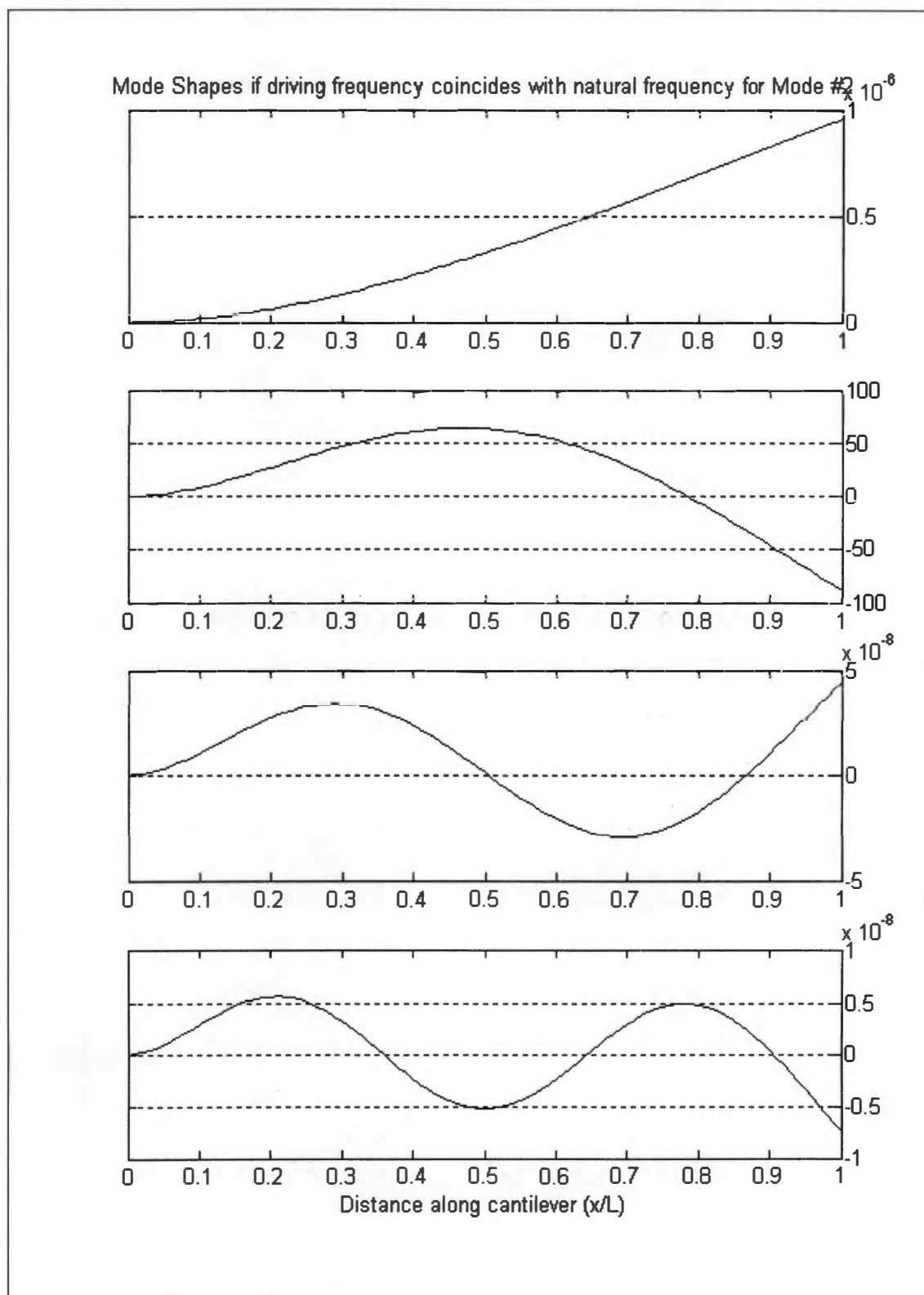


Figure 9: Relative magnitudes of mode shapes if the driving frequency is equal to the second natural frequency.

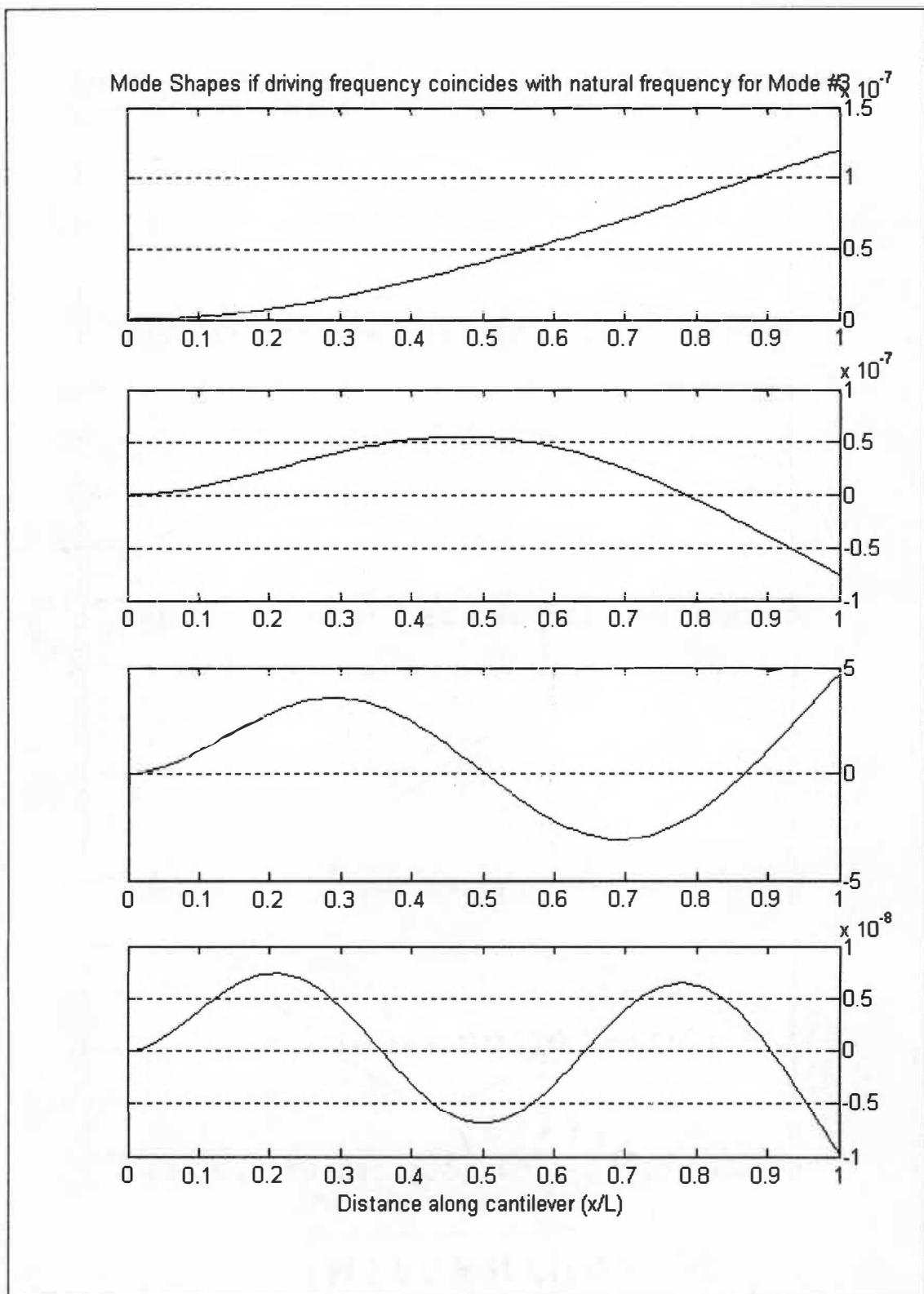


Figure 10: Relative magnitudes of mode shapes if the driving frequency is equal to the third natural frequency.

