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I am submitting herewith a thesis written by Joshua W. Batterson entitled "*Investigation of the Sidewall Boundary Layers in the Bidirectional Vortex Liquid Rocket Engine.*" I have examined the final electronic copy of this thesis for form and content and recommend that it be accepted in partial fulfillment of the requirements for the degree of Master of Science, with a major in aerospace engineering.

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INVESTIGATION OF THE SIDEWALL BOUNDARY LAYERS
IN THE BIDIRECTIONAL VORTEX LIQUID ROCKET ENGINE

A Thesis

Presented for the

Master of Science

Degree

The University of Tennessee

Joshua W. Batterson

December 2007

Dedication

This thesis is dedicated to my loving parents. This is only a small repayment of all that I owe them. It's because of them that I have the values and initiatives to become a person of integrity and veracity.

Acknowledgements

I wish to acknowledge all of those who have guided me toward the completion of my Master of Science degree. Whether it is the support of family and friends or the teachers that have broadened my horizons I am fully aware I wouldn't be here without you. A special thanks to Dr. Joseph Majdalani who presented the idea and advised me through the culmination of the project. I am also deeply grateful for the support received from ORBITEC (FA8650-05-C-2612).

I would like to extend my gratitude to my committee members: Dr. Roy Schulz, Dr. Gary Flandro, and of course my advisor Dr. Joseph Majdalani. I appreciate them for taking the time to mentor me through my degree.

Abstract

To complement previous studies on the bidirectional vortex, we attempt to characterize viscous effects in both the axial and radial directions along the sidewall with standard asymptotic techniques. The actual boundary layer present in the chamber will be a composite of both axial and tangential shearing layers. Since the tangential velocity is completely dominant, we expect the defining characteristics of the composite boundary layer to be congruent with the tangential layer in both thickness and axial invariance. The analytic analysis is outlined first by, the formulation of the boundary layer equations via Prandtl's method. Next, asymptotic techniques are applied to linearize and rigorously truncate the governing equations from PDEs to more manageable ODEs. A scaling transformation is applied to resolve the rapid changes near the wall. Due to the nature of the outer solution, a dependent variable transformation is applied to recover constant boundary conditions. The viscous corrections are matched to the outer solution via Prandtl's matching principle. We see a similar form in all three wall corrections; the axial and radial presented here and the swirl previously formulated. This can be expected to some extent because of the similarity of the asymptotic assumptions and the linearization techniques used in all three cases. Although the assumption that curvature terms can be neglected is never made, they are found to be asymptotically small and the problem then parallels the case of a one-dimensional Cartesian boundary layer. It can be seen that all viscous corrections along the wall are strongly dependent on the value of the vortex Reynolds number, V . This parameter shows up naturally in all vector directions. With the new corrected solutions, other key features of the flowfield can be revisited.

Preface

At the onset of the thesis a brief overview of classic vortex flows is introduced with the emphasis on the swirl velocity component. The focus on swirl velocity should initially seem immaterial since the equations in question are initially decoupled from the tangential velocity component. However, the physical mechanisms controlling the subsequent boundary layer analysis are strongly related to the swirl velocity and therefore a clear understanding of swirl dominated flows in both confined and external flow regimes is essential.

Since the following will discuss viscous treatments of the already published inviscid flow field for the bidirectional vortex liquid rocket engine, the original derivations and solutions will be relatively brief. For a complete understanding of the problem I suggest to the reader exploring the previous publications on this problem. This work is in no way to be considered the *exact* solution of the viscous flow equations. However, this document illustrates the application of key asymptotic techniques with respect to boundary layer type problems in order to obtain a physical analytic *approximation*.

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Nomenclature

a	= chamber radius
A_i	= total inlet (injector) area
b	= chamber outlet radius
l	= chamber aspect ratio, L/a
p	= normalized pressure, $\bar{p}/(\rho U^2)$
$\bar{Q}_i$	= inlet volumetric flow rate
Q_i	= normalized volumetric flow rate, $\bar{Q}_i/(Ua^2) = \sigma^{-1}$
Re	= injection Reynolds number, $Ua/\nu = 1/\varepsilon$
r, z	= normalized radial or axial coordinates, $\bar{r}/a, \bar{z}/a$ (unless otherwise stated)
S	= swirl number, $\pi ab/A_i = \pi\beta\sigma$
$\mathbf{u}$	= normalized velocity $(\bar{u}_r, \bar{u}_z, \bar{u}_\theta)/U$
u_θ	= normalized swirl/spin/tangential velocity, $\bar{u}_\theta/U$
U	= mean inflow swirl velocity, $\bar{u}_\theta(a, L)$
V	= vortex Reynolds number, $Q_i Re(a/L) = (\varepsilon\sigma l)^{-1}$
s	= scaled transformation variable
R, r	= spherical radius coordinate (Bloor and Ingham vortex)
W	= average axial velocity (Bloor and Ingham vortex)

Greek

α	= $\frac{1}{6}\pi^2 - 1$
β	= normalized outlet radius, b/a
δ	= viscous region thickness
ε	= perturbation parameter, $1/Re = \nu/(Ua)$
κ	= inflow parameter, $Q_i/(2\pi l) = (2\pi\sigma l)^{-1}$
δ	= η rescaled wall layer thickness
ν	= kinematic viscosity, μ/ρ
ρ	= density
σ	= modified swirl number, $Q_i^{-1} = S/(\pi\beta)$

ξ = transformation variable

Γ = circulation

Subscripts

i = inlet property

r = radial component

z = axial component

θ = azimuthal component

w, c = wall/core region referring to the tangential velocity boundary layer

z, r = wall region referring to the axial/radial velocity boundary layer

0 = vector magnitude referring to shear stress at the wall

t = total value integrated over the entire surface referring to shear stress at the wall

Superscripts

$\bar{}$ = overbars denote dimensional variables

^{o} = outer (inviscid) solution

$*$ = conformally invariant variables

Abbreviations

BV = bidirectional vortex (liquid rocket engine)

I. Introduction

This thesis proposes viscous corrections to the axial and radial velocity profiles of the bidirectional vortex (BV) liquid rocket engine by following the techniques used to rectify viscosity in the tangential profile (Vyas and Majdalani¹). The bidirectional vortex flow field is not a recent problem. Bidirectional flow in cyclonic separators has been discussed in detail as early as 1987 by Bloor and Ingham.² The current interest in cyclonic, and particularly bidirectional cyclonic motion is sparked by its potential in propulsive applications. Liquid rocket engines utilizing this technology have been experimentally studied by Chiaverini *et al.* at Orbital Technologies with the development of their Vortex Combustion Cold-Wall Chamber (VCCWC).³⁻⁵ Interest continues to grow in this research, as this concept displays superior chamber cooling potential with reduced hardware as well as excellent combustion efficiency. With any good experiment, an accurate analytical model is important to understand the physicality of the experimental results. The forthcoming discussion will describe the analytical models developed not only for rocket propulsion applications but for confined bidirectional swirling motion in general. The previous BV solutions are presented chronologically, starting with the original inviscid solution by Vyas and Majdalani and finishing with the most recent corrections corresponding to the asymptotic treatment of the axial and radial sidewall boundary layers.

A. Swirling Flows

Swirling flows have always been a topic of mathematical and practical interest. Models date back 150 years to Rankine whose solution was first presented in 1858.⁶ Vortex flows are widely known to modify thermal and transport effects in a variety of industrial applications. Cyclonic separators are used in chemical filtration. One specific use is in the enrichment of Uranium. Vortex technology is commonly used in cyclonic furnaces. Well chosen swirl patterns promote increased combustion efficiency by increasing fuel/air mixing as well as increasing residence time.

Atmospheric rotating flows are brought to the forefront of any conversation due to their destructive power and negative publicity. Regardless, atmospheric swirls such as tornadoes, hurricanes, dust devils, waterspouts, and more, can shed light on industry-motivated applications. The scope of this discussion is not limited to atmospheric phenomena. By changing the scale of interest, one can include the very large scales, such as spiral galaxies and, inversely, the very small scales such as wingtip vortices. Irrespective of the size under consideration, all vortex flows seem to share the same intrinsic properties.

Vortex motion can be tripped in a variety of ways. Mechanisms for intentionally inducing vortices are seen in industrial applications. These include, but are not limited to vanes, fans, and rifling. In propulsion, work with vortices has been done to observe the effect of grain scrubbing and residence time in hybrid rockets. In liquid engines swirl motion presents the possibility of more complete mixing by again increasing residence time, but also by promoting a more turbulent flow regime. In addition, recent work has shown that bidirectional vortex motion can provide chamber cooling.^{3-5,7}

B. Historic Models

Regardless of the specific application, an understanding of the historical models is important. Despite the focus of the new research presented here describing the axial and radial profiles, for a complete understanding of vortex flows, we will briefly discuss some classical vortex solutions in order to better understand the characteristic requirements for current and future models. A survey of classical vortex models has been compiled in the first half of the paper by Batterson, Maicke and Majdalani.⁸ It contains a more thorough description of the vortex models briefly noted here.

1. Rankine Combined Vortex (1858)

To illustrate these aforementioned characteristics we recall the original classical vortex solution as presented by Rankine.⁶ His solution is a piecewise continuous function defined in terms of the maximum velocity and its position.

$$\frac{\bar{u}_\theta}{(\bar{u}_\theta)_{\max}} = \begin{cases} \frac{\bar{r}}{\delta}, & \bar{r} \leq \delta \\ \frac{\delta}{\bar{r}}, & \bar{r} > \delta \end{cases} \quad (1.1)$$

Similarly in nondimensional form

$$u_\theta = \frac{\bar{u}_\theta}{(\bar{u}_\theta)_{\max}} = \begin{cases} r, & r \leq 1 \\ \frac{1}{r}, & r > 1 \end{cases} \quad \text{where } r = \frac{\bar{r}}{\delta} \quad (1.2)$$

The condition $\bar{r} \leq \delta$ describes the region known as the *forced core vortex*. This piece of the profile is indicative of solid body rotation. It is defined up to a maximum velocity, $(\bar{u}_\theta)_{\max}$, at a predetermined core thickness, δ . Physically, this area is the region where viscous forces begin to balance and overshadow inertial forces. The region $\bar{r} > \delta$ is known as the *free tail vortex*. This region is inversely proportional to $\bar{r}$, again defined up to a maximum velocity and a core thickness. Figure 1 plots Eq. (1.2). It clearly shows the distinct differences between the forced and free vortex regions.

Despite its simplicity, the Rankine vortex is regularly applied directly or as a first approximation to model a number of atmospheric swirl patterns. It is observed that in typical swirl dominated flows, the velocity in the tangential direction is at least an order of magnitude larger than those in the axial and radial directions. Often these velocities are neglected for the simpler yet still physically meaningful one-dimensional Rankine model.

The piecewise nature of the solution divulges an unphysical cusp where the two regions meet. It is understood that this is an artifact of the definition and in real flow, viscosity will have a smoothing effect. Therefore, the maximum velocity predicted by the model will always overshoot the actual expected maximum. Regardless, it remains a common model to curve fit experimental data taken from tornadoes⁹ and hurricanes.¹⁰ The experimenter accounts for the overshoot by fitting the core and tail regions independently and extracting the core thickness and maximum velocity directly from the experimental data.

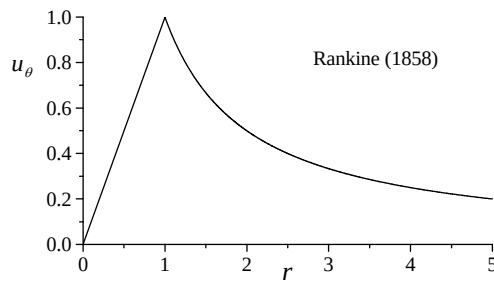


Figure 1 Nondimensional Rankine combined vortex.

2. Lamb-Oseen Vortex (1932)

The Lamb-Oseen vortex introduces time dependence into vortex modeling. This model predicts the situation where no energy source is present to sustain motion. As a result the vortex decays in time due to the presence of shear. As this happens, the maximum velocity decreases and the core thickness increases until the profile is damped to zero. The solution is expressed as

$$\bar{u}_\theta(\bar{r}) = \frac{\Gamma}{2\pi\bar{r}} \left[1 - \exp\left(-\frac{\bar{r}^2}{\delta^2}\right) \right] \quad (1.3)$$

and in nondimensional form

$$u_\theta(r) = \frac{1}{2\pi r} \left[1 - \exp(-r^2) \right] \quad (1.4)$$

$$\text{where } r = \frac{\bar{r}}{\delta} \quad u_\theta(r) = \frac{\bar{u}_\theta(\bar{r})}{\Gamma/\delta}$$

Here $\delta = 2\sqrt{\nu t}$. This represents a self-contained solution where the forced and free vortex regions are combined into a single solution.

3. Burgers-Rott Vortex (1948)

Without further discussion of the Lamb-Oseen vortex, we introduce the Burgers-Rott vortex. These two models are closely related as they are both Gaussian vortex solutions of the incompressible Navier-Stokes equations. The form of the Gaussian solution is that of

$$\bar{u}_\theta = \Gamma G\left(\frac{\bar{r}}{\delta}\right); \quad G(x) = \frac{1}{4\pi} e^{-x^2/4} \quad (1.5)$$

The Burgers-Rott vortex is of the specific form

$$\begin{aligned} \bar{u}_\theta(\bar{r}) &= \frac{\Gamma}{2\pi\bar{r}} \left[1 - \exp\left(-\frac{\bar{r}^2}{\delta^2}\right) \right] \\ \bar{u}_r(\bar{r}) &= -a\bar{r} \\ \bar{u}_z(\bar{z}) &= 2a\bar{z} \end{aligned} \quad (1.6)$$

which, when written nondimensionally, becomes

$$\begin{aligned} u_\theta(r) &= \frac{1}{2\pi r} \left[1 - \exp(-r^2) \right] \\ u_r(r) &= -r \\ u_z(z) &= 2z \end{aligned} \quad (1.7)$$

$$\text{where } (r, z) = \frac{(\bar{r}, \bar{z})}{\delta} \quad u_\theta(r) = \frac{\bar{u}_\theta(\bar{r})}{\Gamma/\delta} \quad u_{r,z} = \frac{\bar{u}_{r,z}}{a\delta}$$

The azimuthal component of Eq. (1.6) exhibits the same representative solution as given by Eq. (1.3). However, in this case, one has $\delta = \sqrt{2\nu/a}$. Unlike the Lamb-Oseen profile, this solution does not display time dependence. Instead, it introduces a suction parameter a . This parameter is a bit of a misnomer. It encompasses the effect of not only suction fields, but any strain field in which an axial flow is induced by pressure differences between regions. In nature, this term can

imitate convective rise in a rotating thunderhead. In propulsion, it can capture the pressure difference between the chamber and the exit plane. Figure 2a,b plots the Lamb-Oseen and Burgers-Rott vortices.

