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I am submitting herewith a dissertation written by Stanley O. Perrine II entitled "Sets of Uniqueness for Vilenkin Series of Bounded Type." I have examined the final copy of this dissertation for form and content and recommend that it be accepted in partial fulfillment of the requirements for the degree of Doctor of Philosophy, with a major in Mathematics.



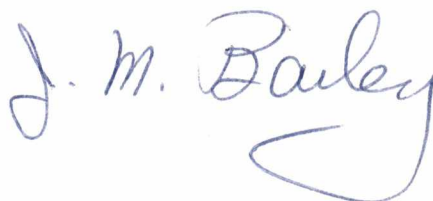
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SETS OF UNIQUENESS FOR VILENKIN
SERIES OF BOUNDED TYPE

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for the
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Stanley O. Perrine II
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DEDICATION

This paper is dedicated to the three greatest influences in my life:

my parents, Warren and Sandy Perrine,

whose guidance and encouragement have brought me to this moment;

and

my wife, Tracy, whose love and support is much greater than I deserve.

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ABSTRACT

This paper explores sets of uniqueness and multiplicity for different collections of Vilenkin series of bounded type. Many of the results are extensions of previously known results from the dyadic setting (Walsh series) to the bounded Vilenkin setting. In all cases, the sequence $\{p_k\}$ is the generating sequence for the Vilenkin group G , and the sequence $\{M_n\}$ is given by $M_0 := 1$ and $M_n := p_1 p_2 \cdots p_n$. Five of the collections of series considered are:

$$\begin{aligned} \mathcal{C} &= \left\{ S \sim \sum a_k \psi_k : \lim_{n \rightarrow \infty} \frac{1}{n} \sum_{k=0}^{n-1} |a_k|^2 = 0 \right\} \\ \mathcal{B} &= \left\{ S \sim \sum a_k \psi_k : \sum_{k=M_n}^{M_{n+1}-1} |a_k|^2 = o(1) \right\} \\ \mathcal{A} &= \left\{ S \sim \sum a_k \psi_k : \lim_{n \rightarrow \infty} \frac{1}{M_n} \sum_{k=0}^{M_n-1} a_k \psi_k(x) = 0 \text{ for all } x \in G \right\} \\ \mathcal{T}_\alpha &= \left\{ S \sim \sum a_k \psi_k : \sum_{k=0}^{\infty} |a_k|^2 [k]^{\alpha-1} < \infty \right\} \text{ for } 0 < \alpha < 1. \end{aligned}$$

where $[k] = M_n$ for all $M_n \leq k < M_{n+1}$,

and

$$\mathcal{T}_\alpha^+ = \{ S \in \mathcal{T}_\alpha : \exists \text{ positive integers } n_0 < n_1 < \cdots \ni S_{n_i}(x) \geq 0 \text{ for a.e. } x \in G \}.$$

The following inclusions are obvious:

$$\mathcal{T}_\alpha^+ \subset \mathcal{T}_\alpha \subset \mathcal{C} \subset \mathcal{A}.$$

Also, the collections $\mathcal{M}_\epsilon$ and $\mathcal{U}_\epsilon^+$ investigated in Chapters 5 and 6 are generalizations of the collections $\mathcal{T}_\alpha$ and $\mathcal{T}_\alpha^+$, respectively.

The main technique used in many of the proofs found herein is fundamental real analysis - measure theory. Because every Vilenkin series has an associated quasi-measure, the group analysis can be performed on the interval $[0,1)$, instead of on the group itself. Also, the orthogonality of the group of Vilenkin characters plays a vital role in these results.

In this dissertation, it is shown that any perfect set of Haar measure zero is a set of multiplicity for $\mathcal{C}$. It is also shown that having α -capacity zero is a necessary and sufficient condition for a set to be a set of uniqueness for the collection $\mathcal{T}_\alpha^+$. Chapters 5 and 6 generalize this last result: one result from Chapter 5 actually shows that the previous theorem holds true for $\mathcal{T}_\alpha$ in place of $\mathcal{T}_\alpha^+$; Chapter 6 introduces a generalized ϵ -capacity, for which it is shown that a set of ϵ -capacity zero is a set of uniqueness for $\mathcal{U}_\epsilon^+$. Finally, it is shown that the small collection $\mathcal{B}$ has a large number of sets of uniqueness, but that there does exist a set of multiplicity for $\mathcal{B}$ which has Haar measure zero.

PREFACE

Harmonic analysis on the dyadic group has been intensely researched for the greater part of the twentieth century. J.L. Walsh's 1923 paper on a complete orthonormal system of functions (later named the *Walsh* system) set the stage for this type of analysis, but the connection with group theory was not yet recognized. Independently, N. Ya. Vilenkin (in 1947) and N. J. Fine (in 1949), realized that the Walsh functions were *characters* for the dyadic group. In his paper, Vilenkin took this idea one (very great) step further, realizing that the dyadic group was a prototype for all infinite, totally disconnected, metrizable, compact abelian groups, and Walsh series were a model for harmonic analysis of these groups. Such groups, now known as a *Vilenkin* groups, are the subject of this dissertation.

Over the past thirty years, one of the most prolific researchers in the area of dyadic harmonic analysis has been William R. Wade. Some of his results have dealt with *sets of uniqueness* for the collection of Walsh series, many times corresponding with the classical theory, and sometimes going completely against it. Beginning in the early 1980's, Kaoru Yoneda began looking at these sets of uniqueness on certain collections of Walsh series having different growth conditions, and also produced a number of results. Some of this work has already been generalized to the Vilenkin group situation (see Grubb), and even further (to certain topological spaces without a group structure). Nevertheless, there are many unsolved problems about uniqueness on Vilenkin groups.

In this dissertation, we shall continue the trend of using work done on the

dyadic group to spark ideas about what might be happening in the more general Vilenkin setting. We will look at a couple of classes of Vilenkin series with differing growth conditions (on the coefficients) and find sets of uniqueness for each one of these classes. It will then be shown that the smallest class of series we examine is not trivially small; i.e. there exists a set of multiplicity for the class which is of (Haar) measure zero. Using some ideas from probability theory to initiate a study of the capacity of sets will lead to results about sets of a certain capacity qualifying as sets of uniqueness. The results not only borrow from probability theory, but also demonstrate techniques from many branches of analysis, including measure-theory and the theory of analysis on (compact) groups. Some of these concepts, especially the idea of quasi-measures (see next chapter) are extremely useful, and should prove applicable to other areas of analysis.

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CHAPTER 1 : BACKGROUND

A *compact abelian group* is an abelian (commutative) group $(G, +)$ together with a topology τ such that (G, τ) is a compact Hausdorff space with continuous group operations (i.e. the maps $(x, y) \rightarrow x + y$ and $x \rightarrow -x$ are continuous maps). A *Vilenkin group* is a compact abelian group which is infinite, totally disconnected, and metrizable. The class of Vilenkin groups which shall be given the attention of this dissertation, the multiplicative Vilenkin groups, can be generated in the following way.

Let $\{p_k\}$ be any (infinite) sequence of positive integers with $p_k \geq 2$ for all k . Recall that the *cyclic group of order $\{p_k\}$* is simply

$$Z_{p_k} = \{0, 1, \dots, p_k - 1\}.$$

The multiplicative Vilenkin group generated by the sequence $\{p_k\}$ is defined to be the direct product over k of the groups Z_{p_k} . The topology on G is given by the product topology, generated by the discrete topologies on each of the Z_{p_k} 's. By Tychonoff's theorem, G is indeed compact. These Vilenkin groups are generalizations of *p-groups* ; i.e. groups which are the direct product of an infinite number of cyclic groups of order p . The most basic of all of these groups is the *Dyadic group*, which is the p -group with $p = 2$.