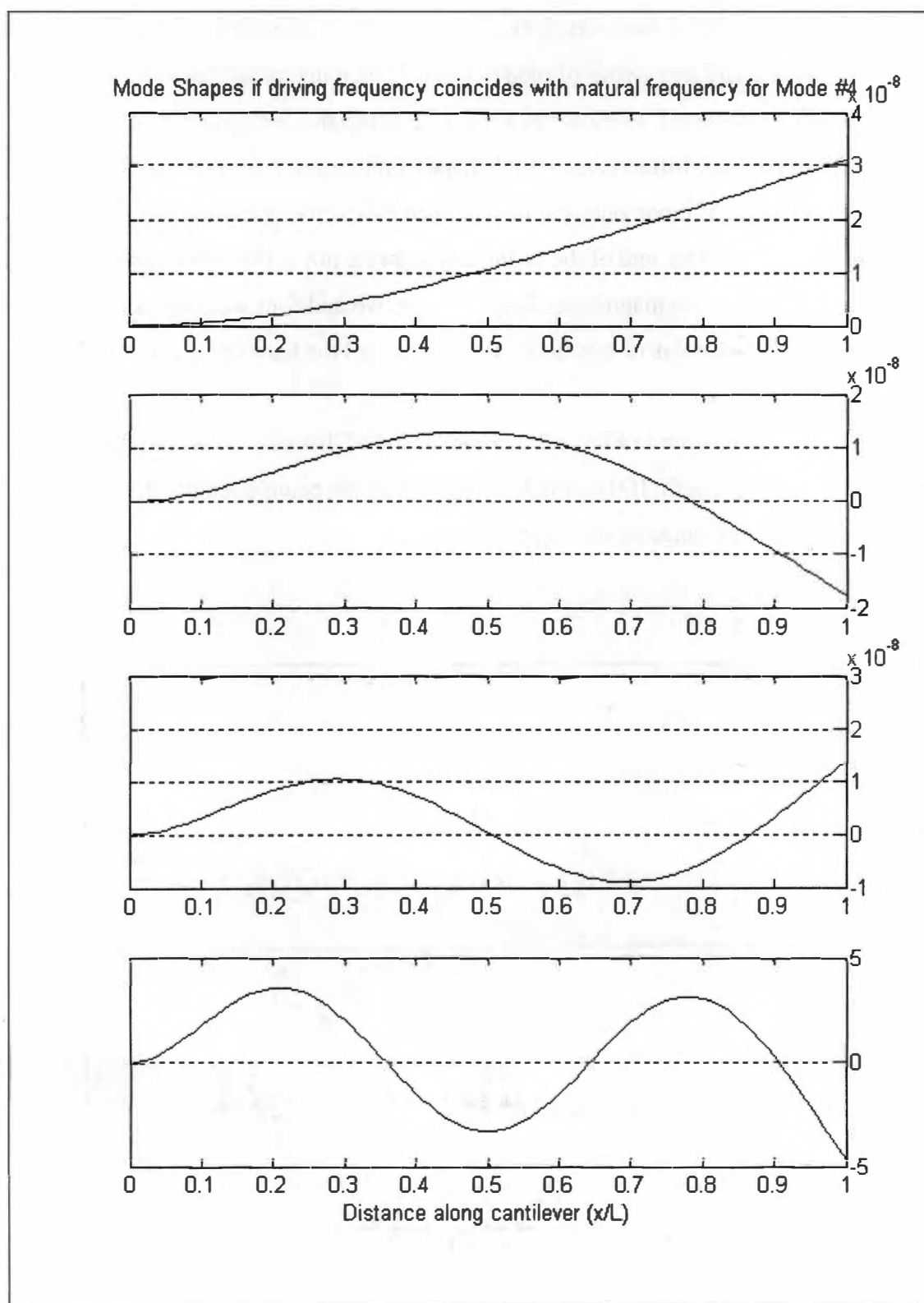


Figure 11: Relative magnitudes of mode shapes if the driving frequency is equal to the fourth natural frequency.

For example, if the beam were struck at 50% of the beam's length (which is very close to a node of mode 3 and anti-nodes of modes 2 and 4) then the magnitudes of the mode shapes would be different, as shown in Figure 12. From this we can see that the third mode contributes very little to the overall shape of the beam. Also note the first mode, since the only node of mode one is at the clamped end of the beam and the only anti-node of mode one is at the free end of the beam, any impulse not at the beam's free end will be a decrease in the relative magnitude. Since, modes two and four were excited near their anti-nodes, then their relative magnitudes are decreased the least among the four modes.

It can also be seen in Figure 2 that modes 2-4 all have nodes within the vicinity of 75% of the beam's length. If the impulse is at 75% of the beam's length, the amplitudes of the mode shapes change as shown in Figure 13.

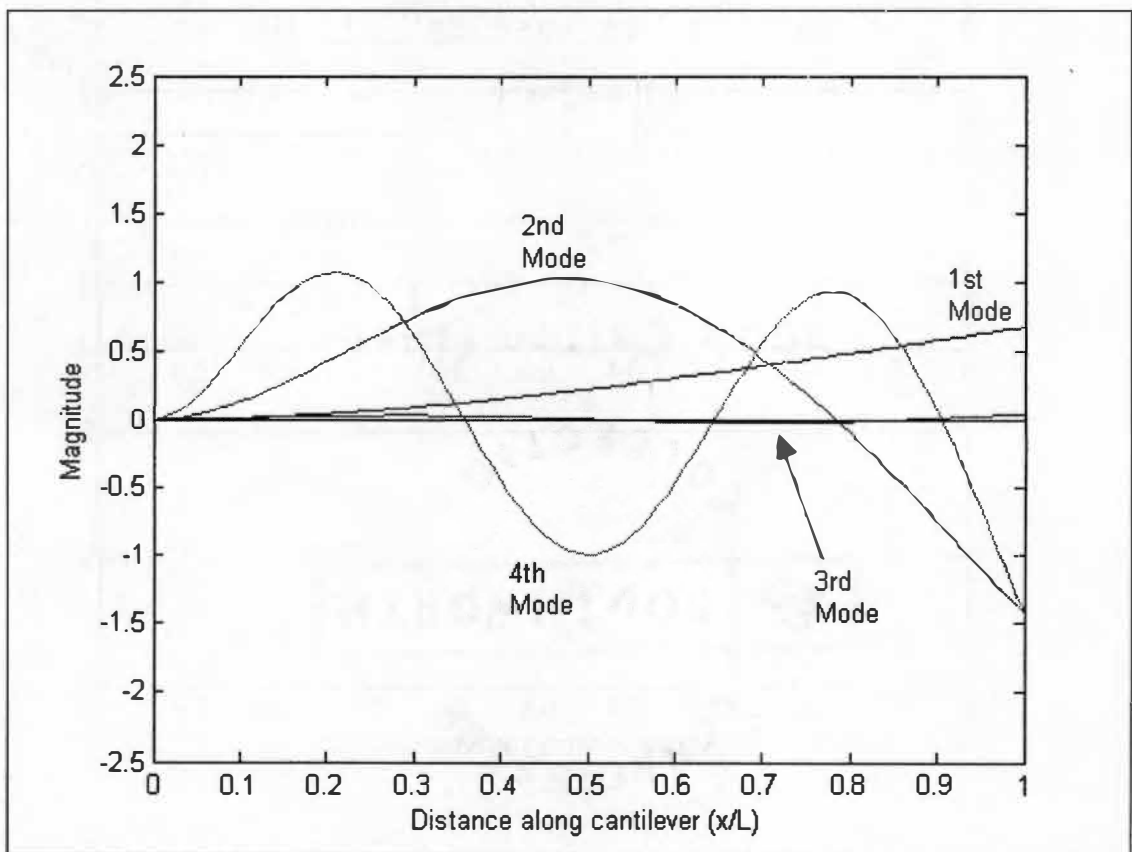


Figure 12: Relative magnitudes of mode shapes if an impulse is applied at $x = 0.5L$ along the beam.