Typical applications of the Lamb-Oseen model involve the situation where a vortex is induced but no energy source is present to sustain motion. For this reason, we often see it used in conjunction with wing tip vortices or vortices shed from lifting bodies,^{11,12} both of which begin to decay once they leave the blunt body. Since it contains temporal dependence it can be utilized when describing the time dependent onset of turbulence. The Burgers-Rott vortex is applicable to a number of steady flow applications. As with the Rankine model, this solution can be used to represent a one-dimensional swirling flow. Examination of the axial and radial components of velocity indicates that both are linear unbounded functions. This insinuates that the vortex in question must be considered to be significantly small compared to the scale of the strain field in which it is present in order to retain physicality of the full three-dimensional solution.

C. Bidirectional Vortex Models

1. Sullivan's Vortex (1959)

The Sullivan vortex is an exact solution to the Navier-Stokes equations in an unbounded domain. The Sullivan structure is characterized by a strong downward motion near its axis of symmetry. Nearing the ground, this downward motion diverges radially and outwardly, eventually reversing direction upwardly. It is commonly referred to as a two-celled vortex due to its bidirectional nature. Because of the strong downdraft and accompanying updraft, this vortex is often described in the context of tornados. Mathematically, the tangential velocity is defined in terms of integral functions. The mathematical model is described as follows¹³

$$\begin{cases} \bar{u}_\theta(\bar{r}) = \frac{\Gamma}{2\pi\bar{r}} \frac{1}{H(\infty)} H\left(\frac{\bar{r}^2}{2\nu/a}\right) = \frac{\Gamma}{2\pi\bar{r}} \frac{1}{H(\infty)} H\left(\frac{\bar{r}^2}{\delta^2}\right) \\ H(x) = \int_0^x e^{f(t)} dt; \quad f(t) = -t + 3 \int_0^t (1 - e^{-y}) \frac{dy}{y} \\ \bar{u}_r(\bar{r}) = -a\bar{r} + \frac{6\nu}{\bar{r}} \left[1 - \exp\left(-\frac{\bar{r}^2}{\delta^2}\right) \right] \\ \bar{u}_z(\bar{r}, \bar{z}) = 2a\bar{z} \left[1 - 3 \exp\left(-\frac{\bar{r}^2}{\delta^2}\right) \right] \end{cases} \quad (1.8)$$

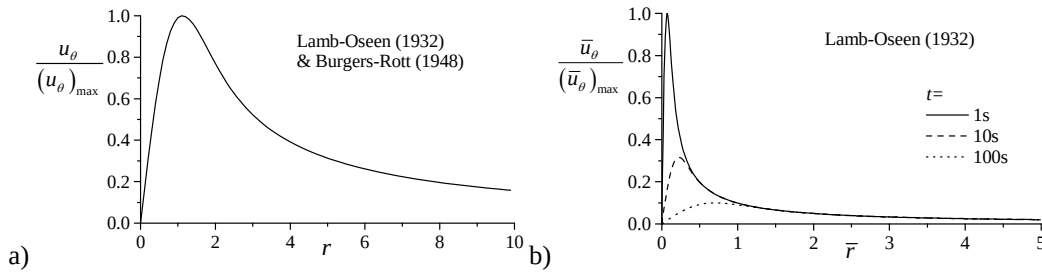


Figure 2 a) Nondimensional Lamb-Oseen and Burgers-Rott vortices and b) dimensional Lamb-Oseen vortex showing temporal decay. All plots are normalized by the maximum velocity at $t = 1s$.

The Sullivan vortex can also be expressed nondimensionally. It takes the form

$$\begin{cases} u_\theta(r) = \frac{1}{2\pi r} \frac{H(r^2)}{H(\infty)} \\ H(x) = \int_0^x e^{f(t)} dt; \quad f(t) = -t + 3 \int_0^t (1 - e^{-y}) \frac{dy}{y} \\ u_r(r) = -r + \frac{3}{r} [1 - \exp(-r^2)] \\ u_z(r, z) = 2z [1 - 3 \exp(-r^2)] \end{cases} \quad (1.9)$$

$$\text{where } (r, z) = \frac{(\bar{r}, \bar{z})}{\delta} \quad u_\theta(r) = \frac{\bar{u}_\theta(\bar{r})}{\Gamma / \delta} \quad u_{r,z} = \frac{\bar{u}_{r,z}}{a\delta}$$

Here $\delta = \sqrt{2\nu/a}$ where a is the suction strength and ν is the viscosity dominated by the eddy viscosity. We see that the integral $f(t)$ appears to diverge, however upon integration we see a negligible contribution for values of $x > 10$. In the tail region one can observe a likening to the Rankine free tail vortex. Figure 3a shows the swirl velocity of the Sullivan vortex.

To observe the bidirectional nature of the Sullivan vortex we examine the axial velocity component. Figure 3b shows the point of flow reversal where the function changes sign. This is referred to as the mantle and can be easily calculated by solving for the zero of the axial velocity equation. It can be shown that the Sullivan vortex mantle is located at $r = \sqrt{\ln(3)} = 1.04815$.

2. Bloor-Ingham Vortex (1987)

The initial work done by Bloor and Ingham² in 1987 is the first look at confined vortex models. It has direct application to cyclonic separators in a conical geometry. Their work is a direct extension of the Polhausen technique used by previous authors in the context of cyclonic separators.¹⁴ It consists of an exact solution of the Euler equations. Inherently, solving non-viscous equations will not rectify all physical boundary conditions. In addition, their original model was essentially insensitive to the inlet flow conditions. Still, their initial results are in agreement with Kelsall's experimental data.¹⁵ The work was furthered by considering the mean flow vorticity to be inversely proportional to the distance from the axis. With some careful assumptions regarding the distribution of the inlet source, they were able to solve the inviscid Bragg-Hawthorne equations in spherical coordinates.² Their vortex is expressed as

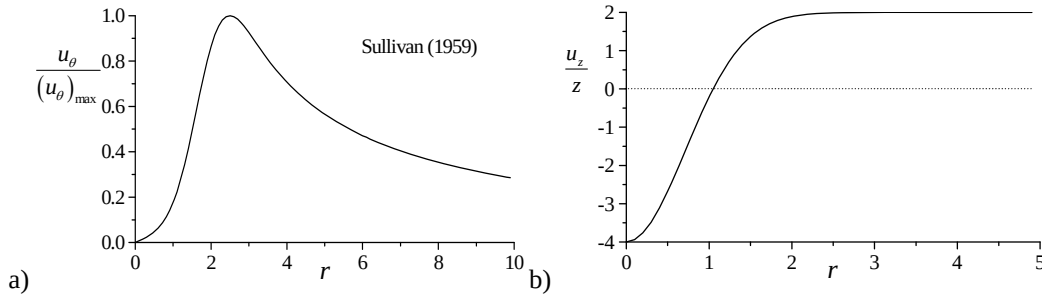


Figure 3 a) Nondimensional tangential and b) axial velocity of the Sullivan vortex. Note the bidirectional nature as the axial component switches from negative to positive.

$$\begin{aligned}\psi &= \sigma R^2 \left\{ \left[\csc^2 \alpha + \ln \left(\frac{1}{2} \tan \alpha \right) - \csc \alpha \cot \alpha \right] \sin^2 \phi - \sin^2 \phi \ln \left(\frac{1}{2} \tan \phi \right) + \cos \phi - 1 \right\} \\ &= \sigma r^2 \left\{ \csc^2 \alpha + \ln \left(\frac{1}{2} \tan \alpha \right) - \csc \alpha \cot \alpha - \ln \left(\frac{1}{2} \tan \phi \right) + \csc \phi \cot \phi - \csc^2 \phi \right\}\end{aligned}\quad (1.10)$$

where the volumetric flow rate through the cyclone is $\bar{Q} = \pi W (a_o^2 - a_i^2)$, with the overbar denoting dimensional variables. Figure 4 shows the original and corresponding geometry and coordinate systems where R and r are the spherical radial coordinates respectively. U is the average swirl velocity at entry and W is the average axial velocity.

3. Vyas-Majdalani Vortex (2003, 2006)

In conformance with historic unidirectional and bidirectional models, the BV is characterized largely by its tangential velocity with the axial and radial components being secondary and of significantly smaller order. Recall that the Rankine and Lamb-Oseen vortices are completely one-dimensional with the three-dimensional Burgers-Rott vortex being swirl dominated in localized regions near the origin. Unlike the Sullivan vortex which diverges outwardly and turns upwardly, the confined nature of this vortex forces the flow to turn inwardly on itself. Previous work on the BV has verified a strong dependence on swirl in the majority of corollaries such as swirling intensity, pressure distribution, and vorticity.

Clearly, the initiative to model such a flow field is due to the recent application of bidirectional swirl to liquid rocket engines.³⁻⁵ It should be stated that without combustion and heat transfer included, this solution remains pertinent to any cold bidirectional flow regime.

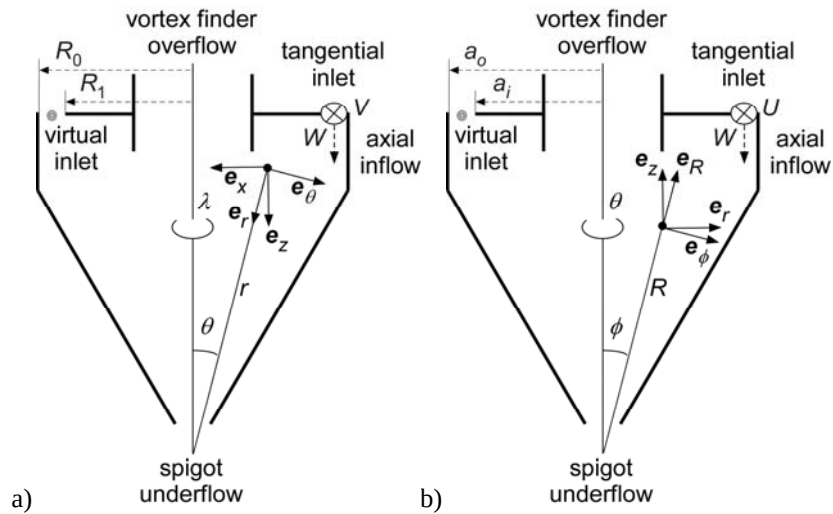


Figure 4 Geometry and coordinates used by a) Bloor and Ingham, and b) corresponding variables and directional units according to the convention defined here.⁸

II. Previous BV Models

A. Exact Inviscid Solution

All previous models were developed in the context of propulsive applications. However, the solutions are applicable to any BV type flows so long as they adhere to the basic geometric configurations of the mathematical model. The following section roughly illustrates the original solution to the BV as presented by Vyas and Majdalani in 2003.^{16,17} This first model is an exact solution to the inviscid Euler equations. The idealized physical representation used consists of a cylindrical tube of arbitrary length L and radius a , having a closed headwall at axial position $\bar{z} = 0$. A partially open downstream endwall is located at a value of $\bar{z} = L$. The chamber surface is assumed to be nonporous. The effects of compressibility at the nozzle are neglected. Essentially the opening at the endwall is considered to be a sink of radius b . In this fashion the analysis strictly deals with the bulk flow regime in the region $0 \leq \bar{z} \leq L$ and $0 \leq \bar{r} \leq a$. The ratio of open flow at the exit plane is defined as $\beta = b/a$ and the chamber aspect ratio is defined as $l = L/a$. See Figs. 5 and 6 for graphic representations.

Oxidizer is injected at high velocity at the base of the cylinder to induce swirl motion that traverses the length of the chamber along the inner surface of the sidewall in the outer vortex region. The assumption of axisymmetry is made and therefore the injectors are assumed to be arbitrary in number as long as the distribution is uniform. At the headwall the flow turns and forms the core vortex which exits at the base. The comparatively low velocity fuel injection at the headwall is neglected. Essentially the inviscid solution seeks to model the flow pattern of the oxidizer only without regard to viscous effects, fuel injection, or combustion reactions.

Three key assumptions are made. As mentioned above, the flow is axisymmetric and the tangential velocity is axially independent. These two assumptions are inherently coupled in the case of axisymmetric, inviscid flow. Finally we assume steady flow. In an actual chamber we expect an entry distance and transient period for fully developed axisymmetric, steady flow.

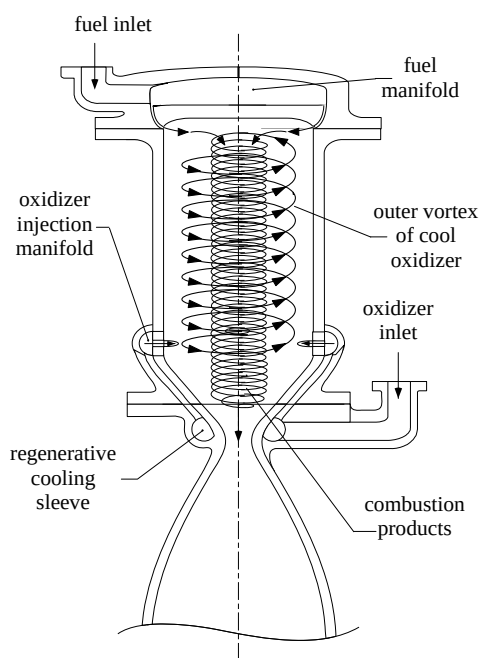


Figure 5 Schematic of ORBITEC's Vortex Combustion Cold Wall Chamber (VCCWC).⁸

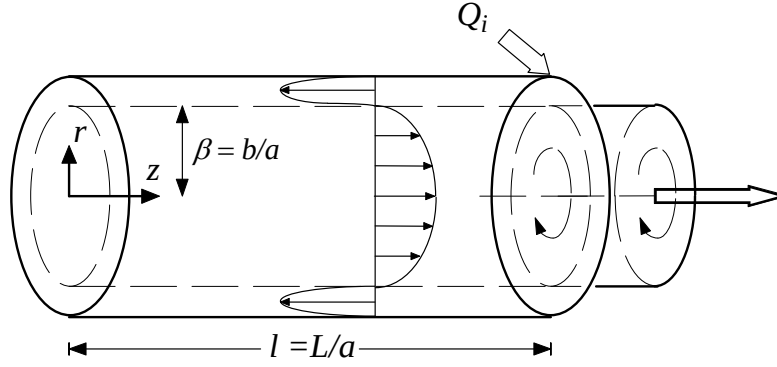


Figure 6 BV diagram.¹⁸

These are both ignored in the spirit of determining a realistic physical model for the bulk flow.

