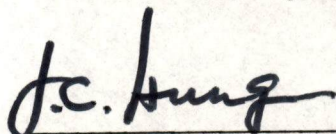


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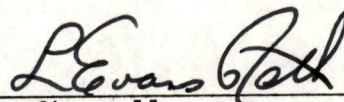


J. C. Hung, Major Professor

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and recommend its acceptance:



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ON GROUP METHOD OF DATA HANDLING

A Thesis
Presented for the
Master of Science
Degree
The University of Tennessee, Knoxville

Morteza Parang

June 1979

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ABSTRACT

The objective of this thesis was to recompute the results of the performance index estimation equation of British economy in 1969, as previously worked by A. G. Ivakhnenko, to verify the computational steps used in applying the Group Method of Data Handling to the above problem. Furthermore, it was intended to make a critical study of the rationale of each computational step used by A. G. Ivakhnenko, and to explore the possibility and application of improved steps as well as a comparative analysis of the final results in both approaches.

The estimation equation was developed and used in predicting the economic performance of Great Britain for the year 1969.

The modified approach in the Group Method of Data Handling was shown to lead to more accurate results as verified by the actual economic performance data available.

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CHAPTER I

INTRODUCTION

In recent years technical improvement in complex system theory is highly in demand to continue the advances toward better methods for system modeling, from which one can, therefore, improve prediction, control, and design optimization. One of the major difficulties in modeling a complex system using conventional modeling methods is the multitudinous and highly complex and unknown interactions between variables in these systems which result in serious problems for such an approach.

Unlike classical control theory which is based on analysis of causes and results, input and output, from which the differential equations of the system can be constructed, a novel and significantly unique approach based on generation and comparison of all possible combinations of input and output has been recently developed for solving complex system problems. This new method, known as "heuristic self-organization," was introduced by the well-known authority on problems of modeling cybernetic systems, the Russian academician A. G. Ivakhnenko. This method is a basis for the polynomial theory of complex systems from which the "polynomial description" (prediction and estimation equations) of a complex system can be found. This approach in prediction and estimation indeed provides a new and powerful tool in the science of complex system modeling. It is also quite unique in that it is a self-adaptive process, that is, a process which acts selectively toward the responses of important variables observed while the system internal parameters are

being changed.¹ Furthermore, with the improvement of the modern day digital computers, it is possible to implement this method of estimation and prediction for systems of any complexity.

The objective of this thesis is to obtain and analyze the results of the polynomial description (or more specifically, the estimation equation) of the British economy in 1969, as previously found by A. G. Ivakhnenko and therefore verify and improve the accuracy of the algorithm computations in the above problem. Further and more important objectives are to make a critical study of the rationale of each computational step used by A. G. Ivakhnenko, and to explore the possibility of applying improved steps as well as conduct an investigation comparing the final results in both approaches.

The outline of the thesis is as follows. A brief discussion of the polynomial theory of complex systems and its application in a system of given differential equation are presented in Chapter II. The application of the new theory based on "Group Method of Data Handling" (GMDH) in a system with known experimental data is studied and its polynomial model investigated.

Chapter III presents a stepwise discussion of the GMDH and its application in predicting a British economic performance index for the year 1969 as performed by A. G. Ivakhnenko. Detailed numerical results at each step in his work are shown along with the final conclusions. At the end, a recomputation of A. G. Ivakhnenko's method is carried to ensure absence of error in his computations.

The most important segment of this work is presented in Chapter IV. In this section a significant modification is shown to greatly improve the results obtained using this method in showing complex system problems. An example of this is clearly demonstrated when this modified method is applied to the British economic performance index example. Here the advantages of the proposed modifications are discussed as reflected in better and more accurate results.

A summary of the thesis is presented in Chapter V along with suggestions for further study.

Finally, important equations and further details related to the GMDH algorithm are presented in Appendix A. Appendix B contains the computer programs with the modified algorithm developed in this work along with all the necessary relevant details.

CHAPTER II

DISCUSSION OF THE POLYNOMIAL THEORY FOR SYSTEM MODELING

I. GENERAL POLYNOMIAL MODELING

The polynomial theory is a powerful method for modeling a complex system. It utilizes mathematical combinatorial methods and a series of "threshold self-selection" based on different criteria such as correlation coefficient, the mean square error criterion, and the diversity-of-data criterion.² Unlike conventional control theory which is based on differential equations the polynomial theory employs prediction and estimation formulas which are in polynomial or graphical form. These formulas, which can adequately describe the complexity of a system, are known as "polynomial descriptions" of such a system.

Presently there are at least three methods whereby one can find the polynomial description of a system: first, by replacing the time derivatives in the differential equations with finite differences (these difference equations which are in polynomial form supply the needed description of the system); second, by "differential fork" method (based on indirect measurement of disturbances)³; and finally, by experimental method using the Group Method of Data Handling (GMDH). All of the above methods can be used for systems of known differential equations. The experimental method (GMDH), however, can also be used for systems of known experimental data where no known or applicable differential equations describing the system exist. This is the most significant feature of GMDH.

The polynomial theory models the input-output relationship of a complex system using a multilayered perceptron-type network structure. Figure 1 illustrates the structure of the overall input-output transformation known as Group Method of Data Handling (GMDH). Each element in the network implements a nonlinear function of its inputs. These functions are the Kolmogorove-Gabor polynomials given as

$$\begin{aligned} \phi = a_0 + \sum_{i=1}^n a_i x_i + \sum_{i=1}^n \sum_{j=1}^n a_i a_j x_i x_j \\ + \sum_{i=1}^n \sum_{j=1}^n \sum_{k=1}^n a_i a_j a_k x_i x_j x_k + \dots \end{aligned} \quad [1]$$

In this work each function implemented is a second-order polynomial of the inputs and each element accepts two inputs. Therefore, as an example, the function implemented by an element in the first layer can be written as

$$\begin{aligned} y = A_2(x) = a_0 + a_1 x_{(1)} + a_2 x_{(2)} + a_3 x_{(1)}^2 \\ + a_4 x_{(2)}^2 + a_5 x_{(1)} x_{(2)} \end{aligned} \quad [2]$$

where $x_{(1)}$ and $x_{(2)}$ are any input elements in the first layer, y is an element of the second layer, and the subscript in A_2 denotes a second-order transformation of the inputs.

Several layers of such "partial" descriptions, an example of which is shown above, and described as

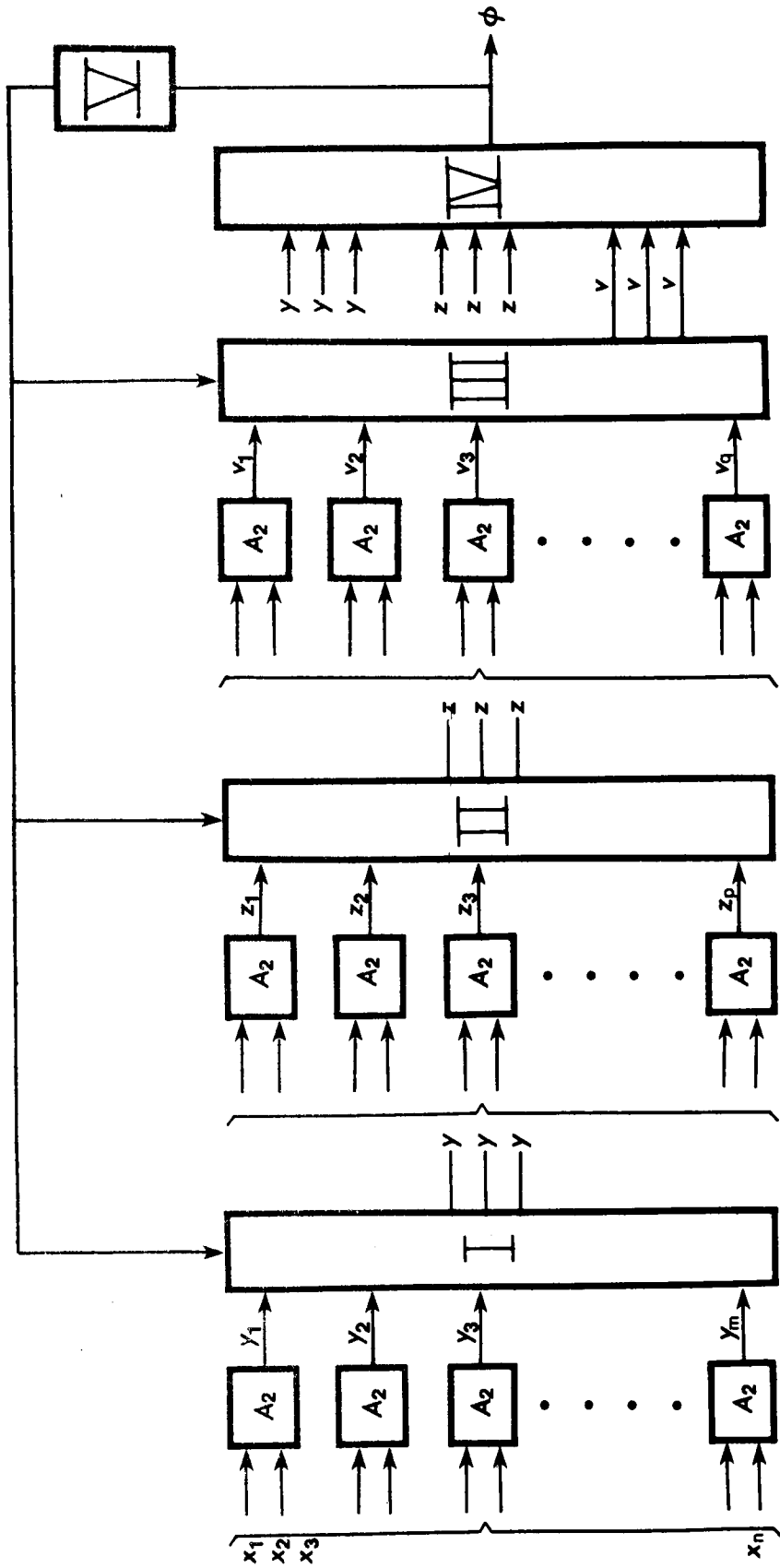


Figure 1. Algorithm of Group Method of Data Handling (GMDH) Using Second Degree Polynomials (I, First Self-selection Threshold; II, Second Self-selection Threshold; III, Third Self-selection Threshold; IV, Selection from All Solutions; V, Threshold Optimization).

$$y_1 = f(x_1, x_2), y_2 = f(x_3, x_4), \dots, y_m = f(x_{n-1}, x_n)$$

$$z_1 = f(y_1, y_2), z_2 = f(y_3, y_4), \dots, z_p = f(y_{m-1}, y_m) \quad [3]$$

$$v_1 = f(z_1, z_2), v_2 = f(z_3, z_4), \dots, v_q = f(z_{p-1}, z_p)$$

(where m , p , and q are the number of pairwise combinations for first, second, and third layer of transformations, respectively, and n is the number of input variables x) will form a "complete" description of the system in the form of

$$\phi = f(x_1, x_2, x_3, \dots, x_n). \quad [4]$$

In the algorithm of GMDH there are a number of "self-selection thresholds." Their purpose is to filter out those elements at each layer which are least useful for predicting the correct output ϕ . Therefore, only those elements whose performance indices exceed the threshold at that layer are allowed to pass to the next layer. The process is repeated where each time a set of new elements (solutions) is selected for generation of the next layer inputs.

We now discuss specifically the application of polynomial theory in a system with known and unknown differential equations.

II. POLYNOMIAL MODEL OF A SYSTEM WITH KNOWN DIFFERENTIAL EQUATION

Polynomial descriptions in systems of known differential equations can be readily developed by replacing the derivatives in a differential equation with finite differences. They can also be obtained by using Group Method of Data Handling with short samples of

experimental data. The above general methods can be used to obtain polynomial descriptions for both linear and nonlinear systems. For input x and output y the prediction equation is

$$y_k = f(y_{k-1}, y_{k-2}, y_{k-3}, \dots, x_{k-1}, x_{k-2}, x_{k-3}, \dots) \quad [5]$$

where the instances of time are denoted as

k = future moment of time

$k - 1$ = instant movement

$k - 2$ = moment one step before

$k - 3$ = moment two steps before

and so on.

The prediction equation, of course, does not include the future values of the input variable among its arguments. But, knowing the present values of all the variables and their values over sufficiently long previous cycles, the output value can be determined over the next cycle.

The estimation equation for input x and output y is

$$y_k = f(y_{k-1}, y_{k-2}, y_{k-3}, \dots, x_k, x_{k-1}, x_{k-2}, x_{k-3}, \dots) \quad [6]$$

where the subscripts k , $k-1$, $k-2$, and so on are defined as in equation [5]. By estimation equation in this context we mean an input-output relationship which enables one to extract information from

noise-computed observations concerning variable interrelationships. Unlike the prediction equation, the estimation equation includes in its arguments the future values of the input. This is, of course, true because the presence of the future signals in the argument as independent variables (thus subject to control) are necessary for the purpose of manipulating the future behavior of the system performance. Knowing the current and the past values of all the variables, as well as the future value of our signal, one can determine the value of the output during the next cycle.

Let us find the polynomial description for a linear first order differential equation of the form

$$\frac{dy(t)}{dt} + A(t)y(t) = B(t)x(t) \quad [7]$$

or

$$\frac{dy}{dt} \approx \frac{\Delta y}{\Delta t} = B(t)x - A(t)y, \quad \Delta t = h$$

the above can be written as

$$\Delta y = hB(t)x - hA(t)y. \quad [8]$$

To obtain the prediction equation we can replace the derivatives in [7] by lag finite difference:

$$y_k - y_{k-1} = hB(t)x_{k-1} - hA(t)y_{k-1}$$

or solving for y_k ,

$$y_k = [1 - hA(t)]y_{k-1} + hB(t)x_{k-1}$$

which is in the form

$$y_k = ay_{k-1} + bx_{k-1} \quad [10]$$

where:

$$a = 1 - hA(t)$$

$$b = hB(t).$$

Equation [10] can be written in a general form as

$$y_{k-n} = ay_{k-(n+1)} + bx_{k-(n+1)} \quad [11]$$

where n is the number of time steps. Using [11], prediction equations of more accuracy can be obtained. For example, the equation for the first five values of n are

$$y_k = ay_{k-1} + bx_{k-1}, \quad n = 0$$

$$y_k = a^2y_{k-2} + bx_{k-1} + abx_{k-2}, \quad n = 1$$

$$y_k = a^3y_{k-3} + bx_{k-1} + abx_{k-2} + a^2bx_{k-3}, \quad n = 2 \quad [12]$$

$$y_k = a^4y_{k-4} + bx_{k-1} + abx_{k-2} + a^2bx_{k-3} + a^3bx_{k-4}, \quad n = 3$$

$$y_k = a^5y_{k-5} + bx_{k-1} + abx_{k-2} + a^2bx_{k-3} + a^3bx_{k-4} + abx_{k-5}, \quad n = 4.$$

To obtain the estimation equation, we replace the derivatives in equation [8] with lead difference:

$$y_k - y_{k-1} = hB(t)x_k - hA(t)y_k \quad [13]$$

or by solving for y_k

$$y_k = \frac{1}{1+hA(t)} y_{k-1} + \frac{hB(t)}{1+hA(t)} x_k$$

which can be expressed in the form

$$y_k = ay_{k-1} + bx_k \quad [14]$$

where:

$$a = \frac{1}{1+hA(t)}$$

$$b = \frac{hB(t)}{1+hA(t)}.$$

Again using recursion formula

$$y_{k-n} = ay_{k-(n+1)} + bx_{k-n} \quad [15]$$

we can obtain the estimation equation for more intervals of previous history (i.e., $T = nh$) where for the first four values of n :

$$y_k = ay_{k-1} + bx_k, \quad n = 0$$

$$y_k = a^2y_{k-2} + bx_k + abx_{k-1}, \quad n = 1$$

$$y_k = a^3y_{k-3} + bx_k + abx_{k-1} + a^2bx_{k-2}, \quad n = 2 \quad [16]$$

$$y_k = a^4y_{k-4} + bx_k + abx_{k-1} + a^2bx_{k-2} + a^3bx_{k-3}, \quad n = 3$$

$$y_k = a^5y_{k-5} + bx_k + abx_{k-1} + a^2bx_{k-2} + a^3bx_{k-3} + a^4bx_{k-4}, \quad n = 4.$$

As was mentioned previously, a distinguishing feature of the derived equations, i.e., prediction and estimation equations, is that in prediction equations there are no future values of the input variable in its argument whereas in estimation equations the reverse is true.

III. POLYNOMIAL MODEL OF A SYSTEM WITH KNOWN EXPERIMENTAL DATA USING GMDH

Contrary to systems of given differential equations where the input-output paths could be determined by the use of equations themselves, in complex systems it is quite difficult to determine and observe the causal connections between inputs and outputs. Therefore, with no apparent input-output paths in such systems, differential equations are not adequate tools for solving complex control problems. The polynomial theory makes it possible to model and obtain solutions for systems of high complexity and is indeed a powerful method for solving such problems.

Algorithm of Group Method of Data Handling was presented in Figure 1, page 6. The starting point of the GMDH algorithm consists of separating the experimental data into two distinct sets, the training and testing sets. This is achieved by the use of variance criterion which will be discussed later. The training set is then used to generate (using Least Square Method) functions in the form of second-order Kolmogorov-Gabor polynomials using two input variables at a time. The function can be written as

$$y = a_0 + a_1x_{(1)} + a_2x_{(2)} + a_3x_{(1)}^2 + a_4x_{(2)}^2 + a_5x_{(1)}x_{(2)}$$

where $x_{(1)}$ and $x_{(2)}$ are dummy parameters for any two input variables in the "first layer of transformation," y is the implemented function for the next layer, and the coefficients a_0 through a_5 are to be determined

using Least Square Method for each combination of input transformation. Next, a process of selection of a finite number of generated functions (i.e., y 's) is carried out by means of the minimum mean-square error criterion, using the testing data (first self-selection threshold). A new training and testing data set can now be produced by means of generated functions y (the details are given in Chapter III). The selected functions are then the basis for generation of a second layer of transformation in which again Least Square Method is employed. The process is then repeated where each time the minimum mean-square error criterion (MMSE) is used in selecting "best" results on the basis of comparison with the testing data. In the final layer of selection only one (optimum) solution is chosen based on MMSE criterion.

This method of separation of experimental data into training and testing data and subsequent filtration process is called "decision regularization." The final unique and optimum solution obtained is known as "regular decision." It should be noted that each succeeding layer in the GMDH algorithm increases by two the degree of the polynomial fit to the input properties, and the degree of the optimal polynomial is proportional to the increasing complexity of the system. Thus one expects increasingly large numbers of polynomial terms for the more complex systems.

CHAPTER III

APPLICATION OF POLYNOMIAL THEORY TO COMPLEX SYSTEM

I. USE OF GMDH IN MODELING BRITISH ECONOMY BY A. G. IVAKHNENKO

A. G. Ivakhnenko's work on application of polynomial theory to the British economy, using Group Method of Data Handling, serves as an illustration of the use of the theory in modeling complex systems. In his example the estimation equation for the performance index of the economy along with the optimization equations for the year 1969 are found. The optimization equations enables us to find the optimal values of certain economic characteristic variables which maximizes the performance index. The relevant economic characteristic variables and the data used in this example are taken from the works of British economists P. C. Parks and F. G. Pyatt.⁴ They have suggested a set of 21 variables closely characterizing the British economy and have furnished their values for the period of 1954-1968.

Figure 2 shows the block diagram of GMDH algorithm used in derivation of the estimation equation. We will proceed with the example of British economy and discuss the details of each block in Figure 2 and the approach taken in each step of the algorithm.

