To the Graduate Council:

I am submitting herewith a dissertation written by Alfred H. Cripps entitled "Green's Theories and Coextensions." I recommend that it be accepted in partial fulfillment of the requirements for the degree of Doctor of Philosophy, with a major in Mathematics.

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# GREEN'S THEORIES AND COEXTENSIONS

A Dissertation

Presented for the

Doctor of Philosophy

Degree

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## DEDICATION

This dissertation is dedicated to the author's mother, Rosa, and to his wife, Shirley.

The author also wishes to express his appreciation and gratitude to Professor J. H. Carruth for his guidance and persistence.

ABSTRACT

In Chapter II, the categories of Green's pairs and Green's theories are defined. The general properties of these categories are investigated, and the two categories are shown to be equivalent. Also the category of Green's pairs are related to the usual Green's quasi-orders while relationships between Green's pairs and other generalizations of Green's relations are developed. This is done by performing a construction using a free semigroup to obtain a coextension with the desired properties. An example is given to show that Green's pairs are more general than the other generalizations studied. Also specific types of Green's pairs and their properties are investigated.

In Chapter III, some of the results of Chapter II are applied in the study of quasi-orders under $\underline{\leq}(H)$ . For each $L$ [$R$] quasi-order under $\underline{\leq}(H)$ , it is shown that there corresponds a functor from $\underline{\leq}L$ [$\underline{\leq}R$] into $\underline{\text{Mon}}$ while for each $L$ [$R$] equivalence under $\underline{\leq}(H)$ , it is shown that there corresponds a functor from $\underline{\leq}L$ [$\underline{\leq}R$] into $\underline{\text{Grp}}$ . Also the $R$ equivalences under $\underline{\leq}(H)$ (viewed as a lattice) are shown to be isomorphic to a lattice contained in the functors from $\underline{\leq}R$ to $\underline{\text{Grp}}$ .

Also in Chapter III, $\underline{\leq}(G)$ and $\underline{\leq}(S)$ coextensions are defined for the category $\overline{\text{GP}}$ . Then the results in the first part of the chapter are used to establish a construction principle using functors from $\underline{\leq}L$ and $\underline{\leq}R$ into $\underline{\text{Mon}}$ . Using this construction principle one is able to construct (reconstruct) all $\underline{\leq}(G)$ and $\underline{\leq}(S)$ coextensions (in the category $\overline{\text{GP}}$ ).

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# CHAPTER I

## PRELIMINARIES

It is assumed that the reader is familiar with the work of Clark and Carruth [3] along with some general knowledge of semigroups. The reader is referred to [3] and [4] for the standard notation and standard results.

The following order theoretic concepts will be needed. Let $X$ denote a set. Then a relation on $X$ is a subset of the Cartesian product $X \times X$. If $\sigma$ is a relation on $X$, then $(x,y) \in \sigma$ is also written as $x\sigma y$. Also $\sigma^{-1}$ denotes the set $\{(y,x):(x,y) \in \sigma\}$. A relation $\sigma$ on $X$ is said to be reflexive if $\{(x,x):x \in X\}$ is contained in $\sigma$. The relation $\sigma$ is said to be symmetric if $(y,x) \in \sigma$ whenever $(x,y) \in \sigma$. Also $\sigma$ is transitive if $(x,z) \in \sigma$ whenever $(x,y) \in \sigma$ and $(y,z) \in \sigma$. An equivalence relation is a relation which is reflexive, symmetric, and transitive. Now if $\sigma$ is an equivalence relation on $X$, then for $x \in X$ $x\sigma$ denotes the set $\{y \in X:(x,y) \in \sigma\}$ and $X/\sigma$ denotes the set $\{x\sigma:x \in X\}$. The map $x \to x\sigma$ from $X$ to $X/\sigma$ is called the natural or projection map of $X$ onto $X/\sigma$.

A quasi-order is a relation which is reflexive and transitive. If $\leq(C)$ denotes a quasi-order on $X$, then the equivalence relation induced by $\leq(C)$ is the relation

$$C = \leq(C) \cap (\leq(C))^{-1} .$$

For $a \in X$, the C-set of "a" is denoted and defined by

$$C(a) = \{x \in X : x \leq (C)a\}$$

and the C-class of "a" is denoted and defined by

$$C_a = \{x \in X : xCa\} .$$

A transformation of a set $X$ is a single-valued mapping of $X$ into $X$. A partial transformation of $X$ is a single-valued mapping of a subset of $X$ into $X$. The domain of a partial transformation, say $f$, is $\{x \in X : (x,y) \in f \text{ for some } y \in X\}$ and is denoted by dom f. The empty transformation is the partial transformation with empty domain. The empty transformation is denoted by "$\square$". The identity transformation on $X$ is a transformation, say $f$, such that $f(x) = x$ for every $x$ in $X$. The identity transformation on $X$ is denoted by $1_X$. For a fixed set $X$, the set of all transformations of $X$ is denoted by $T(X)$ while the set of all partial transformations of $X$ is denoted by $PT(X)$.

A semigroup is a pair $(S,*)$ where $S$ is a set and $*$ is a function from $S \times S$ into $S$ which satisfies the following property: $(x*y)*z = x*(y*z)$ for each $x,y,z \in S$ where $*(u,v)$ is denoted by $u*v$ for $u,v \in S$. Usually the multiplication for a semigroup $(S,*)$ is denoted by juxtaposition, i.e., $x*y = xy$ for $x,y \in S$. Also, in most instances the pair $(S,*)$ will be referred to only by $S$. An identity of a semigroup $S$ is an element, say $e$, of $S$ such that $ex = x = xe$ for all $x$ in $S$. Clearly any identity for $S$ is unique. If a semigroup has an identity, then it will be denoted by "1". An idempotent is an element whose square is itself. Also if

$S$ is a semigroup, then $S^1$ will denote $S$ if $S$ has an identity, otherwise $S^1$ will denote $S$ with an identity adjoined. A monoid is a semigroup with identity while a group, say $G$, is a monoid in which for each element "a" in $G$ there is an element, say $a^{-1}$, in $G$ such that $aa^{-1} = 1 = a^{-1}a$. A semilattice is a commutative semigroup which consists only of idempotents. A subset $T$ of a semigroup $S$ is said to be a subsemigroup of $S$ if $\{st:(s,t) \in T\times T\}$ is a subset of $T$. If, in addition, $T$ is a group then $T$ is said to be a subgroup of $S$. If $A$ is a subset of $S$, then the subsemigroup generated by $A$ is denoted and defined by

$$\langle A\rangle = \cap\{T \subseteq S:T \text{ is a subsemigroup of } S \text{ and } A \subseteq T\}.$$

The semigroup $\langle A\rangle$ can also be found by taking the set of all elements of $S$ that are expressible as finite products of elements of $A$. If $X$ is a set, then the free semigroup generated by $X$, denoted by $F(X)$, is the set of all finite sequences of elements of $X$ where the multiplication is defined by the following: if $(x_1, x_2, \ldots, x_m)$ and $(y_1, y_2, \ldots, y_n)$ are elements of $F(X)$ then their product is $(x_1, x_2, \ldots, x_m, y_1, y_2, \ldots, y_n)$. The elements of $F(X)$ will be referred to as words. Also the notation $(x_1, x_2, \ldots, x_m)$ for a word in $F(X)$ will be shortened to $x_1 \cdot x_2 \ldots x_m$. Hence "$\cdot$" will denote the multiplication in $F(X)$. A left [right] ideal of a semigroup $S$ is a subset $I$ of $S$ such that $SI \subseteq I$ [$IS \subseteq I$]. An ideal of $S$ is a subset of $S$ which is both a left and a right ideal of $S$. Note that the definitions of left ideal, right ideal, and ideal are not the standard ones. They differ in that the subset is not

required to be non-empty. The reason for this is a notational convenience. The domain of the empty transformation is now an ideal. The specific use will be pointed out in Chapter II.

For each element "a" of a semigroup S one associates a transformation $\rho_a$ [$\lambda_a$] of S defined by $x\rho_a = xa$ [$x\lambda_a = ax$] for all x in S. The map $\lambda_a$ [$\rho_a$] is called the inner left [right] translation of S induced by the element a. A transformation $\rho$ of S is called a right translation of S if $x(y\rho) = (xy)\rho$ for all $x,y \in S$. A transformation $\lambda$ of S is called a left translation of S if $(x\lambda)y = (xy)\lambda$ for all $x,y \in S$. Note that an inner right [left] translation is also a right [left] translation. From this point on whenever a map is known to be a left translation, the argument will be written on the right side of the map while the argument of a right translation will be written on the left side of the map. The set of all left translations on the semigroup S will be denoted by $LT(S)$ while the set of all inner left translations on the semigroup S will be denoted by $ILT(S)$. The set of all right translations on the semigroup S will be denoted by $RT(S)$ while the set of all inner right translations on S will be denoted by $IRT(S)$.

A partial right translation of S, say $\alpha$, is a partial transformation of S such that if r and s are in S with $rs, s \in \text{dom } \alpha$, then $(rs)\alpha = r(s\alpha)$. A partial left translation of S, say $\gamma$, is a partial transformation of S such that if r and s are in S with $sr, s \in \text{dom } \gamma$, then $\gamma(sr) = (\gamma s)r$. The set of all partial right translations of S will be denoted by $PRT(S)$ while the set of all partial left translations of S will be denoted by $PLT(S)$.

Note that the identity translation and the empty translation are in both $PRT(S)$ and $PLT(S)$. If $X$ is a set, then note that $PT(X)$ is a monoid under composition and that $T(X)$ is a submonoid of $PT(X)$. Also note that if $S$ is a semigroup, then $LT(S)$, $ILT(S)$, $RT(S)$, and $IRT(S)$ [$PLT(S)$ and $PRT(S)$] are subsemigroups of $T(S)$ [$PT(S)$].

The following definitions are given in [6]. For a semigroup $S$, the usual Green's quasi-orders on $S$ are defined by

$$\leq(L_S) = \{(y,x) \in S\times S : S^1y \subseteq S^1x\}$$
$$\leq(R_S) = \{(y,x) \in S\times S : yS^1 \subseteq xS^1\}$$
$$\leq(H_S) = \leq(L_S) \cap \leq(R_S) .$$

Whenever no confusion is likely in regard to which semigroup is being referred to, the notation $\leq(L_0)$, $\leq(R_0)$, and $\leq(H_0)$ will be used instead of $\leq(L_S)$, $\leq(R_S)$, and $\leq(H_S)$, respectively. If $S$ is a monoid, then the group of units of $S$ is the $H_0$-class of the identity, $H_1$.

The following definitions are taken from [3]. Let $X$ denote a fixed set and let $S$ denote a fixed semigroup. A Green's pair on $X$ is a pair $(\Lambda,P)$ satisfying the following four properties:

(i) $\Lambda$ and $P$ are subsemigroups of $PT(X)$ with arguments written on the right for members of $\Lambda$ and on the left for members of $P$;

(ii) Both $\Lambda$ and $P$ contain the identity transformation of $X$;

(iii) For all $(\lambda,\rho) \in \Lambda\times P$ , $\lambda[\mathrm{dom}\ \lambda \cap \mathrm{dom}\ \rho] \subseteq \mathrm{dom}\ \rho$ and $[\mathrm{dom}\ \lambda \cap \mathrm{dom}\ \rho]\rho \subseteq \mathrm{dom}\ \lambda$ ; and

(iv) For all $(\lambda,\rho) \in \Lambda\times P$ and for all $x \in [\mathrm{dom}\ \lambda \cap \mathrm{dom}\ \rho]$ , $(\lambda x)\rho = \lambda(x\rho)$ .

A Green's pair on $S$ is a Green's pair $(\Lambda,P)$ on the set $X = S$ satisfying the additional conditions that (v) $\Lambda \subseteq PLT(S)$ and $P \subseteq PRT(S)$ . The usual Green's pair $(\Lambda_0,P_0)$ on $S$ is defined by

$$\Lambda_0 = \{\lambda:\lambda \in ILT(S)\} \cup \{1_S\}$$
$$P_0 = \{\rho:\rho \in IRT(S)\} \cup \{1_S\} .$$

For each Green's pair $(\Lambda,P)$ on $X$ the following quasi-orders $\leq(L)$ , $\leq(R)$ , and $\leq(H)$ on $X$ are defined:

(i) $y\leq(L)x$ if $y \in \Lambda x = \{\lambda x:\lambda \in \Lambda$ and $x \in \mathrm{dom}\ \lambda\}$

(ii) $y\leq(R)x$ if $y \in xP = \{x\rho:\rho \in P$ and $x \in \mathrm{dom}\ \rho\}$

(iii) $\leq(H) = \leq(L) \cap \leq(R)$ .

A Green's theory on $X$ is a quadruple $(\leq(L),\leq(R),G,D)$ where $\leq(L)$ and $\leq(R)$ are quasi-orders on $X$ and $G:\leq(L) \to PT(X)$ and $D:\leq(R) \to PT(X)$ are functions satisfying the following four properties:

(i) for $y\leq(L)x$ , the image of $(y,x)$ under $G$ is denoted by $G_y^x$ and satisfies $\mathrm{dom}\ G_y^x = R(x)$ . Dually, for $y\leq(R)x$ , the image of $(y,x)$ under $D$ is denoted by $D_y^x$ and satisfies $\mathrm{dom}\ D_y^x = L(x)$ . Arguments for $G_y^x$ will be

written on the right and arguments for $D_y^x$ will be written on the left.

(ii) Given $x \in X$, $G_x^x z = z$ for all $z \in R(x)$ and $zD_x^x = z$ for all $z \in L(x)$.

(iii) For $z \leq (L) y$ and $y \leq (L) x$, $G_z^y \circ G_y^x = G_z^x$ and for $z \leq (R) y$ and $y \leq (R) x$, $D_y^x \circ D_z^y = D_z^x$.

(iv) For $y \leq (L) x$ and $z \leq (R) x$, $G_y^x z = yD_z^x$.

A Green's theory on $S$ is a Green's theory $(\leq (L), \leq (R), G, D)$ on the set $X = S$ satisfying the additional conditions that for each $y \leq (L) x$, $G_y^x \in PLT(S)$ and for each $y \leq (R) x$, $D_y^x \in PRT(S)$.

Now as pointed out in [3], each Green's pair $(\Lambda, P)$ on $X$ [the semigroup $S$] induces a Green's theory denoted by $A(\Lambda, P)$ or $(\leq (L), \leq (R), G, D)$ on $X$ [$S$] where

(i) $(\leq (L), \leq (R))$ are the quasiorders induced by $(\Lambda, P)$,

(ii) For $y \leq (L) x$ and $z \in R(x)$, $G_y^x z = \lambda z$ where $\lambda$ is any member of $\Lambda$ such that $y = \lambda x$, and

(iii) For $y \leq (R) x$ and $z \in L(x)$, $zD_y^x = z\rho$ where $\rho$ is any member of $P$ such that $y = x\rho$.

Also each Green's theory $(\leq (L), \leq (R), G, D)$ on $X$ induces a Green's pair on $X$ denoted by $B(\leq (L), \leq (R), G, D)$ or $(\Lambda, P)$ where

(i) $\Lambda$ is the subsemigroup of $PT(X)$ generated by $\{G_y^x : y \leq (L) x\} \cup \{1_X\}$ and

(ii) $P$ is the subsemigroup of $PT(X)$ generated by

$$\{D_y^x : y \leq (R) x\} \cup \{1_X\} .$$

Whenever $(\leq(L), \leq(R), G, D)$ is a Green's theory on $X$, this will be denoted by $(X, \leq(L), \leq(R), G, D)$. Also if $(\Lambda, P)$ is a Green's pair on $X$, then this will be denoted by $(X, \Lambda, P)$. Now if $(X, \leq(L), \leq(R), G, D)$ is a Green's theory, then the <u>right Schützenberger monoid</u> of $x$ for $x$ in $X$ is denoted and defined by

$$S_D(x) = \{D_y^x : y \leq (H) x\} .$$

The <u>right Schützenberger group</u> of $x$ for $x$ in $X$ is denoted and defined by

$$G_D(x) = \{D_y^x : yHx\} .$$

Now as shown in [3], for $xLy$, $S_D(y) = S_D(x)$ and $G_D(y) = G_D(x)$; $S_D(x)$ is a monoid and $G_D(x)$ is the group of units of $S_D(x)$. Dually, one has $S_G(x)$ and $G_G(x)$ for each $x \in X$.

A <u>category</u> is a quintuple $C = (O, M, \mathcal{D}, K, \circ)$ where

(i) $O$ is a class whose members are called $C$-objects,

(ii) $M$ is a class whose members are called $C$-morphisms,

(iii) $\mathcal{D}$ and $K$ are functions from $M$ to $O$,

(iv) $\circ$ is a function from $D = \{(f,g) : f, g \in M \text{ and } \mathcal{D}(f) = K(g)\}$; such that the following conditions are satisfied:

(a) If $f \circ g$ is defined, then $\mathcal{D}(f \circ g) = \mathcal{D}(g)$ and $K(f \circ g) = K(f)$;

(b) If $f \circ g$ and $h \circ f$ are defined, then $h \circ (f \circ g) = (h \circ f) \circ g$ ;

(c) For each $C$-object $A$ there exists a $C$-morphism, denote by $e$, such that $\mathcal{D}(e) = A = K(e)$ and

(i) $f \circ e = f$ whenever $(f,e) \in D$

(ii) $e \circ g = g$ whenever $(e,g) \in D$ ;

(d) For any pair $(A,B)$ of $C$-objects, the class $\hom_C(A,B) = \{f : f \in M,\ \mathcal{D}(f) = A \text{ and } K(f) = B\}$ is a set.

If the class of $C$-objects is a set, the $C$ is called a <u>small category</u>. A category $\mathcal{B} = (\mathcal{O}_1, M_1, \mathcal{D}_1, K_1, *)$ is a <u>subcategory</u> of $C$ if

(i) $\mathcal{O}_1 \subseteq \mathcal{O}$ ,

(ii) for $\mathcal{B}$-objects $A$ and $B$, $\hom_{\mathcal{B}}(A,B) \subseteq \hom_C(A,B)$ , and

(iii) for $\mathcal{B}$-objects $A$, $B$, and $C$ with $f \in \hom_{\mathcal{B}}(A,B)$ and $g \in \hom_{\mathcal{B}}(B,C)$ one has $g \circ f = g * f$ .

If $\mathcal{B}$ is a subcategory of $C$ and $\hom_{\mathcal{B}}(A,B) = \hom_C(A,B)$ for each $A,B \in \mathcal{O}_1$ , then $\mathcal{B}$ is a <u>full subcategory</u> of $C$ .

The following categories will be needed in the following chapters:

1) the category <u>Sgp</u> of (algebraic) semigroups and homomorphisms;

2) the category <u>Mon</u> of (algebraic) monoids and identity preserving homomorphisms;

3) the category Grp of algebraic groups and homomorphisms.

Other less familiar categories will be introduced later.

If $C$ denotes the category $(\mathcal{O},\mathcal{M},\mathcal{D},K,\circ)$ and $\mathcal{B}$ denotes the category $(\mathcal{O}_1,\mathcal{M}_1,\mathcal{D}_1,K_1,*)$, then a functor $F$ from $C$ to $\mathcal{B}$ is a function from the class of morphisms of $C$ to the class of morphisms of $\mathcal{B}$ satisfying the following conditions

(i) if $e$ is a $C$-identity, then $F(e)$ is a $\mathcal{B}$-identity and

(ii) $F(f\circ g) = F(f)*F(g)$ whenever $(f,g) \in D$.

The domain of the functor $F$ is $C$ and the codomain of $F$ is $\mathcal{B}$.

Let $F$ and $G$ be functors from $C$ to $\mathcal{B}$. A natural transformation $\Phi$ from $F$ to $G$, denoted by $\Phi:F \to G$, is a function from $\mathcal{O}$ into $\{\hom_{\mathcal{B}}(A,B):A \text{ and } B \text{ are } \mathcal{B}\text{-objects}\}$ such that for each $C$-object $X$, $\Phi(X)$ is in $\hom_{\mathcal{B}}(F(X),G(X))$ and if $X$ and $Y$ are $C$-objects with $f:X \to Y$, then

$$\begin{array}{ccc} F(X) & \xrightarrow{\Phi(X)} & G(X) \\ {\scriptstyle F(f)}\downarrow & \circlearrowright & \downarrow{\scriptstyle G(f)} \\ F(Y) & \xrightarrow{\Phi(Y)} & G(Y) \end{array} .$$

A natural transformation $\Phi:F \to G$ is said to be a natural equivalence if there is a natural transformation $\delta:G \to F$ such that for each $C$-object $A$ one has $\Phi(A)*\delta(A) = 1_{G(A)}$ and $\delta(A)*\Phi(A) = 1_{F(A)}$. A functor $F:C \to \mathcal{B}$ is an equivalence if there is a functor $G:\mathcal{B} \to C$ such that $F\circ G$ is naturally equivalent to the identity functor on $\mathcal{B}$

and $G \circ F$ is naturally equivalent to the identity functor on $C$. If there is an equivalence from $C$ to $B$, then the two categories $C$ and $B$ are said to be <u>equivalent</u>.

## CHAPTER II

## THE CATEGORIES OF GREEN'S PAIRS AND GREEN'S THEORIES

In this chapter the categories of Green's pairs and Green's theories are defined. Certain properties are obtained that will be needed in the next chapter. Also some general properties relating Green's pairs on a semigroup to the usual Green's relations on a semigroup are developed.

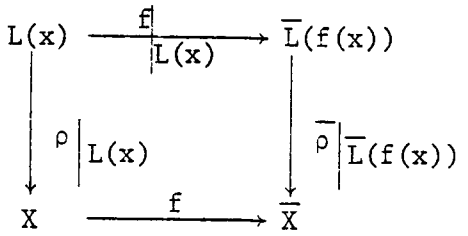
Definition 2.1. Let GPX denote the category consisting of all triples $(X,\Lambda,P)$ where $X$ is a set and $(\Lambda,P)$ is a Green's pair on $X$ and a morphism $F$ from $(X,\Lambda,P)$ to $(\overline{X},\overline{\Lambda},\overline{P})$ is a triple $((X,\Lambda,P), (X,f,\overline{X}), (\overline{X},\overline{\Lambda},\overline{P}))$ where $f:X \to \overline{X}$ is a function from $X$ to $\overline{X}$ satisfying the following two conditions:

(i) for every $\lambda \in \Lambda$ and for every $x \in \text{dom } \lambda$ there exists $\overline{\lambda} \in \overline{\Lambda}$ (depending on both $\lambda$ and $x$) such that the following diagram commutes

$$\begin{array}{ccc} R(x) & \xrightarrow{f\,|\,R(x)} & \overline{R}(f(x)) \\ \downarrow{\scriptstyle \lambda\,|\,R(x)} & & \downarrow{\scriptstyle \overline{\lambda}\,|\,\overline{R}(f(x))} \\ X & \xrightarrow{f} & \overline{X} \end{array}$$

i.e. $\overline{\lambda}f(z) = f(\lambda z)$ for all $z \in R(x)$,

(ii) for every $\rho \in P$ and for every $x \in \text{dom } \rho$ there exists $\overline{\rho} \in \overline{P}$ such that the following diagram commutes




i.e. $f(z\rho) = (f(z))\bar{\rho}$ for all $z \in L(x)$ .

Now if $H,F \in \mathrm{Mor}(GPX)$ , say $H = ((X,\Lambda,P), (X,h,\bar{X}), (\bar{X},\bar{\Lambda},\bar{P}))$ and $F = ((\bar{X},\bar{\Lambda},\bar{P}), (\bar{X},f,\tilde{X}), (\tilde{X},\tilde{\Lambda},\tilde{P}))$ , then $F\circ H \equiv ((X,\Lambda,P), (X,f\circ h,\tilde{X}), (\tilde{X},\tilde{\Lambda},\tilde{P}))$ . (The arguments will be written on the right.)

Now since the precise notation for a morphism is cumbersome, a less precise notation will be adopted. If $F = ((X,\Lambda,P), (X,f,\bar{X}), (\bar{X},\bar{\Lambda},\bar{P})) \in \mathrm{Mor}(GPX)$ , then this will be stated by "$f \in \mathrm{Mor}(GPX)$" or by "$f:(X,\Lambda,P) \to (\bar{X},\bar{\Lambda},\bar{P})$ is in $\mathrm{Mor}(GPX)$ ." Also if $(\Lambda,P)$ and $(\Lambda_1,P_1)$ are Green's pairs for the set $X$ with $((X,\Lambda,P), (X,f,\bar{X}), (\bar{X},\bar{\Lambda},\bar{P}))$ and $((X,\Lambda_1,P_1), (X,f,\bar{X}), (\bar{X},\bar{\Lambda},\bar{P}))$ in $\mathrm{Mor}(GPX)$ , then (with an abuse of notation) "$f$" will stand for both morphisms $f:(X,\Lambda,P) \to (\bar{X},\bar{\Lambda},\bar{P})$ and $f:(X,\Lambda_1,P_1) \to (\bar{X},\bar{\Lambda},\bar{P})$ .

Clearly with the above definition, GPX forms a category.

Definition 2.2. Let GP denote the subcategory of GPX consisting of all $(S,\Lambda,P) \in \mathrm{obj}(GPX)$ such that $S$ is a semigroup, $(\Lambda,P)$ is a Green's pair for the semigroup $S$ , and $f \in \mathrm{Mor}(GP)$ iff $f \in \mathrm{Mor}(GPX)$ and $f \in \mathrm{Mor}(\underline{Sgp})$.

Definition 2.3. Let GTX denote the category of all 5-tuples $(X,\leq(L),\leq(R),G,D)$ where $X$ is a set and $(\leq(L),\leq(R),G,D)$ is a Green's

theory on $X$ , and if $(X,\leq(L),\leq(R),G,D)$ and $(\overline{X},\leq(\overline{L}),\leq(\overline{R}),\overline{G},\overline{D})$ are in obj(GTX) , then a morphism $F$ from $(X,\leq(L),\leq(R),G,D)$ to $(\overline{X},\leq(\overline{L}),\leq(\overline{R}),\overline{G},\overline{D})$ is a triple $((X,\leq(L),\leq(R),G,D), (X,f,\overline{X}), (\overline{X},\leq(\overline{L}),\leq(\overline{R}),\overline{G},\overline{D}))$ where $f:X \to \overline{X}$ is a function from $X$ to $\overline{X}$ such that the following four conditions are satisfied:

(i) $f\times f(\leq(L)) \subseteq \leq(\overline{L})$ ;

(ii) $f\times f(\leq(R)) \subseteq \leq(\overline{R})$ ;

(iii) $f(G^x_y z) = \overline{G}^{f(x)}_{f(y)} f(z)$ for all $(y,x) \in \leq(L)$ and for all $z \in R(x)$ ;

(iv) $f(zD^x_y) = f(z)\overline{D}^{f(x)}_{f(y)}$ for all $(y,x) \in \leq(R)$ and for all $z \in L(x)$ .

Now if $H = ((X,\leq(L),\leq(R),G,D), (X,h,\overline{X}), (\overline{X},\leq(\overline{L}),\leq(\overline{R}),\overline{G},\overline{D}))$ and $F = ((\overline{X},\leq(\overline{L}),\leq(\overline{R}),\overline{G},\overline{D}), (\overline{X},f,\tilde{X}), (\tilde{X},\leq(\tilde{L}),\leq(\tilde{R}),\tilde{G},\tilde{D}))$ are in Mor(GTX) , then $F\circ H = ((X,\leq(L),\leq(R),G,D), (X,f\circ h,\tilde{X}), (\tilde{X},\leq(\tilde{L}),\leq(\tilde{R}),\tilde{G},\tilde{D}))$ . (The arguments will be written on the right.)

Now as in the categories GPX and GP a less precise notation will be adopted by referring to $F = ((X,\leq(L),\leq(R),G,D), (X,f,\overline{X}), (\overline{X},\leq(\overline{L}),\leq(\overline{R}),\overline{G},\overline{D}))$ in Mor(GTX) as just "$f \in$ Mor(GTX)" or "$f:(X,\leq(L),\leq(R),G,D) \to (\overline{X},\leq(\overline{L}),\leq(\overline{R}),\overline{G},\overline{D})$ is in Mor(GTX) ."

Definition 2.4. Let GT denote the subcategory of GTX consisting of all $(S,\leq(L),\leq(R),G,D) \in \text{obj}(GTX)$ such that $S$ is a semigroup, $(\leq(L),\leq(R),G,D)$ is a Green's theory on the semigroup $S$, and $f \in \text{Mor}(GT)$ iff $f \in \text{Mor}(GTX)$ and $f \in \text{Mor}(\underline{Sgp})$.

The above definitions define the categories GP and GT. Several "natural" definitions for the morphisms in the category GP exist. Each has its advantage over the others, but the reason for choosing the above for the definition of a morphism in GP is that the categories GP and GT are equivalent (to be shown later) and the definition is a "middle of the road" definition in terms of restrictions. Also morphisms in GP preserve the semigroup structure and the translation of $L$ and $R$ sets. Hence the $L$ and $R$ quasi-orders are preserved by morphisms in GP.

In the following, the categories GP and GT are shown to be equivalent. That the categories GPX and GTX are equivalent will also follow from the proofs.

Theorem 2.5. If $A:GP \to GT$ is defined by taking $A(S,\Lambda,P)$ as defined in Chapter I for $(S,\Lambda,P) \in \text{obj}(GP)$ and $A(f) = f$ for $f \in \text{Mor}(GP)$, then $A$ is a functor. (Note the abuse of notation; if $f:(S,\Lambda,P) \to (\overline{S},\overline{\Lambda},\overline{P})$, then
$A(f)$ "=" $f:(S,\leq(L),\leq(R),G,D) \to (\overline{S},\leq(\overline{L}),\leq(\overline{R}),\overline{G},\overline{D})$.)

PROOF. By Theorem 5.1 of [3] $A(S,\Lambda,P) \in \text{obj}(GT)$ whenever $(S,\Lambda,P) \in \text{obj}(GP)$. Now let $f:(S,\Lambda,P) \to (\overline{S},\overline{\Lambda},\overline{P})$ be a morphism in GP. It must be shown that $f:(S,\leq(L),\leq(R),G,D) \to (\overline{S},\leq(\overline{L}),\leq(\overline{R}),\overline{G},\overline{D})$ is a morphism in GT. Since $f \in \text{Mor}(GP)$, $f:S \to \overline{S}$ is a homomorphism.