In summary the pre-analysis assumptions consist of

1. Steady
2. Inviscid
3. Axisymmetric
4. Incompressible
5. Nonreactive
6. Swirl velocity is axially independent

1. Normalization

It is helpful in asymptotic treatments to normalize the equations. The choices for normalization are as follows¹⁷

$$z = \frac{\bar{z}}{a}; \quad r = \frac{\bar{r}}{a}; \quad \nabla = a \bar{\nabla} \quad (2.1)$$

$$u_r = \frac{\bar{u}_r}{U}; \quad u_\theta = \frac{\bar{u}_\theta}{U}; \quad u_z = \frac{\bar{u}_z}{U} \quad (2.2)$$

$$p = \frac{\bar{p}}{\rho U^2}; \quad Q_i = \frac{\bar{Q}_i}{U a^2}; \quad Q_o = \frac{\bar{Q}_o}{U a^2} \quad (2.3)$$

Upon application of the new nondimensionalized variables, the governing equations can be expressed in a more generalized form.

2. Formulation of Equations

We begin with the normalized forms of the conservation of mass and momentum equations after all stated assumptions have been assessed.

Continuity

$$\frac{1}{r} \frac{\partial(r u_r)}{\partial r} + \frac{\partial u_z}{\partial z} = 0 \quad (2.4)$$

Momentum

$$u_r \frac{\partial u_r}{\partial r} + u_z \frac{\partial u_r}{\partial z} - \frac{u_\theta^2}{r} = -\frac{\partial p}{\partial r} \quad (2.5)$$

$$u_r \frac{\partial u_\theta}{\partial r} + \frac{u_\theta u_r}{r} = 0 \quad (2.6)$$

$$u_r \frac{\partial u_z}{\partial r} + u_z \frac{\partial u_z}{\partial z} = -\frac{\partial p}{\partial z} \quad (2.7)$$

These equations are subject to the boundary conditions

$$\begin{cases} u_\theta(1)_l = 1; & u_z(r, 0) = 0; & u_r(0, z) = 0 \\ u_r(1, z) = 0; & Q_o = \int_0^{2\pi} \int_0^\beta \mathbf{u}(r, l) \cdot \mathbf{n} r dr d\theta = Q_i \end{cases} \quad (2.8)$$

3. Solution

a) Free Vortex

After examining Eqs. (2.4) through (2.7), the logical starting point becomes blatantly obvious. Equation (2.6) is an ordinary differential equation in this decoupled form

$$\frac{du_\theta}{dr} + \frac{u_\theta}{r} = 0 \quad (2.9)$$

Solution of Eq. (2.9) along with the first boundary condition in Eq. (2.8) readily recovers the classical free vortex solution presented by Rankine.⁶

$$u_\theta = \frac{1}{r} \quad (2.10)$$

b) Axial and Radial Solutions

For the complete solution of the axial and radial profiles see Vyas, Majdalani, and Chiaverini.¹⁶ The following briefly describes the method of solution.

Instead of considering the axial and radial components of the momentum equation directly, first the pressure is eliminated by taking the curl of momentum to recover the vorticity transport equation.

$$\nabla \times (\mathbf{u} \times \boldsymbol{\Omega}) = 0; \quad \frac{\partial(u_r \Omega_\theta)}{\partial r} + \frac{\partial(u_z \Omega_\theta)}{\partial z} = 0 \quad (2.11)$$

$$\boldsymbol{\Omega} = \nabla \times \mathbf{u}; \quad \frac{\partial u_r}{\partial z} - \frac{\partial u_z}{\partial r} = \Omega_\theta \quad (2.12)$$

Equation (2.12) describes the tangential component of vorticity. Before proceeding with the vorticity transport equation, the Stokes stream function ψ is introduced. Even though this is essentially a three-dimensional model, defining the stream function is possible due to the assumption of axisymmetry. The stream function is defined as

$$u_r = -\frac{1}{r} \frac{\partial \psi}{\partial z}; \quad u_z = \frac{1}{r} \frac{\partial \psi}{\partial r} \quad (2.13)$$

By expanding Eq. (2.11) and making the appropriate substitutions according to Eq. (2.13) one is left with

$$-\frac{\partial \psi}{\partial z} \frac{\partial}{\partial r} \left(\frac{\Omega_\theta}{r} \right) + \frac{\partial \psi}{\partial r} \frac{\partial}{\partial z} \left(\frac{\Omega_\theta}{r} \right) = 0 \quad (2.14)$$

The tangential component of vorticity can be defined as $\Omega_\theta = rF[\psi(r, z)]$ where F is a general function of ψ . Since F is arbitrary, a simple linear relation can be assumed without loss in generality, therefore one defines $F = C^2\psi$. The use of a linear relation is inspired by a similar relation in the analysis of the gas motion in a solid rocket motor as presented by Culick.¹⁹ By substituting these new relations into Eq. (2.12), one is left with the linear, separable partial differential equation

$$\frac{\partial^2 \psi}{\partial z^2} + \frac{\partial^2 \psi}{\partial r^2} - \frac{1}{r} \frac{\partial \psi}{\partial r} + C^2 r^2 \psi = 0 \quad (2.15)$$

The boundary conditions become

$$\begin{cases} z = 0; & u_z = 0, & \partial \psi / \partial r = 0 \\ r = 0; & u_r = 0, & \partial \psi / \partial z = 0 \\ r = 1; & u_r = 0, & \partial \psi / \partial z = 0 \\ Q_o = \int_0^{2\pi} \int_0^\beta (\partial \psi / \partial r) dr d\theta = Q_i \end{cases} \quad (2.16)$$

As mentioned previously, Eq. (2.15) is separable. Guided by physical reasoning, the separation constant is determined to be zero and the general form of the stream function is obtained:

$$\psi = (C_1 z + C_2) \left[C_3 \sin\left(\frac{Cr^2}{2}\right) + C_4 \cos\left(\frac{Cr^2}{2}\right) \right] \quad (2.17)$$

From the boundary conditions one can recover the particular solution of the stream function and in turn the axial and radial components. The complete inviscid solution is then defined to be of the form^{16,17}

$$\mathbf{u}(r, z) = -\frac{\kappa}{r} \sin(\pi r^2) \mathbf{e}_r + \frac{1}{r} \mathbf{e}_\theta + 2\pi\kappa z \cos(\pi r^2) \mathbf{e}_z \quad (2.18)$$

where

$$\kappa = \frac{Q_i}{2\pi l} = \frac{1}{2\pi\sigma l} \quad (2.19)$$

4. Results and Discussion

Figure 7 plots the inviscid solution. The graph clearly shows the singular nature of the tangential solution at the core as well as its inability to satisfy the no-slip condition at the wall. A hallmark of all swirling flows, the presence of a forced core vortex must be realized in order to retain a physical model. We expect an inviscid approach to miss this characteristic inherently due to the viscous nature of the forced core region. Since the tangential solution corresponds exactly to the

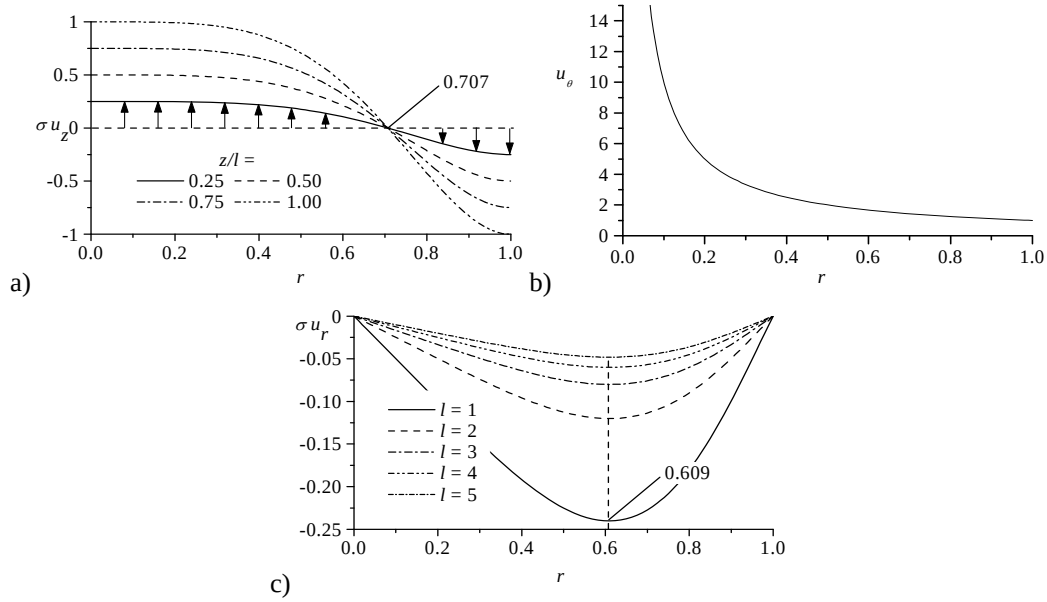


Figure 7 Inviscid a) axial velocity, b) swirl velocity, and c) radial velocity.^{16,17}

Rankine free vortex solution⁶ all classical correlations correspond here as well. For instance the vorticity in the Rankine free tail can be shown to be identically zero. This characteristic will be seen throughout the analysis of the BV and will cause the vorticity to remain confined to the viscous regions only.

Examination of the axial profile shows the flow does indeed change directions at the radial point $\sqrt{2}/2$. As in the Sullivan vortex, this radial position is denoted as the mantle. Computational studies have verified this value to be correct.^{7,20} To clarify, the mantle separates the inner or *core* vortex and outer vortex. The core must not be mistaken with the *forced core* vortex defined by the region where the *tangential* velocity is represented by near solid body rotation. Viscous effects at the wall have not been taken into account and therefore the no-slip condition at the sidewall is not satisfied.

The radial profile satisfies the condition of impermeability at the sidewall. This is an intrinsically necessary condition in the case of a flow perpendicular to the boundary surface. Nevertheless, when examining the first derivative of the radial solution, a discontinuity is observed at the wall. It is reasoned that with the inclusion of viscous effects, both the radial velocity as well as its first derivative would both tend to zero at the wall. The idea is that a component of velocity perpendicular to the boundary will have no contribution to the shear stress on that surface. Therefore the function *and* the first derivative must be zero at the wall.

The inviscid solution presented the first breakthrough toward fully understanding the BV flow field. The assumptions leading to this solution do leave room for error. The first and obvious deviation from real flow has been mentioned and will be rectified to some extent in subsequent sections. That is the effect of viscosity on the flow near the sidewall and near the core. The effect of compressibility in the axial and radial directions can be presumed negligible since the axial and radial velocities are orders of magnitude smaller than in the tangential velocity. Tangential compressibility, however, could potentially be an important aspect to consider in a real flow. The assumption that the tangential velocity is axially independent presents an idealized scenario which at best results in an average flow pattern over a long chamber. The more realistic scenario,

especially for short chambers, remains that large axial variation occurs in an undetermined region near the headwall due to flow reversal, compressibility, viscous effects at the surface, and large suction effects of the core vortex penetrating deep into the bulk flow. These effects have yet to be characterized.

B. Tangential Forced Core and Sidewall Viscous Corrections

A second solution was presented by Vyas and Majdalani^{1,21} in 2003 and 2006 which described viscous effects on the tangential velocity profile near the core and sidewall. As described in the inviscid solution the forced core vortex remains undetermined and therefore the treatment of viscous effects in that region must eliminate the singularity. Simple rescaling of the governing equation recovers the viscous effect at the sidewall and satisfies no-slip. The following briefly illustrates the analysis. The reader is urged to read the original manuscript for a complete understanding of the solution.

For consistency the normalization and assumptions previously stated are also used for this analysis. Here the tangential momentum equation is reconsidered, with the inclusion of second order viscous effects. Asymptotic techniques are applied to linearize the problem. Next two scaling functions are applied to facilitate the capture of viscous effects in their respective areas. Finally, a uniformly valid composite solution is constructed via asymptotic matching principles.