Selection of the Manipulated Variables and Performance Indices

In the model of P. G. Parks et al, there are 21 variables from which 16 are interconnected by differential or algebraic equations. Therefore, any five variables can be chosen for the purpose of

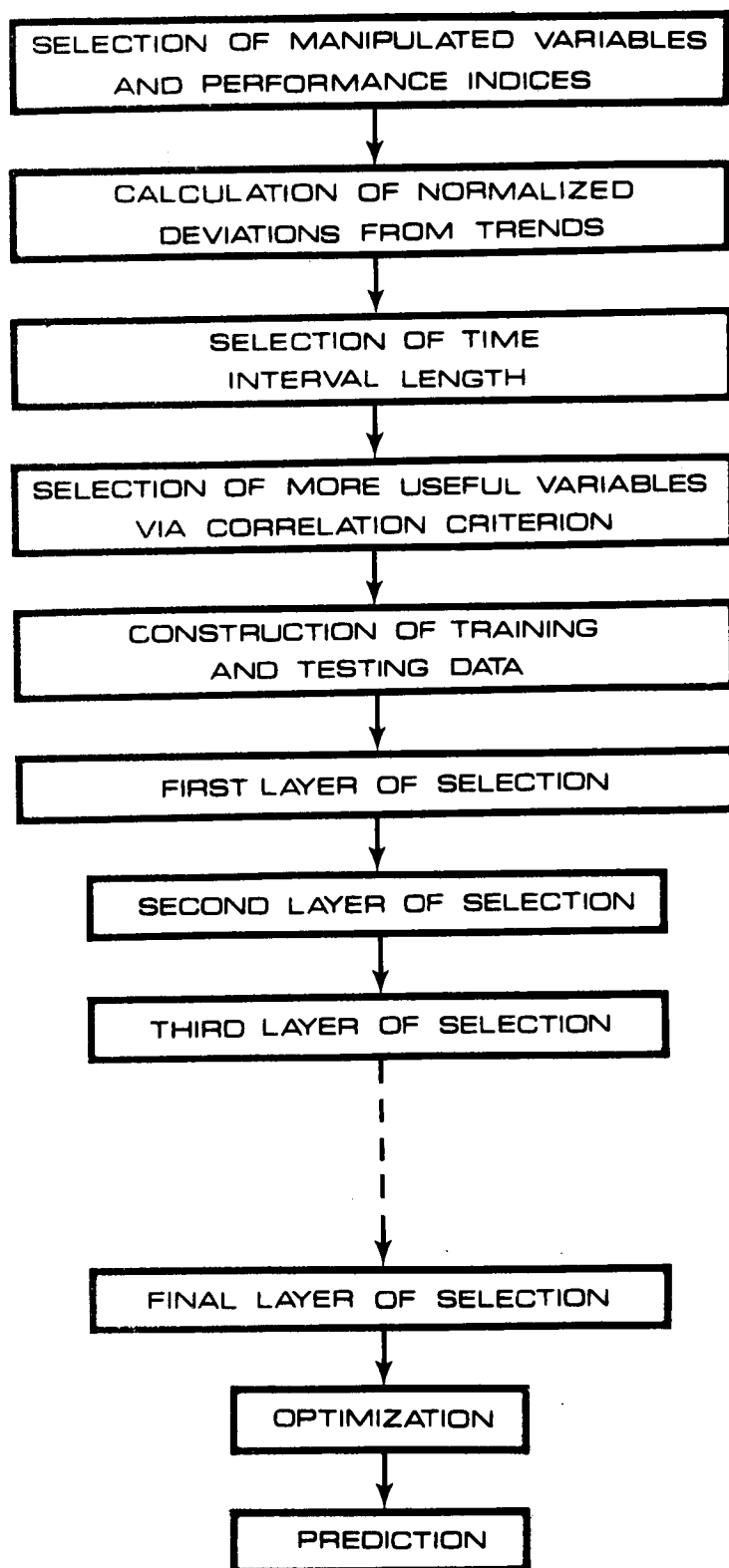


Figure 2. The Block Diagram of GMDH Algorithm.

manipulating. Using the notation V_1 , V_2 , and so on for their designation, they are:

$$V_1 = T_h - \text{taxes on households}$$

$$V_2 = T_c - \text{taxes on companies}$$

$$V_3 = U - \text{the state allowances (e.g., to the unemployed)}$$

$$V_4 = K - \text{total gross capital stock}$$

$$V_5 = g - \text{government expenditures.}$$

The above are, therefore, the selected manipulated variables.

Let us assume an ideal situation (according to P. G. Parks and F. G. Pyatt) such that

$$\phi_1 = 1, \phi_2 = 0, V_3 = 0$$

where ϕ_1 and ϕ_2 are considered as the performance indices of the British economy and defined as

$$\phi_1 = \frac{\omega + \pi}{C_4 K}$$

$$\phi_2 = I_{fg} + I_{fc}$$

with

ω = income from employment

π = gross trading surplus

$$C_4 = \frac{1}{3} \text{ (constant)}$$

I_{fg} = net investment of government abroad

I_{fc} = net investment of companies abroad.

Here $1 - \phi_1$ can be regarded as "the relative excess of production capacity," and ϕ_2 as "the balance-of-payment-index." The above conditions, $\phi_1 = 1$ and $\phi_2 = 0$, signify that the industry is working at full capacity, and $V_3 = 0$ signifies the absence of unemployment. Hence the economic situation can be considered good as ϕ_1 and ϕ_2 approach 1 and 0, respectively, and it is bad as ϕ_1 and ϕ_2 approach 0 and $-\infty$, respectively. The value of the above variables are given in Tables I-III as well as shown in graphical form in Figures 3 through 9.

Calculation of Normalized Deviation

From the preceding section we have a total of seven variables, five of which are manipulated variables ($V_1, \dots, V_5$) and the remaining two (ϕ_1 and ϕ_2) are performance indices. Instead of using the absolute value of each variable throughout the calculations, their normalized deviation from their trends is used (as will be shown shortly). These deviations shall be considered as mutually dependent stationary random processes of time. The trends of variables are calculated using the complete polynomial

$$\phi_1^* = a_0 + a_1 t + a_2 t^2 + a_3 t^3 + \dots + a_m t^m \quad [17]$$

TABLE I
INPUT DATA, 1954-1958

		Years				
Vari- ables	1 (1954)	2 (1955)	3 (1956)	4 (1957)	5 (1958)	
Absolute Values	ϕ_1	0.689	0.698	0.697	0.692	0.674
	ϕ_2	117.	-155.	208.	233.	344.
	V_1	1236.	1330.	1452.	1602.	1696.
	V_2	893.	972.	894	972.	1010.
	V_3	412.	441.	464.	512.	548.
	V_4	64.88×10^3	69.72×10^3	74.8×10^3	79.17×10^3	83.421×10^3
	V_5	3185.	3251.	3515.	3668.	3750.
Trends	ϕ_1^*	0.6851	0.6871	0.6909	0.6965	0.7033
	ϕ_2^*	43.972	93.9577	123.2492	134.3481	129.7768
	V_1^*	1232.3352	1344.4569	1455.2951	1570.0491	1693.9152
	V_2^*	873.3798	944.5949	977.8422	980.4375	959.6966
	V_3^*	375.8002	450.7397	505.0590	544.1065	573.2306
	V_4^*	65593.	70033.	74138.	78044.	81892.
	V_5^*	3171.	3309.	3459.	3625.	3812.
Normalized Deviations from Trends	ϕ_1	0.0048	0.0158	0.0088.	-0.0064	-0.0416.
	ϕ_2	1.6634	-2.6497	0.6876	0.7343	1.6507
	v_1	0.0029	-0.0107	-0.0022	0.0203	0.0012
	v_2	0.0224	0.0290	-0.0857	-0.0086	0.0524
	v_3	0.0963	-0.0216	-0.0812	-0.0590	-0.0440
	v_4	-0.0108	-0.0044	0.0089	0.0144	0.0186
	v_5	0.0042	-0.0176	0.0161	0.0118	-0.0161

TABLE II
INPUT DATA, 1959-1963

		Years				
Vari- ables		6 (1959)	7 (1960)	8 (1961)	9 (1962)	10 (1963)
Absolute Values	ϕ_1	0.697	0.795	0.736	0.724	0.729
	ϕ_2	143.	-265.	-4.	112.	144.
	V_1	1776.	1991.	2249	2458.	2510.
	V_2	968.	715.	813.	981.	857.
	V_3	658.	656.	640.	688.	830.
	V_4	85.56×10^3	88.35×10^3	93.385×10^3	98.98×10^3	104.7×10.3
	V_5	4001.	4248	4589.	4920.	5184.
Trends	ϕ_1^*	0.7107	0.7181	0.7249	0.7305	0.7343
	ϕ_2^*	112.0637	83.7372	47.3257	5.3576	-39.6387
	V_1^*	1832.0917	1989.7769	2172.1671	2384.4616	2631.8579
	V_2^*	922.935	877.4694	830.6147	789.6871	762.0021
	V_3^*	597.7797	623.1022	654.5465	697.4610	757.1941
	V_4^*	85819.	89966.	94471.	99473.	105111.
	V_5^*	4023.	4264.	4538.	4850.	5205.
Normalized Deviations from Trends	ϕ_1	-0.0192	0.0235	0.0153	-0.0088	-0.0072
	ϕ_2	0.2760	-4.1646	-1.0845	19.9046	-3.8759
	v_1	-0.0306	0.0006	0.0353	0.0308	-0.0463
	v_2	0.0488	-0.1851	-0.0212	0.2422	0.1246
	v_3	0.1007	0.0527	-0.0222	-0.0135	0.0961
	v_4	-0.0030	-0.0179	-0.0115	-0.0049	-0.0039
	v_5	-0.0054	-0.0036	0.0112	0.0143	-0.0040

TABLE III
INPUT DATA, 1964-1969

Variables	11 (1964)	12 (1965)	13 (1966)	14 (1967)	15 (1968)	16 (1969)	
Absolute Values	ϕ_1	0.747	0.741	0.716	0.711	0.709	
	ϕ_2	-381.	-50.	64.	-283.	-265.	
	V_1	2801.	3344.	3714.	4051.	4559.	
	V_2	709.	659.	725.	1036.	1085.	
	V_3	813.	919.	1030.	1293.	1520.	
	V_4	111.18×10^3	120.098×10^3	129.58×10^3	136.086×10^3	146.74×10^3	
Trends	V_5	5512.	6043.	6572.	7246.	7702.	
	ϕ_1^*	0.7357	0.7341	0.7289	0.7193	0.7053	0.6857
	ϕ_2^*	-85.1348	-128.6023	-167.5125	-199.3375	-211.549	-231.6183
	V_1^*	2919.5532	3252.7459	3636.6336	4076.4141	4577.2852	5144.4447
	V_2^*	754.8758	775.6239	831.5622	930.0065	1078.2726	1283.6763
	V_3^*	839.0942	948.5097	1090.7896	1271.2805	1495.3326	1768.2937
	V_4^*	111524.	118850.	127229.	136799.	147700.	160071.
	V_5^*	5606.	6058.	6566.	7133.	7765.	8464.
	Normalized Deviations from Trends	ϕ_1	0.0153	0.0093	-0.0149	-0.0118	0.0052
		ϕ_2	3.4752	-0.6112	-1.3820	0.4196	0.1961
V_1		-0.0406	0.0280	0.0212	-0.0062	-0.0039	
V_2		-0.0607	-0.1503	-0.1281	0.1139	0.0062	
V_3		-0.0310	-0.0311	-0.0557	0.0170	0.0164	
V_4		-0.0030	0.0104	0.0184	-0.0052	-0.0065	
V_5	-0.0167	-0.0025	0.0009	0.0157	-0.0080		

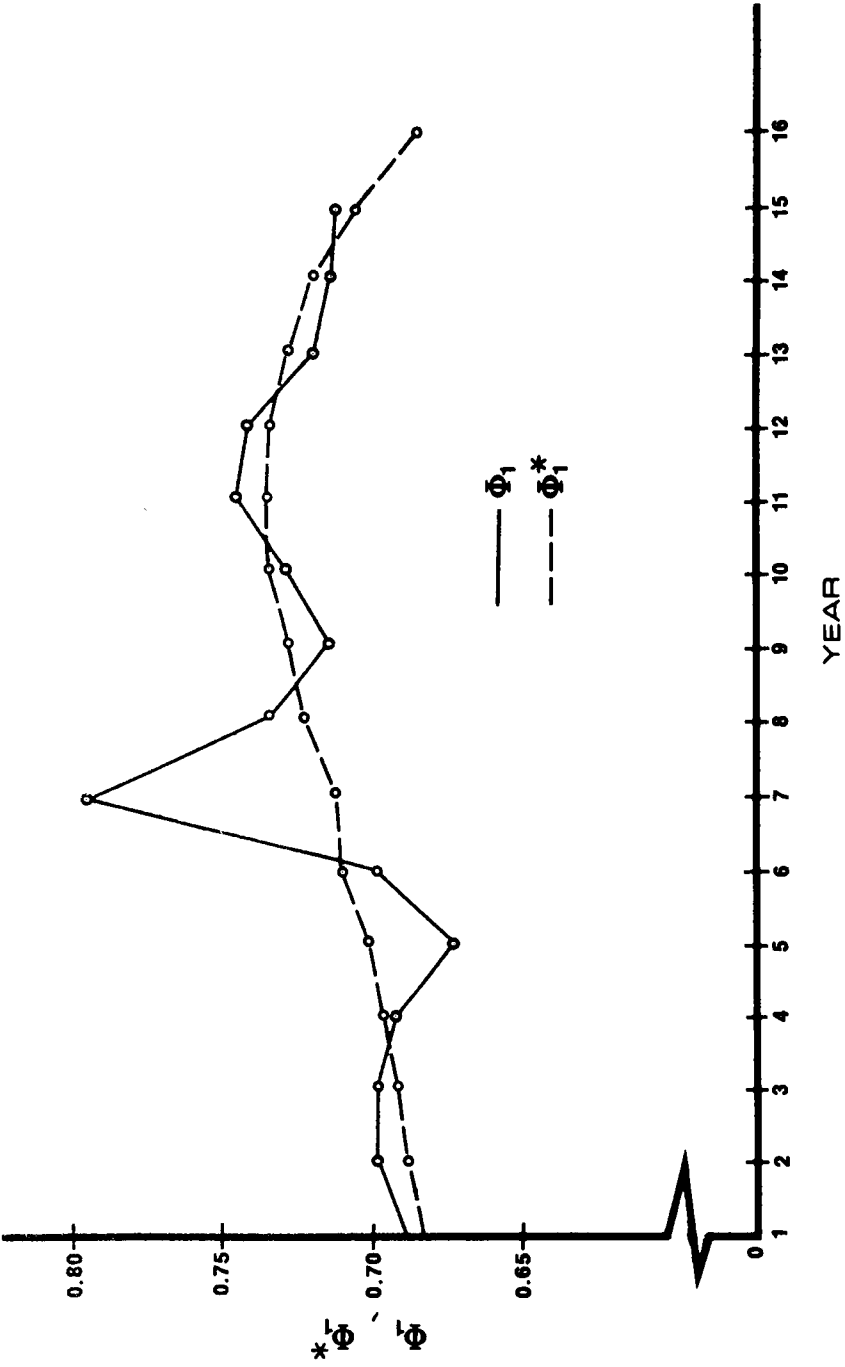


Figure 3. Performance Index ϕ_1 and Trend ϕ_1^* Versus Time (Years).

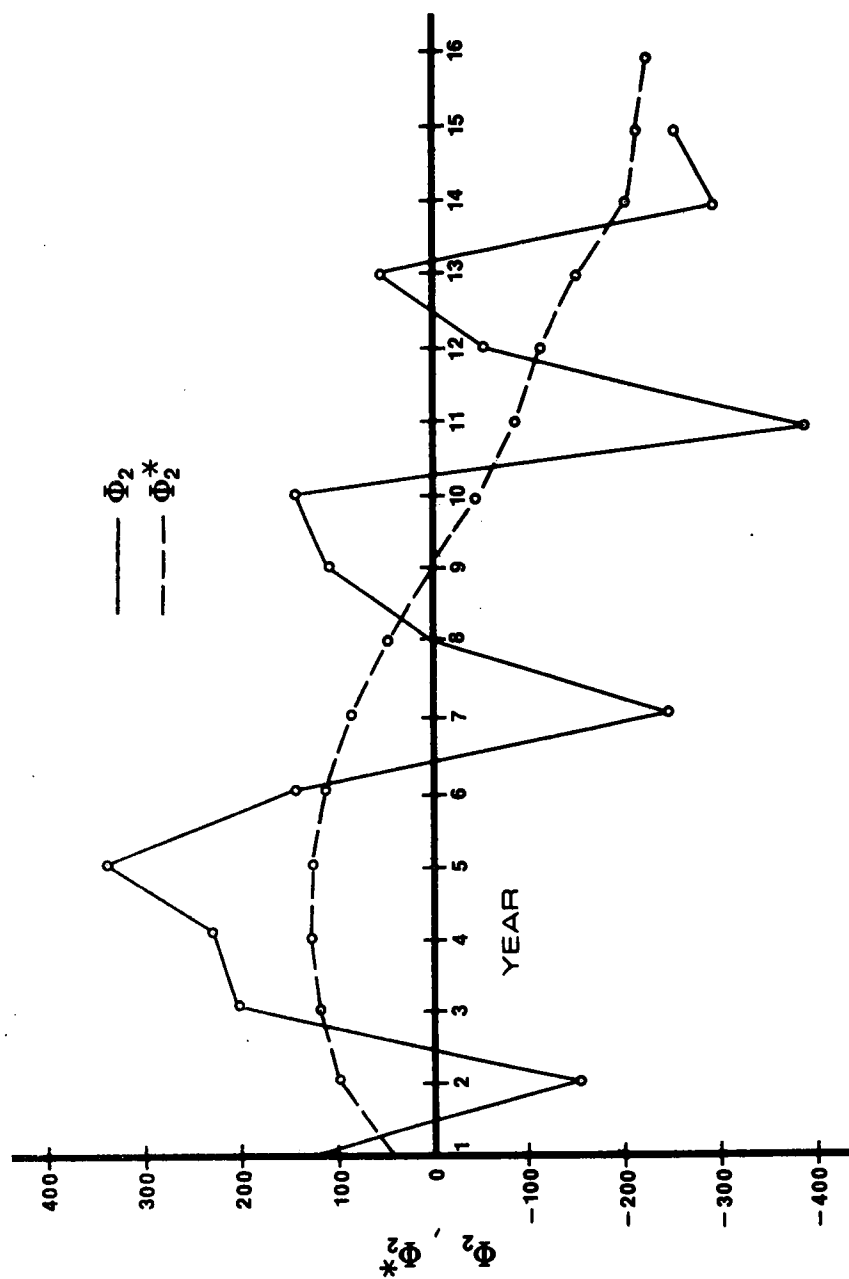


Figure 4. Performance Index ϕ_2 and Trend ϕ_2^* (Millions of Pounds) Versus Time (Years).