Now parts (i) and (iii) of Definition 2.3 will be shown. It will follow dually that parts (ii) and (iv) of Definition 2.3 also hold. Let $(y,x) \in \leq(L)$. Then there exists $\lambda \in \Lambda$ such that $\lambda x = y$ (by the definition of $\leq(L)$). Now since $f \in Mor(GP)$ there exists $\overline{\lambda} \in \overline{\Lambda}$ such that $\overline{\lambda}f(z) = f(\lambda z)$ for all $z \in R(x)$. In particular $\overline{\lambda}f(x) = f(\lambda x) = f(y)$ and it follows from the definition of $\leq(\overline{L})$ that $(f(y),f(x)) \in \leq(\overline{L})$. To show part (iii), let $(y,x) \in \leq(L)$ and $z \in R(x)$. As in the above this gives a $\lambda \in \Lambda$ and a $\overline{\lambda} \in \overline{\Lambda}$ such that $\lambda x = y$ and $\overline{\lambda}f(w) = f(\lambda w)$ for all $w \in R(x)$. Thus $\overline{\lambda}f(z) = f(\lambda z)$. Since $\lambda x = y$, $G_y^x = \lambda|_{R(x)}$ (see Theorem 5.1 of [3]). Also $\overline{\lambda}f(x) = f(y)$ given $\overline{G}_{f(y)}^{f(x)} = \overline{\lambda}|_{\overline{R}(f(x))}$. Therefore $f(G_y^x z) = f(\lambda z) = \overline{\lambda}f(z) = \overline{G}_{f(y)}^{f(x)} f(z)$ and part (iii) of Definition 2.3 follows. Hence $f \in Mor(GT)$.

Clearly since $A(f) = f$, $A$ preserves identities and composition of morphisms. Thus $A$ is a functor.

Lemma 2.6. Let $(S,\Lambda,P),(\overline{S},\overline{\Lambda},\overline{P}) \in obj(GP)$ and let $f:S \to \overline{S}$ be a homomorphism. If

$$\Lambda' = \{\lambda \in \Lambda : \text{if } x \in \text{dom } \lambda, \text{ then there exists } \overline{\lambda} \in \overline{\Lambda} \text{ such that } \overline{\lambda}f(z) = f(\lambda z) \text{ for all } z \in R(x)\}$$

and

$$P' = \{\rho \in P : \text{if } x \in \text{dom } \rho, \text{ then there exists } \overline{\rho} \in \overline{P} \text{ such that } f(z)\overline{\rho} = f(z\rho) \text{ for all } z \in L(x)\},$$

then $(S,\Lambda',P') \in obj(GP)$ and $f:(S,\Lambda',P') \to (\overline{S},\overline{\Lambda},\overline{P})$ is in $Mor(GP)$.

PROOF. To show $(S,\Lambda',P') \in \text{obj}(GP)$, first note that $1_S \in \Lambda' \cap P'$. Now since $(S,\Lambda,P) \in \text{obj}(GP)$ and $(\Lambda',P') \subseteq (\Lambda,P)$, conditions (iii), (iv), and (v) in the definition of a Green's pair on $S$ are satisfied by $(\Lambda',P')$. Now $\Lambda'$ will be shown to be closed under composition. That $P'$ is closed under composition will follow dually. Let $\lambda_1,\lambda_2 \in \Lambda'$ and let $x \in \text{dom } \lambda_2\circ\lambda_1$. Now by the definition of $\Lambda'$, there exist $\overline{\lambda}_1,\overline{\lambda}_2 \in \overline{\Lambda}$ such that $f(\lambda_1 z) = \overline{\lambda}_1 f(z)$ for all $z \in R(x)$ and $f(\lambda_2 z) = \overline{\lambda}_2 f(z)$ for all $z \in R(\lambda_1 x)$. Therefore $f((\lambda_2\circ\lambda_1)z) = f(\lambda_2(\lambda_1 z)) = \overline{\lambda}_2 f(\lambda_1 z) = \overline{\lambda}_2\overline{\lambda}_1 f(z) = (\overline{\lambda}_2\circ\overline{\lambda}_1)f(z)$ for all $z \in R(x)$. Now since $\overline{\lambda}_2\circ\overline{\lambda}_1 \in \overline{\Lambda}$, it follows from the definition of $\Lambda'$ that $\lambda_2\circ\lambda_1 \in \Lambda'$. Thus $\Lambda'$ is closed under composition. Hence $(S,\Lambda',P') \in \text{obj}(GP)$.

Clearly it follows from the choice of $(\Lambda',P')$ that $f:(S,\Lambda',P') \to (\overline{S},\overline{\Lambda},\overline{P})$ is in $\text{Mor}(GP)$.

Theorem 2.7. If $\mathcal{B}:GT \to GP$ is defined by taking $\mathcal{B}(S,\leq(L),\leq(R),G,D)$ as defined in Chapter I for $(S,\leq(L),\leq(R),G,D) \in \text{obj}(GT)$ and $\mathcal{B}(f) = f$ for $f \in \text{Mor}(GT)$, then $\mathcal{B}$ is a functor. (Note the abuse of notation in saying $\mathcal{B}(f)$ "=" $f$.)

PROOF. By Theorem 5.7 of [3] $\mathcal{B}(S,\leq(L),\leq(R),G,D)$ denoted by $(S,\Lambda,P)$ is a Green's pair whenever $(S,\leq(L),\leq(R),G,D) \in \text{obj}(GT)$. Now let $f:(S,\leq(L),\leq(R),G,D) \to (\overline{S},\leq(\overline{L}),\leq(\overline{R}),\overline{G},\overline{D})$ be a morphism in $GT$. It must be shown that $f:(S,\Lambda,P) \to (\overline{S},\overline{\Lambda},\overline{P})$ is a morphism in $GP$. Now $f \in \text{Mor}(GT)$ gives $f:S \to \overline{S}$ is a homomorphism. Thus apply Lemma 2.6 to obtain a Green's pair $(\Lambda',P')$ on $S$ such that $f:(S,\Lambda',P') \to (\overline{S},\overline{\Lambda},\overline{P})$. Now $f \in \text{Mor}(GT)$ also gives

$f(G^x_y z) = \overline{G}^{f(x)}_{f(y)} f(z)$ for all $(y,x) \in \leq(L)$ and $z \in R(x)$. Therefore by the definition of $\Lambda'$, $G^x_y \in \Lambda'$ for all $(y,x) \in \leq(L)$. Now $\Lambda$ is generated by the set $M = \{G^x_y : (y,x) \in \leq(L)\} \cup \{1_S\}$. Hence $\Lambda \subseteq \Lambda'$. But by the definition of $\Lambda'$, $\Lambda' \subseteq \Lambda$. Hence $\Lambda' = \Lambda$. Dually $P' = P$. Therefore $(S,\Lambda',P') = (S,\Lambda,P)$ and $f:(S,\Lambda,P) \to (\overline{S},\overline{\Lambda},\overline{P})$ is in $\mathrm{Mor}(GP)$.

Clearly since $\mathcal{B}(f) = f$, $\mathcal{B}$ preserves identities and composition of morphisms. Hence $\mathcal{B}:GT \to GP$ is a functor.

Theorem 2.8. The categories $GP$ and $GT$ are equivalent.

PROOF. It will be shown that $A \circ \mathcal{B} = 1_{GT}$ and $\mathcal{B} \circ A \cong 1_{GP}$. According to Theorem 5.8 of [3] $A \circ \mathcal{B}(S,\leq(L),\leq(R),G,D) = (S,\leq(L),\leq(R),G,D)$ for all $(S,\leq(L),\leq(R),G,D) \in \mathrm{obj}(GT)$. Therefore since $A \circ \mathcal{B}(f) = f$ for all $f \in \mathrm{Mor}(GT)$, $A \circ \mathcal{B} = 1_{GT}$.

Now to show $\mathcal{B} \circ A \cong 1_{GP}$, let $(S,\Lambda,P) \in \mathrm{obj}(GP)$ and let $(S,\overline{\Lambda},\overline{P}) = \mathcal{B}A(S,\Lambda,P)$. To show $\mathcal{B} \circ A \cong 1_{GP}$, it suffices to show $1_S:(S,\overline{\Lambda},\overline{P}) \to (S,\Lambda,P)$ is an isomorphism. Obviously $1_S:S \to S$ is an isomorphism in $\mathrm{Mor}(\underline{Sgp})$. First Lemma 2.6 will be used to show $1_S:(S,\overline{\Lambda},\overline{P}) \to (S,\Lambda,P)$ is in $\mathrm{Mor}(GP)$. Hence let $(y,x) \in \leq(L)$ and consider $G^x_y$. Now by the definition of $G^x_y$ in $\pi_2 A(S,\Lambda,P)$, there is a $\lambda \in \Lambda$ such that $y = \lambda x$ and $G^x_y = \lambda|_{R(x)}$. Thus $1_S(G^x_y z) = 1_S(\lambda z) = \lambda z = \lambda(1_S z)$ for all $z \in R(x)$. Dually for each $(y,x) \in \leq(R)$, there is a $\rho \in P$ such that $1_S(zD^x_y) = (1_S z)\rho$ for all $z \in L(x)$. Now since $\overline{\Lambda}$ and $\overline{P}$ are generated by the sets $\{G^x_y : (y,x) \in \leq(L)\} \cup \{1_S\}$ and $\{D^x_y : (y,x) \in \leq(R)\} \cup \{1_S\}$ respectively, apply Lemma 2.6 to obtain $1_S:(S,\overline{\Lambda},\overline{P}) \to (S,\Lambda,P)$ is in $\mathrm{Mor}(GP)$. Now

to show $1_S:(S,\Lambda,P) \to (S,\overline{\Lambda},\overline{P})$ is in $Mor(GP)$ (and hence that $1_S:(S,\overline{\Lambda},\overline{P}) \to (S,\Lambda,P)$ is an isomorphism) let $\lambda \in \Lambda$ and $x \in dom\ \lambda$. Let $y = \lambda x$. Then $G_y^x \in \overline{\Lambda}$ and $G_y^x = \lambda\big|_{R(x)}$. Therefore $1_S(\lambda z) = G_y^x(1_S z)$ for all $z \in R(x)$. Dually, if $\rho \in P$ and $x \in dom\ \rho$ there is a $\overline{\rho} \in \overline{P}$ such that $1_S(z\rho) = (1_S z)\overline{\rho}$ for all $z \in L(x)$. Hence by Definition 2.2 $1_S:(S,\Lambda,P) \to (S,\overline{\Lambda},\overline{P})$ is in $Mor(GP)$. This gives $1_S:(S,\overline{\Lambda},\overline{P}) \to (S,\Lambda,P)$ is an isomorphism. Therefore $\mathcal{B}\circ\mathcal{A} \cong 1_{GP}$.

Corollary (of proof) 2.8. $\mathcal{A}\circ\mathcal{B} = 1_{GT}$ and $1_S:(S,\Lambda,P) \to \mathcal{A}\circ\mathcal{B}(S,\Lambda,P)$ is an isomoprhism for all $(S,\Lambda,P) \in obj(GP)$.

Proposition 2.9. If $f:(S,\Lambda,P) \to (\overline{S},\overline{\Lambda},\overline{P})$ is in $Mor(GP)$ and $(S,\Lambda',P') = \mathcal{B}\mathcal{A}(S,\Lambda,P)$, then for each $\lambda' \in \Lambda'$ there is a $\overline{\lambda} \in \overline{\Lambda}$ such that $f(\lambda' x) = \overline{\lambda} f(x)$ for all $x \in dom\ \lambda'$ and dually for each $\rho' \in P'$ there is a $\overline{\rho} \in \overline{P}$ such that $f(x\rho) = f(x)\overline{\rho}$ for all $x \in dom\ \rho'$.

PROOF. By Corollary 2.8 $1_S:(S,\Lambda',P') \to (S,\Lambda,P)$ is an isomorphism. Now consider any $\lambda' \in \Lambda'$. If $\lambda' = G_y^x$ for some $(y,x) \in \leq(L)$ where $(S,\leq(L),\leq(R),G,D) = \mathcal{A}(S,\Lambda,P)$, then $x \in dom\ \lambda'$. Thus $f\circ 1_S:(S,\Lambda',P') \to (\overline{S},\overline{\Lambda},\overline{P})$ in $Mor(GP)$ and $x \in dom\ \lambda'$ imply the existence of some $\overline{\lambda} \in \overline{\Lambda}$ such that $f(\lambda' z) = f(G_y^x z) = \overline{\lambda} f(z)$ for all $z \in R(x) = dom\ \lambda'$. Now if $\lambda' = G_{y_1}^{x_1} \circ \cdots \circ G_{y_n}^{x_n}$ for some $(y_i,x_i) \in \leq(L)$ with $1 \leq i \leq n$, then there exist $\overline{\lambda}_1, \ldots, \overline{\lambda}_n \in \overline{\Lambda}$ such that $f(G_{y_i}^{x_i} z) = \overline{\lambda}_i f(z)$ for all $z \in R(x_i)$ with $1 \leq i \leq n$ (by the first case with $\lambda' = G_y^x$). Therefore if $z \in dom\ \lambda'$, then

$f(\lambda' z) = f(G^{x_1}_{y_1} \circ \cdots \circ G^{x_n}_{y_n} z) = \overline{\lambda}_1 f(G^{x_2}_{y_2} \circ \cdots \circ G^{x_n}_{y_n} z)$

$= \overline{\lambda}_1 \overline{\lambda}_2 f(G^{x_3}_{y_3} \circ \cdots \circ G^{x_n}_{y_n} z) = \cdots = \overline{\lambda}_1 \overline{\lambda}_2 \cdots \overline{\lambda}_n f(z)$ . Let

$\overline{\lambda} = \overline{\lambda}_1 \circ \overline{\lambda}_2 \circ \cdots \circ \overline{\lambda}_n \in \overline{\Lambda}$ , then $f(\lambda' z) = \overline{\lambda} f(z)$ for all $z \in \text{dom } \lambda'$ .

Proposition 2.9 shows that whenever the codomain of a morphism is in the image of the functor $\mathcal{B} \circ \mathcal{A}$ , then a stronger statement may be made pertaining to the Green's pairs involved. This will be used quite often when coextensions are dealt with in Chapter III.

One may view the functor $\mathcal{B} \circ \mathcal{A}$ as giving "the smallest Green's pairs" in the sense that the domains of the partial translations involved are the smallest possible so that the identity morphism in Sgp gives rise to an isomorphism in GP . The next theorem demonstrates a functor from GP to GP that gives "the largest Green's pairs." But first the following lemma is needed.

Lemma 2.10. If $1_S : (S, \Lambda, P) \to (S, \overline{\Lambda}, \overline{P})$ is an isomorphism in Mor(GP) with $\lambda' \in \overline{\Lambda}$ , then $(S, \langle \Lambda \cup \{\lambda'\} \rangle, P) \in \text{obj}(GP)$ and $1_S : (S, \Lambda, P) \to (S, \langle \Lambda \cup \{\lambda'\} \rangle, P)$ is an isomorphism.

PROOF. In order to show $(S, \langle \Lambda \cup \{\lambda'\} \rangle, P) \in \text{obj}(GP)$ , first note that conditions (i), (ii) and (v) in the definition of a Green's pair on S are clearly satisfied.

Now to show conditions (iii) and (iv) hold between $\lambda'$ and members of P , let $\rho \in P$ and $x \in \text{dom } \lambda' \cap \text{dom } \rho$ . Since $1_S : (S, \Lambda, P) \to (S, \overline{\Lambda}, \overline{P})$ is an isomorphism, $\leq(L) = \leq(\overline{L})$ and $\leq(R) = \leq(\overline{R})$ . Thus $x\rho \in xP = R(x) = \overline{R}(x) \subseteq \text{dom } \lambda'$ and $\lambda' x \in \overline{\Lambda} x = \overline{L}(x) = L(x) \subseteq \text{dom } \rho$ .

This gives condition (iii) for $\lambda'$ and members of $P$. Also since $1_S:(S,\Lambda,P) \to (S,\overline{\Lambda},\overline{P})$ is an isomorphism there is a $\rho' \in \overline{P}$ such that $w\rho' = w\rho$ for all $w \in L(x)$. Hence $\lambda'(x\rho) = \lambda'(x\rho') = (\lambda'x)\rho' = (\lambda'x)\rho$ and condition (iv) holds between $\lambda'$ and members of $P$. Since $(\Lambda,P)$ is a Green's pair for $S$, conditions (iii) and (iv) hold for $(\Lambda,P)$. Thus conditions (i)-(v) are satisfied by the generating sets $\Lambda \cup \{\lambda'\}$ and $P$. Hence $(\langle \Lambda \cup \{\lambda'\}\rangle,P)$ is a Green's pair on $S$ according to Lemma 5.3 and Corollary 5.5 of [3].

Now to show $1_S:(S,\Lambda,P) \to (S,\langle \Lambda \cup \{\lambda'\}\rangle,P)$ is an isomorphism, first note that $(\Lambda,P) \subseteq (\langle \Lambda \cup \{\lambda'\}\rangle,P)$ gives $1_S:(S,\Lambda,P) \to (S,\langle \Lambda \cup \{\lambda'\}\rangle,P)$ is a morphism. Now apply Lemma 2.6 with the homomorphism $1_S:S \to S$ and $(S,\langle \Lambda \cup \{\lambda'\}\rangle,P),(S,\Lambda,P) \in \mathrm{obj}(GP)$. Since $\Lambda \cup \{\lambda'\} \subseteq \Lambda'$ (where $\Lambda'$ is defined as in Lemma 2.6) as shown in the above, Lemma 2.6 gives $1_S:(S,\langle \Lambda \cup \{\lambda'\}\rangle,P) \to (S,\Lambda,P)$ is a morphism, and thus an isomorphism.

Theorem 2.11. If $F:GP \to GP$ is defined by $F(S,\Lambda,P) = (S,\overline{\Lambda},\overline{P})$ for $(S,\Lambda,P) \in \mathrm{obj}(GP)$ where $(\overline{\Lambda},\overline{P}) = \cup\{(\Lambda',P'):1_S:(S,\Lambda,P) \to (S,\Lambda',P') \text{ is an isomorphism}\}$ and $F(f) = f$ for $f \in \mathrm{Mor}(GP)$, then $F$ is a functor and $1_S:(S,\Lambda,P) \to F(S,\Lambda,P)$ is an isomorphism.

PROOF. First $F(S,\Lambda,P) = (S,\overline{\Lambda},\overline{P})$ is shown to be in $\mathrm{obj}(GP)$ whenever $(S,\Lambda,P) \in \mathrm{obj}(GP)$. Clearly conditions (ii) and (v) in the definition of a Green's pair are satisfied. Now $\overline{\Lambda}$ is shown to be closed under composition. If $\lambda_1,\lambda_2 \in \overline{\Lambda}$, then there exist Green's

pairs $(\Lambda_1,P_1)$ and $(\Lambda_2,P_2)$ on $S$ such that $\lambda_1 \varepsilon \Lambda_1$ , $\lambda_2 \varepsilon \Lambda_2$ , $1_S:(S,\Lambda_1,P_1) \to (S,\Lambda,P)$ is an isomorphism, and $1_S:(S,\Lambda_2,P_2) \to (S,\Lambda,P)$ is an isomorphism. By Lemma 2.10 $1_S:(S,\langle \Lambda_1 \cup \{\lambda_2\}\rangle,P_1) \to (S,\Lambda,P)$ is an isomorphism. Thus $\lambda_1 \circ \lambda_2 \varepsilon \langle \Lambda_1 \cup \{\lambda_2\}\rangle \subseteq \overline{\Lambda}$ . Hence $\overline{\Lambda}$ is closed under composition. Dually $\overline{P}$ is closed under composition.

Next conditions (iii) and (iv) are shown for $(\overline{\Lambda},\overline{P})$ . Let $(\lambda,\rho) \varepsilon (\overline{\Lambda},\overline{P})$ and $x \varepsilon \text{dom } \lambda \cap \text{dom } \rho$ . Then there are Green's pairs $(\Lambda_1,P_1)$ and $(\Lambda_2,P_2)$ on $S$ such that $\lambda \varepsilon \Lambda_1$ , $\rho \varepsilon P_2$ , $1_S:(S,\Lambda_1,P_1) \to (S,\Lambda,P)$ is an isomorphism, and $1_S:(S,\Lambda_2,P_2) \to (S,\Lambda,P)$ is an isomorphism. By Lemma 2.10 $(S,\langle \Lambda_2 \cup \{\lambda\}\rangle,P_2) \varepsilon \text{obj(GP)}$ . Thus $(\lambda,\rho) \varepsilon (\langle \Lambda_2 \cup \{\lambda\}\rangle,P_2)$ and $(S,\langle \Lambda_2 \cup \{\lambda\}\rangle,P_2) \varepsilon \text{obj(GP)}$ give conditions (iii) and (iv) hold for $(\lambda,\rho)$ , and hence for $(\overline{\Lambda},\overline{P})$ . Therefore $(S,\overline{\Lambda},\overline{P}) \varepsilon \text{obj(GP)}$ .

Now to show $1_S:(S,\Lambda,P) \to (S,\overline{\Lambda},\overline{P})$ is an isomorphism, note first that since $(\Lambda,P) \subseteq (\overline{\Lambda},\overline{P})$ $1_S:(S,\Lambda,P) \to (S,\overline{\Lambda},\overline{P})$ is a morphism. To see that $1_S:(S,\overline{\Lambda},\overline{P}) \to (S,\Lambda,P)$ is a morphism apply Lemma 2.6 and note that $(\overline{\Lambda},\overline{P}) \subseteq (\Lambda',P')$ (where $(\Lambda',P')$ is defined in Lemma 2.6) follows from the definition of $(\overline{\Lambda},\overline{P})$ . Hence $1_S(S,\Lambda,P) \to (S,\overline{\Lambda},\overline{P})$ is an isomorphism.

Now to finish the proof that $F$ is a functor first note that since $F(f) = f$ for $f \varepsilon \text{Mor(GP)}$ , $F$ preserves identities and composition of morphisms. Hence it suffices to show that $F(f) \varepsilon \text{Mor(GP)}$ . Thus let $f(S,\Lambda,P) \to (S_1,\Lambda_1,P_1)$ be in Mor(GP) .

$$\text{Hence } F(S,\Lambda,P) = \underbrace{(S,\overline{\Lambda},\overline{P}) \xrightarrow{1_S} (S,\Lambda,P) \xrightarrow{f} (S_1,\Lambda_1,P_1) \xrightarrow{1_S} (S_1,\overline{\Lambda}_1,\overline{P}_1)}_{F(f)} = F(S_1,\Lambda_1,P_1)$$

and $F(f) \varepsilon \text{Mor(GP)}$ .

Corollary 2.12. $F$ is full and dense.

Corollary 2.13. For a fixed $(S,\Lambda,P) \in \text{obj}(GP)$ the set $\{(\Lambda',P'):1_S:(S,\Lambda,P) \to (S,\Lambda',P') \text{ is an isomorphism}\}$ forms a join semilattice with supremum $(\overline{\Lambda},\overline{P})$ where $(S,\overline{\Lambda},\overline{P}) = F(S,\Lambda,P)$ and join is defined by $(\Lambda_1,P_2) \vee (\Lambda_2,P_2) = (\langle \Lambda_1 \cup \Lambda_2 \rangle, \langle P_1 \cup P_2 \rangle)$ .

The previous results have dealt almost exclusively with the facts pertaining to the categories GPX , GP , GTX , and GT that will be needed in Chapter III. The following results give a more detailed description of how certain Green's pairs are obtained and are related to the usual Green's pair for a semigroup. Also these results show certain relationships exist between certain previously obtained generalizations of Green's relations; see [1], [3], [8], [9], and [12]. In fact, as will be shown, for most applications all of these generalizations (except that of Clark and Carruth) can be described using the generalization given by Anscombre [1]. An example is given to show that the generalization given by Anscombre is not as general as that given by Clark and Carruth [3].

First the following construction is needed.

A construction 2.14. As pointed out in the definition of an ideal, the empty set is allowed as an ideal. Throughout this construction this fact will be used often without mentioning. Now let $(S,\Lambda,P) \in \text{obj}(GP)$ such that $\text{dom}\,\lambda$ is a right ideal of $S$ for each $\lambda \in \Lambda$ and $\text{dom}\,\rho$ is a left ideal of $S$ for each $\rho \in P$ . Let $F$ denote the free semigroup generated by $\Lambda \cup S \cup P$ . The multiplication

in $F$ as stated in Chapter I is denoted by "$\cdot$" while the multiplication in $S$ is denoted by juxtaposition. Define a word $x_1 \cdot x_2 \cdot x_3 \cdot \cdot \cdot x_n$ in $F$ to be <u>reduced</u> iff it has the following three properties:

(i) if $x_i \in S$ for some $1 \leq i \leq n-1$, then $x_{i+1} \notin S$

(ii) if $x_i \in \Lambda$ for some $1 \leq i \leq n-1$, then $x_{i+1} \notin \text{dom } x_i$

(iii) if $x_i \in P$ for some $2 \leq i \leq n$, then $x_{i-1} \notin \text{dom } x_i$.

Let $T = \{z \in F : z \text{ is reduced}\}$. First note that $\Lambda \cup S \cup P \subseteq T$. Now Lemmas 2.15 and 2.16 and Theorem 2.17 will give an associative multiplication (denoted by $*$) on T such that $S$ is embedded in $T$ as a subsemigroup. Theorem 2.26 will show that $\leq(L) = S \times S \cap \leq(L_T)$ and $\leq(R) = S \times S \cap \leq(R_T)$ and thus one will have the existence of a semigroup $T$ containing $S$ as a subsemigroup such that the relations associated with $(\Lambda, P)$ are precisely the restrictions to $S$ of the usual Green's relations on the semigroup $T$.

Before establishing Lemma 2.15, some notation is introduced. First let $\ell(z)$ denote the length of $z$ for $z \in F$ and let $T_n = \{z : z \in T \text{ and } \ell(z) = n\}$ for each $n \in \mathbb{N}$ $(n > 0)$. Define $\pi_1 : T_1 \times T_1 \to T$ by

$$\pi_1(x,y) = \begin{cases} xy & \text{if } xy \in S \\ x \cdot y & \text{otherwise.} \end{cases}$$

More explicity, for $\lambda, \lambda_1, \lambda_2 \in \Lambda$, $s,t \in S$, and $\rho, \rho_1, \rho_2 \in P$:

$$\pi_1(\lambda_1,\lambda_2) = \lambda_1\cdot\lambda_2 \qquad \pi_1(\lambda,\rho) = \lambda\cdot\rho$$

$$\pi_1(\rho_1,\rho_2) = \rho_1\cdot\rho_2 \qquad \pi_1(s,\lambda) = s\cdot\lambda$$

$$\pi_1(\lambda,s) = \begin{cases} \lambda s & \text{if } s \in \operatorname{dom}\lambda \\ \lambda\cdot s & \text{otherwise} \end{cases} \qquad \pi_1(s,t) = st$$

$$\pi_1(s,\rho) = \begin{cases} s\rho & \text{if } s \in \operatorname{dom}\rho \\ s\cdot\rho & \text{otherwise} \end{cases} \qquad \pi_1(\rho,\lambda) = \rho\cdot\lambda$$

$$\pi_1(\rho,s) = \rho\cdot s\ .$$

Lemma 2.15. There is a multiplication $*:T\times T \to T$ which is defined recursively beginning with $\pi_1$ .

PROOF. The author is indebted to C. E. Clark for suggestions which led to the following improved version of the original proof. The proof is based upon the Recursion Theorem. First let

$X = \{f \mid f$ is a partial function from $T\times T$ into $T$ such that

dom $f = T_1\times T_n$ for some $n \in \mathbb{N}\}$ .

Define $F:X \to X$ by $F(f:T_1\times T_n \to T) = (g:T_1\times T_{n+1} \to T)$ where the partial function $g$ is defined by

$$g(x,y_1\cdot y_2\cdot\ \ldots\ \cdot y_{n+1}) = \begin{cases} f(\pi_1(x,y_1),y_2\ \ldots\ y_{n+1}) & \text{if } \pi_1(x,y_1) \in S \\ x\cdot y_1\cdot y_2\ \ldots\ y_{n+1} & \text{otherwise .} \end{cases}$$

Clearly since $f \in X$ , $f(\pi_1(x,y_1),y_2\ \ldots\ y_{n+1}) \in T$ whenever $\pi_1(x,y_1) \in S$ . Hence $g \in X$ and $F$ is a function. Now since $\pi_1:T_1\times T_1 \to T$ is in $X$ , one has by the Recursion Theorem the existence of a map $\eta:\mathbb{N} \to X$ such that $\eta(1) = \pi_1$ and $\eta(n+1) = F(\eta(n))$ . Let $\pi_n = \eta(n)$ for $n \in \mathbb{N}$ and define $m_1 = \bigcup_{n\in\mathbb{N}} \pi_n$ . Clearly since $\operatorname{dom}\pi_i \cap \operatorname{dom}\pi_j = \square$ for $i \neq j$ , $m_1$ is a function from $T_1\times T$ into $T$ . Now let

$Y = \{f \mid f$ is a partial function from $T\times T$ into $T$ such that dom $f = T_n\times T$ for some $n \in \mathbb{N}\}$ .

Define $G:Y \to Y$ by $G(f:T_n\times T \to T) = (g:T_{n+1}\times T \to T)$ where the partial function $g$ is defined by $g(x_1 \ldots x_{n+1})z) = f(x_1 \ldots x_n)m_1(x_{n+1})z))$ . Clearly it follows from the definition of $f$ and $m_1$ that $g$ is in $Y$ . Hence $G$ is a function. Now since $m_1 \in Y$ , one has by the Recursion Theorem the existence of a map $\eta:\mathbb{N} \to Y$ such that $\eta(1) = m$ and $\eta(n+1) = G(\eta(n)$ . Let $m_n = \eta(n)$ and define

$$* = \bigcup_{n\in\mathbb{N}} m_n \ .$$

Thus dom $* = T\times T$ and $*:T\times T \to T$ is the desired multiplication. Written explicitly $* = \pi_1$ on $T_1\times T_1$ and if $z_1,z_2 \in T$ , say $z_1 = x_1 \ldots x_n$ and $z_2 = y_1 \ldots y_m$ with $m,n > 1$ , then

$$z_1*z_2 = \begin{cases} x_1\cdot x_2 \ldots x_n\cdot y_1\cdot y_2 \ldots y_m & \text{if } x_n\cdot y_1 \in T \\ (x_1\cdot x_2 \ldots x_{n-1})*[(x_n*y_1)*(y_2 \ldots y_m)] & \text{otherwise.} \end{cases}$$

Lemma 2.16. The multiplication $*:T\times T \to T$ defined in Lemma 2.15 is associative.