If the sequence $\{p_k\}$ is bounded as $k \rightarrow \infty$, we call G a *bounded multiplicative Vilenkin group* . Otherwise, the group is said to be of *unbounded type* . It will be the bounded case which will be of main interest here, for these bounded groups act similarly to the p -groups. (and thus, to the dyadic group). Many classical results have analogues in the dyadic case, and hence in the bounded

multiplicative Vilenkin case. The unbounded case, however, can be completely different. Some classical theorems which have been proven in the dyadic, p-adic, and bounded Vilenkin cases have been shown *false* in the unbounded case (see Wade [21]).

From the description above, it is clear that the elements of any multiplicative Vilenkin group G are exactly those sequences $(x_0, x_1, \dots)$ whose coordinates satisfy $0 \leq x_k < p_k$, where x_k are integers. In particular, the dyadic group can be thought of as sequences in which every x_k is either 0 or 1.

Addition on the group G is done coordinate-wise, modulo the appropriate p_k of course, and is represented with the symbol $\dot{+}$. (The symbol $\dot{-}$ shall be used to represent 'subtraction', i.e. addition with the additive inverse.) Thus, the sum of two elements $x, y \in G$ is given by

$$x \dot{+} y = ((x_0 + y_0) \bmod p_0, (x_1 + y_1) \bmod p_1, \dots).$$

Clearly, under this definition, the additive identity for the group is $0 = (0, 0, \dots)$. Let e_k represent the element of G having all zeroes as coordinates, except for a 1 in the k^{th} position. It is then possible to write any element $x \in G$ as an infinite sum:

$$x = \sum_{k=0}^{\infty} x_k e_k.$$

Intervals for the group are defined as

$$I_n(x) = \{y \in G : x_k = y_k \text{ for all } k = 0, 1, \dots, n-1\}.$$

These intervals form a base for the product topology G and are sets which are both open and closed in G .

Recall that when dealing with any group, it is also appropriate to talk about its *dual group*, $\widehat{G}$. The elements of $\widehat{G}$, called *characters* of G , are exactly the

complex-valued continuous homomorphisms ψ from G to the circle group $\mathbf{T}$. In other words $|\psi(x)| = 1$ for every element $x \in G$ and, for every $x, y \in G$,

$$\psi(x + y) = \psi(x)\psi(y),$$

with multiplication defined on the circle group $\mathbf{T}$.

From the general theory of locally compact abelian groups (Rudin 1967), there are a couple of important facts that should be noted about the dual group $\widehat{G}$ and its characters ψ . First, since G is compact and metrizable, it is known that $\widehat{G}$ is countable (ibid p. 38). Also, G admits a unique (up to constant multiples) *Haar measure* (ibid. p. 1), and since G is compact, it is customary to define *the* Haar measure on G as the only Haar measure μ such that $\mu(G) = 1$. (Recall: a Haar measure is a translation invariant σ -finite Borel measure.) Finally, the set of characters of G are such that, for any $\psi, \phi \in \widehat{G}$

$$\int_G \psi(x)\phi(x)d\mu(x) = \begin{cases} 1 & ; \text{ if } \psi = \phi \\ 0 & ; \text{ otherwise} \end{cases}$$

In other words, the characters of G form an orthogonal set in $\widehat{G}$ (ibid. p. 10).

The fact that the character group of G is countable leads to the question of an ordering on $\widehat{G}$. Vilenkin (1947) came up with a usable ordering of the characters which has allowed much of this analysis to take place. His ordering is analogous to Paley's ordering of the Walsh system on the dyadic group (Wade [21]). Using a similar approach of defining Rademacher functions on the generators of G (Schipf, Wade, Simon p.1). Vilenkin used the well-known fact that any positive integer n has a unique decomposition of the form

$$n = \sum_{k=0}^{\infty} n_k(M_k)^{-1}$$

where $M_0 := 1$ and $M_k := p_1 p_2 \cdots p_k$ for each positive integer k , and where each n_k is an integer satisfying $0 \leq n_k < p_k$. Using this representation for n , Vilenkin

defined the characters ψ_n from G to $\mathbf{T}$ by

$$(1) \quad \psi_n(x) := \prod_{k=0}^{\infty} \exp\left(\frac{2\pi i x_k n_k}{p_k}\right)$$

where x_k is the k^{th} -coordinate of $x \in G$.

Is it difficult to do much analysis on this abstract group structure? Well, it just so happens that any metrizable compact abelian group (and hence, any Vilenkin group) can be identified with the interval $[0, 1)$ (Freudenthal 1938). Thus, the study of Vilenkin series on G is closely related to the study of orthonormal systems on $[0, 1)$, and thus the Lebesgue integral can be used to perform calculations, instead of the Haar integral. This identification between G and $[0, 1)$ is a natural one. The map to use is a generalization of *Fine's map* (Fine 1949), which he originally used to map the dyadic group to the unit interval.

Let $x = (x_0, x_1, \dots)$ be an element of G , the Vilenkin group generated by the series $\{p_k\}$. Define the map $\varphi : G \rightarrow [0, 1)$ by

$$\varphi(x) = \sum_{k=0}^{\infty} \frac{x_k}{M_{k+1}}.$$

This map is continuous and onto, and 1-1, except at the M_k -adic rational numbers. These numbers have more than one possible expansion: one ending in zeroes and (at least) one other ending with some positive integer repeating. Restrict the map φ to act on G minus the set of M_k -adic rationals which do not end in zeroes; φ is now 1-1. Thus, the inverse of φ also exists, and is onto the group G , minus the aforementioned set. With this slight perturbation, therefore, φ gives an easy way to 'jump back and forth' between the group and the interval while preserving many desired properties.

Given any function $f \in L^1(G)$, define the *Vilenkin-Fourier (or V-F) coefficients* of f as

$$\widehat{f}(k) := \int_G f(x) \overline{\psi_k(x)} d\mu(x) \text{ for all } k \in \mathcal{N}$$

where ψ_k is defined by the Vilenkin ordering found in (1) and μ is the Haar measure on G . Likewise, define the *V-F series* of f as

$$Sf(x) := \sum_{k=0}^{\infty} \widehat{f}(k) \psi_k(x) \text{ for all } x \in G.$$

All the questions previously asked about classical Fourier series can be asked of their Vilenkin counterparts. The answers match up, sometimes, but many times there are significant differences.

The main question that will be addressed in the following chapters concerns the uniqueness of a Vilenkin series; i.e. if a series $S(x)$ converges to an integrable function f , is S the V-F series of f ? The form of this question dealt with in this dissertation is, if a series S converges (in an "appropriate" way) to an $f \in L^1(G)$ off some small set, is S the V-F series of f ? Specifically, the question is: "which sets satisfy the following condition: given a collection $\mathcal{A}$ of Vilenkin series, a set $E \subset G$ is called a *set of uniqueness* for $\mathcal{A}$ if the only series $S \in \mathcal{A}$ whose M_n^{th} -partial sums converge pointwise to zero off of E is the zero series? "

In other words, E is a set of uniqueness for $\mathcal{A}$ if whenever

$$\lim_{n \rightarrow \infty} S_{M_n}(x) = 0 \text{ for all } x \notin E$$

and $S \in \mathcal{A}$, then $S(x) \equiv 0$. A set E which is not a set of uniqueness is called a *set of multiplicity*.

