

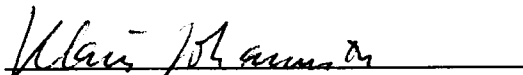
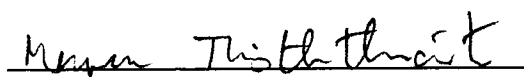
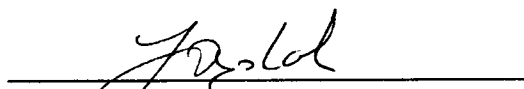
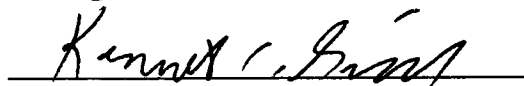
To the Graduate Council:

I am submitting herewith a dissertation written by Yongkuk Kim entitled "Codimension two submanifold decompositions that induce approximate fibrations ." I have examined the final copy of this dissertation for form and content and recommend that it be accepted in partial fulfillment of the requirements for the degree of Doctor of Philosophy, with a major in Mathematics.



Robert J. Daverman, Major Professor

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and recommend its acceptance:


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Associate Vice Chancellor and
Dean of the Graduate School

CODIMENSION TWO SUBMANIFOLD
DECOMPOSITIONS
THAT INDUCE APPROXIMATE
FIBRATIONS

A Dissertation

Presented for the

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Degree

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ABSTRACT

An extensive variety of closed n -manifolds N , called codimension-2 (orientable) fibrators, automatically induce approximate fibrations, in the sense that all proper maps $f: M \rightarrow B$ from any (respectively, orientable) $(n+2)$ -manifold M to a metric space B (equivalently, to a 2-manifold B) such that each $f^{-1}(b)$ has the same homotopy type (or, more generally, the same shape) as N are approximate fibrations. R.J. Daverman showed that closed hopfian manifolds with hopfian fundamental group and nonzero Euler characteristic, as well as closed hopfian manifolds with hyperhopfian fundamental group are codimension-2 orientable fibrators. Using the concept of s-hopfianness, we generalize Daverman's results about codimension-2 orientable fibrators to the orientation-free version as follows;

Closed s-hopfian manifolds with either hyperhopfian fundamental group or hopfian fundamental group and nonzero Euler characteristic are codimension-2 fibrators.

In the second part of this dissertation, we study the behavior of codimension-2 fibrators under the connected sum operation. As a consequence, we will get the neat result on closed 4-manifolds as follows;

Given a closed 4-manifold N which is a nontrivial connected sum of two 4-manifolds N_1 and N_2 , where $\pi_1(N_1)$ and $\pi_1(N_2)$ are hopfian, then N is a codimension-2 fibrator if and only if N is not homotopy equivalent to $RP^4 \# RP^4$.

In the next section we determine which closed n -manifolds ($n \leq 4$) with geometric structure are codimension-2 fibrators. In closing, we mention some unsettled topics.

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1. INTRODUCTION

There is a useful class of mappings called approximate fibrations, namely, those proper mappings which satisfy an approximate version of the homotopy lifting property defining the more familiar class of fibrations. An extensive variety of closed n -manifolds N , called *codimension-2 (orientable) fibrators*, automatically induce approximate fibrations, in the sense that all proper maps $f: M \rightarrow B$ from any (respectively, orientable) $(n+2)$ -manifold M to a metric space B (equivalently, to a 2-manifold B) such that each $f^{-1}(b)$ has the same homotopy type (or, more generally, the same shape) as N are approximate fibrations.

Our concern is the following:

Main question: Which manifolds N are codimension-2 fibrators?

Robert J. Daverman showed that all simply connected manifolds, closed surfaces with nonzero Euler characteristic, and closed manifolds N with $\pi_1(N) \cong \mathbb{Z}_2$ (for example, real projective n -spaces ($n > 1$)), are codimension-2 fibrators. Call a closed manifold N *hopfian* if it is *orientable* and every degree one map $N \rightarrow N$ which induces a π_1 -isomorphism is a homotopy equivalence. And he also showed that closed hopfian manifolds with hopfian fundamental group and nonzero Euler characteristic, as well as closed hopfian manifolds with hyperhopfian fundamental group are codimension-2 *orientable* fibrators.

Suppose that N is a closed n -manifold. If N has a 2-to-1 covering, we consider the covering space $\tilde{N}$ of N corresponding to H , where H is the intersection of all subgroups H_i of index 2 in $\pi_1(N)$, i.e., $H = \bigcap_{i \in I} H_i$ with $[\pi_1(N) : H_i] = 2$ for $i \in I$. Then we see that, by M. Hall's Theorem (for any finitely generated group G , the number of subgroups of G having any fixed finite index is finite), the index set I is finite,

and $\tilde{N}$ is an n -dimensional orientable manifold, which follows from the facts that a (finite) covering space of an n -dimensional orientable manifold is an n -dimensional orientable manifold and any non-orientable manifold has a 2-to-1 orientable covering. Call a closed manifold N *s-hopfian* if N is hopfian when N is orientable and $\tilde{N}$ is hopfian when N is nonorientable. Using this concept, we generalize Daverman's results about codimension-2 orientable fibrators to the orientation-free version as follows; Closed s-hopfian manifolds with either hyperhopfian fundamental group or hopfian fundamental group and nonzero Euler characteristic are codimension-2 fibrators.

The question of whether the collection of codimension 2 fibrators is closed under the Cartesian product operator remains unsolved. Y.H. Im gave an affirmative answer for the case of each N_j ($j = 1, 2, \dots, k$) a closed orientable surface. As an application, we will have an affirmative answer without assuming orientability for any N_j as follows; Any finite product of closed surfaces which are codimension-2 fibrators is a codimension-2 fibration.

On the other hand, surprisingly $RP^3 \# RP^3$ is not a codimension-2 fibration while RP^3 is. In the second part of this dissertation, we study the behavior of codimension-2 fibrators under the connected sum operation. In other words, we discuss the following question;

When is a connected sum of two closed manifolds a codimension-2 fibration?

First we review the basic cases under which connected sums of two closed n -manifolds ($n \geq 3$) are codimension-2 fibrators. Then we show that most closed hopfian n -manifolds $N = N_1 \# N_2$ with $\pi_1(N) \cong \mathbb{Z}_2 * \mathbb{Z}_2$ are codimension-2 orientable fibrators. Next we consider the major non-settled case which is a connected sum N of a simply connected manifold N_1 and a non-simply connected manifold N_2 . We show that if N is s-hopfian

and $\pi_1(N_2)$ is hopfian, and if the simply connected manifold N_1 has a nonzero betti number, then N is a codimension-2 fibration. As a consequence, we will get the complete answer on closed 4-manifolds as follows; Given a closed 4-manifold N which is a nontrivial connected sum of two 4-manifolds N_1 and N_2 , if $\pi_1(N_1)$ and $\pi_1(N_2)$ are hopfian, then N is a codimension-2 fibration, with only one exception $RP^4 \# RP^4$, up to homotopy type.

In the next section we determine which closed n -manifolds ($n \leq 4$) with geometric structure are codimension-2 fibrations. All closed surfaces except the torus and the Klein bottle are known to be codimension-2 fibrations; reformulated in terms of geometric structures, the spherical and hyperbolic surfaces are codimension-2 fibrations but the Euclidean surfaces are not. It is natural to ask whether closed 3-manifolds are codimension-2 fibrations as precisely, unequivocally delineated by their geometric structures; it has negative answer. In [20] Daverman completely determined which closed ORIENTABLE 3-manifolds with geometric structure are codimension-2 ORIENTABLE fibrations, except for the spherical 3-manifolds. First we update his result to orientation-free version. Then we deal with closed 4-manifolds with geometric structure. In closing, we mention some unsettled topics.

2. PRELIMINARIES

Throughout this dissertation, the symbols $\simeq$, $\approx$, and $\cong$ denote homotopy equivalence, homeomorphism, and isomorphism in that order. The symbol χ is used to denote Euler characteristic.

All manifolds are understood to be finite dimensional, connected, metric, and boundaryless. Whenever the presence of boundary is tolerated, the object will be called a manifold with boundary.

Homology and cohomology are computed with integer coefficients unless another coefficient module is mentioned.

A map $f : X \rightarrow Y$ is proper provided that $p^{-1}(C)$ is compact for any compact subset C of Y . Note that p is proper if and only if p is closed, with compact fibers.

A manifold N is said to be closed if N is compact, connected, and boundaryless.

A manifold N is aspherical if $\pi_i(N)$ is trivial for all $i > 1$.

A decomposition G of a metric space X is said to be upper semicontinuous if each $g \in G$ is a compact in X and if, for each $g \in G$ and each open subset U of X containing g , there exists another open subset V of X containing g such that every $g' \in G$ intersecting V is contained in U . It is clear that if G is a usc decomposition of X , then the decomposition map $p : X \rightarrow X/G$ is a proper map.

2.1. APPROXIMATE FIBRATIONS. Approximate fibrations were introduced by D. Coram and P. Duvall [9] as a generalization of Hurewicz fibrations and cell-like maps. A proper map $p : M \rightarrow B$ between locally compact $ANRs$ is called an *approximate fibration* if it has the following approximate homotopy lifting property: Given an open cover ϵ of B , an arbitrary space X , and two maps $g : X \rightarrow M$ and $F : X \times I \rightarrow B$ such that $p \circ g = F_0$, there exists a map $G : X \times I \rightarrow M$ such that $G_0 = g$ and $p \circ G$

is ϵ -close to F . The latter means: for each $z \in X \times I$ there exists an $U_z \in \epsilon$ such that $\{F(z), p \circ G(z)\} \subset U_z$.

Let N^n be a closed manifold. A proper map $p : M \longrightarrow B$ is N^n -like if each fiber $p^{-1}(b)$ is shape equivalent to N . For simplicity, we shall assume that each fiber $p^{-1}(b)$ in an N^n -like map to be an ANR having the homotopy type of N^n .

A significant attraction of approximate fibrations is an associated long exact sequence, comparable to that for fibrations, relating homotopy groups of domain, image and fiber.

Proposition 2.1.1. [9, Corollary 3.5] *If a proper map $p : M \longrightarrow B$ is an approximate fibration, for $b \in B$ and $e \in p^{-1}(b)$, there is a long exact sequence*

$$\cdots \rightarrow \pi_k(p^{-1}(b), e) \rightarrow \pi_k(M, e) \rightarrow \pi_k(B, b) \rightarrow \pi_{k-1}(p^{-1}(b), e) \rightarrow \cdots$$

Proposition 2.1.2. [9] *If a proper map $p : M \longrightarrow B$ is an approximate fibration, then p is N -like for some closed manifold N .*

A proper map $p : M \longrightarrow B$ is a *completely movable* map provided that for each $b \in B$ and each neighborhood U of $p^{-1}(b)$ there is a neighborhood V of $p^{-1}(b)$ in U such that if $p^{-1}(c)$ is in V and W is any neighborhood of $p^{-1}(c)$ in V then there is a homotopy $H : V \times I \rightarrow U$ such that $H(x, 0) = x$ and $H(x, 1) \in W$ for each $x \in V$ and $H(x, t) = x$ for each $x \in p^{-1}(c)$.

Proposition 2.1.3. [10, Proposition 3.6] *Let a proper map $p : M \longrightarrow B$ be N -like. Then p is an approximate fibration if and only if p is completely movable.*

Proposition 2.1.4. [18, Lemma 2.5] *Suppose that $p : M \rightarrow B$ is a proper map defined on an $(n + k)$ -manifold M and $q : M^* \rightarrow M$ is a finite covering. Then p is an approximate fibration if and only if $p \circ q : M^* \rightarrow B$ is.*

2.2. DEGREE OF MAPS. Let N and N' be closed n -manifolds and $f : N \rightarrow N'$ be a map. If both N and N' are orientable, then the *degree of f* is the nonnegative integer d such that the induced endomorphism of $f_* : H_n(N; \mathbb{Z}) \cong \mathbb{Z} \rightarrow H_n(N'; \mathbb{Z}) \cong \mathbb{Z}$ amounts to multiplication by d , up to sign. In general, the *degree mod 2 of f* is the nonnegative integer d such that the induced endomorphism of $f_* : H_n(N; \mathbb{Z}_2) \cong \mathbb{Z}_2 \rightarrow H_n(N'; \mathbb{Z}_2) \cong \mathbb{Z}_2$ amounts to multiplication by d .

