August 13, 1963

To the Graduate Council:

I am submitting herewith a thesis written by Chang-keng Kuo entitled "Cophasal Excitation of Circular Antenna Arrays." I recommend that it be accepted for nine quarter hours of credit in partial fulfillment of the requirements for the degree of Master of Science, with a major in Electrical Engineering.

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Major Professor

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COPHASAL EXCITATION OF CIRCULAR ANTENNA ARRAYS

A Thesis

Presented to

The Graduate Council of

The University of Tennessee

In Partial Fulfillment

of the Requirements for the Degree

Master of Science

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by

Chang-keng Kuo

August 1963

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INTRODUCTION

An antenna array consists of a set of individual radiating elements, each having a current of some amplitude and phase. If the each of the phases is choosen so that there is some direction in space for which the fields from the individual radiators add in phase, the array is said to be cophased, and the excitation is said to be cophasal. This thesis is concerned with some of the characteristics of cophased circular antenna arrays.

By far the most common antenna array is the linear array, in which the elements are equally spaced along a straight line. Such arrays have been the subject of exhaustive studies, and their properties are well understood. Such arrays are usually cophased, and techniques are well known by which the amplitudes of the currents can be choosen to give precise control of side-lobe levels. The method of synthesis using Tchebyscheff polynomials assures an optimum pattern. It is also well known that current distributions can be found for linear arrays that result in extremely high gains. Such super-gain arrays are not cophased, and the high gain is accompanied by undesirable side effects; a very precise control of the currents is required to achieve the desired pattern, and the terminal impedances are highly reactive. These drawbacks are so serious in practice that many antenna engineers have come to be highly suspicious of any antenna array which is not cophased, reasoning in a fallacious manner that if some non-cophased arrays have undesired properties, all such arrays are to be avoided.

All alternative geometry to the linear array is the circle. Stenzel[1] gave the first analysis of radiators equally spaced on the circumference

of a circle. He considered only uniform, cophasal excitation, and his results were restricted to an even number of elements. Knudsen[2] extended this to any number of elements. The uniform, cophasal circular array does not have very desirable patterns, since the first side lobe is about 8 db below the main beam, regardless of the radius, which is intolerably high.

DuHamel[3] and Hickman, Neff, and Tillman[4] developed methods for exciting circular arrays which lead to the synthesis of any desired pattern, and in particular, these methods allow the use of the Tchebyscheff polynomial technique. With these methods the excitation is not cophasal. Hickman and Tillman[5] have shown that the undesirable properties of super-gain arrays do not accompany these designs. It is, however, of considerable interest to determine if amplitude distributions can be found for cophasal excitations which will result in low side lobes, and it is with this problem that this thesis is concerned.

The problem of the radiation pattern of the cophased circular array is approached from a different point of attack than used by Stenzel. Stenzel obtained a rather simple closed form, involving a Bessel function; the present approach gives a Fourier series. Examination of the coefficients indicates why the patterns are unsatisfactory, and suggests methods of improving the result. Finally a method using concentric, cophased rings is outlined, that allows the use of the Tchebyscheff polynomial method.

DERIVATION OF THE FAR-ZONE PATTERN

A single-ring circular array is shown in Fig. 1. There are m identical elements equally spaced around the circumference of a circle of radius ρ. A spherical coordinate system is located with the origin at the center of the circle, and the polar axis is perpendicular to the plane of the circle. The elements are numbered from 1 to m, element 1 lying in the $\phi = 0$ plane. Since the elements are identical, they can be treated as isotropic source, and the resulting array factor multiplied by the element pattern.

The field of the ith element is proportional to

$$F_i = I_i e^{j\beta\rho \sin\theta \cos(\phi - \frac{2\pi}{m} i)} . \tag{1}$$

The pattern of the array is found by summing contributions from all of the elements:

$$F(\theta, \phi) = \sum_{i=1}^{m} I_i e^{j\beta\rho \sin\theta \cos(\phi - \frac{2\pi}{m} i)} . \tag{2}$$

Inspections shows that if I_i is set to

$$I_i = |I_i| e^{-j\beta\rho \sin\theta_0 \cos(\phi_0 - \frac{2\pi}{m} i)} \tag{3}$$

then in the direction $\phi = \phi_0$, $\theta = \theta_0$, (2) becomes

$$F(\theta_0, \phi_0) = \sum_{i=1}^{m} |I_i| . \tag{4}$$

This is the condition called cophasal excitation. Uniform excitation sets

$$|I_i| = I_0 \tag{5}$$

for all i. This is the case analyzed by Stenzel and by Knudsen. The result is

$$F(\theta, \phi) = I_0 \left[J_0(\beta\rho a) + 2\sum_{p=1}^{\infty} j^{mp} J_{mp}(\beta\rho a) \cos mp\delta \right] \tag{6}$$