1. Formulation of Equations

The analysis begins with the introduction of the normalized, viscous, axisymmetric, tangential momentum equation, void of axial dependence.

$$u_r \frac{du_\theta}{dr} + \frac{u_r u_\theta}{r} = \frac{u_r}{r} \frac{d}{dr}(ru_\theta) = \varepsilon \frac{d}{dr} \left[\frac{1}{r} \frac{d}{dr}(ru_\theta) \right]; \quad \varepsilon = \frac{1}{\text{Re}} \quad (2.20)$$

The constant ε takes the place of the kinematic viscosity, ν , in the normalized form. Since $u_\theta = u_\theta(r)$ from the inviscid solution, the governing equation is converted from a PDE to an ODE directly. In an attempt to simplify Eq. (2.20), the natural path is taken and a new variable is defined.

$$\xi = ru_\theta \quad (2.21)$$

Applying this transformation changes Eq. (2.20) into

$$\varepsilon \frac{d}{dr} \left(\frac{1}{r} \frac{d\xi}{dr} \right) - \frac{u_r}{r} \frac{d\xi}{dr} = 0 \quad (2.22)$$

Equation (2.22) remains an exact representation of the viscous tangential momentum equation. Asymptotic methods are now applied beginning with a successive corrections technique. This technique, as is the basis of all asymptotic techniques, targets a first approximation to the solution. Mathematically $u = u_0 + u_1 + \dots$ where a linear basis for the nonlinear solution is recovered in the value of u_0 . In this situation the natural choice for this first approximation is the inviscid solution. In this way u_0 will represent the bulk flow with subsequent terms being small corrections to the overall characteristics. This technique will approximate continuity to leading order and therefore eliminates the need for a coupled system of equations. Since the goal is to calculate the value of u_θ , the successive corrections technique is applied to u_r in an attempt to linearize the problem. Equation (2.22) becomes

$$\varepsilon \frac{d}{dr} \left(\frac{1}{r} \frac{d\xi}{dr} \right) + \frac{\kappa \sin(\pi r^2)}{r^2} \frac{d\xi}{dr} = 0 \quad (2.23)$$

Again, simplicity drives the transformation

$$\eta = \pi r^2; \quad \frac{d}{dr} = 2\pi r \frac{d}{d\eta} \quad (2.24)$$

This finally converts the original equation into a useable governing equation

$$\varepsilon \frac{d^2 \xi}{d\eta^2} + \frac{\kappa \sin \eta}{2\eta} \frac{d\xi}{d\eta} = 0 \quad (2.25)$$

2. Solution

a) *Forced Core Vortex Solution*

In accordance to asymptotic boundary layer techniques, the governing equation must be scaled to the region in question. The standard linear scaling function is applied:

$$s = \frac{\eta}{\delta} \quad (2.26)$$

Scaling Eq. (2.25) recovers

$$\frac{\varepsilon}{\kappa \delta} \frac{d^2 \xi}{ds^2} + \frac{\sin(\delta s)}{2\delta^2 s} \frac{d\xi}{ds} = 0 \quad (2.27)$$

The final linearization comes from approximating the sine term with its Taylor-Series expansion

$$\sin(\delta s) \approx \delta s - \frac{\delta^3 s^3}{3!} + O(\delta^5 s^5) \quad (2.28)$$

Since δ is of the order of the boundary layer thickness, $\delta \ll 1$, only the first term needs to be retained. Backward substitution leaves

$$\frac{\varepsilon}{\kappa \delta} \frac{d^2 \xi}{ds^2} + \frac{1}{2\delta} \frac{d\xi}{ds} = 0 \quad (2.29)$$

Since both terms must be of the same order, the parameter coefficients must balance, thus revealing the distinguished limit

$$\delta \sim \frac{\varepsilon}{\kappa} \quad (2.30)$$

This can be directly substituted into Eq. (2.29) to once again reduce the equation to its most simplified form,

$$\frac{d^2 \xi^i}{ds^2} + \frac{1}{2} \frac{d\xi^i}{ds} = 0 \quad (2.31)$$

The superscript i indicates a fully scaled *inner* variable. This equation governs the uniformly valid solution in the *viscous core* only. It follows that Eq. (2.31) is subject to the following conditions

$$\begin{cases} r = 0, s = 0, \xi^i = 0 \\ r \rightarrow 1, s \rightarrow \infty, \xi^i = \xi^o \end{cases} \quad (2.32)$$

The solution of this equation is straightforward. The value of the constants of integration are determined by direct substitution of the boundary conditions and Prandtl's matching principle. Following Vyas and Majdalani,^{1,21} the forced core solution is found to be

$$\xi^i = \frac{1 - e^{-\frac{V}{4}r^2}}{1 - e^{-\frac{V}{4}}} \rightarrow u_\theta^{core} = \frac{1}{r} \left(\frac{1 - e^{-\frac{V}{4}r^2}}{1 - e^{-\frac{V}{4}}} \right) \quad (2.33)$$

The parameter V is termed the vortex Reynolds number.¹⁶ It is a combination of inviscid flow and geometric parameters introduced by the successive correction technique. Since it is proportional to the Reynolds number, V is assumed to be very large,

$$V = \frac{1}{\varepsilon \sigma l} = \frac{\text{Re } a}{\sigma L} = \frac{2\pi\kappa}{\varepsilon} \quad (2.34)$$

Several key parameters such as the boundary layer thickness, maximum velocity, and others can be directly correlated as functions of the vortex Reynolds number.

b) Sidewall Boundary Layer Solution

As with the viscous core corrections the analysis begins with the introduction of a scaling function to express the governing equation in the correct region.

$$s = \frac{\pi - \eta}{\delta} \quad (2.35)$$

This transformation develops a significantly different form of the governing equation than did Eq. (2.26). The new and more complicated equation is

$$\frac{\varepsilon}{\delta^2} \frac{d^2 \xi}{ds^2} - \frac{\kappa \sin(\pi - \delta s)}{2\delta(\pi - \delta s)} \frac{d\xi}{ds} = 0 \quad (2.36)$$

Again a Taylor-Series of the sine term is used to linearize the problem

$$\begin{aligned} -\frac{\kappa \sin(\pi - s\delta)}{2\delta(\pi - s\delta)} &\sim -\frac{\kappa}{2\delta(\pi - s\delta)} \left(\pi - s\delta - \frac{(\pi - s\delta)^3}{6} \right) \\ &= -\frac{\kappa}{2\delta} \left(1 - \frac{(\pi - s\delta)^2}{6} \right) \sim -\frac{\kappa}{2\delta} \left(1 - \frac{\pi^2}{6} \right) = \frac{\kappa\alpha}{2\delta} \end{aligned} \quad (2.37)$$

This expansion leaves a representative form of the governing equation near the wall:

$$\frac{\varepsilon}{\delta^2} \frac{d^2 \xi}{ds^2} + \frac{\kappa\alpha}{2\delta} \frac{d\xi}{ds} = 0 \quad (2.38)$$

As before the distinguished limit is found to be $\delta = \varepsilon / \kappa$. Reinserting the limit into the previous equation gives the final form of the governing equation for the sidewall boundary layer

$$\frac{d^2 \xi^i}{ds^2} + \frac{\alpha}{2} \frac{d \xi^i}{ds} = 0 \quad (2.39)$$

The boundary conditions at the sidewall become

$$\begin{cases} r=1, s=0, \xi^i=0 \\ r \rightarrow 0, s \rightarrow \infty, \xi^i = \xi^o \end{cases} \quad (2.40)$$

Solution of this equation and satisfaction of the boundary conditions compare to that of the viscous core solution.

$$\xi = \frac{1 - e^{-\frac{V}{4}\alpha(1-r^2)}}{1 - e^{-\frac{V}{4}\alpha}} \rightarrow u_\theta^{wall} = \frac{1}{r} \left[\frac{1 - e^{-\frac{V}{4}\alpha(1-r^2)}}{1 - e^{-\frac{V}{4}\alpha}} \right] \quad (2.41)$$

c) Composite Solution

A uniformly valid composite solution can be recovered using Erdélyi's principle.²² The solution is constructed by imposing each solution (viscous core, inviscid bulk flow, viscous sidewall) on top of one another and subtracting out the overlapping part.

$$u_\theta(r) = u_\theta^{core} + u_\theta^{wall} - \lim_{s \rightarrow \infty} u_\theta^{wall} = \frac{1}{r} \left[\frac{1 - e^{-\frac{V}{4}r^2}}{1 - e^{-\frac{V}{4}}} + \frac{1 - e^{-\frac{V}{4}\alpha(1-r^2)}}{1 - e^{-\frac{V}{4}\alpha}} - 1 \right] \quad (2.42)$$

3. Results and Discussion

In conformance with classical vortex models, the viscous effects do indeed capture the forced core vortex solution. Figures 8a,b show the effects of the vortex Reynolds number on the forced core

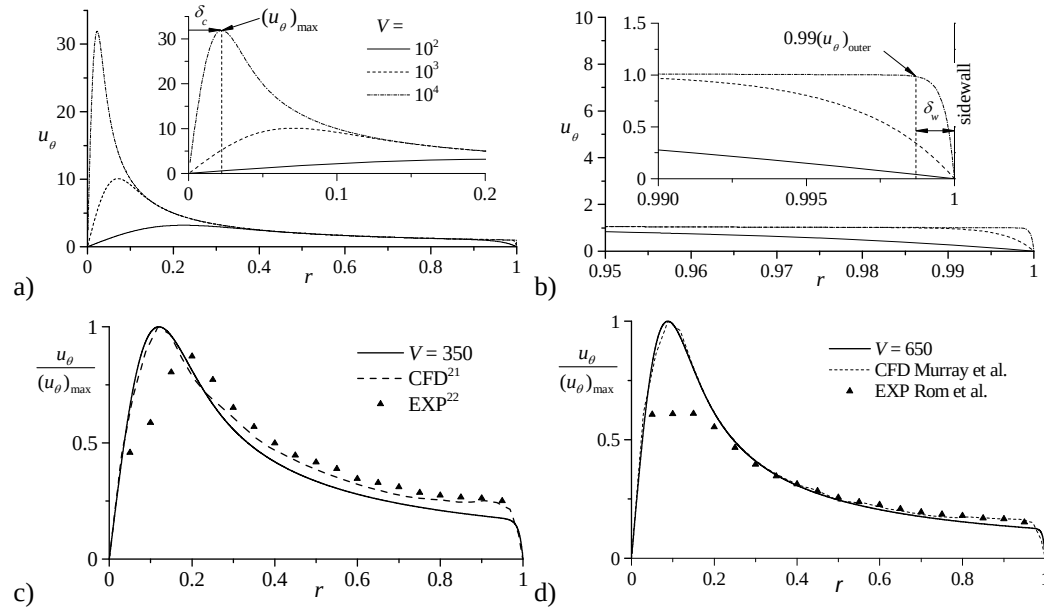


Figure 8 a) Composite solution at different values of V ; b) sidewall boundary layer. Comparison with CFD and experimental data for $V=$ c) 350 and d) 650.^{23,24} The equivalent turbulent vortex Reynolds number can be significantly larger due to the artificial increase in turbulent viscosity.

and sidewall boundary layer respectively. Comparison with experimental and computational data is shown in Fig 8c,d. The experimental data was collected via PIV techniques. Ideally, PIV seed particles should have the same mass as that of the molecules generating the flow. In this case the undershoot of the experimental data when compared with the analytical and computational results is a direct result of the seed particles being larger and heavier than the fluid particles. Since viscous forces dominate in the forced core region, the larger PIV seeds become subject to larger drag forces than the surrounding flow therefore underpredicting the actual flow velocity. We see better agreement to the CFD with increasing vortex Reynolds number. Since the vortex Reynolds number is on the order of κ smaller than the actual Reynolds number, it is expected to be much larger than the specified value in Fig. 8c we turn our attention more toward Fig 8d is more likely to be the physical situation and shows satisfactory agreement.

Several asymptotic assumptions were made in the presentation of these corrections. Fortunately computational and experimental verification confirm that these assumptions do not deviate significantly from nature. As discussed with the inviscid solution, we can expect a divergence from this behavior due to axial dependence near the headwall. Clearly the no-slip cannot be satisfied at the headwall or endwall with an axially independent solution. Vyas and Majdalani examined the pressure profile in the chamber and concluded that it is dominated by the swirl velocity.¹ With the current solution, one could now consider how the pressure gradient will affect the headwall boundary layer; perhaps introducing strong axial dependence away from the wall without deviating from the original assumptions that led to an axially independent solution.

III. Axial and Radial Sidewall Boundary Layer Analysis

The following analysis takes into account the viscous contributions at the sidewall in the remaining vector directions. This is done by deriving the appropriate governing equations and applying asymptotic principles to determine a uniformly valid solution throughout the chamber. The analysis is done in a similar fashion as the tangential velocity, and a common general form of solution results. The following analytic work has appeared in a previous paper.¹⁸

The axial corrections correspond to the classical boundary layer situation where a parallel flow is shearing over a surface. We expect the addition of viscosity at the wall to produce an axial shear stress component, which had been absent in previous studies. Some discussion must be made in order to better visualize the expected effect of viscosity on the radial solution. Clearly, this viscous region is not described with the classic boundary layer profile, but rather should be thought of as a secondary viscous effect. The inviscid solution meets the zeroth order boundary condition at the wall ensuring there is no penetration. However, the current inviscid solution contains a discontinuity with the first derivative. It should be expected that not only the velocity vanishes, but also the acceleration. Since the inviscid solution returns a value of $\partial u_r / \partial r$ at the wall, there remains a component of shear stress. It should not be expected of a perpendicular flow to have a component of shear stress at the wall. Specifically, $\partial u_r / \partial r$ must be zero at this location. Throughout the remainder of the chamber this stress component can be present. Visually we expect viscosity to have a tempering effect on the solution. These ideas are confirmed by visualizing the similarity solution presented by Morgan.^{25,26} The treatment of this perpendicular flow follows the standard boundary layer analysis.

The nomenclature as well as normalization remain consistent with previous work. Also the assumptions of axisymmetry and incompressibility are retained here.

A. Asymptotic Solution

1. Boundary Layer Equations

To begin the analysis, the proper boundary layer equations must be considered. The boundary layer equations can be derived according to Prandtl's method.²⁷ He applies an order of magnitude reduction of the Navier-Stokes equation in order to only retain the contributing terms *in the boundary layer*. In this case we consider the normalized cylindrical Navier-Stokes equations. When deriving boundary layer equations the following conjectures are made²⁸

- In the limit as the Reynolds number ($Re = Ua / \nu$) becomes large, all viscous terms become negligible. This is seen directly in the normalized form of the Navier-Stokes equations.
- The characteristic thickness of the boundary layer, δ , is inversely proportional to some fractional power of the Reynolds number. For typical laminar boundary layers, $\delta \sim \sqrt{1/Re}$
- In the case where ν is small or Re is large the boundary layer thickness becomes significantly small compared to any other characteristic dimension. In this situation curvature terms can be ignored. This idea is similar to the earth appearing flat very close to the surface. This condition can be shown to be small through rigorous analysis or assumed initially without compromising the solution.

For clarity, the boundary layer equations are normalized. The previously defined normalization scheme works well. In fact renormalizing the problem with the axial characteristic velocity and length will recover the same result. For three-dimensional flows, the convention used by Tetervin²⁹ is used to determine the contribution of each term. It is stated that flows and distances perpendicular to the surface must be of order δ while all parallel (or tangent) flows and distances

are order unity. Since $\varepsilon = 1/\text{Re}$ we follow the convention stated before to define the magnitude of ε . Due to the techniques used in the path toward solutions, the momentum equations are primarily considered with asymptotic techniques accounting for continuity.