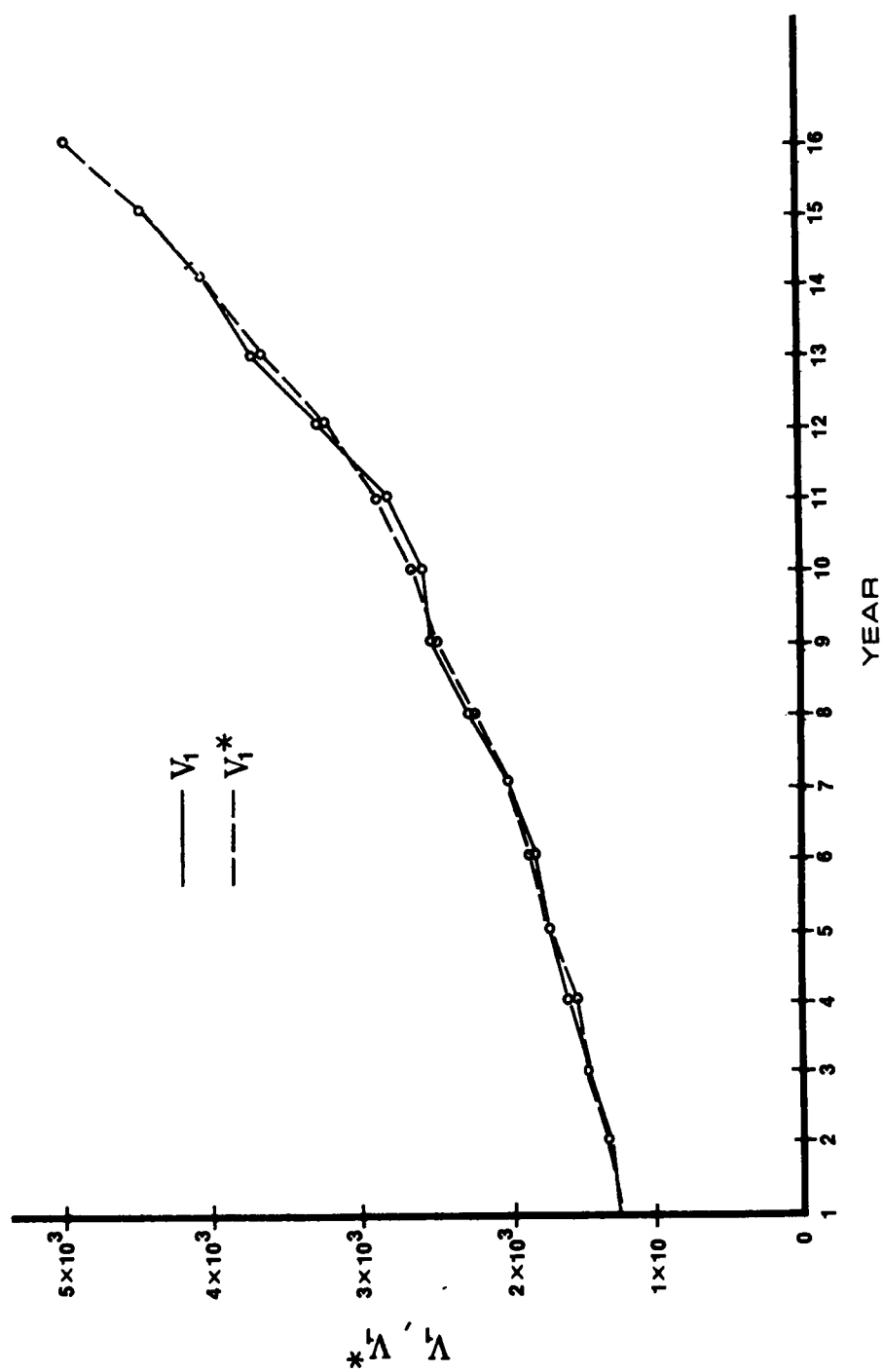


Figure 5. Taxes on Households V_1 and Trend V_1^* (Millions of Pounds) Versus Time (Years).

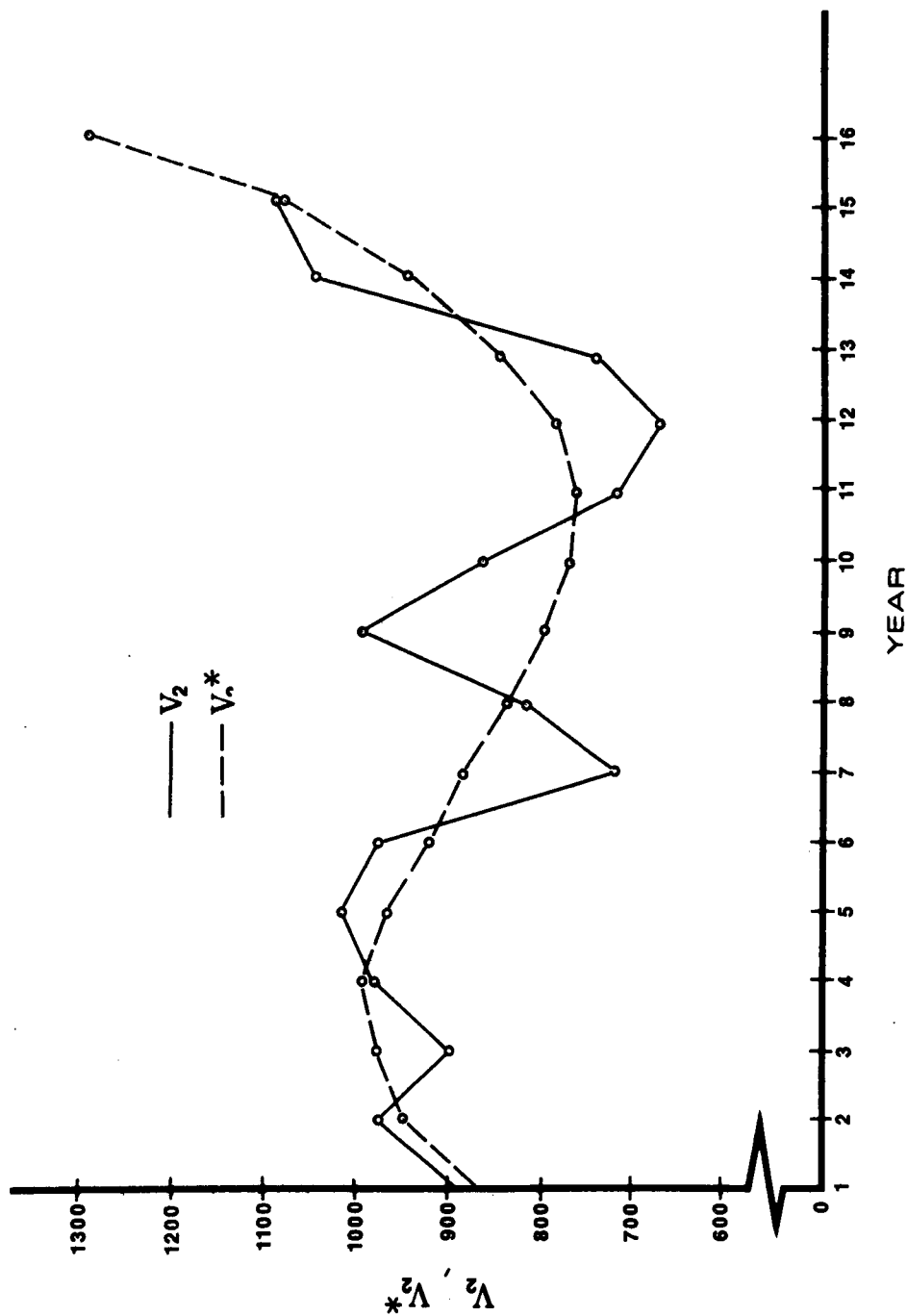


Figure 6. Taxes on Companies V_2 and Trend V_2^* (Millions of Pounds) Versus Time (Years).

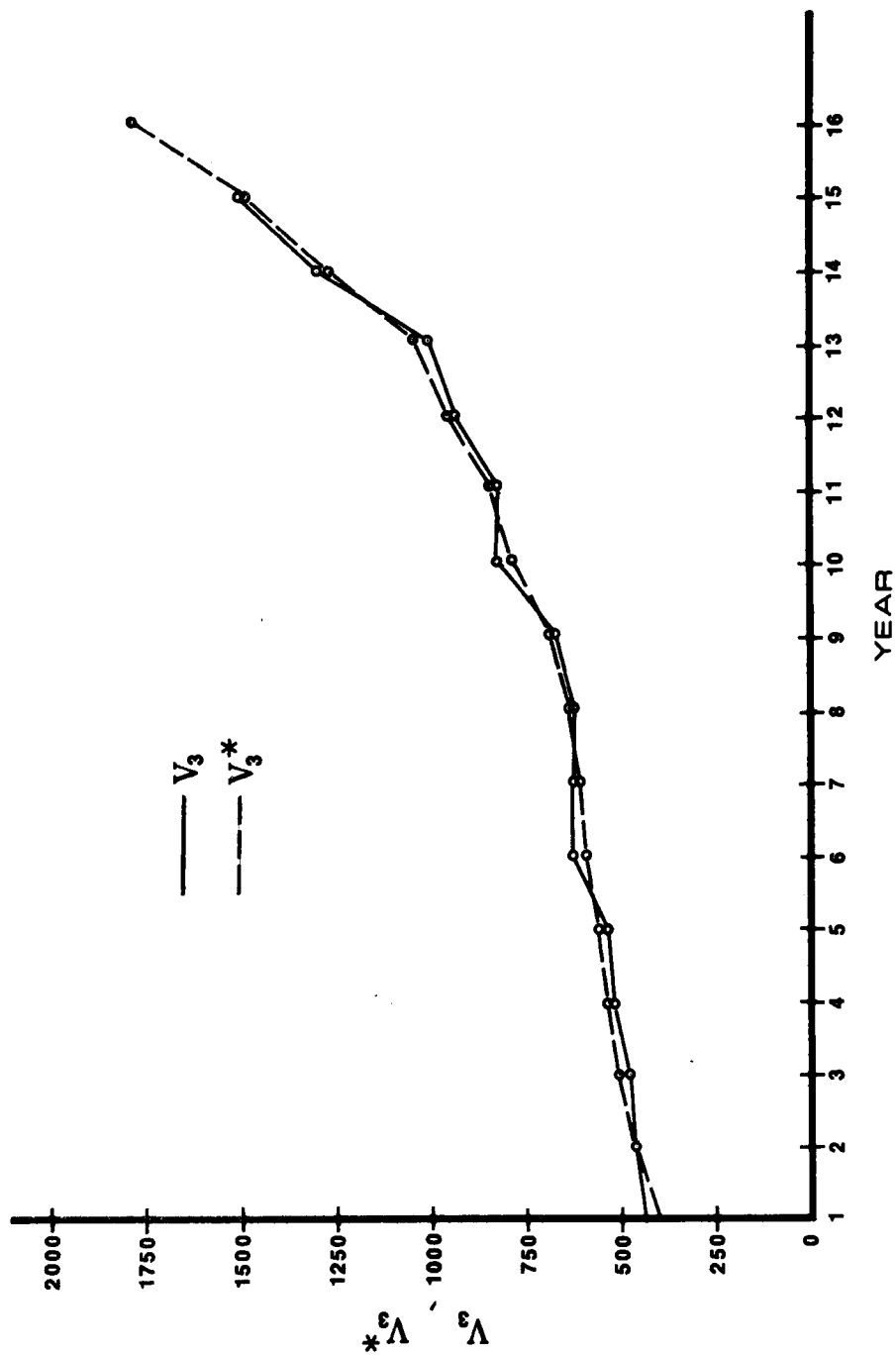


Figure 7. The State Allowances V_3 and Trend V_3^* (Millions of Pounds) Versus Time (Years).

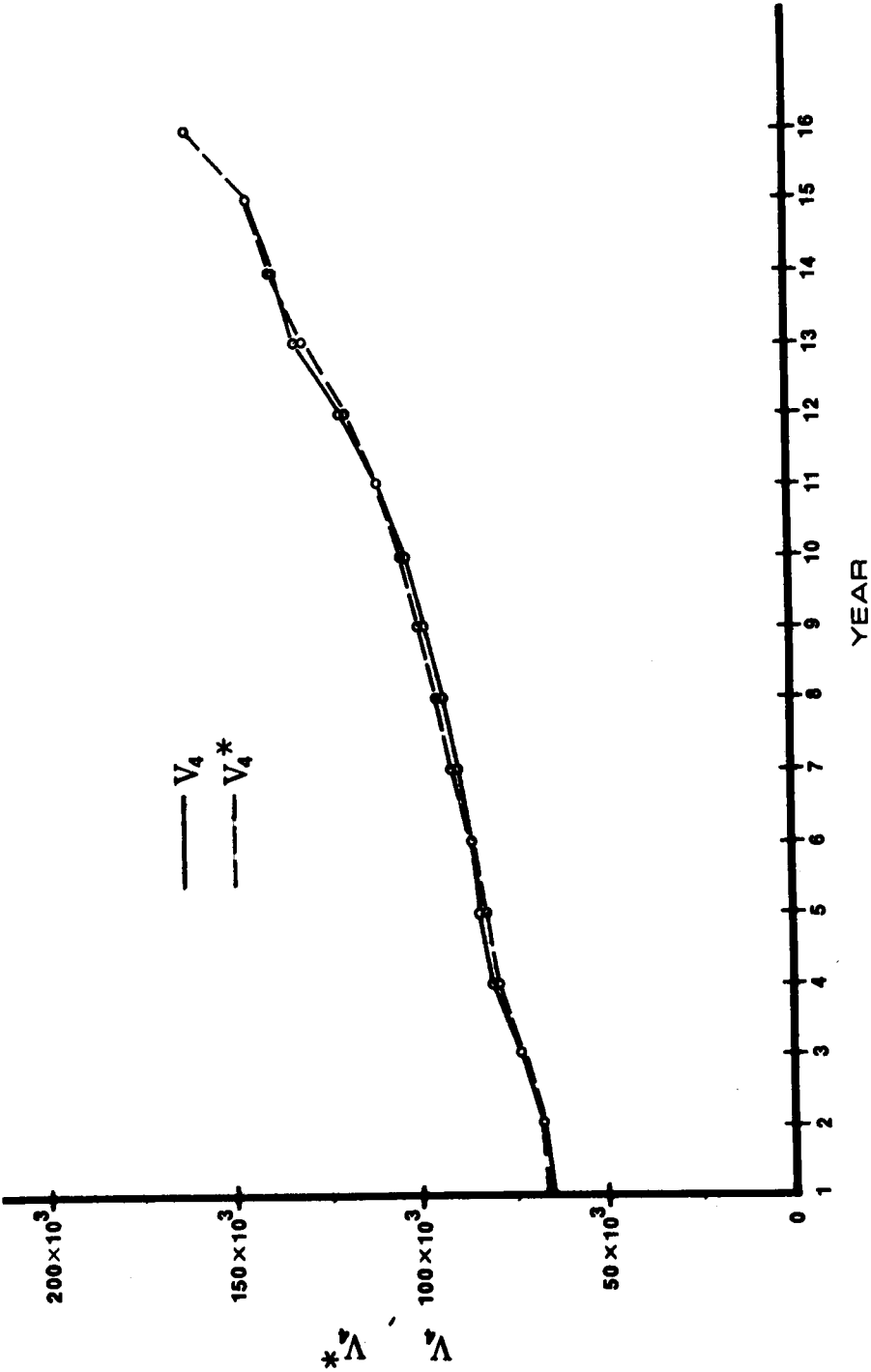


Figure 8. Total Gross Capital Stock V_4 and Trend V_4^* (Millions of Pounds) Versus Time (Years).

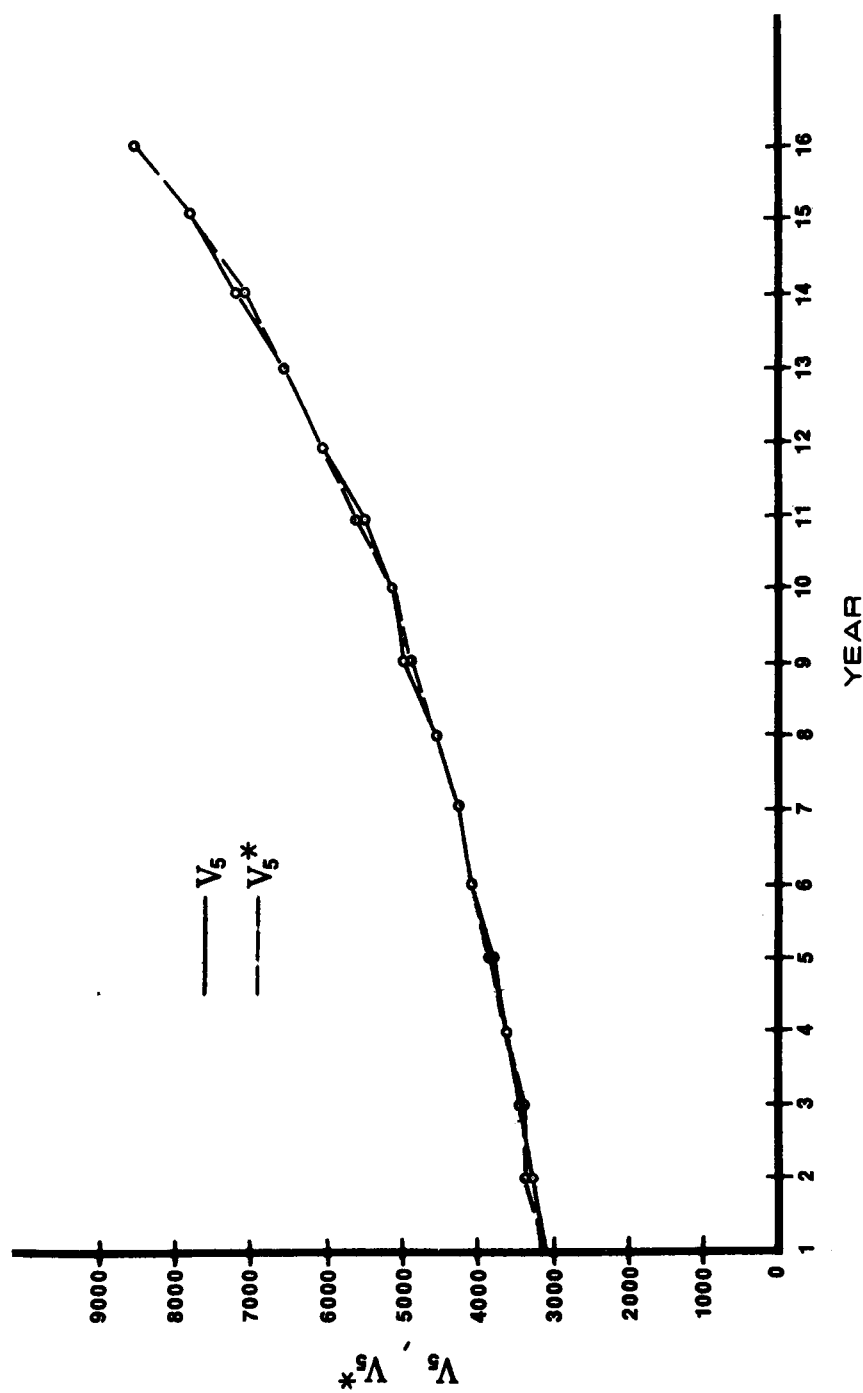


Figure 9. Government Expenditures V_5 and Trend V_5^* (Millions of Pounds) Versus Time (Years).