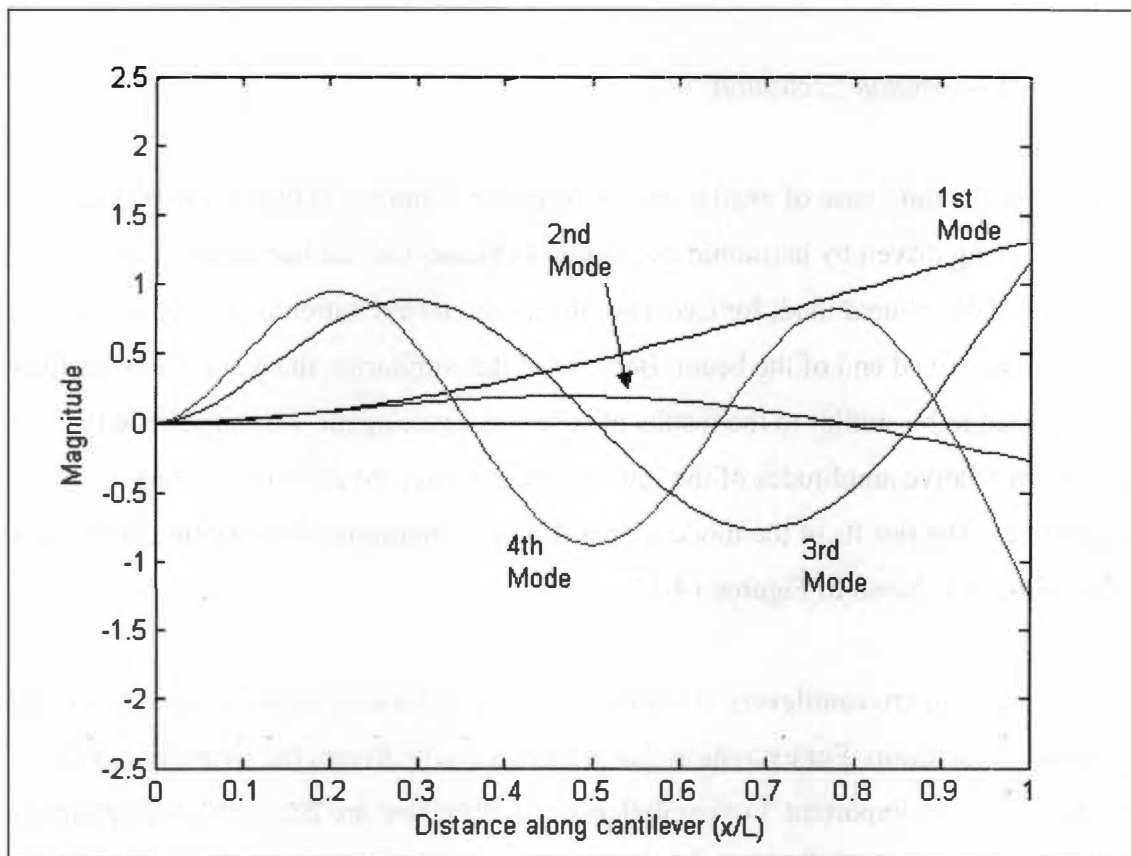


Figure 13: Relative magnitudes of mode shapes if an impulse is applied at $x = 0.75L$ along the beam.

From Figure 13, it can be seen that from an excitation at 75% along the beam's length, the second mode is decreased a great amount compared to the other modes. While the other three modes are decreased, they are all decreased by approximately the same amount. So, if the beam were excited at a point 75% along its length, then the overall response would be similar, although diminished, except for the second mode, which would be almost negligible.

From these two examples, it is determined that the location of the impulse along the beam's length has a direct effect on the relative amplitude of the modes. If the impulse is closer to, or directly on, an anti-node of the mode, then the amplitude of that mode will not vary much from the when the beam were struck at the free end. However, if the impulse is closer to, or directly on a node of the mode, then the amplitude will decrease proportional to how close the impact is to the node.

Case 3 – Base Motion Excitation

For the third case of excitation, the response is similar to that of the first case. Both are being driven by harmonic excitation. For case one, the harmonic excitation is applied to a distributed load; for case two, the harmonic excitation is applied to the base motion of the fixed end of the beam. Because of this similarity, the results for case three are expected to be similar to the results of case one. Once again, we can refer to the maximum relative amplitudes of the four different modes for different driving frequencies. The results of the mode shapes driven by harmonic base motion at different frequencies are shown in Figures 14-17.

For all micro cantilevers, the higher modes can become significant to the overall response of the beam. For excitation that is harmonically driven, the frequency of the excitation is very important. For excitation frequencies that are near natural frequencies of the higher modes, the first mode is still the basic shape of the response, but the higher modes can no longer be ignored. For impact excitation, the location of the impact along the beam's length had a direct effect on the amount each mode is decreased from the basic mode shape when the beam was struck at the free end. If micro cantilevers were used in the study of Brownian motion, in which the location of impacts along the beam cannot be predicted, the higher modes cannot be ignored.

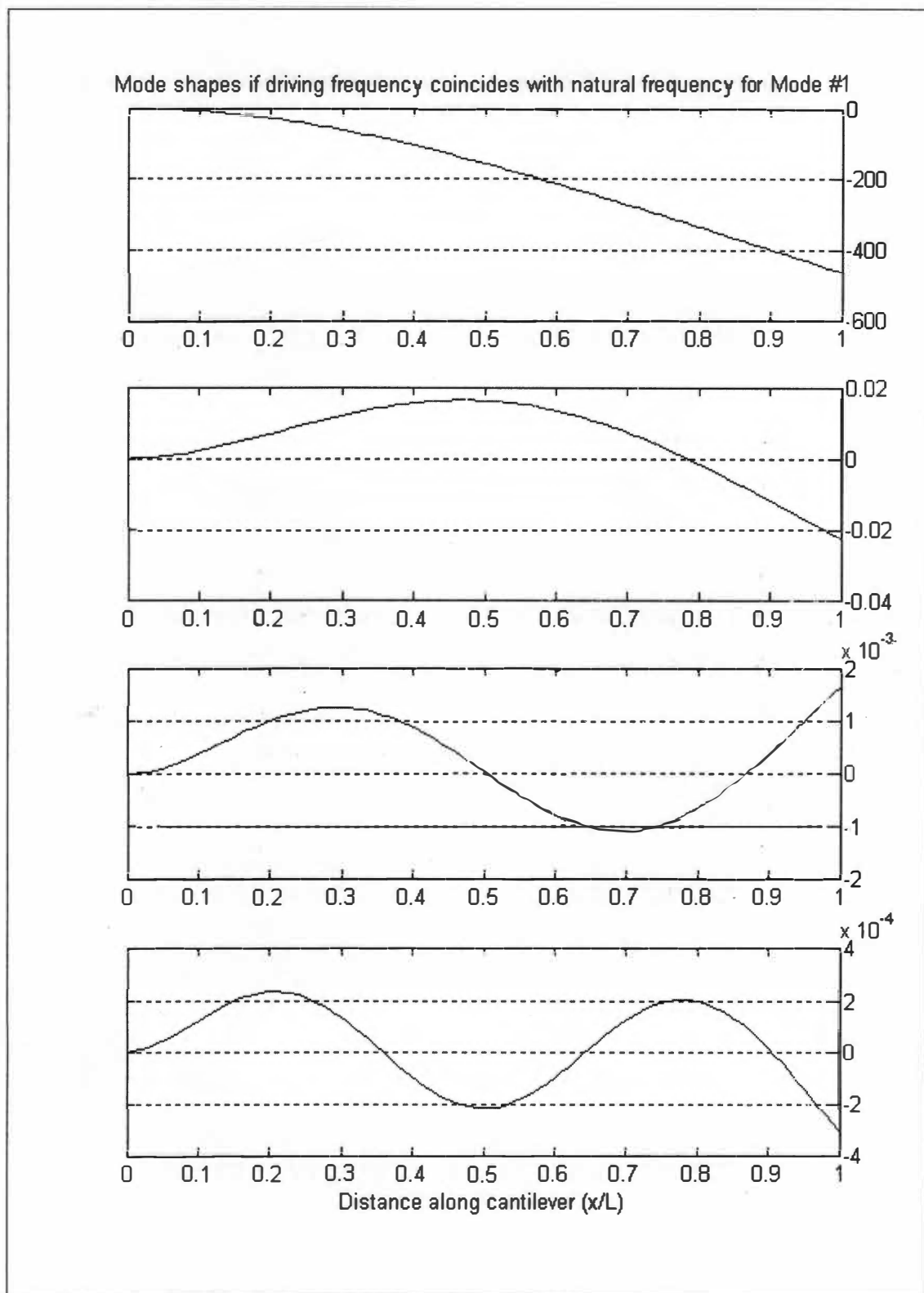


Figure 14: Relative magnitudes of mode shapes if the driving frequency of the base is equal to the first natural frequency.