Clearly, the answer to this question depends on the collection $\mathcal{A}$. If the collection $\mathcal{A}$ is the entire collection of all Vilenkin series, then E is called a *U-set* (correspondingly, an *M-set*), corresponding with the definition used in the classical theory. (i.e. harmonic analysis on the real line.)

Note 1: Since $\sum_{k=0}^{M_n-1} \psi_k(x) \rightarrow 0$ as $n \rightarrow \infty$ for all $x \in G$ except $x = 0$, but $D(x) := \sum_{k=0}^{\infty} \psi_k(x)$ is not the zero series, it is clear that the only U-set for the entire collection of Vilenkin series is the empty set. Thus, the rest of this dissertation will deal with smaller collections of Vilenkin series, and will establish criteria for a set to be a set of uniqueness (or multiplicity) for the given collection of series. It will be shown that if the collection is small enough, there are plenty of uncountable sets of uniqueness.

Note 2: Clearly, this definition for set of uniqueness is sufficient to answer the question of uniqueness posed in the previous paragraph. Suppose E is a U-set. Let $S(x)$ be a Vilenkin series, and suppose that $S(x)$ converges to an integrable function f off the set E . Then, the series $S(x) - Sf(x)$ converges to zero off the set E , where $Sf(x)$ represents the Vilenkin-Fourier series of the function f . Since E is a U-set, the series $S(x) - Sf(x)$ is the zero series, implying that $S(x)$ is the Vilenkin-Fourier series of f .

This chapter concludes with a list of definitions and notation used in the rest of the dissertation. Represent an M_n -adic interval in the group G by $I(j, M_n)$, where

$$x \in I(j, M_n) \text{ if and only if } \sum_{k=0}^{n-1} \frac{x_k}{M_{k+1}} = \frac{j}{M_n},$$

where $j = 0, 1, \dots, M_n - 1$. The M_n -adic interval containing x will be denoted by $I_{M_n}(x)$. By convention, $I(0, 0) \equiv G$.

One of the techniques used throughout this paper will be to look at Vilenkin series in a different light - particularly, a measure-theoretic one. To this end, define a function m_S , where S is any Vilenkin series $S(x) = \sum_{k=0}^{\infty} a_k \psi_k(x)$, to be the real-valued function defined on the M_n -adic intervals as follows:

$$(2) \quad m_S(I) = \lim_{N \rightarrow \infty} \int_I \sum_{k=0}^N a_k \psi_k(x) dx,$$

where the dx , of course, represents the Haar measure on G .

It is clear that the limit in (2) exists and is finite because the sum is finite. Indeed, for large enough values of k , the characters ψ_k alternate values when evaluated over a particular interval $I \subset G$ (see Simon, Wade, Schipp p. 1-2). Since this alternation is sequential through the appropriate roots of unity, and the sum of all the roots of unity for any particular radix is zero, it is clear that, for any interval $I \subset G$,

$$\int_I \psi_k(x) dx = 0$$

for large enough values of k . In other words, for any $x \in G$, there exists an $N \in \mathcal{N}$ (depending on x and n) large enough so that $\int_{I_{M_n}(x)} \psi_k(x) dx = 0$, for all $k > N$. Thus, by switching the sum and the integration in the above definition (since the sum is finite for any x), it is easy to see that for all $x \in G$,

$$m_S(I_{M_n}(x)) = \frac{1}{M_n} \sum_{k=0}^{M_n-1} a_k \psi_k(x).$$

The formula above clearly shows that the function m_S is finitely-subadditive and will henceforth be referred to as a *quasi-measure*.

It is important to realize that the map $S \mapsto m_S$ gives a 1-1 correspondence between the entire collection of Vilenkin series and the set of all quasi-measures on G . This fact will be proven in Proposition 2.4. This proposition will also show that the coefficients of any Vilenkin series S can be defined by integration of the Vilenkin characters against the appropriate quasi-measure. In other words,

$$a_k = \int_G \psi_k(x) dm_S(x),$$

where $dm_S(x)$ is defined by the formula $\int_I dm_S(x) = m_S(I)$. (Recall that each ψ_k is constant on the M_k -adic intervals.) Conversely, given any quasi-measure m on G , the numbers

$$b_k := \int_G \psi_k(x) dm(x)$$

define a Vilenkin series T , where

$$T(x) = \sum_{k=0}^{\infty} b_k \psi_k(x).$$

This series is such that $m = m_T$. This one to one correspondence shows that the study of quasi-measures is equivalent to the study of Vilenkin series. This approach will be used throughout this dissertation.

There are other technicalities and definitions which will be used in one or more of the remaining sections of this paper. However, they will be introduced as needed, so as to proceed with the results. The next chapter contains preliminary results needed for the other chapters, each of which contains generalizations of results by one of the aforementioned authors.

CHAPTER 2 : PRELIMINARIES

Before proceeding with the uniqueness results, it is necessary to extend earlier work done by Fine (1957) and other technical results known in the Dyadic case to the bounded Vilenkin case. Recall that a character series $S(x) = \sum_{k=0}^{\infty} a_k \psi_k(x)$ is called a *Stieltjes series* on G if there exists a measure m such that, for all k

$$a_k = \int_G \overline{\psi_k} dm.$$

The result of Fine's being extended will show a precise way to tell whether or not a series $S(x) = \sum_{k=0}^{\infty} a_k \psi_k(x)$ has a desired growth condition on the coefficients, namely $\frac{1}{n} \sum_{k=0}^{n-1} a_k^2 \rightarrow 0$ as $n \rightarrow \infty$. Recall that a measure m is called *non-atomic* if $m(\{x\}) = 0$ for any singleton point x .

Definition. Given a Vilenkin series $S(x) = \sum_{k=0}^{\infty} a_k \psi_k(x)$, define the *Cesàro means* of S as

$$\sigma_n(x) = \sum_{k=0}^{n-1} \left(1 - \frac{k}{n}\right) a_k \psi_k(x).$$

Proposition 2.1. If $S(x) = \sum_{k=0}^{\infty} a_k \psi_k(x)$, then $S(x)$ is a Vilenkin-Fourier-Stieltjes series for some non-atomic measure m if and only if:

$$(i) \quad \int_G |\sigma_n| dx = O(1) \quad \text{as } n \rightarrow \infty$$

and

$$(ii) \quad \frac{1}{n} \sum_{k=0}^{n-1} a_k^2 \rightarrow 0.$$

A couple of technical lemmas will be needed in order to prove this, the first of which gives a necessary and sufficient condition on the Cèsaro means of a series which guarantees that the series is a Vilenkin-Fourier-Stieltjes series. The second lemma shows that the limit of the averages of the n^{th} partial sums of a Vilenkin-Fourier-Stieltjes series $S(x)$ are equal to the value of the associated measure m evaluated at the point x .

Lemma 2.2. *A necessary and sufficient condition for $S(x) = \sum_{k=0}^{\infty} a_k \psi_k(x)$ to be a Vilenkin-Fourier-Stieltjes series is*

$$\int_G |\sigma_n| dx = O(1) \quad \text{as } n \rightarrow \infty.$$

Proof of Lemma 2.2. Necessity. If $S(x) = \sum_{k=0}^{\infty} a_k \psi_k(x)$ is a Vilenkin-Fourier-Stieltjes series, then there exists a Borel measure m (finite, since G is compact) such that $a_k = \int_G \overline{\psi_k(t)} dm(t)$ for all k . Thus, by the character properties of ψ_k ,

$$\begin{aligned} \sigma_n(x) &= \sum_{k=0}^{n-1} \left(1 - \frac{k}{n}\right) a_k \psi_k(x) \\ &= \int_G \sum_{k=0}^{n-1} \left(1 - \frac{k}{n}\right) \psi_k(x) \overline{\psi_k(t)} dm(t) \\ &= \int_G \sum_{k=0}^{n-1} \left(1 - \frac{k}{n}\right) \psi_k(x \dot{-} t) dm(t) \end{aligned}$$

where $\dot{-}$ is subtraction on the group G . Using Fubini's Theorem, it is clear that

$$(1) \quad \int_G |\sigma_n(x)| dx \leq \int_G \int_G \left| \sum_{k=0}^{n-1} \left(1 - \frac{k}{n}\right) \psi_k(x \dot{-} t) \right| dx |dm(t)|.$$

Thus, it remains only to show that the double integral in (1) is bounded by a constant to show necessity.