PROOF. Let $z_1,z_2,z_3 \in T$ . One must show $(z_1*z_2)*z_3 = z_1*(z_2*z_3)$ . The proof is by induction on $\ell(z_1) + \ell(z_2) + \ell(z_3)$ . (Note that the "empty word" is not in $T$ .) If $\ell(z_1) + \ell(z_2) + \ell(z_3) = 3$ , then the following cases must be checked where $\lambda,\lambda_1,\lambda_2 \in \Lambda$ , $s,s_1,s_2 \in S$ , and $\rho,\rho_1,\rho_2 \in P$ :

$$(\lambda*s)*\rho = \lambda*(s*\rho) \qquad (s*\lambda)*\rho = s*(\lambda*\rho)$$

$$(\lambda*s)*s_1 = \lambda*(s*s_1) \qquad (s*\lambda)*s_1 = s*(\lambda*s_1)$$

$$(\lambda*s)*\lambda_1 = \lambda*(s*\lambda_1) \qquad (s*\lambda)*\lambda_1 = s*(\lambda*\lambda_1)$$

$$(\lambda*\lambda_1)*\rho = \lambda*(\lambda_1*\rho) \qquad (s*s_1)*\rho = s*(s_1*\rho)$$

$$(\lambda*\lambda_1)*s = \lambda*(\lambda_1*s) \qquad (s*s_1)*s_2 = s*(s_1*s_2)$$

$$(\lambda*\lambda_1)*\lambda_2 = \lambda*(\lambda_1*\lambda_2) \qquad (s*s_1)*\lambda_1 = s*(s_1*\lambda_1)$$

$$(\lambda*\rho)*\rho_1 = \lambda*(\rho*\rho_1) \qquad (s*\rho)*\rho_2 = s*(\rho*\rho_2)$$

$$(\lambda*\rho)*s = \lambda*(\rho*s) \qquad (s*\rho)*s_2 = s*(\rho*s_2)$$

$$(\lambda*\rho)*\lambda_1 = \lambda*(\rho*\lambda_1) \qquad (s*\rho)*\lambda_1 = s*(\rho*\lambda_1)$$

$$(\rho*s)*\rho_1 = \rho*(s*\rho_1) \qquad (\rho*\lambda)*\lambda = \rho*(\lambda*\lambda)$$

$$(\rho*s)*s_1 = \rho*(s*s_1) \qquad (\rho*\rho_1)*\rho_2 = \rho*(\rho_1*\rho_2)$$

$$(\rho*s)*\lambda = \rho*(s*\lambda) \qquad (\rho*\rho_1)*s = \rho*(\rho_1*s)$$

$$(\rho*\lambda)*\rho_1 = \rho*(\lambda*\rho_1) \qquad (\rho*\rho_1)*\lambda = \rho*(\rho_1*\lambda)$$

$$(\rho*\lambda)*s = \rho*(\lambda*s)$$

These cases follow straight forwardly from the definition of $*$, the fact that $S$ is a semigroup, and the hypothesis on $(\Lambda,P)$. It should be pointed out that the fact that $\operatorname{dom}\lambda$ is a right ideal and $\operatorname{dom}\rho$ is a left ideal for every $(\lambda,\rho) \in \Lambda\times P$ is used to show $(\lambda*s)*s_1 = \lambda*(s*s_1)$ and $(s*s_1)*\rho = s*(s_1*\rho)$.

Now after dispensing with the initial step of the induction, the next step is broken up into several cases. First assume by way of induction that $(z_1*z_2)*z_3 = z_1*(z_2*z_3)$ whenever $\ell(z_1) + \ell(z_2) + \ell(z_3) < n$ $(n > 1)$ and $z_1,z_2,z_3 \in T$. Let $w_1 = x_1 \ldots x_k$, $w_2 = y_1 \ldots y_\ell$,

and $w_3 = u_1 \ldots u_m$ where $k + \ell + m = n$ and $w_1, w_2, w_3 \in T$. It must be shown that $(w_1 * w_2) * w_3 = w_1 * (w_2 * w_3)$. The following cases will be considered in order:

(i) $x_k \cdot y_1$ and $y_\ell \cdot u_1 \in T$

(ii) $x_k \cdot y_1 \notin T$ and $k, \ell, m > 1$

(a) $y_\ell \cdot u_1 \notin T$

(b) $y_\ell \cdot u_1 \in T$

(iii) $x_k \cdot y_1 \in T$ and $k = 1$, $\ell, m > 1$

(a) $y_\ell \cdot u_1 \notin T$

(b) $y_\ell \cdot u_1 \in T$

(iv) all other cases.

Now consider case (i). By the definition of $*$,

$$\begin{aligned}(w_1 * w_2) * w_3 &= (x_1 \ldots x_k \cdot y_1 \ldots y_\ell) * w_3 \\ &= x_1 \ldots x_k \cdot y_1 \ldots y_\ell \cdot u_1 \ldots u_m \\ &= x_1 \ldots x_k \cdot (y_1 \ldots y_\ell \cdot u_1 \ldots u_m) \\ &= x \ldots x_k \cdot (w_2 * w_3) \\ &= w_1 * (w_2 * w_3) .\end{aligned}$$

Now consider case (ii). Since $x_k \cdot y_1 \notin T$, $x_k * y_1 \in S$. Hence let $s = x_k * y_1$ for some $s \in S$. Thus

$$
\begin{aligned}
(w_1*w_2)*w_3 &= ((x_1 \ldots x_{k-1})*[(x_k*y_1)*(y_2 \ldots y_\ell)])*w_3 \\
&= ((x_1 \ldots x_{k-1})*[s*(y_2 \ldots y_\ell)])*w_3 \\
&= ([(x_1 \ldots x_{k-1})*s]*(y_2 \ldots y_\ell))*w_3 \quad \text{(by the ind. hyp.)} \\
&= [(x_1 \ldots x_{k-1})*s]*((y_2 \ldots y_\ell)*w_3) \quad \text{(by the ind. hyp.)} \\
&= (x_1 \ldots x_{k-1})*[s*((y_2 \ldots y_\ell)*w_3)] \quad \text{(by the ind. hyp.)} \\
&= (x_1 \ldots x_{k-1})*[(s*(y_2 \ldots y_\ell))*w_3] \quad \text{(by the ind. hyp.)} \\
&= (x_1 \ldots x_{k-1})*[[(x_k*y_1)*(y_2 \ldots y_\ell)]*w_3] \\
&= (x_1 \ldots x_{k-1})*[[x_k*(y_1 \cdot y_2 \ldots y_\ell)]*w_3] \quad \text{(by the def. of *)} \\
&= (x_1 \ldots x_{k-1})*[x_k*[(y_1 \ldots y_\ell)*w_3]] \quad \text{(by the ind. hyp.)} .
\end{aligned}
$$

Now to finish case (ii), consider the two subcases. First (a). If $y_\ell \cdot u_1 \notin T$, then $y_\ell * u_1 \in S$. Hence let $t = y_\ell * u_1$ for some $t \in S$. Therefore as in the above, one has the following

$$
\begin{aligned}
(w_1*w_2)*w_3 &= (x_1 \ldots x_{k-1})*[x_k*[w_2*w_3]] \quad \text{(from above)} \\
&= (x_1 \ldots x_{k-1})*[x_k*[(y_1 \ldots y_{\ell-1})*[(y_\ell*u_1)*(u_2 \ldots u_m)]]] \\
&= (x_1 \ldots x_{k-1})*[x_k*[(y_1 \ldots y_{\ell-1})*[t*(u_2 \ldots u_m)]]] \\
&= [(x_1 \ldots x_{k-1})*x_k]*[(y_1 \ldots y_{\ell-1})*[t*(u_2 \ldots u_m)]] \\
&= w_1*[(y_1 \ldots y_{\ell-1})*[(y_\ell*u_1)*(u_2 \ldots u_m)]] \\
&= w_1*[(y_1 \ldots y_\ell)*(u_1 \ldots u_m)] \quad \text{(by def. of *)} \\
&= w_1*(w_2*w_3) .
\end{aligned}
$$

Now in case (b),

$$
\begin{aligned}
(w_1*w_2)*w_3 &= (x_1 \ldots x_{k-1})*[x_k*[w_2*w_3]] \\
&= (x_1 \ldots x_{k-1})*[x_k*[y_1 \ldots y_\ell \cdot u_1 \ldots u_m]] \\
&= (x_1 \ldots x_{k-1})*[(x_k*y_1)*(y_2 \ldots y_\ell \cdot u_1 \ldots u_m)] \\
&\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \text{(by def. of *)} \\
&= (x_1 \ldots x_k)*(y_1 \ldots y_\ell \cdot u_1 \ldots u_m) \qquad \text{(by def. of *)} \\
&= w_1*(w_2*w_3) \ .
\end{aligned}
$$

Hence in case (ii), $(w_1*w_2)*w_3 = w_1*(w_2*w_3)$ .

Now consider case (iii) (a). If $y_\ell \cdot u_1 \notin T$ , then let $t = y_\ell \cdot u_1$ for some $t \in S$ . Thus

$$
\begin{aligned}
(w_1*w_2)*w_3 &= (x_1*(y_1 \ldots y_\ell))*w_3 \\
&= (x_1 \cdot y_1 \ldots y_\ell)*w_3 \\
&= (x_1 \cdot y_1 \ldots y_\ell)*(u_1 \ldots u_m) \\
&= (x_1 \cdot y_1 \ldots y_{\ell-1})*[(y_\ell*u_1)*(u_2 \ldots u_m)] \\
&= (x_1 \cdot y_1 \ldots y_{\ell-1})*[t*(u_2 \ldots u_m)] \\
&= (x_1*(y_1 \ldots y_{\ell-1}))*[t*(u_2 \ldots u_m)] \\
&= x_1*((y_1 \ldots y_{\ell-1})*[t*(u_2 \ldots u_m)]) \\
&= x_1*((y_1 \ldots y_\ell)*(u_1 \ldots u_m)) \\
&= w_1*(w_2*w_3) \ .
\end{aligned}
$$

Now case (iii) (b) is just a subcase of case (i). Hence in case (iii), $(w_1*w_2)*w_3 = w_1*(w_2*w_3)$ .

Now all the other cases such as $x_k \cdot y_1 \in T$ with $k = 1$ , $\ell = 1$ , $m > 1$ , etc. follow from similar arguments or are subcases of the above cases. Hence the proofs are omitted. Therefore

$(w_1*w_2)*w_3 = w_1*(w_2*w_3)$ whenever $\ell(w_1) + \ell(w_2) + \ell(w_3) = n$ . Thus $*$ is associative.

The above construction gives the following theorem.

Theorem 2.17. If $(S,\Lambda,P) \in \text{obj}(GP)$ such that $\text{dom}\ \lambda$ is a right ideal of $S$ for all $\lambda \in \Lambda$ and $\text{dom}\ \rho$ is a left ideal of $S$ for all $\rho \in P$ , then the set $T$ constructed in 2.14 along with the multiplication $*$ defined in Lemma 2.15 form a semigroup containing $S$ as a subsemigroup.

Corollary 2.18. If $h:F \to T$ is defined by

$$h(x_1 \ . \ . \ . \ x_n) = \begin{cases} x_1* \ . \ . \ . \ *x_n & \text{if } n > 1 \\ x_1 & \text{otherwise} , \end{cases}$$

then $h$ is a homomorphism onto $T$.

PROOF. Since $*$ is an associative multiplication from $T\times T$ into $T$ and $x_i \in \Lambda \cup S \cup P \subseteq T$ for $1 \leq i \leq n$ , one has $h$ is a well-defined function from $F$ onto $T$ . (Note $h\big|_T = 1_T$ .) Now by the associativity of $*$ ,

$$\begin{aligned} h((x_1 \ . \ . \ . \ x_n)\cdot(y_1 \ . \ . \ . \ y_m)) &= h(x_1 \ . \ . \ . \ x_n\cdot y_1 \ . \ . \ . \ y_m) \\ &= x_1* \ . \ . \ . \ *x_n*y_1* \ . \ . \ . \ y_m \\ &= (x_1* \ . \ . \ . \ *x_n)*(y_1* \ . \ . \ . \ *y_m) \\ &= h(x_1 \ . \ . \ . \ x_n)*h(y_1 \ . \ . \ . \ y_m) . \end{aligned}$$

Thus $h$ is a homomorphism. (Note that it is assumed that the "empty word" is not in $F$ .)

Corollary 2.19. Let $C$ denote the congruence generated by the relations

$\{(\lambda\cdot s,\lambda s):\lambda \in \Lambda \text{ and } s \in \operatorname{dom}\lambda\} \cup \{(s\cdot\rho,s\rho):\rho \in P \text{ and } s \in \operatorname{dom}\rho\}$ in $F$. Then $T \overset{\sim}{=} F/C$.

PROOF. By Corollary 2.18 the map $h:F \to T$ is a homomorphism onto $T$. Clearly it follows from the definition of $h$ and that of $*$ that $\ker h = C$. Thus $T \overset{\sim}{=} F/\ker h = F/C$.

Theorem 2.17 and its proof are generalizations of a result obtained by Tamura and Graham (see Theorem 1 in [11]) where they answered the following question posed in [4]: If $S$ is a semigroup, then what are necessary and sufficient conditions on $S$ so that there is a semigroup $T$ containing $S$ as an ideal of $T$ and every [left, right] translation of $S$ is induced by some inner [left, right] translation of $T$. Also the construction in 2.14 is a generalization of that presented in [10].

Let $\overline{\overline{GP}}$ denote the full subcategory of $GP$ whose objects consist of all $(S,\Lambda,P) \in \operatorname{obj}(GP)$ such that $\operatorname{dom}\lambda$ is a right ideal (possibly empty) of $S$ for each $\lambda \in \Lambda$ and $\operatorname{dom}\rho$ is a left ideal (possibly empty) of $S$ for each $\rho \in P$. If one considers a fixed $(S,\Lambda,P) \in \operatorname{obj}(\overline{\overline{GP}})$ and the semigroup $T$ constructed in 2.14, then the following corollary shows that the quasi-orders $\underline{\leq}(L)$ and $\underline{\leq}(R)$ on $S$ can be induced by inner translations on $T$.

Corollary 2.20. If $(S,\Lambda,P) \in \text{obj}(\overline{\overline{GP}})$, then there is a Green's pair $(\Lambda',P')$ on $T$ (as constructed in 2.14) such that

$$\leq(L) = S\times S \cap \leq(L') \quad \text{and}$$
$$\leq(R) = S\times S \cap \leq(R') .$$

PROOF. Let $\Lambda'$ be the subsemigroup generated by $\{\lambda:\lambda \in \Lambda\} \cup \{1_T\}$ in $T^1$ and let $P'$ be the subsemigroup generated by $\{\rho:\rho \in P\} \cup \{1_T\}$ in $T^1$. Let $a \in S$. Then clearly $L(a) = \Lambda a \subseteq \Lambda' a$. Hence $L(a) \subseteq S \cap \Lambda' a = S \cap L'(a)$. Now if $z \in S \cap L'(a)$, then $z = \lambda_1 * \ldots * \lambda_n * a$ for some $\lambda_1, \ldots, \lambda_n \in \Lambda$. Now as will be shown in Lemma 2.25 $\lambda_1 * \ldots \lambda_n * a = \lambda_1(\ldots(\lambda_n a)\ldots)$ which is in $\Lambda a$. Thus $z \in L(a)$ and $S \cap L'(a) \subseteq L(a)$. Therefore $L(a) = S \cap L'(a)$ for every $a \in S$. Hence $\leq(L) = S\times S \cap \leq(L')$. Dually one has $\leq(R) = S\times S \cap \leq(R')$.

Now with the previous construction completed, one is able to compare the generalizations obtained by Wallace [12], Anscombre [1], Marki and Steinfeld [8], and Pastijn [9]. First for the sake of completeness, the definition given by Anscombre is stated in the case of semigroups.

Definition 2.21 [Anscombre]. Let $S$ be a subsemigroup of the monoid $D$. Then a pair $(\Lambda,P)$ is said to be a generalized Green's pair (as defined by Anscombre) on $S$ if $(\Lambda,P)$ satisfies the following three properties:

(i) $\Lambda$ is a subsemigroup of $\mathcal{ILT}(D)$ and $P$ is a subsemigroup of $\mathcal{IRT}(D)$,

(ii) both $\Lambda$ and $P$ contain the identity, and

(iii) for all $a \in S$, for all $\lambda \in \Lambda$, and for all $\rho \in P$, if $\lambda a \in S$ and $a\rho \in S$, then $\lambda a\rho \in S$.

Clearly a Green's pair $(\Lambda,P)$ as defined by Anscombre is also a Green's pair as defined by Clark and Carruth. One also has the following result.

Theorem 2.22. If $(\leq(L),\leq(R))$ is a pair of quasi-orders on a semigroup $S$, then the quasi-orders $\leq(L)$ and $\leq(R)$ are induced by a Green's pair $(\Lambda,P)$ with $(S,\Lambda,P) \in \text{obj}(\overline{\overline{GP}})$ iff they are induced by a pair $(\Lambda',P')$ as defined by Anscombre.

PROOF. If $(\leq(L),\leq(R))$ is induced by a pair $(\Lambda,P)$ with $(S,\Lambda,P) \in \text{obj}(\overline{\overline{GP}})$, then by Corollary 2.20 one has the existence of a pair $(\Lambda',P')$ on a semigroup $T^1$ which clearly satisfies the definition given by Anscombre. Now as noted earlier each pair $(\Lambda',P')$ satisfying Anscombre's definition is also a Green's pair.

Now let $\overline{GP}$ denote the full subcategory of $GP$ whose objects consist of all $(S,\Lambda,P) \in \text{obj}(GP)$ such that $L(x)$ is a left ideal of $S$ for each $x \in S$ and $R(x)$ is a right ideal of $S$ for each $x \in S$. By Theorem 7.9 in [3], one has $\overline{GP}$ is a subcategory of $\overline{\overline{GP}}$, i.e., $L(x)$ a left ideal of $S$ for each $x \in S$ implies $\text{dom}\,\rho$ is a left ideal for all $\rho \in P$ and dually for $R$ and $\Lambda$. Now in examples 8.13, 8.15, and 8.16 in [3], Green's pairs are described that show in general that those generalizations in [1], [8], [9], and [12] (as far as the

quasi-orders are concerned) are subcategories of $\overline{\overline{GP}}$ . (Actually the Green's pairs describing these generalizations are given. It follows that these Green's pairs satisfy the necessary conditions to be in $\overline{\overline{GP}}$ .) To be more specific the generalizations described by Marki and Steinfeld [8] and Pastijn [9] can be realized by Green's pairs which are in the category $\overline{GP}$ . Now in view of Theorem 2.22 one sees that (as far as the quasi-orders are concerned) the generalizations in [8], [9], and [12] are special cases of that described by [1].

Now one might ask the following question: If $(\underline{\leq}(L),\underline{\leq}(R))$ is a pair of quasi-orders obtained from a pair $(\Lambda,P)$ with $(S,\Lambda,P) \in \text{obj}(GP)$ , then can $(\underline{\leq}(L),\underline{\leq}(R))$ be realized by a pair $(\Lambda',P')$ with $(S,\Lambda',P') \in \text{obj}\ (\overline{\overline{GP}})$ ? That is, is the generalization given by Anscombre as general as that given by Clark and Carruth? The answer in general is no. The following example demonstrates this fact.

Example 2.23. Let $S$ be the semigroup formed by the following Cayley table.

| | e | f | g | a |
|---|---|---|---|---|
| e | e | e | e | e |
| f | f | f | f | f |
| g | g | g | g | g |
| a | e | e | f | e |

Let $\Lambda = \{1_S\}$ and $P = \{1_S,\rho,\square\}$ where $\rho$ is the map $\{a \mapsto g\}$ . Note that $\rho^2 = \square$ and $a \notin S^2$ . Thus $\rho \in PRT(S)$ . This gives $(S,\Lambda,P) \in \text{obj}(GP)$ . Also note that dom $\rho$ is not a left ideal of $S$ .

Now if the quasi-orders $\leq(L)$ and $\leq(R)$ on $S$ can be realized from a pair $(\Lambda',P')$ on $S$ satisfying 2.21, then one has the existence of a semigroup $T$ containing $S$ such that the map $\rho$ can be considered as an inner right translation of $T$. Hence extending $\rho$ to a translation on $S$, one has

$e = eg = e(a\rho) = (ea)\rho = e\rho = (aa)\rho = a(a\rho) = ag = f$. But $e \neq f$. Therefore there does not exist such a semigroup $T$.

Now the above example also shows that in general one cannot do the construction in 2.14 because as noted in the example $\operatorname{dom}\rho$ is not a left ideal of $S$.

As noted before, the category $\overline{GP}$ is a subcategory of $\overline{\overline{GP}}$. Hence one can apply Corollary 2.20 to each object of $\overline{GP}$. However in the category $\overline{GP}$ one has even a stronger result holding which will be demonstrated by Theorem 2.26.

Lemma 2.24. Let $(S,\Lambda,P) \in \operatorname{obj}(\overline{GP})$ and let $T$ denote the semigroup constructed in 2.14. If $(z_1 \ldots z_n)*s = t$ for some $z_1 \ldots z_n \in T$ and $s,t \in S$, then $z_1,z_2,\ldots,z_n \in \Lambda \cup S$.

PROOF. The proof is by induction on the length of $z_1 \ldots z_n$. If $n = 1$, then $z_1*s = t$ where $s,z_1 \in T_1$. Hence by the definition of $*$, $* = \pi_1$ on $T_1$. Thus from the definition of $\pi_1$, one has $z_1 \in \Lambda \cup S$.

Now assume $n > 1$ and whenever $(z_1 \ldots z_k)*x = y$ for some $z_1 \ldots z_k \in T$, $k < n$, and $x,y \in S$, then $z_1,\ldots,z_k \in \Lambda \cup S$. If $(z_1 \ldots z_n)*s = t$, then

$$t = (z_1 \ldots z_n)*s = \begin{cases} z_1 \ldots z_n \cdot s & \text{if } z_n \cdot s \in T \\ (z_1 \ldots z_{n-1})*(z_n*s) & \text{otherwise .} \end{cases}$$

But $z_1 \ldots z_n \cdot s \notin S$ . Hence $t \in S$ gives $z_n \cdot s \notin T$ . Therefore if $x = z_n*s$ , then $x \in S$, $t = (z_1 \ldots z_{n-1})*x$ , and by the induction hypothesis $z_1, z_2, \ldots, z_{n-1} \in \Lambda \cup S$ . Now by the first case with $n = 1$ , one has $z_n*s = x \in S$ gives $z_n \in \Lambda \cup S$ . Hence $z_1, \ldots, z_n \in \Lambda \cup S$ .

Lemma 2.25. Let $(S,\Lambda,P) \in \text{obj}(\overline{GP})$ and let $T$ denote the semigroup constructed in 2.14. If $(z_1 \ldots z_n)*s = t$ for some $z_1 \ldots z_n \in T$ , $z_1, z_2, \ldots, z_n \in \Lambda \cup S$ , and $s,t \in S$ , then $(z_1 \ldots z_n)*s = z_1(z_2(\ldots(z_n s)\ldots))$ .

PROOF. The proof is by induction on the length of $z_1 \ldots z_n$ . If $n = 1$ , then $z_1*s = t$ with $z_1 \in \Lambda \cup S$ . Thus by the definition of $*$ , $z_1*s = \pi_1(z_1,s) = z_1 s$ since $t \in S$ .

Now let $n > 1$ and assume $(z_1 \ldots z_k)*x = z_1(z_2(\ldots(z_k x)\ldots))$ for all $k < n$ and $z_1 \ldots z_k \in T$ such that $z_1, z_2, \ldots, z_k \in \Lambda \cup S$ and $(z_1 \ldots z_k)*x$ , $x \in S$ . If $(z_1 \ldots z_n)*s = t$ , then $z_n \cdot s \notin T$ since otherwise $t = z_1 \ldots z_n \cdot s \notin S$ . Hence $t = (z_1 \ldots z_{n-1})*(z_n*s)$ and $z_n*s \in S$ . Let $x = z_n*s$ and apply the induction hypothesis to obtain $t = (z_1 \ldots z_{n-1})*x$ $= z_1(\ldots(z_{n-1}x)\ldots) = z_1(\ldots(z_{n-1}(z_n*s))\ldots)$ $= z_1(\ldots(z_{n-1}(z_n s))\ldots)$ .

Theorem 2.26. Let $(S,\Lambda,P) \in \text{obj}(GP)$. Then $L(x)$ is a left ideal for each $x \in S$ and $R(x)$ is a right ideal for each $x \in S$ iff there is a semigroup $T$ containing $S$ such that

$$\leq(L) = (S\times S) \cap \leq(L_T) \quad \text{and}$$
$$\leq(R) = (S\times S) \cap \leq(R_T)$$

where $\leq(L_T)$ and $\leq(R_T)$ are the usual quasi-orders on $T$.

PROOF. The implication is shown first. Note that in this case $(S,\Lambda,P) \in \text{obj}(\overline{GP})$ and Theorem 2.17 can be applied to give the semigroup $T$.

Now it is shown that $\leq(L) = (S\times S) \cap \leq(L_T)$. Let $(y,x) \in \leq(L)$. Hence there is a $\lambda \in \Lambda$ such that $y = \lambda x$. Thus $y = \lambda * x$ by the definition of $*$. Therefore $(y,x) \in \leq(L_T)$ and $\leq(L) \subseteq (S\times S) \cap \leq(L_T)$. Now if $(y,x) \in (S\times S) \cap \leq(L_T)$, then there is some $z_1 \ldots z_n \in T$ such that $y = (z_1 \ldots z_n)*x$. By Lemma 2.24, $z_1, \ldots, z_n \in \Lambda \cup S$. Now by Lemma 2.25, $y = z_1(z_2(\ldots(z_{n-1}(z_n x))\ldots))$. Also since $L(x)$ is a left ideal for each $x \in S$, one has by Theorem 7.9 of [3] $\leq(L_S) \subseteq \leq(L)$. Hence if $u\leq(L_S)v$, then $u\leq(L)v$. Therefore $y = z_1(z_2(\ldots(z_{n-1}(z_n x))\ldots))$ along with $\leq(L_S) \subseteq \leq(L)$ gives the following string:

$$
\begin{aligned}
y &\leq(L) z_1(z_2(\ .\ .\ .((z_{n-1}(z_n x))\ .\ .\ .)) \\
&\leq(L) z_2(\ .\ .\ .(z_{n-1}(z_n x)\ .\ .\ .) \\
&\ \ \vdots \\
&\leq(L) z_{n-1}(z_n x) \\
&\leq(L) z_n x \\
&\leq(L) x\ .
\end{aligned}
$$

Now by the transtivity of $\leq(L)$, $y \leq(L)x$. Therefore $\leq(L) \supseteq (S\times S) \cap \leq(L_T)$ and hence $\leq(L) = S\times S \cap \leq(L_T)$. Dually $\leq(R) = S\times S \cap \leq(R_T)$.

Now to show the converse let $x \in S$, then $\leq(L) = (S\times S) \cap \leq(L_T)$ gives $L(x) = S \cap T^1x$ for each $x \in S$. Since $S$ is a subsemigroup of $T$ and $T^1x$ is a principal left ideal of $T$, $L(x)$ is clearly a left ideal of $S$. Dually $R(x)$ is a right ideal of $S$ for each $x \in S$.

As a result of Theorem 2.26 one can draw the following conclusion. First consider the usual quasi-orders $\leq(L_0)$ and $\leq(R_0)$ on a semigroup $S$. If $(\Lambda,P)$ is a Green's pair on $S$ so that $(\leq(L_0),\leq(R_0)) \subseteq (\leq(L),\leq(R))$, then by Theorem 7.9 of [3] $(S,\Lambda,P) \in \text{obj}(\overline{GP})$. Hence the results of Theorem 2.26 follow, that is, the only way to enlarge the usual quasi-orders by a Green's pair is to have a larger semigroup containing the original semigroup. Therefore as far as enlarging the usual quasi-orders, the generalization given by Anscombre is equivalent to that given by Clark and Carruth.

# CHAPTER III

## COEXTENSIONS

In this chapter the notation and theory of Green's pairs are applied to the development of coextensions. In particular the results obtained by Leech [7] are generalized to Green's pairs while extending Leech's results in general. This also incorporates certain results obtained by Bowman [2] and Grillet [5].

The first part of the chapter deals with relations and their relationship to the $\leq(H)$-quasi-order.

Fix $(X,\Lambda,P) \in \mathrm{obj}(GPX)$ . Let $\leq R$ denote the small category with $\mathrm{obj}(\leq R) = X$ and for every $x,y \in X$ $\mathrm{Hom}(x,y)$ is $\{(y,x)\}$ if $(y,x) \in \leq(R)$ and empty otherwise. If $(z,y),(y,x) \in \mathrm{Mor}(\leq R)$ , then $(z,y)\circ(y,x) = (z,x)$ . Dually let $\leq L$ denote the small category with $\mathrm{obj}(\leq L) = X$ and for every $x,y \in X$ , $\mathrm{Hom}(x,y)$ is $\{(y,x)\}$ if $(y,x) \in \leq(L)$ and empty otherwise. If $(z,y),(y,x) \in \mathrm{Mor}(\leq L)$ , then $(z,y)\circ(y,x) = (z,x)$ .

For a pair of subsets $(\Lambda,P)$ of $PT(X)$ , a relation $U$ on $X$ is said to be $(\Lambda,P)$ <u>compatible</u> if for all $\rho \in P$ and for all $(y,x) \in U$ such that $y,x \in \mathrm{dom}\ \rho$ , $(y\rho,x\rho) \in U$ and if for all $\lambda \in \Lambda$ and for all $(y,x) \in U$ such that $y,x \in \mathrm{dom}\ \lambda$ , $(\lambda y,\lambda x) \in U$ . A quasi-order $\leq(C)$ on $(X,\Lambda,P)$ is an $[L,R]$ <u>$H$ quasi-order</u> provided $\leq(C)$ is $[(\Lambda',\{1_X\}),(\{1_X\},P')]$ $(\Lambda',P')$ compatible where $(X,\Lambda',P') = B\circ A(X,\Lambda,P)$ . An $[L,R]$ $H$ quasi-order $\leq(C)$ on $(S,\Lambda,P) \in \mathrm{obj}(GP)$ is a $[L,R]$ <u>quasi-congruence</u> on $(S,\Lambda,P)$ if $\leq(C)$

is also $[(\Lambda_0,\{1_S\}),(\{1_S\},P_0)]$ $(\Lambda_0,P_0)$ compatible. Hence note that $\leq(C)$ is a quasi-congruence on $(S,\Lambda,P)$ iff $\leq(C)$ is $(\Lambda' \cup \Lambda_0, P' \cup P_0)$ compatible.

A [left, right] congruence on $(S,\Lambda,P)$ is a $[L,R]$ quasi-congruence on $(S,\Lambda,P)$ which is also an equivalence relation on $S$. An $[L,R]$ $H$ quasi-order $\leq(C)$ on $(X,\Lambda,P)$ is said to be an $[L,R]$ $H$ quasi-order under $\leq(H)$ if $\leq(C) \subseteq \leq(H)$. As stated in Chapter I, let $C = \leq(C) \cap (\leq(C))^{-1}$ whenever $\leq(C)$ is a quasi-order. Let $C(x)$ denote the $C$-set of $x$ for $x \in X$ and let $C_x$ denote the $C$-class of $x$ for $x \in X$.

Observations. If $\leq(C)$ is a $(\Lambda,P)$ compatible quasi-order on $X$, then $C$ is $(\Lambda,P)$ compatible. This follows from the fact that if $\leq(C)$ is $(\Lambda,P)$ compatible, then $(\leq(C))^{-1}$ is $(\Lambda,P)$ compatible and thus $C = \leq(C) \cap (\leq(C))^{-1}$ is $(\Lambda,P)$ compatible. Note also that if $\leq(C) \subseteq \leq(H)$, then $C = \leq(C) \cap (\leq(C))^{-1} \subseteq \leq(H) \cap (\leq(H))^{-1} = H$.