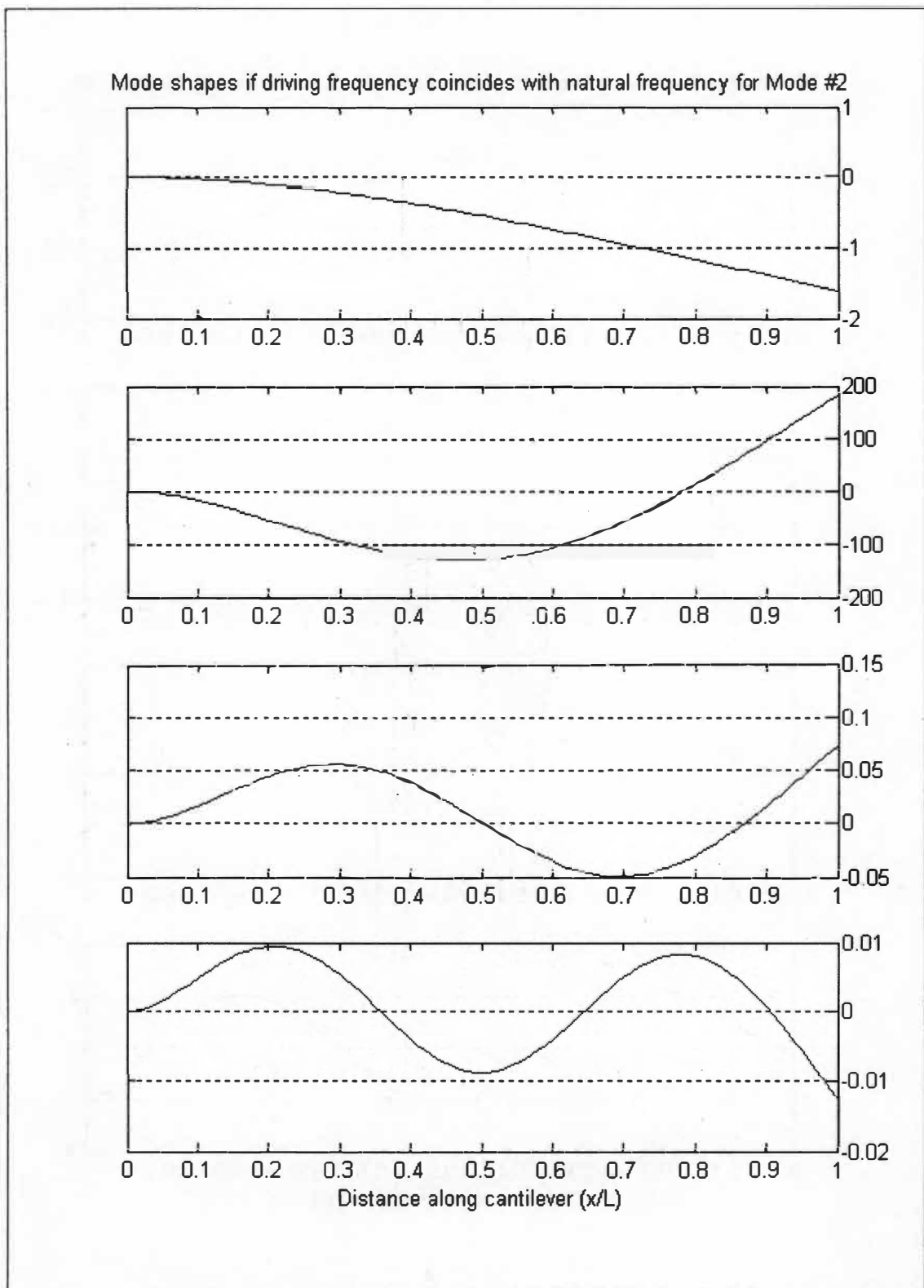


Figure 15: Relative magnitudes of mode shapes if the driving frequency of the base is equal to the second natural frequency.

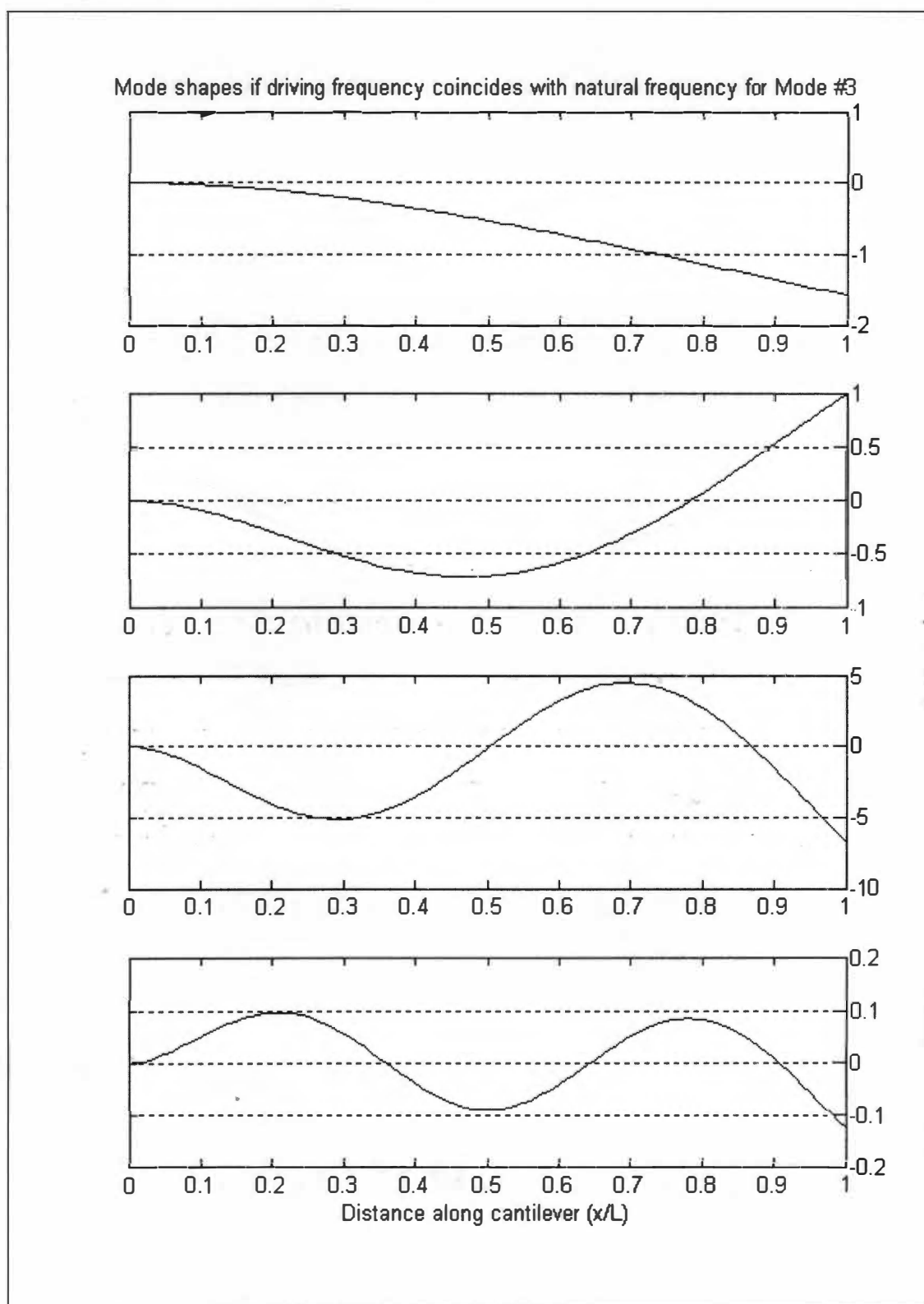


Figure 16: Relative magnitudes of mode shapes if the driving frequency of the base is equal to the third natural frequency.