Recall that the Haar measure on G is a translation invariant measure, and thus $\psi_k(x-t)$ can be rewritten as $\psi_k(x)$, using a change of variables, without changing the dx . Hence, the integrand in the inside integral is nothing more than the (Vilenkin-) Fejér kernel.

$$\int_G |\sigma_n(x)| dx \leq \int_G \int_G |K_n(x)| dx |dm(t)|$$

However, by a result of Yano's (generalized by Pal and Simon to bounded Vilenkin systems), the L^1 -norm of $K_n(x) = 1$. Hence, by the fact that G is compact,

$$\int_G \int_G |K_n(x)| dx |dm(t)| = |m(G)| \equiv C < \infty$$

and the result is obvious.

Sufficiency. Assume that $\int_G |\sigma_n| = O(1)$ as $n \rightarrow \infty$ holds. Let $\mathcal{C}(G)$ be the space of all continuous functions on G with supremum norm. Define functionals Λ_n on $\mathcal{C}(G)$ by

$$(\Lambda_n f)(x) = \int_G \sigma_n(x) f(x) dx.$$

Clearly, for each n , Λ_n is a linear functional. By the condition on $\int_G |\sigma_n|$, the sequence $\{\Lambda_n\}$ is also uniformly norm bounded. Thus, since $\mathcal{C}(G)$ is separable for any bounded Vilenkin group G , Thm 11.29 of Rudin (Real and Complex Analysis) shows that there exists a subsequence of functionals $\{\Lambda_{n_j}\}$ such that

$$\Lambda_{n_j}(f) \rightarrow \Lambda(f) \quad \text{for all } f \in \mathcal{C}(G),$$

where Λ is linear and bounded in norm by $\sup_n \|\Lambda_n\|$. Thus, the Riesz Representation Theorem guarantees the existence of a unique finite regular Borel measure m on G such that $\Lambda(f) = \int_G f dm$ for every $f \in \mathcal{C}(G)$. Hence, it is easy to see that as $j \rightarrow \infty$,

$$\int_G \sigma_{n_j}(x) f(x) dx \longrightarrow \int_G f(x) dm(x).$$

Substituting $f = \overline{\psi_k}$ into the formula above yields

$$\begin{aligned}
\int_G \overline{\psi_k} dm &= \lim_{j \rightarrow \infty} \int_G \sigma_{n_j}(x) \overline{\psi_k(x)} dx \\
&= \lim_{j \rightarrow \infty} \int_G \sum_{\ell=0}^{n_j} \left(1 - \frac{\ell}{n_j}\right) a_k \psi_\ell(x) \overline{\psi_k(x)} dx \\
&= \lim_{j \rightarrow \infty} \sum_{\ell=0}^{n_j} \left(1 - \frac{\ell}{n_j}\right) a_k \int_G \psi_\ell(x) \overline{\psi_k(x)} dx \\
&\text{(by orthogonality)} = \lim_{j \rightarrow \infty} \left(1 - \frac{k}{n_j}\right) a_k \\
&= a_k.
\end{aligned}$$

Hence, $S(x) = \sum_{k=0}^{\infty} a_k \psi_k(x)$ is a Vilenkin-Fourier-Stieltjes series. $\square$ *Lemma 1.*

Lemma 2.3. *If $S(x) = \sum_{k=0}^{\infty} a_k \psi_k(x)$ is a Vilenkin-Fourier-Stieltjes series with associated measure m , then for every $x \in G$*

$$\lim_{n \rightarrow \infty} \frac{1}{n} \sum_{k=0}^{n-1} a_k \psi_k(x) = m(\{x\}).$$

Proof of Lemma 2.3. Because $S(x)$ is a Vilenkin-Fourier-Stieltjes series, it is clear that

$$\begin{aligned}
\sum_{k=0}^{n-1} a_k \psi_k(x) &= \sum_{k=0}^{n-1} \int_G \overline{\psi_k(t)} \psi_k(x) dm(t) \\
&= \int_G \sum_{k=0}^{n-1} \psi_k(x) \overline{\psi_k(t)} dm(t).
\end{aligned}$$

To get a handle on the partial sums, it is necessary only to look at the integral of the Vilenkin-Dirichlet kernel

$$D_n(x-t) = \sum_{k=0}^{n-1} \psi_k(x) \overline{\psi_k(t)}.$$

First, examine the case $x = t$. By the character properties of ψ_k , it is clear that

$$\frac{1}{n} D_n(x \dot{-} x) = \frac{1}{n} \sum_{k=0}^{n-1} |\psi_k(x)|^2 = 1.$$

If $x \neq t$, the only fact needed is that the Haar measure dx is translation-invariant.

$$\begin{aligned} \int_G D_n(x \dot{-} t) dx &= \sum_{k=0}^{n-1} \int_G \psi_k(x) \overline{\psi_k(t)} dx \\ &= \sum_{k=0}^{n-1} \int_G \psi_k(x \dot{-} t) dx \\ &= \sum_{k=0}^{n-1} \int_G \psi_k(x) dx \\ &= 1. \end{aligned}$$

since $\int_G \psi_k(x) dx = 0$ for all $k \neq 0$.

Because $(\frac{1}{n})D_n(x \dot{-} t)$ is bounded by 1 for all n , and equals 1 when $t = x$, the Lebesgue Dominated Convergence Theorem shows

$$\begin{aligned} \lim_{n \rightarrow \infty} \frac{S_n(x)}{n} &= \int_G \lim_{n \rightarrow \infty} \frac{D_n(x \dot{-} t)}{n} dm(t) \\ &= \int_{\{x\}} 1 dm(t) \\ &= m(\{x\}). \quad \square \text{ Lemma 2.} \end{aligned}$$

It is now possible to proceed with the proof of the extension of Fine's result dealing with the coefficients of a Vilenkin-Fourier-Stieltjes series associated with a non-atomic measure.

Proof of Proposition 2.1. By Lemma 2.2, condition (i) of the proposition is equivalent to S being a Stieltjes series of some measure m . Thus, for each k , it is clear that $|a_k| \leq \int_G |\psi_k| |dm| = |m|(G) \equiv C < \infty$ implies that $\left| \frac{S_n(x)}{n} \right| \leq \tilde{C}$

also. Hence, applying the Lebesgue Dominated Convergence Theorem along with Lemma 2.3 shows

$$\begin{aligned}\lim_{n \rightarrow \infty} \int_G \frac{S_n(x)}{n} dm(x) &= \int_G m(\{x\}) dm(x) \\ &= \sum_{x \in G} m^2(\{x\}).\end{aligned}$$

Notice that because $m(G)$ is finite and G is infinite (and totally disconnected), the above sum has at most countable-many non-zero terms. It remains only to establish a relationship between the above sum and condition (ii) to complete the proof.