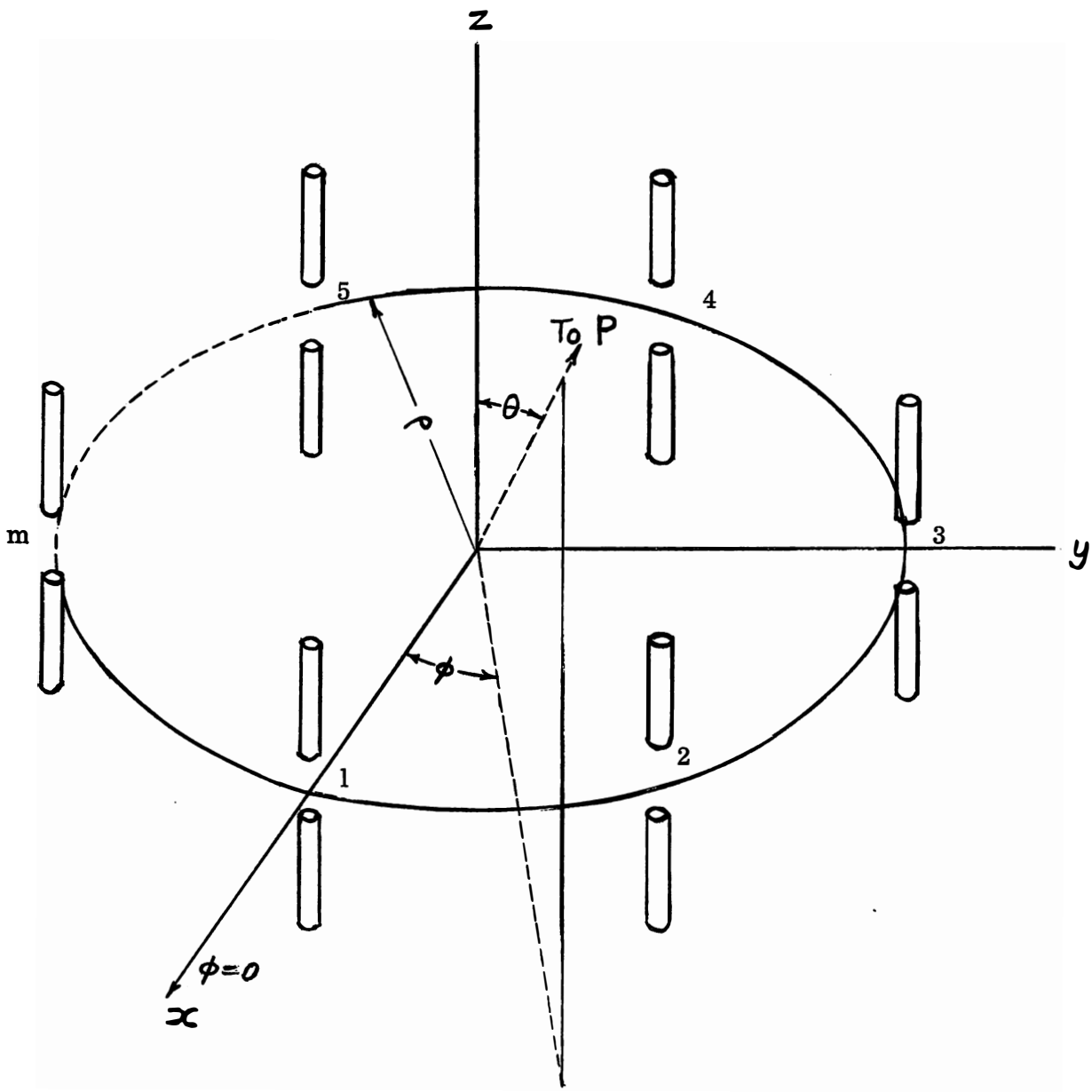


Fig. 1. The Coordinate and the Arrangement of the Array.

in which

$$a = \left[(\sin\theta\cos\phi - \sin\theta_0\cos\phi_0)^2 + (\sin\theta\sin\phi - \sin\theta_0\sin\phi_0)^2\right]^{\frac{1}{2}}$$

$$\delta = \cos^{-1}\frac{1}{a}(\sin\theta\cos\phi - \sin\theta_0\cos\phi_0).$$

For the important practical case of $\theta_0 = \theta = \pi/2$, and $\phi_0 = 0$, the leading term is

$$F(\tfrac{\pi}{2}, 0) = I_0 J_0 (2\beta\rho\sin\tfrac{\phi}{2}), \tag{7}$$

the remaining terms being negligble if m is large enough. In practice, m is chosen so as to make the first term in the sum small compared to the Bessel function in (7).

We will now obtain an alternative form for the pattern. The general method of Hickman, Neff, and Tillman is to resolve the currents into symmetrical components by

$$I^{(n)} = \frac{1}{m}\sum_{i=1}^{m} I_i e^{-j\frac{2\pi}{m}ni}. \tag{8}$$

The pattern of each component is then shown to be

$$F^{(n)}(\theta, \phi) = j^n I^{(n)} J_n(\beta\rho\sin\theta)\, e^{jn\phi} + \sum_{K=1}^{\infty}\Big[$$

$$j^{mk-n} I^{(n)} e^{-j(mk-n)\phi} J_{mk-n}(\beta\rho\sin\theta)$$

$$+ j^{mk+n} I^{(n)} e^{j(mk+n)\phi} J_{mk+n}(\beta\rho\sin\theta)\Big]. \tag{9}$$

As above, all of the terms in the summation can be neglected. The total pattern is then

$$F(\theta, \phi) = \sum_{n=1}^{m} F^{(n)}(\theta, \phi). \tag{10}$$

For simplicity, we will restrict the following treatment to the case of $\theta_0 = \pi/2$, $\phi_0 = 0$. Substituting (3) and (5) into (8),

$$I^{(n)} = \frac{1}{m} \sum_{i=1}^{m} I_0 e^{-j\beta\rho \cos \frac{2\pi}{m} i} e^{-j \frac{2\pi}{m} ni}. \tag{11}$$

The first exponential is expanded into the series

$$e^{-j\beta\rho \cos \frac{2\pi}{m} i} = J_0 (\beta\rho) + 2 \sum_{k=1}^{\infty} (-1)^k j^k J_k (\beta\rho) \cos \frac{2\pi}{m} ki.$$

Inserting this into (11) and reversing the finite and infinite sums,

$$I^{(n)} = \frac{I_0}{m} \left\{ J_0 (\beta\rho) \sum_{i=1}^{m} e^{-j \frac{2\pi}{m} ni} + 2 \sum_{k=1}^{\infty} \left[(-1)^k j^k J_k (\beta\rho) \sum_{i=1}^{m} e^{-j \frac{2\pi}{m} ni} \cos k \frac{2\pi}{m} i \right] \right\}. \tag{12}$$

There are two forms of sums in (12), both of which can be reduced to geometric progressions 1 - m.

$$\sum_{i=1}^{m} e^{-j \frac{2\pi}{m} ni} = \frac{\sin \left(\frac{-m}{2} \frac{2\pi}{m} n \right)}{\sin \left(-\frac{1}{2} \frac{2\pi}{m} n \right)} e^{-j \frac{m+1}{2} \frac{2\pi}{m} n}$$

$$= \frac{\sin n\pi}{\sin \frac{n}{m}\pi} e^{-j \frac{n(m+1)}{m}\pi} \tag{13}$$

because of geometric progressions. But $\sin n\pi = 0$ $(n = 1 \sim m)$ therefore

$\sum_{i=1}^{m} e^{-j \frac{2\pi}{m} ni} = 0$ unless $\sin \frac{n}{m}\pi = 0$. It follows that

$$\sum_{i=1}^{m} e^{-j \frac{2\pi}{m} ni} = 0 \qquad \text{at } n \neq 0,$$

$$= m \qquad \text{at } n = 0. \tag{14}$$

The other sum is

$$\sum_{i=1}^{m} e^{-j\frac{2\pi}{m}ni} \cos k\frac{2\pi}{m}i$$

$$= \frac{1}{2}\sum_{i=1}^{m} e^{-j\frac{2\pi}{m}ni}\left[e^{j\frac{2k\pi i}{m}} + e^{-j\frac{2k\pi i}{m}}\right]$$

$$= \frac{1}{2}\sum_{i=1}^{m}\left[e^{-j\frac{2\pi}{m}(n+k)i} + e^{-j\frac{2\pi}{m}(n-k)i}\right]. \tag{15}$$

But

$$\sum_{i=1}^{m} e^{-j\frac{2\pi}{m}(n+k)i} = \frac{\sin\pi(n+k)}{\sin\frac{\pi}{m}(n+k)}\, e^{-j\frac{(m+1)(n+k)}{m}\pi}$$

$$= 0 \qquad \text{at } n + k \neq lm,$$

$$= m \qquad \text{at } n + k = lm \tag{16}$$

where l is any integer. Since the values of k are from 1 to ∞, the values of l must be positive.

$$\sum_{i=1}^{m} e^{-j\frac{2\pi}{m}(n-k)i} = \frac{\sin\pi(n-k)}{\sin\frac{\pi}{m}(n-k)}\, e^{-j\frac{\pi(m+1)}{m}(n-k)}$$

$$= 0 \qquad \text{at } n - k \neq l'm$$

$$= m \qquad \text{at } n - k \neq l'm \tag{17}$$

where l' must be any negative integer.