In the following $R$ quasi-orders under $\leq(H)$ are investigated. These results will be used later in the chapter. Recall from Chapter I that $S_D(x) = \{D_y^x : y \leq(H) x\}$ and $G_D(x) = \{D_y^x : yHx\}$.

Lemma 3.1. If $\leq(C)$ is an $R$ quasi-order on $(X,\Lambda,P) \in \text{obj}(GPX)$ with $\leq(C) \subseteq \leq(H)$, then for a fixed $x \in X$ the following sets are equal:

(a) $A = \{g \in S_D(x) : C(x)g \subseteq C(x)\}$

(b) $B = \{g \in S_D(x) : xg \in C(x)\}$.

PROOF. Certainly $A \subseteq B$ . Now to show $B \subseteq A$ , let $y \in C(x)$ and let $g \in B$ . Since $(y,x) \in \leq(C) \subseteq \leq(H)$ and $\leq(C)$ is an $R$ quasi-order, $(yg,xg) \in \leq(C)$ . Now $(yg,xg) \in \leq(C)$ and $(xg,x) \in \leq(C)$ give $(yg,x) \in \leq(C)$ . Therefore $C(x)g \subseteq C(x)$ and $g \in A$ . Thus $B \subseteq A$ .

Let ${}_C\mathcal{R}(x)$ denote the equal sets A and B of Lemma 3.1. If no confusion is likely, the subscript "C" will be suppressed.

Corollary 3.2. With the hypothesis of Lemma 3.1, $x\mathcal{R}(x) = C(x)$ for all $x \in X$ .

PROOF. By the definition of $\mathcal{R}(x)$ , $x\mathcal{R}(x) \subseteq C(x)$ . Now if $y \in C(x)$ , then $y \leq (H) x$ . Therefore $D_y^x \in S_D(x)$ (by the definition of $S_D(x)$ ) and $xD_y^x = y$ . Hence $D_y^x \in \mathcal{R}(x)$ and $x\mathcal{R}(x) \supseteq C(x)$ . Thus $x\mathcal{R}(x) = C(x)$ .

Corollary 3.3. If $\leq(U)$ is an $R$ quasi-order on $(X,\Lambda,P) \in \text{obj}(GPX)$ with $\leq(U) \subseteq \leq(H)$ , then for a fixed $x \in X$ the following sets are equal:

(a) $A = \{g \in G_D(x) : U(x)g = U(x)\}$

(b) $B = \{g \in G_D(x) : U_x g = U_x\}$

(c) $C = \{g \in G_D(x) : U(x)g \subseteq U(x)\}$

(d) $D = \{g \in S_D(x) : U_x g \subseteq U_x\}$

(e) $E = \{g \in S_D(x) : xg \in U_x\}$ .

PROOF. First note that $B \subseteq D \subseteq E$ and $A \subseteq C$. Now apply Lemma 3.1 with $\leq(C) = U$ to get $D = E$. Now if $g \in D$, then $U_x g \subseteq U_x$. Since $U \subseteq H$ and $U_x g \subseteq U_x$, $(x,xg) \in H$. Thus $g \in G_D(x)$. Now $U_x g \subseteq U_x$ gives $U_x = U_x g g^{-1} \subseteq U_x g^{-1} \subseteq U_x$ where the last containment follows from the fact that $U$ is an $R$ quasi-order under $H$. Now $U_x g^{-1} = U_x$ gives $U_x = U_x g^{-1} g = U_x g$ and $g \in B$. Therefore $D = B$. Similarly $A = C$.

Now to show $A \subseteq D$, let $g \in A$. Then $U_x g = (U(x) \cap U^{-1}(x))g \subseteq U(x)g \cap U^{-1}(x)g = U(x) \cap U^{-1}(x) = U_x$. Hence $g \in D$ and $A \subseteq D$. To show $E \subseteq C$, let $g \in E$. Thus as noted above this gives $g \in G_D(x)$ and hence $g \in \{h \in G_D(x) : xh \in U(x)\} \subseteq \{h \in S_D(x) : xh \in U(x)\}$. Now apply Lemma 3.1 with $\leq(C) = \leq(U)$ to obtain $\{h \in S_D(x) : xh \in U(x)\} = \{h \in S_D(x) : U(x)h \subseteq U(x)\}$. Therefore $g \in \{h \in S_D(x) : U(x)h \subseteq U(x)\}$ and since $g \in G_D(x)$, $g \in C$. Thus it has been shown that $E \subseteq C \subseteq A \subseteq D = B = E$.

Let ${}_U\mathcal{R}_x$ denote the equal sets in Corollary 3.3. If no confusion is likely, the subscript "U" will be suppressed.

Corollary 3.4. With the hypothesis of Corollary 3.3, $x\mathcal{R}_x = U_x$ for all $x \in X$.

PROOF. Observe that ${}_C\mathcal{R}_x = {}_C\mathcal{R}(x)$ for an equivalence $\leq(C)$ and apply Corollary 3.2 with $\leq(C) = U$.

Theorem 3.5. If $\leq(C)$ is an $R$ quasi-order on $(X,\Lambda,P) \in \mathrm{obj}(GPX)$ with $\leq(C) \subseteq \leq(H)$, then $\leq(C)$ induces a functor

${}_C\mathcal{r}(_):\leq R \to \underline{\text{Mon}}$ defined by ${}_C\mathcal{r}(x) = \{g \in S_D(x): xg \in C(x)\}$ and if $(z,x) \in \text{Mor}(\leq R)$, then ${}_C\mathcal{r}(z,x)(D_y^x) = D^z_{yD_z^x}$ for $D_y^x \in {}_C\mathcal{r}(x)$.

PROOF. Suppress the "C" and denote ${}_C\mathcal{r}$ by $\mathcal{r}$. First note that $D_x^x$ (the identity of $S_D(x)$) is in $\mathcal{r}(x)$ for all $x \in X$. Now to show $\mathcal{r}(x)$ is a semigroup, let $D_y^x, D_z^x \in \mathcal{r}(x) \subseteq S_D(x)$. Then $D_y^x \circ D_z^x \in S_D(x)$. Now $C(x)(D_y^x \circ D_z^x) \subseteq C(x)D_z^x \subseteq C(x)$ by Lemma 3.1. Therefore $D_y^x \circ D_z^x \in \mathcal{r}(x)$ by Lemma 3.1. Thus $\mathcal{r}(x)$ is a monoid (since $D_x^x \in \mathcal{r}(x)$) for every $x \in X$.

Now to show $\mathcal{r}(z,x)(D_y^x) \in \mathcal{r}(z)$ for $(z,x) \in \text{Mor}(\leq R)$ and $D_y^x \in \mathcal{r}(x)$, note that $(y,x) = (xD_y^x, x) \in \leq(C)$. Now since $\leq(C)$ is an $\mathcal{R}$ quasi-order under $\leq(H)$ and $(y,x) \in \leq(C)$, $(yD_z^x, z) = (yD_z^x, xD_z^x) \in \leq(C) \subseteq \leq(H)$. Hence $D^z_{yD_z^x} \in S_D(x)$. Since $zD^z_{yD_z^x} = yD_z^x \in C(z)$, $D^z_{yD_z^x} \in \mathcal{r}(z)$.

To show $\mathcal{r}(z,x)$ is a morphism in $\underline{\text{Mon}}$, first note that $\mathcal{r}(z,x)(D_x^x) = D^z_{xD_z^x} = D_z^z$. Thus $\mathcal{r}(z,x)$ is identity preserving. To show $\mathcal{r}(z,x)$ is a homomorphism, let $D_y^x, D_w^x \in \mathcal{r}(x)$. Now

$$\begin{aligned}
\mathcal{r}(z,x)(D_y^x \circ D_w^x) &= \mathcal{r}(z,x)(D^x_{yD_w^x}) \\
&= D^z_{yD_w^xD_z^x} \\
&= D^z_{yD^x_{wD_z^x}} \\
&= D^z_{yD_z^x \circ D^z_{wD_z^x}} \\
&= D^z_{yD_z^x \circ D^z_{wD_z^x}} \\
&= \mathcal{r}(z,x)(D_y^x) \circ \mathcal{r}(z,x)(D_w^x).
\end{aligned}$$

To show $\mathcal{R}$ preserves composition of morphisms, let $(z,x),(u,z) \in \mathrm{Mor}(\underline{\leq}R)$ and let $D^x_y \in \mathcal{R}(x)$. Thus

$$\begin{aligned}\mathcal{R}((u,z)\circ(z,x))(D^x_y) &= \mathcal{R}(u,x)(D^x_y)\\ &= D^u_{yD^x_u}\\ &= D^u_{yD^x_zD^z_u}\\ &= \mathcal{R}(u,z)(D^z_{yD^x_z})\\ &= \mathcal{R}(u,z)(\mathcal{R}(z,x)(D^x_y)).\end{aligned}$$

Therefore $\mathcal{R}$ preserves composition of morphisms. Also note that $\mathcal{R}(x,x)(D^x_y) = D^x_{yD^x_x} = D^x_y$ and $\mathcal{R}(x,x) = 1_{\mathcal{R}(x)}$. Thus $\mathcal{R}:\underline{\leq}R \to \underline{\mathrm{Mon}}$ is a functor.

Corollary 3.6. If $\underline{\leq}(U)$ is an $R$ quasi-order on $(X,\Lambda,P)$ with $\underline{\leq}(U) \subseteq \underline{\leq}(H)$, then $U = \underline{\leq}(U) \cap (\underline{\leq}(U))^{-1}$ induces a functor ${}_u\mathcal{R}_- :\underline{\leq}R \to \underline{\mathrm{Grp}}$ where ${}_u\mathcal{R}_x = \{g \in S_D(x) : xg \in U_x\}$ and ${}_u\mathcal{R}(z,x)(D^x_y) = D^z_{yD^x_z}$ for $D^x_y \in {}_u\mathcal{R}_x$.

PROOF. Apply Theorem 3.5 with $\underline{\leq}(C) = U$. This gives ${}_u\mathcal{R}_- :\underline{\leq}R \to \underline{\mathrm{Mon}}$. Hence it suffices to show ${}_u\mathcal{R}_x$ is a group for each $x \in X$. Let $D^x_y \in {}_u\mathcal{R}_x$. Thus $(y,x) \in U \subseteq H$ and $D^x_y \in G_D(x)$. Hence $(D^x_y)^{-1} \in G_D(x)$ and $(x,x(D^x_y)^{-1}) = (y(D^x_y)^{-1},x(D^x_y)^{-1}) \in U$ (since $U$ is an $R$ quasi-order). This gives $(D^x_y)^{-1} \in {}_u\mathcal{R}_x$ by the definition of ${}_u\mathcal{R}_x$. Hence ${}_u\mathcal{R}_x$ is a group.

Note that an equivalent formulation of ${}_u\mathcal{r}_x$ in Corollary 3.6 may be obtained by taking any of the sets in Corollary 3.3. Hence each of these sets are groups. Also note that if one begins with an equivalence relation $U$ on $X$ which is contained in $H$ and $U$ is a $R$ quasi-order, then Corollary 3.6 may be applied. In particular it follows as a corollary of Corollary 3.6 that the results of Corollary 3.6 hold for any quasi-congruence under $H$ when one considers a fixed $(S,\Lambda,P) \in \text{obj}(GP)$ . This fact will be used later in this chapter.

Theorem 3.7. If $\leq(C)$ is an $H$ quasi-order on $(X,\Lambda,P)$ with $\leq(C) \subset \leq(H)$ , then $\leq(C)$ induces a functor ${}_C\mathcal{l}(_):\underline{\leq L} \to \underline{\text{Mon}}$ defined by ${}_C\mathcal{l}(x) = {}_C\mathcal{r}(x)$ for all $x \in \text{obj}(\underline{\leq L})$ and if $(z,x) \in \text{Mor}(\underline{\leq L})$ , then $\mathcal{l}(z,x)(D^x_y) = D^z_{zD^x_y}$ for $D^x_y \in {}_C\mathcal{l}(x)$ .

PROOF. The proof is dual to that of Theorem 3.5 in the following sense. If $D^x_y \in {}_C\mathcal{l}(x)$ and $(z,x) \in \text{Mor}(\underline{\leq L})$ , then by the dual of Theorem 3.5 $G^z_{G^x_{zy}}$ is in $G_G(z)$ (i.e., $G^z_{G^x_{zy}}$ is the image of $G^x_y$ under the dual of $\mathcal{r}(z,x)$ ). Hence $zD^x_y = G^x_{zy} \leq(H) z$ . Now if one considers the functorial isomorphism $D^x_y \xmapsto{\theta(x)} G^x_y$ from $G_D(x)$ to $G_G(x)$ and lets ${}_C\overline{\mathcal{l}}(_)$ denote the dual functor of ${}_C\mathcal{r}(_)$ , then one has the following commutative diagram for $(w,z),(z,x) \in \text{Mor}(\underline{\leq L})$ (with "C" suppressed):

$$\begin{array}{ccccc}
\mathcal{l}(x) & & \mathcal{l}(x) & & \mathcal{l}(x) \\
\| & & \| & & \| \\
\mathcal{r}(x) & \xrightarrow{\mathcal{l}(z,x)} & \mathcal{r}(z) & \xrightarrow{\mathcal{l}(w,z)} & \mathcal{r}(w) \\
{\scriptstyle\theta(x)}\downarrow & & & & \uparrow{\scriptstyle\theta(x)^{-1}} \\
\overline{\mathcal{l}}(x) & \xrightarrow{\overline{\mathcal{l}}(z,x)} & \overline{\mathcal{l}}(z) & \xrightarrow{\overline{\mathcal{l}}(w,z)} & \overline{\mathcal{l}}(w)
\end{array} \quad .$$

Using this diagram, it easily follows that ${}_C\mathcal{l}(_)$ is a functor.

Corollary 3.8. If $\leq(U)$ is an $H$ quasi-order on $(X,\Lambda,P)$ with $\leq(U) \subseteq \leq(H)$, then $U = \leq(U) \cap (\leq(U))^{-1}$ induces a functor ${}_u\ell_- : {\leq}L \to \underline{Grp}$ defined by ${}_u\ell_x = {}_u r_x$ and ${}_u\ell(z,x)(D^x_y) = D^z_{zD^x_y}$ for $D^x_y \in {}_u\ell_x$.

PROOF. The proof is similar to that of Corollary 3.6 and is omitted.

Proposition 3.9. Let $\leq(C)$ be an $H$ quasi-order under $\leq(H)$. Let ${}_C\ell_- : {\leq}L \to \underline{Grp}$ be the functor in Corollary 3.8 induced by $C$. If $(z,w) \in C$ and $x \in X$ such that $(z,x) \in Mor(\leq L)$, then $\ell(z,x) = \ell(w,x)$.

PROOF. Since $(z,w) \in C \subseteq H$ and $(z,x) \in \leq(L)$, $(w,x) \in \leq(L)$. Now if $D^x_y \in \ell_x$, then $\ell(z,x)(D^x_y) = D^z_{zD^x_y}$ and $\ell(w,x)(D^x_y) = D^w_{wD^x_y}$. To show $D^z_{zD^x_y} = D^w_{wD^x_y}$, it suffices to show $zD^w_{wD^x_y} = zD^x_y$. Now

$$\begin{aligned} zD^w_{wD^x_y} &= (G^w_z w)D^w_{wD^x_y} \\ &= G^w_z(wD^w_{wD^x_y}) \\ &= G^w_z(wD^x_y) \\ &= (G^w_z w)D^x_y \\ &= zD^x_y . \end{aligned}$$

Therefore $\ell(z,x) = \ell(w,x)$.

The next results leading to Theorem 3.13 establish a relationship between $R$ quasi-orders under $\leq(H)$ and functors from ${\leq}R$ into $\underline{Grp}$.

Proposition 3.10. Let $\leq(C)$ be an $R$ quasi-order under $\leq(H)$ for $(X,\Lambda,P) \in \text{obj}(GPX)$. If $\underline{\pi} : \underline{\leq R} \to \underline{Grp}$ is the functor induced by $C$ and if $u: \underline{\leq R} \to \underline{Grp}$ is a subfunctor of $\underline{\pi}$, then $u$ is a normal subfunctor of $\underline{\pi}$.

PROOF. Since $u$ is a subfunctor of $\underline{\pi}$, $u(x)$ is a subgroup of $\pi_x$ for each $x \in X$. Hence to show $u$ is a normal subfunctor of $\underline{\pi}$, one must show $D_z^x\, u(x)(D_z^x)^{-1} \subseteq u(x)$ for all $D_z^x \in \pi_x$. Let $D_y^x \in u(x)$ and $D_z^x \in \pi_x$. Hence $(z,x),(y,x) \in C$. Now by Lemma 6.5 part (b) of [3], $D^z_{zD^x_y} = D^x_y$. Thus $D^x_y = D^z_{zD^x_y} \in \pi_z$ and $\pi(x,z)(D^z_{zD^x_y}) = D^x_{zD^x_yD^z_x}$. Now

$$\begin{aligned} D^x_z D^x_y (D^x_z)^{-1} &= D^x_{zD^x_y}(D^x_z)^{-1} \\ &= D^x_{zD^x_y} D^z_x \\ &= D^x_{zD^x_y}\, D^x_{xD^z_x} \\ &= D^x_{(zD^x_y)D^x_{xD^z_x}} \\ &= D^x_{(zD^x_y)D^z_x} \\ &= \pi(x,z)(D^z_{zD^x_y}) \\ &= \pi(x,z)(D^x_y)\ . \end{aligned}$$

Since $u$ is a subfunctor of $\underline{\pi}$ and $D^x_y \in u(x)$, $D^x_z D^x_y (D^x_z)^{-1} = \pi(x,z)(D^x_y) = u(x,z)(D^x_y) \in u(x)$. Therefore $D^x_z D^x_y (D^x_z)^{-1} \in u(x)$.

Lemma 3.11. Let $\leq(C_1)$ and $\leq(C_2)$ be $R$ quasi-orders in $(X,\Lambda,P)$ with $\leq(C_1) \subseteq \leq(C_2) \subseteq \leq(H)$. If ${}_1\mathcal{r}$ and ${}_2\mathcal{r}$ are the functors induced by $\leq(C_1)$ and $\leq(C_2)$ respectively, then ${}_1\mathcal{r} \leq {}_2\mathcal{r}$ (${}_1\mathcal{r}$ is a subfunctor of ${}_2\mathcal{r}$) with equality holding iff $\leq(C_1) = \leq(C_2)$.

PROOF. Since $x\ {}_1\mathcal{r}(x) = C_1(x)$ and $x\ {}_2\mathcal{r}(x) = C_2(x)$, one has $\leq(C_1) \subseteq \leq(C_2)$ implies $x\ {}_1\mathcal{r}(x) = C_1(x) \subseteq C_2(x)$. Thus by Lemma 3.1 part (b), ${}_1\mathcal{r}(x) \subseteq {}_2\mathcal{r}(x)$. Now if $D_y^x \in {}_1\mathcal{r}(x) \subseteq {}_2\mathcal{r}(x)$, then ${}_1\mathcal{r}(z,x)(D_y^x) = D^z_{yD^z_x} = {}_2\mathcal{r}(z,x)(D_y^x)$ for all $(z,x) \in \mathrm{Mor}(\leq R)$. Hence ${}_1\mathcal{r}(z,x) = {}_2\mathcal{r}(z,x)\big|_{{}_1\mathcal{r}(x)}$ and ${}_1\mathcal{r} \leq {}_2\mathcal{r}$.

Conversely if ${}_1\mathcal{r}(x) \leq {}_2\mathcal{r}(x)$, then $C_1(x) = x\ {}_1\mathcal{r}(x) \subseteq x\ {}_2\mathcal{r}(x) = C_2(x)$ and $C_1 \subseteq C_2$. Now the rest of the Lemma follows.

Lemma 3.12. Let $\leq(C)$ be an $R$ quasi-order on $(X,\Lambda,P)$ with $\leq(C) \subseteq \leq(H)$. If $D_y^x \in \mathcal{r}(x)$ and $D_z^x \in G_D(x)$, then $(D_z^x)^{-1}D_y^xD_z^x = \mathcal{r}(z,x)(D_y^x)$ and $(D_z^x)^{-1}\mathcal{r}(x)D_z^x = \mathcal{r}(z)$.

PROOF. First note that $(D_z^x)^{-1}D_y^xD_z^x = D_x^zD_y^xD_z^x = D_x^zD_{yD_z^x}^x = D^z_{yD_z^x} = \mathcal{r}(z,x)(D_y^x)$. Hence $(D_z^x)^{-1}\mathcal{r}(x)D_z^x \subseteq \mathcal{r}(z)$. Now apply this with $D_x^z$ and $\mathcal{r}(z)$ to get $(D_x^z)^{-1}\mathcal{r}(z)D_x^z \subseteq \mathcal{r}(x)$. Hence $\mathcal{r}(z) = (D_z^x)^{-1}(D_x^z)^{-1}\mathcal{r}(z)D_x^zD_z^x \subseteq (D_z^x)^{-1}\mathcal{r}(x)D_z^x \subseteq \mathcal{r}(z)$. Therefore $(D_z^x)^{-1}\mathcal{r}(x)D_z^x = \mathcal{r}(z)$.

Theorem 3.13. Let $G$ denote the maximal $R$ equivalence on $(X,\Lambda,P) \in \mathrm{obj}(GPX)$ contained in $H$. Then the mapping $C \mapsto {}_C\mathcal{r}$ is an

order isomorphism of the lattice of $R$ equivalences under $H$ with the lattice of normal subfunctors of ${}_G\mathcal{A}_-$ .

PROOF. According to Lemma 3.11 the mapping $C \to {}_C\mathcal{A}_-$ is a one-to-one order preserving map from the lattice of $R$ equivalences under $G$ into the lattice of subfunctors of ${}_G\mathcal{A}_-$ . By Proposition 3.10, a subfunctor of ${}_G\mathcal{A}_-$ is also a normal subfunctor of ${}_G\mathcal{A}_-$ . Hence to finish the proof of the theorem it suffices to show the mapping $C \mapsto {}_C\mathcal{A}_-$ is surjective. Thus let ${}_1\mathcal{A} \leq {}_G\mathcal{A}$ . We proceed to show that the sets $\{x\cdot{}_1\mathcal{A}(x)\}$ for $x \in X$ form a partition of $X$ which is an $R$ equivalence on $(X,\Lambda,P)$ . If $y \in x\cdot{}_1\mathcal{A}(x) \cap z\cdot{}_1\mathcal{A}(z)$ for some $x,z \in X$ , then $y = xD_y^x = zD_y^z$ with $D_y^x \in {}_1\mathcal{A}(x) \subseteq {}_G\mathcal{A}_x$ and $D_y^z \in {}_1\mathcal{A}(z) \subseteq {}_G\mathcal{A}_z$ . Thus $xGyGz$ and $D_z^x \in {}_G\mathcal{A}_x$ . Now by Lemma 3.12 if $D_u^x \in {}_1\mathcal{A}(x) \subseteq {}_G\mathcal{A}_x$ , then $(D_z^x)^{-1}D_u^xD_z^x = {}_G\mathcal{A}(z,x)(D_u^x) = {}_1\mathcal{A}(z,x)(D_u^x) \in {}_1\mathcal{A}(z)$ . Now as in Lemma 3.12, it follows that $(D_z^x)^{-1}{}_1\mathcal{A}(x)D_z^x = {}_1\mathcal{A}(z)$ . Therefore $z\cdot{}_1\mathcal{A}(z) = (xD_z^x)((D_z^x)^{-1}{}_1\mathcal{A}(x)D_z^x) = x\ {}_1\mathcal{A}(x)D_z^x$ . Now $y \in z\cdot{}_1\mathcal{A}(z)$ and $y \in x\cdot{}_1\mathcal{A}(x)$ gives $y \in x{}_1\mathcal{A}(x) \cap x\ {}_1\mathcal{A}(x)D_z^x$ . Since ${}_1\mathcal{A}(x) \subseteq {}_G\mathcal{A}_x \subseteq G_D(x)$ , ${}_1\mathcal{A}(x)D_z^x \subseteq G_D(x)$ , and $x\cdot{}_1\mathcal{A}(x) \cap x\ {}_1\mathcal{A}(x)D_z^x \neq \square$ , ${}_1\mathcal{A}(x) \cap {}_1\mathcal{A}(x)D_z^x \neq \square$ . Hence ${}_1\mathcal{A}(x) = {}_1\mathcal{A}(x)D_z^x$ and $z\cdot{}_1\mathcal{A}(z) = x\cdot{}_1\mathcal{A}(x)$ . Therefore the sets $\{x\cdot{}_1\mathcal{A}(x)\}$ for $x \in X$ form a partition of $X$ .

Now to show this partition is an $R$ equivalence, let $(w,x) \in \leq(R)$ and $u \in x\cdot{}_1\mathcal{A}(x)$ . Thus $(uD_w^x, xD_w^x) = (uD_w^x, w)$ and $uD_w^x = (xD_u^x)D_w^x = w({}_G\mathcal{A}(w,x)(D_u^x)) = w({}_1\mathcal{A}(w,x)(D_u^x)) \in w\ {}_1\mathcal{A}(w)$ . Hence $uD_w^x \in w\cdot{}_1\mathcal{A}(w)$ and it follows that the partition is an $R$ equivalence. Let $C$ denote this $R$ equivalence.

To show ${}_C\mathcal{r}$ is ${}_1\mathcal{r}$, note that for $x \in X$, $x\ {}_1\mathcal{r}(x) = C_x = x\ {}_C\mathcal{r}_x$ where the first equality is by the definition of $C$ and the second is by Corollary 3.4. Thus ${}_1\mathcal{r}(x) = {}_C\mathcal{r}_x$. Now ${}_1\mathcal{r}(z,x) = {}_G\mathcal{r}(z,x)\big|_{{}_1\mathcal{r}(x)} = {}_G\mathcal{r}(z,x)\big|_{{}_C\mathcal{r}_x} = {}_C\mathcal{r}(z,x)$ for $(z,x) \in \mathrm{Mor}(\underline{\leq}R)$. Thus ${}_1\mathcal{r} = {}_C\mathcal{r}$ and the mapping $C \mapsto {}_C\mathcal{r}$ is surjective.

Now note that the pair $(\mathcal{l},\mathcal{r})$ of functors described in the previous results would be more precisely stated as the pair of functors associated with $D$. Hence one could also associate a pair of functors with $G$ in a dual fashion.

Thus far in this chapter the results have dealt only with the category GPX. However for each of these results one has a corresponding corollary in the category GP where one replaces statements such as "$\underline{\leq}(C)$ an $R$ quasi-order under $\underline{\leq}(H)$" by "$\underline{\leq}(c)$ an $R$ quasi-congruence under $\underline{\leq}(H)$." Hence in what follows the previous results will be applied both in the category GPX and GP even though they have not been explicitly stated.

The following results obtain an analogy to the Main Homomorphism Theorem in Sgp for the categories GPX and GP. Once this has been obtained, the reverse situation is investigated, i.e., the case of coextensions. First some definitions are needed.

Let $(X,\Lambda,P),(\overline{X},\overline{\Lambda},\overline{P}) \in \mathrm{obj}(\mathrm{GPX})$ [$\mathrm{obj}(\mathrm{GP})$]; a surmorphism $f:(X,\Lambda,P) \to (\overline{X},\overline{\Lambda},\overline{P})$ is a morphism in GPX [GP] such that $f\times f(\underline{\leq}(L)) = \underline{\leq}(\overline{L})$ and $f\times f(\underline{\leq}(R)) = \underline{\leq}(\overline{R})$. If $C$ is an $H$ equivalence (an $H$ quasi-order which is an equivalence) on $(X,\Lambda,P)$, then let $\pi:X \to X/C$ denote the projection of $X$ onto $X/C$. In the following

let $\mathcal{B}\circ\mathcal{A}(X,\Lambda,P)$ be denoted by $(X,\Lambda',P')$ . For each $\lambda' \in \Lambda'$ define $\underline{\lambda}':\pi(\text{dom } \lambda') \to X/C$ by $\underline{\lambda}'(\pi x) = \pi(\lambda' x)$ for all $x \in \text{dom } \lambda'$ . Dually, for each $\rho' \in P'$ define $\underline{\rho}':\pi(\text{dom } \rho') \to X/C$ by $(\pi x)\underline{\rho}' = \pi(x\rho')$ for all $x \in \text{dom } \rho'$ . Let $\Lambda/C$ denote the set $\{\underline{\lambda}':\lambda' \in \Lambda'\}$ , and let $P/C$ denote the set $\{\underline{\rho}':\rho' \in P'\}$ . Whenever $(S,\Lambda,P) \in \text{obj}(GP)$ and $C$ is a congruence on $(S,\Lambda,P)$ , $(S/C,\Lambda/C,P/C)$ is obtained in exactly the same fashion.

Lemma 3.14. Let $\leq(C)$ be an $H$ quasi-order on $(X,\Lambda,P) \in \text{obj}(GPX)$ with $\leq(C) \subseteq \leq(H)$ . Then $(X/C,\Lambda/C,P/C) \in \text{obj}(GPX)$ and $\pi:(X,\Lambda,P) \to (X/C,\Lambda/C,P/C)$ is a surmorphism.

PROOF. First $(X/C,\Lambda/C,P/C)$ is shown to be in $\text{obj}(GPX)$ . To show each $\underline{\lambda}' \in \Lambda/C$ is well-defined suppose $\pi x = \pi y$ for some $x,y \in X$ and $x \in \text{dom } \lambda'$ . Hence $xCy$ and $\lambda' x C \lambda' y$ , i.e, $\pi(\lambda' x) = \pi(\lambda' y)$ , since $C$ is an $H$ equivalence. Thus $\underline{\lambda}'(\pi y) = \pi(\lambda' y) = \pi(\lambda' x) = \underline{\lambda}'(\pi x)$ . Hence each $\underline{\lambda}' \in \Lambda/C$ is well-defined. Dually, each $\underline{\rho}' \in P/C$ is well-defined. Now note that (i) and (ii) in the definition of a Green's pair are clearly satisfied. Now to show parts (iii) and (iv) let $(\underline{\lambda}',\underline{\rho}') \in \Lambda' \times P'$ and let $\pi x \in \text{dom } \underline{\lambda}' \cap \text{dom } \underline{\rho}'$ , then $C \subseteq H$ implies $x \in \pi^{-1}(\pi x) \subseteq \text{dom } \lambda'$ and $x \in \pi^{-1}(\pi x) \subseteq \text{dom } \rho'$ . Thus $x \in \text{dom } \lambda' \cap \text{dom } \rho'$ and $(\lambda' x)\rho' = \lambda'(x\rho')$ . Therefore $(\underline{\lambda}'(\pi x)\underline{\rho}') = (\pi(\lambda' x))\underline{\rho}' = \pi((\lambda' x)\rho') = \pi(\lambda'(x\rho')) = \underline{\lambda}'(\pi(x\rho')) = \underline{\lambda}'((\pi x)\underline{\rho}')$ . Hence conditions (iii) and (iv) are satisfied and $(X/C,\Lambda/C,P/C) \in \text{obj}(GPX)$ .