At the onset we seek to develop a prediction of the subsequent mathematical solutions based on physical intuition. Typical two-dimensional flat plate type boundary layers are characterized primarily by the magnitude of free stream velocity and characteristic length. Also it is shown that the thickness of the boundary layer increases along the length of the plate. Now considering the BV, we observe a three-dimensional boundary layer where the actual boundary layer will be a composite of the contributions of tangential and axial velocities. Evaluating this idea further we can deduce some key concepts. This is a swirl dominated flow with the contribution of axial velocity being very small. Therefore we expect the composite boundary layer to mimic the already calculated tangential boundary layer. Asymptotically they should be equivalent. Therefore, we must take care that the axial layer calculations are consistent with this idea. Along with this, the tangential velocity is assumed to be axially independent. Therefore we can assume axial independence of the composite and subsequently axial boundary layers. Unlike flat plate boundary layers, no growth along the length of the chamber could be possible.

2. Axial Corrections

Based on Prandtl's ideas, we proceed to truncate the cylindrical Navier-Stokes equations. For the axial sidewall boundary layer, we have

$$u_r \frac{\partial u_z}{\partial r} + u_z \frac{\partial u_z}{\partial z} = -\frac{\partial p}{\partial z} + \varepsilon \left(\frac{\partial^2 u_z}{\partial r^2} + \frac{1}{r} \frac{\partial u_z}{\partial r} + \frac{\partial^2 u_z}{\partial z^2} \right) \quad (3.1)$$

$$\delta \frac{1}{\delta} \quad 1 \quad 1 \quad 1 \quad \delta^2 \left(\frac{1}{\delta^2} \quad \frac{1}{\delta} \frac{1}{\delta} \quad 1 \right)$$

Equation (3.1) clearly illustrates the order of magnitude of each term. By truncating at order δ the boundary layer equation becomes

$$u_r \frac{\partial u_z}{\partial r} + u_z \frac{\partial u_z}{\partial z} = -\frac{\partial p}{\partial z} + \varepsilon \left(\frac{\partial^2 u_z}{\partial r^2} + \frac{1}{r} \frac{\partial u_z}{\partial r} \right) \quad (3.2)$$

We consider the two crucial boundary conditions

$$\begin{cases} u_z(1, z) = 0 \\ \lim_{r \rightarrow 0} u_z(r, z) = u_z^o \end{cases} \quad (3.3)$$

These conditions will ensure that no-slip is satisfied and that the boundary layer seamlessly mates with the outer solution. Note that u_z^o corresponds to the solution that must be recovered as $u_z(r, z)$ approaches the main stream flow velocity profile.

It is seen that curvature terms are retained after the order of magnitude reduction is completed. This goes to show that for low speed, highly viscous flow regimes where the boundary layer thickness can be large at the wall, curvature terms do indeed contribute. We expect that to not be the case in the BV and therefore the third conjecture can be applied with confidence. We do not eliminate these terms directly in an effort to analytically show their relative smallness.

The pressure gradient is extracted from the inviscid solution in accordance with boundary layer theory:

$$\frac{\partial p}{\partial z} = -4\pi^2 \kappa^2 z \quad (3.4)$$

From experience we know that the inflow parameter κ is substantially small. We could truncate the pressure gradient term immediately; however, as with the curvature terms, it will be shown to naturally fall out of the equation.

Led by physical reasoning we state that $\partial/\partial r \gg \partial/\partial z$: derivatives with respect to the axial direction are neglected. This has been suggested by Conlisk³⁰ and others to seek an average solution throughout the chamber. This will significantly simplify Eq. (3.2). Lastly, in an attempt to linearize the problem we apply the successive corrections technique described previously by substituting in the leading order (inviscid) radial solution. This recovers Eq. (3.2) in the compacted form

$$\varepsilon \frac{1}{r} \frac{\partial}{\partial r} \left(r \frac{\partial u_z}{\partial r} \right) + \frac{\kappa}{r} \sin(\pi r^2) \frac{\partial u_z}{\partial r} = -4\pi^2 \kappa^2 z \quad (3.5)$$

At this juncture it becomes useful to apply the transformation used by Vyas and Majdalani.¹ We define

$$\eta = \pi r^2; \quad \frac{\partial}{\partial r} = 2\pi r \frac{\partial}{\partial \eta} \quad (3.6)$$

Direct substitution into Eq. (3.5) leads to

$$\varepsilon \left(\frac{\partial^2 u_z}{\partial \eta^2} + \frac{1}{\eta} \frac{\partial u_z}{\partial \eta} \right) + \frac{\kappa}{2\eta} \sin(\eta) \frac{\partial u_z}{\partial \eta} = -\frac{\pi \kappa^2 z}{\eta} \quad (3.7)$$

The next step in asymptotic analysis of boundary layers is to seek a scaling transformation that will more easily confront the rapid changes near the wall. We know that as $r \rightarrow 1$; $\eta \rightarrow \pi$. Therefore we state the appropriate scaling transformation to be

$$s = \frac{\pi - \eta}{\delta}; \quad \eta = \pi - s\delta \quad (3.8)$$

It should be mentioned that this transformation exactly parallels the one utilized with the tangential sidewall boundary layer. It remains the logical linear scaling choice. Applying this transformation we arrive one step closer to the governing equation scaled to the boundary layer:

$$\frac{\varepsilon}{\delta^2} \left(\frac{\partial^2 u_z}{\partial s^2} - \frac{\delta}{\pi - s\delta} \frac{\partial u_z}{\partial s} \right) - \frac{\kappa}{2\delta(\pi - s\delta)} \sin(\pi - s\delta) \frac{\partial u_z}{\partial s} = -\frac{\pi \kappa^2 z}{\pi - s\delta} \quad (3.9)$$

The sine term can be expanded according to Eq. (2.37). With this substitution we obtain the linearized form of the governing equation. Namely,

$$\frac{\varepsilon}{\delta^2} \left(\frac{\partial^2 u_z}{\partial s^2} - \frac{\delta}{\pi - s\delta} \frac{\partial u_z}{\partial s} \right) + \frac{\kappa \alpha}{2\delta} \frac{\partial u_z}{\partial s} = -\frac{\pi \kappa^2 z}{\pi - s\delta} \quad (3.10)$$

Immediately it can be seen that this correction will have a similar form to that of the tangential boundary layer due to the commonality in the linearization techniques. In order to balance corresponding terms, the distinguished limit is again determined to be $\delta \sim \varepsilon/\kappa$. Upon

substitution, Eq. (3.10) becomes the uniformly valid governing equation *in the boundary layer* once terms of $O(\varepsilon)$ are dismissed. One is left with

$$\frac{\partial^2 u_z}{\partial s^2} + \frac{\alpha}{2} \frac{\partial u_z}{\partial s} = 0 \quad (3.11)$$

With corresponding boundary conditions

$$\begin{cases} u_z(0, z) = 0 \\ \lim_{s \rightarrow \infty} u_z(s, z) = u_z^o \end{cases} \quad (3.12)$$

At this point one could proceed directly with success, nevertheless there remains one mathematical pitfall left to overcome. It is easily shown that direct application of Prandtl's or van Dyke's³¹ matching principles fall short once Erdélyi's principle of composite expansions is applied to obtain the final solution. The problem lies in determining the overlapping terms to be subtracted out. When one solution is trigonometric, the overlap will be bounded by the properties of the trigonometric function. For instance the limit of the cosine function becomes undefined as the independent variable tends toward infinity. Clearly, we cannot utilize the matching principles when one function becomes undefined. Likewise if we examine the solution as it tends toward the physical boundary, then the boundary conditions cannot be satisfied once Erdélyi's principle is applied. This is why we choose the limit type boundary condition. Even though it is the logical choice, it is not so easily applied. By examining the outer solution, essentially the value which must be satisfied when leaving the boundary layer, we realize it is not a constant value, but rather a function of the same independent variable as the boundary layer solution itself, $u_z^o = u_z^o(s, z)$. Mathematically, we cannot handle this without matching. For this reason we introduce a final transformation motivated by the inviscid solution:

$$u_z = \xi_z(s) \pi z \cos(\pi - 2\pi s/V) \quad (3.13)$$

The application of this transformation yields

$$\begin{aligned} -\pi z \cos(2\pi s/V) \frac{d^2 \xi}{ds^2} + \left(\frac{4\pi^2 z \sin(2\pi s/V)}{V} - \frac{\alpha}{2} \pi z \cos(2\pi s/V) \right) \frac{d\xi}{ds} \\ + \left(\alpha \frac{\pi^2 z \sin(2\pi s/V)}{V} + \frac{4\pi^3 z \cos(2\pi s/V)}{V^2} \right) \xi = 0 \end{aligned} \quad (3.14)$$

Immediately we recognize that $1/V$ can be assumed to be small and therefore negligible. This leads to the final form of the governing equation, namely,

$$\frac{d^2 \xi_z}{ds^2} + \frac{\alpha}{2} \frac{d\xi_z}{ds} = 0 \quad (3.15)$$

with transformed boundary conditions

$$\begin{cases} \xi_z(0) = 0 \\ \lim_{s \rightarrow \infty} \xi_z(s) = \xi_z^{(o)} = 2\kappa \end{cases} \quad (3.16)$$

Unmistakably, the boundary condition is now a constant and the boundary layer solution can be arrived at. Upon partial integration of this set, the solution is found to be

$$\xi_z(s) = 2\kappa \left(1 - e^{-\frac{1}{2}\alpha s}\right) \quad (3.17)$$

Rewriting Eq. (3.17) in terms of the original variables, the axial solution including the viscous correction becomes

$$u_z(r, z) = 2\pi\kappa z \cos(\pi r^2) \left[1 - e^{-\frac{V}{4}\alpha(1-r^2)}\right] \quad (3.18)$$

where $V = 2\pi\kappa/\varepsilon$ is the same vortex Reynolds number as previously defined. Note that the exact form of the tangential sidewall correction is recovered.

3. Radial Corrections

As with previous solutions we begin by considering the viscous momentum equation in an attempt to formulate the appropriate boundary layer equation,

$$\begin{aligned} u_r \frac{\partial u_r}{\partial r} - \frac{u_\theta^2}{r} + u_z \frac{\partial u_r}{\partial z} = -\frac{\partial p}{\partial r} + \varepsilon \left(\frac{\partial^2 u_r}{\partial r^2} + \frac{1}{r} \frac{\partial u_r}{\partial r} + \frac{\partial^2 u_r}{\partial z^2} - \frac{u_r}{r^2} \right) \\ \delta \quad 1 \quad \frac{1}{\delta} \quad 1 \quad \delta \quad \frac{1}{\delta} \quad \delta^2 \left(\frac{1}{\delta} \quad \frac{1}{\delta} \quad 1 \quad \delta \quad \frac{1}{\delta} \right) \end{aligned} \quad (3.19)$$

All previous statements regarding order of magnitude analysis apply here. Unlike classical shearing boundary layers, truncation must retain not only leading order and first order terms, but also second order terms in order to produce a meaningful equation. This physical manifestation corroborates the idea that viscous effects in the radial direction are secondary in nature. This reduction corresponds to

$$\varepsilon \left(\frac{\partial^2 u_r}{\partial r^2} + \frac{1}{r} \frac{\partial u_r}{\partial r} - \frac{u_r}{r^2} \right) - u_r \frac{\partial u_r}{\partial r} + \frac{u_\theta^2}{r} - u_z \frac{\partial u_r}{\partial z} = \frac{\partial p}{\partial r} \quad (3.20)$$

subject to

$$\begin{cases} u_r(1) = 0 \\ \lim_{r \rightarrow 0} u_r(r) = u_r^o \end{cases} \quad (3.21)$$

The question can be raised: since the assumption that the first derivative must go to zero is made, why does it not show up in the boundary conditions? Initially, we set these conditions in order to maintain the integrity of the inviscid solution. It can be easily shown that either condition could be used, as the other condition is always self satisfied.

Again we assert that axial changes are insignificant and truncate axial derivatives. Since the final radial solution will be void of axial dependence, the truncation made in lieu of this assumption transforms the governing equation from a PDE directly to an ODE. Next the pressure gradient is calculated from the inviscid profile¹⁶

$$\frac{\partial p}{\partial r} = \frac{\kappa^2}{r^3} \sin^2(\pi r^2) \left[1 - 2\kappa\pi r^3 \cot(\pi r^2)\right] + \frac{u_\theta^2}{r} \quad (3.22)$$

Direct substitution immediately cancels the u_θ^2/r terms. The remaining pressure gradient is $O(\kappa^2)$. Imposing the inviscid solution on the viscous profile via successive corrections techniques leads to

$$\varepsilon \left[\frac{1}{r} \frac{d}{dr} \left(r \frac{du_r}{dr} \right) + \frac{\kappa}{r^3} \sin(\pi r^2) \right] + \frac{\kappa}{r} \sin(\pi r^2) \frac{du_r}{dr} = O(\kappa^2) \quad (3.23)$$

Again the transformation $\eta = \pi r^2$ is employed. With careful substitution and factorization, Eq. (3.23) becomes

$$\varepsilon \left[\frac{d^2 u_r}{d\eta^2} + \frac{1}{\eta} \frac{du_r}{d\eta} + \frac{\kappa \sqrt{\pi}}{4\eta^{5/2}} \sin(\eta) \right] + \frac{\kappa}{2\eta} \sin(\eta) \frac{du_r}{d\eta} = \frac{O(\kappa^2)}{4\pi\eta} \quad (3.24)$$

In order to capture the profile in the near wall region, the corresponding linear scaling transformation (Eq. (3.8)) is again introduced. Afterward, the sine term is expanded according to Eq. (2.37) such that

$$\begin{aligned} \frac{\varepsilon}{\delta^2} \left[\frac{d^2 u_r}{ds^2} - \frac{\delta}{\pi - s\delta} \frac{du_r}{ds} + \frac{\delta^2 \kappa \sqrt{\pi}}{4(\pi - s\delta)^{5/2}} \sin(\pi - s\delta) \right] \\ + \frac{\kappa}{2\delta} \left(\frac{\pi^2}{6} - 1 \right) \frac{du_r}{ds} = \frac{1}{4\pi(\pi - s\delta)} O(\kappa^2) \end{aligned} \quad (3.25)$$