Substituting the results of (16) and (17) into (15), we get

$$\sum_{i=1}^{m} e^{-j\frac{2\pi}{m}ni} \cos k\frac{2\pi}{m}i$$

$$= \frac{1}{2}m \qquad \text{at } k = n$$

$$= \frac{1}{2}m \qquad \text{at } k = m - n$$

$$= \frac{1}{2}m \qquad \text{at } k = m + n$$

$$= \frac{1}{2}m \qquad \text{at } k = 2m - n$$

$$= \frac{1}{2}m \qquad \text{at } k = 2m + n\,. \tag{18}$$

Finally, combining all of the parts, the sequence current $I^{(n)}$ is found to be

$$I^{(n)} = \frac{I_0}{m}\Big[(-1)^n j^n J_n(\beta\rho) + (-1)^{m-n} j^{m-n} J_{m-n}(\beta\rho)$$

$$+ (-1)^{m+n} j^{m+n} J_{m+n}(\beta\rho) + (-1)^{2m-n} j^{2m-n} J_{2m-n}(\beta\rho)$$

$$+ (-1)^{2m+n} j^{2m+n} J_{2m+n}(\beta\rho) + \ldots\Big] Xm$$

$$= (-1)^n I_0 j^n J_n(\beta\rho) + \sum_{p=1}^{\infty}\Big[(-1)^{pm-n} I_0 j^{pm-n} J_{pm-n}(\beta\rho)$$

$$+ (-1)^{pm+n} I_0 j^{pm+n} J_{pm+n}(\beta\rho)\Big] \tag{19}$$

In (19) the leading term is the principal term; all of the terms in the summation are small. If m is increased without limit, the residuals become negligible.

So, in the case of m very large, equation (11) becomes

$$I^{(n)} = (-1)^n I_0 j^n J_n(\beta\rho). \tag{20}$$

The field pattern due to nth sequence current is

$$F^{(n)}(\phi) = I^{(n)} j^{n} e^{jn\phi} J_n(\beta\rho). \tag{21}$$

There are m possible sets of sequence currents, and any m consecutive values of n are satisfactory. Let the values of n be

$$-\frac{m-1}{2}, -\frac{m-3}{2}, \ldots\ldots -2, -1, 0, 1, 2, \ldots\ldots \frac{m-1}{2}.$$

Consequently, the total field pattern is:

$$F(\phi) = \sum_{n=-\frac{m-1}{2}}^{\frac{m-1}{2}} F^{(n)}(\phi)$$

$$= I^{(0)} J_{(0)}(\beta\rho) + I^{(1)} j e^{j\phi} J_1(\beta\rho) + I^{(-1)} j^{-1} e^{-j\phi} J_{-1}(\beta\rho)$$

$$+ I^{(2)} j^{2} e^{j2\phi} J_2(\beta\rho) + I^{(-2)} j^{(-2)} e^{-j2\phi} J_{-2}(\beta\rho)$$

$$+ \ldots\ldots$$

$$+ I^{(n)} j^{n} e^{jn\phi} J_n(\beta\rho) + I^{(-n)} j^{-n} e^{-jn\phi} J_{-n}(\beta\rho)$$

$$+ \ldots\ldots$$

$$= I_0 J_0^{2}(\beta\rho) + \sum_{n=1}^{\frac{m-1}{2}} \left[I^{(n)} j^{n} e^{jn\phi} J_n(\beta\rho) + I^{(-n)} j^{-n} e^{-jn\phi} J_{-n}(\beta\rho) \right]$$

$$= I_0 J_0^{2}(\beta\rho) + \sum_{n=1}^{\frac{m-1}{2}} (-1)^{n} j^{2n} I_0 J_n^{2}(\beta\rho) \left[e^{jn\phi} + e^{-jn\phi} \right]$$

$$= I_0 J_0^2(\beta\rho) + 2I_0 \sum_{n=1}^{\frac{m-1}{2}} J_n^2(\beta\rho) \cos n\phi \,. \tag{22}$$

As before, if m is large, (22) also may be written as:

$$F(\phi) = I_0 J_0^2(\beta\rho) + 2I_0 J_1^2(\beta\rho) \cos\phi + 2I_0 J_2^2(\beta\rho) \cos 2\phi$$

$$+ \ldots\ldots$$

$$= I_0 J_0^2(\beta\rho) + 2\sum_{n=1}^{\infty} J_n^2(\beta\rho) \cos n\phi \,. \tag{23}$$

This is the desired alternative form to (7). Their equivalence is shown in the Appendix.

A typical pattern for a small array, $\beta\rho = 2$, is shown in Fig. 2. The high side-lobe level is clearly evident.