Proposition 2.2.1. [47] *If $\theta : L \rightarrow N$ is a d -fold covering map, $d > 0$, then the degree of θ is d and $\chi(L) = d\chi(N)$.*

Proposition 2.2.2. [52, Lemma 15.12] *Let X and Y be closed orientable n -manifolds. If $f : X \rightarrow Y$ has degree one, then $f_\# : \pi_1(X) \rightarrow \pi_1(Y)$ is an epimorphism.*

Remark. In Proposition 2.2.2, we cannot replace the condition $\deg f = 1$ by $\deg f = 1 \pmod{2}$ (see [4]).

Proposition 2.2.3. *Let X and Y be closed orientable n -manifolds and $f : X \rightarrow Y$ a map with non-zero degree t . Then,*

- 1) $\beta_i(X) \geq \beta_i(Y)$ for all i .
- 2) *If there exists a j such that $(|TH_j(Y)|, t) = 1$, where $TH_j(Y)$ is the torsion subgroup of $H_j(Y)$, then $H_j(Y)$ is isomorphic to a subgroup of $H_j(X)$.*
- 3) *If there exists a torsion group $H_j(Y)$ such that $(|H_j(Y)|, t) = 1$, then $f^* : H^{n-j}(Y) \rightarrow H^{n-j}(X)$ is injective and $f_* : H_j(X) \rightarrow H_j(Y)$ is surjective.*

4) For any prime number s with $(s, t) = 1$, $f^*: H^{n-j}(Y; \mathbb{Z}_s) \rightarrow H^{n-j}(X; \mathbb{Z}_s)$ is injective and $f_*: H_j(X; \mathbb{Z}_s) \rightarrow H_j(Y; \mathbb{Z}_s)$ is surjective.

5) If $t = 1$, then $f_*: H_i(X) \rightarrow H_i(Y)$ is surjective for all i .

Proof. Let Γ_X and Γ_Y be the orientation classes of X and Y , respectively. Let G be an abelian group. Define $g: H^{n-i}(X; G) \rightarrow H^{n-i}(Y; G)$ by $g = D_Y^{-1} \circ f_* \circ D_X$, where D_X and D_Y are the Poincaré duality maps. Consider the following commutative diagram

$$\begin{array}{ccccc} H^{n-i}(X; G) & \xleftarrow{f^*} & H^{n-i}(Y; G) & \xlongequal{\quad} & H^{n-i}(Y; G) \\ D_X \downarrow \cong & & \cap t \Gamma_Y \downarrow & & \times t \downarrow \\ H_i(X; G) & \xrightarrow{f_*} & H_i(Y; G) & \xrightarrow[\cong]{D_Y^{-1}} & H^{n-i}(Y; G). \end{array}$$

Then, since $g(f^*(a)) = D_Y^{-1} f_* D_X f^*(a) = D_Y^{-1} f_*(f^*(a) \cap \Gamma_X) = D_Y^{-1}(a \cap f_*(\Gamma_X)) = D_Y^{-1}(a \cap \Gamma_Y)(\deg f) = ta$, f^* is monic mod torsion, so we have that $\beta_i(Y) \leq \beta_i(X)$.

2) and 3) follow immediately from the fact that $(g \circ f^*) = (\times t) \circ \text{Id}: H^{n-i}(Y) \rightarrow H^{n-i}(Y)$ is an injection.

4) In the above diagram, since $(s, t) = 1$, we see that $\cap t \Gamma_Y: H^{n-i}(Y; \mathbb{Z}_s) \rightarrow H_i(Y; \mathbb{Z}_s)$ is an isomorphism. This proves 4).

5) is obvious. □

2.3. HOPFIAN MANIFOLDS AND HOPFIAN GROUPS. A group Γ is said to be *hopfian* if every epimorphism $f: \Gamma \rightarrow \Gamma$ is necessarily an isomorphism. A finitely presented group Γ is said to be *hyperhopfian* if every homomorphism $f: \Gamma \rightarrow \Gamma$ with $f(\Gamma)$ normal and $\Gamma/f(\Gamma)$ cyclic is an isomorphism(onto). By definition, hyperhopfian groups are hopfian. A group Γ is said to be *residually finite* if for any non-trivial element x of Γ there is a homomorphism f from Γ onto a finite group K such that $f(x) \neq 1_K$.

Residually finite groups versus hopfian groups : [1], [2], [59], [70], and [71].

1. Every finitely generated residually finite group is hopfian.
2. If G is finitely generated and G contains a residually finite group (hopfian group) of finite index, then G is residually finite (hopfian, respectively).
3. If G is a residually finite group, then so is every subgroup of G . But (finitely generated) hopfian groups can contain non-hopfian subgroups (of finite index).
4. The free product of two finitely generated residually finite groups (hopfian groups) is again residually finite (hopfian, respectively).
5. The direct product of two finitely generated residually finite groups is again residually finite. This is not true for hopfian groups. However, we don't know whether it is for finitely presented hopfian groups.
6. Polycyclic groups are residually finite. In particular, finitely generated nilpotent groups are residually finite.

Call a closed manifold N *hopfian* if it is *orientable* and every degree one map $N \longrightarrow N$ which induces a π_1 -isomorphism is a homotopy equivalence.

The next result summarizes useful information connecting hopfian manifolds and hopfian fundamental groups.

Proposition 2.3.1. [22, Theorem 2.2] or [50] *A closed, orientable n -manifold N is a hopfian manifold if any one of the following conditions holds:*

- 1) $n \leq 4$;
- 2) $\pi_1(N)$ contains a nilpotent subgroup of finite index;
- 3) $\pi_i(N)$ is trivial for $1 < i < n - 1$.

Remark. Whether all closed orientable manifolds are hopfian is related to a famous old problem of Hopf (cf. [50]).

Suppose that N is a closed n -manifold and a proper map $p : M \longrightarrow B$ is N -like. Let G be the set of all fibers, i.e., $G = \{p^{-1}(b) : b \in B\}$. Put

$C = \{p(g) \in B : g \in G \text{ and there exist a neighborhood } U_g \text{ of } g \text{ in } M \text{ and a retraction } R_g : U_g \longrightarrow g \text{ such that } R_g|_{g'} : g' \longrightarrow g \text{ is a degree one map for all } g' \in G \text{ with } g' \in G \text{ in } U_g\}$, and

$C' = \{p(g) \in B : g \in G \text{ and there exist a neighborhood } U_g \text{ of } g \text{ in } M \text{ and a retraction } R_g : U_g \longrightarrow g \text{ such that } R_g|_{g'} : g' \longrightarrow g \text{ is a degree one mod 2 map for all } g' \in G \text{ with } g' \in G \text{ in } U_g\}$.

Call C the *continuity set of p* and C' the *mod 2 continuity set of p* . D. Coram and P. Duvall [11] showed that C and C' are dense, open subsets of B .

Proposition 2.3.2. [22, Theorem 2.1] *If N is a closed hopfian n -manifold with hopfian $\pi_1(N)$ and $p : M \rightarrow B$ is an N -like proper map defined on an orientable $(n+2)$ -manifold, then p is an approximate fibration over its continuity set.*

2.4. CODIMENSION-2 FIBRATORS.

Definition. A closed n -manifolds N is a codimension-2 (orientable) fibration if any N -like proper map $f : M \rightarrow B$ from any (respectively, orientable) $(n+2)$ -manifold M to a metric space B (equivalently, to a 2-manifold B) is an approximate fibration.

The following is basic for investigating codimension-2 fibrators.

Proposition 2.4.1. [11, Proposition 2.8] *Let $p : M \longrightarrow B$ be a map from an orientable $(n + 2)$ -manifold M onto a metric space B such that each $p^{-1}(b)$ is an orientable n -manifold up to shape. Then B is a 2-manifold and $D = B \setminus C$ is locally finite in B , where C represents the continuity set of p . Moreover, if either M or some $p^{-1}(b)$ are nonorientable, B is a 2-manifold with boundary (possibly empty) and $D' = (\text{int } B) \setminus C'$ is locally finite in B , where C' represents the mod 2 continuity set of p .*

Proposition 2.4.2. [18] *Closed manifolds N with $|\pi_1(N)| \leq 2$ are codimension-2 fibrators. Hence all simply connected manifolds and any real projective n -space RP^n are codimension-2 fibrators.*

Remark. Daverman and Kim [31] generalize this result as follows;
Closed manifolds N with finite $\pi_1(N)$ and 2-group $H_1(N)$ are codimension-2 fibrators.

Proposition 2.4.3. [22] *Let N be a closed hopfian manifold. If N has either hyperhopfian fundamental group, or hopfian fundamental group and nonzero Euler characteristic, then N is a codimension-2 orientable fibrator.*

Recently N. Chinen show that some codimension-2 orientable fibrators are codimension-2 fibrators as follows;

Proposition 2.4.4. [4] *Let N be a closed n -manifold and a proper map $p : M \rightarrow B$ defined on an $(n + 2)$ -manifold M be N -like. If N has no 2-to-1 covering, for any $b \in B$, $p^{-1}(b)$ has a small orientable neighborhood in M . Hence, codimension-2 orientable fibrators which do not have a 2-to-1 covering are codimension-2 fibrators.*

2.5. MANIFOLDS THAT FAIL TO BE CODIMENSION-2 FIBRATORS. S^1 and $RP^n \# RP^n$ fail to be codimension-2 fibrators because of the following fact:

Proposition 2.5.1. [18] *Any closed manifold that cyclically covers itself (nontrivially) fails to be a codimension-2 fibrator. Hence any mapping torus of some periodic homeomorphism defined on a closed manifold fails to be a codimension-2 fibrator.*

Remark. Recently, Daverman presented non-codimension-2 fibrators among closed manifolds with finite fundamental group, for example, $L(p, q) \times S^3$ (p is odd), where $L(p, q)$ is any lens space (see [27]).

Remark. At the time of writing this dissertation, the existence of codimension-2 orientable fibrators which fail to be codimension-2 fibrators, is still unknown.

Proposition 2.5.2. *Let N_1^n and N_2^m be closed manifolds. If N_1 is not a codimension-2 fibrator, $N_1 \times N_2$ is not a codimension-2 fibrator.*

Proof. Take an N_1 -like proper map $p : M^{n+2} \rightarrow B^2$ which fails to be an approximate fibration. Then the composition map $M^{n+2} \times N_2^m \xrightarrow{\text{projection}} M^{n+2} \xrightarrow{p} B^2$ fails to be an approximate fibration. □

3. S-HOPFIAN MANIFOLDS AS CODIMENSION-2 FIBRATORS

Let N be a closed n -manifold which has a 2-to-1 covering. Consider the covering space $\tilde{N}$ of N corresponding to H , where $H = \bigcap_{i \in I} H_i$ with $[\pi_1(N) : H_i] = 2$ for $i \in I$. Then we see that, by M. Hall's Theorem (for any finitely generated group G , the number of subgroups of G having any fixed finite index is finite), the index set I is finite, and $\tilde{N}$ is an n -dimensional orientable manifold, which follows from the facts that a (finite) covering space of an n -dimensional orientable manifold is an n -dimensional orientable manifold and any non-orientable manifold has a 2-to-1 orientable covering.

To get some orientation-free version of Proposition 2.4.3, we use the following definition, which is a technically improved definition of the strongly hopfian concept introduced in [64].

Definition. A closed manifold N is *s-hopfian* if N is hopfian when N is orientable, and $\tilde{N}$ is hopfian when N is non-orientable, where $\tilde{N}$ is the covering space of N corresponding to $H = \bigcap_{i \in I} H_i$ with $I = \{i : [\pi_1(N) : H_i] = 2\}$.