where * is used to designate the trend of each variable. Using GMDH, it is possible to obtain these trends for any high degree m of the complete polynomial. In practice we must not go beyond a certain optimum degree (this corresponds to MMSE of very small magnitude), since for large values of m , the curves of trends will be too close to the original variable curves, therefore making it impossible to calculate the deviations. However, for sufficiently smooth curves and fairly small number of arguments, the trends polynomial of high degree can be used to predict the variables in the next year from a single variable. For small degrees as well as excessively large degrees of m , the prediction accuracy decreases. In our case (small m) the trends are calculated without the use of GMDH. Instead, ordinary regression analysis is used, and the degree of trend polynomial is selected by optimization based on comparison of the results of a number of calculations (see [5]). These calculations show that the number of data points should be approximately five times larger than the degree of the trend polynomial. Proceeding with this assumption, we note that 15 interpolation points (Tables I-III, pp. 19-21) exist, therefore, we choose $m = 3$. The cubic trend polynomial is now

$$\phi_1^* = a_0 + a_1 t + a_2 t^2 + a_3 t^3. \quad [18]$$

Using least squares method we can determine the coefficients of the third-degree trend polynomial (equation [9]) such as to minimize the sum of the square of the deviations between the expected and actual observations. Given N number of data points, this can be expressed as

$$S = \sum_{i=1}^N [\phi_{1i} - (a_0 + a_1 t_i + a_2 t_i^2 + a_3 t_i^3)]^2. \quad [19]$$

In order to minimize S , the partial derivatives of S with respect to all coefficients are set equal to zero. The result is

$$\begin{aligned} \frac{\partial S}{\partial a_0} &= \sum_{i=1}^N 2 [\phi_{1i} - (a_0 + a_1 t_i + a_2 t_i^2 + a_3 t_i^3)](-1) = 0 \\ \frac{\partial S}{\partial a_1} &= \sum_{i=1}^N 2 [\phi_{1i} - (a_0 + a_1 t_i + a_2 t_i^2 + a_3 t_i^3)](-t_i) = 0 \\ \frac{\partial S}{\partial a_2} &= \sum_{i=1}^N 2 [\phi_{1i} - (a_0 + a_1 t_i + a_2 t_i^2 + a_3 t_i^3)](-t_i^2) = 0 \\ \frac{\partial S}{\partial a_3} &= \sum_{i=1}^N 2 [\phi_{1i} - (a_0 + a_1 t_i + a_2 t_i^2 + a_3 t_i^3)](-t_i^3) = 0. \end{aligned} \quad [20]$$

By rearranging the above terms, we obtain the normal equations in matrix form:

$$\begin{bmatrix} N & \sum_{i=1}^N t_i & \sum_{i=1}^N t_i^2 & \sum_{i=1}^N t_i^3 \\ \sum_{i=1}^N t_i & \sum_{i=1}^N t_i^2 & \sum_{i=1}^N t_i^3 & \sum_{i=1}^N t_i^4 \\ \sum_{i=1}^N t_i^2 & \sum_{i=1}^N t_i^3 & \sum_{i=1}^N t_i^4 & \sum_{i=1}^N t_i^5 \\ \sum_{i=1}^N t_i^3 & \sum_{i=1}^N t_i^4 & \sum_{i=1}^N t_i^5 & \sum_{i=1}^N t_i^6 \end{bmatrix} \begin{bmatrix} a_0 \\ a_1 \\ a_2 \\ a_3 \end{bmatrix} = \begin{bmatrix} \sum_{i=1}^N \phi_{1i} \\ \sum_{i=1}^N t_i \phi_{1i} \\ \sum_{i=1}^N t_i^2 \phi_{1i} \\ \sum_{i=1}^N t_i^3 \phi_{1i} \end{bmatrix} \quad [21]$$

Using the data from Tables I-III, pp. 19-21, with the set of 15 interpolation points ($N = 15$), the normal equations can be written as

$$9.309 = 15a_0 + 120a_1 + 1240a_2 + 14400a_3$$

$$81.164 = 120a_0 + 1240a_1 + 14400a_2 + 178312a_3$$

[22]

$$866.740 = 1240a_0 + 14400a_1 + 178312a_2 + 2299200a_3$$

$$10254.801 = 14400a_0 + 178312a_1 + 2299200a_2 + 30482920a_3.$$

Solving the above simultaneous equations for the coefficients a_0 through a_3 will yield

$$a_0 = 0.6873, \quad a_1 = -0.0033$$

$$a_2 = 0.0018, \quad a_3 = -0.0001.$$

Therefore, the trend polynomial of ϕ_1 is

$$\phi_1^* = 0.6873 - 0.0033t + 0.0018t^2 - 0.0001t^3. \quad [23]$$

In a similar manner the above approach can be used to determine the trend of the other variables ϕ_2 , V_1 , V_2 , V_3 , V_4 , and V_5 . These variables and their trends are shown in Figures 3 through 9, pp. 22-28, and their values are listed in Tables I-III, pp. 19-21. The normalized

deviations of the variables from their trends are defined as

$$\begin{aligned}\phi_1 &= \frac{\phi_1 - \phi_1^*}{\phi_1^*}, \quad \phi_2 = \frac{\phi_2 - \phi_2^*}{\phi_2^*} \\ v_1 &= \frac{v_1 - v_1^*}{v_1^*}, \quad v_2 = \frac{v_2 - v_2^*}{v_2^*} \\ v_3 &= \frac{v_3 - v_3^*}{v_3^*}, \quad v_4 = \frac{v_4 - v_4^*}{v_4^*} \\ v_5 &= \frac{v_5 - v_5^*}{v_5^*}.\end{aligned}\tag{24}$$

As previously mentioned, the above normalized deviations are considered as mutually dependent stationary random processes of time.

Selection of Time Interval Length

The choice of the length of time interval T is based on consideration of accuracy desired in the results. Details of this selection are not crucial to the present problem and the reader is referred to [6]. Therefore, the purpose of continuity in the text it suffices to mention that the minimum length of the interval is bounded by the nature of correlation functions between the input and output variables. For the following application the interval $T = 4$ is chosen as in [6]. The above selection is regarded as the "first self-selection threshold," i.e., the first attempt on selecting a threshold to be used in the algorithm.

From the preceding section we have a total of seven variables (five manipulated variables and two performance indices). Having chosen the time interval $T = 4$, there are five instances of time for each variable or 35 possible instances of time for all seven variables. These can be expressed as

$$\phi_{1k}, \phi_{1(k-1)}, \phi_{1(k-2)}, \phi_{1(k-3)}, \phi_{1(k-4)}$$

$$\phi_{2k}, \phi_{2(k-1)}, \phi_{2(k-2)}, \phi_{2(k-3)}, \phi_{2(k-4)}$$

$$v_{1k}, v_{1(k-1)}, v_{1(k-2)}, v_{1(k-3)}, v_{1(k-4)}$$

$$v_{2k}, v_{2(k-1)}, v_{2(k-2)}, v_{2(k-3)}, v_{2(k-4)} \quad [25]$$

$$v_{3k}, v_{3(k-1)}, v_{3(k-2)}, v_{3(k-3)}, v_{3(k-4)}$$

$$v_{4k}, v_{4(k-1)}, v_{4(k-2)}, v_{4(k-3)}, v_{4(k-4)}$$

$$v_{5k}, v_{5(k-1)}, v_{5(k-2)}, v_{5(k-3)}, v_{5(k-4)},$$

where as before, the instances of time are denoted as

k = future moment of time

$k - 1$ = instant moment

$k - 2$ = moment one step before

$k - 3$ = moment two step before

$k - 4$ = moment three step before.

From Chapter II, we recall the estimation equation for input x and output y is

$$y_k = f(y_{k-1}, y_{k-2}, y_{k-3}, \dots, x_k, x_{k-1}, x_{k-2}, x_{k-3}, \dots).$$

Therefore, from equation [6], the estimation equation for ϕ_1 is

$$\begin{aligned} \phi_{1k} = & f[\phi_1(k-1), \phi_1(k-2), \phi_1(k-3), \phi_1(k-4), \\ & \phi_2(k-1), \phi_2(k-2), \phi_2(k-3), \phi_2(k-4), \\ & v_{1k}, v_{1(k-1)}, v_{1(k-2)}, v_{1(k-3)}, v_{1(k-4)}, \\ & v_{2k}, v_{2(k-1)}, v_{2(k-2)}, v_{2(k-3)}, v_{2(k-4)}, \\ & v_{3k}, v_{3(k-1)}, v_{3(k-2)}, v_{3(k-3)}, v_{3(k-4)}, \\ & v_{4k}, v_{4(k-1)}, v_{4(k-2)}, v_{4(k-3)}, v_{4(k-4)}, \\ & v_{5k}, v_{5(k-1)}, v_{5(k-2)}, v_{5(k-3)}, v_{5(k-4)}] \end{aligned} \quad [26]$$

where the elements in the argument of the above equation are all possible input variables. By examining equation [26], we can note that all of the past, present, and future values of the variables are among the argument of the estimation equation with the exception of future values of the performance indices ϕ_1 , and ϕ_2 . This is true,

since ϕ_1 and ϕ_2 are those variables to be estimated for the future. Thus, obviously, their future values do not appear in the argument of the estimation equation [26]. For the sake of brevity, the estimation equation for ϕ_2 and the prediction equation for ϕ_1 and ϕ_2 are omitted here and can be found in Appendix A.

Selection of the More Useful Variables Via Correlation Criterion

From the preceding section we have a total of 33 arguments in the estimation equation. To shorten and ease the calculations we only retain 14 arguments. These include five future variables v_{1k} through v_{5k} (variables with index k are future variables where k denotes future moment of time) and nine other variables which have the strongest correlation with the output variable ϕ_1 . This is the "second self-selection threshold" obtained using correlation criterion. Using the general correlation coefficient formula

$$k_{\phi r}(\tau) = \frac{\sum_{i=1}^{i=n-\tau} \sum_{j=1+\tau}^{j=n} \phi_i v_j}{\left(\sum_{i=1}^{i=n-\tau} \phi_i^2 \sum_{j=1+\tau}^{j=n} v_j^2 \right)^{1/2}} \quad [27]$$

we can calculate the correlation coefficients of the variables ϕ and v for any time shift $\tau = 1, 2, 3, \dots$ where n is the number of data points. It can be shown that the 14 variables chosen by this criterion are⁶

$$\begin{array}{ll}
x_1 = v_{1k} & x_8 = \phi_1(k-2) \\
x_2 = v_{2k} & x_9 = v_2(k-2) \\
x_3 = v_{3k} & x_{10} = v_4(k-2) \\
x_4 = v_{4k} & x_{11} = v_1(k-3) \\
x_5 = v_{5k} & x_{12} = \phi_2(k-4) \\
x_6 = v_1(k-1) & x_{13} = v_2(k-4) \\
x_7 = v_3(k-1) & x_{14} = v_5(k-4)
\end{array} \quad [28]$$

where they are denoted by x_1 through x_{14} . The values of the above variables are shown in Table IV.

There are two central points to be emphasized at this stage. First, some of the variables (primarily the more ancient ones) chosen and listed above by A. G. Ivakhnenko cannot be determined due to the dependence on data for the period prior to 1954. This problem arises due to interdependence of variables, some of which (e.g., $v_{2(k-4)}$) require data on the period of time not under consideration (e.g., 1953, and so on). This problem can also be present when absence of data in the selected time period (in this case 1954-1968) exist. The approach used to remedy the lack of data in [6] is to assume a value of zero for all missing variables. This is presumably based

TABLE IV
THE 14 SELECTED VARIABLES

n	Year	Variable					
		$x_1 = V_{1k}$	$x_2 = V_{2k}$	$x_3 = V_{3k}$	$x_4 = V_{4k}$	$x_5 = V_{5k}$	$x_6 = V_{1(k-1)}$
	1969						-0.0039
1	1968	-0.0039	0.0062	0.0164	-0.0065	-0.0080	-0.0062
2	1967	-0.0062	0.1139	0.0170	-0.0052	0.0157	0.0212
3	1966	0.0212	-0.1281	-0.0557	0.0184	0.0009	0.0280
4	1965	0.0280	-0.1503	-0.0311	0.0104	-0.0025	-0.0406
5	1964	-0.0406	-0.0607	-0.0310	-0.0030	-0.0167	-0.0463
6	1963	-0.0463	0.1246	0.0961	-0.0039	-0.0040	0.0308
7	1962	0.0308	0.2422	-0.0135	-0.0049	0.0143	0.0353
8	1961	0.0353	-0.0212	-0.0222	-0.0115	0.0112	0.0006
9	1960	0.0006	-0.1851	0.0527	-0.0179	-0.0036	-0.0306
10	1959	-0.0306	0.0488	0.1007	-0.0030	-0.0054	0.0012
11	1958	0.0012	0.0524	-0.0440	0.0186	-0.0161	0.0203
12	1957	0.0203	-0.0086	-0.0590	0.0144	0.0118	-0.0022
13	1956	-0.0022	-0.0857	-0.0812	0.0089	0.0161	-0.0170

TABLE IV (Continued)

n	Year	Variable											
		$x_7^=V_3(k-1)$	$x_8^=\phi_1(k-2)$	$x_9^=V_2(k-2)$	$x_{10}^=V_4(k-2)$	$x_{11}^=V_3(k-3)$	$x_{12}^=2(k-4)$	$x_{13}^=V_2(k-4)$	$x_{14}^=V_5(k-4)$				
	1969	0.0164	-0.0118	0.1139	-0.0052	0.0212	-0.6112	-0.1503	-0.0025				
1	1968	0.0170	-0.0149	-0.1281	0.0184	0.0280	3.4752	-0.0607	-0.0167				
2	1967	-0.0557	0.0093	-0.1503	0.0104	-0.0406	-3.8759	0.1246	-0.0040				
3	1966	-0.0311	0.0153	-0.0607	-0.0030	-0.0463	19.9048	0.2422	0.0143				
4	1965	-0.0310	-0.0072	0.1246	-0.0039	0.0308	-1.0845	-0.0212	0.0112				
5	1964	0.0961	-0.0088	0.2422	-0.0049	0.0353	-4.1646	-0.1851	-0.0036				
6	1963	-0.0135	0.0153	-0.0212	-0.0115	0.0006	0.2760	0.0488	-0.0054				
7	1962	-0.0222	0.0235	-0.1851	-0.0179	-0.0306	1.6507	0.0524	-0.0161				
8	1961	0.0527	-0.0192	0.0488	-0.0030	0.0012	0.7343	-0.0086	0.0118				
9	1960	0.1007	-0.0416	0.0524	0.0186	0.0203	0.6876	-0.0857	0.0161				
10	1959	-0.0440	-0.0064	-0.0086	0.0144	-0.0022	-2.6497	0.0290	-0.0176				
11	1958	-0.0590	0.0088	-0.0857	0.0089	-0.0107	1.6634	0.0224	0.0042				
12	1957	-0.0812	0.0158	0.0290	-0.0044	0.0029	----	----	----				
13	1956	-0.0216	0.0048	0.0224	-0.0108	----	----	----	----				

on the assertion that beyond the origin of time chosen (i.e., prior to 1954) the trend of the variables--the starred quantities--perfectly fit the original input variables. Thus, for example,

$$v_1 = \frac{V_1 - V_1^*}{V_1^*} = 0$$

for time before 1954 and so on. This approach and its above inherent assumption is seen to be both unnecessary and unjustified. We will return to this matter in more detail in the next chapter.

Second, the future variables in equation [28], i.e., x_1 , x_2 , x_3 , x_4 and x_5 , are excluded from the selection criterion (correlation criterion) based on the assumption that their existence in the estimation equation of ϕ_1 is necessary. This necessity is presumed since these five variables are future signals, i.e, their existence in the equation is necessary for the purpose of manipulation of the performance index value for optimizing the future behavior of the economic system. As will be shown in this work (Chapter III), the exclusion of these variables from the criterion used for the selection of other variables (correlation criterion) is both unwarranted and undesirable. Indeed, for the example of English economy, it will be shown that the extension of correlation criterion to the selection of these future variables leads to more accurate results. The values of the above selected variables from 1954 to 1968 are shown in Table IV, where each year is designated by a number.

Thus with the selection of the 14 variables the estimation equation now becomes

$$\phi_{1k} = f(x_1, x_2, x_3, x_4, x_5, x_6, x_7, x_8, x_9, x_{10}, x_{11}, x_{12}, x_{13}, x_{14})$$

[29]

Construction of the Training and Testing Data

The graphs of the original variables and their trends are shown in Figures 3 through 9, pp. 22-28, and their values are listed in Tables I-III, pp. 19-21. Since we have input data for 15 years, and having chosen the time interval length equal to four years, we are left with 13 points of interpolation signifying data for 13 years. These points are divided into two unequal sets on the basis of their variance. Ten points with larger variations are chosen as one set, i.e., the training data, and the remaining points are used for testing data. This method is known as decision regularization. Training data is strictly used for interpolation, i.e., calculation of the polynomial coefficients. And, the testing data are used for the purpose of filtration or rejection of the unwanted results. As an example, let us determine the variations of the point 5.

$$\begin{aligned} D^2 &= \frac{1}{n}(x_1^2 + x_2^2 + x_3^2 + \dots + x_n^2) \\ &= \frac{1}{14}[(-.0406)^2 + (-0.0607)^2 + (-0.0310)^2 + \dots \\ &\quad + (-0.0036)^2] = 17.456. \end{aligned}$$

The importance of the decision regularization method in GMDH lies in the fact that it incorporates those interpolation points with

the largest variance in the training data. The reason is that small variations of the interpolation points would only cause small variations of the result (polynomial coefficients). In this way the choice of the training data is made (which consists of points with larger variations) for the purpose of interpolation. Naturally, with this choice a more stable solution, or so-called regularized decision, is achieved. On the other hand, if one chooses to use the original data (13 points) for interpolation, the solutions become unstable, i.e., small variations of the original data yield large variations of the result. Therefore, by regularization method this problem can be avoided.

An optimum distribution of the points among the training and testing data can be obtained by a number of trials. In our case, it is found⁶ that the prediction accuracy of the variables will be the greatest by choosing ten points in the training data and three points in the testing data. These two sets of data are shown in Tables V and VI, respectively, along with the variance for each point.