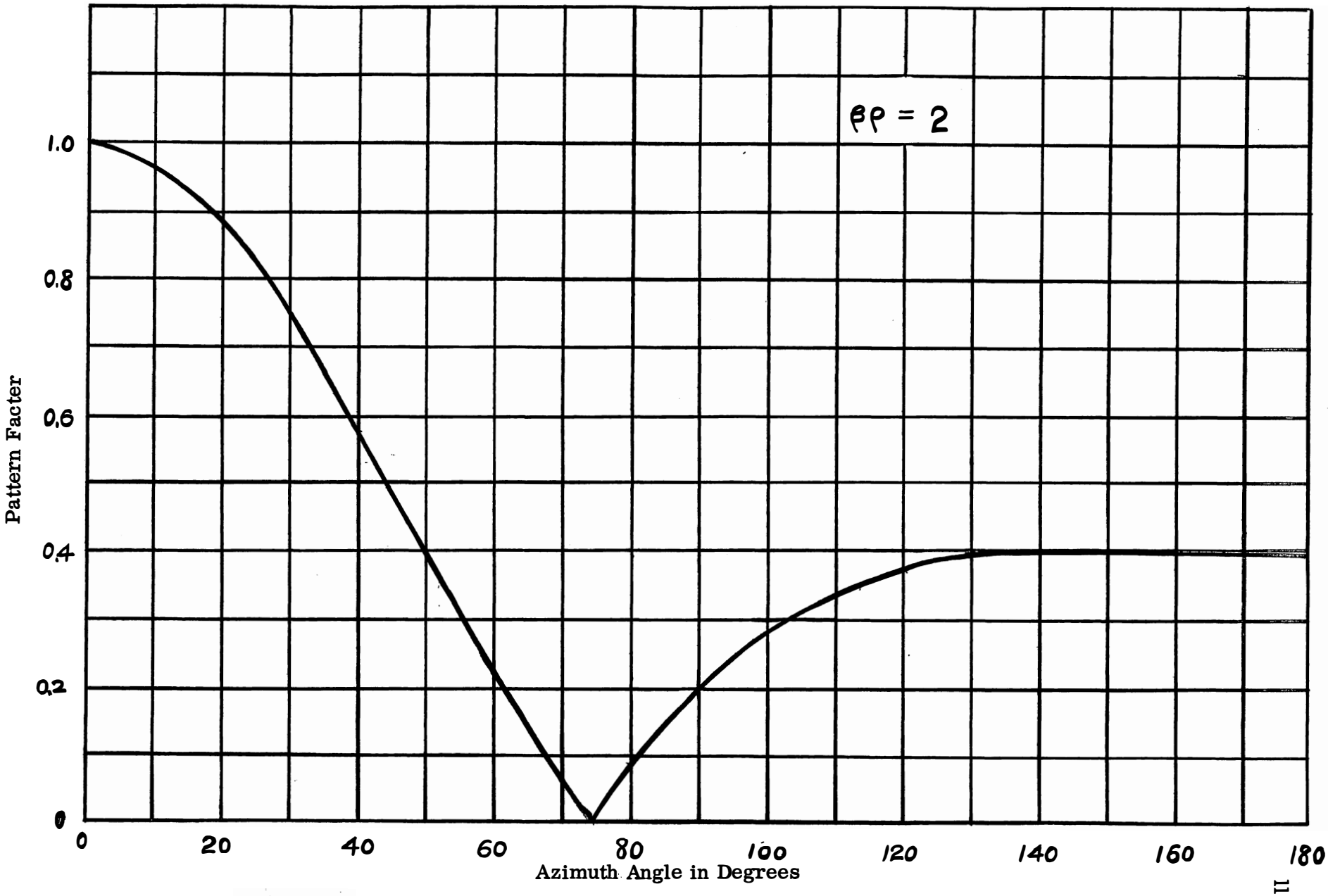


Fig. 2. Azimuthal Pattern of the Ordinary Cophasal Excitation in Circular Array.

RADIATION FROM A COPHASED CURRENT SHEET

In the preceeding section it was seen that the pattern of a circular array could be approximated by

$$F(\phi) = I_0 J_0 (2\beta\rho \sin \phi / 2)$$

$$= I_0 J_0^2 (\beta\rho) + 2J_1^2 (\beta\rho) \cos \phi$$

$$+ 2J_2^2 (\beta\rho) \cos 2\phi + \ldots \ldots \quad (1)$$

These forms were obtained by neglecting higher order terms in series expansions. A much simpler development, based on a current sheet, is given in this section. The result given in (1) is the limit as the number of elements, m, became infinitely large. But an infinite number of elements is physically equivalent to a circular sheet of current, and it should be possible to obtain (1) directly as the pattern for such a sheet.

Such a current sheet is seen in Fig. 3. The current can be written as

$$I(\bar{\phi}) = \sum_{n=0}^{\infty} \epsilon_n I_n \cos n\bar{\phi} . \quad (2)$$

(2) is the Fourier expansion of $I(\bar{\phi})$ and contains only cosine terms, since $I(\bar{\phi})$ must be even if the radiation pattern is to be symmetrical about $\phi = 0$. In general, the Fourier coefficients I_n can be complex. The pattern due to any single term of (2) is

$$F_n(\phi) = \int_0^{2\pi} I_n \cos n\bar{\phi}\, e^{j\beta\rho\cos(\phi - \bar{\phi})}\, d\bar{\phi}. \quad (3)$$

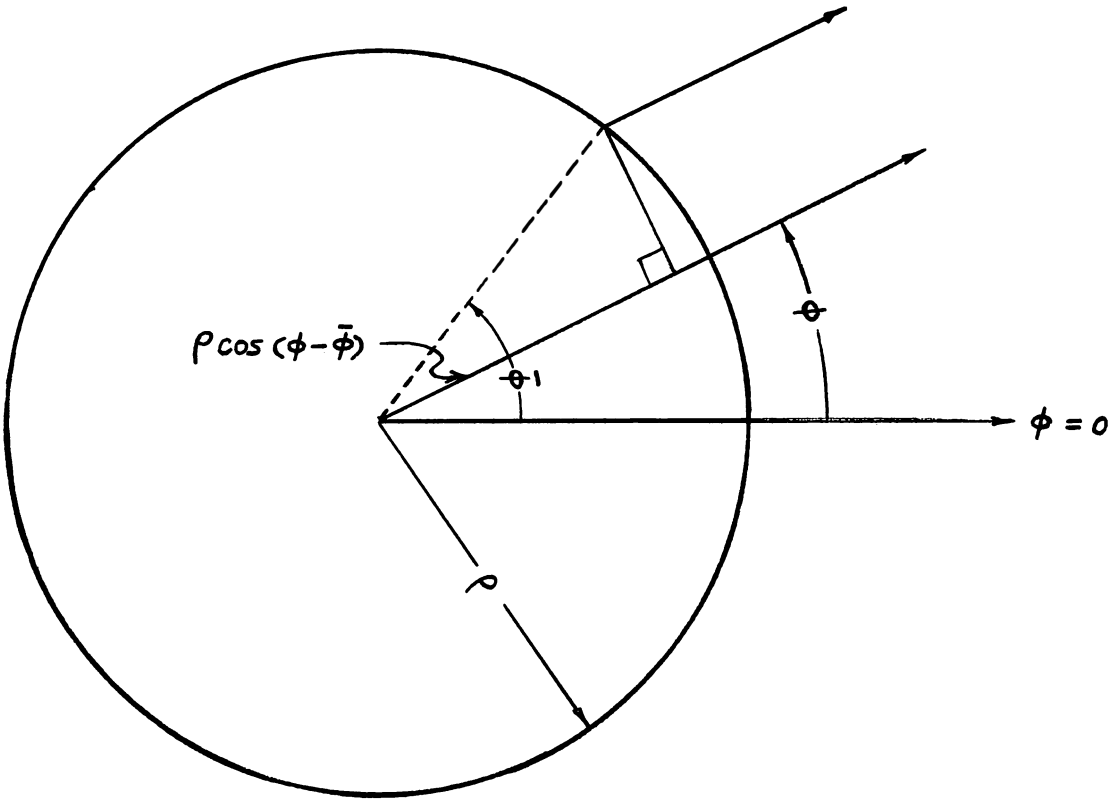


Fig. 3. Circular Current Showing Symbols and Geometry to Find Far Field Pattern.