From now on, we reserve the symbols $\tilde{N}$ and H for the above meanings.

Note that in the above definition, we do not hypothesize the hopfianness of $\tilde{N}$ when N is orientable.

Examples of s-hopfian manifolds.

- 1) Every closed manifold with abelian fundamental group;
- 2) Every closed manifold with finite fundamental group;
- 3) Every closed aspherical manifold;
- 4) Every closed n -manifold ($n \leq 4$).

3.1. MANIFOLDS N WITH HYPERHOPFIAN $\pi_1(N)$.

Lemma 3.1.1. *Let N be a closed manifold. Suppose $f : N \rightarrow N$ induces the isomorphism $f_* : H_1(N; \mathbb{Z}_2) \rightarrow H_1(N; \mathbb{Z}_2)$. Then $f_{\#}^{-1}(H) = H$, where $f_{\#} : \pi_1(N) \rightarrow \pi_1(N)$. In particular, if $f : N \rightarrow N$ has degree one mod 2, then $f_{\#}^{-1}(H) = H$.*

Proof. It suffices to show that

$$H = \ker \left[\pi_1(N) \xrightarrow[\text{Hurewicz map}]{h} H_1(N) \xrightarrow{g} H_1(N; \mathbb{Z}_2) \right]$$

from considering the following diagram

$$\begin{array}{ccc} \pi_1(N) & \xrightarrow{f_{\#}} & \pi_1(N) \\ \downarrow & & \downarrow \\ H_1(N) & & H_1(N) \\ \downarrow & & \downarrow \\ H_1(N; \mathbb{Z}_2) & \xrightarrow[\cong]{f_*} & H_1(N; \mathbb{Z}_2). \end{array}$$

Let H_i be a subgroup of $\pi_1(N)$ with index 2. Since $\pi_1(N)/H_i \cong \mathbb{Z}_2$, which is cyclic (and so is abelian), the commutator subgroup $[\pi_1(N), \pi_1(N)]$ of $\pi_1(N)$ must be contained in H_i for all i . It follows from the fact $\ker h = [\pi_1(N), \pi_1(N)]$ that $h(H_i) \equiv \overline{H_i}$ is index 2 in $H_1(N)$.

Here, note

$$\ker g = \text{im}[H_1(N) \xrightarrow{\times 2} H_1(N)]$$

which is the intersection of all subgroups of $H_1(N)$ with index 2. Hence $\bigcap \overline{H_i} \subset \ker g$.

Actually we also see that $\ker g \subset \bigcap \overline{H_i}$. □

Lemma 3.1.2. *Let N be a closed manifold. Suppose that $f : \pi_1(N) \rightarrow \pi_1(N)$ is a homomorphism whose induced action on $H_1(N; \mathbb{Z}_2)$ is an automorphism (i.e., $f(H) \subset$*

H and the natural map $f': \pi_1(N)/H \rightarrow \pi_1(N)/H$ is an isomorphism). Then

1) f is an epimorphism if and only if $(f | H): H \rightarrow H$ is an epimorphism.

2) f is an isomorphism if and only if $(f | H): H \rightarrow H$ is an isomorphism.

Proof. First, note that f' is an isomorphism if and only if $g \circ f$ is an epimorphism and $H = f^{-1}(H) = \ker(g \circ f)$, where $g: \pi_1(N) \rightarrow \pi_1(N)/H$.

If $\pi_1(N) = H \cup g_1H \cup \dots \cup g_kH$, then $\pi_1(N)/H = \{H, g_1H, \dots, g_kH\}$.

Proof of (1). Suppose f is an epimorphism. Since $f^{-1}(H) = H$, we have $H = ff^{-1}(H) = f(H)$, i.e., $(f | H): H \rightarrow H$ is an epimorphism. Conversely, suppose $(f | H): H \rightarrow H$ is an epimorphism, i.e., $H = f(H)$. Let $y \in \pi_1(N)$ but $y \notin H$. Then $y \in g_iH$ for some i . Since f' is an isomorphism, there exists uniquely a g_jH such that $f(g_j)H = f'(g_jH) = g_iH$. So $y = f(g_j)h$ for some $h \in H$. But since $H = f(H)$, there exists $h' \in H$ such that $f(h') = h$. Put $x = g_jh'$. Then $f(x) = f(g_jh') = f(g_j)f(h') = f(g_j)h = y$.

Proof of (2). Suppose f is an isomorphism. Then, by 1), $f(H) = H = f^{-1}(H)$, so $(f | H)$ is an isomorphism. Conversely, suppose $(f | H): H \rightarrow H$ is an isomorphism. Then, by 1), f is an epimorphism. If $x \in \ker f$, then $f(x) = e \in H$. But since $f^{-1}(H) = H$, $x \in H$, and so $x = e$. $\square$

The following recent result of Chinen give us useful information about manifolds with hyperhopfian fundamental group being codimension-2 fibrators.

Lemma 3.1.3. [5, Theorem 5.3] *Let N be a closed n -manifold with hyperhopfian fundamental group. If every N -like proper map $p: M^{n+2} \rightarrow B^2$ from an $(n+2)$ -manifold M onto a 2-manifold B is an approximate fibration over C' , where C' denotes the mod 2 continuity set of p , then N is a codimension-2 fibrator.*

The next lemma plays an important role in dealing with s-hopfian manifolds.

Lemma 3.1.4. *Let N be a closed s-hopfian n -manifold with hopfian fundamental group. Let a proper map $p : M \rightarrow B$ defined on an $(n + 2)$ -manifold M be N -like. Then, p is an approximate fibration over some dense open subset O of the mod 2 continuity set C' of p and $C' \setminus O$ is locally finite.*

Proof. Let $G = \{p^{-1}(b) \equiv g_b : b \in B\}$. Fix $g_0 \in G$ with $p(g_0) \in C'$. Take a neighborhood $U(\subset C')$ of $p(g_0)$ such that $p^{-1}(U)$ retracts to g_0 , and take a smaller connected neighborhood V of $p(g_0)$ such that $p^{-1}(V)$ deformation-retracts to g_0 in $p^{-1}(U)$. Call this retraction $R : p^{-1}(V) \rightarrow g_0$. Then, we have an epimorphism $R_{\#} : \pi_1(p^{-1}(V)) \rightarrow \pi_1(g_0)$.

If N has no 2-to-1 covering, in light of Proposition 2.4.4, we may assume that V is orientable. By Proposition 2.3.2 and Proposition 2.4.1, p is an approximate fibration over some dense open subset O_V of V and $V \setminus O_V$ is locally finite.

Now assume that N has a 2-to-1 covering. Take the covering map $q : V^* \rightarrow p^{-1}(V)$ corresponding to $R_{\#}^{-1}(H)$. Since $[\pi_1(p^{-1}(V)) : R_{\#}^{-1}(H)] = [\pi_1(g_0) : H] < \infty$, q is finite.

Claim: For all $g \in G$ with $g \subset p^{-1}(V)$, $q^{-1}(g)$ and $\tilde{N}$ have the same homotopy type.

First, appealing to the method of the proof of claim in [4, Proposition 3.5], we see that for all $g \in G$ with $g \subset p^{-1}(V)$, $q^{-1}(g)$ is connected. Now, since $R \mid g : g \rightarrow g_0$ has degree one mod 2, by Lemma 3.1.1, we have $(R \mid g)_{\#}^{-1}(H) = H$ and $(R_{\#} \circ i_{\#})(H) = (R \mid g)_{\#}(H) \subset H$. Here note $i_{\#}(H) \subset R_{\#}^{-1}(H)$, for $(R_{\#} \circ i_{\#})(H) \subset H$.

Clearly $(q \mid g_C^*)_{\#}(\pi_1(g_C^*)) \cong K \subset H$ since $(R \mid g)_{\#}^{-1}(H) = H$ and $R_{\#}(R_{\#}^{-1}(H)) = H$.

Take the covering space $f : N_H \rightarrow g$ corresponding to H and let $i : g \rightarrow p^{-1}(V)$ be the inclusion.

Consider

$$\begin{array}{ccccc} & & & & V^* \\ & & & & \downarrow q \\ N_H & \xrightarrow{f} & g & \xrightarrow{i} & p^{-1}(V) \end{array}$$

Since $i_{\#}(f_{\#}(\pi_1(N_H))) = i_{\#}(H) \subset R_{\#}^{-1}(H)$, we have a unique lifting $\lambda : N_H \rightarrow V^*$ of $i \circ f$ so that $i \circ f = q \circ \lambda$. It follows from the fact of $\lambda(N_H) \subset q^{-1} \circ i(g)$ that $\lambda(N_H) \subset g_C^*$. Hence we have $H \subset K$. Consequently, we have $H = K$. This proves the claim.

Set $G^* = \{g^* \equiv q^{-1}(g) : g \in G \text{ with } g \subset p^{-1}(V)\}$, and let $p^* : V^* \rightarrow V^*/G^* \equiv B^* = V$ be the decomposition map. Consider

$$\begin{array}{ccc} V^* & \xrightarrow{p^*} & B^* \\ q \downarrow & & \parallel \\ p^{-1}(V) & \xrightarrow{p} & V. \end{array}$$

By Proposition 2.4.1, we see that the continuity set $C(p^*)$ of p^* is dense open in V , and $V \setminus C(p^*)$ is locally finite. So it is enough to show that p^* is an approximate fibration over the continuity set $C(p^*)$ of p^* . Fix $b \in C(p^*)$. Carefully take a small neighborhood $W(\subset C(p^*))$ of b and a retraction $R_b : p^{-1}(W) \rightarrow g_b$. Let $R_b^* : W^* \equiv q^{-1}(p^{-1}(W)) \rightarrow g_b^*$ be the lifting of R_b .

For any $a \in W$, consider the following

$$\begin{array}{ccccc} g_a^* & \longrightarrow & W^* & \xrightarrow{R_b^*} & g_b^* \\ q| \downarrow & & q| \downarrow & & q| \downarrow \\ g_a & \longrightarrow & p^{-1}(W) & \xrightarrow{R_b} & g_b. \end{array}$$

Claim: $(R_b^* |)$ is a homotopy equivalence; then it follows from [22, Theorem 2.1]

and [9] that $p^* = p \circ q$ is an approximate fibration over the continuity set $C(p^*)$ of p^* .

Case I : N is orientable.

Since $(R_b^* |) : g_a^* \rightarrow g_b^*$ is a map of degree one, $(R_b |) : g_a \rightarrow g_b$ has degree one. The hopfian hypotheses of N and $\pi_1(N)$ yield that $(R_b |)$ is a homotopy equivalence. In particular, $(R_b |)_\# : \pi_1(g_a) \rightarrow \pi_1(g_b)$ is an isomorphism. By Lemma 3.1.2, we have that $(R_b^* |)_\# : \pi_1(g_a^*) \rightarrow \pi_1(g_b^*)$ is an isomorphism. Moreover, since for $i \geq 2$, the homomorphism

$$\pi_i(g_a^*) \cong \pi_i(g_a) \xrightarrow[(R_b|)_\#]{\cong} \pi_i(g_b) \cong \pi_i(g_b^*)$$

is an isomorphism. By Whitehead Theorem, we have that $(R_b^* |)$ is a homotopy equivalence.

Case II : N is nonorientable.