Claim. $\int_G \frac{S_n(x)}{n} dm(x) = \lim_{N \rightarrow \infty} \frac{1}{n} \int_G S_n(x) \sigma_N(x) dx$ for each $n \in \mathcal{N}$.

Proof of Claim. First, Lemma 2.2 implies that S is a Vilenkin-Fourier-Stieltjes series if and only if the Cesàro sums are uniformly bounded in L^1 -norm. Hence, by using the definition of the σ_N 's,

$$\begin{aligned}\lim_{N \rightarrow \infty} \frac{1}{n} \int_G S_n(x) \sigma_N(x) dx &= \lim_{N \rightarrow \infty} \int_G \frac{S_n(x)}{n} \sum_{k=0}^{N-1} \left(1 - \frac{k}{N}\right) a_k \psi_k(x) dx \\ (\text{by } S \text{ a V-F-Stieltjes series}) &= \lim_{N \rightarrow \infty} \int_G \frac{S_n(x)}{n} \sum_{k=0}^{N-1} \left(1 - \frac{k}{N}\right) \int_G \overline{\psi_k(t)} dm(t) \psi_k(x) dx \\ (\text{by Fubini}) &= \lim_{N \rightarrow \infty} \iint_G \frac{S_n(x)}{n} \sum_{k=0}^{N-1} \left(1 - \frac{k}{N}\right) \psi_k(x-t) dx dm(t) \\ (\text{by translation-invariance of } dx) &= \lim_{N \rightarrow \infty} \iint_G \frac{S_n(x)}{n} \sum_{k=0}^{N-1} \left(1 - \frac{k}{N}\right) \psi_k(x) dx dm(t) \\ (2) \quad &= \lim_{N \rightarrow \infty} \frac{1}{n} \iint_G \sum_{\ell=0}^{n-1} a_\ell \psi_\ell(x) \sum_{k=0}^{N-1} \left(1 - \frac{k}{N}\right) \psi_k(x) dx dm(t).\end{aligned}$$

Now, because N is going to infinity, for any positive integer n , the integer N will eventually be larger than n . Thus, the sum over k above can be broken (for $N > n$), with (2) becoming

$$(3) \quad \lim_{N \rightarrow \infty} \frac{1}{n} \int_G \sum_{\ell=0}^{n-1} \left\{ \sum_{k=0}^{n-1} + \sum_{k=n}^{N-1} \right\} \left(1 - \frac{k}{N}\right) a_\ell \left(\int_G \psi_\ell(x) \psi_k(x) dx \right) dm(t)$$

Further examining the second piece of the sum over k in (3) yields

$$\sum_{k=n}^{N-1} \left(1 - \frac{k}{N}\right) \int_G \psi_k(x) dx = 0,$$

because $\int_G \psi_k(x) dx = 0$ for all $k \neq 0$. Thus, the final equation in (3) reduces to

$$\lim_{N \rightarrow \infty} \frac{1}{n} \int_G S_n(x) \sigma_N(x) dx = \lim_{N \rightarrow \infty} \frac{1}{n} \int_G \sum_{\ell=0}^{n-1} \sum_{k=0}^{n-1} \left(1 - \frac{k}{N}\right) a_\ell \left(\int_G \psi_\ell(x) \psi_k(x) dx \right) dm(t).$$

Use the orthogonality of the characters ψ to get

$$\begin{aligned} \lim_{N \rightarrow \infty} \frac{1}{n} \int_G S_n(x) \sigma_N(x) dx &= \lim_{N \rightarrow \infty} \frac{1}{n} \int_G \sum_{k=0}^{n-1} \left(1 - \frac{k}{N}\right) a_k \psi_k(x) dm(t) \\ &= \lim_{N \rightarrow \infty} \int_G \frac{S_n(x)}{n} \left(1 - \frac{k}{N}\right) dm(t) \\ &= \int_G \frac{S_n(x)}{n} dm(x) \quad \square \textit{ Claim.} \end{aligned}$$

Use the claim, the definition of the Cèsaro means (the σ_n 's), Fubini, and the fact that S is a Vilenkin-Fourier-Stieltjes series to rearrange the n^{th} partial sums.

$$\begin{aligned} \int_G \frac{S_n(x)}{n} dm(x) &= \lim_{N \rightarrow \infty} \frac{1}{n} \int_G S_n(x) \sigma_N(x) dx \\ &= \lim_{N \rightarrow \infty} \frac{1}{n} \sum_{j=0}^{N-1} \left(1 - \frac{j}{N}\right) a_j \int_G S_n(x) \psi_j(x) dx \\ &= \lim_{N \rightarrow \infty} \frac{1}{n} \sum_{j=0}^{N-1} \left(1 - \frac{j}{N}\right) \int_G \overline{\psi_j(t)} dm(t) \int_G S_n(x) \psi_j(x) dx \\ &= \lim_{N \rightarrow \infty} \frac{1}{n} \sum_{j=0}^{N-1} \left(1 - \frac{j}{N}\right) \iint_G S_n(x) \overline{\psi_j(t-x)} dx dm(t). \end{aligned}$$

Haar measure's translation invariance once again allows for a change of variables.

$$\lim_{N \rightarrow \infty} \frac{1}{n} \sum_{j=0}^{N-1} \left(1 - \frac{j}{N}\right) \iint_G S_n(x) \overline{\psi_j(t-x)} dx dm(t)$$

$$= \lim_{N \rightarrow \infty} \frac{1}{n} \sum_{j=0}^{N-1} \left(1 - \frac{j}{N}\right) \iint_G S_n(x) \overline{\psi_j(x)} dx dm(t).$$

Do the same type of manipulation as before to account for the $S_n(x)$.

$$\begin{aligned} & \lim_{N \rightarrow \infty} \frac{1}{n} \sum_{j=0}^{N-1} \left(1 - \frac{j}{N}\right) \iint_G S_n(x) \overline{\psi_j(x)} dx dm(t) \\ &= \lim_{N \rightarrow \infty} \frac{1}{n} \sum_{j=0}^{N-1} \left(1 - \frac{j}{N}\right) \iint_G \sum_{k=0}^{n-1} a_k \psi_k(x) \overline{\psi_j(x)} dx dm(t) \\ &= \lim_{N \rightarrow \infty} \frac{1}{n} \sum_{k=0}^{n-1} \sum_{j=0}^{N-1} \left(1 - \frac{j}{N}\right) \iiint_G \overline{\psi_k(t)} dm(t) \psi_k(x) \overline{\psi_j(x)} dx dm(t). \end{aligned}$$

Use Fubini once again to get

$$= \lim_{N \rightarrow \infty} \frac{1}{n} \sum_{k=0}^{n-1} \sum_{j=0}^{N-1} \left(1 - \frac{j}{N}\right) \iiint_G \psi_k(x) \overline{\psi_j(x)} dx \overline{\psi_k(t)} dm(t) dm(t).$$

By orthogonality, this last expression reduces to

$$\begin{aligned} &= \lim_{N \rightarrow \infty} \frac{1}{n} \sum_{k=0}^{n-1} \left(1 - \frac{k}{N}\right) \int_G \left(\int_G \overline{\psi_k(t)} dm(t) \right) \overline{\psi_k(t)} dm(t) \\ &= \lim_{N \rightarrow \infty} \frac{1}{n} \sum_{k=0}^{n-1} \left(1 - \frac{k}{N}\right) a_k^2. \end{aligned}$$

(Recall that $m(G) = 1$ for any $m \in \mathcal{M}(G)$).