First Layer of Selection

As we know, from earlier sections, there are a total of 14 variables and 13 interpolation points. Ten of these 13 interpolation points are in the training data, and the remaining three points are in the testing data. By combining the 14 x variables two at a time using the second degree Kolmogorov-Gabor polynomial

$$y_n = a_{0n} + a_{1n}x_b + a_{2n}x_c + a_{3n}x_b^2 + a_{4n}x_c^2 + a_{5n}x_bx_c \quad [30]$$

TABLE V
THE TRAINING DATA

n	3	5	2	1	10	7	11	4	8	9
D^2	396.288	17.456	15.080	12.100	7.0381	2.826	2.786	1.220	0.5480	0.5350
ϕ_1	-0.0149	0.0153	-0.0118	0.0052	-0.0192	-0.0088	-0.0416	0.0093	0.0153	0.0235
x_1	0.0212	-0.0406	-0.0062	-0.0039	-0.0306	0.0308	0.0012	0.0280	0.0353	0.0006
x_2	-0.1281	-0.0607	0.1139	0.0062	0.0488	0.2422	0.0524	-0.1503	-0.0212	-0.1851
x_3	-0.0557	-0.0310	0.0170	0.0164	0.1007	-0.0135	-0.0440	-0.0311	-0.0222	0.0527
x_4	0.0184	-0.0030	-0.0052	-0.0065	-0.0030	-0.0049	0.0186	0.0104	-0.0115	-0.0179
x_5	0.0009	-0.0167	0.0157	-0.0080	-0.0054	0.0143	-0.0161	-0.0025	0.0112	-0.0036
x_6	0.0280	-0.0463	0.0212	-0.0062	0.0012	0.0353	0.0203	-0.0406	0.0006	-0.0306
x_7	-0.0311	0.0961	-0.0557	0.0170	-0.0440	-0.0222	-0.0590	-0.0310	0.0527	0.1007
x_8	0.0153	-0.0088	0.0093	-0.0149	-0.0064	0.0235	0.0088	-0.0072	-0.0192	-0.0416
x_9	-0.0607	0.2422	-0.1503	-0.1281	-0.0086	-0.1851	-0.0857	0.1246	0.0488	0.0524
x_{10}	-0.0030	-0.0049	0.0104	0.0184	0.0144	-0.0179	0.0089	-0.0039	-0.0030	0.0186
x_{11}	-0.0463	0.0353	-0.0406	0.0280	-0.0022	-0.0306	-0.0107	0.0308	0.0012	0.0203
x_{12}	19.9048	-4.1646	-3.8759	3.4752	-2.6497	1.6507	1.6634	-1.0845	0.7343	0.6876
x_{13}	0.2422	-0.1851	0.1246	-0.0607	0.0290	0.0524	0.0224	-0.0212	-0.0086	-0.0857
x_{14}	0.0143	-0.0036	-0.0040	-0.0167	-0.0176	-0.0161	0.0042	0.0112	0.0118	0.0161

TABLE VI
THE TESTING DATA

n	6	13	12
D^2	0.1080	0.0160	0.0130
ϕ_1	-0.0072	0.0088	-0.0064
x_1	-0.0463	-0.0022	0.0203
x_2	0.1246	-0.0857	-0.0086
x_3	0.0961	-0.0812	-0.0590
x_4	-0.0039	0.0089	0.0144
x_5	-0.0040	0.0161	0.0118
x_6	0.0308	-0.0170	-0.0022
x_7	-0.0135	-0.0216	-0.0812
x_8	0.0153	0.0048	0.0158
x_9	-0.0212	0.0224	0.0290
x_{10}	-0.0115	-0.0108	-0.0044
x_{11}	0.0006	---	0.0029
x_{12}	0.2760	---	---
x_{13}	0.0488	---	---
x_{14}	-0.0054	---	---

(where $n = 1, 2, 3, \dots, 91$ and b and c are the indices of variables x for all combinations. Table VII shows the change of index of variables y for different combinations of the variables x . For example, combinations of x_4 and x_7 would yield y_{39}), we obtain a total of 91 equations. The coefficients of all 91 equations can now be found using the least squares method. The normal equations in matrix form are:

$$\begin{bmatrix}
 N & \sum x_{bi} & \sum x_{ci} & \sum x_{bi}^2 & \sum x_{ci}^2 & \sum x_{bi}x_{ci} \\
 \sum x_{bi} & \sum x_{bi}^2 & \sum x_{bi}x_{ci} & \sum x_{bi}^3 & \sum x_{bi}x_{ci}^2 & \sum x_{bi}^2x_{ci} \\
 \sum x_{ci} & \sum x_{bi}x_{ci} & \sum x_{ci}^2 & \sum x_{bi}^2x_{ci} & \sum x_{ci}^3 & \sum x_{bi}x_{ci}^2 \\
 \sum x_{bi}^2 & \sum x_{bi}^3 & \sum x_{bi}^2x_{ci} & \sum x_{bi}^4 & \sum x_{bi}^2x_{ci}^2 & \sum x_{bi}^3x_{ci} \\
 \sum x_{ci}^2 & \sum x_{bi}x_{ci}^2 & \sum x_{ci}^3 & \sum x_{bi}^2x_{ci}^2 & \sum x_{ci}^4 & \sum x_{bi}x_{ci}^3 \\
 \sum x_{bi}x_{ci} & \sum x_{bi}^2x_{ci} & \sum x_{bi}x_{ci}^2 & \sum x_{bi}^3x_{ci} & \sum x_{bi}x_{ci}^3 & \sum x_{bi}^2x_{ci}^2
 \end{bmatrix}
 \begin{bmatrix}
 a_{0n} \\
 a_{1n} \\
 a_{2n} \\
 a_{3n} \\
 a_{4n} \\
 a_{5n}
 \end{bmatrix}
 =
 \begin{bmatrix}
 \sum y_{ni} \\
 \sum x_{bi}y_{ni} \\
 \sum x_{ci}y_{ni} \\
 \sum x_{bi}^2y_{ni} \\
 \sum x_{ci}^2y_{ni} \\
 \sum x_{bi}x_{ci}y_{ni}
 \end{bmatrix}
 \quad [31]$$

where the above summations are from $i = 1$ to N . The training data is now used in [31] resulting in six normal equations involving the six unknown coefficients. By substitution of testing set variables in every equation for each point we thus obtain three values of y for every pair of testing variables substituted. Therefore, a total of 91 sets of testing points are obtained. We wish to choose 14 sets of the above among the 91 available sets. This is accomplished by calculating the deviation of each set (composed of three values of y) from

TABLE VII
NUMERATION OF INDICES

i	1	2	3	4	5	6	7	8	9	10	11	12	13	14
j														
1		1	2	3	4	5	6	7	8	9	10	11	12	13
2			14	15	16	17	18	19	20	21	22	23	24	25
3				26	27	28	29	30	31	32	33	34	35	36
4					37	38	39	40	41	42	43	44	45	46
5						47	48	49	50	51	52	53	54	55
6							56	57	58	59	60	61	62	63
7								64	65	66	67	68	69	70
8									71	72	73	74	75	76
9										77	78	79	80	81
10											82	83	84	85
11												86	87	88
12													89	90
13														91

Note: This table furnishes appropriate subscripts for selected variables in each layer, i.e., $y_{x_i x_j}$, $z_{y_i y_j}$, ..., (e.g., $y_{x_5 x_3} = y_{27}$ and $z_{y_{12} y_5} = z_{53}$...).

the performance index ϕ_1 using the following error equation

$$\epsilon = \frac{\Delta}{\sum_{i=1}^3 \phi_1^2} \cdot 100 = \frac{\sum_{i=1}^3 (\phi_i - y_i)^2}{\sum_{i=1}^3 \phi_i^2} \cdot 100. \quad [32]$$

We choose 14 smallest values of Δ from among the 91 obtained by the use of equation [32]. Thus the choice of 14 Δ 's makes available the 14 equations for the variable y . The 14 pairs of testing variables x used in that calculation, which resulted in 14 values of y , will be our new testing set. Having already obtained the values of y from the above equations for the testing data of the next layer, we proceed to obtain these values (using the 14 equations) for the training data. Thus every pair of x 's in the training data is now substituted in the chosen 14 y equations giving rise to a new set of 14 values for y , each containing a value for each point (10 values for 10 points).

Therefore, we have completed the first layer of selection by obtaining the 14 sets of variables y which is now itself composed of a testing set (using testing values of x) and a training set (using training values of x). These new sets of data (variable y) are to be used for the next layer of selection.

Second Layer of Selection

As in the first layer, the 14 chosen y variables are combined two at a time to give 91 polynomials for the second layer. As before, these are written in the form

$$z_n = b_{0n} + b_{1n}y_b + b_{2n}y_c + b_{3n}y_b^2 + b_{4n}y_c^2 + b_{5n}y_by_c \quad [33]$$

where again, $n = 1, 2, 3, \dots, 91$ and b and c are the indices for all 91 combinations (refer to Table VII). We proceed with the calculation of the coefficients of all the equations and estimate the accuracy of each variable z in the exact manner used for variable y . Having constructed the new training and testing data of the variable z , we can proceed in the same manner with the calculation in the next layer and the layers thereafter. Each layer is designated by a new variable and is given as:

Layer	Variable	Layer	Variable
0	x	4	w
1	y	5	p
2	z	6	q
3	v		

Last Layer of Selection and the Final Results

In the last layer of selection (in this case, the third layer) instead of choosing 14 variables, only one variable with the smallest mean square error is chosen which is the final answer. Table VIII shows the selected variables in each layer with their mean square error. We note that in the third layer the variable with the smallest mean square error is v_{45} with $\Delta = 4.6 \times 10^{-6}$. As we proceed with different layers, we also note that the percent change in mean square error decreases consistently. However, its rate of decreasing diminishes for each new layer, that is, the mean square error becomes more "stable."

TABLE VIII
CHANGE OF MEAN SQUARE ERROR BETWEEN LAYERS

First Layer	Δ	Second Layer	Δ	Third Layer	Δ	Fourth Layer	Δ
y_8	0.0002018	z_3	0.0001203	v_1	0.0000361	w_{10}	0.0000086
y_{23}	0.000077	z_{17}	0.0000295	v_{14}	0.0002333	w_{17}	0.0000859
y_{26}	0.0028734	z_{31}	0.0004168	v_{26}	0.0000650	w_2	0.0000391
y_{42}	0.0001649	z_{37}	0.0000527	v_{45}	0.0000046	w_{38}	0.0000049
y_{37}	0.000214	z_{49}	0.0000492	v_{47}	0.0000077	w_{37}	0.0000140
y_{58}	0.0002541	z_{62}	0.0000248	v_{38}	0.0000267	w_{61}	0.0000038
y_{65}	0.0004655	z_{39}	0.0001265	v_{68}	0.0000562	w_{39}	0.0000238
y_{72}	0.0001619	z_{40}	0.0000809	v_{57}	0.0000345	w_{57}	0.0000075
y_{80}	0.0001437	z_{58}	0.0001100	v_8	0.0000681	w_{79}	0.0000179
y_{83}	0.0001177	z_{21}	0.0001258	v_{42}	0.0000195	w_{83}	0.0000079
y_{82}	0.0002175	z_{22}	0.0000796	v_{52}	0.0000081	w_{87}	0.0000091
y_{79}	0.0001454	z_{23}	0.0000839	v_{53}	0.0000208	w_{44}	0.0000054
y_{24}	0.0001718	z_{87}	0.0001116	v_{54}	0.0000105	w_{12}	0.0000138
y_{85}	0.0003389	z_{76}	0.0001689	v_{13}	0.0001046	w_{91}	0.0000410

It is noteworthy to emphasize that further filtration proceeding with more selection of layers would tend to result in "degeneration." Thus, it is important that an optimum number of layer selection be made where stability is achieved but "degeneration" is avoided. In this example, the final solution is selected in the third layer (i.e., v_{45} with $\Delta = 4.6 \times 10^{-6}$) which gives rise to the estimation equation for ϕ_1 as

$$\begin{aligned}\phi_1 = v_{45} &= .003323 + 0.646507z_{87} + 0.526509z_{37} \\ &\quad - 16.453265z_{87}z_{37} + 0.865475z_{87}^2 + 0.180493z_{37}^2 \\ z_{87} &= 0.008973 - 0.156455y_{82} + 1.251193y_{24} - 39.357736y_{82}y_{24} \\ &\quad + 8.323526y_{82}^2 - 17.029865y_{24}^2 \\ z_{37} &= 0.007554 + 1.215627y_{37} - 0.093074y_{42} - 134.128761y_{37}y_{42} \\ &\quad + 62.837984y_{37}^2 + 82.155422y_{42}^2 \\ y_{82} &= -0.105098 - 0.279247x_{10} + 0.502991x_{11} + 6.177137x_{10}x_{11} \\ &\quad + 14.975779x_{10}^2 + 8.386482x_{11}^2 \quad [34] \\ y_{24} &= -0.009200 - 0.075002x_2 - 0.075860x_{13} + 0.283867x_2x_{13} \\ &\quad + 0.276377x_2^2 + 0.126828x_{13}^2 \\ y_{37} &= -0.002446 - 0.900065x_4 - 0.041400x_5 + 78.041982x_4x_5 \\ &\quad + 19.375319x_4^2 - 1.912654x_5^2 \\ y_{42} &= 0.00463879 - 0.945760x_4 - 0.728913x_{10} - 109.743571x_4x_{10} \\ &\quad - 8.106316x_4^2 - 53.126352x_{10}^2.\end{aligned}$$

Optimization Equations

From the preceding section, the estimation equation of the variable ϕ_1 can be written in the general form as

$$\phi_1 = f(c, v_1, v_2, v_3, v_4, v_5) \quad [35]$$

where C is a constant. For maximizing the performance index (in the case of ideal economic condition $\phi_1 = 1$), the derivative of the above equation [35] with respect to each manipulated variable is set equal to zero. These are given as

$$\begin{aligned} \frac{\partial \phi_1}{\partial v_1} = 0, \quad \frac{\partial \phi_1}{\partial v_2} = 0, \\ \frac{\partial \phi_1}{\partial v_3} = 0, \quad \frac{\partial \phi_1}{\partial v_4} = 0, \quad \frac{\partial \phi_1}{\partial v_5} = 0. \end{aligned} \quad [36]$$

Using the above equations we can find the optimal values of the manipulated variables which maximizes the performance index of the British economy for the year 1969.

Prediction for 1969

Given the manipulated variables for the year 1969, the control equation [34] can be used to predict the performance index for that year. From [6], the values of the 1969 manipulated variables are given as

$$\begin{aligned} x_1 &= 0.01196539 & x_3 &= 0.04597296 & x_5 &= 0.0408790 \\ x_2 &= 0.1243898 & x_4 &= 0.04441776 \end{aligned}$$

By substituting the above values in the estimation equation [34], we can obtain ϕ_1 as

$$\phi_1 = 0.0177.$$

From Tables I-III, pp. 19-21, ϕ_1^* (trend of ϕ) is

$$\phi_1^* = 0.6857.$$

Calculating the performance index ϕ_1 from the normalized deviations equation [24], the predicted value for ϕ_1 can be obtained using

$$\phi_1 = \phi_1 \phi_1^* + \phi_1^* \quad [37]$$

Substituting the values of ϕ_1^* and ϕ_1 in the above relation we obtain

$$\phi_1 = 0.6857(1 + 0.0177),$$

Therefore

$$\phi_1 = 0.6978$$

which is the predicted value of ϕ_1 for the year 1969. The actual value of the performance index ϕ_1 for 1969 is given as

$$\phi_1 = 0.6910.$$

By observing the above results we note that with the use of the estimation equation in predicting the performance index an accurate result with extremely low error of 0.98% can be obtained.

II. RECOMPUTATION OF THE A. G. IVAKHNENKO EXAMPLE: THE BRITISH ECONOMY

In the following section we will proceed with a recomputation of the British economy example using A. G. Ivakhnenko's method. The reason for this is to check and ensure absence of any miscomputations or errors in his solutions. We will use the block diagram of the GMDH algorithm shown in Figure 2, p. 16, with a brief explanation of the algorithm procedure. However, before we proceed with the recomputation, it is important to note that the first three blocks of the GMDH algorithm (Figure 2, p. 16), namely, selection of manipulated variables and performance indices, calculation of normalized deviations, and selection of the prehistory interval length are omitted. These steps are omitted to simplify the calculations as well as emphasize the recomputation of the more significant features of the problem. The first step is omitted because the related variables are simply suggested by the economical aspect of the problem. The second step (calculation of normalized deviations) is omitted and A. G. Ivakhnenko's calculated values are chosen directly. This greatly simplifies comparison of results with A. G. Ivakhnenko's values since we begin the calculation with his third degree polynomial fit to the data and thus the recomputation of this problem will show the deviation from A. G. Ivakhnenko's values in the more significant steps of the problem. The third step (selection of the prehistory interval) is chosen also for reasons of convenience in

comparison of the two calculations. Therefore, we will start with the fourth block of the GMDH algorithm block diagram of Figure 2, p. 16.

Selection of the More Useful Variables Via Correlation Criterion

From estimation equation (equation [26]) we have a total of 33 arguments from which 14 most useful arguments are chosen which have the strongest correlation with the output variable ϕ_1 . These include five future variables ($v_{1k}, v_{2k}, v_{3k}, v_{4k}, v_{5k}$) and nine other variables with the strongest correlation with the output variable. As previously mentioned, the above future variables are excluded from the selection criterion used by A. G. Ivakhnenko. The remaining nine variables with their correlation coefficient are found as

$$K\phi_1 v_{1(k-1)} = 0.6256 \quad K\phi_1 v_{2(k-2)} = 0.5676 \quad K\phi_1 \phi_{2(k-4)} = 0.4807$$

$$K\phi_1 v_{3(k-1)} = -0.6638 \quad K\phi_1 v_{4(k-2)} = 0.4526 \quad K\phi_1 v_{2(k-4)} = 0.5231$$

$$K\phi_1 \phi_{1(k-2)} = 0.6474 \quad K\phi_1 v_{1(k-3)} = -0.6925 \quad K\phi_1 v_{5(k-4)} = -0.6867$$

Hence, the 14 chosen variables are

$$\begin{array}{lll} x_1 = v_{1k} & x_6 = v_{1(k-1)} & x_{11} = v_{1(k-3)} \\ x_2 = v_{2k} & x_7 = v_{3(k-1)} & x_{12} = \phi_{2(k-4)} \\ x_3 = v_{3k} & x_8 = \phi_{1(k-2)} & x_{13} = v_{2(k-4)} \quad [38] \\ x_4 = v_{4k} & x_9 = v_{2(k-2)} & x_{14} = v_{5(k-4)} \\ x_5 = v_{5k} & x_{10} = v_{4(k-2)} & \end{array}$$

where they are denoted by x_1 through x_{14} . The above selected variables are exactly similar to the variables selected by A. G. Ivakhnenko (equation [28]). These variables along with the values for different years are shown in Table IV, p. 37, where each year is designated by a number. Having chosen the 14 variables, the estimation equation now becomes, as before,

$$\phi_{1k} = f(x_1, x_2, x_3, x_4, x_5, x_6, x_7, x_8, x_9, x_{10}, x_{11}, x_{12}, x_{13}, x_{14}).$$

Construction of the Training and Testing Data

There are 13 points of interpolation signifying data for 13 years. In a similar manner discussed earlier, these data points are divided into two sets, training set and testing set on the basis of their variance. Training data includes ten remaining three points. The variations for each point is calculated and are found as

$D_1^2 = 12.100$	$D_7^2 = 2.826$
$D_2^2 = 15.080$	$D_8^2 = 0.548$
$D_3^2 = 396.288$	$D_9^2 = 0.535$
$D_4^2 = 1.220$	$D_{10}^2 = 7.0381$
$D_5^2 = 17.456$	$D_{11}^2 = 2.786$
$D_6^2 = 0.108$	$D_{12}^2 = 0.013$
	$D_{13}^2 = 0.016$

The above variations are the same as found by A. G. Ivakhnenko. The two data sets are shown in Table V, p. 42, and Table VI, p. 43.

First Layer of Selection

By combining the 14 x variables two at a time using the second degree Kolmogorov-Gabor polynomial

$$y_n = a_{0n} + a_{1n}x_b + a_{2n}x_c + a_{3n}x_b^2 + a_{4n}x_c^2 + a_{5n}x_bx_c$$

(where $n = 1, 2, 3, \dots, 91$ and b and c are the indices of variables x for all 91 combinations), a combination of 91 equations can likewise be obtained. The training data is now used in the Least Square Method to find all the coefficients (a 's) of the 91 equations (y 's) from which we choose only 14 variables using the testing data.