Since $b \in C(p^*)$, by Proposition 2.2.2, we have an epimorphism $(R_b^* |)_\# : \pi_1(g_a^*) \rightarrow \pi_1(g_b^*)$. By Lemma 3.1.2, $(R_b |)_\# : \pi_1(g_a) \rightarrow \pi_1(g_b)$ is an epimorphism. But since $\pi_1(g_a) \cong \pi_1(g_b) \cong \pi_1(N)$ is hopfian, $(R_b |)_\# : \pi_1(g_a) \rightarrow \pi_1(g_b)$ is an isomorphism. Again, by Lemma 3.1.2, $(R_b^* |)_\# : \pi_1(g_a^*) \rightarrow \pi_1(g_b^*)$ is an isomorphism. Now, since $q^{-1}(p^{-1}(a)) \simeq q^{-1}(p^{-1}(b)) \simeq \tilde{N}$ is hopfian and $(R_b^* |)_\#$ is an isomorphism, $(R_b^* |)$ is a homotopy equivalence. $\square$

Remark. The usefulness of above Lemma can be described as follows;

Let N be a closed s-hopfian manifold with hopfian $\pi_1(N)$. Verification that an N -like map $p : M^{n+2} \rightarrow B^2$ is an approximate fibration over the mod 2 continuity set C' of p reduces to the case where $C' = R^2$, p is an approximate fibration over $R^2 \setminus 0$, and there is a strong deformation retraction $R : p^{-1}(C') \rightarrow p^{-1}(0)$. Then $R|_{p^{-1}(x)}$, $x \neq 0$ gives rise to a map $f : N \rightarrow N$, and the essential step is to prove f is a homotopy equivalence.

Theorem 3.1.5. *Let N be a closed s -hopfian manifold with hyperhopfian $\pi_1(N)$. Then N is a codimension-2 fibration.*

Proof. Let a proper map $p : M \rightarrow B$ defined on an $(n+2)$ -manifold M be N -like. In light of Lemma 3.1.4, we may assume that p is an approximate fibration over $C' \setminus \{x\}$ for some $x \in C'$. Then, by copying the proof of [22, Lemma 5.1] or [5, Lemma 5.1], we see that p is an approximate fibration over the mod 2 continuity set C' of p . Now by Lemma 3.1.3, N is a codimension-2 fibration. $\square$

Corollary 3.1.1. *A closed n -manifold N with hyperhopfian $\pi_1(N)$ is a codimension-2 fibration if any one of the following conditions holds:*

- 1) $\pi_1(N)$ is finite;
- 2) $\pi_1(N)$ is nilpotent;
- 3) $\pi_i(N)$ is trivial for $1 < i < n - 1$, in particular, N is aspherical.

Corollary 3.1.2. *Let N^3 be a closed 3-manifold with finite $\pi_1(N)$. If $\pi_1(N)$ has no cyclic direct factor, then N is a codimension-2 fibration.*

Proof. By [22, Theorem 4.7], $\pi_1(N)$ is hyperhopfian. $\square$

Remark. The existence of codimension-2 (orientable) non-fibrations among closed 3-manifolds with finite fundamental group is still unknown.

3.2. MANIFOLDS N WITH HOPFIAN $\pi_1(N)$ AND $\chi(N) \neq 0$.

Theorem 3.2.1. *Closed s -hopfian n -manifolds N with hopfian fundamental group and nonzero Euler characteristic are codimension-2 fibrations.*

Proof. By Proposition 2.4.3 and Proposition 2.4.4, we may assume that N has a 2-to-1 covering. Let a proper map $p : M^{n+2} \rightarrow B^2$ from an $(n+2)$ -manifold M^{n+2}

onto a 2-manifold B with boundary be N -like, and $G = \{p^{-1}(b) : b \in B\}$.

Claim I : p is an approximate fibration over the mod 2 continuity set C' of p .

In light of Lemma 3.1.4, we localize the situation so that C' is an open disk containing $b_0 = p(g_0)$ and p is an approximate fibration over $C' \setminus b_0$. Also we may assume that $R : p^{-1}(C') \rightarrow g_0$ is a strong deformation retraction. Take a covering $q : M^* \rightarrow p^{-1}(C')$ corresponding to $R_{\#}^{-1}(H) = H$. Then as in the proof of Lemma 3.1.4, we see that, for all $g \in G$ with $g \subset p^{-1}(C')$, $q^{-1}(g)$ and $\tilde{N}$ have the same homotopy type. Now we follow the method of the proof in [22, Theorem 5.10], then we have $(R^* | g^*)_* : H_1(g^*) \rightarrow H_1(g_0^*)$ is an epimorphism, where R^* is a lifting of R . By [22, Lemma 5.2], $(R |)^*$ has degree one. For any $g \in G$ with $p(g) \in C'$, consider

$$\begin{array}{ccc} \tilde{N} \simeq g^* & \xrightarrow{(R|)^*} & g_0^* \simeq \tilde{N} \\ q| \downarrow & & q| \downarrow \\ N \simeq g & \xrightarrow{(R|)} & g_0 \simeq N. \end{array}$$

Subcase 1: N is orientable.

Then $(R |)$ has degree one. The hopfianness of N and $\pi_1(N)$ give us that $(R |)$ is a homotopy equivalence, so that $(R |)^*$ is.

Subcase 2: N is nonorientable.

Since $(R |)^*$ has degree one, by Proposition 2.2.2, we have an epimorphism $(R |)_{\#}^* : \pi_1(g^*) \rightarrow \pi_1(g_0^*)$. By Lemma 3.1.2, $(R |)_{\#} : \pi_1(g) \rightarrow \pi_1(g_0)$ is an epimorphism. But since $\pi_1(N)$ is hopfian, $(R |)_{\#}$ is an isomorphism. Again by Lemma 3.1.2, we have an isomorphism $(R |)_{\#}^* : \pi_1(g^*) \rightarrow \pi_1(g_0^*)$. By the hopfianness of $\tilde{N}$, $(R |)^*$ is a homotopy equivalence.

Claim II: $\text{int} B = C'$.

Let $D' = (\text{int} B) \setminus C'$. Assume that $D' \neq \emptyset$. Let $b_0 \in D'$. In light of Proposition 2.4.1, we localize the situation so that $\text{int} B$ is an open disk containing $b_0 = p(g_0)$ and p is an approximate fibration over $(\text{int} B) \setminus b_0$. Also we may assume that $R : p^{-1}(\text{int} B) \longrightarrow g_0$ is a strong deformation retraction. Take a covering $q : M^* \longrightarrow p^{-1}(\text{int} B)$ corresponding to $R_{\#}^{-1}(H) = H$. By an argument similar to the proof of claims in the Lemma 3.1.4, we see that $g_0^* \equiv q^{-1}(g_0)$ is connected and has the homotopy type of $\tilde{N}$. Since $p|_{p^{-1}((\text{int} B) \setminus g_0)} : p^{-1}((\text{int} B) \setminus g_0) \longrightarrow (\text{int} B) \setminus b_0$ is an approximate fibration, for any $g, g' (\neq g_0) \in G$ in $p^{-1}(\text{int} B)$, their components $q^{-1}(g)_C \equiv g_C^*$ and $q^{-1}(g')_C \equiv g_C'^*$ have the same homotopy type by Proposition 2.1.2. And since M^* is orientable, by [22, Proposition 2.9] g_C^* and g_0^* are orientable.

Now as before we follow the method of the proof in [22, Theorem 5.10], then we have $(R^*|_{g_C^*})_* : H_1(g_C^*) \longrightarrow H_1(g_0^*)$ is an epimorphism, where R^* is a lifting of R . By [22, Lemma 5.2'], $R^*|_{g_C^*}$ has a positive degree. It follows from Proposition 2.2.3 that $\beta_i(g_C^*) \geq \beta_i(g_0^*)$ for each i .

Now, for $g (\neq g_0) \in G$, let $i : g \longrightarrow p^{-1}(\text{int} B)$ be the inclusion map. Set $\overline{H} = i_{\#}^{-1}(q_{\#}(\pi_1(M^*)) \cap i_{\#}(\pi_1(g))) = i_{\#}^{-1}(H \cap i_{\#}(\pi_1(g)))$ and $K = (q|_{g_C^*})_{\#}(\pi_1(g_C^*))$. Then we can easily see that $K \subset \overline{H}$ and $H \subset \overline{H}$. And by [73, Proposition 11.1], we have $\overline{H} \subset K$, i.e., $K = \overline{H}$.

Now, let's examine the induced map $i_{\#}$ case by case.

Case 1: $i_{\#}$ is an epimorphism.

Then, since $\pi_1(g) = \pi_1(p^{-1}(\text{int} B))$ is hopfian, $i_{\#}$ is an isomorphism. So we have that for all $g \in G$ with $p(g) \in \text{int} B$, g^* is connected and has the homotopy type of $\tilde{N}$. By the same proof as Claim I, p is an approximate fibration over $\text{int} B$.

Case 2: $i_{\#}$ is not onto.

Subcase 1: $H = \overline{H} = K$.

Then for all $g \in G$ with $p(g) \in \text{int } B$, g^* is connected and has the homotopy type of $\tilde{N}$. By the same reason of the proof in Claim I, p is an approximate fibration over $\text{int } B$.

Subcase 2 : $H \subset \overline{H} = K$ but $H \neq \overline{H}$.

We'll show that this case cannot happen. Take the covering map $N_{\overline{H}} \longrightarrow g$ corresponding to $\overline{H}$, and take the covering map $N_H \longrightarrow g$ corresponding to H . Consider

$$\begin{array}{ccc} & N_{\overline{H}} & \\ & \downarrow & \\ N_H & \longrightarrow & g \end{array}$$

Since $H \subset \overline{H}$ and $H \neq \overline{H}$, we have a $d-1$ covering map $N_H \longrightarrow N_{\overline{H}}$ with $d \geq 2$. By the facts of $g_0^* \simeq \tilde{N} \approx N_H$ and $N_{\overline{H}} \approx N_K \approx g_C^*$ with $d \geq 2$, we see that $\beta_i(g_0^*) \geq \beta_i(g_C^*)$ for each i (from [42, Corollary 1]) and $\chi(g_0^*) = d\chi(g_C^*)$ with $d \geq 2$. But since we already have that $\beta_i(g_C^*) \geq \beta_i(g_0^*)$ for each i , $\chi(g_C^*) = \chi(g_0^*) = d\chi(g_C^*)$ with $d \geq 2$, which gives the contradiction $\chi(\tilde{N}) = \chi(g_0^*) = 0$.

Claim III: $\partial B = \emptyset$.

Suppose not. Then there exist $a_0 \in \partial B$, a neighborhood U of a_0 in B , and a deformation retract $R : p^{-1}(U) \longrightarrow p^{-1}(a_0)$ such that

- (1) $U \approx$ the upper half plane $\{(x, y) \in E^2 \mid y \geq 0\}$,
- (2) $A = (\partial B) \cap U$ is an open arc, and
- (3) for all $a \in A$, $R|_{p^{-1}(a)} : p^{-1}(a) \longrightarrow p^{-1}(a_0)$ is a homotopy equivalence (see [18] or [61]).

Take the covering map $q : M^* \longrightarrow p^{-1}(U)$ corresponding to H . Then by another argument similar to the proof in Lemma 3.1.4, we have that for all $a \in A$, $q^{-1}(p^{-1}(a))$ is

connected and $q^{-1}(p^{-1}(a)) \simeq q^{-1}(p^{-1}(a_0)) \simeq \tilde{N}$. And, since p is an approximate fibration over $p^{-1}(\text{int } U)$, for all $b, b' \in \text{int } U$, $q^{-1}(p^{-1}(b))_C \simeq q^{-1}(p^{-1}(b'))_C \simeq (\text{say}) N^*$, where $q^{-1}(p^{-1}(b))_C$ and $q^{-1}(p^{-1}(b'))_C$ are components of $q^{-1}(p^{-1}(b))$ and $q^{-1}(p^{-1}(b'))$, respectively. Hence, by the fact of M^* is orientable and [18, Proposition 2.9], we see that for all $b \in \text{int } U$ and for all $a \in A$, the components $q^{-1}(p^{-1}(b))_C$ of $q^{-1}(p^{-1}(b))$ and $q^{-1}(p^{-1}(a))$ are orientable. Therefore, if $G^* = \{q^{-1}(p^{-1}(b))_C, q^{-1}(p^{-1}(a)) \mid b \in \text{int } U, a \in A\}$ is the usc decomposition of M^* , then by Proposition 2.4.1, $B^* = M^*/G^*$ is a 2-manifold without boundary.