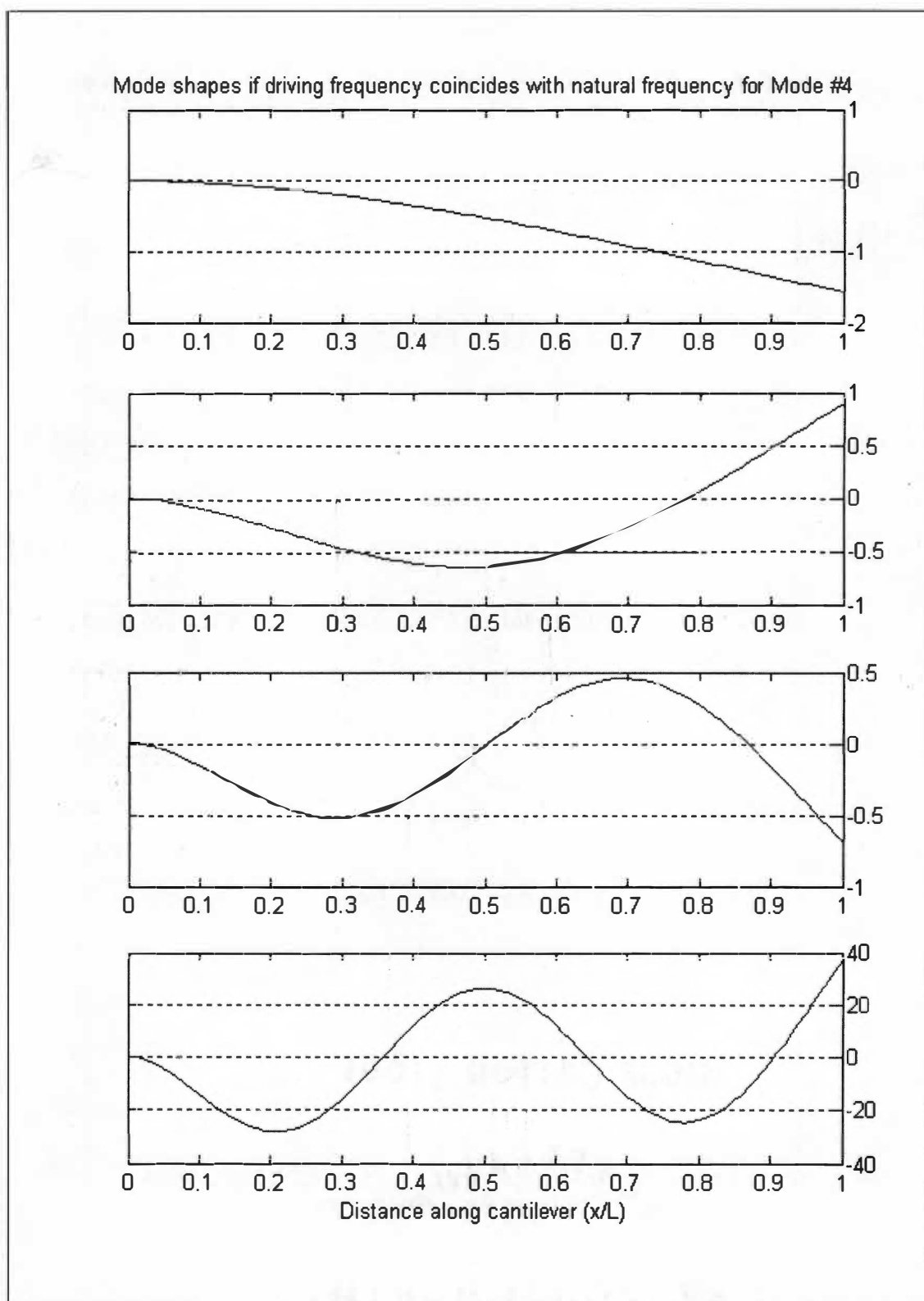


Figure 17: Relative magnitudes of mode shapes if the driving frequency of the base is equal to the fourth natural frequency.

VII. Conclusions

As was shown previously, simple vibration beam theory can be used to moderately estimate the environmental damping characteristics of a beam while confirming the mass and stiffness properties already calculated for the beam. This is the case for beams ranging in size from a few meters long to a few microns long. Regardless of size, the principles used to derive the mode shapes and equations of motion for a cantilever beam remain the same. However, one must keep in mind that this theory is only accurate for the first few modes. This is because the equation modeling the transverse vibration of a beam, Eq. (11), ignores the effects of shear deformation and rotary inertia. For our study, only the first four modes were considered. When the beam was struck at the free end in the experimental setup, the imaginary peaks of the first four modes give a good idea of the overall contribution of each mode to the total response. While the fourth mode has an imaginary peak of a mere -8.8×10^{-6} m/N, the first mode has an imaginary peak of -39.9×10^{-3} m/N. From this, it is shown that the first mode contributes the most to the overall response, the second mode contributes less than the first mode to the overall response, the third mode contributes less than the second mode to the overall response, and so on. While this is the general case, there are many cases where the higher modes simply cannot be ignored. As discussed early, under certain conditions of excitation, some of the modes may be greatly amplified while others are almost nonexistent.

For this study, a reasonably sized cantilever beam was experimentally used to validate the equations determined from the theoretical analysis. It was found that the modal mass and modal stiffness calculated analytically were comparatively accurate to the modal mass and modal stiffness determined experimentally. It should be noted that the error seemed to increase with each higher mode. This is most likely due to the experimental setup not being completely isolated from any noise from the outside environment. Different experimental locations and setups were tried to minimize any noise. Another factor that could have contributed to the increasing error is the limitations

of the equipment used. For the final results, reasonable data was achieved, albeit, with some noise.

As stated before, this study was prompted by a need to better understand the forced vibration of micro cantilever systems. One often finds cantilevers modeled as single degree of freedom systems due to the very minor additions the higher modes usually make to the total response. However, in many cases this is not adequate. The higher modes become important under certain common excitations and cannot be ignored. In order for the assumed behavior of a micro cantilever to be accurate, all of the beam's modal components must be considered.

There are many applications of technology today that use micro cantilever systems or could benefit from using micro cantilever systems. Whether the use of a micro cantilever is for scanning surfaces on the atomic level or arrays of micro cantilevers used to extract power from being excitation by Brownian motion, the understanding of the forced vibration of these micro cantilevers is paramount. The response of each mode and its contribution to the total response of a beam must be fully understood in order to use them to their full advantage. If the behavior of the modal components of a micro cantilever beam can be accurately predicted from a number of different common excitations, then it is possible to use these systems to their maximum potential.

There are many aspects of this study that can be examined further for any future studies. In this thesis, an impulse at an arbitrary distance along the length of the beam was studied, ignoring any variation along the width of the beam. One consideration to examine is the effect of an impulse at some arbitrary distance along the length *and* width of the beam. Theory of vibrating plates could be used to determine complex mode shapes and how they vary across the surface of the beam. Another issue to consider is the effect of the impact of many particles undergoing Brownian motion. As a result of the randomness of Brownian motion, many particles together will, in general, produce a driving mechanism having a flat frequency spectrum (i.e. white noise). Also, the damping

was modeled as solely environmental. Another study might include an improved model of damping to include other types of damping. Experimentally, for this study, the data acquired was acceptable, but was affected by different magnitudes of error for different modes. Future studies could include more experimental data using different equipment and setups in order to reduce the error included in higher modes.

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Appendices

Appendix A

The setup of this experiment using a Brüel & Kjær Dual-Channel Signal Analyzer Type 2034 for this experiment is described below. The Brüel & Kjær Impact Hammer Type 8202 was attached to input channel A, and the Brüel & Kjær Accelerometer Type 4339 was attached to input channel B. A picture of the impact hammer and accelerometer is included in Figure A.1. The maximum voltages for channels A and B were set to 800 millivolts and 8 volts, respectively. For both channels, both filters were on. For channel A, the calibration units were set to 4.00 millivolts per Newton. For channel B, the calibration units were set to 9.81 millivolts per unit gravity. The plot was set to show either the real or imaginary part of the frequency response integrated twice for position since acceleration was the input. The averaging was set to take ten linear averages. Each data acquisition cycle was set to trigger on when channel A was +0.1% of the maximum value. The weighting for channel A was set to a transient weighting factor with a length of approximately 100 milliseconds with no shift. The weighting for channel B was set to a rectangular weighting factor. For each particular mode, a different frequency span and zoom were set.

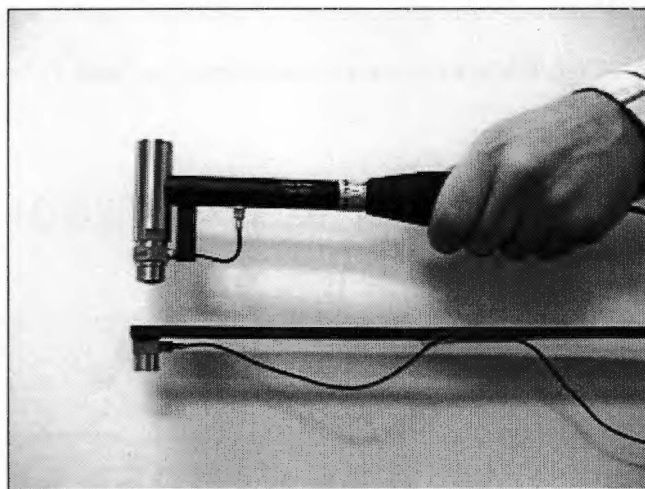


Figure A.1: A photo taken of the impact hammer and accelerometer used.