As before, the exponential is expanded, giving

$$F_n(\phi) = \int_0^{2\pi} I_n \cos n\bar{\phi} \left[J_0(\beta\rho) + 2\sum_{k=1}^{\infty} j^k J_k(\beta\rho) \cos k(\phi - \bar{\phi}) \right] d\bar{\phi} . \qquad (4)$$

Interchanging the order of summation and integration,

$$F_n(\phi) = I_n \left[J_0(\beta\rho) \int_0^{2\pi} \cos n\bar{\phi}\, d\bar{\phi} + 2\sum_{k=1}^{\infty} j^k J_k(\beta\rho) \int_0^{2\pi} \cos n\bar{\phi} \cos k(\phi - \bar{\phi}) \right] d\bar{\phi} . \qquad (5)$$

Now,

$$\int_0^{2\pi} \cos n\bar{\phi}\, d\bar{\phi} = 2\pi, \qquad n = 0$$

$$= 0 . \qquad n \neq 0$$

Also,

$$\int_0^{2\pi} \cos n\bar{\phi} \cos k(\phi - \bar{\phi})\, d\bar{\phi} = \pi \cos n\phi \qquad n = k$$

$$= 0, \qquad n \neq k.$$

Then

$$F_n(\phi) = I_n 2\pi j^n J_n(\beta\rho) \cos n\phi. \qquad (6)$$

It is seen that each Fourier component of the current sheet is related only to the corresponding Fourier component of the pattern. That is

$$I(\phi) = \sum_{n=0}^{\infty} \epsilon_n I_n \cos n\phi, \qquad (7)$$

$$F(\phi) = \sum_{n=0}^{\infty} \epsilon_n A_n \cos n\bar{\phi}, \tag{8}$$

$$A_n = 2\pi j^n J_n(\beta\rho) I_n. \tag{9}$$

These simple results hold for any current sheet for which $I(\bar{\phi})$ is an even function.

To obtain the pattern for cophasal excitation it is necessary only to obtain the expansion of $I(\bar{\phi})$ into a Fourier series. For uniform, cophasal excitation, $\phi_0 = 0$, $\theta_0 = \pi/2$

$$I(\bar{\phi}) = I_0 e^{-j\beta\rho\cos\bar{\phi}}. \tag{10}$$

But an expansion for this current is

$$I(\bar{\phi}) = I_0 \left[J_0(\beta\rho) + 2\sum_{n=1}^{\infty} (-1)^n j^n J_n(\beta\rho) \cos n\bar{\phi}\right].$$

It follows immediately that

$$A_n = 2\pi j^n J_n(\beta\rho)\left[(-1)^n j^n J_n(\beta\rho)\right]$$

$$= 2\pi J_n^2(\beta\rho). \tag{11}$$

The pattern of the current sheet is then

$$F(\phi) = 2\pi I_0 \left[J_0^2(\beta\rho) + 2\sum_{n=1}^{\infty} J_n^2(\beta\rho) \cos n\phi\right]. \tag{12}$$

This is the same form as (1). The element currents leading to (1) were I_0/m, so the total current is I_0. The current sheet has a total current

$$\int_0^{2\pi} |I(\bar{\phi})| \, d\bar{\phi} = 2\pi I_0,$$

so the two results are identical.

Some conclusions can now be drawn concerning the cophased circular array:

1. The pattern will contain significant terms only for $n \leq \beta\rho$, since $J_n(\beta\rho)$ is small if $n > \beta\rho$. Thus the series for the pattern converges very rapidly.
2. If $\beta\rho$ is near any root of a Bessel function, the corresponding term in the pattern will be small. This accounts for the poor behavior of arrays with $\beta\rho = 2$ or $\beta\rho = 3$, $J_0(2.403) = 0$.

This last item suggests a way of improving such small arrays. An element at the center of the array can be used to change the constant term, and thus much improve the pattern.

SYNTHESIS OF DESIRED PATTERN

The analysis of the preceding section leads to a simple synthesis method to give patterns with equal prescribed side-lobes. In the nth degree Tchebyscheff polynomial $T_n(Z)$, let Z be

$$Z = a \cos \phi + b \tag{1}$$

choose a and b so that $a + b = Z_0$; $-a + b = -1$, and as ϕ varies from 0 to π, Z varies from Z_0 to - 1. The pattern is $T_n(Z_0)$ at $\phi = 0$, and the side-lobe level is $T_n(Z_0)$, since $|T_n(Z)| \leq 1$, for $|Z| \leq 1$. Expansion of $T_n(a \cos \phi + b)$, leads to A_n, and consequently I_n.

In general, the Tchebyscheff polynomial may be reduced to the form of

$$T_n(X) = A_0 + A_1 \cos \phi + A_2 \cos 2\phi + \ldots + A_n \cos n\phi . \tag{2}$$

We may now combine the patterns of one or more concentric current sheets with that of an antenna situated at the center of the array circle, and from the total radiating pattern, a form like that of (2) can be obtained. Then, by the method of comparing coefficients, the currents in each sheet required to produce the desired pattern of the form of (2) can be found.

The method will be described below with some examples.

Example 1. Obtain the currents required to produce the pattern of

$$T_2(X) = A_0 + A_1 \cos \phi + A_2 \cos 2\phi.$$

Solution: As shown in Fig. 4 (a), two current sheets and an isolated antenna will be used. The isolated antenna situated at the center of the circle carries