Now to show $\pi$ is a surmorphism, first note that by the definition of $(X/C,\Lambda/C,P/C)$ and $\pi:X \to X/C$, $\pi:(X,\Lambda',P') \to (X/C,\Lambda/C,P/C)$ is a morphism in GPX. Now $\pi(L'(x)) = \pi(\Lambda'x) = \Lambda/C(\pi x)$ and $\pi(R'(x)) = (\pi x)P/C$ for all $x \in X$. Therefore $\pi:(X,\Lambda',P') \to (X/C,\Lambda/C,P/C)$ is a surmorphism. Since $1_X:(X,\Lambda,P) \to (X,\Lambda',P') = \mathcal{B}\circ\mathcal{A}(X,\Lambda,P)$ is an isomorphism, $\pi = \pi\circ 1_X:(X,\Lambda,P) \to (X/C,\Lambda/C,P/C)$ is a surmorphism.

Proposition 3.15. Let $\underline{\leq}(C)$ be a quasi-congruence on $(S,\Lambda,P) \in \text{obj}(GP)$ with $\underline{\leq}(C) \subseteq \underline{\leq}(H)$, then $(S/C,\Lambda/C,P/C) \in \text{obj}(GP)$ and $\pi:(S,\Lambda,P) \to (S/C,\Lambda/C,P/C)$ is a surmorphism.

PROOF. Since $\pi:(S,\Lambda,P) \to (S/C,\Lambda/C,P/C)$ has already been shown to be a surmorphism in GPX by Lemma 3.14, it suffices to show that condition (v) in the definition of a Green's pair for a semigroup holds and that $\pi:S \to S/C$ is a homomorphism. Clearly $\pi$ is a homomorphism. Now if $\pi x,(\pi y)(\pi x) \in \text{dom}\ \underline{\rho}'$ for some $\underline{\rho}' \in P/C$, then $x,yx \in \text{dom}\ \rho'$ since $C \subseteq H$. Thus $((\pi y)(\pi x))\underline{\rho}' = (\pi(yx))\underline{\rho}' = \pi((yx)\rho') = \pi(y(x\rho')) = (\pi y)(\pi(x\rho')) = (\pi y)((\pi x)\underline{\rho}')$. Dually, the condition follows for $\Lambda/C$. Hence $(S/C,\Lambda/C,P/C) \in \text{obj}(GP)$.

Proposition 3.16. If $p:(X,\Lambda,P) \to (\overline{X},\overline{\Lambda},\overline{P})$ is a surmorphism in Mor(GPX) and if $C_p$ is the relation $\{(x,y) \in X\times X:p(x) = p(y)\}$ in $X$ induced by $p:X \twoheadrightarrow \overline{X}$, then $C_p$ is an $H$ equivalence on $(X,\Lambda,P)$. If $C_p \subseteq H$, then $(X/C_p,\Lambda/C_p,P/C_p) \cong (\overline{X},\overline{\Lambda},\overline{P})$.

PROOF. To show $C_p$ is an $H$ equivalence on $(X,\Lambda,P)$ it suffices to show $C_p$ is $(\Lambda',P')$ compatible since clearly $C_p$ is an equivalence on $X$. If $xC_py$ and $x,y \in \text{dom } \lambda'$ for some $\lambda' \in \Lambda'$, then $p(x) = p(y)$ and there is a $\overline{\lambda} \in \overline{\Lambda}$ such that $p(\lambda'z) = \overline{\lambda}p(z)$ for all $z \in \text{dom } \lambda'$ (see Proposition 2.9). Hence $p(\lambda'x) = \overline{\lambda}p(x) = \overline{\lambda}p(y) = p(\lambda'y)$ and $\lambda'xC_p\lambda'y$. Dually if $xC_py$ and $x,y \in \text{dom } \rho'$ for some $\rho' \in P'$, then $x\rho'C_py\rho'$. Hence $C_p$ is $(\Lambda',P')$ compatible.

Now if $C_p \subseteq H$, then by Lemma 3.14 $(X/C_p,\Lambda/C_p,P/C_p) \in \text{obj}(GPX)$. Let $\pi:X \to X/C_p$ denote the projection, and let $\phi:X/C_p \to \overline{X}$ be the induced isomorphism. Now to show $\phi:(X/C_p,\Lambda/C_p,P/C_p) \to (\overline{X},\overline{\Lambda},\overline{P})$ is an isomorphism one must show $\phi^{-1},\phi \in \text{Mor}(GPX)$. Let $\underline{\lambda}' \in \Lambda/C_p$. Thus $\underline{\lambda}'(\pi z) = \pi(\lambda'z)$ for all $z \in \text{dom } \lambda'$. Since $p \in \text{Mor}(GPX)$ with $p:(X,\Lambda,P) \to (\overline{X},\overline{\Lambda},\overline{P})$, there is a $\overline{\lambda} \in \overline{\Lambda}$ such that $\overline{\lambda}p(z) = p(\lambda'z)$ for all $z \in \text{dom } \lambda'$. Hence $\phi(\underline{\lambda}'(\pi z)) = \phi(\pi(\lambda'z)) = p(\lambda'z) = \overline{\lambda}(pz) = \overline{\lambda}(\phi(\pi z))$ for all $\pi z \in \text{dom } \underline{\lambda}'$. Dually $\phi((\pi z)\underline{\rho}') = (\phi\pi z)\overline{\rho}$ for all $\pi z \in \text{dom } \underline{\rho}'$. Thus $\phi \in \text{Mor}(GPX)$.

Now to show $\phi^{-1} \in \text{Mor}(GPX)$, let $\overline{\lambda} \in \overline{\Lambda}$ and $y \in \text{dom } \overline{\lambda}$. Since $p:X \twoheadrightarrow \overline{X}$, there is an $x \in X$ such that $p(x) = y$. Since $p \times p(\underline{\leq}(L)) = \underline{\leq}(\overline{L})$, there is a $\lambda' \in \Lambda'$ such that $p(\lambda'x) = \overline{\lambda}y = \overline{\lambda}p(x)$. Now $p \in \text{Mor}(GPX)$ implies there is a $\lambda_1 \in \overline{\Lambda}$ such that $p(\lambda'z) = \lambda_1 p(z)$ for all $z \in R(x)$. In particular, $\overline{\lambda}p(x) = p(\lambda'x) = \lambda_1 p(x)$ and by the proof of Theorem 5.1 of [3] $\overline{\lambda}z = \lambda_1 z$ for every $z \underline{\leq}(\overline{R})p(x)$. Therefore $p(\lambda'u) = \lambda_1 p(u) = \overline{\lambda}p(u)$ for every $u \underline{\leq}(R)x$. Hence

$\phi^{-1}(\overline{\lambda}p(u)) = \phi^{-1}(p(\lambda'u)) = (\phi^{-1}\circ p)(\lambda'u) = \pi(\lambda'u) = \underline{\lambda}'(\pi u) = \underline{\lambda}'(\phi^{-1}p(u))$ for every $u\leq(R)x$ . The dual condition follows for $\overline{P}$ . Thus $\phi^{-1} \in \mathrm{Mor}(GPX)$ and $\phi$ is an isomorphism in $\mathrm{Mor}(GPX)$ .

Corollary 3.17. If $p:(S,\Lambda,P) \to (\overline{S},\overline{\Lambda},\overline{P})$ is a surmorphism in $\mathrm{Mor}(GP)$ and if $C_p$ is the congruence on $S$ induced by $p:S \twoheadrightarrow \overline{S}$ , then $C_p$ is a congruence on $(S,\Lambda,P)$ . If $C_p \subseteq H$ , then $(S/C_p,\Lambda/C_p,P/C_p) \cong (\overline{S},\overline{\Lambda},\overline{P})$ .

PROOF. Since $p:S \twoheadrightarrow \overline{S}$ is a homomorphism, $C_p$ is a congruence on $S$ . Now by Proposition 3.16, $C_p$ is $(\Lambda',P')$ compatible. Hence $C_p$ is a congruence on $(S,\Lambda,P)$.

Now assume $C_p \subseteq H$ . The induced mapping $\Phi:S/C_p \to \overline{S}$ is an isomorphism in $\mathrm{Mor}(\underline{\mathrm{Sgp}})$ . Now Proposition 3.16 and $\Phi$ an isomorphism in $\mathrm{Mor}(\underline{\mathrm{Sgp}})$ give $\Phi,\Phi^{-1} \in \mathrm{Mor}(GP)$ . Thus $\Phi$ is an isomorphism in $\mathrm{Mor}(GP)$ .

Corollary 3.18. If $p:(S,\Lambda,P) \to (\overline{S},\overline{\Lambda},\overline{P})$ is a surmorphism in $\mathrm{Mor}(GPX)$ with $(S,\Lambda,P),(\overline{S},\overline{\Lambda},\overline{P}) \in \mathrm{obj}(GPX)$ [$\mathrm{Mor}(GP)$ with $(S,\Lambda,P),(\overline{S},\overline{\Lambda},\overline{P}) \in \mathrm{obj}(GP)$] and if $C_p = \Delta$ (the diagonal of $S\times S$ , i.e., $p$ is one-to-one), then $(S,\Lambda,P) \cong (\overline{S},\overline{\Lambda},\overline{P})$ .

PROOF. By Proposition 3.16 [Corollary 3.17] $(S/C_p,\Lambda/C_p,P/C_p) \cong (\overline{S},\overline{\Lambda},\overline{P})$ . But clearly $C_p = \Delta$ implies $(S/C_p,\Lambda/C_p,P/C_p) \cong (S,\Lambda,P)$ . Hence $(S,\Lambda,P) \cong (\overline{S},\overline{\Lambda},\overline{P})$ .

The following begins a description of certain types of coextensions. One type corresponds to groups and one corresponds to

monoids. The results will be broken into essentially four major parts. These are (a) the set theoretical results needed to construct certain coextensions, (b) the semigroup results needed to construct certain coextensions, (c) the set theoretical results used to reconstruct certain coextensions, and (d) the semigroup results used to reconstruct certain coextensions.

In the following fix $(X,\Lambda_1,P_1) \in \text{obj}(GPX)$ and let $(X,\Lambda,P) = B\circ A(X,\Lambda_1,P_1)$. Let $GPX_X$ denote the subcategory of $GPX$ with $((\overline{X},\overline{\Lambda},\overline{P}),p) \in \text{obj}(GPX_X)$ iff $p$ is a surmorphism from $(\overline{X},\overline{\Lambda},\overline{P})$ to $(X,\Lambda_1,P_1)$ where $(\overline{X},\overline{\Lambda},\overline{P}) \in \text{obj}(GPX)$ and $f \in \text{Mor}(GPX_X)$ iff $f:(\overline{X},\overline{\Lambda},\overline{P}) \to (\tilde{X},\tilde{\Lambda},\tilde{P})$ is in $\text{Mor}(GPX)$ with $\tilde{p}\circ f = p$ where $((\tilde{X},\tilde{\Lambda},\tilde{P}),\tilde{p}) \in \text{obj}(GPX_X)$. The category $GPX_X$ will be referred to as the <u>coextensions</u> of $(X,\Lambda_1,P_1)$. If $((\overline{X},\overline{\Lambda},\overline{P}),p) \in \text{obj}(GPX_X)$, then $(\overline{X},\overline{\Lambda},\overline{P})$ is said to be a $\leq(G)$-coextension [$\leq(S)$-coextension] of $(X,\Lambda,P)$ iff $C_p \subseteq \overline{H}$ [there is a mapping denoted by "—", from $X$ to $\overline{X}$ which preserves the $\leq(L)$ and $\leq(R)$ quasi-orders, $p(\overline{x}) = x$, and $(C_p)_{\overline{x}} \subseteq \overline{H}(\overline{x})$ for all $x \in X$]. Let $\leq(G_X)$ [$\leq(S_X)$] denote the full subcategory of $GPX_X$ whose objects consist of all $\leq(G)$-coextensions [$\leq(S)$-coextensions] of $(X,\Lambda_1,P_1)$. Note that the definition of a $\leq(G)$-coextension may be rephrased to "there is a mapping '—' from $X$ to $\overline{X}$ such that $p(\overline{x}) = x$ and $(C_p)_{\overline{x}} \subseteq \overline{H}_{\overline{x}}$ for all $x \in X$." Hence the similarity of the two definitions can be more readily seen. In particular one has that each $\leq(G)$-coextension is also a $\leq(S)$-coextension. Note also that if $p:(\overline{X},\overline{\Lambda},\overline{P}) \to (X,\Lambda_1,P_1)$ is a surmorphism, then since $1_X:(X,\Lambda_1,P_1) \to (X,\Lambda,P)$ is an isomorphism $p:(\overline{X},\overline{\Lambda},\overline{P}) \to (X,\Lambda,P)$ is also a surmorphism. This fact will be used often without mentioning.

A pair of functors $\ell:\underline{\leq}L \to \underline{\text{Mon}}$ and $r:\underline{\leq}R \to \underline{\text{Mon}}$ is said to be compatible if for all $x$ in $X$, $\ell(x) = r(x)$. A factor system for $\ell$ is a function

$$\alpha:B \to \bigcup_{x\in X} \ell(x)$$

where

$$B = \{(\lambda,x):\lambda \in \Lambda \text{ and } x \in \text{dom } \lambda\} \cup \{(x,\rho):\rho \in P \text{ and } x \in \text{dom } \rho\}$$

such that $\alpha(a,b) \in \ell(ab)$ for all $(a,b) \in B$. Let $B_1$ denote the set $\{(\lambda,x):\lambda \in \Lambda \text{ and } x \in \text{dom } \lambda\}$, and let $B_2$ denote the set $\{(x,\rho):\rho \in P \text{ and } x \in \text{dom } \rho\}$.

If $((\tilde{X},\tilde{\Lambda}_1,\tilde{P}_1),p) \in \text{obj}(\underline{\leq}(G_X))$ $[\text{obj}(\underline{\leq}(S_X))]$, then $(\mathcal{B}\circ A(\tilde{X},\tilde{\Lambda}_1,\tilde{P}_1),p) \equiv ((\tilde{X},\tilde{\Lambda},\tilde{P}),p)$ is in $\text{obj}(\underline{\leq}(G_X))$ $[\text{obj}(\underline{\leq}(S_X))]$ (since $1_{\tilde{X}}:(\tilde{X},\tilde{\Lambda}_1,\tilde{P}_1) \to (\tilde{X},\tilde{\Lambda},\tilde{P})$ is an isomorphism) and a lifting from $(X,\Lambda,P)$ to $(\tilde{X},\tilde{\Lambda},\tilde{P})$ is a mapping, denoted by "~", from $X \cup \Lambda \cup P$ to $\tilde{X} \cup \tilde{\Lambda} \cup \tilde{P}$ with the following three properties.

(a) for all $x \in X$, $p(\tilde{x}) = x$ (with $\tilde{x} \in \tilde{X}$) and
$(C_p)_{\tilde{x}} \subseteq \tilde{H}_{\tilde{x}}$ ["~" preserves $\underline{\leq}(L)$ and $\underline{\leq}(R)$ and
$(C_p)_{\tilde{x}} \subseteq \tilde{H}(\tilde{x})]$

(b) if $\lambda \in \Lambda$ and $\lambda = G^{u}_{v}$, then $\tilde{\lambda} = \tilde{G}^{\tilde{u}}_{\tilde{v}}$; if $\lambda \in \Lambda$ and $\lambda = G^{u_1}_{v_1}\circ G^{u_1}_{v_2}\circ \cdots \circ G^{u_n}_{v_n}$, then $\tilde{\lambda} = \tilde{G}^{\tilde{u}_1}_{\tilde{v}_2}\circ \cdots \circ \tilde{G}^{\tilde{u}_n}_{\tilde{v}_n}$

(c) if $\rho \in P$ and $\rho = D^{u}_{v}$, then $\tilde{\rho} = \tilde{D}^{\tilde{u}}_{\tilde{v}}$; if $\rho \in P$ and $\rho = D^{u_1}_{v_1}\circ \cdots \circ D^{u_n}_{v_n}$, then $\tilde{\rho} = \tilde{D}^{\tilde{u}_1}_{\tilde{v}_1}\circ \cdots \circ \tilde{D}^{\tilde{u}_n}_{\tilde{v}_n}$.

Proposition 3.19. If $((\tilde{X},\tilde{\Lambda},\tilde{P})p) \in obj(\underline{\leq}(G_X))$ , then $(p\times p)^{-1}[\underline{\leq}(L)] = \underline{\leq}(\tilde{L})$ and $(p\times p)^{-1}[\underline{\leq}(R)] = \underline{\leq}(\tilde{R})$ .

PROOF. Since $p$ is a surmorphism, $(p\times p)[\underline{\leq}(\tilde{L})] = \underline{\leq}(L)$ . Hence $\underline{\leq}(\tilde{L}) \subseteq (p\times p)^{-1}[\underline{\leq}(L)]$ . Now if $(u,v) \in (p\times p)^{-1}[\underline{\leq}(L)]$ , then $(p(u),p(v)) \in \underline{\leq}(L)$ . Since $(p\times p)[\underline{\leq}(\tilde{L})] = \underline{\leq}(L)$ , there is a pair $(w,z) \in \underline{\leq}(\tilde{L})$ such that $p(w) = p(u)$ and $p(z) = p(v)$ . Now $C_p \subseteq \tilde{H}$ and $p(w) = p(u)$ and $p(z) = p(v)$ implies $u\tilde{H}w \underline{\leq}(\tilde{L})z\tilde{H}v$ . Therefore $u \underline{\leq}(\tilde{L})v$ . Hence $(p\times p)^{-1}[\underline{\leq}(L)] \subseteq \underline{\leq}(\tilde{L})$ and $(p\times p)^{-1}[\underline{\leq}(L)] = \underline{\leq}(\tilde{L})$ . Dually $(p\times p)^{-1}[\underline{\leq}(R)] = \underline{\leq}(\tilde{R})$ .

Clearly Proposition 3.19 implies the existence of a lifting from $(X,\Lambda,P)$ to $(\tilde{X},\tilde{\Lambda},\tilde{P})$ whenever $((\tilde{X},\tilde{\Lambda},\tilde{P}),p) \in obj(\underline{\leq}(G_X))$ and $(\tilde{X},\tilde{\Lambda},\tilde{P}) = B\circ A(\tilde{X},\tilde{\Lambda},\tilde{P})$ . Also note in view of the definition of a lifting one may consider a lifting "~" from $(X,\Lambda,P)$ to $(\overline{X},\overline{\Lambda},\overline{P})$ as consisting of a functor from $\underline{\leq}L$ to $\underline{\leq}\tilde{L}$ and a functor from $\underline{\leq}R$ to $\underline{\leq}\tilde{R}$ . That is, for each object $x$ of $\underline{\leq}L = X =$ objects of $\underline{\leq}R$ , $\tilde{x}$ is in $\tilde{X} = obj(\underline{\leq}\tilde{L}) = obj(\underline{\leq}\tilde{R})$ and the definition of a lifting insures that composition of morphisms are preserved. Hence Proposition 3.19 insures the existence of a functor "~" .

The 5-tuple $(X,\Lambda,P,\ell,r,\alpha)$ will denote a system where $\ell:\underline{\leq}L \to \underline{Mon}$ , $r:\underline{\leq}R \to \underline{Mon}$ is a compatible pair of functors with $\alpha$ a factor system for $\ell$ . $\overline{X}$ will denote the set $\bigcup_{a\in X}\{\{a\}\times\ell(a)\}$ .

Definition 3.20. Consider the system $(X,\Lambda,P,\ell,r,\alpha)$ . For each $x \in X$ and $h \in \ell(x)$ , if $x \in dom\ \rho$ with $\rho \in P$ , then define $\overline{\lambda}_h$ by

$$\overline{\lambda}_h(x\rho,g) = (x\rho,\pi(x\rho,x)(h)g) \quad \text{for all} \quad (x\rho,g) \in X .$$

For each $\lambda \in \Lambda$ define $\overline{\lambda}$ by

$$\overline{\lambda}(x,g) = (\lambda x,\alpha(\lambda,x)\ell(\lambda x,x)(g)) \quad \text{for each}$$
$$(x,g) \in \overline{X} \quad \text{with} \quad x \in \operatorname{dom} \lambda .$$

Let $\Lambda_1' = \{\overline{\lambda}_h : h \in \ell(x) \text{ and } x \in X\}$ and $\Lambda_2' = \{\overline{\lambda} : \lambda \in \Lambda\}$. Let $\overline{\Lambda}$ denote the semigroup generated by $\Lambda_1' \cup \Lambda_2' \cup \{1_{\overline{X}}\}$ in $PT(\overline{X})$, i.e.,

$$\overline{\Lambda} = \langle \Lambda_1' \cup \Lambda_2' \cup \{1_{\overline{X}}\} \rangle .$$

Dually, for each $x \in X$ and for each $h \in \pi(x)$, if $x \in \operatorname{dom} \lambda$ with $\lambda \in \Lambda$, then define $\overline{\rho}_h$ by

$$(\lambda x,g)\overline{\rho}_h = (\lambda x,g\ell(\lambda x,x)(h)) \quad \text{for each} \quad (\lambda x,g) \in \overline{X} .$$

For each $\rho \in P$ define $\overline{\rho}$ by

$$(x,g)\overline{\rho} = (x\rho,\pi(x\rho,x)(g)\alpha(x,\rho)) \quad \text{for each}$$
$$(x,g) \in \overline{X} \quad \text{with} \quad x \in \operatorname{dom} \rho .$$

Let $P_1' = \{\overline{\rho}_h : h \in \pi(x) \text{ and } x \in X\}$ and $P_2' = \{\overline{\rho} : \rho \in P\}$. Let $\overline{P}$ denote the semigroup generated by $P_1' \cup P_2' \cup \{1_{\overline{X}}\}$ in $PT(\overline{X})$, i.e.,

$$\overline{P} = \langle P_1' \cup P_2' \cup \{1_{\overline{X}}\} \rangle .$$

Let $(\overline{X},\overline{\Lambda},\overline{P})$ denote the triple described above.

The next three lemmas determine necessary and sufficient conditions on the triple $(\overline{X},\overline{\Lambda},\overline{P})$ so that $(\overline{X},\overline{\Lambda},\overline{P}) \in \operatorname{obj}(GPX)$.

Lemma 3.21. With the notation of Definition 3.20, one has for each $\overline{\lambda} \in \Lambda_2'$, $\overline{\rho} \in P_2'$, and $(y,h) \in \overline{X}$ such that $(y,h) \in \operatorname{dom} \overline{\lambda} \cap \operatorname{dom} \overline{\rho}$, $(\overline{\lambda}(y,h))\overline{\rho} = \overline{\lambda}((y,h)\overline{\rho})$ iff the following condition holds:

$$(3.22) \qquad \begin{aligned} &\alpha(\lambda,y\rho)\ell(\lambda y\rho,y\rho)[\pi(y\rho,y)[h]\alpha(y,\rho)] \\ &= \pi(\lambda y\rho,\lambda y)[\alpha(\lambda,y)\ell(\lambda y,y)[h]]\alpha(\lambda y,\rho) \end{aligned}$$

for all $\lambda \in \Lambda$, for all $y \in X$, and for all $\rho \in P$ such that $y \in \operatorname{dom} \lambda \cap \operatorname{dom} \rho$.

PROOF. Using the notation described in the lemma, one has

$$\begin{aligned} (\overline{\lambda}(y,h))\overline{\rho} &= (\lambda y,\alpha(\lambda,y)\ell(\lambda y,y)[h])\overline{\rho} \\ &= (\lambda y\rho,\pi(\lambda y\rho,\lambda y)[\alpha(\lambda,y)\ell(\lambda y,y)[h]]\alpha(\lambda y,\rho)) \ . \end{aligned}$$

Also one has

$$\begin{aligned} \overline{\lambda}((y,h)\overline{\rho}) &= \overline{\lambda}(y\rho,\pi(y\rho,y)[h]\alpha(y,\rho)) \\ &= (\lambda y\rho,\alpha(\lambda,y\rho)\ell(\lambda y\rho,y\rho)[\pi(y\rho,y)[h]\alpha(y,\rho)]) \ . \end{aligned}$$

Hence $(\overline{\lambda}(y,h))\overline{\rho} = \overline{\lambda}((y,h)\overline{\rho})$ iff 3.22 is true.

Lemma 3.23. For every $\overline{\lambda}_h \in \Lambda_1'$, $\overline{\rho}_k \in P_1'$, and $(y,g) \in \overline{X}$ such that $(y,g) \in \operatorname{dom} \overline{\lambda}_h \cap \operatorname{dom} \overline{\rho}_k$, one has $(\overline{\lambda}_h(y,g))\overline{\rho}_k = \overline{\lambda}_h((y,g)\overline{\rho}_k)$.

PROOF. Since $(y,g) \in \operatorname{dom} \overline{\lambda}_h$, there is a $\rho \in P$ and an $x \in \operatorname{dom} \rho$ such that $y = x\rho$ and $h \in \ell(x)$. Also $(y,g) \in \operatorname{dom} \overline{\rho}_k$

implies there is a $\lambda \in \Lambda$ and a $z \in \text{dom } \lambda$ such that $y = \lambda z$ and $k \in \ell(z)$. Hence

$$\begin{aligned}(\overline{\lambda}_h(y,g))\overline{\rho}_k &= (y,r(x\rho,x)[h]g)\overline{\rho}_k \\ &= (y,r(x\rho,x)[h]g\ell(\lambda z,z)[k]) \\ &= \overline{\lambda}_h(y,g\ell(\lambda z,z)[k]) \\ &= \overline{\lambda}_h((y,g)\overline{\rho}_k) \ .\end{aligned}$$

Lemma 3.24. For every $\overline{\lambda} \in \Lambda_2'$, $\overline{\rho}_h \in P_1'$, and $(y,g) \in \overline{X}$ such that $(y,g) \in \text{dom } \overline{\lambda} \cap \text{dom } \overline{\rho}_h$, one has $(\overline{\lambda}(y,g))\overline{\rho}_h = \overline{\lambda}((y,g)\overline{\rho}_h)$.

PROOF. Since $(y,g) \in \text{dom } \overline{\rho}_h$ there is a $\gamma \in \Lambda$ and an $x \in \text{dom } \gamma$ such that $y = \gamma x$ and $h \in \ell(x)$. Thus $\lambda y = \lambda\gamma x$ and

$$\begin{aligned}(\overline{\lambda}(y,g))\overline{\rho}_h &= (\lambda y,\alpha(\lambda,y)\ell(\lambda y,y)[g])\overline{\rho}_h \\ &= (\lambda y,\alpha(\lambda,y)\ell(\lambda y,y)[g]\ell(\lambda\gamma x,x)[h]) \\ &= (\lambda y,\alpha(\lambda,y)\ell(\lambda y,y)[g]\ell(\lambda y,y)[\ell(y,x)[h]]) \\ &= (\lambda y,\alpha(\lambda,y)\ell(\lambda y,y)[g\ell(y,x)[h]]) \\ &= \overline{\lambda}(y,g\ell(y,x)[h]) \\ &= \overline{\lambda}((y,g)\overline{\rho}_h) \ .\end{aligned}$$

Theorem 3.25. If $(\ell:\underline{\leq}L \to \underline{\text{Mon}}$, $r:\underline{\leq}R \to \underline{\text{Mon}})$ is a pair of compatible functors with a factor system $\alpha$, then the triple $(\overline{X},\overline{\Lambda},\overline{P})$ as defined in 3.20 is in obj(GPX) iff condition (3.22) is satisfied. Also if $(\overline{X},\overline{\Lambda},\overline{P}) \in \text{obj}(GPX)$, then $((\overline{X},\overline{\Lambda},\overline{P}),\pi_1) \in \text{obj}(GPX_X)$ where $\pi_1:\overline{X} \to X$ is the first coordinate projection.

PROOF. If $(\overline{X},\overline{\Lambda},\overline{P}) \in \text{obj}(GPX)$, then by Lemma 3.21 condition (3.22) is satisfied. Now if condition (3.22) is satisfied, then

according to Lemma 5.3 and Corollary 5.5 of [3] it suffices to show that conditions (iii) and (iv) hold between the generating sets of $\overline{\Lambda}$ and $\overline{P}$. Lemmas 3.21, 3.23, 3.24, and the dual of 3.24 do just that. Hence $(\overline{X},\overline{\Lambda},\overline{P}) \in \text{obj}(GPX)$ whenever 3.22 is satisfied.

Clearly it follows from the definition of $\overline{X}$ and $\pi_1$, that $\pi_1:\overline{X} \to X$ is a surmorphism in $\text{Mor}(\underline{\text{Set}})$. Also by the definition of $(\overline{\Lambda},\overline{P})$ and by Lemma 2.6, it follows that $\pi_1:(\overline{X},\overline{\Lambda},\overline{P}) \to (X,\Lambda,P)$ is a surmorphism in $\text{Mor}(GPX)$.

Now that some construction principles have been investigated, the reverse situation is considered. That is, if one starts with a $\underline{\leq}(G)$-coextension, then can this coextension be reconstructed by the above construction?

Lemma 3.26. Let $((\tilde{X},\tilde{\Lambda}_1,\tilde{P}_1),p) \in \text{obj}(\underline{\leq}(G_X))$ and let $\underline{\leq}(C)$ be an $H$ quasi-order under $\underline{\leq}(\tilde{H})$ with $C = C_p$. (Note $\underline{\leq}(C)$ may equal $C_p$.) Let "~" denote a lifting from $(X,\Lambda,P)$ to $(\tilde{X},\tilde{\Lambda},\tilde{P}) = \mathcal{B}\circ\mathcal{A}(\tilde{X},\tilde{\Lambda}_1,\tilde{P}_1)$. (Note such a lifting exists by Lemma 3.19.) Then if $\tilde{\ell}(_):\underline{\leq}\tilde{L} \to \underline{\text{Mon}}$ is the functor defined in Theorem 3.7 by the $H$ quasi-order $\underline{\leq}(C)$ and if $\ell:\underline{\leq}L \to \underline{\text{Mon}}$ is defined by $\ell(x) = \tilde{\ell}(\tilde{x})$ for all $x \in X$ and $\ell(z,x) = \tilde{\ell}(\tilde{z},\tilde{x})$ for all $(z,x) \in \underline{\leq}(L)$, then $\ell$ is a functor.

PROOF. As noted earlier one may view the restriction of a lifting to the objects of $\underline{\leq}L$ as a functor from $\underline{\leq}L$ into $\underline{\leq}\tilde{L}$. Hence $\ell:\underline{\leq}L \to \underline{\text{Mon}}$ is just the composition of the functor $\tilde{}:\underline{\leq}L \to \underline{\leq}\tilde{L}$ and the functor $\tilde{\ell}:\underline{\leq}\tilde{L} \to \underline{\text{Mon}}$.

Lemma 3.27. (Using the notation of Lemma 3.26.) If $\tilde{\pi}(_):\underline{\leq}\tilde{R} \to \underline{Mon}$ is the functor defined in Theorem 3.5 by the $H$ quasi-order $\underline{\leq}(C)$ and if $\pi:\underline{\leq}R \to \underline{Mon}$ is defined by $\pi(x) = \tilde{\pi}(\tilde{x})$ for all $x \in X$ and $\pi(z,x) = \tilde{\pi}(\tilde{z},\tilde{x})$ for all $(z,x) \in \underline{\leq}(R)$, then $\pi$ is a functor.