Second and Remaining Layers of Selection

The new generated set of y variables (which itself is composed of training and testing data) is now used to generate a set of new variables z by the same process which led to the y variables. We proceed in the same manner with the transformations of the next layer and the layers thereafter. In the last layer of selection (third layer) we only select one variable with smallest mean square error which is the final answer. Table IX shows the results of our recomputations with all the selected variables in each layer and their mean square error. Using the results obtained, the estimation equation for the output variable ϕ_1 is now found to be

TABLE IX

RECOMPUTED VALUES OF CHANGE OF MEAN SQUARE ERROR BETWEEN LAYERS

First Layer	Δ	Second Layer	Δ	Third Layer	Δ
y_8	0.0006749	z_3	0.0004774	v_1	0.0002316
y_{23}	0.0000770	z_{17}	0.0000280	v_{14}	0.0002267
y_{26}	0.0028734	z_{31}	0.0004116	v_{26}	0.0000634
y_{42}	0.0001649	z_{37}	0.0000522	v_{45}	0.0000053
y_{37}	0.0002140	z_{49}	0.0004907	v_{47}	0.0000056
y_{58}	0.0001488	z_{62}	0.0000282	v_{38}	0.0000272
y_{65}	0.0004655	z_{39}	0.0001252	v_{68}	0.0000564
y_{72}	0.0001629	z_{40}	0.0000809	v_{57}	0.0000347
y_{80}	0.0001437	z_{58}	0.0000283	v_8	0.0001454
y_{83}	0.0001177	z_{21}	0.0001224	v_{42}	0.0000196
y_{82}	0.0002175	z_{22}	0.0000813	v_{52}	0.0000080
y_{79}	0.0001454	z_{23}	0.0000853	v_{53}	0.0000207
y_{24}	0.0001718	z_{87}	0.0001152	v_{54}	0.0000095
y_{85}	0.0003389	z_{76}	0.0001674	v_{13}	0.0005882

$$\phi_1 = v_{45} = 0.005282 + 0.645242z_{87} + 0.528138z_{37}$$

$$- 16.878164z_{87}z_{37} + 0.978970z_{87}^2 + 0.511868z_{37}^2$$

$$z_{87} = 0.009100 - 0.168078y_{72} + 1.264174y_{24} - 40.012862y_{82}y_{24}$$

$$+ 8.875416y_{82}^2 - 17.485793y_{24}^2$$

$$z_{37} = -0.007578 + 1.217107y_{37} - 0.097732y_{42} - 134.227888y_{37}y_{42}$$

$$+ 62.460627y_{37}^2 + 82.788614y_{42}^2$$

$$y_{82} = -0.010510 - 0.279248x_{10} + 0.502992x_{11} + 6.177137x_{10}x_{11}$$

$$+ 14.975779x_{10}^2 + 8.386482x_{11}^2 \quad [39]$$

$$y_{24} = -0.009201 - 0.075002x_2 - 0.075861x_{13} + 0.283867x_2x_{13}$$

$$+ 0.276377x_2^2 + 0.126828x_{13}^2$$

$$y_{37} = -0.002446 - 0.900066x_4 - 0.041400x_5 + 78.041982x_4x_5$$

$$+ 19.375320x_4^2 - 1.912654x_5^2$$

$$y_{42} = 0.004639 - 0.945761x_4 - 0.728914x_{10} - 109.743571x_4x_{10}$$

$$- 8.106317x_4^2 - 53.126353x_{10}^2.$$

By comparing the results of Table IX to that of Table VIII, p. 48, we note the similarity of the results in each selection layer. This similarity also appears in our final answer (estimation equation of ϕ_1) when compared to equation [34] of the last section. There are, however, small numerical differences between the coefficients in the two equations as it is evident from this comparison.

Prediction for 1969

The values of the 1969 manipulated variables given as [6]

$$x_1 = 0.01196539$$

$$x_3 = -0.04597296$$

$$x_2 = 0.1243898$$

$$x_4 = 0.04441776$$

$$x_5 = 0.0408790$$

can now be substituted in the ϕ_1 estimation equation. Thus following earlier analysis we find the predicted value of the performance index ϕ_1 for the year 1969 to be

$$\phi_1 = 0.6890.$$

The result obtained for ϕ_1 compares well with the A. G. Ivakhnenko calculated value. The small difference between the two results is probably due to errors associated with numerical roundoff, and so on and is of no serious concern. Therefore, we can conclude

with reasonable accuracy that the application of A. G. Ivakhnenko's method to the British economy example as performed in Reference 6 is free of error associated with numerical or computational steps. However, it remains to be seen that basic modifications in the algorithm introduced into A. G. Ivakhnenko's method will result in greatly more accurate results. These improvements in the accuracies of the solutions to be obtained in the following chapter are therefore not of numerical nature but rather of fundamental improvements in the method itself.

CHAPTER IV

MODIFIED GMDH ALGORITHM FOR THE EXAMPLE OF BRITISH ECONOMY

I. CONSTRUCTION OF TRAINING AND TESTING DATA AND SELECTION OF LAYERS

In this chapter a modified approach to the previous work of A. G. Ivakhnenko on application of the polynomial theory in determining the estimation equation of the British economy for the year 1969 is studied. There are two major modifications in the algorithm of the Group Method of Data Handling. First, as opposed to the nine arguments chosen in the previous work, we select 14 arguments in the estimation equation of the economic index which have the strongest correlations with the output variable (performance index). Second the absence of data information associated with some of the variables in the argument of the estimation equation are considered as missing data and are not assumed zero as is done in A. G. Ivakhnenko's application. The above approaches are discussed in detail in this chapter as well as the final results which are presented at the end of Chapter IV. The computer algorithms are programmed in FORTRAN IV language, and are given in Appendix B.

As in the previous chapter, we proceed with the algorithm of the GMDH to obtain the estimation equation of the performance index ϕ_1 of the British economy for the year 1969. The block diagram of the GMDH algorithm is shown in Figure 2, p. 16. A more brief description of the steps in the block diagram is discussed in the following.

Selection of the Manipulated Variables and Performance Indices

As before⁶, we choose the following variables for the purpose of manipulation. These are denoted by V_1 , V_2 , and so on, and are given as

$$V_1 = T_h - \text{taxes on households}$$

$$V_2 = T_c - \text{taxes on companies}$$

$$V_3 = U - \text{the state allowances (e.g., to the unemployed)}$$

$$V_4 = K - \text{total gross capital stock}$$

$$V_5 = g - \text{government expenditures.}$$

The above are, therefore, regarded as manipulated variables. Similarly, the following variables are chosen as the performance indices of the British economy. They are given as

$$\Phi_1 = \frac{\omega + \pi}{C_4 K}$$

$$\Phi_2 = I_{fg} + I_{fc}$$

where

ω = income from employment

π = gross trading surplus

$$C_4 = \frac{1}{4} \text{ (constant)}$$

I_{fg} = net investment of government abroad

I_{fc} = net investment of companies abroad.

Therefore, we have a total of seven variables which consist of five manipulated variables ($V_1, V_2, \dots, V_5$) and two performance indices (ϕ_1 and ϕ_2).

Calculation of Normalized Deviations of Variables from Their Trends

The trends of the above selected variables are calculated using the third degree polynomial

$$\phi_1^* = a_0 + a_1 t + a_2 t^2 + a_3 t^3. \quad [40]$$

By applying the method of least squares, and data from Tables I-III, pp. 19-21, in equation [40], we find a system of four normal Gaussian equations with four coefficients of equation [40] as unknown. Thus,

$$9.309 = 15a_0 + 120a_1 + 1240a_2 + 14400a_3$$

$$81.164 = 120a_0 + 1240a_1 + 14400a_2 + 178312a_3$$

$$866.740 = 1240a_0 + 14400a_1 + 178312a_2 + 2299200a_3$$

$$10254.801 = 14400a_0 + 178312a_1 + 2299200a_2 + 30482920a_3.$$

The simultaneous solution to the above equations will give us the coefficients of the equation [40] as

$$\begin{aligned} a_0 &= 0.6873 & a_2 &= 0.0018 \\ a_1 &= -0.0033 & a_3 &= -0.0001. \end{aligned}$$

Equation [40] now becomes

$$\phi_1^* = 0.6873 - 0.0033t + 0.0018t^2 - 0.0001t^3.$$

As mentioned in Chapter III, by using the above approach, the trend of the other variables can be found in a similar manner.

Having calculated the trends of each variable, their normalized deviations from their trends can be determined by the relations

$$\phi_1 = \frac{\phi_1 - \phi_1^*}{\phi_1^*}$$

$$v_1 = \frac{V_1 - V_1^*}{V_1^*}$$

$$v_3 = \frac{V_3 - V_3^*}{V_3^*}$$

$$\phi_2 = \frac{\phi_2 - \phi_2^*}{\phi_2^*}$$

$$v_2 = \frac{V_2 - V_2^*}{V_2^*}$$

$$v_4 = \frac{V_4 - V_4^*}{V_4^*}$$

$$v_5 = \frac{V_5 - V_5^*}{V_5^*}$$

The values of the above variables and their trends, as well as their normalized deviations, are listed in Tables I-III, pp. 19-21.

Selection of Time Interval Length

Similar to the preceding chapter, the length of the time interval is chosen equal to $T = 4$, which can be regarded as our first self-selection threshold. From equation [6], the estimation equation for ϕ_1 can be written as

$$\begin{aligned}
\phi_{1k} = f[& \phi_1(k-1), \phi_1(k-2), \phi_1(k-3), \phi_1(k-4), \\
& \phi_2(k-1), \phi_2(k-2), \phi_2(k-3), \phi_2(k-4), \\
& v_{1k}, v_{1(k-1)}, v_{1(k-2)}, v_{1(k-3)}, v_{1(k-4)}, \\
& v_{2k}, v_{2(k-1)}, v_{2(k-2)}, v_{2(k-3)}, v_{2(k-4)}, \\
& v_{3k}, v_{3(k-1)}, v_{3(k-2)}, v_{3(k-3)}, v_{3(k-4)}, \\
& v_{4k}, v_{4(k-1)}, v_{4(k-2)}, v_{4(k-3)}, v_{4(k-4)}, \\
& v_{5k}, v_{5(k-1)}, v_{5(k-2)}, v_{5(k-3)}, v_{5(k-4)}]
\end{aligned}
\tag{41}$$

with the subscripts defined as

- k = future moment of time
- $k - 1$ = present moment
- $k - 2$ = moment one step before
- $k - 3$ = moment two steps before
- $k - 4$ = moment three steps before.

Selection of More Useful Arguments Via Correlation Criterion

There are 33 variables in the argument of the estimation equation [40] shown on Table X. By inspecting Table X, we note that the 1956 values of all variables with moment two steps before (i.e.,

TABLE X
TRANSFORMATION OF INPUT DATA TO THE SET OF VARIABLES

Year	Variable						
	ϕ_{1k}	ϕ_{2k}	v_{1k}	v_{2k}	v_{3k}	v_{4k}	v_{5k}
1968	0.005200	0.195100	0.003900	0.006200	0.016400	-0.006500	-0.008000
1967	-0.011800	0.419600	-0.006200	0.113900	0.017000	-0.005200	0.015700
1966	-0.014900	-1.382000	0.021200	-0.128100	-0.055700	0.018400	0.000900
1965	0.009300	-0.611200	0.028000	-0.150300	-0.031100	0.010400	-0.002500
1964	0.015300	3.475200	-0.040600	-0.060700	-0.031000	-0.003000	-0.016700
1963	-0.007200	-3.875900	-0.046300	0.124600	0.096100	-0.003900	-0.004000
1962	-0.008800	19.904600	0.030800	0.242200	-0.013500	-0.004900	0.014300
1961	0.015300	-1.084500	0.035300	-0.021200	-0.022200	-0.011500	0.011200
1960	0.023500	-4.164600	0.000600	-0.185100	0.052700	-0.017900	-0.003600
1959	-0.019200	0.276000	-0.030600	0.048800	0.100700	-0.003000	-0.005400
1958	-0.041600	1.650700	0.001200	0.052400	-0.044000	0.018600	-0.016100
1957	-0.006400	0.734300	0.020300	-0.008600	-0.059000	0.014400	0.011800
1956	0.008800	0.687600	-0.002200	-0.085700	-0.081200	0.008900	0.016100

TABLE X (Continued)

Year	Variable						
	$\phi_1(k-1)$	$\phi_2(k-1)$	$v_1(k-1)$	$v_2(k-1)$	$v_3(k-1)$	$v_4(k-1)$	$v_5(k-1)$
1968	-0.011800	0.419600	-0.006200	0.113900	0.017000	-0.005200	0.015700
1967	-0.014900	-1.382000	0.021200	-0.128100	-0.055700	0.018400	0.000900
1966	0.009300	-0.611200	0.028000	-0.150300	-0.031100	0.010400	-0.002500
1965	0.015300	3.475200	-0.040600	-0.060700	-0.031000	-0.003000	-0.016700
1964	-0.007200	-3.875900	-0.046300	0.124600	0.096100	-0.003900	-0.004000
1963	-0.008800	19.904600	0.030800	0.242200	-0.013500	-0.004900	0.014300
1962	0.015300	-1.084500	0.035300	-0.021200	-0.022200	-0.011500	0.011200
1961	0.023500	-4.164600	0.000600	-0.185100	0.052700	-0.017900	-0.003600
1960	-0.019200	0.276000	-0.030600	0.048800	0.100700	-0.003000	-0.005400
1959	-0.041600	1.650700	0.001200	-0.052400	-0.044000	-0.018600	-0.016100
1958	-0.006400	0.734300	0.020300	-0.008600	-0.059000	0.014400	0.011800
1957	0.008800	0.687600	-0.002200	-0.085700	-0.081200	0.008900	0.016100
1956	0.015800	-2.649700	-0.010700	0.029000	-0.021600	-0.004400	-0.017600

TABLE X (Continued)

Year	Variable						
	$\phi_1(k-2)$	$\phi_2(k-2)$	$v_1(k-2)$	$v_2(k-2)$	$v_3(k-2)$	$v_4(k-2)$	$v_5(k-2)$
1968	-0.014900	-1.382000	0.021200	-0.128100	-0.055700	0.018400	0.000900
1967	0.009300	-0.611200	0.028000	-0.150300	-0.031100	0.010400	-0.002500
1966	0.015300	3.475200	-0.040600	-0.060700	-0.031000	-0.003000	-0.016700
1965	-0.007200	-3.875900	-0.046300	0.124600	0.096100	-0.003900	-0.004000
1964	-0.008800	19.904600	0.030800	0.242200	-0.013500	-0.004900	0.014300
1963	0.015300	-1.084500	0.035300	-0.021200	-0.022200	-0.011500	0.011200
1962	0.023500	-4.164600	0.000600	-0.185100	0.052700	-0.017900	-0.003600
1961	-0.019200	0.276000	-0.030600	0.048800	0.100700	-0.003000	-0.005400
1960	-0.041600	1.650700	0.001200	0.052400	-0.044000	0.018600	-0.016100
1959	-0.006400	0.734800	0.020300	-0.008600	-0.059000	0.014400	0.011800
1958	0.008800	0.687600	-0.002200	-0.085700	-0.081200	0.008900	0.016100
1957	0.015800	-2.649700	-0.010700	0.029000	-0.021600	-0.004400	-0.017600
1956	0.004800	1.663400	0.002900	0.022400	0.096300	-0.01080	0.004200

TABLE X (Continued)

Year	Variable						
	$\phi_1(k-3)$	$\phi_2(k-3)$	$v_1(k-3)$	$v_2(k-3)$	$v_3(k-3)$	$v_4(k-3)$	$v_5(k-3)$
1968	0.009300	-0.611200	0.028000	-0.150300	-0.031100	0.010400	-0.002500
1967	0.015300	3.475200	-0.040600	-0.060700	-0.031000	-0.003000	-0.016700
1966	-0.007200	-3.875900	-0.046300	0.124600	0.096100	-0.003900	-0.004000
1965	-0.008800	19.904600	0.030800	0.242200	-0.013500	-0.004900	0.014300
1964	0.015300	-1.084500	0.035300	-0.021200	-0.022200	-0.011500	0.011200
1963	0.023500	-4.164600	0.000600	-0.185100	0.052700	-0.017900	-0.003600
1962	-0.019200	0.276000	-0.030600	0.048800	0.100700	-0.003000	-0.005400
1961	-0.041600	1.650700	0.001200	0.052400	-0.044000	0.018600	-0.016100
1960	-0.006400	0.734300	0.020300	-0.008600	-0.059000	0.014400	0.011800
1959	0.008800	0.687500	-0.002200	-0.085700	-0.081200	0.008900	0.016100
1958	0.015800	-2.649700	-0.010700	0.029000	-0.021600	-0.004400	-0.017600
1957	0.004800	1.663400	0.002900	0.022400	0.096300	-0.010800	0.004200
1956	--	--	--	--	--	--	--

TABLE X (Continued)

Year	Variable						
	$\phi_1(k-4)$	$\phi_2(k-4)$	$V_1(k-4)$	$V_2(k-4)$	$V_3(k-4)$	$V_4(k-4)$	$V_5(k-4)$
1968	0.015300	3.475200	-0.040600	-0.060700	-0.031000	-0.003000	-0.016700
1967	-0.007200	-3.875900	-0.046300	0.124600	0.096100	-0.003900	-0.004000
1966	-0.008800	19.904600	0.030800	0.242200	-0.013500	-0.004900	0.014300
1965	0.015300	-1.084500	0.035300	-0.021200	-0.022200	-0.011500	0.011200
1964	0.023500	-4.164600	0.000600	-0.185100	0.052700	-0.017900	-0.003600
1963	-0.019200	0.276000	-0.030600	0.048800	0.100700	-0.003000	-0.005400
1962	-0.041600	1.650700	0.001200	0.052400	-0.044000	0.018600	-0.016100
1961	-0.008400	0.734300	0.020300	-0.008600	-0.059000	0.014400	0.011800
1960	0.008800	0.687600	-0.002200	-0.085700	-0.081200	0.008900	0.016100
1959	0.015800	-2.649700	-0.010700	0.029000	-0.021600	-0.004400	-0.017600
1958	0.004800	1.663400	0.002900	0.022400	0.096300	-0.010800	0.004200
1957	--	--	--	--	--	--	--
1956	--	--	--	--	--	--	--

$\phi_1(k-3), \phi_2(k-3), \dots, v_5(k-3)$) as well as the 1956 and 1957 values of those variables with moment three steps before (i.e., $\phi_1(k-4), \phi_2(k-4), \dots, v_5(k-4)$) are unavailable. As mentioned in the last chapter, this absence of information is due to its dependence on data for the period prior to 1954.

To remedy the lack of data in the previous work, a value of zero was assumed by A. G. Ivakhnenko for all missing variables. However, there are no convincing rationalization for replacing the unavailable data with a zero value. Furthermore, this assumption would be bound to introduce errors in the final solution. It is also important to state at this time that such assumption is also unnecessary for the final solution of the problem as will be shown shortly. Therefore, throughout the calculations in the modified method we avoid assuming any specific value for the missing data but, rather all data related to this unavailable information (and therefore unknown) are discarded.