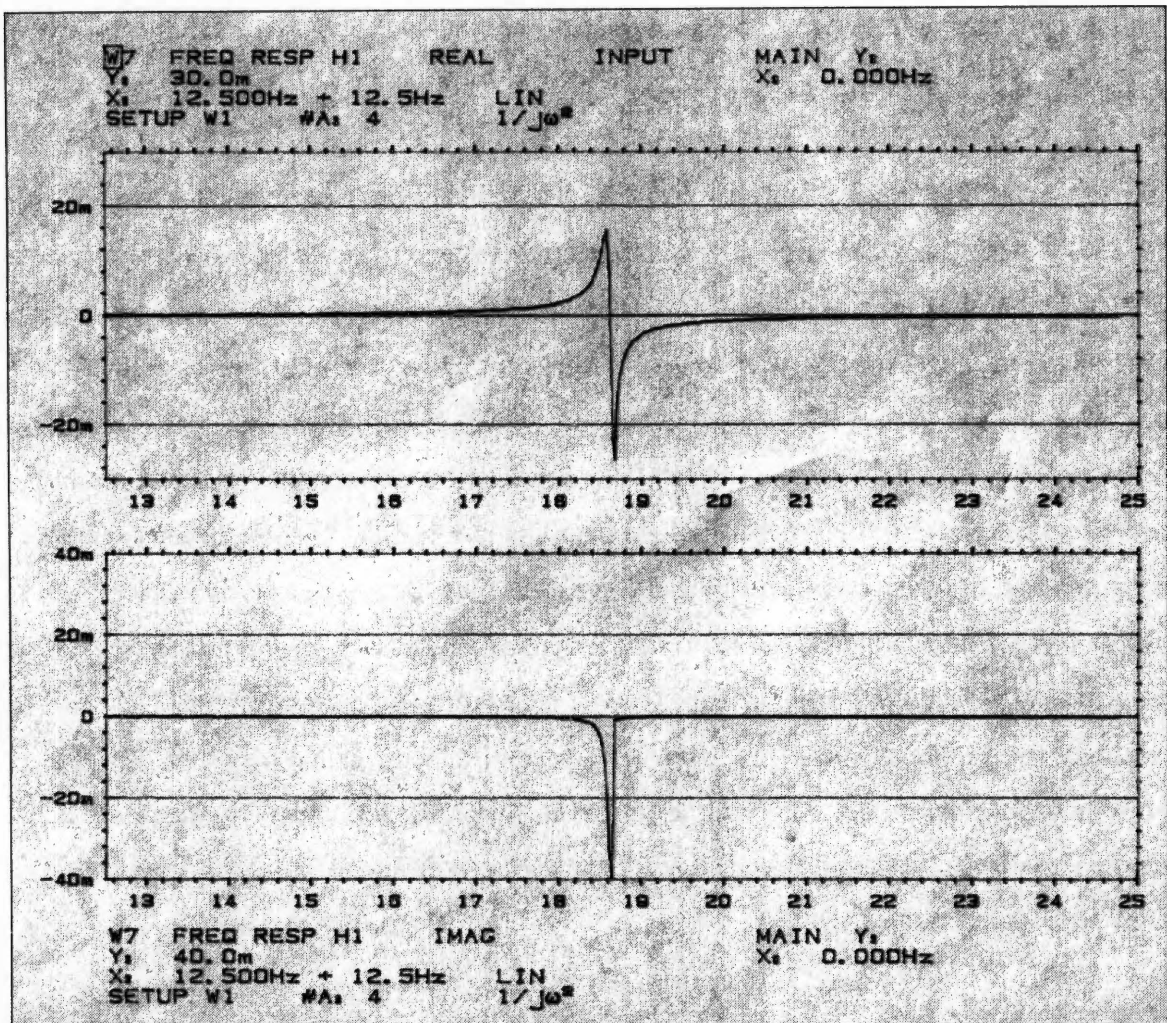


Figure A.2: Frequency response data for mode 1

Real part crosses x-axis at: 18.625 Hz

Imaginary peak at natural frequency is: -0.0399 m/N

Location of local maximum of real part is: 18.593 Hz

Location of local minimum of real part is: 18.656 Hz

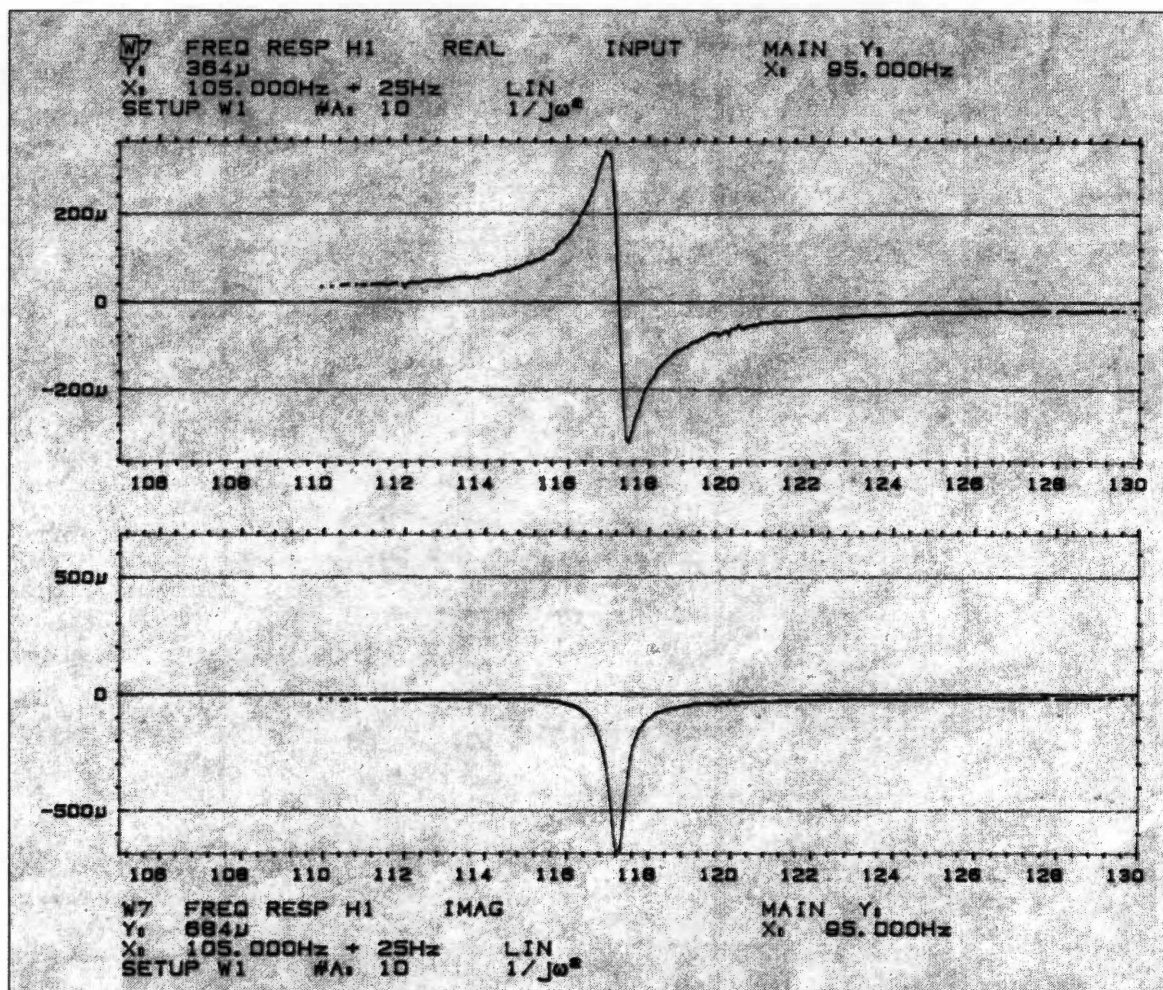


Figure A.3: Frequency response data for mode 2

Real part crosses x-axis at: 117.25 Hz

Imaginary peak at natural frequency is: -0.000692 m/N

Location of local maximum of real part is: 116.937 Hz

Location of local minimum of real part is: 117.500 Hz

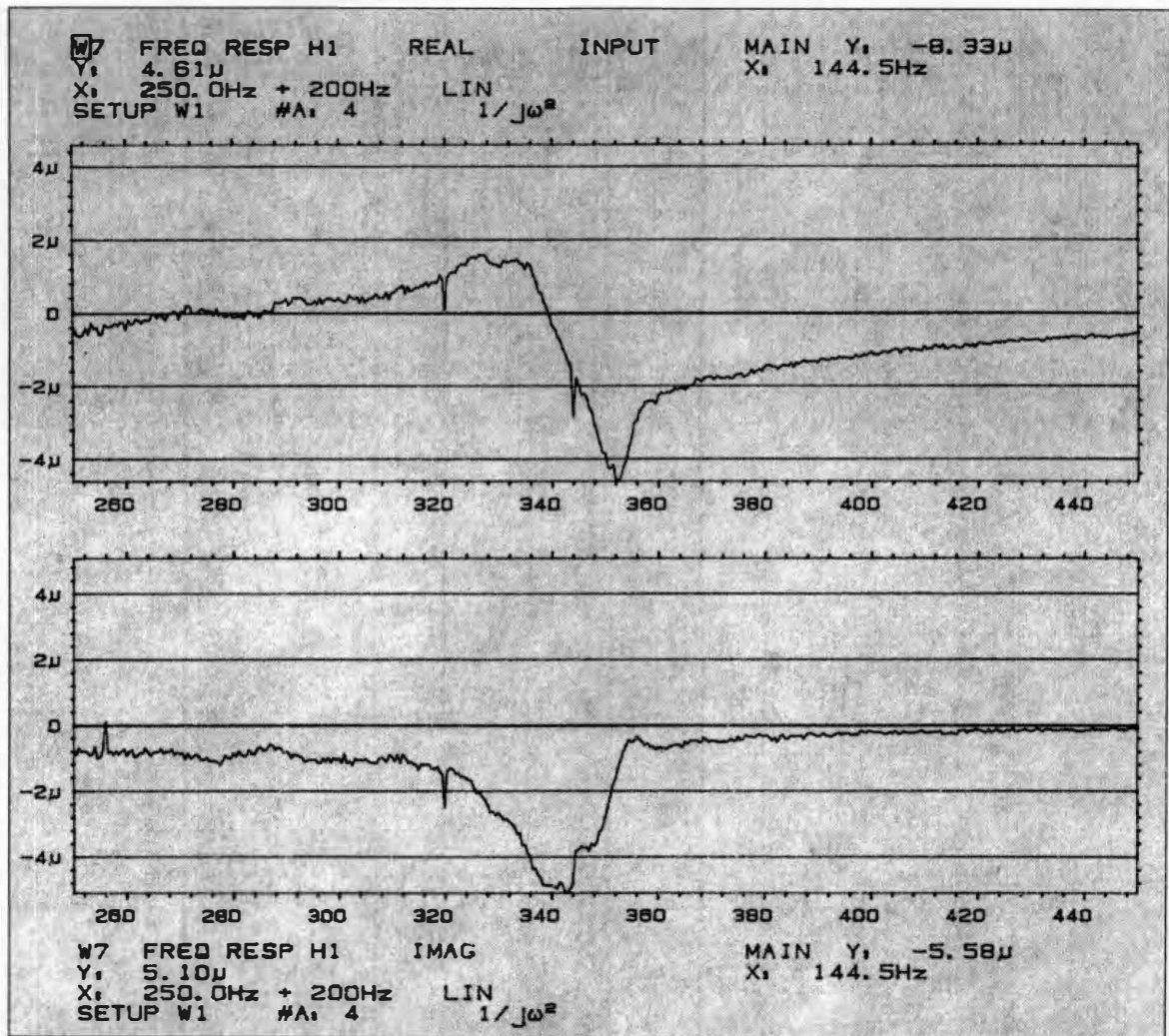


Figure A.4: Frequency response data for mode 3

Real part crosses x-axis at: 339 Hz

Imaginary peak at natural frequency is: -0.0000051 m/N

Location of local maximum of real part is: 327.0 Hz

Location of local minimum of real part is: 352.5 Hz

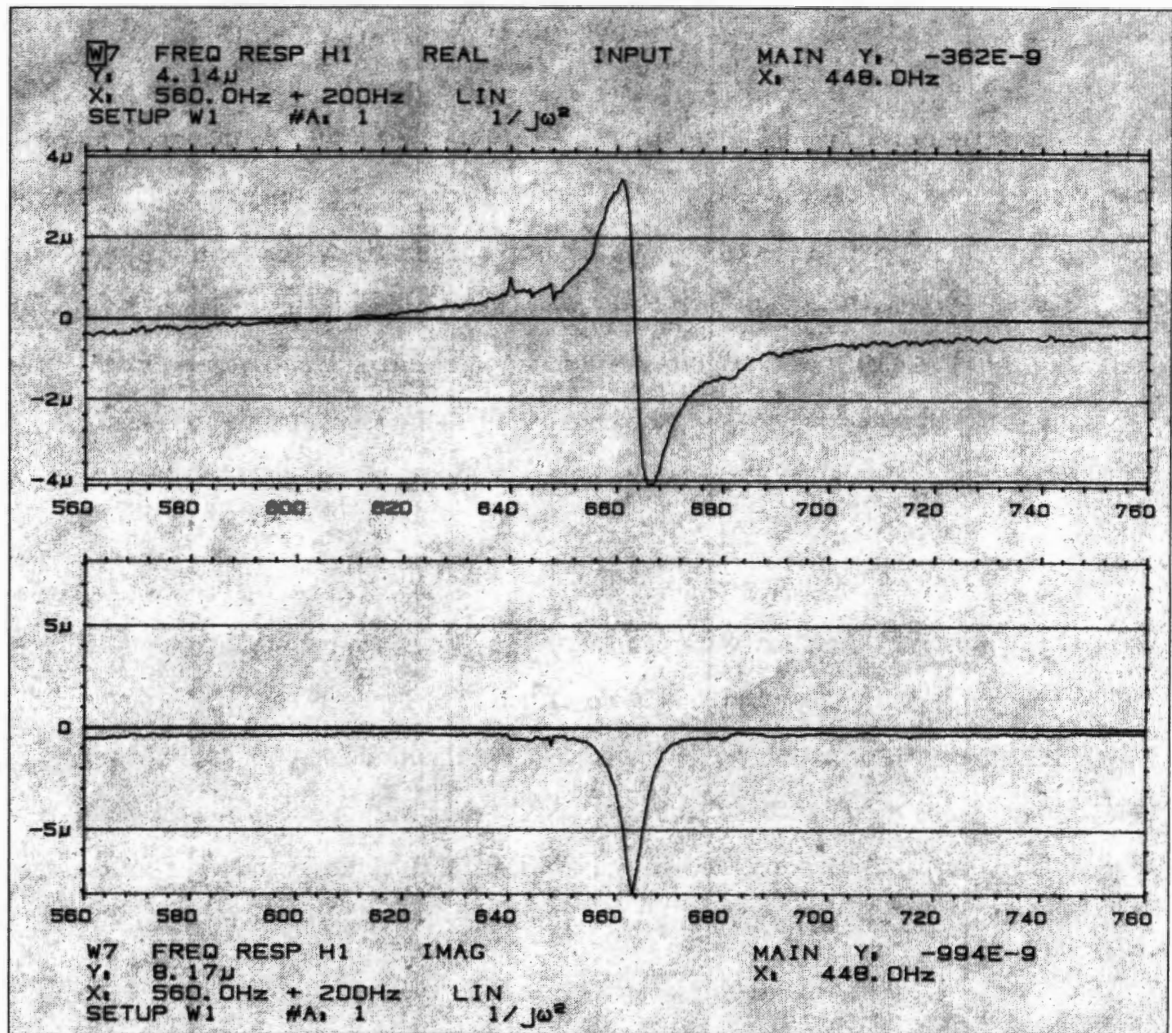


Figure A.5: Frequency response data for mode 4

Real part crosses x-axis at: 663 Hz

Imaginary peak at natural frequency is: -0.0000088 m/N

Location of local maximum of real part is: 660.5 Hz

Location of local minimum of real part is: 667 Hz

Appendix B

Sometimes it is beneficial to discretize a continuous system so that can be more easily understood. This is especially true when the structure is geometrically complex. As a supplemental exercise to the experimental portion of this study, the cantilever was discretized using the displacement of four particular points in the beam as shown in Figure