Let $p^* : M^* \longrightarrow B^*$ be the decomposition map and C^* be its continuity set. Since $p^*(q^{-1}(p^{-1}(A)))$ is homeomorphic to an open arc and $B^* \setminus C^*$ is locally finite in B^* , there is a point $a^* \in p^*(q^{-1}(p^{-1}(A))) \cap C^*$. So we have a map $N^* \simeq q^{-1}(p^{-1}(b))_C \longrightarrow q^{-1}(p^{-1}(a)) \simeq \tilde{N}$ with degree one, where $p^*(q^{-1}(p^{-1}(a))) = a^*$ and for some $b \in \text{int } U$. Hence we have $\beta_i(N^*) \geq \beta_i(\tilde{N})$ for each i .

For $g \in G$ with $p(g) = b$, let $i : g \longrightarrow p^{-1}(U)$ be the inclusion map. Set $\overline{H} = i_{\#}^{-1}(q_{\#}(\pi_1(M^*)) \cap i_{\#}(\pi_1(g))) = i_{\#}^{-1}(H \cap i_{\#}(\pi_1(g)))$ and $K = (q \mid g_C^*)_{\#}(\pi_1(g_C^*))$, where $g_C^* = q^{-1}(p^{-1}(b))_C$. Then we can easily see that $K \subset \overline{H}$, $H \subset \overline{H}$. But since g^* has two (or more) components, $H \neq \overline{H}$. By [73, Proposition 11.1], we have $K = \overline{H}$.

Now we take the covering map $N_{\overline{H}} \longrightarrow g$ corresponding to $\overline{H}$, and take the covering map $N_H \longrightarrow g$ corresponding to H . And since $H \subset \overline{H}$ and $H \neq \overline{H}$, we have a $d - 1$ covering map $\tilde{N} \simeq N_H \longrightarrow N_{\overline{H}} \approx N_K \simeq N^*$ with $d \geq 2$, so we have for each i , $\beta_i(\tilde{N}) \geq \beta_i(N^*)$ and $\chi(\tilde{N}) = d\chi(N^*)$. As before $\chi(N^*) = \chi(\tilde{N}) = d\chi(N^*)$, which gives a contradiction $\chi(\tilde{N}) = 0$. $\square$

Corollary 3.2.1. *Let N^n be a closed n -manifold with $\chi(N) \neq 0$. Then N is a codimension-2 fibrator if any one of the following conditions holds:*

- 1) $\pi_1(N)$ is abelian;
- 2) $\pi_1(N)$ is hopfian and $\pi_i(N) = 0$ for $1 < i < n - 1$;
- 3) $\pi_1(N)$ is hopfian and $n = 4$;
- 4) [4] $\pi_1(N)$ is finite;
- 5) [18] $n = 2$.

Corollary 3.2.2. *Closed 4-manifolds N^4 with finite $\pi_1(N)$ are codimension-2 fibrators.*

Proof. If N is orientable, then $\chi(N) = 2 + \beta_2(N) > 0$. Now assume that N is nonorientable. Then since the orientable 4-manifold $\tilde{N}$ has a finite fundamental group, $\chi(\tilde{N}) \neq 0$. So $\chi(\tilde{N}) = [\pi_1(N) : \pi_1(\tilde{N})] \chi(N)$ implies that $\chi(N) \neq 0$. $\square$

Note. In Corollary 3.2.2, we cannot replace the condition of finite $\pi_1(N)$ by finite $H_1(N)$, because of $RP^4 \# RP^4$. However, the next corollary tells us which closed 4-manifolds N with finite $H_1(N)$ are codimension-2 fibrators.

Corollary 3.2.3. *Let N be a closed 4-manifold with hopfian $\pi_1(N)$ and finite $H_1(N)$. If N is orientable or $H_1(N; \mathbb{Z}_2) \cong \mathbb{Z}_2$, then N is a codimension-2 fibrator.*

Proof. If N is orientable, then $\chi(N) = 2 + \beta_2(N) \neq 0$. Now assume that N is nonorientable and $H_1(N; \mathbb{Z}_2) \cong \mathbb{Z}_2$. Then since $H_1(N; \mathbb{Z}_2) \cong H^3(N; \mathbb{Z}_2) \cong \text{Hom}(H_3(N), \mathbb{Z}_2) \oplus \text{Ext}(H_2(N), \mathbb{Z}_2)$ and the torsion subgroup of $H_3(N)$ is isomorphic to $\mathbb{Z}_2$, $\beta_3(N) = 0$. Hence we have that $\chi(N) = 1 - 0 + \beta_2(N) - 0 + 0 = 1 + \beta_2(N) \neq 0$. $\square$

Remark. In Theorem 3.2.1, we cannot omit the condition $\chi(N) \neq 0$, since S^1 and $RP^3 \# RP^3$ are not codimension-2 fibrators. Also see the following Proposition.

Proposition 3.2.2. *Let N be a closed orientable 4-manifold with $\chi(N) = 0$. If $\pi_1(N)$ is free abelian, N fails to be a codimension-2 fibrator.*

Proof. N is homotopy equivalent to $S^3 \times S^1$, $S^2 \times S^1 \times S^1$, or $S^1 \times S^1 \times S^1 \times S^1$ ([54]). □

3.3. PRODUCTS OF CODIMENSION-2 FIBRATORS. The question of whether the collection of codimension 2 fibrators is closed under the Cartesian product operator remains unsolved. Now, let us consider the following question:

Question: *Is any finite product of codimension-2 fibrators a codimension-2 fibration? i.e., If $N_1, N_2, \dots, N_k$ are closed manifolds which are codimension-2 fibrators, is $N_1 \times N_2 \times \dots \times N_k$ a codimension-2 fibration?*

The answer is not yet settled. But the answer is *yes* for the case of each N_j ($j = 1, 2, \dots, k$) a closed orientable surface ([61] and [60]). Here, we have an affirmative answer without assuming orientability for any N_j as follows;

Corollary 3.3.1. *Any finite product of closed surfaces which are codimension-2 fibrators is a codimension-2 fibration.*

Proof. Let $N_1, N_2, \dots, N_k$ be closed surfaces which are codimension-2 fibrators, and $N \equiv N_1 \times N_2 \times \dots \times N_k$. First, note that for all $j = 1, \dots, k$, $\chi(N_j) \neq 0$, for the torus and Klein bottle are the only examples of non-codimension-2 fibrators (see [18]). So we have $\chi(N) \neq 0$. Moreover, $\pi_1(N) \cong \bigoplus_{j=1}^k \pi_1(N_j)$ is residually finite. Hence it suffices to show that N is s-hopfian. If N has no 2-to-1 covering, then N must be orientable, so that each N_j is orientable. In [61] and [60], Y. Im took care of this case. Hence we consider the case that N has a 2-to-1 covering. Since N is of the form *products of RP^2 $\times$ products of S^2 $\times$ products of closed surfaces which are neither RP^2 nor S^2* , $\tilde{N}$ must be of the form *products of S^2 $\times$ products of S^2 $\times$ products of closed surfaces which are aspherical*. Hence $\tilde{N}$ is hopfian, which follows from the fact

that any finite product of simply connected manifolds and aspherical closed manifolds with hopfian fundamental groups is hopfian (see [62]). $\square$

Remark. Recently there has been progress about the problem of products of fibrators (see [30]).

4. CONNECTED SUMS OF MANIFOLDS

In this chapter, we study the behavior of codimension-2 fibrators under the connected sum operation. If N is closed orientable surface, then either $N \approx S^2$ or $N \approx M_1 \# \cdots \# M_g$, where each $M_i = S^1 \times S^1$ and $2g = 2 - \chi(N)$. And if N is closed nonorientable surface, then $N \approx M'_1 \# \cdots \# M'_g$, where each $M'_i = RP^2$ and $g = 2 - \chi(N)$. Moreover, we have seen that N is a codimension-2 fibration if and only if $\chi(N) \neq 0$. So all manifolds we deal with in this chapter have dimension $n \geq 3$.

We use $N_1^n \# N_2^n$ to denote a connected sum of two n -manifolds N_1 and N_2 ($n \geq 3$), where neither N_1 nor N_2 is a homotopy sphere. Then N is of the form $K_1 \cup K_2$, where $K_1 \cap K_2 = S^{n-1}$ and $K_1 = N_1 \setminus (\text{int } B^n)$, $K_2 = N_2 \setminus (\text{int } B^n)$. We reserve the notations K_1 and K_2 to mean the above throughout this chapter.

4.1. THE BASIC CASES.

Lemma 4.1.1. [22, Corollary 4.12] *If G_1, G_2 are nontrivial, finitely generated groups such that $G_2 \not\cong \mathbb{Z}_2$ and $G_1 * G_2$ is hopfian, then $G_1 * G_2$ is hyperhopfian.*

Proposition 4.1.2. *Let $N = N_1 \# N_2$ be a closed s -hopfian manifold. Then N is a codimension-2 fibration if any one of the following conditions holds:*

- 1) $\pi_1(N_1)$ and $\pi_1(N_2)$ are hyperhopfian;
- 2) $\pi_1(N_1)$ and $\pi_1(N_2)$ are nontrivial hopfian, and $\pi_1(N) \neq \mathbb{Z}_2 * \mathbb{Z}_2$.

Proof. First, we see that $\pi_1(N) \cong \pi_1(N_1) * \pi_1(N_2)$ by the van Kampen Theorem so that $\pi_1(N)$ is hopfian, because the free product of two finitely generated hopfian groups is again hopfian [41]. It follows from Lemma 4.1.1 and Theorem 3.1.5 that N is a codimension-2 fibration. □

Corollary 4.1.1. *Let $N = N_1 \# N_2$ be a closed 3-manifold with hopfian $\pi_1(N_i)$ for $i = 1, 2$. Then N is a codimension-2 fibrator, with only one exception $\pi_1(N) \cong \mathbb{Z}_2 * \mathbb{Z}_2$. Hence, if N has a connected sum decomposition consisting of virtually Haken manifolds and/or those with finite or cyclic fundamental group and if $\pi_1(N) \not\cong \mathbb{Z}_2 * \mathbb{Z}_2$, then N is a codimension-2 fibrator.*

Consider the connected sum $N \equiv RP^n \# RP^n \# \cdots \# RP^n$, (k -copies of real projective n -space RP^n). If $k = 1$, then $\pi_1(N) \cong \mathbb{Z}_2$, so by Proposition 2.4.2, N is a codimension-2 fibrator. And if $k > 2$, then by Lemma 4.1.1, $\pi_1(N)$ is hyperhopfian. But since N is s-hopfian, by Theorem 3.1.5, N is a codimension-2 fibrator. However, surprisingly when $k = 2$, N is not a codimension-2 fibrator, for it cyclically cover itself (nontrivially). Therefore, it is worthwhile to consider the following question; When is a connected sum of two closed manifolds a codimension-2 fibrator?

Let us begin with a technical Lemma.

Lemma 4.1.3. *Let $p : M^{n+2} \rightarrow R^2$ be a proper map defined on an orientable $(n + 2)$ -manifold such that p is an approximate fibration over $R^2 \setminus \{0\}$. Suppose that $R : M \rightarrow p^{-1}(0) \equiv g_0$ is a strong deformation retraction. Put $G = \{p^{-1}(x) : x \in R^2\}$.*

- 1) (cf. [22, Lemma 5.2]) If for any $g \in G$, $(R \mid g)_* : H_1(g) \rightarrow H_1(g_0)$ is an epimorphism, then $(R \mid g)$ has degree one.*
- 2) If p is N -like for some orientable n -manifold N with finite $H_1(N)$, then the degree of $(R \mid g)$ equals $[H_1(g_0) : (R \mid g)_*(H_1(g))]$.*

Proof of 1). We verify that $(R \mid g)$ is degree one map by checking it gives a cohomology isomorphism between $H^n(g_0) \cong H_2(M, M \setminus g_0)$ and $H^n(g) \cong H_2(M, M \setminus g)$ and applying the universal coefficient theorem to obtain the same for homology. The proof

of [22, Lemma 5.2] shows that $H_2(M, M \setminus g_0) \rightarrow H_2(R^2, R^2 \setminus 0)$ is an isomorphism. The large step depends on showing p provides an epimorphism $H_2(M, M \setminus g) \rightarrow H_2(R^2, R^2 \setminus p(g))$. To find the epimorphism, examine the homology ladder

$$\begin{array}{ccccc} \mathbb{Z} \cong H_2(M, M \setminus g) & \xrightarrow{\partial} & H_1(M \setminus g) & \longrightarrow & H_1(M) \\ q_* \downarrow & & p'_* \downarrow & & \\ \mathbb{Z} \cong H_2(R^2, R^2 \setminus p(g)) & \xrightarrow{\cong} & H_1(R^2 \setminus p(g)) & & \end{array}$$

Since for any $g \in G$, $(R \mid g)_* : H_1(g) \rightarrow H_1(g_0)$ is an epimorphism, we see that $H_1(M \setminus g) = i_*(H_1(g')) + im\partial$, where $i : g' \rightarrow M \setminus g$ denotes inclusion. As point inverses of p are connected, p'_* is an epimorphism. With $i_*(H_1(g)) \subset \ker p'_*$, diagram chasing yields that p'_* carries $im\partial$ isomorphically onto $H_1(R^2 \setminus p(g))$.