In order to simplify and shorten the calculations, 14 variables among the 33 variables are selected which have the strongest correlation with the output variable ϕ_{1k} . The correlation coefficients of all the variables with respect to ϕ_{1k} are calculated (using equation [27]) and are shown in Table XI. The 14 largest (in absolute value) correlation coefficients are found to be

TABLE XI

THE CORRELATION COEFFICIENTS OF ALL VARIABLES WITH THE PERFORMANCE INDEX ϕ_1

Correlation Coefficients				
$K\phi_1\phi_1(0) = 1.000000$	$K\phi_1\phi_1(1) = 0.241010$	$K\phi_1\phi_1(2) = -0.654241$	$K\phi_1\phi_1(3) = -0.404238$	$K\phi_1\phi_1(4) = 0.250546$
$K\phi_1\phi_2(0) = -0.181014$	$K\phi_1\phi_2(1) = -0.191640$	$K\phi_1\phi_2(2) = 0.255911$	$K\phi_1\phi_2(3) = 0.313768$	$K\phi_1\phi_2(4) = -0.209562$
$K\phi_1v_1(0) = 0.070243$	$K\phi_1v_1(1) = -0.693535$	$K\phi_1v_1(2) = -0.079321$	$K\phi_1v_1(3) = 0.609822$	$K\phi_1v_1(4) = 0.124905$
$K\phi_1v_2(0) = -0.518582$	$K\phi_1v_2(1) = 0.097665$	$K\phi_1v_2(2) = 0.566199$	$K\phi_1v_2(3) = 0.078827$	$K\phi_1v_2(4) = -0.583906$
$K\phi_1v_3(0) = -0.024420$	$K\phi_1v_3(1) = 0.754786$	$K\phi_1v_3(2) = 0.504630$	$K\phi_1v_3(3) = -0.233954$	$K\phi_1v_3(4) = -0.548806$
$K\phi_1v_4(0) = -0.569158$	$K\phi_1v_4(1) = -0.691796$	$K\phi_1v_4(2) = -0.058038$	$K\phi_1v_4(3) = 0.340147$	$K\phi_1v_4(4) = 0.229692$
$K\phi_1v_5(0) = 0.139255$	$K\phi_1v_5(1) = -0.345824$	$K\phi_1v_5(2) = -0.324907$	$K\phi_1v_5(3) = 0.416898$	$K\phi_1v_5(4) = 0.278849$

$$K\phi_1 v_2(0) = -0.518582$$

$$K\phi_1 v_2(2) = 0.566199$$

$$K\phi_1 v_4(0) = -0.569158$$

$$K\phi_1 v_3(2) = 0.504630$$

$$K\phi_1 v_1(1) = -0.693535$$

$$K\phi_1 \phi_1(3) = -0.404238$$

$$K\phi_1 v_3(1) = 0.754786$$

$$K\phi_1 v_1(3) = 0.609822$$

$$K\phi_1 v_4(1) = -0.691796$$

$$K\phi_1 v_5(3) = 0.416898$$

$$K\phi_1 v_5(1) = -0.345824$$

$$K\phi_1 v_2(4) = -0.583906$$

$$K\phi_1 \phi_1(2) = -0.654241$$

$$K\phi_1 \phi_3(4) = -0.548806.$$

Therefore, from above, the 14 selected variables are

$$x_1 = v_{2k}$$

$$x_6 = v_{5(k-1)}$$

$$x_{11} = v_{1(k-3)}$$

$$x_2 = v_{4k}$$

$$x_7 = \phi_{1(k-2)}$$

$$x_{12} = v_{5(k-3)}$$

[42]

$$x_3 = v_{1(k-1)}$$

$$x_8 = v_{2(k-2)}$$

$$x_{13} = v_{2(k-4)}$$

$$x_4 = v_{3(k-1)}$$

$$x_9 = v_{3(k-2)}$$

$$x_{14} = v_{3(k-4)}$$

$$x_5 = v_{4(k-1)}$$

$$x_{10} = \phi_{1(k-3)}$$

where they are denoted by x_1 through x_{14} . The values of the above selected variables for each corresponding year are shown in Table XII where each year is designated by a number for ease in reference (e.g., 1968, 1967, and so on, corresponding to points 1, 2, and so on, respectively). From the above we can summarize our approach and its difference from the previous work as follows. First, those variables whose values at certain years cannot be determined are not assumed zero as it has been done in the earlier work. These unknown variables are simply not considered in the calculation procedure. Second, the future variables x_1, x_2, x_3, x_4 , and x_5 in equation [41] are included in the selection criterion (correlation criterion) whereas they were excluded in A. G. Ivakhnenko's work. The extension of the correlation criterion to the selection of these future variables, as it will be shown, will lead to more accurate results.

Thus, having selected the 14 variables, the estimation equation can now be written as

$$\phi_{1k} = f(x_1, x_2, x_3, x_4, x_5, x_6, x_7, x_8, x_9, x_{10}, x_{11}, x_{12}, x_{13}, x_{14}).$$

Construction of the Training and Testing Data

As mentioned before, we have chosen the time interval T equal to four years with input data available for 15 years. This would only leave us 13 points of interpolation (signifying data for 13 years). The above points are now divided into two sets on the basis of their variance. That is, ten points with larger variations are chosen as one set (training data) and three points with less variations as another set

TABLE XII

THE 14 SELECTED VARIABLES IN THE MODIFIED APPROACH

n Year	Variable									
	$x_1 = v_{2k}$	$x_2 = v_{4k}$	$x_3 = v_1(k-1)$	$x_4 = v_3(k-1)$	$x_5 = v_4(k-1)$	$x_6 = v_5(k-1)$	$x_7 = \phi_1(k-2)$			
1969			-0.0039	0.0164	-0.0065	-0.0080	-0.0118			
1 1968	0.006200	-0.006500	-0.006200	0.017000	-0.005200	0.015700	-0.014900			
2 1967	0.113900	-0.005200	0.021200	-0.055700	0.018400	0.000900	0.009300			
3 1966	-0.128100	0.018400	0.028000	-0.031100	0.010400	-0.002500	0.015300			
4 1965	-0.150300	0.010400	-0.040600	-0.031000	-0.003000	-0.016700	-0.007200			
5 1964	-0.060700	-0.003000	-0.046300	0.096100	-0.003900	-0.004000	-0.008800			
6 1963	0.124600	-0.003900	0.030800	-0.013500	-0.004900	0.014300	0.015300			
7 1962	0.242200	-0.004900	0.035300	-0.022200	-0.011500	0.011200	0.023500			
8 1961	-0.021200	-0.011500	0.000600	0.052700	-0.017900	-0.003600	-0.019200			
9 1960	-0.185100	-0.017900	-0.030600	0.100700	-0.003000	-0.005400	-0.041600			
10 1959	0.048800	-0.003000	0.001200	-0.044000	0.018600	-0.016100	-0.006400			
11 1958	0.052400	0.018600	0.020300	-0.059000	0.014400	0.011800	0.008800			
12 1957	-0.008600	0.014400	-0.002200	-0.081200	0.008900	0.016100	0.015800			
13 1956	-0.085700	0.008900	-0.010700	-0.021600	-0.004400	-0.017600	0.004800			

TABLE XII (Continued)

n	Year	Variable							
		$x_8^v_2(k-2)$	$x_9^v_3(k-2)$	$x_{10}^{\phi}_1(k-3)$	$x_{11}^v_1(k-3)$	$x_{12}^v_5(k-3)$	$x_{13}^v_2(k-4)$	$x_{14}^v_3(k-4)$	
	1969	0.1139	0.0170	-0.0149	0.0212	0.0009	-0.1503	-0.0311	
1	1968	-0.128100	-0.055700	0.009300	0.028000	-0.002500	-0.060700	-0.031000	
2	1967	-0.150300	-0.031100	0.015300	-0.040600	-0.016700	0.124600	0.096100	
3	1966	-0.060700	-0.031000	-0.007200	-0.046300	-0.004000	0.242200	-0.013500	
4	1965	0.124600	0.096100	-0.008800	0.030800	0.014300	-0.021200	-0.022200	
5	1964	0.242200	-0.013500	0.015300	0.035300	0.011200	-0.185100	0.052700	
6	1963	-0.021200	-0.022200	0.023500	0.000600	-0.003600	0.048800	0.100700	
7	1962	-0.185100	0.052700	-0.019200	-0.030600	-0.005400	0.052400	-0.044000	
8	1961	0.048800	0.100700	-0.041600	0.001200	-0.016100	-0.008600	-0.059000	
9	1960	0.052400	-0.044000	-0.006400	0.020300	0.011800	-0.085700	-0.081200	
10	1959	-0.008600	-0.059000	0.008800	-0.002200	0.016100	0.029000	-0.021600	
11	1958	-0.085700	-0.081200	0.015800	-0.010700	-0.017600	0.022400	0.096300	
12	1957	0.029000	-0.021600	0.004800	0.002900	0.004200	-----	-----	
13	1956	0.022400	0.096300	-----	-----	-----	-----	-----	

(testing data). Training data which consists of points with larger variations are used for interpolation and the testing data are used to filter the unwanted results and therefore only allow those results with certain minimum square error to pass through the different layers of calculations. The training and testing data (for the first layer) along with the variance for each point are shown in Tables XIII and XIV.

First Layer of Selection

By combining the 14 selected variables two at a time into the second degree polynomial

$$y_n = a_{0n} + a_{1n}x_b + a_{2n}x_c + a_{3n}x_b^2 + a_{4n}x_c^2 + a_{5n}x_bx_c \quad [43]$$

(where $n = 1, 2, 3, \dots, 91$ and b and c are the indices of variables x for all combinations) a total of 91 equations can be obtained. The coefficients of each equation is determined using the training data in the least squared method. Having determined the coefficients of all 91 equations, we proceed with selection of 14 new variables (y variables) which have the smallest mean-square error using the testing data. The mean square error equation [32] is given in Chapter III. In a similar fashion as before, we construct a new set of training and testing data of the variable y for the next layer. The reader can find the detailed procedure of the above calculations by referring to Chapter III.

Second Layer of Selection

The 14 chosen y variables from the first layer are now combined two at a time to give rise to 91 polynomials of the form

TABLE XIII

THE TRAINING DATA OF THE MODIFIED APPROACH

n	5	7	3	2	9	4	11	6	13	1
D^2	0.008048	0.007449	0.006039	0.004818	0.004760	0.003756	0.002271	0.002226	0.002015	0.001853
ϕ_1	0.015300	-0.008800	-0.014900	-0.011800	0.023500	0.009300	-0.041600	-0.007200	0.008800	0.005200
x_1	-0.060700	0.242200	-0.128100	0.113900	-0.185100	-0.150300	0.052400	0.124600	-0.085700	0.006200
x_2	-0.003000	-0.004900	0.018400	-0.005200	-0.017900	0.010400	0.018600	-0.003900	0.008900	-0.006500
x_3	-0.046300	0.035300	0.028000	0.021200	-0.030600	-0.040600	0.020300	0.030800	-0.010700	-0.006200
x_4	0.096100	-0.022200	-0.031100	-0.055700	0.100700	-0.031000	-0.059000	-0.013500	-0.021600	0.017000
x_5	-0.003900	-0.011500	0.010400	0.018400	-0.003000	-0.003000	0.014400	-0.004900	-0.004400	-0.005200
x_6	-0.004000	0.011200	-0.002500	0.000900	-0.005400	-0.016700	0.011800	0.014300	-0.017600	0.015700
x_7	-0.008800	0.023500	0.015300	0.009300	-0.041600	-0.007200	0.008800	0.015300	0.004800	-0.014900
x_8	0.242200	-0.185100	-0.060700	-0.150300	0.052400	0.124600	-0.085700	-0.021200	0.022400	-0.128100
x_9	-0.013500	0.052700	-0.031000	-0.031100	-0.044000	0.096100	-0.081200	-0.022200	0.096300	-0.055700
x_{10}	0.015300	-0.019200	-0.007200	0.015300	-0.006400	-0.008800	0.015800	0.023500	----	0.009300
x_{11}	0.035300	-0.030600	-0.046300	-0.040600	0.020300	0.030800	-0.010700	0.000600	----	0.028000
x_{12}	0.011200	-0.005400	-0.004000	-0.016700	0.011800	0.014300	-0.017600	-0.003600	----	-0.002500
x_{13}	-0.185100	0.052400	0.242200	0.124600	-0.085700	-0.021200	0.022400	0.048800	----	-0.060700
x_{14}	0.052700	-0.044000	-0.013500	0.096100	-0.081200	-0.022200	0.096300	0.100700	----	-0.031000

TABLE XIV
THE TESTING DATA OF THE MODIFIED APPROACH

n	8	12	10
D^2	0.001581	0.00735	0.000727
ϕ_1	0.015300	-0.006400	-0.019200
x_1	-0.021200	-0.008600	0.048800
x_2	-0.011500	0.014400	-0.008000
x_3	0.000600	-0.002200	0.001200
x_4	0.052700	-0.081200	-0.044000
x_5	-0.017900	0.008900	0.018600
x_6	-0.003600	0.016100	-0.016100
x_7	-0.019200	0.015800	-0.006400
x_8	0.048800	0.029000	-0.008600
x_9	0.100700	-0.021600	-0.059000
x_{10}	-0.041600	0.004800	0.008800
x_{11}	0.001200	0.002900	-0.002200
x_{12}	-0.016100	0.004200	0.016100
x_{13}	-0.008600	----	0.029000
x_{14}	-0.059000	---	-0.021600

$$z_n = b_{0n} + b_{1n}y_b + b_{2n}y_c + b_{3n}y_b^2 + b_{4n}y_c^2 + b_{5n}y_by_c$$

where $n = 1, 2, 3, \dots, 91$ and b and c are indices for all 91 combinations. As in the first layer, the coefficients of the polynomials are found and the new training and testing data as well as a new set of 14 variables are determined. These variables are now used to generate a new set of variables for the next layer. This process is repeated for successive layers where each new set of variables are designated by a new letter given as

LAYER	LETTER
0	x
1	y
2	z
3	v
4	w

Last Layer of Selection and Final Results of the Calculation

The selected variables in each layer with their mean-square error are shown in Table XV. By comparing the smallest mean-square errors in the third and fourth layer, we note that the rate of the error between the two layers is decreasing as expected, but it is, however, still considerably large in magnitude. This implies that the mean-square error is not yet "stabilized" up to the fourth layer, i.e., the rate of error is not insignificant up to this layer. Therefore, one can produce more accurate results by proceeding to obtain higher selection layer (e.g.,

TABLE XV

CHANGE OF MEAN-SQUARE ERROR BETWEEN LAYERS OF THE MODIFIED APPROACH

First layer	ϵ	Second layer	ϵ	Third layer	ϵ	Fourth layer	ϵ
y_{45}	4.47431	z_{30}	1.095600	v_{78}	0.161430	w_{17}	0.063334
y_{46}	8.834774	z_{57}	1.144468	v_9	0.232433	w_{68}	0.076342
y_{40}	29.412479	z_{19}	1.293705	v_{11}	0.367150	w_5	0.095009
y_{82}	32.473402	z_{72}	1.654102	v_{18}	0.367637	w_8	0.104612
y_{84}	32.866643	z_{40}	2.470713	v_{13}	0.410046	w_2	0.119186
y_8	35.285090	z_{22}	3.009660	v_{38}	0.448512	w_9	0.177789
y_{65}	35.883633	z_{28}	3.128142	v_{44}	0.462679	w_{43}	0.221002
y_{80}	42.882022	z_{35}	4.698042	v_{56}	0.530817	w_{65}	0.246048
y_{69}	60.533011	z_{17}	5.762532	v_{14}	0.767256	w_{16}	0.294057
y_{26}	63.664981	z_5	5.821862	v_{82}	0.810535	w_{66}	0.350545
y_{88}	65.180483	z_{73}	5.870333	v_5	0.935414	w_{88}	0.373984
y_{67}	66.263669	z_1	6.973783	v_{23}	1.028649	w_{61}	0.405270
y_{75}	68.788000	z_{64}	8.019300	v_{51}	1.160906	w_{63}	0.407522
y_{64}	69.079807	z_{10}	8.905840	v_{70}	1.231811	w_{34}	0.462873

fifth, and so forth) until the decreasing rate of error is observed to be insignificant. Once this is established, care has to be taken to cease the further advance to higher layers if "degeneration" is to be avoided.

In this work, however, we chose to seek the final solution in the fourth layer for two reasons. First, with the advantage of a priori knowledge of the performance index value for the year 1969 (0.691) in our example, we can test the predictability of the above value by using the estimation equation obtained. This test shows a sufficiently accurate result, to be discussed shortly, suggesting further layer selection is unnecessary. Second, any further selection of layers would immensely complicate the format of the solution (the estimation equation) making the algebra intractable. This is true since it can be shown that the polynomial power of the estimation equation increases as 2^n where n is the number of layers selected. Thus, we choose to avoid complication in the algebra, especially since the results obtained in the fourth layer are already sufficiently accurate for our purposes here.

In the last layer (fourth layer) therefore we only choose one variable (instead of 14 which is needed to proceed to the next layer) with the smallest mean-square error. Hence, from Table XV with $\Delta = 3.817 \times 10^{-7}$ is our final selection. The ϕ_1 estimation equation for 1969 now can be written as

$$\phi_{1k} = \omega_{17} = 0.001978885259501 + 0.606600771465935v_9$$

$$+ 0.391148533411297v_{38} - 310.286644416773900v_9^2$$

$$- 318.29145494758270v_{38}^2 + 626.961773659335400v_9v_{38}$$

$$v_9 = 0.000047178382955 + 0.730305613977386z_{30}$$

$$+ 0.292538735554196z_5 + 12.158801336990270z_{30}^2$$

$$+ 1.733418978051304z_5^2 - 14.581008232822570z_{30}z_5$$

[44]

$$v_{38} = -0.000658802151406 + 0.829099233210119z_{72}$$

$$+ 0.238096181660696z_{22} + 31.324035734531210z_{72}^2$$

$$+ 50.435644726442600z_{22}^2 - 83.383005545258130z_{72}z_{22}$$

$$z_{30} = 0.002664769611207 + 0.732612256482162y_{40}$$

$$+ 0.469098874402267y_{80} + 3.015173516205834y_{40}^2$$

$$- 43.802546566414430y_{80}^2 + 34.469616385268690y_{40}y_{80}$$

$$\begin{aligned}
 z_5 = & 0.003591123391012 + 0.617691800358136y_{45} \\
 & + 0.329656071760010y_8 + 7.939353143165313y_{45}^2 \\
 & - 34.402255649754160y_8^2 + 13.338077009622740y_{45}y_8
 \end{aligned}$$

$$\begin{aligned}
 z_{72} = & 0.005106175767657 + 0.650228636066707y_{80} \\
 & + 0.637770787645995y_{26} - 59.025480974982830y_{80}^2 \\
 & - 11.997765022871650y_{26}^2 + 63.650534262351470y_{80}y_{26}
 \end{aligned}$$

[44]

$$\begin{aligned}
 z_{22} = & -0.003340075920798 + 1.038040765155978y_{46} \\
 & + 0.014897747521114y_{88} + 14.391384008845190y_{46}^2 \\
 & + 48.681749438286370y_{88}^2 - 50.901597370639370y_{46}y_{88}
 \end{aligned}$$

$$\begin{aligned}
 y_{40} = & 0.001407424014086 + 0.277938116978547x_4 \\
 & + 0.041568404959746x_8 - 0.351203872711045x_4^2 \\
 & + 0.328152347357771x_8^2 - 1.573484824636260x_4x_8
 \end{aligned}$$

$$\begin{aligned}
 y_{80} = & 0.005438915023777 + 0.170815008832557x_9 \\
 & - 0.090501419424639x_{13} - 2.301661149159258x_9^2 \\
 & + 0.169139614143966x_{13}^2 + 0.706074349687476x_9x_{13}
 \end{aligned}$$

$$\begin{aligned}
 y_{45} = & 0.005473031768987 + 0.403180587902069x_4 \\
 & + 0.025196645577524x_{13} - 2.636995612342247x_4^2 \\
 & - 0.257718512045965x_{13}^2 - 0.619583776474826x_4x_{13}
 \end{aligned}$$

[44]

$$\begin{aligned}
 y_8 = & 0.010713563544486 - 0.081967965564293x_1 \\
 & + 0.223237404590501x_9 - 0.151596349246819x_1^2 \\
 & - 2.733668396748627x_9^2 + 0.519164063340392x_1x_9
 \end{aligned}$$

$$\begin{aligned}
 y_{26} = & 0.005419121159147 - 0.157961079220824x_3 \\
 & + 0.339868423684117x_4 + 1.094202912082982x_3^2 \\
 & - 0.421993113236902x_4^2 + 6.273850292544349x_3x_4
 \end{aligned}$$

$$\begin{aligned}
y_{46} = & 0.0.005798589437667 + 0.330960623235850x_4 \\
& - 0.018614502573788x_{14} - 1.683109932807356x_4^2 \\
& - 0.868554087335519x_{14}^2 - 0.497112376103686x_4x_{14} \quad [44]
\end{aligned}$$

$$\begin{aligned}
y_{88} = & -0.005182005671914 + 0.402292330429017x_{11} \\
& - 0.127626329025794x_{14} + 5.322612115297766x_{11}^2 \\
& - 0.200602941570570x_{14}^2 - 0.566431182738324x_{11}x_{14}.
\end{aligned}$$

Thus, the above is the final estimation equation obtained by our modified approach in this work. The above is to be compared to equation [34] in Chapter III, which is the result of A. G. Ivakhnenko's work. It should be noted in passing that we have not rounded the coefficient values to six decimal places (as in A. G. Ivakhnenko's work) but have preferred to leave them in their final form of up to 15 decimal places.