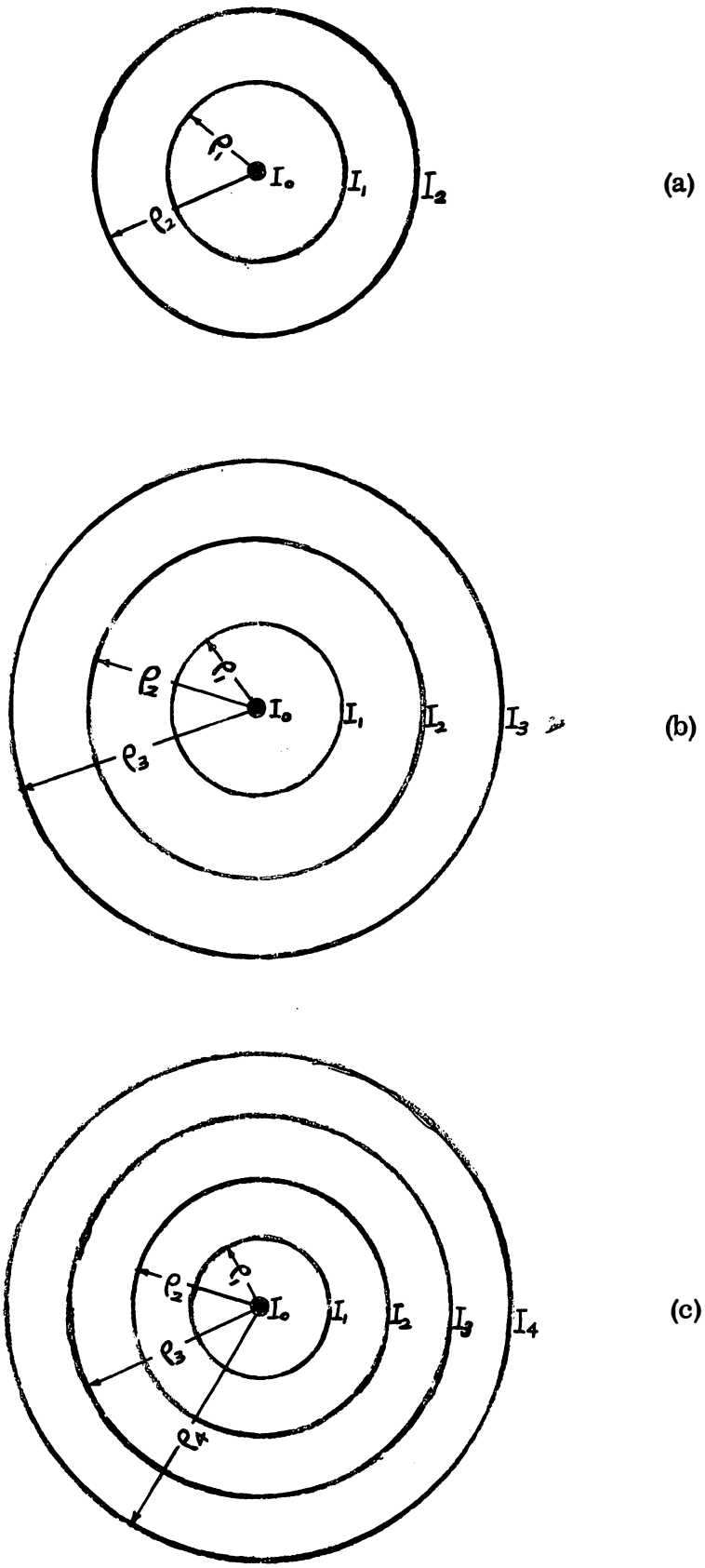


Fig. 4. The Current Sheets Combined with a Center Element.

a current of I_0; the currents of the currents sheet are I_1 and I_2 respectively. Let F_{I_0} be the pattern due to I_0, F_{I_1} be the pattern due to I_1, F_{I_2} be the pattern due to I_2, $F(\phi)$ be the total combined pattern. Then,

$$F(\phi) = F_{I_0} + F_{I_1} + F_{I_2}$$

$$F(\phi) = I_0 + 2\pi I_1\left[J_0^2(\beta\rho_1) + 2J_1^2(\beta\rho_1)\cos\phi + 2J_2^2(\beta\rho_1)\cos 2\phi + \cdots\right]$$

$$+ 2\pi I_2\left[J_0^2(\beta\rho_2) + 2J_1^2(\beta\rho_2)\cos\phi + 2J_2^2(\beta\rho_2)\cos 2\phi + \cdots\right]$$

$$= I_0 + 2\pi\left[I_1J_0^2(\beta\rho_1) + I_2J_0^2(\beta\rho_2)\right]$$

$$+ 4\pi\left[I_1J_1^2(\beta\rho_1) + I_2J_1^2(\beta\rho_2)\right]\cos\phi$$

$$+ 4\pi\left[I_1J_2^2(\beta\rho_1) + I_2J_2^2(\beta\rho_2)\right]\cos 2\phi$$

$$+ \cdots\cdots \tag{3}$$

Neglecting terms for $n > 2$

$$\left.\begin{aligned} I_0 + 2\pi\left[I_1J_0^2(\beta\rho_1) + I_2J_0^2(\beta\rho_2)\right] &= A_0 \\ 4\pi\left[I_1J_1^2(\beta\rho_1) + I_2J_1^2(\beta\rho_2)\right] &= A_1 \\ 4\pi\left[I_1J_2^2(\beta\rho_1) + I_2J_2^2(\beta\rho_2)\right] &= A_2 \end{aligned}\right\} \tag{4}$$

Therefore if the desired side-lobe level is given, A_0, A_1, A_2 can be found; then, from these values and (4), I_0, I_1 and I_2 can be found. This gives the desired pattern in the cophasal excitation.

For example, if $\beta\rho_1 = 1$, $\beta\rho_2 = 2$, then,

$$J_0(\beta\rho_1) = 0.7652$$

$$J_0(\beta\rho_2) = 0.2239$$

$$J_1(\beta\rho_1) = 0.4401$$

$$J_1(\beta\rho_2) = 0.5767$$

$$J_2(\beta\rho_1) = 0.1149$$

$$J_2(\beta\rho_2) = 0.3528$$

From (4)

$$\left.\begin{aligned} I_0 + 2\pi\left[0.587\, I_1 + 0.050 I_2\right] &= A_0 \\ 4\pi\left[0.194\, I_1 + 0.333\, I_2\right] &= A_1 \\ 4\pi\left[0.013\, I_1 + 0.125\, I_2\right] &= A_2 \end{aligned}\right\} \qquad (5)$$

a) for 20 db side lobe level;

$$T_2(X) = 2.7 + 4.5 \cos \phi + 2.8 \cos 2\phi$$

$$A_0 = 2.7$$

$$A_1 = 4.5$$

$$A_2 = 2.8$$

from these values and (5), we get

$$I_0 = 7.62$$

$$2\pi I_1 = -9.42$$

$$2\pi I_2 = 12.25.$$

These can be normalized with I_0 as reference.

$$I_0 = 1$$

$$2\pi I_1 = -1.24$$

$$2\pi I_2 = 1.61$$

b) for 25 db side lobe level,

$$A_0 = 5.37$$

$$A_1 = 8.36$$

$$A_2 = 4.25$$

and

$$I_0 = 9.94$$

$$2\pi I_1 = -9.33$$

$$2\pi I_2 = 17.98$$

or in normalized value,

$$I_0 = 1$$

$$2\pi I_1 = -0.94$$

$$2\pi I_2 = 1.81$$

c) for 30 db side lobe level

$$A_0 = 7.5$$

$$A_1 = 7.5$$

$$A_2 = 6.25$$

and

$$I_0 = 23.1$$

$$2\pi I_1 = -28.9$$

$$2\pi I_2 = -28.2$$

or in normalized values,

$$I_0 = 1$$

$$2\pi I_1 = -1.25$$

$$2\pi I_2 = 1.24$$

The procedure is readily extended. A $T_3(\phi)$ pattern requires terms through n = 3, and thus needs three current sheets. Tables I, II, and III summarize the results of calculations like those above for $T_2(\phi)$, $T_3(\phi)$ and $T_4(\phi)$ patterns, and 20 db, 25 db, and 30 db side-lobe levels.