A.6. In this case, the continuous system is represented by a four-degree of freedom system. The basic problem here is to establish the local mass, stiffness, and damping matrices so that the beam can be represented in terms of discrete masses, springs, and dashpots.

The mass, stiffness, and damping for all the modes can be found using the direct transfer function (frequency response) as described in this thesis. The cross transfer function gives the information needed in order to get the mode shapes, and in essence, the modal matrix. The modal matrix can be derived for the beam by recording the imaginary peaks of the frequency response and hitting the beam at different points. The frequency response for a system is the summation of the frequency of each mode due to a force as shown below.

$$H(\omega) = \frac{X_1(a)}{K_1} \left[\frac{1}{1 - r_1^2 + j2\zeta_1 r_1} \right] + \frac{X_2(a)}{K_2} \left[\frac{1}{1 - r_2^2 + j2\zeta_2 r_2} \right] + \dots$$

Now, the equation above can be rewritten as a series of complex responses multiplied by a scaling factor.

$$H_j(\omega) = \Phi_{1j} \left[\frac{1}{1 - r_1^2 + j2\zeta_1 r_1} \right] + \Phi_{2j} \left[\frac{1}{1 - r_2^2 + j2\zeta_2 r_2} \right] + \dots$$

A subscript k has been added to indicate that this is the response of due to a force at the k th location. For a discrete system, it is possible to obtain a set of k frequency responses

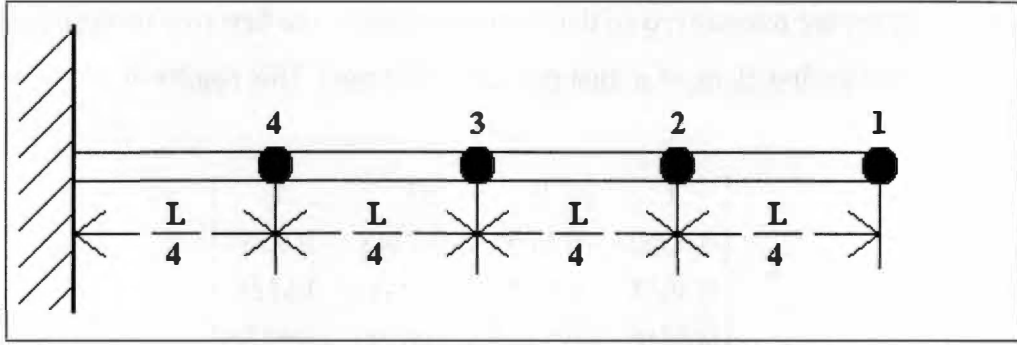


Figure A.6: A continuous beam discretized into four equally spaced points.