Clearly, as $N \rightarrow \infty$, this last sum converges to $\frac{1}{n} \sum_{k=0}^{n-1} a_k^2$. Therefore, the first statements along with this latest computation give the following relationship.

$$\frac{1}{n} \sum_{k=0}^{n-1} a_k^2 = \sum_{x \in G} m^2(\{x\})$$

It is clear from this relationship that m is a non-atomic measure if and only if $\frac{1}{n} \sum_{k=0}^{n-1} a_k^2 \rightarrow 0$ as $n \rightarrow \infty$. The proposition is proven. $\square$

A second result which will be used throughout this dissertation is one that shows a relationship between the collection of all Vilenkin series and the quasi-measures, introduced in Chapter 1. As it turns out, there is a one to one

correspondence between the Vilenkin series and these quasi-measures. Let ℓ^0 represent the space of all infinite sequences. (i.e. $\bar{a} \in \ell^0$ is $\bar{a} = (a_0, a_1, \dots)$, where a_k is a number, in this case, corresponding to a coefficient of a Vilenkin series.)

Proposition 2.4. *Let $\mathbf{QM}(G)$ represent the set of all quasi-measures on the group G . For $\nu \in \mathbf{QM}(G)$, define the series $S\nu$ as*

$$S\nu(x) = \sum_{k=0}^{\infty} \widehat{\nu(k)} \psi_k(x),$$

where $\widehat{\nu(k)} = \int_G \psi_k(x) d\nu(x)$. Then, the map from $\mathbf{QM}(G)$ to ℓ^0 given by

$$(*) \quad \nu \longmapsto S\nu$$

is one to one, onto, and linear: i.e. given any Vilenkin series S , there exists a $m_S \in \mathbf{QM}(G)$ such that

$$\widehat{m_S(k)} = a_k \text{ and } S_{M_n}(x) = M_n m_S(I_{M_n}(x)).$$

Proof. Let $S_n\nu$ denote the n^{th} -partial sum of the series $S\nu$. Recall that the n^{th} -order Vilenkin-Dirichlet kernel of a series S is defined to be

$$D_n(x) := \sum_{k=0}^{n-1} \psi_k(x).$$

Therefore, for any positive integer n ,

$$\begin{aligned} S_n\nu(x) &= \sum_{k=0}^{n-1} \widehat{\nu(k)} \psi_k(x) \\ &= \int_G \sum_{k=0}^{n-1} \psi_k(x) \overline{\psi_k(t)} d\nu(t) \\ &= \int_G D_n(x \cdot t) d\nu(t). \end{aligned}$$

The Paley Lemma, originally done for series on the Dyadic group (Wade, Schipp, and Simon p.7) and extended to Vilenkin groups by Vilenkin (1947), states

$$D_{M_n}(x) = \begin{cases} M_n & ; \text{ if } x \in I_{M_n}(0) \\ 0 & ; \text{ otherwise.} \end{cases}$$

Thus, the above calculations together with the extension of the Paley Lemma show

$$S_{M_n} \nu(x) = M_n \nu(I_{M_n}(x))$$

for every $x \in G$, implying $(*)$ is 1-1 and linear.

To show that $(*)$ is onto, let (a_n) be any sequence of numbers. Define the series S on G by

$$S(x) = \sum_{k=0}^{\infty} a_k \psi_k(x).$$

The M_n^{th} partial sums of the Vilenkin series S are the Vilenkin polynomials $S_{M_n}(x) = \sum_{k=0}^{M_n-1} a_k \psi_k(x)$. Define a set function ν on the algebra generated by the M_n -adic intervals of G by

$$\nu(I) := \lim_{n \rightarrow \infty} \int_I S_{M_n}(x) dx.$$

This limit exists because $\int_I \psi_k(x) dx = 0$ for all $k > M_{n_0}$, where M_{n_0} is given by

$$|I| = \frac{1}{M_{n_0}}.$$

Thus, if M_{n_0} is as above, it is clear that

$$\nu(I) = \int_I S_{M_{n_0}}(x) dx,$$

implying that $\widehat{\nu(k)} = a_k$ for all k , and $\nu \in \mathbf{QM}(G)$. □

The final of these preliminary results is a fundamental lemma which is needed for many of the theorems found in the remaining chapters. This lemma shows

a relationship between the coefficients of a Vilenkin series and its related quasi-measure. This lemma is a very technical result, and requires a good deal of computation. However, the real machinery behind the proof is simply the orthogonality of the characters of G and the simple fact that the sum of the n^{th} roots of unity is zero for any positive integer $n \geq 2$. A definition is required before stating the lemma.

Definition 2.5. For any quasi-measure m on G , define the unitary difference of m for an interval $I(j, M_n)$ as

$$\Delta m(I(j, M_n)) := \sum_{\alpha=0}^{p_{n+1}-1} \epsilon^{\frac{2\pi i \alpha}{p_{n+1}}} m(I(jp_{n+1} + \alpha, M_{n+1})).$$

Lemma 2.6. Let $S(x) = \sum_{k=0}^{\infty} a_k \psi_k(x)$ be a Vilenkin series, and let m_S denote the quasi-measure associated with S . Then,

$$(4) \quad \sum_{k=M_n}^{M_{n+1}-1} |a_k|^2 = M_n \sum_{j=0}^{M_n-1} |\Delta m_S(I(j, M_n))|^2$$

for all positive integers n .

Proof of 2.6. The following calculation uses simple finite sum manipulations along with the homomorphism properties of the ψ_k 's.

$$\begin{aligned} & M_n \sum_{j=0}^{M_n-1} |\Delta m_S(I(j, n))|^2 \\ &= M_n \sum_{j=0}^{M_n-1} \left| \sum_{\alpha=0}^{p_{n+1}-1} \exp\left(\frac{2\pi i \alpha}{p_{n+1}}\right) \frac{1}{M_{n+1}} \sum_{k=0}^{M_{n+1}-1} a_k \psi_k\left(\frac{jp_{n+1} + \alpha}{M_{n+1}}\right) \right|^2 \\ &= \frac{1}{M_{n+1}p_{n+1}} \sum_{j=0}^{M_n-1} \left| \sum_{k=0}^{M_{n+1}-1} a_k \psi_k\left(\frac{jp_{n+1}}{M_{n+1}}\right) \sum_{\alpha=0}^{p_{n+1}-1} \exp\left(\frac{2\pi i \alpha}{p_{n+1}}\right) \psi_k\left(\frac{\alpha}{M_{n+1}}\right) \right|^2 \\ (5) \quad &= \frac{1}{M_{n+1}p_{n+1}} \sum_{j=0}^{M_n-1} \left| \sum_{k=M_n}^{M_{n+1}-1} a_k \psi_k\left(\frac{jp_{n+1}}{M_{n+1}}\right) \sum_{\alpha=0}^{p_{n+1}-1} \exp\left(\frac{2\pi i \alpha}{p_{n+1}}\right) \psi_k\left(\frac{\alpha}{M_{n+1}}\right) \right|^2. \end{aligned}$$

The last equality holds because of the following: each ψ_k is piece-wise constant on the appropriate intervals of G , and all $\psi_k = 1$ on the first appropriate interval. Specifically, $\psi_n \equiv 1$ on $I_{M_n}(0)$. Thus, for $k < M_n$ and $\alpha = 0, 1, \dots, p_{n+1} - 1$, it holds that

$$\psi_k \left(\frac{\alpha}{M_{n+1}} \right) \equiv 1.$$

Thus, since the sum of the p_{n+1}^{th} roots of unity always equals zero, the sum in (5) clearly holds.