To complete the above discussion, the simple equations directly relating A and I are given below, both in the case of

(1) $F(\phi) = A_0 + A_1 \cos\phi + A_2 \cos 2\phi$ and

(2) $F(\phi) = A_0 + A_1 \cos\phi$

(1) $F(\phi) = A_0 + A_1 \cos\phi + A_2 \cos 2\phi.$

Here two currents sheets are used.

$$\left.\begin{aligned} I_0 + 2\pi\left[I_1 J_0^2(\beta\rho_1) + I_2 J_0^2(\beta\rho_2)\right] &= A_0 \\ 4\pi\left[I_1 J_1^2(\beta\rho_1) + I_2 J_1^2(\beta\rho_2)\right] &= A_1 \\ 4\pi\left[I_1 J_2^2(\beta\rho_1) + I_2 J_2^2(\beta\rho_2)\right] &= A_2 \end{aligned}\right\} \quad (6)$$

Table I. The values of the currents in pattern of $T_2(X)$

	20 db		25 db		30 db	
A_0	2.7		5.37		7.50	
A_1	4.5		8.36		7.50	
A_2	2.8		4.25		6.25	
	original value	normalized value	original value	normalized value	original value	normalized value
I_0	7.62	1	9.94	1	23.1	1
$2\pi I_1$	- 9.42	-1.24	- 9.33	-0.94	- 28.9	- 1.25
$2\pi I_2$	12.25	1.61	17.98	1.81	- 28.2	1.24

Table II. The values of the currents in pattern of $T_3(X)$

	20 db		25 db		30 db	
A_0	0.91		3.52		8.61	
A_1	4.70		6.12		14.84	
A_2	1.64		4.32		9.52	
A_3	2.05		2.62		4.17	
	original value	normalized value	original value	normalized value	original value	normalized value
I_0	-28.8	1	-19.24	1	-6.1	1
$2\pi I_1$	51.4	-1.79	38.60	-2.28	23.4	-3.67
$2\pi I_2$	-29.0	1.00	-21.70	1.28	-11.9	1.95
$2\pi I_3$	15.7	-0.55	16.95	-0.88	23.7	-3.72

Table III. The values of the currents in pattern of $T_4(X)$

	20 db		25 db		30 db	
A_0	1.48		3.94		11.20	
A_1	3.56		7.30		20.30	
A_2	2.40		5.88		15.12	
A_3	1.80		3.90		8.80	
A_4	1.74		2.44		3.85	
	original value	normalized value	original value	normalized value	original value	normalized value
I_0	51.33	1	85.8	1	58.4	1
$2\pi I_1$	-78.50	-1.53	-150.0	-1.75	-106.1	-1.82
$2\pi I_2$	98.70	1.92	105.0	1.23	106.2	1.82
$2\pi I_3$	-181.50	-3.52	-50.0	-0.58	-35.8	-0.61
$2\pi I_4$	23.20	0.45	27.4	0.32	76.7	1.31

$$I_1 = \frac{A_1 J_1^2(\beta\rho_2) - A_2 J_1^2(\beta\rho_2)}{4\pi\left[J_1^2(\beta\rho_1)\, J_2^2(\beta\rho_2) - J_1^2(\beta\rho_2)\, J_2^2(\beta\rho_1)\right]}$$

$$I_2 = \frac{A_2 J_1^2(\beta\rho_1) - A_1 J_2^2(\beta\rho_1)}{4\pi\left[J_1^2(\beta\rho_1)\, J_2^2(\beta\rho_2) - J_1^2(\beta\rho_2)\, J_2^2(\beta\rho_1)\right]}$$

$$I_0 = A_0 - 2\pi J_0^2(\beta\rho_1)\, I_1 - 2\pi J_0^2(\beta\rho_2)\, I_2 \qquad (7)$$

(2) $F(\phi) = A_0 + A_1 \cos\phi$.

Here only one current sheet is used

$$I_0 + 2\pi I_1 J_0^2(\beta\rho) = A_0$$

$$4\pi I_1 J_1^2(\beta\rho) = A_1 \qquad (8)$$

$$I_1 = \frac{A_1}{4\pi J_1^2(\beta\rho)}$$

$$I_0 = A_0 - \frac{A_1 J_0^2(\beta\rho)}{2 J_1^2(\beta\rho)} \qquad (9)$$

CONCLUSION

It is obvious from the above discussion, the disadvantages of the high side-lobe level in the case of a single circular array or current sheet used alone may be improved by the method of combining several current sheets with an isolated antenna put at the center of the circle. By this method, any desired side-lobe level can be obtained still using the cophasal excitation.

Equations (8) and (9) give the approximately similar pattern of first degree of the Tchebyscheff equation; where as equations (6) and (7) give that of second degree. If the higher degree are required, of course the procedure described above can be used.

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REFERENCES

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APPENDIXES

APPENDIX A

In this appendix it will be shown that the two forms $J_0(2\beta\rho \sin \phi/2)$ and $J_0^2(\beta\rho) + 2J_1^2(\beta\rho) \cos \phi + 2J_2^2(\beta\rho) \cos 2\phi + \ldots$ are identical.

From the triangle with sides x, y and included angle α as shown in the figure,

$$r^2 = x^2 + y^2 - 2xy \cos\alpha \tag{1}$$

$$u = r \cos \delta = r \cos (\beta - \phi + \alpha)$$

$$= r \cos \left[\phi - (\alpha + \beta)\right] \tag{2}$$

also,

$$u = y \cos (\phi - \alpha) + x \cos (\pi - \phi)$$

$$= -x \cos \phi + y \cos (\phi - \alpha) \tag{3}$$

(2) and (3) give

$$e^{jr \cos \left[\phi - (\alpha + \beta)\right]} = e^{jy \cos (\phi - \alpha)} \cdot e^{-jx \cos \phi} \tag{4}$$

Now, integrating both sides of (4) with respect to ϕ from 0 to 2π, the result is:

$$2\pi J_0(r) = 2\pi J_0(x)\, J_0(y) + 4\pi \sum_{n=1}^{\infty} J_n(x)\, J_n(y) \cos n\alpha. \tag{5}$$