that include k number of modes. A modal matrix can then be assembled from all the different multiplication factors obtained as shown below.

$$X = \begin{bmatrix} \Phi_{11} & \Phi_{21} & \cdots & \Phi_{k1} \\ \Phi_{12} & \Phi_{22} & \cdots & \Phi_{k1} \\ \vdots & \vdots & \ddots & \vdots \\ \Phi_{1k} & \Phi_{2k} & \cdots & \Phi_{kk} \end{bmatrix}$$

It is then usually best to put the columns in relative terms by dividing each column by the first element in that column. Experimentally, it is easiest to obtain the scaling factor for each mode based upon the peak of the imaginary response. To obtain a modal matrix for a four-degree of freedom system, first, the beam is tapped at the free end and the imaginary peaks for all four modes are recorded. Then, the beam is tapped at 25% of the beam's length from the free end and the imaginary peaks for all four modes are recorded. This is also repeated at 50% of the beam's length and 25% of the beam's length from the fixed end. After all the different modal peaks at all the different locations have been recorded, they are put into matrix form. For this study, the data was recorded as follows

$$X = \begin{bmatrix} -0.0381 & -0.0000947 & -0.00000958 & -0.00000705 \\ -0.025 & 0.0000161 & 0.00000713 & 0.00000672 \\ -0.0116 & 0.0000809 & -0.000000518 & -0.00000601 \\ -0.00326 & 0.0000527 & -0.00000865 & 0.00000434 \end{bmatrix}$$

Next, the columns are normalized so that every element in the first row is one (i.e. divide each column by the first element in that particular column). This results in

$$X = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0.6562 & -0.1700 & -0.7443 & -0.9532 \\ 0.3045 & -0.8543 & 0.0541 & 0.8525 \\ 0.0856 & -0.5565 & 0.9029 & -0.6156 \end{bmatrix}$$

which is the experimental modal matrix for the beam. This can be compared to the theoretical modal matrix for a cantilever beam of four equally spaced points shown below.

$$X = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0.6577 & -0.1350 & -0.5815 & -0.6214 \\ 0.3395 & -0.7137 & 0.0197 & 0.7071 \\ 0.0973 & -0.4173 & 0.7245 & -0.6852 \end{bmatrix}$$

From this comparison, it is shown that the first two modes (columns) are reasonably accurate while the third and fourth modes (columns) have more error. Referring back to the discrete system, we can write out the equation of motion for a discrete system as

$$[m]\ddot{y} + [c]\dot{y} + [k]y = [f]$$

where the 'y' is the local coordinate, and the lowercase m , c , and k , are the local mass, local stiffness, and local damping, respectively. Then, a substitution can be made that is similar to Eq. (12), which yields

$$[m][X]\ddot{h} + [c][X]\dot{h} + [k][X]h = [f]$$

where ' η ' is the modal coordinate, and the capital M , C , and K , are the modal mass, modal damping, and modal stiffness, respectively. The next step is to multiply everything by the transpose of the modal matrix. This is similar to what was done in Eq. (13). This is done to uncouple the different modal equations. Since the modes are orthogonal to each other, when it is multiplied by itself it is one; when it is multiplied by different mode, the product is zero. This results in the following equation

$$[X]^T [m][X]\ddot{\eta} + [X]^T [c][X]\dot{\eta} + [X]^T [k][X]\eta = [X]^T [f]$$

The above equation is similar to Eq. (15), which can be rewritten as

$$[M]\ddot{\eta} + [C]\dot{\eta} + [K]\eta = [F]$$

The two equations of motion mentioned in this appendix are equal (just written in terms of different coordinates). A set of equations can be written linking the modal mass, modal stiffness, and modal damping, with the local mass, local stiffness, and local damping, respectively as

$$\begin{aligned} [M] &= [X]^T [m][X] \\ [C] &= [X]^T [c][X] \\ [K] &= [X]^T [k][X] \\ [F] &= [X]^T [f] \end{aligned}$$

Rewriting the above equations to isolate the local terms yields the following equations

$$\begin{aligned} [m] &= [X]^{-T} [M][X]^{-1} \\ [c] &= [X]^{-T} [C][X]^{-1} \\ [k] &= [X]^{-T} [K][X]^{-1} \\ [f] &= [X]^{-T} [F] \end{aligned}$$

This now gives the equations needed to use the modal mass, modal stiffness, modal damping, and the modal matrix to find the local mass, local stiffness, and local damping. Shown below are the calculations for the local mass, local stiffness, and local damping realizing that the modal mass, modal stiffness, and modal damping must be put into diagonal form for the matrix algebra.

$$\begin{aligned}
 [m] &= \begin{bmatrix} 2.2346 & 1.3841 & -0.0642 & 0.2126 \\ 1.3841 & 4.9664 & 1.3079 & -0.0693 \\ 1.3841 & 1.3079 & 5.0839 & -0.1502 \\ 0.2126 & -0.0693 & -0.1502 & 4.7957 \end{bmatrix} \times 10^{-12} \\
 [c] &= \begin{bmatrix} 0.7557 & -1.2443 & -0.2945 & 1.9928 \\ -1.2443 & 2.2833 & 0.1418 & -3.0962 \\ -0.2945 & 0.1418 & 1.3037 & -2.1103 \\ 1.9928 & -3.0962 & -2.1103 & 8.0027 \end{bmatrix} \times 10^{-6} \\
 [k] &= \begin{bmatrix} 0.6838 & -1.4351 & 0.9705 & -0.3086 \\ -1.4351 & 3.6333 & -3.6217 & 2.0062 \\ 0.9705 & -3.6217 & 6.1967 & -5.5210 \\ -0.3086 & 2.0062 & -5.5210 & 7.8893 \end{bmatrix} \times 10^3
 \end{aligned}$$

where the mass is in units of kilograms, damping is in units of kg/sec (N•sec/m), and stiffness is in units of N/m. Obviously, these numbers are somewhat incorrect since the mass matrix is not diagonal. However, since there are errors with any experimental data acquisition, only the important parts can be individually considered. For example, in the mass matrix, all of the elements of the diagonal are relatively close except for the first element, which is approximately slightly less than half of the other three. This makes sense considering how the locations of the mass lumps were chosen. The second, third, and fourth local mass are equal because they are located in the middle parts of the beam, whereas, the first local mass is half of the other three because it is located on the end. Notice, also, that the other elements of the mass matrix are not zero as suspected although they are small compared to the diagonal elements.

It is hard to observe the inherent nature of the system from an analysis of the numbers included in the stiffness matrix. However, it would be more useful to make use of the flexibility matrix. The relationship between the stiffness matrix and the flexibility matrix is shown below.

$$\begin{aligned}\bar{f} &= [k]\bar{y} \\ [\alpha]\bar{f} &= \bar{y} \\ [\alpha] &= [k]^{-1}\end{aligned}$$

Taking the inverse of the stiffness matrix results in a flexibility matrix shown below.

$$[\alpha] = \begin{bmatrix} 0.0402 & 0.0253 & 0.0111 & 0.0029 \\ 0.0253 & 0.0169 & 0.0079 & 0.0022 \\ 0.0111 & 0.0079 & 0.0044 & 0.0015 \\ 0.0029 & 0.0022 & 0.0015 & 0.0007 \end{bmatrix}$$

This can be interpreted as the deflection of the four different points as a result of loading the beam with a unit load at the four different points. So, the rows represent the deflection of the different points (i.e. row 1 are the deflections of point 1, row 2 are the deflections of point 2, etc.) and the columns represent at which point the unit load was placed to obtain deflection (i.e. column 1 are the deflection of all the point if the load is at point 1, column 2 are the deflection of all the point if the load is at point 2, etc.). Using this method, one could essentially setup a four-degree of freedom system to mimic the behavior of these deflections at the different points.

Next the damping matrix obtained is considered. Damping is a dynamic term that could be external (environmental) or internal (structural). It is impossible to tell the damping characteristics of the beam from the damping matrix shown. Therefore, in order to model this system with dashpots, a trial-and-error method would need to be implemented to create an accurate model that results in a similar damping matrix.

Vita

Aaron Lee Echols was born in Nashville, TN on June 22, 1980. He was raised in Hendersonville, TN. He went to grade school at Davidson Academy in Nashville, TN and went to junior school at T. W. Hunter Middle School in Hendersonville, TN. He graduated from Beech Senior High School in 1998. From there, he went to Tennessee Technological University in Cookeville, TN where he received his Bachelor of Science in Mechanical Engineering. Aaron received his Master of Science in Mechanical Engineering from the University of Tennessee in Knoxville in 2005.

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