Specify a closed disk $D \subset R^2$ containing $0, p(g)$ in its interior. One see that what is need, the desired isomorphism between $H^n(g_0)$ and $H^n(g)$, in the diagram below.

$$\begin{array}{ccccc} H^n(g_0) & \longleftarrow & H^n(p^{-1}(D)) & \longrightarrow & H^n(g) \\ \downarrow \cong & & \downarrow \cong & & \downarrow \cong \\ H_2(M, M \setminus g_0) & \longleftarrow & H_2(M, M \setminus p^{-1}(D)) & \longrightarrow & H_2(M, M \setminus g) \\ \downarrow \cong & & \downarrow \cong & & \downarrow \text{onto} \\ H_2(R^2, R^2 \setminus 0) & \xleftarrow{\cong} & H_2(R^2, R^2 \setminus D) & \xrightarrow{\cong} & H_2(R^2, R^2 \setminus p(g)). \end{array}$$

Proof of 2). Apply [9, Corollary 3.5] to represent $\pi_1(M \setminus g_0)$ as an extension

$$1 \longrightarrow \pi_1(g) \xrightarrow{i_\#} \pi_1(M \setminus g_0) \longrightarrow \mathbb{Z} \longrightarrow 1,$$

where $(R|g)^*$ is the lifting of $(R|g)$. Because g_0 has the homotopy type of a codimension-2 compactum from M , the inclusion $M \setminus g_0 \rightarrow M$ induces an epimorphism $j_\# : \pi_1(M \setminus g_0) \rightarrow \pi_1(M)$. Hence we have that $\pi_1(g_0)/(R|g)_\#(\pi_1(g)) \cong \pi_1(M)/j_\#i_\#(\pi_1(g))$ is cyclic, so that $(R|g)_\#(\pi_1(g))$ contains the commutator subgroup $[\pi_1(g_0) : \pi_1(g_0)]$ of $\pi_1(g_0)$. Now, take a covering $q : g_0^* \rightarrow g_0$ corresponding to $(R \mid g)_\#(\pi_1(g))$. Then

consider

$$\begin{array}{ccc} g & \xrightarrow[\text{degree}=1]{(R|g)^*} & g_0^* \\ \parallel & & \downarrow q \\ g & \xrightarrow{(R|g)} & g_0. \end{array}$$

So we have that the degree of $(R|g)$ equals the degree of q , which is $[\pi_1(g_0) : (R|g)_\#(\pi_1(g))] = [H_1(g_0) : (R|g)_*(H_1(g))]$. $\square$

Note. In 1) of the above lemma, we do not assume that all fibers have the same homotopy type.

4.2. THE CASE OF $\pi_1(N) = \mathbb{Z}_2 * \mathbb{Z}_2$. Proposition 4.1.2 yields one notable exception the case of $\pi_1(N) = \mathbb{Z}_2 * \mathbb{Z}_2$. However, the next Proposition says that most closed hopfian n -manifolds $N = N_1 \# N_2$ with $\pi_1(N) \cong \mathbb{Z}_2 * \mathbb{Z}_2$ are codimension-2 orientable fibrators.

Lemma 4.2.1. *Let Γ be a generated by a and b . Then Γ has at most three subgroups of index 2, which are (1) $\langle b, aba^{-1}, a^2 \rangle$, (2) $\langle a, bab^{-1}, b^2 \rangle$ and (3) $\langle ab, a^2ba^{-1}, a^2 \rangle$.*

Proposition 4.2.2. *Let $N = N_1 \# N_2$ be a closed hopfian n -manifold with $\pi_1(N_i) \cong \mathbb{Z}_2$ for $i = 1, 2$. Then N is a codimension-2 orientable fibrator if any one of the following conditions holds:*

- 1) *there are prime numbers $s, t > 2$ (not necessarily distinct) such that $H_j(N_1; \mathbb{Z}_s) \neq 0$ and $H_k(N_2; \mathbb{Z}_t) \neq 0$ for some indices j, k with $0 < j, k < n$.*
- 2) *$|H_j(N_1; \mathbb{Z}_2)| > |H_j(N_2; \mathbb{Z}_2)|$ and $|H_k(N_2; \mathbb{Z}_2)| > |H_k(N_1; \mathbb{Z}_2)|$ for some indices j, k with $0 < j, k < n$.*

Proof. Let $p : M^{n+2} \rightarrow B^2$ defined on an orientable $(n+2)$ -manifold M be an N -like proper map. Since N is hopfian with hopfian $\pi_1(N)$, p is an approximate

fibration over the continuity set C of p by Proposition 2.3.2. Now, as before in light of Proposition 2.4.1 we localize to the situation in which $B = E^2$ and p is an approximate fibration over $C = E^2 \setminus 0$. Use g_0 to represent $p^{-1}(0)$, which is a strong deformation retract of M under a retraction $R: M \rightarrow g_0$. Put $G = \{p^{-1}(b) : b \in B\}$. Denote $R|_g$ by R_g for $g \in G$. Assume that $(R_g)_\#(\pi_1(g)) \neq \pi_1(g_0)$. Take the covering $q: g_0^* \rightarrow g_0$ corresponding to $(R_g)_\#(\pi_1(g))$. Then there is a lifting $\overline{R_g}: g \rightarrow g_0^*$ of R_g so that $q \circ \overline{R_g} = R_g$. Here note that by Lemma 4.1.3, $\deg \overline{R_g} = 1$ so that $(\overline{R_g})_*: H_1(g) \rightarrow H_1(g_0^*)$ is onto. Hence $H_1(g_0^*)$ should be finite. And since $(\deg \overline{R_g}) = 2$ (see [12] for the proof) and $(\deg q)(\deg \overline{R_g}) = \deg R_g$, we have $\deg q = 2$. Since $\pi_1(N) \cong \langle a, b | a^2 = b^2 = 1 \rangle$, by Lemma 4.2.1, the subgroups of index 2 in $\pi_1(N)$ are (1) $\langle b, aba^{-1} | a^2 = b^2 = 1 \rangle$, (2) $\langle a, bab^{-1} | a^2 = b^2 = 1 \rangle$ and (3) $\langle ab, ba^{-1} | a^2 = b^2 = 1 \rangle$. From the fact that g_0^* has the homotopy type of N^* , where N^* is a 2-to-1 covering space of N , we see that g_0^* is formed either by two copies of K_1 and $q^{-1}(K_2)$ or by $q^{-1}(K_1)$ and two copies of K_2 or by a 2-to-1 covering space L of N with $H_1(L) = \mathbb{Z}$. Since $H_1(g_0^*)$ is finite, the third case can not happen. Without loss of generality, we assume that g_0^* is of the first form.

Proof of 1). By Proposition 2.2.3, we see that $(R_g)_*$, $(\overline{R_g})_*$ and q_* are surjective on $\mathbb{Z}_s$ -homology. Consider the following commutative diagram

$$\begin{array}{ccc} H_j(g; \mathbb{Z}_s) & \xrightarrow[\text{onto}]{(\overline{R_g})_*} & H_j(g_0^*; \mathbb{Z}_s) \\ \parallel & & q_* \downarrow \text{onto} \\ H_j(g; \mathbb{Z}_s) & \xrightarrow[\text{onto}]{(R_g)_*} & H_j(g_0; \mathbb{Z}_s). \end{array}$$

Since $H_j(g; \mathbb{Z}_s) \cong H_j(g_0; \mathbb{Z}_s)$ is hopfian, $(R_g)_*$ is an isomorphism, so $(\overline{R_g})_*: H_j(g_0; \mathbb{Z}_s) \rightarrow H_j(g_0^*; \mathbb{Z}_s)$ is an isomorphism.

On the other hand, by Proposition 2.2.3 again, we have that $q_*: H^{n-j}(g_0; \mathbb{Z}_s) \rightarrow$

$H^{n-j}(g_0^*; \mathbb{Z}_s)$ and $q_*^*: H^{n-j}(K_2; \mathbb{Z}_s) \rightarrow H^{n-j}(q^{-1}(K_2); \mathbb{Z}_s)$ are injective so that $H_j(K_2; \mathbb{Z}_s)$ and $H_j(g_0; \mathbb{Z}_s)$ are isomorphic to subgroups of $H_j(q^{-1}(K_2); \mathbb{Z}_s)$ and $H_j(g_0^*; \mathbb{Z}_s)$, respectively. Since $H_j(K_1; \mathbb{Z}_s)$ is non-trivial, $|H_j(g_0; \mathbb{Z}_s)| < |H_j(g_0^*; \mathbb{Z}_s)|$, which gives a contradiction.

Proof of 2). Since $\deg \overline{R_g} = 1$, we have an epimorphism $\overline{(R_g)}_* : H_j(g; \mathbb{Z}_2) \rightarrow H_j(g_0^*; \mathbb{Z}_2)$. But since $|H_j(N_1; \mathbb{Z}_2)| > |H_j(N_2; \mathbb{Z}_2)|$, as before we have a contradiction from $|H_j(g_0; \mathbb{Z}_2)| \leq |H_j(g_0^*; \mathbb{Z}_2)|$. $\square$

Question. Can we get some orientation-free version of Proposition 4.2.2?

Corollary 4.2.1. *Let $N \# N$ be a closed hopfian n -manifold ($n \geq 4$) with hopfian $\pi_1(N)$. Then $N \# N$ is a codimension-2 orientable fibrator if and only if $N \# N$ is not a 2-to-1 covering of itself.*

Proof. $\implies$: Proposition 2.5.1.

$\impliedby$: Now we may assume that $\pi_1(N) \cong \mathbb{Z}_2$, otherwise $N \# N$ is a codimension-2 (orientable) fibrator. Observing the proof of Proposition 4.2.2, for all $s > 1$ and any $1 < j < n$, we have an epimorphism $H_j(N \# N; \mathbb{Z}_s) \rightarrow H_j(N \# N \# \overline{N}; \mathbb{Z}_s)$, where $\overline{N}$ is the universal covering of N . This implies that $H_j(\overline{N}; \mathbb{Z}_s)$ is trivial for any $s > 1$ and any $1 < j < n$ so that $\overline{N}$ must be the homotopy sphere. Hence $N \# N \# \overline{N} \approx N \# N \# S^n \approx N \# N$ is a 2-to-1 covering of $N \# N$. $\square$

Corollary 4.2.2. *Let $RP^n \# N$ be a closed hopfian n -manifold ($n \geq 4$). If $H_j(N)$ is nontrivial for some index $0 < j$ (even) $< n$, then $RP^n \# N$ is a codimension-2 orientable fibrator.*

Proof. Explore the proof of Proposition 4.2.2. $\square$

4.3. THE MAJOR NONSETTLED CASES. Proposition 4.1.2 and Proposition 4.2.2 make us consider the major non-settled case :

given a closed n -manifold $N = N_1 \# N_2$ ($n \geq 4$), what if $\pi_1(N_1)$ is trivial and $\pi_1(N_2)$ is not?