As before, the distinguished limit is determined to be $\delta \sim \varepsilon/\kappa$. Upon reinsertion of the distinguished limit into Eq. (3.25), we reduce the governing equation to the appropriate boundary layer form by truncating at $O(\varepsilon)$. The result is

$$\frac{d^2 u_r}{ds^2} + \frac{\alpha}{2} \frac{du_r}{ds} = 0 \quad (3.26)$$

We recover similar boundary conditions

$$\begin{cases} u_r(0) = 0 \\ \lim_{s \rightarrow \infty} u_r(s) = u_r^o \end{cases} \quad (3.27)$$

As with the axial solution, the problem with matching occurs here as well. Therefore, we must deploy another transformation in which the boundary conditions can be formally characterized. Namely,

$$u_r = \xi_r(s) \sin(\pi - 2\pi s/V) (1 - 2s/V)^{-1/2} \quad (3.28)$$

Once the derivatives with respect to the new dependent variable are handled, the following equation is derived

$$\begin{aligned} \frac{\sin(2\pi s/V)}{\sqrt{1-2s/V}} \frac{d^2 \xi}{ds^2} + 2 \left(\frac{2\pi \cos(2\pi s/V)}{V\sqrt{1-2s/V}} + \frac{\sin(2\pi s/V)}{V(1-2s/V)^{3/2}} + \frac{\alpha \sin(2\pi s/V)}{2\sqrt{1-2s/V}} \right) \frac{d\xi}{ds} \\ + \left(\alpha \frac{\pi \cos(2\pi s/V)}{V\sqrt{1-2s/V}} + \frac{\alpha}{2} \frac{\sin(2\pi s/V)}{V(1-2s/V)^{3/2}} + \frac{4\pi \cos(2\pi s/V)}{V^2(1-2s/V)^{3/2}} \right. \\ \left. + \frac{3\sin(2\pi s/V)}{V^2(1-2s/V)^{5/2}} - \frac{4\pi^2 \sin(2\pi s/V)}{V^2\sqrt{1-2s/V}} \right) \xi = 0 \end{aligned} \quad (3.29)$$

It is seen that we can truncate terms of $O(1/V)$, this leaves the equation

$$\frac{d^2 \xi_r}{ds^2} + \alpha \frac{d \xi_r}{ds} = 0 \quad (3.30)$$

and the boundary conditions

$$\begin{cases} \xi_r(0) = 0 \\ \lim_{s \rightarrow \infty} \xi_r(s) = -\kappa \end{cases} \quad (3.31)$$

Notice the slight difference with the coefficient in Eq. (3.31) compared to that of Eq. (3.26) and Eq. (2.39). The effect will become more evident as the solution continues. $\xi_r(s)$ is satisfied by the function

$$\xi_r(s) = -\kappa [1 - \exp(-s\alpha)] \quad (3.32)$$

In terms of the original variables the solution is expressed as

$$u_r(r) = -\frac{\kappa}{r} \sin(\pi r^2) \left[1 - \exp\left(-\frac{V}{2} \alpha (1 - r^2)\right) \right] \quad (3.33)$$

4. Results and Discussion

a) Axial Velocity Profile

Figures 9a,b indeed show the classic boundary layer type profile we expect where the bulk flow inviscid solution is obtained outside the boundary layer while the no-slip condition is satisfied at the wall. The graphics describe the new solution with two key parameters. We can see the effect of axial position in Fig. 9a and the effect of vortex Reynolds number in Fig. 9b. Since the inviscid axial solution increases in magnitude as it traverses the chamber, we see the increased magnitude of the velocity. More interesting is the effect of vortex Reynolds number. With increasing V , the slope of the boundary layer increases. Subsequent changes in shear stress can be predicted as well. In the limiting case as $V \rightarrow \infty$, the viscosity term becomes unity and the inviscid solution is recovered exactly. In the opposite case as $V \rightarrow 0$, corresponding to the situation of no mean inflow velocity, the solution disappears.

b) Radial Velocity Profile

To iterate, the viscous equation is derived by keeping second order terms, corresponding to the idea that viscous effects in the perpendicular direction will be secondary in nature. This idea explains the very small deviations from the original inviscid solution. By examining the scale of Fig. 9c we can see that even for a small vortex Reynolds number, the inviscid solution is quickly recovered. More importantly, the expected tempering effect of viscosity is visualized. This has a direct effect on the shear stress vector at the wall.

c) Boundary Layer Thicknesses

An analytic solution for the boundary layer thickness as a function of vortex Reynolds number can be calculated by defining the thickness to be the distance from the wall, to where 99% of the bulk flow is recovered. The thickness equations can be written as

$$\delta_z = 1 - \sqrt{\frac{4 \ln(0.01)}{V\alpha} + 1}; \quad \delta_r = 1 - \sqrt{\frac{2 \ln(0.01)}{V\alpha} + 1} \quad (3.34)$$

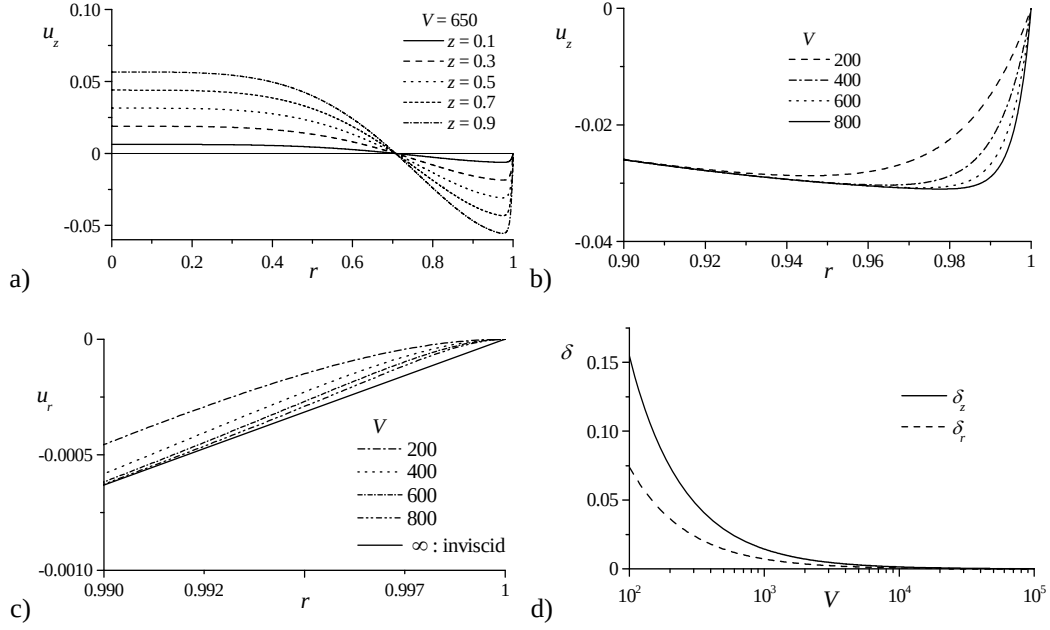


Figure 9 a) Axial velocity profile with variations in axial position, $V = 650$; b) axial boundary layer profile with variations in vortex Reynolds number, $z = 0.3$; c) near wall radial velocity profile with respect to varying vortex Reynolds number; and d) comparison between the radial and axial boundary layers.

Figure 9d plots the thickness as a function of vortex Reynolds number for both boundary layers. Notice that for very large V , the thicknesses are almost indistinguishable and asymptotically equal. Inversely, as the vortex Reynolds number decreases we see a divergence of the radial thickness to a value half that of the axial and tangential parts. This supports the idea that the governing equation is essentially an order of magnitude smaller. Table 1 quantifies the thicknesses of all previously calculated viscous regions.

Mathematically, we see both axial and tangential equations balance the diffusive term with the radial contribution to the convective term. This balance becomes apparent in both boundary layer calculations by examining the distinguished limit. To compare the boundary layer thickness as predicted by the distinguished limit we can insert values of κ and ε . With values of $\kappa = 0.01$ and $\varepsilon = 0.0001$, the calculated thicknesses are $\delta_z = 0.02299$ and $\delta_r = 0.01143$ while the distinguished limit predicts $\delta = 0.01$. Both predict the same order of magnitude which is the goal of determining the distinguished limit.

d) Wall Stress Tensor

The wall shear stress can now be evaluated according to the following relations

$$\begin{aligned}
 \tau_{\theta\theta} &= 2\varepsilon \left(\frac{1}{r} \frac{\partial u_\theta}{\partial \theta} + \frac{u_r}{r} \right) \\
 &= -\frac{\varepsilon^2 V}{\pi r^2} \sin(\pi r^2) \left(1 - e^{-\frac{V}{2} a(1-r^2)} \right) = -\frac{2\varepsilon \kappa}{r^2} \sin(\pi r^2) \left(1 - e^{-\frac{V}{2} a(1-r^2)} \right)
 \end{aligned} \tag{3.35}$$

Table 1 Comparison to previously determined boundary layer thicknesses.

V	δ_z	δ_r	δ_c^*	δ_w^*
200	0.07415	0.03636	0.15852	0.07415
400	0.03636	0.01801	0.11209	0.03636
600	0.02409	0.01197	0.09152	0.02409
800	0.01801	0.00896	0.07926	0.01801
1000	0.01438	0.00716	0.07089	0.01438

*as defined by Vyas and Majdalani¹

$$\begin{aligned}
 \tau_{rr} &= 2\varepsilon \frac{\partial u_r}{\partial r} = \varepsilon^2 V \left[\begin{aligned} &-2 \cos(\pi r^2) \left(1 - e^{-\frac{V}{2}\alpha(1-r^2)} \right) + \frac{1}{\pi r^2} \sin(\pi r^2) \left(1 - e^{-\frac{V}{2}\alpha(1-r^2)} \right) \\ &+ \frac{V}{\pi} \alpha \sin(\pi r^2) e^{-\frac{V}{2}\alpha(1-r^2)} \end{aligned} \right] \\
 &= 2\pi\varepsilon\kappa \left[\begin{aligned} &-2 \cos(\pi r^2) \left(1 - e^{-\frac{V}{2}\alpha(1-r^2)} \right) + \frac{1}{\pi r^2} \sin(\pi r^2) \left(1 - e^{-\frac{V}{2}\alpha(1-r^2)} \right) \\ &+ \frac{V}{\pi} \alpha \sin(\pi r^2) e^{-\frac{V}{2}\alpha(1-r^2)} \end{aligned} \right] \quad (3.36)
 \end{aligned}$$

$$\tau_{zz} = 2\varepsilon \frac{\partial u_z}{\partial z} = 2V\varepsilon^2 \cos(\pi r^2) \left(1 - e^{-\frac{V}{4}\alpha(1-r^2)} \right) = 4\pi\kappa\varepsilon \cos(\pi r^2) \left(1 - e^{-\frac{V}{4}\alpha(1-r^2)} \right) \quad (3.37)$$

$$\begin{aligned}
 \tau_{r\theta} &= \varepsilon \left[\frac{1}{r} \frac{\partial u_r}{\partial \theta} + r \frac{\partial}{\partial r} \left(\frac{u_\theta}{r} \right) \right] = \varepsilon \left[\frac{\partial u_\theta}{\partial r} - \frac{u_\theta}{r} \right] \\
 &= \frac{\varepsilon}{r^2} \left[\frac{\frac{1}{2} e^{-\frac{V}{4}r^2} (4 + r^2 V)}{1 - e^{-\frac{V}{4}}} + \frac{\frac{1}{2} e^{-\frac{V}{4}\alpha(1-r^2)} [4 - r^2 V \alpha] - 1 - e^{-\frac{V}{4}\alpha}}{1 - e^{-\frac{V}{4}\alpha}} - \coth\left(\frac{V}{8}\right) \right] \\
 &\approx \frac{\varepsilon}{2r^2} \left\{ e^{-\frac{V}{4}r^2} (4 + r^2 V) + e^{-\frac{V}{4}\alpha(1-r^2)} [4 - r^2 V \alpha] - 4 \right\} \quad (3.38)
 \end{aligned}$$

$$\tau_{\theta z} = \varepsilon \left(\frac{1}{r} \frac{\partial u_z}{\partial \theta} + \frac{\partial u_\theta}{\partial z} \right) = 0 \quad (3.39)$$

$$\begin{aligned}
\tau_{zr} &= \varepsilon \left(\frac{\partial u_z}{\partial r} + \frac{\partial u_r}{\partial z} \right) = \varepsilon \frac{\partial u_z}{\partial r} \\
&= \frac{\varepsilon^2 r z V}{2} \left[-V \cos(\pi r^2) \alpha e^{-\frac{V}{4}\alpha(1-r^2)} - 4\pi \sin(\pi r^2) \left(1 - e^{-\frac{V}{4}\alpha(1-r^2)} \right) \right] \\
&= \varepsilon r z \pi \kappa \left[-V \cos(\pi r^2) \alpha e^{-\frac{V}{4}\alpha(1-r^2)} - 4\pi \sin(\pi r^2) \left(1 - e^{-\frac{V}{4}\alpha(1-r^2)} \right) \right]
\end{aligned} \tag{3.40}$$

Since the emphasis is to examine the shear stress near the wall as a mechanism for roll torque, we take the limit of each stress function as it approaches the wall; $r \rightarrow 1$.

$$\begin{cases} (\tau_{rr} = \tau_{\theta\theta} = \tau_{zz} = \tau_{\theta z})|_{\text{wall}} = 0 \\ \tau_{r\theta}^{(w)} = -\frac{\varepsilon V \alpha}{2} = \pi \kappa \alpha \\ \tau_{zr}^{(w)} = \frac{\varepsilon^2 z V^2 \alpha}{2} = \frac{2\pi^2 \kappa^2 z \alpha}{\varepsilon} \end{cases} \tag{3.41}$$

Compared to the work by Vyas and Majdalani¹ we see some differences with the previous predictions. For instance τ_{zz} and τ_{rr} have been determined to be exactly rather than approximately zero. Conversely, τ_{zr} is now defined to have a value. As before the value of $\tau_{r\theta}$ dominates the shear stress component while all other terms are zero or second order in κ . The magnitude of shear stress becomes

$$\tau_0^{(w)} = \sqrt{(\tau_{r\theta}^{(w)})^2 + (\tau_{zr}^{(w)})^2} = \pi \alpha \kappa \sqrt{1 + 4\pi^2 \kappa^2 z^2} = \frac{1}{2} \alpha \varepsilon V \sqrt{1 + \varepsilon^2 V^2 z^2} \tag{3.42}$$