PROOF. By restricting the lifting "~" to the objects of $\underline{\leq}R$, one has $\tilde{}:\underline{\leq}R \to \underline{\leq}\tilde{R}$ is a functor. Hence $\pi$ is the composition of the functor $\tilde{}:\underline{\leq}R \to \underline{\leq}\tilde{R}$ and the functor $\tilde{\pi}:\underline{\leq}\tilde{R} \to \underline{Mon}$.

Lemma 3.28. (Using the notation of Lemma 3.26.) If $\alpha:B \to \bigcup_{x\in X} \ell(x)$ is defined by $\alpha(\lambda,x) = \tilde{D}^{\widetilde{\lambda x}}_{\tilde{\lambda}\tilde{x}}$ for all $(\lambda,x) \in B_1$ and $\alpha(x,\rho) = \tilde{D}^{\widetilde{x\rho}}_{\tilde{x}\tilde{\rho}}$ for all $(x,\rho) \in B_2$, then $\alpha$ is a factor system for $\ell$.

PROOF. To show $\alpha$ is a factor system for $\ell$, it suffices to show $\tilde{D}^{\widetilde{\lambda x}}_{\tilde{\lambda}\tilde{x}} \in \tilde{\ell}(\widetilde{\lambda x})$ for each $(\lambda,x) \in B_1$. Dually, the condition for $B_2$ also follows. Now $\tilde{D}^{\widetilde{\lambda x}}_{\tilde{\lambda}\tilde{x}}$ will be in $\tilde{\ell}(\widetilde{\lambda x})$ if (by the definition of $\tilde{\ell}(\widetilde{\lambda x})$) $\tilde{\lambda}\tilde{x} \underline{\leq}(C)\widetilde{\lambda x}$. In particular if $p(\tilde{\lambda}\tilde{x}) = p(\widetilde{\lambda x})$, i.e., if $\tilde{\lambda}\tilde{x}C_p\widetilde{\lambda x}$.

First consider the case where $\tilde{\lambda} = \tilde{G}^{\tilde{u}}_{\tilde{v}}$. (Hence $\lambda = G^u_v \in \Lambda$ for some $(v,u) \in \underline{\leq}(L)$.) Since $p \in Mor(\underline{\leq}(G_X))$, there is a $\lambda' \in \Lambda$ such that $p(\tilde{G}^{\tilde{u}}_{\tilde{v}}w) = \lambda' p(w)$ for all $w \in \tilde{R}(\tilde{u})$. In particular $\lambda' u = \lambda' p(\tilde{u}) = p(\tilde{G}^{\tilde{u}}_{\tilde{v}}\tilde{u}) = p(\tilde{v}) = v = G^u_v u = \lambda u$ and by the proof of Theorem 5.8 of [3], $\lambda' x = \lambda x$ for all $x \in R(u)$. Hence one has $p(\tilde{\lambda}w) = \lambda' p(w) = \lambda p(w)$ for all $\tilde{R}(\tilde{u})$. Therefore

$$(3.29) \qquad p(\tilde{G}^{\tilde{u}}_{\tilde{v}} w) = G^{u}_{v} p(w) \quad \text{for all} \quad w \in \tilde{R}(\tilde{u}) .$$

Now if $x \in \text{dom } \lambda$, then $p(\tilde{\lambda}\tilde{x}) = \lambda p(\tilde{x}) = \lambda x = p(\widetilde{\lambda x})$.

Now consider the case where $\tilde{\lambda} = G^{\tilde{u}_1}_{\tilde{v}_1} \circ \cdots \circ G^{\tilde{u}_n}_{\tilde{v}_n}$ and $\tilde{x} \in \text{dom } \tilde{\lambda}$. Then by (3.29) one has

$$\begin{aligned} p(\tilde{\lambda}\tilde{x}) &= p(\tilde{G}^{\tilde{u}_1}_{\tilde{v}_1} \circ \cdots \circ \tilde{G}^{\tilde{u}_n}_{\tilde{v}_n} \tilde{x}) \\ &= G^{u_1}_{v_1} p(\tilde{G}^{\tilde{u}_2}_{\tilde{v}_2} \circ \cdots \tilde{G}^{\tilde{u}_n}_{\tilde{v}_n} \tilde{x}) \\ &\quad \vdots \\ &= G^{u_1}_{v_1} \circ \cdots \circ G^{u_n}_{v_n} p(\tilde{x}) \\ &= G^{u_1}_{v_1} \circ \cdots \circ G^{u_n}_{v_n} x \\ &= p(\widetilde{\lambda x}) \end{aligned}$$

where $\lambda = G^{u_1}_{v_1} \circ \cdots \circ G^{u_n}_{v_n}$.

Note that as a result of the proof of Lemma 3.28 one has that

$$(3.30) \qquad p(\tilde{\lambda} w) = \lambda p(w) \quad \text{for all} \quad \lambda \in \Lambda \quad \text{and for all} \quad p(w) \in \text{dom } \lambda .$$

Lemma 3.31. The functors and the factor system defined in Lemmas 3.26, 3.27, and 3.28, $(\ell, r, \alpha)$, satisfy condition (3.22).

PROOF. Let $(\lambda, \rho) \in (\Lambda, P)$ and $y \in X$ such that $y \in \text{dom } \lambda \cap \text{dom } \rho$. Let $\tilde{D}^{\tilde{y}}_{u} \in \ell(y)$. Now

$$\alpha(\lambda,y\rho)\ell(\lambda y\rho,y\rho)[r(y\rho,y)[h]\alpha(y,\rho)] = \tilde{D}^{\widetilde{\lambda y\rho}}_{\tilde{\lambda}\widetilde{y\rho}}\ \ell(\lambda y\rho,y\rho)[r(y\rho,y)[\tilde{D}^{\tilde{y}}_{u}]\tilde{D}^{\widetilde{y\rho}}_{\tilde{y}\tilde{\rho}}]$$

$$= \tilde{D}^{\widetilde{\lambda y\rho}}_{\tilde{\lambda}\widetilde{y\rho}}\ \ell(\lambda y\rho,y\rho)[\tilde{D}^{\widetilde{y\rho}}_{u\tilde{D}^{\tilde{y}}_{\widetilde{y\rho}}}\ \tilde{D}^{\widetilde{y\rho}}_{\tilde{y}\tilde{\rho}}]$$

$$= \tilde{D}^{\widetilde{\lambda y\rho}}_{\tilde{\lambda}\widetilde{y\rho}}\ \ell(\lambda y\rho,y\rho)[\tilde{D}^{\widetilde{y\rho}}_{u\tilde{D}^{\tilde{y}}_{\widetilde{y\rho}}\tilde{D}^{\widetilde{y\rho}}_{\tilde{y}\tilde{\rho}}}]$$

$$= \tilde{D}^{\widetilde{\lambda y\rho}}_{\tilde{\lambda}\widetilde{y\rho}}\ \ell(\lambda y\rho,y\rho)[\tilde{D}^{\widetilde{y\rho}}_{u\tilde{D}^{\tilde{y}}_{\tilde{y}\tilde{\rho}}}]$$

$$= \tilde{D}^{\widetilde{\lambda y\rho}}_{\tilde{\lambda}\widetilde{y\rho}}\ \tilde{D}^{\widetilde{\lambda y\rho}}_{\widetilde{\lambda y\rho}\tilde{D}^{\widetilde{y\rho}}_{u\tilde{D}^{\tilde{y}}_{\tilde{y}\tilde{\rho}}}}$$

and

$$(\tilde{\lambda}\widetilde{y\rho})\tilde{D}^{\widetilde{\lambda y\rho}}_{\widetilde{\lambda y\rho}}\ \tilde{D}^{\widetilde{y\rho}}_{u\tilde{D}^{\tilde{y}}_{\tilde{y}\tilde{\rho}}} = \tilde{G}^{\widetilde{\lambda y\rho}}_{\tilde{\lambda}\widetilde{y\rho}}\ (\widetilde{\lambda y\rho}\ \tilde{D}^{\widetilde{y\rho}}_{u\tilde{D}^{\tilde{y}}_{\tilde{y}\tilde{\rho}}})$$

$$= (\tilde{G}^{\widetilde{\lambda y\rho}}_{\tilde{\lambda}\widetilde{y\rho}}\widetilde{\lambda y\rho})\tilde{D}^{\widetilde{y\rho}}_{u\tilde{D}^{\tilde{y}}_{\tilde{y}\tilde{\rho}}}$$

$$= (\tilde{\lambda}\widetilde{y\rho})\tilde{D}^{\widetilde{y\rho}}_{u\tilde{D}^{\tilde{y}}_{\tilde{y}\tilde{\rho}}}$$

$$= \tilde{\lambda}(\widetilde{y\rho}\tilde{D}^{\widetilde{y\rho}}_{u\tilde{D}^{\tilde{y}}_{\tilde{y}\tilde{\rho}}})$$

$$= \tilde{\lambda}(u\tilde{D}^{\tilde{y}}_{\tilde{y}\tilde{\rho}})$$

$$= \tilde{\lambda}(\tilde{G}^{\tilde{y}}_{u}(\tilde{y}\tilde{\rho}))$$

$$= \tilde{\lambda}((\tilde{G}^{\tilde{y}}_{u}\tilde{y})\tilde{\rho})$$

$$= \tilde{\lambda}(u\tilde{\rho})\ .$$

This gives that the left side of condition (3.22) reduces to $\tilde{D}^{\widetilde{\lambda y\rho}}_{\tilde{\lambda}(u\tilde{\rho})}$ .

Now consider the right hand side of (3.22) ,

$$\mathcal{r}(\lambda y\rho,\lambda y)[\alpha(\lambda,y)\mathcal{l}(\lambda y,y)[h]]\alpha(\lambda y,\rho)$$

$$= \mathcal{r}(\lambda y\rho,\lambda y)[\alpha(\lambda,y)]\mathcal{r}(\lambda y\rho,\lambda y)[\mathcal{l}(\lambda y,y)[h]]\alpha(\lambda y,\rho)$$

$$= \mathcal{r}(\lambda y\rho,\lambda y)[\tilde{D}^{\widetilde{\lambda y}}_{\widetilde{\lambda y}}]\mathcal{r}(\lambda y\rho,\lambda y)[\tilde{D}^{\widetilde{\lambda y}}_{\widetilde{\lambda y}\tilde{D}^{\tilde{y}}_{u}}]\tilde{D}^{\widetilde{\lambda y\rho}}_{\widetilde{\lambda y\rho}}$$

$$= \tilde{D}^{\widetilde{\lambda y\rho}}_{(\tilde{\lambda y})\tilde{D}^{\widetilde{\lambda y}}_{\lambda y\rho}}\ \tilde{D}^{\widetilde{\lambda y\rho}}_{(\widetilde{\lambda y}\tilde{D}^{\tilde{y}}_{u})\tilde{D}^{\widetilde{\lambda y}}_{\widetilde{\lambda y\rho}}}\ \tilde{D}^{\widetilde{\lambda y\rho}}_{\widetilde{\lambda y\rho}}$$

and

$$((\tilde{\lambda y})\tilde{D}^{\widetilde{\lambda y}}_{\widetilde{\lambda y\rho}})\tilde{D}^{\widetilde{\lambda y\rho}}_{(\widetilde{\lambda y}\tilde{D}^{\tilde{y}}_{u})\tilde{D}^{\widetilde{\lambda y}}_{\widetilde{\lambda y\rho}}}\ \tilde{D}^{\widetilde{\lambda y\rho}}_{\widetilde{\lambda y\rho}} = (\tilde{\lambda y})\tilde{D}^{\widetilde{\lambda y}}_{(\widetilde{\lambda y}\tilde{D}^{\tilde{y}}_{u})\tilde{D}^{\widetilde{\lambda y}}_{\widetilde{\lambda y\rho}}}\ \tilde{D}^{\widetilde{\lambda y\rho}}_{\widetilde{\lambda y\rho}}$$

$$= (\tilde{\lambda y})\tilde{D}^{\widetilde{\lambda y}}_{\widetilde{\lambda y}\tilde{D}^{\tilde{y}}_{u}}\ \tilde{D}^{\widetilde{\lambda y}}_{\widetilde{\lambda y\rho}}\ \tilde{D}^{\widetilde{\lambda y\rho}}_{\widetilde{\lambda y\rho}}$$

$$= (\tilde{\lambda y})\tilde{D}^{\widetilde{\lambda y}}_{\widetilde{\lambda y}\tilde{D}^{\tilde{y}}_{u}}\ \tilde{D}^{\widetilde{\lambda y}}_{\widetilde{\lambda y\rho}}$$

$$= \tilde{G}^{\widetilde{\lambda y}}_{\tilde{\lambda y}}(\widetilde{\lambda y}\tilde{D}^{\tilde{y}}_{u})\tilde{D}^{\widetilde{\lambda y}}_{\widetilde{\lambda y\rho}}$$

$$= ((\tilde{G}^{\widetilde{\lambda y}}_{\tilde{\lambda y}}\widetilde{\lambda y})\tilde{D}^{\tilde{y}}_{u})\tilde{D}^{\widetilde{\lambda y}}_{\widetilde{\lambda y\rho}}$$

$$= ((\tilde{\lambda y})\tilde{D}^{\tilde{y}}_{u})\tilde{D}^{\widetilde{\lambda y}}_{\widetilde{\lambda y\rho}}$$

$$= (\tilde{\lambda}(\tilde{y}\tilde{D}^{\tilde{y}}_{u}))\tilde{D}^{\widetilde{\lambda y}}_{\widetilde{\lambda y\rho}}$$

$$= (\tilde{\lambda u})\tilde{D}^{\widetilde{\lambda y}}_{\widetilde{\lambda y\rho}}$$

$$= \tilde{\lambda}(u\tilde{D}^{\widetilde{\lambda y}}_{\widetilde{\lambda y\rho}})$$

$$= \tilde{\lambda}\tilde{G}^{\widetilde{\lambda y}}_{u}(\widetilde{\lambda y\rho})$$

$$= \tilde{\lambda}((\tilde{G}^{\widetilde{\lambda y}}_{u}\widetilde{\lambda y})\tilde{\rho})$$

$$= \tilde{\lambda}(u\tilde{\rho})\ .$$

Hence the right hand side of (3.22) reduces to $\tilde{D}^{\lambda y \rho}_{\tilde{\lambda}(u\rho)}$ . Therefore condition (3.22) is satisfied by $(\ell,r,\alpha)$ .

Theorem 3.32. If $(\tilde{X},\tilde{\Lambda}_1,\tilde{P}_1),p) \in obj(\leq(G_X))$ with $\leq(C)$ an $H$ quasi-order under $\leq(\tilde{H})$ such that $C = C_p$ , then the functors and the factor system defined in Lemmas 3.26, 3.27, and 3.28 give rise to a triple $(\overline{X},\overline{\Lambda},\overline{P})$ such that $(\overline{X},\overline{\Lambda},\overline{P})$ is in obj(GPX) , $((\overline{X},\overline{\Lambda},\overline{P}),\pi_1) \in obj(GPX_X)$ where $\pi_1:\overline{X} \to X$ is the first coordinate projection, and there is a surmorphism $\Phi_1:(\overline{X},\overline{\Lambda},\overline{P}) \to (\tilde{X},\tilde{\Lambda}_1,\tilde{P}_1)$ .

PROOF. By Lemma 3.31 and Theorem 3.25 the triple $(\overline{X},\overline{\Lambda},\overline{P})$ is in obj(GPX) . Now again by Theorem 3.25 $((\overline{X},\overline{\Lambda},\overline{P}),\pi_1) \in obj(GPX_X)$ . Now it follows from Lemma 3.34 (to be proven) that the map $\Phi_1:(\overline{X},\overline{\Lambda},\overline{P}) \to (\tilde{X},\tilde{\Lambda},\tilde{P})$ , $(x,\tilde{D}^{\tilde{x}}_u) \mapsto u$ , is a surmorphism. Thus $(\overline{X},\overline{\Lambda},\overline{P}) \xrightarrow{\Phi_1} (\tilde{X},\tilde{\Lambda},\tilde{P}) \xrightarrow{1_{\tilde{X}}} (\tilde{X},\tilde{\Lambda}_1,\tilde{P}_1)$ gives $\Phi_1:(\overline{X},\overline{\Lambda},\overline{P}) \to (\tilde{X},\tilde{\Lambda}_1,\tilde{P}_1)$ is a surmorphism.

Corollary 3.33. If $((\tilde{X},\tilde{\Lambda}_1,\tilde{P}_1),p) \in obj(\leq(G_X))$ , then there is a pair of functors $\underline{\ell}:\leq L \to \underline{Grp}$ , $\underline{r}:\leq R \to \underline{Grp}$ and a factor system $\alpha$ for $\underline{\ell}$ such that the triple $(\overline{X},\overline{\Lambda},\overline{P})$ (as defined in Definition 3.20) is in obj(GPX) , $((\overline{X},\overline{\Lambda},\overline{P}),\pi) \in obj(\leq(G_X))$ where $\pi:\overline{X} \to X$ is the first coordinate projection, and there is an isomorphism $\Phi:(\overline{X},\overline{\Lambda},\overline{P}) \to (\tilde{X},\tilde{\Lambda}_1,\tilde{P}_1)$ such that $p \circ \Phi = \pi$ .

PROOF. Apply Theorem 3.32 with the $H$ quasi-order $\leq(C)$ equal $C_p$ . This gives a pair of functors $\ell(_):\leq L \to \underline{Mon}$ and $r(_):\leq R \to \underline{Mon}$ (obtained as in Lemmas 3.26 and 3.27). But as noted in Corollary 3.6 and Corollary 3.8, whenever the $H$ quasi-order $\leq(C)$ is an $H$

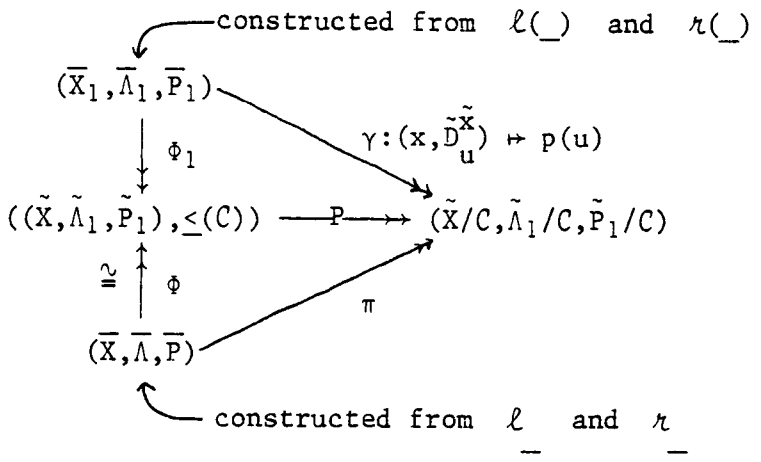
equivalence the codomain of $\tilde{\ell}$ and $\tilde{r}$ becomes $\underline{\text{Grp}}$. Hence the codomain of $\ell(_)$ and $r(_)$ is also $\underline{\text{Grp}}$ and thus denoted by $\ell$ and $r$, respectively. Now Theorem 3.32 also gives $(\overline{X},\overline{\Lambda},\overline{P}) \in \text{obj}(\text{GPX})$, $((\overline{X},\overline{\Lambda},\overline{P}),\pi) \in \text{obj}(\text{GPX}_X)$, and that $\Phi:(\overline{X},\overline{\Lambda},\overline{P}) \in \text{obj}(\tilde{X},\tilde{\Lambda}_1,\tilde{P}_1)$ is a surmorphism. To finish the proof of the Corollary it suffices to show $C_\pi \subseteq \overline{H}$, $\Phi$ is an isomorphism, and $p\circ\Phi = \pi$.

First to show $C_\pi \subseteq \overline{H}$ note that for $a \in X$ $\{a\}\times\ell_a \subseteq \overline{H}_{(a,1)}$. (The maps $\{\overline{\lambda}_h : h \in \ell_a\}$ give $\{a\}\times\ell_a \subseteq \overline{L}_{(a,1)}$ and the maps $\{\overline{\rho}_h : h \in \ell_a\}$ give $\{a\}\times\ell_a \subseteq \overline{R}_{(a,1)}$.) Now by the definition of $\pi$, $\pi(a,\tilde{D}_u^{\tilde{a}}) = a$ for each $(a,\tilde{D}_u^{\tilde{a}}) \in \overline{X}$. Hence $(C_\pi)_{(a,1)} = \{a\}\times\ell_a$ for each $a \in X$. Thus $C_\pi \subseteq \overline{H}$.

Now to show $\Phi:(\overline{X},\overline{\Lambda},\overline{P}) \to (\tilde{X},\tilde{\Lambda}_1,\tilde{P}_1)$, $(x,\tilde{D}_u^{\tilde{x}}) \mapsto u$, is an isomorphism it suffices (since $\Phi$ is known to be a surmorphism) by Corollary 3.18 to show $\Phi$ is one-to-one. But if $\Phi(x,\tilde{D}_u^{\tilde{x}}) = \Phi(y,\tilde{D}_v^{\tilde{y}})$ for some $(x,\tilde{D}_u^{\tilde{x}}),(y,\tilde{D}_v^{\tilde{y}}) \in \overline{X}$, then $u = v$ and $\tilde{x}C_p u = v C_p \tilde{y}$. Hence $\tilde{x}C_p\tilde{y}$, i.e., $x = p(\tilde{x}) = p(\tilde{y}) = y$. But "~" is a mapping from $X$ to $\tilde{X}$. Thus $x = y$ gives $\tilde{x} = \tilde{y}$ and $(x,\tilde{D}_u^{\tilde{x}}) = (y,\tilde{D}_v^{\tilde{y}})$. Therefore $\Phi$ is one-to-one.

Now to show $p\circ\Phi = \pi$, let $(x,\tilde{D}_u^{\tilde{x}}) \in \overline{X}$ and note that $p\circ\Phi(x,\tilde{D}_u^{\tilde{x}}) = p(u) = p(\tilde{x}) = x = \pi(x,\tilde{D}_u^{\tilde{x}})$.

It is interesting to note that if one merely starts with $(\tilde{X},\tilde{\Lambda}_1,\tilde{P}_1) \in \text{obj}(\text{GPX})$ and an $H$ quasi-order $\leq(C)$ under $\leq(H)$, then Theorem 3.32 and Corollary 3.33 allow one to construct the following:




The following lemma was used in the proof of Theorem 3.32; the proof of the lemma however was postponed until now.

Lemma 3.34. Let $((\tilde{X},\tilde{\Lambda}_1,\tilde{P}_1),p) \in \mathrm{obj}(\leq(G_X))$ with $\leq(C)$ an $H$ quasi-order under $\leq(\tilde{H})$ such that $C = C_p$. Let $(\overline{X},\overline{\Lambda},\overline{P})$ be the triple constructed from the functors and the factor system defined in Lemmas 3.26, 3.27, and 3.28. Then the mapping $\Phi:(\overline{X},\overline{\Lambda},\overline{P}) \to (\tilde{X},\tilde{\Lambda},\tilde{P})$ defined by $\Phi(x,\tilde{D}_u^{\tilde{X}}) = u$ is a surmorphism.
(Note $(\tilde{X},\tilde{\Lambda},\tilde{P}) = \mathcal{B}\circ\mathcal{A}(\tilde{X},\tilde{\Lambda}_1,\tilde{P}_1)$ .)

PROOF. To show $\Phi \in \mathrm{Mor}(GPX)$ it suffices to show $(\Lambda_1' \cup \Lambda_2', P_1' \cup P_2') \subseteq (\Lambda',P')$ where $\Lambda_1'$, $\Lambda_2'$, $P_1'$, and $P_2'$ are defined in Definition 3.20 and $(\Lambda',P')$ is defined in Lemma 2.6. (Actually a corollary to Lemma 2.6 for the category $GPX$ is what is used.)

First consider $\overline{\lambda} \in \Lambda_2'$ for $\lambda \in \Lambda$ and let $(x,k) \in \mathrm{dom}\,\overline{\lambda}$. It will be shown that $\Phi(\overline{\lambda}z) = \tilde{\lambda}\Phi(z)$ for all $z \leq(\overline{R})(x,k)$ where $\tilde{\lambda}$ is the lifting of $\lambda$. If $z \leq(\overline{R})(x,k)$, then $z = (x\rho,\tilde{D}_u^{\widetilde{x\rho}})$ for some $\rho \in P$ and $\tilde{D}_u^{\widetilde{x\rho}} \in \ell(x\rho)$ (since $\pi\times\pi \leq(\overline{R}) = \leq(R)$ ). Hence

$$\Phi(\bar{\lambda}z) = \Phi(\bar{\lambda}(x\rho,\tilde{D}_u^{\widetilde{x\rho}})$$

$$= \Phi(\lambda x\rho, \alpha(\lambda,x\rho)\ell(\lambda x\rho,x\rho)[\tilde{D}_u^{\widetilde{x\rho}}])$$

$$= \Phi(\lambda x\rho, \tilde{D}_{\tilde{\lambda}\widetilde{x\rho}}^{\widetilde{\lambda x\rho}} \tilde{D}_{\widetilde{\lambda x\rho}\tilde{D}_u^{\widetilde{x\rho}}}^{\widetilde{\lambda x\rho}})$$

$$= (\tilde{\lambda}\widetilde{x\rho})\tilde{D}_{\widetilde{\lambda x\rho}\tilde{D}_u^{\widetilde{x\rho}}}^{\widetilde{\lambda x\rho}}$$

$$= \tilde{G}_{\tilde{\lambda}\widetilde{x\rho}}^{\widetilde{\lambda x\rho}}(\widetilde{\lambda x\rho}\tilde{D}_u^{\widetilde{x\rho}})$$

$$= (\tilde{G}_{\tilde{\lambda}\widetilde{x\rho}}^{\widetilde{\lambda x\rho}}\widetilde{\lambda x\rho})\tilde{D}_u^{\widetilde{x\rho}}$$

$$= (\tilde{\lambda}\widetilde{x\rho})\tilde{D}_u^{\widetilde{x\rho}}$$

$$= \tilde{\lambda}(\widetilde{x\rho}\tilde{D}_u^{\widetilde{x\rho}})$$

$$= \tilde{\lambda}u$$

$$= \tilde{\lambda}\Phi(x\rho,\tilde{D}_u^{\widetilde{x\rho}})$$

$$= \tilde{\lambda}\Phi(z) \ .$$

Now consider the case with $\bar{\lambda}_h \in \Lambda_1'$ and $h \in \ell(x)$ for some $x \in X$. Let $z \leq (\bar{R})(x,k)$ with $z,(x,k) \in \bar{X}$. Thus $z = (x\rho,\tilde{D}_u^{\widetilde{x\rho}})$ for some $\rho \in P$ and $\tilde{D}_u^{\widetilde{x\rho}} \in \ell(x\rho)$. Since $h \in \ell(x)$, let $h = \tilde{D}_y^{\tilde{x}}$ for some $y \leq (C)\tilde{x}$. Hence

$$\Phi(\overline{\lambda}_h(x\rho,\tilde{D}_u^{\widetilde{x\rho}})) = \Phi(x\rho,\hbar(x\rho,x)[h]\tilde{D}_u^{\widetilde{x\rho}})$$

$$= \Phi(x\rho,\tilde{D}_{yD_{\widetilde{x\rho}}^{\tilde{x}}}^{\widetilde{x\rho}}\ \tilde{D}_u^{\widetilde{x\rho}})$$

$$= (y\tilde{D}_{\widetilde{x\rho}}^{\tilde{x}})\tilde{D}_u^{\widetilde{x\rho}}$$

$$= y\tilde{D}_u^{\tilde{x}}$$

$$= \tilde{G}_y^{\tilde{x}}u$$

$$= \tilde{G}_y^{\tilde{x}}\Phi(x\rho,\tilde{D}_u^{\widetilde{x\rho}})$$

$$= \tilde{G}_y^{\tilde{x}}\Phi(z)\ .$$

Therefore if $\gamma = \tilde{G}_y^{\tilde{x}} \in \tilde{\Lambda}$, then $\Phi(\overline{\lambda}_h z) = \gamma\Phi(z)$ for all $z \underline{<} (\overline{R})(x,k)$. Hence one has $\Lambda_1' \cup \Lambda_2' \subseteq \Lambda'$. Dually one has $P_1' \cup P_2' \subseteq P'$. Thus $\Phi \in \mathrm{Mor}(GPX)$.

Now to finish the proof that $\Phi$ is a surmorphism, it is shown that $\Phi\times\Phi\ \underline{<}(\overline{L}) = \underline{<}(\tilde{L})$. Let $(v,u) \in \underline{<}(\tilde{L})$ and let $z = p(v)$ and $x = p(u)$. Thus $\tilde{z}\tilde{H}v \underline{<}(\tilde{L})u\tilde{H}\tilde{x}$. Let $k = \tilde{D}_v^{\tilde{z}}$, $\tilde{\lambda} = \tilde{G}_z^{\tilde{x}}$, $h = \tilde{D}_{\tilde{x}}^{u}$, and $\lambda = G_z^x$. Note that $k^{-1}$ and $h^{-1}$ exist. Now consider $\overline{\lambda}_k \circ \overline{\lambda} \circ \overline{\lambda}_h(x,\tilde{D}_u^{\tilde{x}})$.

$$\begin{aligned}
\overline{\lambda}_k(\overline{\lambda}(\overline{\lambda}_h(x,\tilde{D}_u^{\tilde{x}}) &= \overline{\lambda}_k(\overline{\lambda}(x,\tilde{D}_{\tilde{x}}^{u}\tilde{D}_u^{\tilde{x}})) \\
&= \overline{\lambda}_k(\overline{\lambda}(x,1)) \\
&= \overline{\lambda}_k(\lambda x,\alpha(\lambda,x)) \\
&= \overline{\lambda}_k(z,\tilde{D}_{\tilde{\lambda}\tilde{x}}^{\widetilde{\lambda x}}) \\
&= \overline{\lambda}_k(z\tilde{D}_{\tilde{z}}^{\tilde{z}}) \\
&= \overline{\lambda}_k(z,1) \\
&= (z,\tilde{D}_v^{\tilde{z}}) \ .
\end{aligned}$$

Therefore $(z,\tilde{D}_v^{\tilde{z}}) \leq(\overline{L})(x,\tilde{D}_u^{\tilde{x}})$ and $\Phi\times\Phi[(z,\tilde{D}_v^{\tilde{z}}),(x,\tilde{D}_u^{\tilde{x}})] = (v,u)$ . Thus $\Phi\times\Phi \leq(\overline{L}) = \leq(\tilde{L})$ . Dually, $\phi\times\Phi \leq(\overline{R}) = \leq(\tilde{R})$ . Hence $\Phi:(\overline{X},\overline{\Lambda},\overline{P}) \to (\tilde{X},\tilde{\Lambda},\tilde{P})$ is a surmorphism.