II. OPTIMIZATION AND PREDICTION

Optimization

The estimation equation [44] can be written now in the general form

$$\phi_{1k} = f(x_1) = f(v_{2k}) \quad [45]$$

where ϕ_{1k} is only a function of the manipulated variable v_{2k} . This

is because the value of all other variables for 1969, i.e., $x_3, x_4, x_8, x_9, x_{11}, x_{13}$, and x_{14} are known (to be discussed in next section). For overall maximization of the performance index ϕ_1 (recall that for good economic situation $\phi_1 \rightarrow 1$, where $0 < \phi_1 < 1$), the optimal values of the manipulated variable v_{2k} can be obtained by setting the derivative of equation [45] to zero. Thus

$$\frac{d\phi_{1k}}{dv_{2k}} = 0.$$

Using the above equation we are able to find optimal values of the manipulated variable v_{2k} which maximizes the performance index ϕ_{1k} of the British economy for the year 1969. Therefore, by choosing the optimal values of v_{2k} one obtains the ability to manipulate the performance index value for necessary and desirable economic changes.

Prediction for 1969

We now wish to examine the accuracy of the estimation equation [44]. This is achieved by utilization of equation [44] to predict the performance index ϕ_1 for the year 1969. This is possible since the manipulated variable x_1 is fortunately available and therefore a check and comparison of A. G. Ivakhnenko's work with our approach is realizable. From Table XII, p. 74, the 1969 values of variables appearing in equation [44] are found to be

$$x_3 = -0.0039$$

$$x_{11} = 0.0212$$

$$x_4 = 0.0164$$

$$x_{13} = -0.1503$$

$$x_8 = 0.1139$$

$$x_{14} = -0.0311$$

$$x_9 = 0.0170$$

The value of the 1969 manipulated variable x_1 is also known and quoted in [6] to be

$$x_1 = -0.1243898.$$

By substituting the above values into estimation equation [44], we obtain ϕ_1 as

$$\phi_1 = 0.0074.$$

From Tables I-III, pp. 19-21, ϕ_1^* (trend of ϕ_1) is

$$\phi_1^* = 0.6857.$$

The performance index ϕ_1 can now be found (equation [27])

$$\phi_1 = \phi_1 \phi_1^* + \phi_1^*$$

which by substituting ϕ_1 and ϕ_1^* will give

$$\phi_1 = 0.6857 (1 + 0.0074)$$

or

$$\phi_1 = 0.6908.$$

The result of the unmodified work of A. G. Ivakhnenko as given in Chapter III is a value of 0.6978 (see also [6]).

By comparing the above predicted results of ϕ_1 for 1969 by both methods with the actual value of the performance index given in [6] as

$$\phi_1 = 0.6910$$

we can indeed conclude that our modified approach yields better and more accurate results. The predicted value of performance index obtained by our suggested modified approach tends to confirm the correctness of the assumptions made and thus the better and more superior nature of the present approach is established.

CHAPTER V

CONCLUSIONS

I. SUMMARY OF THE THESIS

The objective of this work was to investigate and compare the results of A. G. Ivakhnenko's work on application of the polynomial theory to systems of known experimental data and develop a more accurate modified method. The application of the above theory to the British economy, as a complex system, was demonstrated by using the Group Method of Data Handling. For this purpose the previous work of A. G. Ivakhnenko and his GMDH algorithm was discussed in detail. The application of this work to the British economy and the results obtained were elaborated in a stepwise fashion.

In Chapter IV, a modified approach to the GMDH algorithm used previously was developed where the major emphasis was placed on certain alterations in the algorithm. Specifically, the present work did not exclude any input variable from the selection pool to which the correlation criterion was applied. Also in this approach we avoided the assumption of zero value imposed on all the input data which were not available. Chapter IV includes the detailed steps and final results of this modified approach. A comparison of the results shows a more accurate value for the performance index as compared to the A. G. Ivakhnenko work. This confirms the correctness of the assumptions used in this approach and shows it as a valid modification to be used if better results are to be achieved.

II. SUGGESTIONS FOR FURTHER STUDY

Further research in this work could be focused on several areas. An obvious area is the further application of the method to other problems of interest both in social-economic spheres or in natural science areas. Such problems could include complex problems as far apart as industrial processes or environmentally related decisions in our modern societies to such biologically involved problems as modelling of brain.

Another area of research could be directed toward a better or more accurate functional form of the partial description polynomials (Kolmogorov-Gabor) such as y (equation [1]) for the first layer or a similar equation for z in the second layer and so on. For example, an expansion in third or higher degrees for these polynomials and their effect on the final solution could be investigated. Finally, a rational method for the construction of the testing and training data and the optimum number of data in each could be developed. The present choice of three testing points in testing data and ten points in training data in A. G. Ivakhnenko's work is based on a trial and error selection process leading to a better and more accurate result. A rational criterion for this selection would then have to be developed eliminating completely the need for a trial and error procedure.

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APPENDIXES

APPENDIX A

PREDICTION AND ESTIMATION EQUATIONS

The general form of the prediction and estimation equation of performance index ϕ can be formulated as follows. From equation [5] in Chapter II we recall the general prediction equation for input x and output y to be:

$$y_k = f(y_{k-1}, y_{k-2}, y_{k-3}, \dots, x_{k-1}, x_{k-2}, x_{k-3}, \dots).$$

From the above, the general prediction equation for the variable ϕ with time shift $T = 4$ can be written as (see also [7])

$$\begin{aligned} \phi_k = & f[\phi_1(k-1), \phi_1(k-2), \phi_1(k-3), \phi_1(k-4), \\ & \phi_2(k-1), \phi_2(k-2), \phi_2(k-3), \phi_2(k-4), \\ & v_1(k-1), v_1(k-2), v_1(k-3), v_1(k-4), \\ & v_2(k-1), v_2(k-2), v_2(k-3), v_2(k-4), \\ & v_3(k-1), v_3(k-2), v_3(k-3), v_3(k-4), \\ & v_4(k-1), v_4(k-2), v_4(k-3), v_4(k-4), \\ & v_5(k-1), v_5(k-2), v_5(k-3), v_5(k-4)]. \end{aligned}$$

The general estimation equation for input x and output y was given in equation [6] in Chapter II as:

$$y_k = f(y_{k-1}, y_{k-2}, y_{k-3}, \dots, x_k, x_{k-1}, x_{k-2}, x_{k-3}, \dots).$$

The above leads to the general estimation equation for the performance index ϕ as:

$$\phi_k = f[\phi_1(k-1), \phi_1(k-2), \phi_1(k-3), \phi_1(k-4),$$

$$\phi_2(k-1), \phi_2(k-2), \phi_2(k-3), \phi_2(k-4),$$

$$v_{1k}, v_{1(k-1)}, v_{1(k-2)}, v_{1(k-3)}, v_{1(k-4)},$$

$$v_{2k}, v_{2(k-1)}, v_{2(k-2)}, v_{2(k-3)}, v_{2(k-4)},$$

$$v_{3k}, v_{3(k-1)}, v_{3(k-2)}, v_{3(k-3)}, v_{3(k-4)},$$

$$v_{4k}, v_{4(k-1)}, v_{4(k-2)}, v_{4(k-3)}, v_{4(k-4)},$$

$$v_{5k}, v_{5(k-1)}, v_{5(k-2)}, v_{5(k-3)}, v_{5(k-4)}].$$

APPENDIX B

COMPUTER PROGRAMS

The computer programs developed for the calculation of polynomial coefficients of the estimation equation are presented in the following pages. The main program is followed by the subroutine for the inversion of the matrices. Note that the programs presented here generate the results of any one layer of transformation. Therefore, successive use of these programs are necessary for generation of the final solution. These programs are written in FORTRAN IV language.

```

C      THIS PROG. COMPUTES THE COEFFS. OF THE PARTIAL DESCRIPTIONS
C      IN EACH LAYER OF TRANSFORMATION (USING LSM) AND GENERATES A
C      NEW TRAINING AND TESTING DATA FOR THE SUBSEQUENT LAYER.
      IMPLICIT INTEGER*4(I-N),REAL*8(A-H,O-Z)
      DIMENSION XB(20),XC(20),F(20),PXB(40),PXC(40),SXBXC(20),SXCXB(20),
1SSXBXC(20),SFXB(20),SFXC(20),C(20,20),B(20),A(20),D(20,20),Y(100),
2XRTS(10),XCTS(10),FTS(10),X(20,20),XTS(20,20),DELTA(100),PHI(100),
3EFRCF(100),YTS(10)
C      NP1 IS THE NO. OF TRAINING DATA INTERPOLATION POINTS. UNKNOWN
C      DATA POINTS ARE SET = 0.
      NP1 = 0
C      READ PHI(1) FROM TRAINING DATA.
      READ(5,11) (F(I),I=1,5)
      READ(5,11) (F(I),I=6,NP1)
      DO 9 K = 1,14
C      READ VARIABLES FROM TRAINING DATA (SET UNKNOWN = 0).
      READ(5,11) (X(I,K),I=1,5)
      READ(5,11) (X(I,K),I=6,NP1)
9      CONTINUE
C      READ PHI(1) FROM TESTING DATA.
      READ(5,12) (FTS(I),I=1,3)
C      READ VARIABLES FROM TESTING DATA (SET UNKNOWN = 0).
      READ(5,12) ((XTS(I,K),I=1,3),K=1,14)
11      FORMAT(5F14.10)
12      FORMAT(3F14.10)
      WRITE(6,61)
61      FORMAT(1H1)
C      FORM XB AND XC.
      LP = 1
      DO 5 LL = 1,13
      MCM = LL + 1
      DO 5 JJ = MCM,14
      DO 6 MP = 1,NP1
      XB(MP) = X(MP,LL)
      XC(MP) = X(MP,JJ)
6      CONTINUE
      DO 7 KCP = 1,3
      XBTS(KCP) = XTS(KCP,LL)
      XCTS(KCP) = XTS(KCP,JJ)
7      CONTINUE
C      FOR UNKNOWN (ZERO) VARIABLES, ITS PAIR IS SET = TO ZERO.
      DO 8 I = 1,NP1
      IF(XB(I).EQ.0.0) XC(I)=0.0
      IF(XC(I).EQ.0.0) XB(I)=0.0
8      CONTINUE
C      FORM SUMS OF POWERS OF XB AND XC.
      DO 20 K = 1,4
      PXB(K) = 0.0
      PXC(K) = 0.0
      DO 20 I = 1,NP1
      PXB(K) = PXB(K)+XB(I)**K
      PXC(K) = PXC(K)+XC(I)**K
20      CONTINUE
C      FORM SUMS OF POWERS OF XB*XC.
      DO 30 K = 1,3
      SXBXC(K) = 0.0
      SXCXB(K) = 0.0
      DO 30 I = 1,NP1
      SXBXC(K) = SXBXC(K)+XB(I)*(XC(I))**K

```

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      SXCXB(K) = SXCXB(K)+XC(I)*(XB(I))**K
30  CONTINUE
      DO 35 K = 1,2
        SSXBXC(K) = 0.0
      DO 40 I = 1,NP1
40  SSXBXC(K) = SSXBXC(K)+(XB(I)*XC(I))**K
35  CONTINUE
C    FORM SUMS OF POWERS OF PHI(1)*XB AND PHI(1)*XC.
      DO 45 K = 1,2
        SFXB(K) = 0.0
        SFXC(K) = 0.0
      DO 45 I = 1,NP1
        SFXB(K) = SFXB(K)+F(I)*(XB(I)**K)
        SFXC(K) = SFXC(K)+F(I)*(XC(I)**K)
45  CONTINUE
C    FORM SUMS OF PHI(1).
      SF = 0.0
      DO 50 I = 1,NP1
        SF = SF + F(I)
50  CONTINUE
C    FORM SUMS OF PHI(1)*XB*XC.
      SFXBXC = 0.0
      DO 51 I = 1,NP1
        SFXBXC = SFXBXC + (F(I)*XB(I)*XC(I))
51  CONTINUE
C    GENERATING NORMAL MATRIX C.
      C(1,1) = DFLQAT(NP1)
      C(2,1) = PXB(1)
      C(3,1) = PXC(1)
      C(4,1) = PXB(2)
      C(5,1) = PXC(2)
      C(6,1) = SSXBXC(1)
      C(2,2) = PXB(2)
      C(3,2) = C(6,1)
      C(4,2) = PXB(3)
      C(5,2) = SXBXC(2)
      C(6,2) = SXCXB(2)
      C(3,3) = PXC(2)
      C(4,3) = C(6,2)
      C(5,3) = PXC(3)
      C(6,3) = C(5,2)
      C(4,4) = PXB(4)
      C(5,4) = SSXBXC(2)
      C(6,4) = SXCXB(3)
      C(5,5) = PXC(4)
      C(6,5) = SXBXC(3)
      C(6,6) = C(5,4)
      DO 55 I = 1,5
        DO 55 J = 1,5
          L = I + J
          IF(L.GE.7) GO TO 55
          C(I,L) = C(L,I)
55  CONTINUE
C    GENERATING RIGHT SIDE MATRIX B.
      B(1) = SF
      B(2) = SFXB(1)
      B(3) = SFXC(1)
      B(4) = SFXB(2)
      B(5) = SFXC(2)
      B(6) = SFXBXC
C    INVERT MATRIX C.

```

```

CALL INVERS (C,D,6)
DO 60 I = 1,6
  A(I) = 0.0
DO 65 J = 1,6
65 A(I) = A(I)+D(I,J)*B(J)
60 CONTINUE
  WRITE(6,79)LL,JJ
79 FORMAT(32X,'***** FOR COMBINATION OF ARGUMENTS X',I2,' AND X',I2,'
1 *****')
  WRITE(6,70) (A(K),K=1,3)
  WRITE(6,71) (A(K),K=4,6)
70 FORMAT(// (5X,'COEFFICIENTS A(K) :',10X,3F21.15))
71 FORMAT(// (34X,3F21.15)//)
  DO 72 I = 1,3
  YTS(I)=A(1)+A(2)*XBTS(I)+A(3)*XCTS(I)+A(4)*(XBTS(I)**2)+A(5)*(XCTS
1 (I)**2)+A(6)*XBTS(I)*XCTS(I)
  IF(XBTS(I).EQ.0.0) YTS(I)=9E10
  IF(XCTS(I).EQ.0.0) YTS(I)=9E10
72 CONTINUE
  DO 78 I = 1,NP1
  Y(I) = A(1)+A(2)*XB(I)+A(3)*XC(I)+A(4)*(XB(I)**2)+A(5)*(XC(I)**2)+
1 A(6)*XB(I)*XC(I)
  IF(XB(I).EQ.0.0) Y(I)=9E10
  IF(XC(I).EQ.0.0) Y(I)=9E10
78 CONTINUE
  WRITE(6,73)LP
73 FORMAT(50X,'*****Y',I2,'(K)*****')
  WRITE(6,174) (Y(I),I=1,5)
  WRITE(6,74) (Y(I),I=6,NP1)
174 FORMAT(// (5(8X,F14.10)))
74 FORMAT(// (5(8X,F14.10))//)
  WRITE(6,77)LP
77 FORMAT(50X,'***** YTS',I2,'(K) *****')
  WRITE(6,102) (YTS(I),I=1,3)
102 FORMAT(// (35X,3(F14.10,5X)))
  DELTA(LP) = 0.0
  PHI(LP) = 0.0
  DO 75 I = 1,3
  IF(XBTS(I).EQ.0.0) GO TO 85
  IF(XCTS(I).EQ.0.0) GO TO 85
  CCNST = 1.0
  GO TO 86
85 CCNST = 0.0
86 DELTA(LP) = DELTA(LP)+CCNST*((FTS(I)-YTS(I))**2)
  PHI(LP) = PHI(LP)+CCNST*(FTS(I)**2)
75 CONTINUE
C   CALCULATE ERROR.
  ERROR(LP) = ((DELTA(LP))/(PHI(LP)))*100.0
  WRITE(6,76)LP,DELTA(LP),LP,ERROR(LP)
76 FORMAT(// (30X,'DELTA',2X,I2,'=',F14.10,9X,'ERROR',2X,I2,'=',F14.6))
  WRITE(6,200)
200 FORMAT(3X,115'-')
  LP = LP + 1
  5 CONTINUE
C   SORT OUT ERROR IN DESCENDING ORDER.
  NA = 90
  DO 3 J = 1,90
  MAP = J
  MA = J + 1
  DO 2 I = MA,91
  2 IF(ERROR(I).GT.ERROR(MAP)) MAP=I

```

```
      TEMP = ERROR(J)
      ERROR(J) = ERROR(MAP)
3    ERROR(MAP) = TEMP
      WRITE(6,100)
100  FORMAT(35X,'ERROR IN DESCENDING ORDER')
      WRITE(6,4) (ERROR(I),I=1,91)
4    FORMAT(50X,F14.6)
      STOP
      END
```

```

SUBROUTINE INVERS (A,B,N)
IMPLICIT INTEGER*4(I-N),REAL*8(A-H,O-Z)
DIMENSION A(20,20),B(20,20)
EPS = 1E-15
DO 6 I = 1,N
DO 5 J = 1,N
IF(I-J)4,3,4
3 B(I,J) = 1.0
GO TO 5
4 B(I,J) = 0.0
5 CONTINUE
6 CONTINUE
DEL = 1.0
DO 45 K = 1,N
IF (K-N)12,30,30
12 IMAX = K
AMAX = DABS(A(K,K))
KPI = K+1
DO 20 I = KPI,N
IF(AMAX-DABS(A(I,K))) 15,20,20
15 IMAX = I
AMAX =DABS(A(I,K))
20 CONTINUE
IF(IMAX-K)25,30,25
25 DO 29 J = 1,N
ATMP = A(IMAX,J)
A(IMAX,J) = A(K,J)
A(K,J) = ATMP
BTMP = B(IMAX,J)
B(IMAX,J) = B(K,J)
29 B(K,J) = BTMP
DEL = -DEL
30 CONTINUE
IF(DABS(A(K,K))-EPS)93,93,35
35 DEL = A(K,K)*DEL
DIV = A(K,K)
DO 38 J = 1,N
A(K,J) = A(K,J)/DIV
38 B(K,J) = B(K,J)/DIV
DO 43 I = 1,N
AMULT = A(I,K)
IF(I-K)39,43,39
39 DO 42 J = 1,N
A(I,J) = A(I,J)-AMULT*A(K,J)
42 B(I,J) = B(I,J)-AMULT*B(K,J)
43 CONTINUE
45 CONTINUE
99 RETURN
93 WRITE(6,113)K
GO TO 99
113 FORMAT(5X,'SINGULAR MATRIX FOR K =',I2)
END

```

VITA

The author was born in Tehran, Iran, on July 31, 1953. He attended and graduated from Norman High School, Norman, Oklahoma in May 1971. He entered the University of Oklahoma at Norman, Oklahoma in July, 1971 and was graduated with a Bachelor of Science degree in Electrical Engineering in July, 1975. He was accepted to the Graduate School of the University of Tennessee at Knoxville, Tennessee in September, 1975 to begin study toward the Master of Science degree in Electrical Engineering. His academic experience includes holding various positions as a teaching aid and tutor in undergraduate mathematics courses during 1977-1978.

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