Figure 10 appropriately visualizes each shear stress component according to the radial or axial position. The total shear force over the surface of the sidewalls can be calculated by integrating the magnitude of shear stress over the surface of the chamber:

$$\begin{aligned}
F_0^{(w)} &= \pi \alpha \kappa \int_0^{2\pi} \int_0^l \sqrt{1 + (\varepsilon z V)^2} dz d\theta \\
&= 2\pi^2 \alpha \kappa \int_0^l \sqrt{1 + (\varepsilon z V)^2} dz = \pi^2 \alpha \kappa l \left[\sqrt{1 + (\varepsilon l V)^2} + \frac{\sinh^{-1}(\varepsilon l V)}{\varepsilon l V} \right]
\end{aligned} \tag{3.43}$$

The roll torque is a direct result of the tangential shear stress. It can be predicted by

$$\bar{T}_\theta = \pi \alpha Q_i \rho U^2 a^3 = (\frac{1}{6} \pi^3 - \pi) \rho \bar{Q}_i U a = 2.02612 \dot{m}_i U a = 1.01306 \dot{m}_i U D \tag{3.44}$$

e) Pressure Solution

We learn from viscous flow theory that pressure gradients can be calculated directly from the inviscid solution by way of Euler's equations for the case of thin boundary layers. This cannot take this at face value. The singularity of the purely inviscid solution at the core will propagate into the pressure distribution. Therefore, at the very least, the viscous forced core corrections must be included when determining pressure profiles. The boundary layer corrections can be included to modify the profile slightly. In order to predict the pressure profile up to leading order we insert the current viscous solution into the inviscid Euler equations. This combines the effect of the viscous solution and the accepted inviscid borne pressure calculations by truncating the complete Navier-Stokes equation at leading order. We calculate

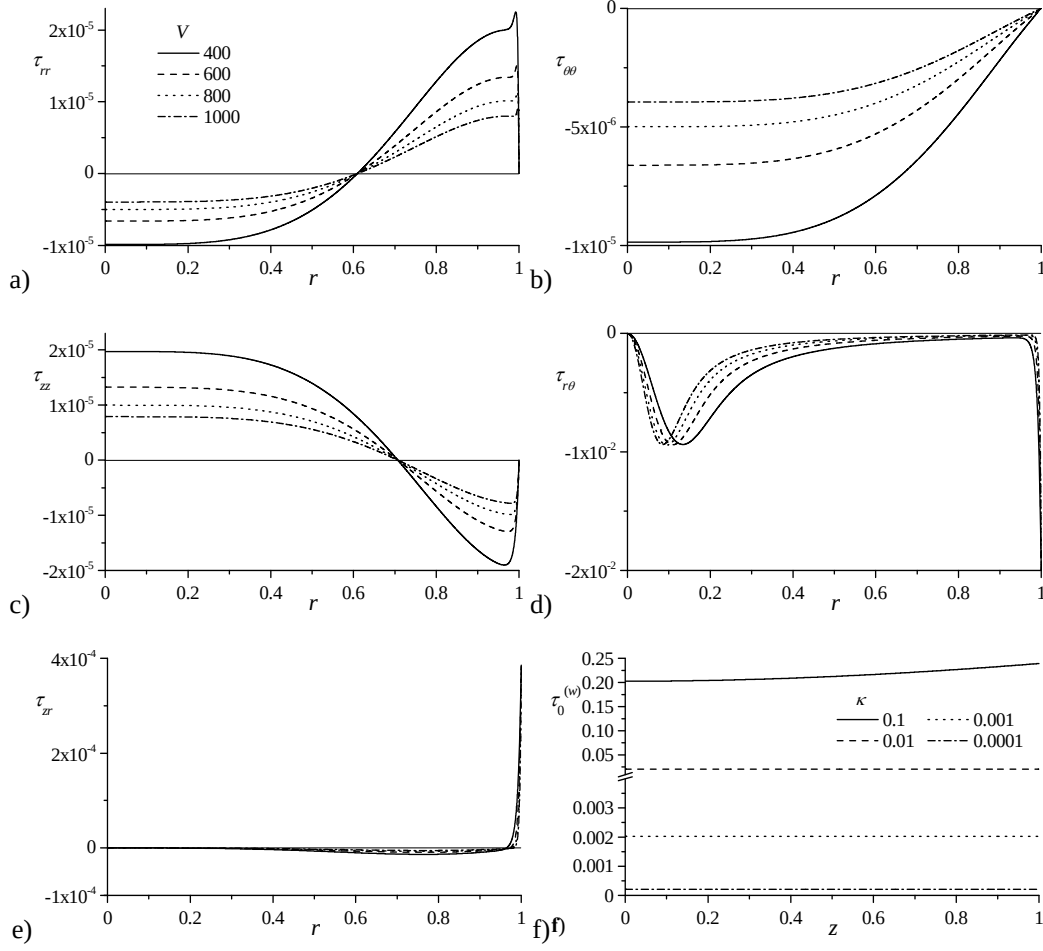


Figure 10. Plots of shear stress with respect to radial position at an arbitrary axial position of $z = 0.3$, $\kappa = 0.01$ and a) τ_{rr} , b) $\tau_{\theta\theta}$, c) τ_{zz} , d) $\tau_{r\theta}$, e) τ_{zr} , and f) $\tau_0^{(w)}$.

$$\frac{\partial p}{\partial r} = \frac{u_\theta^2}{r} - u_r \frac{\partial u_r}{\partial r} - u_z \frac{\partial u_r}{\partial z} \quad (3.45)$$

From Eq. (3.45) the radial pressure gradient can be calculated from,

$$\begin{aligned} \frac{\partial p}{\partial r} = & \kappa^2 r^{-1} \sin(\pi r^2) \left[1 - e^{-\frac{1}{2}\alpha V(1-r^2)} \right] \\ & \left\{ \left[r^{-2} \sin(\pi r^2) - 2\pi \cos(\pi r^2) \right] \left[1 - e^{-\frac{1}{2}\alpha V(1-r^2)} \right] + \alpha V \sin(\pi r^2) e^{-\frac{1}{2}\alpha V(1-r^2)} \right\} \\ & + \frac{1}{r^3} \left[\frac{1 - e^{-\frac{V}{4}r^2}}{1 - e^{-\frac{V}{4}}} + \frac{1 - e^{-\frac{V}{4}\alpha(1-r^2)}}{1 - e^{-\frac{V}{4}\alpha}} - 1 \right]^2 \end{aligned} \quad (3.46)$$

With the knowledge of κ being small we can represent the pressure gradient as

$$\frac{\partial p}{\partial r} \approx \frac{1}{r^3} \left[1 - e^{-\frac{V}{4}r^2} - e^{-\frac{V}{4}\alpha(1-r^2)} \right]^2 \quad (3.47)$$

Also, we seek to determine the axial gradient

$$\frac{\partial p}{\partial z} = -u_r \frac{\partial u_z}{\partial r} - u_z \frac{\partial u_z}{\partial z} \quad (3.48)$$

Simple examination of the right hand side of Eq. (3.48) shows that the pressure gradient is also $O(\kappa^2)$. In remaining consistent with the radial gradient being truncated at $O(\kappa^2)$, we find the axial gradient to be noncontributing. With the help of symbolic software, Eq. (3.47) can be integrated³² up to a constant of integration to be determined by the head wall pressure. The pressure distribution becomes

$$p = p_0 - \frac{1}{2r^2} \left\{ \left(1 - e^{-\frac{1}{4}Vr^2} \right)^2 + \left[1 - e^{-\frac{1}{4}\alpha V(1-r^2)} \right]^2 - 1 + 2e^{-\frac{1}{4}V[\alpha+(1-\alpha)r^2]} \right\} \\ - \frac{V}{4} \left\{ \text{Ei}\left(-\frac{1}{2}Vr^2\right) - \text{Ei}\left(-\frac{1}{4}Vr^2\right) + \alpha e^{-\frac{1}{4}\alpha V} \left[\text{Ei}\left(\frac{1}{4}\alpha Vr^2\right) - \text{Ei}\left(\frac{1}{4}\alpha V\right) \right] \right\} \\ - \alpha e^{-\frac{1}{2}\alpha V} \left[\text{Ei}\left(\frac{1}{2}\alpha Vr^2\right) - \text{Ei}\left(\frac{1}{2}\alpha V\right) \right] \quad (3.49)$$

where

$$\text{Ei}(x) = \int_{-x}^{\infty} \left(e^{-t} / t \right) dt = \gamma + \ln|x| + \sum_{m=1}^{\infty} \frac{x^m}{m!m}; \quad \gamma = 0.5772156649 \text{ (Euler's constant)} \quad (3.50)$$

After asymptotic reduction of the pressure gradient, it is seen that it relies completely on the tangential velocity. The overshadowing contribution of the azimuthal velocity in swirl dominated flows is once again shown. Asymptotic reduction causes the solution to be axially invariant. Figure 11a shows the effect of the viscous forced core vortex. Clearly the core region contains the highest pressure gradient. Figure 11b shows the pressure profile. Note that the vortex Reynolds number is only significant in the forced core region. Beyond this region, the profile is for all intents and purposes independent of V .

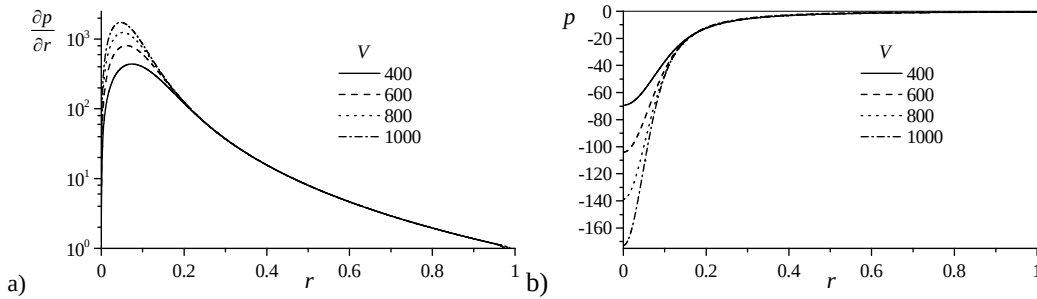


Figure 11 Radial variations in a) pressure gradient and the b) pressure distribution.¹⁸

f) Vorticity and Circulation

Vorticity

The axisymmetric vorticity is given by

$$\mathbf{\Omega} = \nabla \times \mathbf{u} = -\frac{\partial u_\theta}{\partial z} \mathbf{e}_r + \left(\frac{\partial u_r}{\partial z} - \frac{\partial u_z}{\partial r} \right) \mathbf{e}_\theta + \left(\frac{\partial u_\theta}{\partial r} + \frac{u_\theta}{r} \right) \mathbf{e}_z \quad (3.51)$$

The formal expansion of the derivatives leads to the vector solution

$$\begin{aligned} \mathbf{\Omega} = \pi r z \kappa \left\{ \alpha V e^{-\frac{1}{4}\alpha V(1-r^2)} \cos(\pi r^2) + 4\pi \left[1 - e^{-\frac{1}{4}\alpha V(1-r^2)} \right] \sin(\pi r^2) \right\} \mathbf{e}_\theta \\ + \frac{V}{2} \left[\frac{e^{-\frac{1}{4}Vr^2}}{1 - e^{-\frac{1}{4}V}} - \frac{\alpha e^{-\frac{1}{4}\alpha V(1-r^2)}}{1 - e^{-\frac{1}{4}\alpha V}} \right] \mathbf{e}_z \end{aligned} \quad (3.52)$$

We see that the tangential component is order κ and therefore we expect the chief contribution to vorticity to be restricted to the axial component. As previously stated by Vyas and Majdalani¹ the axial vorticity can be accurately represented by

$$\Omega_z \simeq \frac{1}{2} V \left[e^{-\frac{1}{4}Vr^2} - \alpha e^{-\frac{1}{4}\alpha V(1-r^2)} \right] \quad (3.53)$$

A note will be made about the classical Rankine vortex.⁶ This classic solution is defined by a piecewise continuous function where the core is governed by solid body rotation and the free region is inversely proportional to the radius. Based on Eq. (1.2),

$$u_\theta = \frac{\bar{u}_\theta}{(\bar{u}_\theta)_{\max}} = \begin{cases} r, & r \leq 1 \\ \frac{1}{r}, & r > 1 \end{cases} \quad \text{where } r = \frac{\bar{r}}{\delta} \quad (3.54)$$

We see the blatant commonality between the Rankine solution and the inviscid bulk flow of the BV. It can be shown that the free vortex region of the Rankine model is derivable from a potential function. By definition the vorticity of a potential is zero, $\nabla \times \nabla A = 0$. Therefore all axial vorticity must be confined to the viscous regions. Similarly, boundary layers are accepted as a source of vorticity as a means to satisfy the no-slip condition.

Figure 12a represents the axial vorticity plots. As expected via the Rankine analysis, the bulk flow is vorticity free. The graphics show that vorticity is restricted to the viscous regions. Figure 12b represents the contribution to the vorticity vector by the tangential component. The region of contribution is again located exclusively in the viscous region with the most significant contribution in the area of maximum axial velocity: the endwall. To reinforce the statement earlier, the maximum magnitude of axial vorticity dwarfs that of the tangential component.

Circulation

The circulation is commonly defined as the line integral of the tangential component of velocity around a closed curve. However, by means of Stokes' Theorem, we can directly relate the circulation to the vorticity via

$$\Gamma = \iint_A (\nabla \times \mathbf{u}) \cdot \mathbf{n} \, dA \quad (3.55)$$

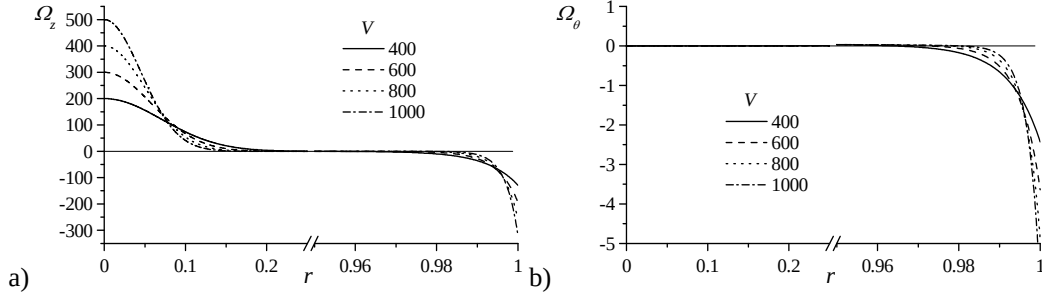


Figure 12 Vorticity profiles: a) axial vorticity $z = 0.3, \kappa = 0.01$; and b) tangential vorticity at $z = 0.3, \kappa = 0.01$.