Since $\psi_k \left(\frac{\alpha}{M_{n+1}} \right) = \exp \left(\frac{-2\pi i \alpha}{M_{n+1}} \right)$ for all $M_n \leq k < M_{n+1}$ and for every $\alpha = 0, 1, \dots, p_{n+1} - 1$, (5) reduces even further.

$$\begin{aligned} (5) &= \frac{1}{M_n} \sum_{j=0}^{M_n-1} \left| \sum_{k=M_n}^{M_{n+1}} a_k \psi_k \left(\frac{j p_{n+1}}{M_{n+1}} \right) \right|^2 \\ &= \frac{1}{M_n} \sum_{j=0}^{M_n-1} \left| \sum_{k=M_n}^{M_{n+1}} a_k \psi_k \left(\frac{j}{M_n} \right) \right|^2 \\ &= \frac{1}{M_n} \sum_{j=0}^{M_n-1} \sum_{k=M_n}^{M_{n+1}-1} \sum_{\ell=M_n}^{M_{n+1}-1} a_k \overline{a_\ell} \psi_k \left(\frac{j}{M_n} \right) \overline{\psi_\ell \left(\frac{j}{M_n} \right)} \\ (6) \quad &= \frac{1}{M_n} \sum_{k=M_n}^{M_{n+1}-1} \sum_{\ell=M_n}^{M_{n+1}-1} a_k \overline{a_\ell} \sum_{j=0}^{M_n-1} \psi_k \left(\frac{j}{M_n} \right) \overline{\psi_\ell \left(\frac{j}{M_n} \right)}. \end{aligned}$$

This last sum looks closely related to the orthogonality condition of the ψ_k 's, namely

$$\int_G \psi_k(x) \overline{\psi_\ell(x)} dx = \begin{cases} 1 & ; \text{ if } k = \ell \\ 0 & ; \text{ otherwise.} \end{cases}$$

However, it turns out that the condition above is exactly this orthogonality condition. As was stated earlier, each of the ψ_k 's are constant on the appropriate I_{M_k} 's. Thus,

$$\frac{1}{M_n} \sum_{j=0}^{M_n-1} \psi_k \left(\frac{j}{M_n} \right) \overline{\psi_\ell \left(\frac{j}{M_n} \right)} = \int_G \psi_k(x) \overline{\psi_\ell(x)} dx.$$

and (6) yields the desired result.

$$\begin{aligned}
& \frac{1}{M_n} \sum_{k=M_n}^{M_{n+1}-1} \sum_{\ell=M_n}^{M_{n+1}-1} a_k \overline{a_\ell} \sum_{j=0}^{M_n-1} \psi_k \left(\frac{j}{M_n} \right) \overline{\psi_\ell \left(\frac{j}{M_n} \right)} \\
&= \frac{1}{M_n} \sum_{k=M_n}^{M_{n+1}-1} a_k \overline{a_k} M_n \\
&= \sum_{k=M_n}^{M_{n+1}-1} |a_k|^2.
\end{aligned}$$

CHAPTER 3 : SETS OF MULTIPLICITY FOR THE COLLECTION $\mathcal{C}$

Recall that a *perfect* set is a closed set with no isolated points. In the dyadic case, Shneider (1949) proved that the Cantor middle-halves set is a U-set for the Walsh series, using convergence of the n -partial sums instead of the 2^n -partial sums. Thus, not every perfect set of Haar measure zero is an M-set (a set of multiplicity for the collection of all bounded Vilenkin series). The purpose of this chapter is to show that if a strong enough growth condition is imposed on the coefficients, then every perfect set of (Haar) measure zero in G is a set of multiplicity. This result is an extension of Yoneda's work (1983) to the bounded Vilenkin case, and uses the M_n -partial sums, a weaker condition than that of Shneider. Thus, it shall be shown that *all* perfect sets of Haar measure zero are sets of multiplicity for the given set of Vilenkin character series.

Theorem 3.1. *Let G be a Vilenkin group of bounded type. If E is a perfect set of Haar measure zero in G , then E is a set of multiplicity for the collection*

$$\mathcal{C} = \left\{ S \sim \sum_{k=0}^{\infty} a_k \psi_k(x) : \lim_{n \rightarrow \infty} \frac{1}{n} \sum_{k=0}^{n-1} |a_k|^2 = 0 \right\}.$$

Proof. Let $E_n = \bigcup_{I \cap E \neq \emptyset} I(j, n)$ for all $n = 0, 1, \dots$. Clearly, $E = \bigcap_{n=1}^{\infty} E_n$. Define $N_n(j)$ to be the number of k satisfying

$$I(k, M_{n+1}) \cap I(j, M_n) \cap E \neq \emptyset$$

for all $n = 0, 1, \dots$; $j = 0, 1, \dots, M_n - 1$; and $k = 0, 1, \dots, M_{n+1} - 1$. Define the measure m_S on G to be $m_S(G) = 1$, and for every $\ell = 0, 1, \dots, p_n - 1$, such

that $I(p_n + \ell, M_{n+1}) \cap E \neq \emptyset$, define

$$m_S(I(p_n + \ell, M_{n+1})) = \frac{1}{N_n(j)} m_S(I(j, M_n)).$$

Define $m_S(I) = 0$ otherwise. Clearly, m_S is a positive Borel measure on G which is 0 off E , and thus the associated Vilenkin series $S(x)$ is a non-zero series which is zero off the set E . Since E is a perfect set, every $x \in E$ is such that $I_{M_n}(x) \cap E$ always contains (at least) 2 points of E for any positive integer n . (This is because x cannot be an isolated point.) Hence, for large enough n ,

$$m_S(I_{M_{n+1}}(x)) \leq \frac{1}{2} m_S(I_{M_n}(x)) \quad \text{for all } x \in E.$$

Therefore, it is clear that $\lim_{n \rightarrow \infty} m_S(I_{M_n}(x)) = 0$ for all $x \in E$. It is, however, also clear by definition that $m_S(I_{M_n}(x)) = 0$ for all $x \notin E$ for large enough n .

Thus,

$$m_S(I_{M_n}(x)) = \frac{1}{M_n} \sum_{k=0}^{M_n-1} a_k \psi_k(x) = 0 \quad \text{for all } x \notin E$$

for large enough n . E is therefore a set of multiplicity for $\mathcal{C}$ if the chosen $S(x)$ is in $\mathcal{C}$. Clearly, by the above, for any x not in E .

$$\lim_{n \rightarrow \infty} \frac{1}{M_n} \sum_{k=0}^{M_n-1} a_k \psi_k(x) = \lim_{n \rightarrow \infty} m_S(I_{M_n}(x)) = 0$$

and thus

$$\lim_{n \rightarrow \infty} m_S(I_{M_n}(x)) = 0 \quad \text{for all } x \in G.$$

Hence, m_S is a non-atomic measure and by Proposition 2.1, $S \in \mathcal{C}$. $\square$

CHAPTER 4 : THE COLLECTION $\mathcal{T}_\alpha^+$

This chapter makes a connection between the uniqueness question and the capacity of sets (a gauge of the size of sets which is more delicate than Lebesgue (or Haar) measure). Our theorems extend previous results done on the dyadic group to the more general setting of (bounded) Vilenkin groups. The collection of series for this chapter is slightly smaller than that of the previous chapter, so we shall concentrate on finding sets of *uniqueness* for this smaller collection. (Note that if a collection $\mathcal{A} \subset \mathcal{B}$, then a set of multiplicity for $\mathcal{A}$ is also one for $\mathcal{B}$.) Beginning with a characterization of the capacities of sets modeled after the paper of L.H. Harper (1966), the framework will be put in place in order to proceed with the uniqueness question. This question was first addressed by Wade (1975) (while working on a problem suggested by V.L. Shapiro in 1964) and further generalized by Yoneda (1983 and 1988). Yoneda's generalizations will be further addressed in Chapters 5 and 6.