Proposition 4.3.1. *Let $N = N_1 \# N_2$ be a s -hopfian closed n -manifold with trivial $\pi_1(N_1)$. If either $\pi_1(N_2)$ is hyperhopfian or $|\pi_1(N_2)| \leq 2$, then N is a codimension-2 fibrator.*

Proposition 4.3.2. *Let $N = N_1 \# N_2$ be a s -hopfian closed n -manifold with trivial $\pi_1(N_1)$ and hopfian $\pi_1(N_2)$. If $\beta_j(N_1) \neq 0$ for some j ($0 < j < n$), then N is a codimension-2 fibrator.*

Proof. 1) By the van Kampen Theorem, we see that $\pi_1(N) \cong \pi_1(K_2) \cong \pi_1(N_2)$, which is hopfian. 2) Using the Mayer-Vietoris sequence, we see that $\beta_j(K_1) = \beta_j(N_1) \neq 0$. Let $p : M \longrightarrow B$ be an N -like proper map defined on an $(n+2)$ -manifold M . Put $G = \{p^{-1}(b) : b \in B\}$.

Case I : N is orientable.

Subcase I: M is orientable.

Again in light of Proposition 2.4.1, we localize to the situation in which $B = E^2$ and p is an approximate fibration over $C = E^2 \setminus 0$. Use g_0 to represent $p^{-1}(0)$, which is a strong deformation retract of M under a retraction $R : M \rightarrow g_0$. By Lemma 4.1.3, it suffices to show that the restriction of R induces an epimorphism $H_1(g) \rightarrow H_1(g_0)$ for $g \neq g_0$.

Take the covering $q : M^* \rightarrow M$ corresponding to $i_*(\pi_1(g))$, where $i : g \rightarrow M$ is the inclusion. Denote the restriction map $R|_g$ by R_g .

Subcase 1: $[\pi_1(M) : i_{\#}(\pi_1(g))] = t < \infty$.

Suppose that $t \neq 1$. Then consider

$$\begin{array}{ccc} & q^{-1}(g_0) \cong g_0^* & \\ & \downarrow q|_{g_0^*} & \\ g & \xrightarrow{R_g} & g_0. \end{array}$$

Since g_0^* is connected and $\pi_1(g_0^*) \cong (R_g)_{\#}(\pi_1(g)) \cong i_{\#}(\pi_1(g))$, we have a lifting $\overline{R_g}$ of R_g so that $(q|_{g_0^*}) \circ \overline{R_g} = R_g$. By Lemma 4.1.3, the degree of $\overline{R_g}$ is one so that we have $\beta_j(g) \geq \beta_j(g_0^*)$ from Proposition 2.2.3. Also we see that $\beta_j(g_0^*) \geq \beta_j(g_0)$ (from [42, Corollary 1]). Since $\beta_j(g) = \beta_j(g_0)$, we have $\beta_j(g_0^*) = \beta_j(g_0)$.

Now, identify g_0 with $K_1 \cup K_2$. Put $(q|_{g_0})^{-1}(K_1)_C = (L_1)_C$, $(q|_{g_0})^{-1}(K_2) = L_2$, where $(q|_{g_0})^{-1}(K_1)_C$ is a component of $(q|_{g_0})^{-1}(K_1)$. Then, since $(q|_{L_2}) : L_2 \rightarrow K_2$ is a finite covering, we have $\beta_j(L_2) \geq \beta_j(K_2)$ (from [42, Corollary 1]). But since $t > 1$, we have $\beta_j(g_0^*) = t\beta_j((L_1)_C) + \beta_j(L_2) = t\beta_j(K_1) + \beta_j(L_2) > \beta_j(K_1) + \beta_j(K_2) = \beta_j(g_0)$, which gives a contradiction.

Subcase 2: $[\pi_1(M) : i_{\#}(\pi_1(g))] = \infty$.

If $(R_g)_* : H_1(g) \rightarrow H_1(g_0)$ is not an epimorphism, then by the argument in the proof of Proposition 2.4.3, we have that $q^{-1}(g_0)$ has finitely generated homology, which is impossible because $q^{-1}(g_0)$ contains infinitely many copies of K_1 and $\beta_j(K_1) \neq 0$.

Assume that N has a 2-to-1 covering.

Claim : p is an approximate fibration over the mod 2 continuity set C' of p .

In light of Lemma 3.1.4, we can localize the situation so that C' is an open disk containing $b_0 = p(g_0)$ and p is an approximate fibration over $C' \setminus b_0$. Also we may assume that $R : p^{-1}(C') \rightarrow g_0$ is a strong deformation retraction. Take the covering $q : M^* \rightarrow p^{-1}(C')$ corresponding to $R_{\#}^{-1}(H) = H$, where $H = \bigcap_{i \in I} H_i$ with $I = \{i : [\pi_1(N) : H_i] = 2\}$. Then for $g \in G$ with $p(g) \in C'$, $q^{-1}(g) \simeq N_H$, where N_H

is the covering of H corresponding to H . Put $G^* = \{q^{-1}(g) : g \in G \text{ with } p(g) \in C'\}$ and let $p^* : M^* \rightarrow M^*/G^*$ be the decomposition map. Here note $M^*/G^* = C'$. By the same reason of the Subcase I, we see that the mod 2 continuity set C' of p equals the continuity set $C(p^*)$ of p^* . Take any $x \in C'$ and carefully take a neighborhood $W_x \subset C'$ of x such that $R_x : p^{-1}(W_x) \rightarrow g_x \equiv p^{-1}(x)$ is a strong deformation retraction. For $g \in G$ with $p(g) \in W_x$, let $\overline{(R_x |)} : q^{-1}(g) \rightarrow q^{-1}(g_x)$ be a map such that $(q |) \circ \overline{(R_x |)} = (R_x |) \circ (q |)$. Then the degree of $(R_x |) =$ the degree of $\overline{(R_x |)} = 1$ so that W_x is contained in the continuity set $C(p)$ of p . But since N is hopfian, p is an approximate fibration over W_x . It follows from the movability argument [9] that p is an approximate fibration over C' .

Claim : p is an approximate fibration over $(\text{int} B)$ and $\partial B = \emptyset$.

Again we localize that $\text{int} B$ is an open disk containing $b_0 = p(g_0)$ and p is an approximate fibration over $\text{int} B \setminus b_0$. Also assume that $R : \text{int} B \rightarrow g_0$ is a strong deformation retraction. Take the covering $q : M^* \rightarrow p^{-1}(\text{int} B)$ corresponding to $R_{\#}^{-1}(H) = H$, where $H = \bigcap_{i \in I} H_i$ with $I = \{i : [\pi_1(N) : H_i] = 2\}$. For $g (\neq g_0) \in G$, let $i : g \rightarrow p^{-1}(C')$ be the inclusion map. Set $K = (q | g_C^*)_{\#}(\pi_1(g_C^*))$ and $\overline{H} = i_{\#}^{-1}(q_{\#}(\pi_1(M^*)) \cap i_{\#}(\pi_1(g))) = i_{\#}^{-1}(H \cap i_{\#}(\pi_1(g)))$, where g_C^* is a component of $q^{-1}(g)$. The argument similar to Theorem 3.2.1 gives the following facts:

- 1) $H \subset \overline{H} = K$;
- 2) we only need check the case of $H \neq \overline{H}$;
- 3) in case $H \neq \overline{H}$, we have $\beta_i(N_{\overline{H}}) = \beta_i(g_C^*) = \beta_i(g_0^*) = \beta_i(N_H)$ for all i , where $\bar{q} : N_{\overline{H}} \rightarrow N$ and $\tilde{q} : N_H \rightarrow N$ are the covering maps corresponding to $\overline{H}$ and H , respectively. But since $\beta_j((\tilde{q})^{-1}(K_2)) \geq \beta_j((\bar{q})^{-1}(K_2))$, we see that $\beta_j(N_H) = [\pi_1(N) : H] \beta_j(K_1) + \beta_j((\tilde{q})^{-1}(K_2)) > [\pi_1(N) : \overline{H}] \beta_j(K_1) + \beta_j((\bar{q})^{-1}(K_2)) = \beta_j(N_{\overline{H}})$, which gives a contradiction.

Now, we will show that $\partial B = \emptyset$. Suppose $\partial B \neq \emptyset$. Then there exist $a_0 \in \partial B$, a neighborhood U of a_0 in B , and a deformation retraction $R : p^{-1}(U) \longrightarrow p^{-1}(a_0)$ such that

- (1) $U \approx$ the upper half plane $\{(x, y) \in E^2 \mid y \geq 0\}$,
- (2) $A = (\partial B) \cap U$ is an open arc, and
- (3) for all $a \in A$, $R \mid p^{-1}(a) : p^{-1}(a) \longrightarrow p^{-1}(a_0)$ is a homotopy equivalence.

Take the covering $q : M^* \rightarrow p^{-1}(U)$ corresponding to $R_{\#}^{-1}(H)$, where $H = \bigcap_{i \in I} H_i$ with $I = \{i : [\pi_1(N) : H_i] = 2\}$. For $g \in G$ with $p(g) = b$, let $i : g \longrightarrow p^{-1}(U)$ be the inclusion map. Set $\bar{H} = i_{\#}^{-1}(q_{\#}(\pi_1(M^*)) \cap i_{\#}(\pi_1(g))) = i_{\#}^{-1}(H \cap i_{\#}(\pi_1(g)))$ and $K = (q \mid g_C^*)(\pi_1(g_C^*))$. Take the covering map $\bar{q} : N_{\bar{H}} \rightarrow N$ corresponding to $\bar{H}$. Then, since g^* has two (or more) components, we have a $d - 1$ covering map $N_H \longrightarrow N_{\bar{H}} \approx N_K$ with $d > 1$ and also have a degree one map $N_K \simeq q^{-1}(p^{-1}(b))_C \longrightarrow q^{-1}(p^{-1}(a)) \simeq N_H$ (see [64, lemma 3.2] for more detail), where $\tilde{q} : N_H \rightarrow N$ is the covering map corresponding to H . By Proposition 2.2.3, we have that $\beta_j(N_H) \geq \beta_j(N_K) = \beta_j(N_{\bar{H}}) \geq \beta_j(N_H)$, i.e., $\beta_j(N_H) = \beta_j(N_H)$. But as before we see that $\beta_j(N_H) > \beta_j(N_{\bar{H}})$, which gives a contradiction.

Case II: N is nonorientable.

Then M must be nonorientable [18].

Claim : p is an approximate fibration over the mod 2 continuity set C' of p .

With the same setting of Subcase II in Case I, since N_H is hopfian, p^* is an approximate fibration over W_x so that p is.

Claim : p is an approximate fibration over $(\text{int} B)$ and $\partial B = \emptyset$.

The proof is just a copy of Subcase II in Case I. □

Corollary 4.3.1. *Let $N = N_1 \# N_2$ be a closed n -manifold with trivial $\pi_1(N_1)$ and finite or abelian $\pi_1(N_2)$. If $\beta_j(N_1) \neq 0$ for some j ($0 < j < n$), then N is a codimension-2 fibrator.*

Corollary 4.3.2. *Let $N = N_1 \# N_2$ be a s -hopfian closed n -manifold with trivial $\pi_1(N_1)$ and hopfian $\pi_1(N_2)$. If $\chi(N_2) \neq 0$, then N is a codimension-2 fibrator.*

Proof. If $\beta_j(N_1) \neq 0$ for some j ($0 < j < n$), we are done. So assume that $\beta_j(N_1) = 0$ for all j ($0 < j < n$). Then since n must be even, $\chi(N_1) = 2$, so we have that $\chi(N) = \chi(K_1) + \chi(K_2) - \chi(S^{n-1}) = \chi(N_1) - 1 + \chi(N_2) - 1 - 0 = 2 - 1 + \chi(N_2) - 1 - 0 = \chi(N_2) \neq 0$. Consequently, we have that N is an n -manifold with hopfian $\pi_1(N)$ and $\chi(N) \neq 0$. By Theorem 3.2.1, N is a codimension-2 fibrator. $\square$

All previous results give us a complete answer to the question concerning connected sums and codimension-2 fibrators in closed 4-manifolds.