The area to be considered is that of the cross-section of the chamber. For this calculation we consider the axial component of vorticity integrated over the circular area A :

$$\Gamma = \iint_A \frac{Vr}{2} \left[\frac{e^{-\frac{V}{4}r^2}}{1 - e^{-\frac{V}{4}}} - \frac{\alpha e^{-\frac{V}{4}\alpha(1-r^2)}}{1 - e^{-\frac{V}{4}\alpha}} \right] dr d\theta; \quad \begin{cases} 0 \leq r \leq 1, \\ 0 \leq \theta \leq 2\pi \end{cases} \quad (3.56)$$

Upon integration we determine $\Gamma = 0$. This represents a unique cancellation of the products of vorticity in the core region with those in the sidewall region.

g) *Swirling Intensity*

In order to predict the mixing potential with respect to various initial parameters we evaluate the swirling intensity as defined by Chang and Dhir.³³

$$\Omega_s = \frac{1}{4} \left(\int_0^{1/\sqrt{2}} ru_z dr \right)^{-2} \int_0^{1/\sqrt{2}} ru_z u_\theta dr \quad (3.57)$$

Symbolic programming evaluates this integral as

$$\tilde{\Omega} = -\frac{\pi\sqrt{\pi}(1-i)}{4\sqrt{2}} \frac{1}{\kappa z} \left\{ \frac{\operatorname{erf}\left[\frac{1}{4}(1+i)\sqrt{4\pi-iV}\right]}{\sqrt{4\pi-iV}} + \frac{\operatorname{erfi}\left[\frac{1}{4}(1+i)\sqrt{4\pi+iV}\right]}{\sqrt{4\pi+iV}} - \frac{(1+i)}{\sqrt{\pi}} C(1) \right\} \quad (3.58)$$

where small terms are dropped. Here $C(1) \approx 0.779893$ where $C(x)$ is the Fresnel integral defined by

$$C(x) = \int_0^x \cos\left(\frac{1}{2}\pi r^2\right) dr \quad (3.59)$$

We can describe the complete form of the swirling intensity by Eq. (3.58). Despite the imaginary representation of this equation, it actually returns values in the real number space. Perhaps the most important aspect to understand when considering swirling intensity is the axial position of highest intensity, and therefore best mixing potential. An accurate approximation of the axial profile can be stated simply as $\Omega_s \approx 0.866/(\kappa z)$. In this direction large changes in the vortex Reynolds number correspond to small changes in swirling intensity and therefore can be approximated as vortex Reynolds number independent as $V \rightarrow \infty$. These plots are seen in Fig 13.

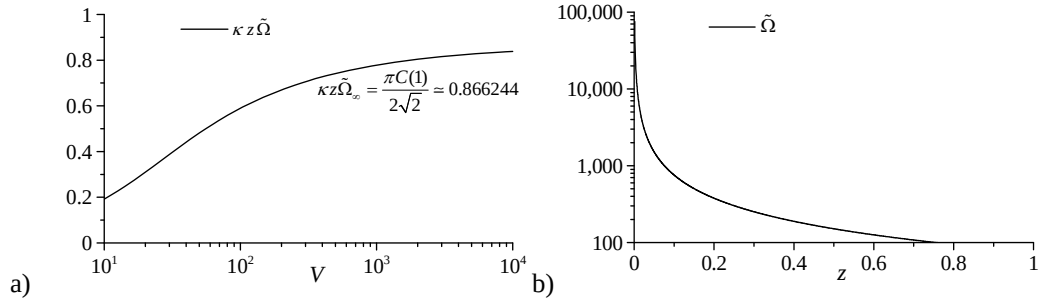


Figure 13 Swirling intensity with respect to a) vortex Reynolds number and b) axial position, $\kappa = 0.01$ and $V = 650$.

Figure 13a,b compare swirling intensity with respect to vortex Reynolds number and axial position respectively. Figure 13b best illustrates the fact that mixing is greatest near the headwall. In propulsive applications, this assures that a headwall placement of fuel injectors should correspond to the best mixing of oxidizer and fuel and therefore the highest combustion efficiency.

IV. Conclusion

This work extends the solutions of the bidirectional vortex. The viscous corrections presented here originate from the exact inviscid solution. We are able to further the work by applying asymptotic techniques to capture viscous corrections at the core in the tangential direction and along the sidewall in all three vector directions. In this study specifically, we capture the viscous effects that will enforce the no-slip condition in the axial direction as well as observe the tempering of the radial profile. Upon completion of the viscous corrections, we are able to calculate several other corollaries such as boundary layer thicknesses, leading order pressure solutions, vorticity, and shear stress.

All analytic solutions are carried out in a similar manner with similar assumptions and scaling parameters applied to the full Navier-Stokes equations. The appropriate boundary layer equations are obtained through rigorous order of magnitude truncation. Several key transformations of both the *independent* and *dependent* variables are applied in order to both simplify the equations as well as allow matching techniques to be applied. Both axial and radial solutions closely resemble that of the tangential part. This result is due largely to the method of linearization used for each.

The observation which must be restated is that the axial boundary layer *must* parallel that of the tangential solution. This is because the composite layer will be strongly dependent on the tangential velocity. In conformance with the assumptions regarding the swirl velocity, the axial boundary layer must be axially invariant and of the same order as the tangential. Mathematically this is shown in both analyses by the balance between the radial contribution to the convective and the diffusive terms.

With respect to the shear stresses, the axial and radial corrections help to refine the stress tensor in several of its elements. The $\tau_{zr}^{(w)}$ term and, hence, the total shear are seen to exhibit a small axial dependence which cannot be manifested in the absence of friction parallel to the wall. We also determine that $\tau_{rr}^{(w)}$ and $\tau_{zz}^{(w)}$ are not asymptotically small but strictly zero. Overall, only secondary contributions to the resultant shear force are realized. The total roll torque remains dominated by the tangential component. Its dimensional form is evaluated and found to depend on the fluid moment couple, a product of the mass flow rate into the chamber, the circumferential velocity, and the chamber diameter. It is interesting that the differential analysis leads to a similar prediction of torque as that obtained from a global, control-volume approach.

We see a negligible contribution of the newly formulated boundary layers in the calculation of the remaining corollaries. Approximate representations of the pressure profile, vorticity, and swirling intensity encompass only the contributions of the tangential velocity. These solutions almost entirely restore the work done by Vyas and Majdalani.¹ They reinforce the concept that swirl velocity dominates the analytic modeling of the bidirectional vortex.

References

References

- ¹Vyas, A. B., and Majdalani, J., "Characterization of the Tangential Boundary Layers in the Bidirectional Vortex Thrust Chamber," AIAA Paper 2006-8888, 9-12 July 2006.
- ²Bloor, M. I. G., and Ingham, D. B., "The Flow in Industrial Cyclones," *Journal of Fluid Mechanics*, Vol. 178, 1987, pp. 507-519.
- ³Chiaverini, M. J., Malecki, M. J., Sauer, J. A., and Knuth, W. H., "Vortex Combustion Chamber Development for Future Liquid Rocket Engine Applications," AIAA Paper 2002-2149, July 2002.
- ⁴Chiaverini, M. J., Malecki, M. J., Sauer, J. A., Knuth, W. H., and Majdalani, J., "Vortex Thrust Chamber Testing and Analysis for O₂-H₂ Propulsion Applications," AIAA Paper 2003-4473, July 2003.
- ⁵Chiaverini, M. J., Malecki, M. M., Sauer, J. A., Knuth, W. H., and Hall, C. D., "Testing and Evaluation of Vortex Combustion Chamber for Liquid Rocket Engines," JANNAF 2002.
- ⁶Rankine, *A Manual of Applied Mechanics*, 9th ed., C. Griffin and Co., London, UK, 1858.
- ⁷Fang, D., Majdalani, J., and Chiaverini, M. J., "Hot Flow Model of the Vortex Cold Wall Liquid Rocket," AIAA Paper 2004-3676, July 2004.
- ⁸Batterson, J. W., Maicke, B., and Majdalani, J., "Advancements in Theoretical Models of Confined Vortex Flowfields," JANNAF Joint Propulsion Conference JANNAF 2007.
- ⁹Bertato, M., Giaiotti, D. B., Mansato, A., and Stel, F., "An Interesting Case of Tornado in Friuli-Northeastern Italy.," *Atmospheric Research*, Vol. 7-68, 2003, pp. 3-21.
- ¹⁰Mallen, K. J., Montgomery, M. T., and Wang, B., "Reexamining the near-Core Radial Structure of the Tropical Cyclone Primary Circulation: Implications for Vortex Resiliency," *Journal of the Atmospheric Sciences*, Vol. 62, No. 2, 2005, pp. 408-425.
- ¹¹Pérez-Saborid, M., Herrada, M. A., Gómez-Barea, A., and Barrero, A., "Downstream Evolution of Unconfined Vortices: Mechanical and Thermal Aspects," *Journal of Fluid Mechanics*, Vol. 471, No. 1, 2002, pp. 51-70.
- ¹²Devenport, W. J., Rife, M. C., Liapis, S. I., and Follen, G. J., "The Structure and Development of a Wing Tip Vortex," *Journal of Fluid Mechanics*, Vol. 312, No. 1, 1996, pp. 67-106.
- ¹³Sullivan, R. D., "A Two-Cell Vortex Solution of the Navier-Stokes Equations," *Journal of the Aerospace Sciences*, 1959, pp. 767-768.
- ¹⁴Bloor, M. I. G., and Ingham, D. B., "Theoretical Investigation of the Flow in a Conical Hydrocyclone," *Transactions of the Institution of Chemical Engineers*, Vol. 51, 1973, pp. 36-41.
- ¹⁵Kelsall, D. F., "A Study of Motion of Solid Particles in a Hydraulic Cyclone," *Transactions of the Institution of Chemical Engineers*, Vol. 30, 1952, pp. 87-103.
- ¹⁶Vyas, A. B., Majdalani, J., and Chiaverini, M. J., "The Bidirectional Vortex. Part 1: An Exact Inviscid Solution," AIAA Paper 2003-5052, July 2003.
- ¹⁷Vyas, A. B., and Majdalani, J., "Exact Solution of the Bidirectional Vortex," *AIAA*, Vol. 44, No. 10, 2006.
- ¹⁸Batterson, J. W., and Majdalani, J., "On the Boundary Layers of the Bidirectional Vortex," AIAA Fluid Dynamics Conference AIAA Paper 2007-4123, 2007.
- ¹⁹Culick, F. E. C., "Rotational Axisymmetric Mean Flow and Damping of Acoustic Waves in a Solid Propellant Rocket," *Journal of Propulsion and Power*, Vol. 5, No. 6, 1989, pp. 657-664.
- ²⁰Fang, D., Majdalani, J., and Chiaverini, M. J., "Simulation of the Cold-Wall Swirl Driven Combustion Chamber," AIAA Paper 2003-5055, July 2003.
- ²¹Vyas, A. B., Majdalani, J., and Chiaverini, M. J., "The Bidirectional Vortex. Part 2: Viscous Core Corrections," AIAA Paper 2003-5053, July 2003.
- ²²Erdélyi, A., *Asymptotic Expansions*, Dover, New York, NY, 1956.
- ²³Murray, A. L., Gudgen, A. J., Chiaverini, M. J., Sauer, J. A., and Knuth, W. H., "Numerical Code Development for Simulating Gel Propellant Combustion Processes," JANNAF 2004.

- ²⁴Rom, C. J., Anderson, M. H., and Chiaverini, M. J., "Cold Flow Analysis of a Vortex Chamber Engine for Gelled Propellant Combustor Applications," AIAA Paper 2004-3359, July 2004.
- ²⁵Morgan, A. J. A., "The Reduction of One of the Number of Independent Variables in Some Systems of Partial Differential Equations," *Quarterly Journal of Math*, Vol. 2, 1952.
- ²⁶Morgan, A. J. A., "Possible Similarity Solutions of the Laminar, Incompressible, Boundary-Layer Equations," *ASME*, Vol. 80, 1958.
- ²⁷Prandtl, L., "Zur Berechnung Der Grenzschichten," *ZAMM*, No. 18, 1938, pp. 77-82.
- ²⁸Schlichting, H., "Lecture Series: Boundary Layer Theory Part 1 - Laminar Flow," NACA, 1949.
- ²⁹Tetervin, N., "Boundary-Layer Momentum Equations for Three-Dimensional Flow," edited by N. A. C. f. Aeronautics NACA, 17 Nov, 1947, Vol. Technical Note, p. 27.
- ³⁰Conlisk, A. T., "Source-Sink Flows in a Rapidly Rotating Annulus," Dissertation, Purdue, 1978.
- ³¹Van Dyke, M., *Perturbation Methods in Fluid Mechanics*, The Parabolic Press, Stanford, CA, 1975.
- ³²Wolfram, S., *Mathematica. A System for Doing Mathematics on Computer*. Addison Wesley, Reading, MA, 1988.
- ³³Chang, F., and Dhir, V. K., "Mechanisms of Heat Transfer Enhancement and Slow Decay of Swirl in Tubes with Tangential Injection," *International Journal of Heat and Fluid Flow*, Vol. 16, No. 2, 1995, pp. 78-87.

Vita

Joshua Will Batterson was born on April 11th, 1983 to Richard and Mary Batterson of Huntington, Indiana. After finishing at Huntington North high school in 2001, he began his undergraduate studies at Tri-State University in mechanical engineering. In 2005 he completed his tenure at Tri-State and began pursuing his graduate education at the University of Tennessee Space Institute. Josh Batterson received his Master of Science in Aerospace Engineering from the University of Tennessee, Knoxville in November of 2007. His focus was on propulsion with an emphasis on vortex technology in liquid rocket engines.

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