In looking back at Theorem 3.32 and Corollary 3.33 one has that these enable one to reconstruct an isomorphic copy of a known $\leq(G)$-coextension. Now the following theorem determines necessary and sufficient conditions on a compatible pair of functors with codomain Grp so that one may construct a $\leq(G)$-coextension from a given $(X,\Lambda_1,P_1) \in \text{obj}(GPX)$ .

Theorem 3.35. If $(\ell:\leq L \to \underline{\text{Grp}}\ ,\ r:\leq R \to \underline{\text{Grp}})$ is a pair of compatible functors with a factor system $\alpha$ , then $((\overline{X},\overline{\Lambda},\overline{P}),\pi) \in \text{obj}(\leq(G_X))$ iff condition (3.22) holds. ( $\pi:\overline{X} \to \overline{X}$ is the first coordinate projection.)

PROOF. First note that if $((\overline{X},\overline{\Lambda},\overline{P}),\pi) \in \text{obj}(\underline{\leq}(G_X))$ , then $(\overline{X},\overline{\Lambda},\overline{P}) \in \text{obj}(GPX)$ . Hence by Theorem 3.25 condition (3.22) is satisfied.

Now in the proof of Corollary 3.33 it is shown that $((\overline{X},\overline{\Lambda},\overline{P}),\pi) \in \text{obj}(\underline{\leq}(G_X)$ whenever $(\overline{X},\overline{\Lambda},\overline{P}) \in \text{obj}(GPX)$ .

The observant reader will note that if the triple $(\ell,r,\alpha)$ satisfies condition (3.22) with the codomain of $\ell$ being Mon , then one obtains a coextension of $(X,\Lambda_1,P_1)$ . If one adds the condition that the mapping $x \mapsto \overline{x} = (x,1) \in \overline{X}$ from $X$ to $\overline{X}$ preserves the quasi-orders $\underline{\leq}(L)$ and $\underline{\leq}(R)$ , then one obtains a $\underline{\leq}(S)$-coextension. This will be shown by Theorem 3.41. The next topic to be investigated is the analogy of the work done for $\underline{\leq}(G)$-coextensions applied to $\underline{\leq}(S)$-coextensions.

The following lemma insures the existence of a lifting for $\underline{\leq}(S)$-coextensions.

Lemma 3.36. If $((\tilde{X},\tilde{\Lambda}_1,\tilde{P}_1),p) \in \text{obj}(\underline{\leq}(S_X))$ with $\mathcal{B}\circ\mathcal{A}(\tilde{X},\tilde{\Lambda},\tilde{P}) = (\tilde{X},\tilde{\Lambda},\tilde{P})$ , then there is a lifting "~" from $(X,\Lambda,P)$ to $(\tilde{X},\tilde{\Lambda},\tilde{P})$ and $p(\tilde{\lambda}z) = \lambda p(z)$ for all $z \in \text{dom}\ \tilde{\lambda}$ with $\lambda \in \Lambda$ and $p(z\tilde{\rho}) = p(z)\rho$ for all $z \in \text{dom}\ \tilde{\rho}$ with $\rho \in P$ .

PROOF. By the definition of $((\tilde{X},\tilde{\Lambda},\tilde{P}),p) \in \text{obj}(\underline{\leq}(S_X))$ , there is a mapping "~" from $X$ to $\tilde{X}$ such that $(C_p)_{\tilde{x}} \subseteq \tilde{H}(\tilde{x})$ , $p(\tilde{x}) = x$ , and $\tilde{}$ preserves the $\underline{\leq}(L)$ and $\underline{\leq}(R)$ quasi-orders. If $G_v^u \in \Lambda$ , then $(v,u) \in \underline{\leq}(L)$ . Now since $\tilde{}$ preserves the $\underline{\leq}(L)$ quasi-order, $(\tilde{v},\tilde{u}) \in \underline{\leq}(\tilde{L})$ . Hence $\tilde{G}_{\tilde{v}}^{\tilde{u}}$ exists. It follows that "~" is a lifting from $(X,\Lambda,P)$ to $(\tilde{X},\tilde{\Lambda},\tilde{P})$ .

Now to show the second part of the lemma, first let $\lambda = G^u_v \in \Lambda$. Hence $\tilde{G}^{\tilde{u}}_{\tilde{v}} = \tilde{\lambda} \in \tilde{\Lambda}$ and since $p \in \mathrm{Mor}(\leq(S_X))$, there is a $\lambda' \in \Lambda$ such that $p(\tilde{G}^{\tilde{u}}_{\tilde{v}}w) = p(\tilde{\lambda}w) = \lambda' p(w)$ for every $w \in \tilde{R}(\tilde{u})$. In particular $\lambda' u = \lambda' p(\tilde{u}) = p(\tilde{G}^{\tilde{u}}_{\tilde{v}}\tilde{u}) = p(\tilde{v}) = v = G^u_v u$ and by the proof of Theorem 5.8 of [3] $\lambda' x = G^u_v x$ for all $x \in R(u)$. Hence $p(\tilde{G}^{\tilde{u}}_{\tilde{v}}z) = \lambda' p(z) = G^u_v p(z)$ for all $z \in \mathrm{dom}\,(\tilde{G}^{\tilde{u}}_{\tilde{v}}) = \tilde{R}(\tilde{u})$ and $p(\tilde{\lambda}z) = \lambda p(z)$ for all $z \in \mathrm{dom}\,\tilde{\lambda}$. In the event that $\lambda = G^{u_1}_{v_1} \circ \cdots \circ G^{u_n}_{v_n}$ with $z \in \mathrm{dom}\,\tilde{\lambda}$, one can apply the above to obtain

$$
\begin{aligned}
p(\tilde{\lambda}z) &= p(\tilde{G}^{\tilde{u}_1}_{\tilde{v}_1} \circ \cdots \circ \tilde{G}^{\tilde{u}_n}_{\tilde{v}_n} z) \\
&= G^{u_1}_{v_1} p(\tilde{G}^{\tilde{u}_2}_{\tilde{v}_2} \circ \cdots \circ \tilde{G}^{\tilde{u}_n}_{\tilde{v}_n} z) \\
&\;\;\vdots \\
&= G^{u_1}_{v_1} \circ \cdots \circ G^{u_n}_{v_n} p(z) \\
&= \lambda p(z) .
\end{aligned}
$$

Thus for $\lambda \in \Lambda$ and $z \in \mathrm{dom}\,\tilde{\lambda}$, $p(\tilde{\lambda}z) = \lambda p(z)$. Dually for $\rho \in P$ and $z \in \mathrm{dom}\,\tilde{\rho}$, $p(z\tilde{\rho}) = p(z)\rho$.

Lemma 3.36 enables one to construct a $\leq(S)$-coextension in a manner similar to that of a $\leq(G)$-coextension. That is, the basic properties needed to construct a $\leq(G)$-coextension are that

(i) to each $\mathcal{C}$-class there is a corresponding submonoid of the Schützenberger monoid, and

(ii) there is a lifting of $(X,\Lambda,P)$.

Lemma 3.36 establishes property (ii), while the following lemmas establish property (i).

Lemma 3.37. Let $((\tilde{X},\tilde{\Lambda}_1,\tilde{P}_1)p) \in \text{obj}(\leq(S_X))$ and let "~" denote a lifting from $(X,\Lambda,P)$ to $(\tilde{X},\tilde{\Lambda},\tilde{P}) = \mathcal{B}\circ\mathcal{A}(\tilde{X},\tilde{\Lambda}_1,\tilde{P}_1)$. If $\mathcal{R}:\leq R \to \underline{\text{Mon}}$ is defined by $\mathcal{R}(x) = \{\tilde{D}_u^{\tilde{x}}: u C_p \tilde{x}\}$ for every $x \in X$ and $\mathcal{R}(z,x)(\tilde{D}_u^{\tilde{x}}) = \tilde{D}_{u\tilde{D}_{\tilde{z}}^{\tilde{x}}}^{\tilde{z}}$ for every $(z,x) \in \leq(R)$, then $\mathcal{R}$ is a functor.

PROOF. First note that if $(u,\tilde{x}) \in C_p$, then since $((\tilde{X},\tilde{\Lambda}_1,\tilde{P}_1),p) \in \text{obj}(\leq(S_X))$ $u \in \tilde{H}(\tilde{x})$. Thus $\tilde{D}_u^{\tilde{x}} \in S_D(\tilde{x})$ and $\mathcal{R}(x) \subseteq S_D(\tilde{x})$. Also note that $\tilde{D}_{\tilde{x}}^{\tilde{x}}$ (the identity of $S_D(\tilde{x})$) is in $\mathcal{R}(x)$ for every $x \in X$.

To show $\mathcal{R}(x)$ is a semigroup for $x \in X$, let $\tilde{D}_y^{\tilde{x}},\tilde{D}_z^{\tilde{x}} \in \mathcal{R}(x)$. Then $(y,\tilde{x}),(z,\tilde{x}) \in C_p$. Since $C_p$ is an $H$ equivalence (see Proposition 3.16), $(y\tilde{D}_z^{\tilde{x}},z) = (y\tilde{D}_z^{\tilde{x}},\tilde{x}\tilde{D}_z^{\tilde{x}}) \in C_p$. Now $(y\tilde{D}_z^{\tilde{x}},z) \in C_p$ and $(z,\tilde{x}) \in C_p$ give $(y\tilde{D}_z^{\tilde{x}},\tilde{x}) \in C_p$ and $y\tilde{D}_z^{\tilde{x}} \leq(\tilde{H})\tilde{x}$. Therefore $\tilde{D}_y^{\tilde{x}}\circ\tilde{D}_z^{\tilde{x}} = \tilde{D}_{y\tilde{D}_z^{\tilde{x}}}^{\tilde{x}} \in \mathcal{R}(x)$ and $\mathcal{R}(x)$ is a monoid for each $x \in X$.

If $(z,x) \in \text{Mor}(\leq R)$ and $\tilde{D}_y^{\tilde{x}} \in \mathcal{R}(x)$, then $(y,\tilde{x}) \in C_p$. Hence $(y\tilde{D}_{\tilde{z}}^{\tilde{x}},\tilde{z}) = (y\tilde{D}_{\tilde{z}}^{\tilde{x}},\tilde{x}\tilde{D}_{\tilde{z}}^{\tilde{x}})$ is in $C_p$. Therefore $\tilde{D}_{y\tilde{D}_{\tilde{z}}^{\tilde{x}}}^{\tilde{z}} \in \mathcal{R}(z)$ and $\mathcal{R}(z,x)$ is a mapping from $\mathcal{R}(x)$ to $\mathcal{R}(z)$. Now note that $\mathcal{R}(z,x)(\tilde{D}_{\tilde{x}}^{\tilde{x}}) = \tilde{D}_{\tilde{x}\tilde{D}_{\tilde{z}}^{\tilde{x}}}^{\tilde{z}} = \tilde{D}_{\tilde{z}}^{\tilde{z}}$ and $\mathcal{R}(z,x)$ takes the identity of $\mathcal{R}(x)$ to the identity of $\mathcal{R}(z)$. To show $\mathcal{R}(z,x)$ is a homomorphism, let $\tilde{D}_y^{\tilde{x}},\tilde{D}_w^{\tilde{x}} \in \mathcal{R}(x)$. Then

$$\begin{aligned}
\mathcal{h}(z,x)\tilde{D}^{\tilde{x}}_{y}\cdot\tilde{D}^{\tilde{x}}_{w}) &= \mathcal{h}(z,x)(\tilde{D}^{\tilde{x}}_{y\tilde{D}^{\tilde{x}}_{w}}) \\
&= \tilde{D}^{\tilde{z}}_{(y\tilde{D}^{\tilde{x}}_{w})\tilde{D}^{\tilde{x}}_{\tilde{z}}} \\
&= \tilde{D}^{\tilde{z}}_{(y\tilde{D}^{\tilde{x}}_{\tilde{z}}\tilde{D}^{\tilde{z}}_{w})\tilde{D}^{\tilde{x}}_{\tilde{z}}} \\
&= \tilde{D}^{\tilde{z}}_{y\tilde{D}^{\tilde{x}}_{\tilde{z}}(\tilde{D}^{\tilde{z}}_{w}\tilde{D}^{\tilde{x}}_{\tilde{z}})} \\
&= \tilde{D}^{\tilde{z}}_{y\tilde{D}^{\tilde{x}}_{\tilde{z}}}\ \tilde{D}^{\tilde{z}}_{w}\tilde{D}^{\tilde{x}}_{\tilde{z}} \\
&= \tilde{D}^{\tilde{z}}_{y\tilde{D}^{\tilde{x}}_{\tilde{z}}}\ \tilde{D}^{\tilde{z}}_{w\tilde{D}^{\tilde{x}}_{\tilde{z}}} \\
&= \mathcal{h}(z,x)(\tilde{D}^{\tilde{x}}_{y})\cdot\mathcal{h}(z,x)(\tilde{D}^{\tilde{x}}_{w}) \ .
\end{aligned}$$

Therefore $\mathcal{h}(z,x)(\tilde{D}^{\tilde{x}}_{y}\cdot\tilde{D}^{\tilde{x}}_{w}) = \mathcal{h}(z,x)(\tilde{D}^{\tilde{x}}_{y})\cdot\mathcal{h}(z,x)(\tilde{D}^{\tilde{x}}_{w})$ and $\mathcal{h}(z,x):\mathcal{h}(x) \to \mathcal{h}(z)$ is a homomorphism.

To show $\mathcal{h}$ preserves composition of morphisms, let $(z,x),(u,z) \in \mathrm{Mor}(\underline{\leq}R)$ and let $\tilde{D}^{\tilde{x}}_{y} \in \mathcal{h}(x)$ . Thus

$$\begin{aligned}
\mathcal{h}[(u,z)\circ(z,x)](\tilde{D}^{\tilde{x}}_{y}) &= \mathcal{h}(u,x)(\tilde{D}^{\tilde{x}}_{y}) \\
&= \tilde{D}^{\tilde{u}}_{y\tilde{D}^{\tilde{x}}_{\tilde{u}}} \\
&= \tilde{D}^{\tilde{u}}_{y\tilde{D}^{\tilde{x}}_{\tilde{z}}\tilde{D}^{\tilde{z}}_{\tilde{u}}} \\
&= \mathcal{h}(u,z)(\tilde{D}^{\tilde{z}}_{y\tilde{D}^{\tilde{x}}_{\tilde{z}}}) \\
&= \mathcal{h}(u,z)[\mathcal{h}(z,x)(\tilde{D}^{\tilde{x}}_{y})] \ .
\end{aligned}$$

Hence $\mathcal{h}$ preserves composition of morphisms. Now note that $\mathcal{h}(x,x)(\tilde{D}^{\tilde{x}}_{y}) = \tilde{D}^{\tilde{x}}_{y\tilde{D}^{\tilde{x}}_{\tilde{x}}} = \tilde{D}^{\tilde{x}}_{y}$ and $\mathcal{h}(x,x)$ is the identity. Hence $\mathcal{h}$ preserves identities. Thus $\mathcal{h}:\underline{\leq}R \to \underline{\mathrm{Mon}}$ is a functor.

Lemma 3.38. Let $((\tilde{X},\tilde{\Lambda}_1,\tilde{P}_1),p) \in \text{obj}(\underline{\leq}(S_X))$ and let "~" denote a lifting from $(X,\Lambda,P)$ to $(\tilde{X},\tilde{\Lambda},\tilde{P}) = \mathcal{B}\circ\mathcal{A}(\tilde{X},\tilde{\Lambda}_1,\tilde{P}_1)$. If $\ell:\underline{\leq}L \to \underline{\text{Mon}}$ is defined by $\ell(x) = r(x)$ (as $r$ is defined in Lemma 3.37) for every $x \in X$ and $\ell(z,x)(\tilde{D}_u^{\tilde{x}}) = \tilde{D}_{\tilde{z}\tilde{D}_u^{\tilde{x}}}^{\tilde{z}}$ for every $(z,x) \in \underline{\leq}(L)$, then $\ell$ is a functor.

PROOF. The proof is dual to that of Lemma 3.37 in the same way as Theorem 3.7 was dual to Theorem 3.5 and is thus omitted.

Lemma 3.39. (Using the notation of Lemma 3.38.) If $\alpha:B \to \bigcup_{x\in X}\ell(x)$ is defined by $\alpha(\lambda,x) = \tilde{D}_{\tilde{\lambda}\tilde{x}}^{\widetilde{\lambda x}}$ for all $(\lambda,x) \in B_1$ and $\alpha(x,\rho) = \tilde{D}_{\tilde{x}\tilde{\rho}}^{\widetilde{x\rho}}$ for all $(x,\rho) \in B_2$, then $\alpha$ is a factor system for $\ell$, $\alpha(G_u^x,x) = 1$ for all $(u,x) \in \underline{\leq}(L)$, and $\alpha(x,D_u^x) = 1$ for all $(u,x) \in \underline{\leq}(R)$.

PROOF. To show $\alpha$ is a factor system for $\ell$, one must show $\tilde{D}_{\tilde{\lambda}\tilde{x}}^{\widetilde{\lambda x}} \in \ell(\lambda x)$, i.e., $\tilde{\lambda}\tilde{x}C_p\widetilde{\lambda x}$, for each $(\lambda,x) \in B_1$. But by Lemma 3.36, $p(\tilde{\lambda}\tilde{x}) = \lambda p(\tilde{x}) = \lambda x = p(\widetilde{\lambda x})$ for all $x \in \text{dom}\,\lambda$. Thus $\tilde{\lambda}\tilde{x}C_p\widetilde{\lambda x}$.

To show $\alpha(G_u^x,x) = 1$ for each $(u,x) \in \underline{\leq}(L)$, let $\lambda = G_u^x$. Then $\tilde{\lambda} = \tilde{G}_{\tilde{u}}^{\tilde{x}}$ and $\tilde{\lambda}\tilde{x} = \tilde{u} = \widetilde{\lambda x}$. Therefore $1 = \tilde{D}_{\tilde{\lambda}\tilde{x}}^{\widetilde{\lambda x}} = \alpha(G_u^x,x)$.

Lemma 3.40. The functors and the factor system $(\ell,r,\alpha)$ defined in Lemmas 3.37, 3.38, and 3.39 satisfy condition (3.22).

PROOF. The proof is the same as that of Lemma 3.31 and is omitted.

Theorem 3.41. If $(\ell:\underline{\leq}L \to \underline{\text{Mon}}$, $r:\underline{\leq}R \to \underline{\text{Mon}})$ is a pair of compatible functors with a factor system $\alpha$, then

$((\overline{X},\overline{\Lambda},\overline{P}),\pi) \in \text{obj}(\underline{\leq}(S_X))$ [where $(\overline{X},\overline{\Lambda},\overline{P})$ is defined in Definition 3.20] iff condition (3.22) is satisfied along with the following condition:

(3.42) the mapping $x \mapsto \overline{x} = (x,1)$ from $(X,\Lambda,P)$ to $(\overline{X},\overline{\Lambda},\overline{P})$ preserves $\underline{\leq}(L)$ and $\underline{\leq}(R)$ .

Conversely, if $((\tilde{X},\tilde{\Lambda}_1,\tilde{P}_1),p) \in \text{obj}(\underline{\leq}(S_X))$ , then the functors and factor system $(\ell,r,\alpha)$ defined in Lemmas 3.37, 3.38, and 3.39 give rise to a triple $(\overline{X},\overline{\Lambda},\overline{P})$ such that $((\overline{X},\overline{\Lambda},\overline{P}),\pi) \in \text{obj}(\underline{\leq}(S_X))$ and there is an isomorphism $\Phi:(\overline{X},\overline{\Lambda},\overline{P}) \to (\tilde{X},\tilde{\Lambda}_1,\tilde{P}_1)$ such that $p\circ\Phi = \pi$ .

PROOF. To prove the first part of the theorem, first consider a triple $(\ell,r,\alpha)$ which satisfies conditions (3.22) and (3.42). By Theorem 3.25 $((\overline{X},\overline{\Lambda},\overline{P}),\pi) \in \text{obj}(GPX_X)$ . Now consider the map "$\overline{\ }$" from $X$ to $\overline{X}$ defined by $x \mapsto \overline{x} = (x,1) \in \overline{X}$ . To show $((\overline{X},\overline{\Lambda},\overline{P}),\pi) \in \text{obj}(\underline{\leq}(S_X))$ it suffices to show $(C_\pi)_{\overline{x}} \subseteq \overline{H}(\overline{x})$ for all $x \in X$ (since by (3.42) "$\overline{\ }$" is known to preserve $\underline{\leq}(L)$ and $\underline{\leq}(R)$ ). Clearly $(C_\pi)_{\overline{x}}$ is the set $\{x\}\times\ell(x)$ . If $(x,u) \in (C_\pi)_{\overline{x}}$ , then $\overline{x}\overline{\rho}_u = (x,1)\overline{\rho}_u = (x,u)$ , $\overline{\lambda}_u\overline{x} = \overline{\lambda}_u(x,1) = (x,u)$ , and $(x,u) \underline{\leq}(\overline{H})\overline{x}$ . Hence $(C_\pi)_{\overline{x}} \subseteq \overline{H}(\overline{x})$ for all $x \in X$ . Therefore $((\overline{X},\overline{\Lambda},\overline{P}),\pi) \in \text{obj}(\underline{\leq}(S_X))$ .

Assume $((\overline{X},\overline{\Lambda},\overline{P}),\pi) \in \text{obj}(\underline{\leq}(S_X))$ . Then $(\overline{X},\overline{\Lambda},\overline{P}) \in \text{obj}(GPX)$ and by Theorem 3.25 condition 3.22 is satisfied. Also since $((\overline{X},\overline{\Lambda},\overline{P}),\pi) \in \text{obj}(\underline{\leq}(S_X))$ , there is a mapping, say "~" , from $X$ to $\overline{X}$ which preserves the quasi-orders $\underline{\leq}(L)$ and $\underline{\leq}(R)$ . If $(z,w) \in \underline{\leq}(L)$ , then $(\tilde{z},\tilde{w}) \in \underline{\leq}(\overline{L})$ . Thus let $\tilde{z} = (z,u)$ for some

$u \in \ell(z)$ and let $\tilde{w} = (w,v)$ for some $v \in \ell(w)$. Since $(C_\pi)_{\tilde{x}} \subseteq \overline{H}(\tilde{x})$ for all $x \in X$, $(w,1) \leq (\overline{H})\tilde{w}$ and $(z,1) \leq (\overline{H})\tilde{z}$. Next note that $\tilde{w} = (w,v) = \overline{\lambda}_v(w,1) = \overline{\lambda}_v\overline{w}$ and $\tilde{w} \leq (\overline{L})\overline{w}$. Therefore $\overline{z} \leq (\overline{H})\tilde{z} \leq (\overline{L})\tilde{w} \leq (\overline{L})\overline{w}$ and $\overline{z} \leq (\overline{L})\overline{w}$. Hence "$\overline{\ }$" preserves the quasi-order $\leq(L)$. Dually "$\overline{\ }$" preserves the quasi-order $\leq(R)$. Thus condition (3.42) is satisfied and $((\overline{X},\overline{\Lambda},\overline{P}),\pi) \in \text{obj}(\underline{\leq}(S_X))$ iff conditions (3.22) and (3.42) are satisfied.

Now to prove the second part of the theorem, let $((\tilde{X},\tilde{\Lambda}_1,\tilde{P}_1),p) \in \text{obj}(\underline{\leq}(S_X))$. By Lemma 3.40 the triple $(\ell,r,\alpha)$ satisfies condition (3.22). By Lemma 3.39 $\alpha(G_u^x,x) = 1$ for every $(u,x) \in \leq(L)$ and $\alpha(x,D_u^x) = 1$ for every $(u,x) \in \leq(R)$. Thus for $(u,x) \in \leq(L)$, one must show $(u,1) \leq (\overline{L})(x,1)$ in order to demonstrate condition (3.42). But if $\lambda = G_u^x$, then $\overline{\lambda}(x,1) = (\lambda x, \alpha(\lambda,x)) = (u,\alpha(G_u^x,x)) = (u,1)$. Therefore $(u,1) \leq (\overline{L})(x,1)$. Dually if $(u,x) \in \leq(R)$, then $(\overline{u},\overline{x}) \in \leq(\overline{R})$. Thus by the first part of this theorem, $(\overline{X},\overline{\Lambda},\overline{P}),\pi) \in \text{obj}(\underline{\leq}(S_X))$.

Now if $\Phi:(\overline{X},\overline{\Lambda},\overline{P}) \to (\tilde{X},\tilde{\Lambda}_1,\tilde{P}_1)$ is defined by $\Phi(x,\tilde{D}_u^{\tilde{x}}) = u$, then a proof similar to that of Lemma 3.34 shows $\Phi$ is a surmorphism. Also as in the proof of Corollary 3.33 (since $C_p$ is an $H$ equivalence) $\Phi$ is one-to-one. Hence by Corollary 3.18 $\Phi$ is an isomorphism. To show $p \circ \Phi = \pi$, note that for $(x,\tilde{D}_u^{\tilde{x}}) \in \overline{X}$,

$$p \circ \Phi(x,\tilde{D}_u^{\tilde{x}}) = p(u) = p(\tilde{x}) = x = \pi(x,\tilde{D}_u^{\tilde{x}}).$$

So far the results dealing with quasi-orders and coextensions have dealt strictly with the category GPX. The purpose of these set theoretical results has been to establish certain properties

independent of the semigroup structure. In the following a semigroup structure will be created for the coextension results. Also the previous results and definitions (unless otherwise stated) will now be applied as though they have been explicitly stated for the category GP . For example, Theorem 3.5 will be applied when $\leq(C)$ is a quasi-congruence under $\leq(H)$ where as it has been stated only in the case $\leq(C)$ is an $H$ quasi-order under $\leq(H)$ (but clearly it generalizes to quasi-congruences).

In the following fix $(S,\Lambda_1,P_1) \in \text{obj}(GP)$ . In order to induce a multiplication on the set $\overline{S}$ (as defined in 3.20) the two conditions

(i) $L(x)$ is a left ideal of $S$ for all $x \in S$ and

(ii) $R(x)$ is a right ideal of $S$ for all $x \in S$

will be assumed. That is, it will be assumed that $(S,\Lambda_1,P_1) \in \text{obj}(\overline{GP})$ . Let $B\circ A(S,\Lambda_1,P_1)$ be denoted by $(S,\Lambda,P)$ . Let $\overline{GP}_S$ denote the subcategory of $\overline{GP}$ with $(T,\Lambda_2,P_2),p) \in \text{obj}(\overline{GP}_S)$ iff $p:(T,\Lambda_2,P_2) \to (S,\Lambda_1,P_1)$ is a surmorphism in $\text{Mor}(\overline{GP})$ and $f \in \text{Mor}(\overline{GP}_S)$ iff $f \in \text{Mor}(GPX_S)$ and $f \in \text{Mor}(\underline{Sgp})$ . Let $\leq(G_S)$ $[\leq(S_S)]$ denote the full subcategory of $\overline{GP}_S$ whose objects consist of all $\leq(G)$-coextensions $[\leq(S)$-coextensions$]$ of $(S,\Lambda_1,P_1)$ .

A <u>factor system</u> (in the case of $\overline{GP}_S$) for a functor $\ell:\leq L \to \underline{Mon}$ is a function $\alpha:(S\times S \cup B) \to \bigcup_{s\in S} \ell(s)$ such that $\alpha\big|_B$ is a factor system in $GPX_S$ and $\alpha(x,y) \in \ell(xy)$ for all $(x,y) \in S\times S$ .

Definition 3.43. Let $(\overline{S},\overline{\Lambda},\overline{P})$ denote the triple defined in Definition 3.20 by the system $(S,\Lambda,P,\ell,r,\alpha)$. Define a multiplication on $\overline{S}$ by the following:

$$(x,g)(y,h) = (xy, r(xy,x)[g]\alpha(x,y)\ell(xy,y)[h])$$
$$\text{for all } (x,g),(y,h) \in \overline{S}.$$

If $(\ell:{}_{\leq}L \to \underline{\text{Mon}}$, $r:{}_{\leq}R \to \underline{\text{Mon}})$ is a compatible pair of functors with $(S,\Lambda,P) \in \overline{GP}$, then $(S,\Lambda,P) \in \overline{GP}$ gives $Sx \subseteq L(x)$ and $xS \subseteq R(x)$ for all $x \in S$. Thus for $y \in S$, $xy \leq (L)y$ and $xy \leq (R)x$. Hence $r(xy,x)$ and $\ell(xy,y)$ are defined for each $(x,y) \in S\times S$. Therefore the multiplication defined in Definition 3.43 does exist. In general it is not associative, but the following lemma provides necessary and sufficient conditions for the multiplication to be associative.

Lemma 3.44. The set $\overline{S}$ with the multiplication defined in 3.43 is a semigroup iff the following condition holds:

(3.45) $$\alpha(x,yz)\ell(xyz,yz([r(yz,y)[h]\alpha(y,z)]$$
$$= r(xyz,xy)[\alpha(x,y)\ell(xy,y)[h]]\alpha(xy,z)$$
$$\text{for all } x,y,z \in S \text{ and for all } h \in \ell(y).$$

PROOF. Leech [7] proves this lemma in the case the codomain of $\ell$ is Grp. It easily follows in the case the codomain of $\ell$ is Mon. To show this let $(x,g),(y,h),(z,k) \in \overline{S}$. A direct computation gives

$$(x,g)[(y,h)(z,k)]$$

$$= (xyz, r(xyz,x)[g]\alpha(x,yz)\ell(xyz,yz)[r(yz,y)[h]\alpha(x,y)]\ell(xyz,z)[k])$$

and

$$[(x,g)(y,h)](z,k)$$

$$= (xyz, r(xyz,x)[g]r(xyz,xy)[\alpha(x,y)\ell(xy,y)[h]]\alpha(xy,z)\ell(xyz,z)[k]) .$$

Thus if $A = \alpha(x,yz)\ell(xyz,yz)[r(yz,y)[h]\alpha(y,z)]$ and $B = r(xyz,xy)[\alpha(x,y)\ell(xy,y)[h]]\alpha(xy,z)$, then $(x,g)[(y,h)(z,k)] = (xyz, r(xyz,x)[g]A\ell(xyz,z)[k])$ and $[(x,g)(y,h)](z,k) = (xyz, r(xyz,x)[g]B\ell(xyz,z)[k])$. Now if condition (3.45) is satisfied, then $A = B$ and the multiplication on $\overline{S}$ is associative. Conversely if $\overline{S}$ is associative then $(xyz,A) = (x,1)[(y,h)(z,1)] = [(x,1)(y,h)](z,1) = (xyz,B)$. Hence $A = B$ and condition 3.45 is satisfied.

Now that a semigroup structure can be easily placed on $\overline{S}$, the conditions needed for $(\overline{S},\overline{\Lambda},\overline{P}) \in \text{obj}(\overline{GP})$ is investigated.