Theorem 4.3.3. *Let $N = N_1 \# N_2$ be a closed 4-manifold with hopfian $\pi_1(N_i)$ for $i = 1, 2$. Then N is a codimension-2 fibrator, with only one exception $RP^4 \# RP^4$ up to homotopy type.*

Proof. First note that N is a s -hopfian manifold with hopfian $\pi_1(N)$. Due to Proposition 2.4.2 and Proposition 4.1.2, we only need consider the following two cases.

Case 1) : $\pi_1(N_1)$ is trivial and $\pi_1(N_2)$ is non-trivial.

Then $H_2(N_1)$ must be a free group so that $\beta_2(N_1) \neq 0$, otherwise N_1 is the homotopy sphere.

Case 2): $\pi_1(N_i) = \mathbb{Z}_2$ for each $i = 1, 2$.

Note that $\chi(U_i) = 2 + \beta_2(U_i) \geq 2$, where U_i is the universal covering of N_i , so that

$\chi(N_i) \geq 1$, for $\chi(U_i) = 2\chi(N_i)$. Since $N_1 \# N_2 \not\cong RP^4 \# RP^4$, $N_1 \not\cong RP^4$ or $N_2 \not\cong RP^4$. Say $N_1 \not\cong RP^4$. Then $\chi(N_1) \neq 1$ (see [56]). So $\chi(N) = \chi(N_1) - 1 + \chi(N_2) - 1 > 0$. $\square$

At this point, one may ask: What if $\beta_i(N_1) = 0$ for $0 < i < n$? In other words, given a closed n -manifold $N = N_1 \# N_2$ with simply connected N_1 , when for $0 < i < n$, $\beta_i(N_1) = 0$, is N a codimension-2 (orientable) fibrator?

Applying the method in the proof of Proposition 4.2.2, we have a partial answer as follows;

Proposition 4.3.4. *Let $N = N_1 \# N_2$ be a closed hopfian n -manifold with trivial $\pi_1(N_1)$ and hopfian $\pi_1(N_2)$. Suppose that for all i ($0 < i < n$), $\beta_i(N_1) = 0$. If there is a prime number e such that $H_1(N_2; \mathbb{Z}_e) = 0$ and $H_j(N_1; \mathbb{Z}_e) \neq 0$ for some j ($0 < j < n$), then N is a codimension-2 orientable fibrator. Moreover, if $e = 2$, then N is a codimension-2 fibrator.*

5. MANIFOLDS WITH GEOMETRIC STRUCTURE

In this section we determine which n -manifolds ($n \leq 4$) with geometric structure are codimension-2 fibrators. The cases of dimension 1 and 2 can be completely determined as follows;

DIMENSION 1.

There is a unique geometry: that of the real Euclidean line E^1 . The 1-sphere S^1 is the only closed 1-manifold, which is not a codimension-2 fibration, for it covers itself cyclically.

DIMENSION 2. There are three 2-dimensional geometries; those of the 2-sphere S^2 , the Euclidean plane E^2 and the hyperbolic plane H^2 .

Closed surfaces with geometric structure of S^2 or H^2 are codimension-2 fibrators, for those have positive or negative, respectively, Euler characteristics. And there are only two closed surfaces with geometric structure of E^2 , namely the torus $S^1 \times S^1$ and the Klein bottle K , which are the only non-codimension-2 fibrators among closed surfaces.

5.1. 3-MANIFOLDS WITH GEOMETRIC STRUCTURE. There are eight 3-dimensional geometries; E^3 , H^3 , S^3 , $S^2 \times E^1$, $H^2 \times E^1$, $SL_2(R)$, Nil , Sol . In [20] Daverman determined which closed ORIENTABLE 3-manifolds with geometric structure are codimension-2 ORIENTABLE fibrators, except spherical 3-manifold. In this section we update his result to orientation-free version in light of Theorem 3.1.5.

A geometric 3-manifold M is a compact orientable 3-manifold such that N is either a Haken manifold, a hyperbolic 3-manifold, a Seifert manifold, a 3-ball or a connected sum of those 3-manifolds. A famous conjecture of Thurston is that all compact orientable 3-manifolds are geometric 3-manifolds.

All closed 3-manifolds with geometric structure of $SL_2(R)$ are codimension-2 fibrators, as they have hyperhopfian fundamental groups ([20, Lemma 5.1]). Moreover, since all closed orientable 3-manifolds with Sol structure have hyperhopfian fundamental group ([20, Theorem 7.2]), they are codimension-2 fibrators. And having only four manifolds such as $S^2 \times S^1$, $S^2 \tilde{\times} S^1$, $RP^2 \times S^1$ and $RP^3 \# RP^3$, all closed 3-manifolds with geometric structure modelled on $S^2 \times E^1$ fail to be codimension-2 fibrators.

Lemma 5.1.1. [98] *Suppose that N is a geometric 3-manifold which is not covered by S^3 . Then any nonzero degree proper map $f : N \rightarrow N$ induces an injective homomorphism $f_{\#} : \pi_1(N) \rightarrow \pi_1(N)$.*

Proposition 5.1.2. *Every closed orientable hyperbolic 3-manifold N is a codimension-2 fibrator.*

Proof. If N has no 2-to-1 covering, being a codimension-2 orientable fibrator [20, Theorem 3.3], N is a codimension-2 fibrator. Moreover if N fibers over S^1 , then $\pi_1(N)$ is hyperhopfian [20, Corollary 3.2], so that N is a codimension-2 fibrator.

Now assume that N has a 2-to-1 covering and does not fiber over S^1 . Let $p : M \rightarrow B$ be an proper N -like map defined on a 5-manifold. Since $\pi_1(N)$ is cohopfian [20], using Lemma 5.1.1, we can easily see that p is an approximate fibration over the mod 2 continuity set C' of p .

Claim: p is an approximate fibration over $\text{int}B$.

Reduce the situation so that $\text{int}B = R^2$ and p is an approximate fibration over $R^2 \setminus 0$.

We also may have a strong deformation $R : M \rightarrow p^{-1}(0)$. We only need to check that for any $x \in R^2$ with $x \neq 0$, $R|_g : p^{-1}(x) \rightarrow p^{-1}(0)$ has nonzero degree.

As before, take the covering M_H of M corresponding to $R_{\#}^{-1}(H) \cong H$ and then consider the following

$$\begin{array}{ccccc} N_K & \xrightarrow{i^*} & M_H & \xrightarrow{R^*} & N_H \\ \downarrow & & \downarrow & & \downarrow \\ p^{-1}(x) & \xrightarrow{i} & M & \xrightarrow{R} & p^{-1}(0). \end{array}$$

Note that $[\pi_1(N_H) : (R^* \circ i^*)_{\#}(\pi_1(N_K))] < \infty$, otherwise $[\pi_1(p^{-1}(0)) : (R \circ i)_{\#}(\pi_1(p^{-1}(x)))] = \infty$, which implies that N would fiber over S^1 .

Applying the method of the proof of Lemma 4.1.3, we see that $R^* \circ i^*$ has nonzero degree so that $R \circ i$ has.

Claim: the boundary ∂B of B is empty.

Just mimic the proof of the above claim. □

5.2. 4-MANIFOLDS WITH GEOMETRIC STRUCTURE. The classification of 4-dimensional geometries in the sense of Thurston was first achieved by R. Filipkiewicz [44]. It turned out there are 19 maximal 4-dimensional geometries: and one (F^4) of them is not realizable by any closed manifold. So we only consider following eighteen 4-dimensional geometries:

♠ NON-ASPHERICAL GEOMETRIES

- Compact models: S^4 , CP^2 and $S^2 \times S^2$.
- Modelled on $S^2 \times R^2$ or $S^3 \times R$: $S^2 \times E^2$, $S^2 \times H^2$ and $S^3 \times E^1$.

♠ ASPHERICAL GEOMETRIES:

- Type A: E^4 , Nil^4 , $Nil^3 \times E^1$, $Sol_{m,n}^4$, Sol_0^4 and Sol_1^4 .
- Type B: $H^3 \times E^1$, $SL_2(R) \times E^1$, $H^2 \times E^2$, $H^2 \times H^2$, H^4 and $H^2(C)$.

Now let us determine which 4-manifolds with geometric structure are codimension-2 fibrators. While every closed surface has a residually finite fundamental group (so is hopfian), some closed 4-manifold may have a non-hopfian fundamental group, since any finitely presented group can be realized by a closed orientable 4-manifold. If a closed manifold has a non-hopfian fundamental group, we don't have any tool to show that it is a codimension-2 fibration. We will check whether given closed 4-manifolds with given geometric structure have hopfian fundamental groups.

THE GEOMETRY OF COMPACT MODELS : S^4 , CP^2 AND $S^2 \times S^2$.

There are only eleven closed 4-manifolds of this type [56]. Having finite fundamental group, all closed 4-manifolds with compact geometric structure are codimension-2 fibrators by Corollary 3.2.2.

THE GEOMETRY OF MODELLED ON $S^2 \times R^2$ OR $S^3 \times R$.

- The Geometry of $S^2 \times H^2$.

Having a finite covering $S^2 \times B$, where B is a closed orientable hyperbolic surface ([55], [56] and [57]), closed 4-manifolds with geometric structure of $S^2 \times H^2$ have negative Euler characteristics and residually finite fundamental groups so that they are codimension-2 fibrators.

- The Geometry of $S^2 \times E^2$.

Having a finite covering $S^2 \times S^1 \times S^1$ [56], closed 4-manifolds with geometric structure of $S^2 \times E^2$ have zero Euler characteristics and residually finite fundamental groups. $S^2 \times S^1 \times S^1$ is obviously not a codimension-2 fibration by Proposition 2.5.2.

An $S^1 \times S^1$ -bundle over RP^2 which does not fiber over S^1 or an $RP^2 \# RP^2$ -bundle

over RP^2 which does not fiber over S^1 are codimension-2 fibrators, for each have hyperhopfian fundamental group (all possible groups are $(\mathbb{Z}_2 * \mathbb{Z}_2) * (\mathbb{Z}_2 * \mathbb{Z}_2)$ and $(\mathbb{Z}_2 * \mathbb{Z}_2) * (\mathbb{Z} \oplus \mathbb{Z}_2)$, $\mathbb{Z} * (\mathbb{Z}_2 * \mathbb{Z}_2)$, respectively [57]).

- The Geometry of $S^3 \times E^1$.

Closed 4-manifolds N with geometric structure $S^3 \times E^1$ have a finite covering $S^3 \times S^1$ [56]. So $\chi(N) = 0$ and $\pi_1(N)$ is residually finite. $S^3 \times S^1$ is obviously not a codimension-2 fibration by Proposition 2.5.2.

THE GEOMETRY OF TYPE A.

Every closed 4-manifold N with geometric structure of Type A has a residually finite $\pi_1(N)$ and $\chi(N) = 0$. Which manifolds are codimension-2 fibrators is still open.

THE GEOMETRY OF TYPE B.

Every closed 4-manifold N with geometric structure of either $H^2 \times H^2$, H^4 or $H^2(C)$ is a codimension-2 fibration, as $\chi(N) > 0$ ([97], [67, Lemma 1]).

In closing, we mention the following unsettled topics.

Question 1. *If N and N' are closed s -hopfian manifolds, then is $N \times N'$ a closed s -hopfian manifold?*

Question 2. *What conditions on a closed manifold are necessary for being a codimension-2 fibration? What if $\chi(N) \neq 0$? For a closed (orientable) 4-manifold with abelian $\pi_1(N)$, N is a codimension-2 fibration iff $\chi(N) \neq 0$?*

Question 3. *Let N be a closed orientable n -manifold. If N is a codimension-2 orientable fibration, then is N a codimension-2 fibration?*

What if $n = 3$?

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