Thus

$$J_0 \left(\sqrt{x^2 + y^2 - 2xy \cos\alpha}\right)$$

$$= \sum_{n=0}^{\infty} \epsilon_n J_n(x)\, J_n(y) \cos n\alpha . \tag{6}$$

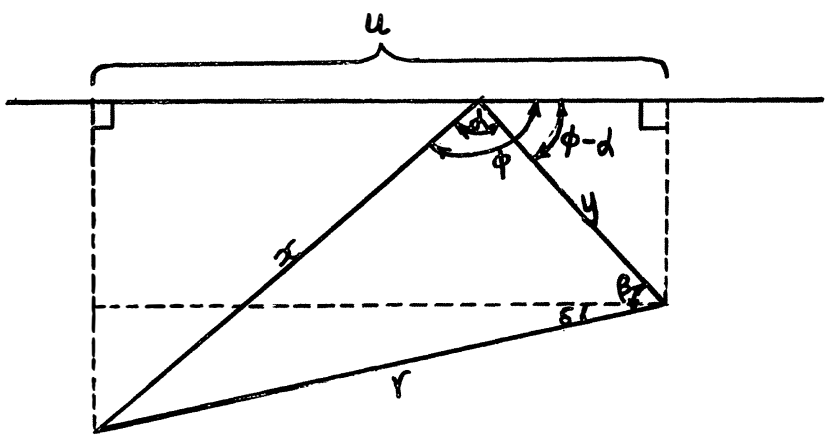
Let x = y = a, and we get finally,

$$J_0\left(2a \sin \frac{\alpha}{2}\right)$$

(7)

$$= \sum_{n=0}^{\infty} \epsilon_n J_n^2(a) \cos n\alpha$$

which is the desired result.



APPENDIX B

In this appendix a complete design will be carried out. Let a co-phased array be required with a $T_2(\phi)$ pattern having 20 db side-lobe level.

Two concentric rings will be used, $\beta\rho_1 = 1$ and $\beta\rho_2 = 2$, as shown in Fig. 5. The form for discrete elements must be used. To eliminate residuals,

$$J_{m_1}(2\beta\rho_1) \ll J_0(2\beta\rho_1) ,$$

$$J_{m_2}(2\beta\rho_2) \ll J_0(2\beta\rho_2) ,$$

which are met reasonably well by $m_1 = 5$, and $m_2 = 7$. The center element current is 1.0, and the total ring currents are -1.24 and 1.61. They are thus: $I_1 = 1\angle 0$, $I_2 = I_5 = -0.248 \angle -17.7$, $I_3 = I_4 = -0.248 \angle 46.3$, $I_6 = 0.23 \angle -114.6$, $I_7 = I_{12} = 0.23 \angle -58.8$, $I_8 = I_{11} = 0.23 \angle 25.7$ and $I_9 = I_{10} = 0.23 \angle 103$.

The impedance of the ith element is

$$Z_i = \frac{\sum_{k=0}^{12} I_k Z_{ik}}{I_i}$$

this gives:

$$Z_0 = 49 \angle 49.3^\circ$$

$$Z_1 = 119.6 \angle 156.2^\circ$$

$$Z_2 = Z_5 = 158.2 \angle 153^\circ$$

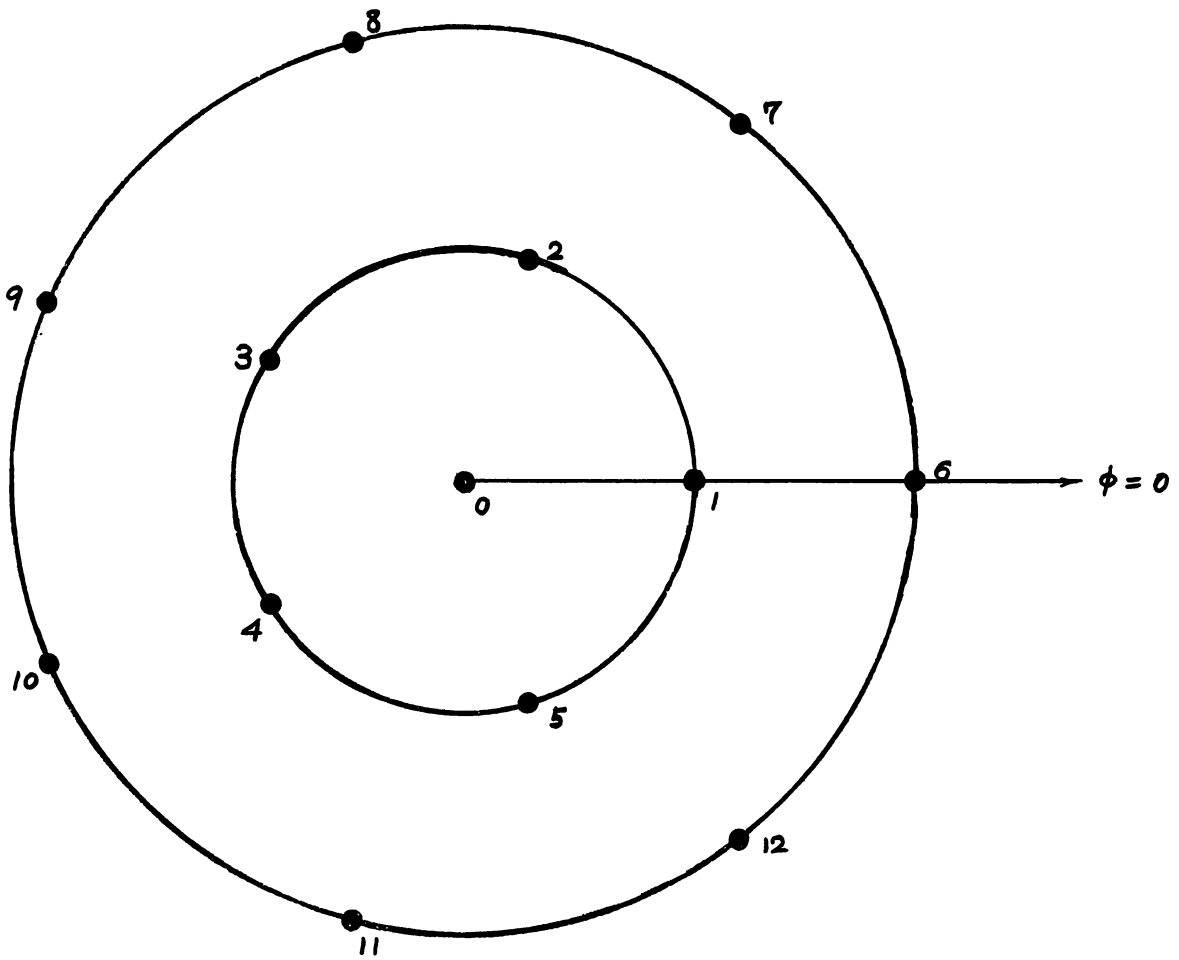


Fig. 5. The Arrangement of the Elements in Appendix B.

$Z_3 = Z_4 = 88.9 \angle 109.4^{\circ}$

$Z_6 = 234 \angle 34.8^{\circ}$

$Z_7 = Z_{12} = 157 \angle 0.2^{\circ}$

$Z_8 = Z_{11} = 130 \angle 2.7^{\circ}$

$Z_9 = Z_{10} = 10.3 \angle -151.5^{\circ}$

All of the pertinent design information is thus available.