Lemma 3.46. For each $\overline{\rho} \in P_2'$ and for all $(y,h),(x,g) \in \overline{S}$ such that $(x,g)\cdot(y,h),(y,h) \in \text{dom}\ \overline{\rho}$, one has $(x,g)[(y,h)\overline{\rho}] = [(x,g)\cdot(y,h)]\overline{\rho}$ iff the following condition is satisfied:

(3.47) $$\alpha(xy\rho,y\rho)\ell(xy\rho,y\rho)[r(y\rho,y)[h]\alpha(y,\rho)]$$
$$= r(xy\rho,xy)[\alpha(x,y)\ell(xy,x)[h]]\alpha(xy,\rho)$$

for all $x,y \in S$ and $\rho \in P$ such that $y \in \text{dom}\ \rho$.

PROOF. Let $(y,h),(x,g)\cdot(y,h) \in \operatorname{dom} \overline{\rho}$. Then

$$(x,g)[(y,h)\overline{\rho}] = (x,g)(y\rho, r(x\rho,x)[h]\alpha(x,\rho))$$
$$= (xy\rho, r(xy\rho,x)[g]\alpha(x,y\rho)\ell(xy\rho,y\rho)[r(y\rho,y)[h]\alpha(y,\rho)])$$

and

$$[(x,g)(y,h)]\overline{\rho} = (xy, r(xy,x)[g]\alpha(x,y)\ell(xy,y)[h])\overline{\rho}$$
$$= (xy\rho, r(xy\rho,xy)[r(xy,x)[g]\alpha(x,y)\ell(xy,y)[h]]\alpha(xy,\rho))$$
$$= (xy\rho, r(xy\rho,x)[g]r(xy\rho,xy)[\alpha(x,y)\ell(xy,y)[h]]\alpha(xy,\rho)) .$$

Now if condition (3.47) is satisfied, then clearly $[(x,g)(y,h)]\overline{\rho} = (x,g)[(y,h)\overline{\rho}]$. Also if for all $(y,h),(x,g) \in \overline{S}$ such that $(x,g)\cdot(y,h),(y,h) \in \operatorname{dom} \overline{\rho}$ one has $(x,g)[(y,h)\overline{\rho}] = [(x,g)\cdot(y,h)]\overline{\rho}$, then consider the case with $(x,g) = (x,1)$. Thus $r(xy\rho,x)[g] = r(xy\rho,x)[1] = 1$ and condition (3.47) follows.

Lemma 3.48. For each $\overline{\rho}_k \in P_1'$ and for every $(y,h),(x,g) \in \overline{S}$ such that $(x,g)\cdot(y,h),(y,h) \in \operatorname{dom} \overline{\rho}_k$, one has $(x,g)\cdot[(y,h)\overline{\rho}_k] = [(x,g)(y,h)]\overline{\rho}_k$.

PROOF. If $(y,h) \in \operatorname{dom} \overline{\rho}_k$, then there is a $\lambda$ in $\Lambda$ and $z \in S$ such that $\lambda z = y$ and $k \in \ell(z)$. Thus

$$
\begin{aligned}
(x,g)[(y,h)\overline{\rho}_k] &= (x,g)[(\lambda z,h)\overline{\rho}_k] \\
&= (x,g)(\lambda z,h\ell(y,z)[k]) \\
&= (xy,\pi(xy,x)[g]\alpha(x,y)\ell(xy,y)[h\ell(y,z)[k]]) \\
&= (xy,\pi(xy,x)[g]\alpha(x,y)\ell(xy,y)[h]\ell(xy,z)[k]) \\
&= (xy,\pi(xy,x)[g]\alpha(x,y)\ell(xy,y)[h])\overline{\rho}_k \\
&= [(x,g)\cdot(y,h)]\overline{\rho}_k \ .
\end{aligned}
$$

Conditions (3.22), (3.45), (3.47), and the dual of condition (3.47) may be combined into the following condition:

(3.49) $\quad \alpha(a,bc)\ell(abc,bc)[\pi(bc,b)[h]\alpha(b,c)]$

$$= \pi(abc,ab)[\alpha(a,b)\ell(ab,b)[h]]\alpha(ab,c)$$

for all $(a,b,c) \in (S\times S\times S) \cup (B_1\times S) \cup (S\times B_2) \cup (\Lambda\times B_2 \cap B_1\times P)$ and $h \in \ell(b)$ .

Theorem 3.50. Consider the system $(S,\Lambda,P,\ell,\pi,\alpha)$ . $(\overline{S},\overline{\Lambda},\overline{P}) \in \mathrm{obj}(\overline{GP})$ iff condition (3.49) is satisfied. Also if $(\overline{S},\overline{\Lambda},\overline{P}) \in \mathrm{obj}(\overline{GP})$ , then $((\overline{S},\overline{\Lambda},\overline{P}),\pi)) \in \mathrm{obj}(\overline{GP}_S)$ .

PROOF. If $(\overline{S},\overline{\Lambda},\overline{P}) \in \mathrm{obj}(\overline{GP})$ , then by Lemma 3.21 condition (3.22) is satisfied, by Lemma 3.44 condition (3.45) is satisfied, and by Lemma 3.46 (and its dual) condition (3.47) (and its dual) are satisfied. Thus $(\overline{S},\overline{\Lambda},\overline{P}) \in \mathrm{obj}(\overline{GP})$ gives condition (3.49) .

Conversely, suppose condition (3.49) is satisfied. By Lemmas 3.21, 3.44, 3.46, and the dual of 3.46, one has $(\overline{S},\overline{\Lambda},\overline{P}) \in \mathrm{obj}(GP)$ . To show $(\overline{S},\overline{\Lambda},\overline{P}) \in \mathrm{obj}(\overline{GP})$ one must show that for each $(y,h) \in \mathrm{dom}\ \overline{\rho}$ one has $(x,g)\cdot(y,h) \in \mathrm{dom}\ \overline{\rho}$ for every $(x,g) \in \overline{S}$ and for each $(y,h) \in \mathrm{dom}\ \overline{\rho}_k$ one has $(x,g)\cdot(y,h) \in \mathrm{dom}\ \overline{\rho}_k$ for every

$(x,g) \in \overline{S}$ . If $(y,h) \in \operatorname{dom} \overline{\rho}$ , then $y \in \operatorname{dom} \rho$ . Thus $xy \in \operatorname{dom} \rho$ and $(xy,w) \in \operatorname{dom} \overline{\rho}$ for all $w \in \ell(xy)$ . Therefore $(x,g)\cdot(y.h) = (xy, r(xy,x)[g]\alpha(x,y)\ell(xy,y)[h]) \in \operatorname{dom} \overline{\rho}$ . Next if $(y,h) \in \operatorname{dom} \overline{\rho}_k$ , then there is a $z \in S$ and $\lambda \in \Lambda$ such that $k \in \ell(z)$ and $y = \lambda z$ . Consider $G^y_{xy} \in \Lambda$ and note that $xy = G^y_{xy} y = G^y_{xy} \lambda z$ . Hence by the definition of $\overline{\rho}_k$ , $(xy,w) \in \operatorname{dom} \overline{\rho}_k$ for all $w \in \ell(xy) = ((G^y_{xy}\lambda)z)$ . Therefore $(x,g)\cdot(y,h) \in \operatorname{dom} \overline{\rho}_k$ . Thus $(\overline{S},\overline{\Lambda},\overline{P}) \in \operatorname{obj}(\overline{GP})$ .

Finally, if $(\overline{S},\overline{\Lambda},\overline{P}) \in \operatorname{obj}(\overline{GP})$ , then Theorem 3.25 gives $(\overline{S},\overline{\Lambda},\overline{P}) \in \operatorname{obj}(GPX_S)$ . But clearly $\pi:\overline{S} \to S$ is a homomorphism. Hence $((\overline{S},\overline{\Lambda},\overline{P}),\pi) \in \operatorname{obj}(\overline{GP}_S)$ .

Now that the construction principles have been investigated, the reconstruction of a $\underline{\leq}(G)$-coextension of $(S,\Lambda_1,P_1)$ is considered. This will follow closely the previous reconstruction done for the category $GPX_X$ .

Lemma 3.51. Let $((\tilde{S},\tilde{\Lambda}_1,\tilde{P}_1),p) \in \operatorname{obj}(\underline{\leq}(G_S))$ and let $\underline{\leq}(C)$ be an $H$ quasi-congruence under $\underline{\leq}(\tilde{H})$ with $C = C_p$ . Let "~" denote a lifting from $(S,\Lambda,P)$ to $(\tilde{S},\tilde{\Lambda},\tilde{P}) = B\circ A(\tilde{S},\tilde{\Lambda}_1,\tilde{P}_1)$ and let $\ell:\underline{\leq}L \to \underline{\text{Mon}}$ be the functor induced by $\underline{\leq}(C)$ in Lemma 3.26. If $\alpha:((S\times S) \cup B) \to \bigcup_{s\in S} \ell(s)$ is defined by $\alpha(x,y) = \tilde{D}^{\widetilde{xy}}_{\tilde{x}\tilde{y}}$ for all $(x,y) \in S\times S$ and $\alpha\big|_B$ is defined as in Lemma 3.28, then $\alpha$ is a factor system for $\ell$ .

PROOF. To show $\alpha$ is a factor system for $\ell$ with $(S,\Lambda,P) \in \operatorname{obj}(\overline{GP})$ , it suffices to show $\tilde{D}^{\widetilde{xy}}_{\tilde{x}\tilde{y}} \in \ell(xy)$ for all

$(x,y) \in S\times S$ . Now $D^{\widetilde{xy}}_{\widetilde{xy}}$ will be in $\ell(xy)$ if $p(\widetilde{\tilde{x}\tilde{y}}) = p(\widetilde{xy})$ since $C_p = C \subseteq \leq(C)$ . But $p(\widetilde{xy}) = xy = p(\tilde{x})p(\tilde{y}) = p(\tilde{x}\tilde{y})$ . Therefore $D^{\widetilde{xy}}_{\widetilde{xy}} \in \ell(xy)$ and $\alpha$ is a factor system for $\ell$ .

Lemma 3.52. (With the notation of Lemma 3.51.) Let $\alpha$ be the factor system described in Lemma 3.51 along with $\ell$ and $r$ the functors described in Lemmas 3.26 and 3.27, respectively. Then the triple $(\ell,r,\alpha)$ satisfies condition (3.49).

PROOF. To show $(\ell,r,\alpha)$ satisfies condition (3.49) it suffices to show condition 3.45 since Lemma 3.31 shows the rest of 3.49. Now the proof that 3.45 is satisfied is the same computation as given in the proof of Lemma 3.31 where the triple $(\lambda,y,\rho)$ in the proof of Lemma 3.31 is replaced by the triple $(x,y,z)$ . Thus the proof is omitted.

Theorem 3.53. If $((\tilde{S},\tilde{\Lambda}_1,\tilde{P}_1),p) \in obj(\leq(G_S))$ with $\leq(C)$ an $H$ quasi-congruence under $\leq(\tilde{H})$ such that $C = C_p$ , then the functors and the factor system defined in Lemmas 3.26, 3.27, and 3.51 give rise to a triple $(\overline{S},\overline{\Lambda},\overline{P})$ such that $(\overline{S},\overline{\Lambda},\overline{P})$ is in $obj(\overline{GP})$ , $((\overline{S},\overline{\Lambda},\overline{P}),\pi_1) \in obj(\overline{GP}_S)$ where $\pi_1:\overline{S} \to S$ is the first coordinate projection, and there is a surmorphism $\Phi_1:(\overline{S},\overline{\Lambda},\overline{P}) \to (\tilde{S},\tilde{\Lambda}_1,\tilde{P}_1)$ .

PROOF. Lemma 3.52 shows that the triple $(\ell,r,\alpha)$ satisfies condition (3.49). Hence by Theorem 3.50, $((\overline{S},\overline{\Lambda},\overline{P}),\pi_1) \in obj(\overline{GP}_S)$ . Now by Theorem 3.32, the map $\Phi_1:(\overline{S},\overline{\Lambda},\overline{P}) \to (\tilde{S},\tilde{\Lambda}_1,\tilde{P}_1)$ defined by $(x,D^{\tilde{x}}_{\tilde{u}}) \mapsto u$ is a surmorphism in GPX . Now it follows from Lemma 3.18

in [7] that the map $\Phi_1:\overline{S} \to S$ is a homomorphism. Hence $\Phi_1:(\overline{S},\overline{\Lambda},\overline{P}) \to (\tilde{S},\tilde{\Lambda}_1,\tilde{P}_1)$ is a surmorphism in $\mathrm{Mor}(\overline{GP})$ .

Corollary 3.54. If $((\tilde{S},\tilde{\Lambda}_1,\tilde{P}_1),p) \in \mathrm{obj}(\underline{\leq}(G_S))$ , then there is a pair of functors $\underline{\ell}:\underline{\leq}L \to \underline{\mathrm{Grp}}$ , $\underline{r}:\underline{\leq}R \to \underline{\mathrm{Grp}}$ and a factor system $\alpha$ for $\underline{\ell}$ such that the triple $(\overline{S},\overline{\Lambda},\overline{P})$ is in $\mathrm{obj}(\overline{GP})$ , $((\overline{S},\overline{\Lambda},\overline{P}),\pi) \in \mathrm{obj}(\underline{\leq}(G_S))$ , and there is an isomorphism $\Phi:(\overline{S},\overline{\Lambda},\overline{P}) \to (\tilde{S},\tilde{\Lambda}_1,\tilde{P}_1)$ such that $p\circ\Phi = \pi$ .

PROOF. Apply Theorem 3.53 with the quasi-congruence $\underline{\leq}(C)$ equal $C_p$ . This gives a pair of functors $\underline{\ell}:\underline{\leq}L \to \underline{\mathrm{Mon}}$ and $\underline{r}:\underline{\leq}R \to \underline{\mathrm{Mon}}$ and a factor system $\alpha$ for $\underline{\ell}$ . Since the triple $(\underline{\ell},\underline{r},\alpha\big|_B)$ is the triple obtained in Theorem 3.32, one may apply Corollary 3.33 to obtain the codomain of $\underline{\ell}$ and $\underline{r}$ is $\underline{\mathrm{Grp}}$ and $\Phi:(\overline{S},\overline{\Lambda},\overline{P}) \to (\tilde{S},\tilde{\Lambda}_1,\tilde{P}_1)$ is an isomorphism in $\mathrm{Mor}(GPX_S)$ . Now by Theorem 3.53 $(\overline{S},\overline{\Lambda},\overline{P}) \in \mathrm{obj}(\overline{GP})$ and $\phi$ is a homomorphism. Hence $\Phi:(\overline{S},\overline{\Lambda},\overline{P}) \to (\tilde{S},\tilde{\Lambda}_1,\tilde{P}_1)$ is an isomorphism in $\mathrm{Mor}(\overline{GP}_S)$ . Therefore since $((\tilde{S},\tilde{\Lambda}_1,\tilde{P}_1),p) \in \mathrm{obj}(\underline{\leq}(G_S))$ , $((\overline{S},\overline{\Lambda},\overline{P}),p\circ\Phi) \in \mathrm{obj}(\underline{\leq}(G_S))$ . But by Corollary 3.33, $p\circ\Phi = \pi$ . Thus $((\overline{S},\overline{\Lambda},\overline{P}),\pi) \in \mathrm{obj}(\ (G_S))$ .

Theorem 3.55. If $(\underline{\ell}:\underline{\leq}L \to \underline{\mathrm{Grp}}$ , $\underline{r}:\underline{\leq}R \to \underline{\mathrm{Grp}})$ is a pair of compatible functors with a factor system $\alpha$ , then $(\overline{S},\overline{\Lambda},\overline{P}) \in \mathrm{obj}(\overline{GP})$ iff $((\overline{S},\overline{\Lambda},\overline{P}),\pi) \in \mathrm{obj}(\underline{\leq}(G_S))$ iff condition (3.49) holds.

PROOF. By Theorem 3.50, $(\overline{S},\overline{\Lambda},\overline{P}) \in \mathrm{obj}(\overline{GP})$ iff condition (3.49) holds. Hence to finish the proof it suffices to show that if $(\overline{S},\overline{\Lambda},\overline{P}) \in \mathrm{obj}(\overline{GP})$ , then $((\overline{S},\overline{\Lambda},\overline{P}),\pi) \in \mathrm{obj}(\underline{\leq}(G_S))$ . Thus assume $(\overline{S},\overline{\Lambda},\overline{P}) \in \mathrm{obj}(\overline{GP})$ . By the second part of Theorem 3.35,

$((\overline{S},\overline{\Lambda},\overline{P}),\pi)$ is a $\leq(\mathcal{G})$-coextension of $(S,\Lambda,P)$ in the category GPX and so $\mathcal{C}_\pi \subseteq \overline{\mathcal{H}}$. Therefore since $(\overline{S},\overline{\Lambda},\overline{P}) \in \text{obj}(\overline{\mathcal{GP}})$ and $\mathcal{C}_\pi \subseteq \overline{\mathcal{H}}$, $((\overline{S},\overline{\Lambda},\overline{P}),\pi) \in \text{obj}(\leq(\mathcal{G}_S))$.

Theorem 3.56. Let $(\ell,r,\alpha)$ be a triple such that $\overline{S}$ is a semigroup with codomain of $\ell$ being Grp. (Thus $\alpha$ need only be a function from $S\times S$ into $\bigcup_{s\in S} \ell(s)$ with $\alpha(x,y) \in \ell(xy)$ for all $(x,y) \in S\times S$.) If $\mathcal{B}\circ\mathcal{A}(S,\Lambda_1,P_1) = \mathcal{B}\circ\mathcal{A}(S,\Lambda_0,P_0)$, then there is a Green's pair $(\overline{\Lambda}',\overline{P}')$ for $\overline{S}$ such that $((\overline{S},\overline{\Lambda}',\overline{P}'),\pi) \in \text{obj}(\leq(\mathcal{G}_S))$.

PROOF. Let $\overline{\Lambda}_0 = \{\lambda_z : z \in \overline{S}^1\}$ and $\overline{P}_0 = \{\rho_z : z \in \overline{S}^1\}$. Let $\overline{\Lambda}'$ be the semigroup generated by $\overline{\Lambda}_0 \cup \Lambda_1'$ in $\mathcal{PT}(\overline{S})$, and let $\overline{P}'$ be the semigroup generated by $\overline{P}_0 \cup P_1'$ in $\mathcal{PT}(\overline{S})$. Clearly $(\overline{\Lambda}',\overline{P}')$ satisfies (i) and (ii) in the definition of a Green's pair. Now each of the pairs $(\overline{\Lambda}_0,P_1')$, $(\Lambda_1',\overline{P}_0)$, $(\Lambda_1',P_1')$, $(\overline{\Lambda}_0,\overline{P}_0)$ satisfies (iii), (iv), and (v) in the definition of a Green's pair for the following reasons: the first by Lemma 3.48; the second by the dual of Lemma 3.48; the third by Lemma 3.23; and the fourth by the fact that $\overline{S}$ is a semigroup. Thus since $\overline{\Lambda}_0 \cup \Lambda_1'$ generates $\overline{\Lambda}'$ and $\overline{P}_0 \cup P_1'$ generates $\overline{P}'$, the above shows $(\overline{S},\overline{\Lambda}',\overline{P}') \in \text{obj}(\mathcal{GP})$. Now since $(\overline{\Lambda}_0,\overline{P}_0) \subseteq (\overline{\Lambda}',\overline{P}')$, $(\overline{S},\overline{\Lambda}',\overline{P}') \in \text{obj}(\overline{\mathcal{GP}})$.

Clearly $\pi:\overline{S} \to S$ is a homomorphism onto $S$. Thus $\pi:(\overline{S},\overline{\Lambda}_0,\overline{P}_0) \to (S,\Lambda_0,P_0)$ is a morphism in $\text{Mor}(\overline{\mathcal{GP}})$. Now $\mathcal{B}\circ\mathcal{A}(S,\Lambda_1,P_1) = \mathcal{B}\circ\mathcal{A}(S,\Lambda_0,P_0)$ implies $\leq(\mathcal{L}) = \leq(\mathcal{L}_0)$ and $\leq(\mathcal{R}) = \leq(\mathcal{R}_0)$. Therefore $\pi$ a homomorphism from $\overline{S}$ onto $S$ gives $\pi:(\overline{S},\overline{\Lambda}_0,\overline{P}_0) \to (S,\Lambda_1,P_1)$ is a surmorphism in $\text{Mor}(\overline{\mathcal{GP}})$. Clearly for

each $\gamma \in \Lambda_1'$, $\pi(\gamma z) = 1_S \cdot \pi(z)$ for all $z \in \text{dom } \gamma$ and dually for each $\delta \in P_2'$, $\pi(z\delta) = (\pi z)1_S$ for all $z \in \text{dom } \delta$. Now it follows that $(\Lambda_1' \cup \overline{\Lambda}_0, P_1' \cup \overline{P}_0) \subseteq (\Lambda', P')$ where $(\Lambda', P')$ is defined in Lemma 2.6. Hence by Lemma 2.6 $\pi: (\overline{S}, \overline{\Lambda}', \overline{P}') \to (S, \Lambda_1, P_1)$ is a surmorphism. Now since $(\Lambda_1', P_1') \subseteq (\overline{\Lambda}', \overline{P}')$, one has that $C_\pi \subseteq \overline{H}'$. Therefore $((\overline{S}, \overline{\Lambda}', \overline{P}'), \pi) \in \text{obj}(\leq(G_S))$.

This completes the results dealing with $\leq(G)$-coextensions. The following deals with $\leq(S)$-coextensions.

Lemma 3.57. Let $((\tilde{S}, \tilde{\Lambda}_1, \tilde{P}_1), p) \in \text{obj}(\leq(S_S))$ and let "~" denote a lifting from $(S, \Lambda, P)$ to $(\tilde{S}, \tilde{\Lambda}, \tilde{P}) = \mathcal{B} \circ \mathcal{A}(\tilde{S}, \tilde{\Lambda}_1, \tilde{P}_1)$. Let $(\ell, r)$ be the pair of functors defined in Lemmas 3.37 and 3.38. If $\alpha: (S \times S) \cup B \to \bigcup_{s \in S} \ell(s)$ is defined by $\alpha(x,y) = \tilde{D}^{\widetilde{xy}}_{\tilde{x}\tilde{y}}$ for all $x,y \in S$ and $\alpha\big|_B$ is defined as in Lemma 3.39, then $\alpha$ is a factor system for $\ell$, $\alpha(G^x_u, x) = 1$ for all $(u,x) \in \leq(L)$, $\alpha(x, D^x_u) = 1$ for all $(u,x) \in \leq(R)$, and $(\ell, r, \alpha)$ satisfy condition (3.49).

PROOF. According to Lemma 3.39 $\alpha\big|_B$ is a factor system for $\ell$ (with $(S, \Lambda_1, P_1) \in \text{obj}(GPX)$), $\alpha(G^x_u, 1) = 1$ for all $(u,x) \in \leq(L)$, and $\alpha(x, D^x_u) = 1$ for all $(u,x) \in \leq(R)$. To show $\alpha$ is a factor system for $\ell$ (with $(S, \Lambda_1, P_1) \in \text{obj}(\overline{GP})$) it suffices to show $\alpha(x,y) \in \ell(xy)$ for all $x,y \in S$. Note that for $(x,y) \in S \times S$, $p(\widetilde{xy}) = xy = p(\tilde{x})p(\tilde{y}) = p(\tilde{x}\tilde{y})$. Thus $\widetilde{xy} C_p \tilde{x}\tilde{y}$ and by the definition of $\ell$, $\tilde{D}^{\widetilde{xy}}_{\tilde{x}\tilde{y}} \in \ell(xy)$. Hence $\alpha$ is a factor system for $\ell$.

The proof that condition (3.49) is satisfied by $(\ell, r, \alpha)$ is the same as that of Lemma (3.52) and is omitted.

Theorem 3.58. If $(\ell:\underline{\leq}L \to \underline{\text{Mon}}\ ,\ r:\underline{\leq}R \to \underline{\text{Mon}})$ is a pair of compatible functors with a factor system $\alpha$, then $((\overline{S},\overline{\Lambda},\overline{P}),\pi) \in \text{obj}(\underline{\leq}(S_S))$ iff conditions (3.49) and (3.42) are satisfied.

Conversely, if $((\tilde{S},\tilde{\Lambda}_1,\tilde{P}_1),p) \in \text{obj}(\underline{\leq}(S_S))$ and $(\ell,r,\alpha)$ is the triple described in Lemma 3.57, then the triple $(\overline{S},\overline{\Lambda},\overline{P})$ induced by $(\ell,r,\alpha)$ is in $\text{obj}(\overline{GP}_S)$ with the surmorphism $\pi$ and the map $\Phi:(\overline{S},\overline{\Lambda},\overline{P}) \to (\tilde{S},\tilde{\Lambda}_1,\tilde{P}_1)$, $(x,\tilde{D}_u^{\tilde{x}}) \mapsto u$, is an isomorphism such that $p\circ\Phi = \pi$.

PROOF. If the triple $(\ell,r,\alpha)$ satisfies conditions (3.49) and (3.42), then by the first part of Theorem 3.41, $((\overline{S},\overline{\Lambda},\overline{P}),\pi) \in \text{obj}(\underline{\leq}(S_S))$ when one considers the category GPX. Hence to show $((\overline{S},\overline{\Lambda},\overline{P}),\pi) \in \text{obj}(\underline{\leq}(S_S))$ in the category $\overline{GP}$, it suffices to show $(\overline{S},\overline{\Lambda},\overline{P}) \in \text{obj}(\overline{GP})$ and $\pi:\overline{S} \to S$ is a homomorphism. By Theorem 3.50 $(\overline{S},\overline{\Lambda},\overline{P}) \in \text{obj}(\overline{GP})$ and clearly $\pi$ is a homomorphism. Hence $((\overline{S},\overline{\Lambda},\overline{P}),\pi) \in \text{obj}(\underline{\leq}(S_S))$.

For the converse of the first part of the theorem consider $((\overline{S},\overline{\Lambda},\overline{P}),\pi) \in \text{obj}(\underline{\leq}(S_S))$. By the first part of Theorem 3.41, $(\ell,r,\alpha)$ satisfies condition (3.42). Now since $(\overline{S},\overline{\Lambda},\overline{P}) \in \text{obj}(\overline{GP})$, Theorem 3.50 implies condition (3.49) is also satisfied.

For the second part of the theorem, let $((\tilde{S},\tilde{\Lambda}_1,\tilde{P}_1),p) \in \text{obj}(\underline{\leq}(S_S))$. By Lemma 3.57, the triple $(\ell,r,\alpha)$ satisfies condition (3.49). Now by the second part of Theorem 3.41 the triple $(\ell,r,\alpha\big|_B)$ (and hence $(\ell,r,\alpha)$ ) satisfies condition (3.42). Again by Theorem 3.41 the map $\Phi$ from $(\overline{S},\overline{\Lambda},\overline{P})$ to $(\tilde{S},\tilde{\Lambda}_1,\tilde{P}_1)$ is an isomorphism in $\text{Mor}(\text{GPX}_S)$. But as noted in Theorem 3.53 when

$(\overline{S},\overline{\Lambda},\overline{P})$ and $(\tilde{S},\tilde{\Lambda}_1,\tilde{P}_1)$ are objects in $\overline{GP}$, $\Phi$ is a homomorphism. Hence $\Phi$ is an isomorphism in $\mathrm{Mor}(\overline{GP}_S)$. This also gives $((\overline{S},\overline{\Lambda},\overline{P}),\pi) \in \mathrm{obj}(\underline{\leq}(S_S))$.

Theorem 3.59. Let $(\ell,r,\alpha)$ be a triple such that $\overline{S}$ is a semigroup with codomain of $\ell$ being Mon. (Thus $\alpha$ need only be a function from $S\times S$ into $\bigcup_{s\in S} \ell(s)$ with $\alpha(x,y) \in \ell(xy)$ for all $(x,y) \in S\times S$ and satisfy condition (3.45).) If $\mathcal{B}\circ\mathcal{A}(S,\Lambda_1,P_1) = \mathcal{B}\circ\mathcal{A}(S,\Lambda_0,P_0)$ and $\alpha(x,y)$ is in the group of units of $\ell(xy)$ for all $x,y \in S$, then there is a Green's pair $(\overline{\Lambda}',\overline{P}')$ on $\overline{S}$ such that $((\overline{S},\overline{\Lambda}',\overline{P}'),\pi) \in \mathrm{obj}(\underline{\leq}(S_S))$.

PROOF. Let $(\overline{\Lambda}',\overline{P}')$ denote the pair defined in the proof of Theorem 3.56. As shown in the proof of Theorem 3.56, $(\overline{S},\overline{\Lambda}',\overline{P}') \in \mathrm{obj}(\overline{GP})$ and $((\overline{S},\overline{\Lambda}',\overline{P}'),\pi) \in \mathrm{obj}(\overline{GP}_S)$. Thus to show $((\overline{S},\overline{\Lambda}',\overline{P}'),\pi) \in \mathrm{obj}(\underline{\leq}(S_S))$, it suffices to show that for the map $x \mapsto \overline{x} = (x,1)$ from $S$ to $\overline{S}$, "$\bar{}$" preserves the quasi-orders $\underline{\leq}(L)$ and $\underline{\leq}(R)$ and $(C_\pi)_{\overline{x}} \subseteq \overline{H}'(\overline{x})$ for all $x \in S$.

First note that $(C_\pi)_{\overline{x}} = \{x\}\times\ell(x)$. As it was pointed out in the proof of Theorem 3.56, $(\Lambda_1',P_1') \subseteq (\overline{\Lambda}',\overline{P}')$ implies $\{x\}\times\ell(x) \subseteq \overline{H}'(x,1) = \overline{H}'(\overline{x})$. Therefore $(C_\pi)_{\overline{x}} \subseteq \overline{H}'(\overline{x})$ for all $x \in S$. If $y\underline{\leq}(L)x$, then $\mathcal{B}\circ\mathcal{A}(S,\Lambda_1,P_1) = \mathcal{B}\circ\mathcal{A}(S,\Lambda_0,P_0)$ gives $y = zx$ for some $z \in S$. Thus one has $\lambda_{\overline{z}}\overline{x} = (z,1)\cdot(x,1) = (zx,\alpha(z,x)) = (y,\alpha(z,x))$. By assumption, $\alpha(z,x)$ is in the group of units of $\ell(zx) = \ell(y)$. Thus let $h = \alpha(z,x)^{-1} \in \ell(y)$. Hence $\overline{\lambda}_h \in \Lambda_1' \subseteq \overline{\Lambda}'$ and $\overline{\lambda}_h(y,\alpha(z,x)) = (y,h\alpha(z,x)) = (y,1) = \overline{y}$. Therefore $\overline{\lambda}_h\cdot\lambda_{\overline{z}}\overline{x} = \overline{y}$ and

since $\overline{\lambda}_h \circ \lambda_{\overline{z}} \in \overline{\Lambda}'$ , $\overline{y} \leq(\overline{L}')\overline{x}$ . Hence "$\overline{\ }$" preserves $\leq(L)$ and dually "$\overline{\ }$" preserves $\leq(R)$ . Thus $((\overline{S},\overline{\Lambda}',\overline{P}'),\pi) \in \text{obj}(\leq(S_S))$ .

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