Let E be a (closed) subset of G . Let $\mathcal{M}(E)$ represent the set of all non-negative, Borel measures of norm 1 on G supported on E . For any $\nu \in \mathcal{M}(E)$ and $0 < \alpha < 1$, define the *energy integral* (or *potential function*), which was first used in probability theory, as

$$I(\nu) = \iint_{G \times G} \Phi_\alpha(x - y) d\nu(x) d\nu(y)$$

where $\Phi_\alpha(x) = |x|^{-\alpha}$. Clearly, this function is non-negative, and thus the integral always exists, although it may equal infinity. This integral is used to define what is meant by the capacity of a set.

For fixed α , $0 < \alpha < 1$, the α -capacity of a set E is defined to be the number C , where

$$C = V^{-\frac{1}{\alpha}},$$

and $V = \inf_{\nu} I(\nu)$. If $V = \infty$ (i.e. $I(\nu)$ is infinite for every $\nu \in \mathcal{M}(E)$), then E is said to be of α -capacity zero.

There is a large amount of material available on α -capacity as defined above, but the kernel Φ_{α} is unsuitable for analysis on totally disconnected groups. Thus, the first step is to change the kernel Φ_{α} to something which works better for the totally disconnected group G , without changing the concept of α -capacity zero.

For $x \in G$, define $\{x\} = \frac{1}{M_n}$, where n is the number of components of x before the first non-zero component (i.e. $x = (x_1, x_2, \dots) : x_1 = \dots = x_n = 0$, but $x_{n+1} \neq 0$). For convenience, define $\{0\} = 0$. Clearly, then, a new kernel, defined as:

$$K_{\alpha}(x) = \{x\}^{-\alpha}$$

is a non-negative function which is continuous everywhere (except at 0). (Continuity stems from the fact that $x \rightarrow \{x\}$ is a continuous function from $G \rightarrow [0, 1]$, since the inverse image of $\{x\}$ under this map is the set $G_n = \{x \in G : x_1 = \dots = x_n = 0\}$, which is a basic open neighborhood in the Vilenkin system.)

Note: the author toyed with the idea of defining $\{x\} = \frac{x_{n+1}}{M_n}$, since the kernel above will still yield the same sets of capacity zero as the original Φ_{α} with this definition of $\{x\}$. However, the calculations proved more difficult. One may want to define the kernel this way in order to gain better approximations or possibly even to aid in extending the ideas which follow to the unbounded Vilenkin case.

$K_{\alpha}(x)$ clearly has all of the desired properties of the original kernel; it remains only then to show that both kernels define the same sets of α -capacity zero. To that end, first observe that, since $x_k = 0$ for all $k = 1, 2, \dots, n$, it is possible

to rewrite

$$\{x\} = \frac{p_{n+1}}{x_{n+1}} \sum_{k=1}^{n+1} \frac{x_k}{M_k}$$

if $\{x\} = \frac{1}{M_n}$. Also, recall that the distance between any two points $x, y \in G$ is given by

$$\sum_{k=0}^{\infty} \frac{|x_k - y_k|}{M_k}.$$

Using these two facts yields.

$$\begin{aligned} |x| &= \sum_{k=1}^{\infty} \frac{x_k}{M_k} = \sum_{k=1}^n \frac{x_k}{M_k} + \sum_{k=n+1}^{\infty} \frac{x_k}{M_k} \\ &= \frac{x_{n+1}}{p_{n+1}} \{x\} + \sum_{k=n+1}^{\infty} \frac{x_k}{M_k}. \end{aligned}$$

Because the infinite sum $\sum_{k=1}^{\infty} \frac{x_k}{M_k}$ converges, the following claim makes sense.

Claim 1.

$$2\{x\} = \frac{2}{M_n} \geq \sum_{k=n+2}^{\infty} \frac{x_k}{M_k}$$

Proof of Claim 1. This is a simple calculation. Recall that $x_k < p_k$ for any k .

Therefore,

$$\begin{aligned}
(M_n) \sum_{k=n+2}^{\infty} \frac{x_k}{M_k} &< (M_n) \sum_{k=n+1}^{\infty} \frac{p_{k+1}}{M_{k+1}} \\
&= \sum_{k=n+1}^{\infty} \frac{M_n}{M_k} \\
&= \sum_{k=n+1}^{\infty} \frac{1}{p_{n+1} \cdots p_k} \\
&= \sum_{j=1}^{\infty} \frac{1}{p_{n+1} \cdots p_{j+n}} \\
&\leq \sum_{j=1}^{\infty} \frac{1}{2^{(j+n)-(n+1)}} \\
&= \sum_{j=1}^{\infty} \frac{1}{2^{j-1}} = 2 \quad \square \text{ Claim 1.}
\end{aligned}$$

Hence,

$$\begin{aligned}
|x| &= \frac{x_{k+1}}{p_{n+1}} \{x\} + \sum_{k=n+2}^{\infty} \frac{x_k}{M_k} \\
&\leq \left(\frac{x_{k+1}}{p_{n+1}} + 2 \right) \{x\} \\
&\leq 3 \{x\}
\end{aligned}$$

From the definitions, it is easy to see that for arbitrary x , $\{x\} \leq |x|$, and thus the following comparison of the two kernels clearly holds.

$$\Phi_{\alpha}(x) \leq 3^{\alpha} K_{\alpha}(x) \leq 3^{\alpha} \Phi_{\alpha}(x)$$

for any α between 0 and 1. This demonstrates that E is a set of α -capacity zero for K_{α} if and only if it is one for Φ_{α} .

Why go to such trouble to changing the kernel from Φ_{α} to K_{α} ? The main reason at this point is that the Vilenkin-Fourier coefficients for $K_{\alpha}(x)$ are much

easier to calculate than for $\Phi_\alpha(x)$. Begin by defining sets $G_n := I_n(0)$ for $n \geq 1$. It is clear that G_n is the set

$$G_n = \{x \in G : x_1 = \cdots = x_n = 0\}$$

For convenience, define $G_0 = \{x \in G : x_1 \neq 0\}$.

Because the Vilenkin system is more complicated than the Walsh system, it is necessary to further break apart these basic sets into simpler subsets $G_{n,j} = \{x \in G_n : x_{n+1} = j\}$. Clearly, the $G_{n,j}$'s are disjoint, and $\bigcup_j G_{n,j} = G_n$ for each n . Thus, using these ideas, it is again possible to rewrite $\{x\}$ in a convenient way.

Note: $G_{n,0} = G_{n+1}$. Therefore, in the discussion that follows, the case $j = 0$ will often be omitted in the n^{th} case, as it falls under the general $(n+1)^{\text{th}}$ case.

Claim 2. If $\{x\} = \frac{1}{M_n}$, the following conditions hold:

$$(1) \quad (x + \delta^{k,j}) \notin G_{k-1,j} \quad \text{for all } k = 1, 2, \dots, n ; j = 1, 2, \dots, p_k - 1$$

$$(2) \quad (x + \delta^{n+1,x_{n+1}}) \in G_{n+1,x_{n+1}} ; (x + \delta^{n+1,j}) \notin G_{n+1,j} \quad \text{for all } j \neq x_{n+1}$$

and

$$(3) \quad (x + \delta^{k,j}) \notin G_{k,j} \quad \text{for all } k > n+1 : j = 1, 2, \dots, p_k - 1$$

where $\delta^{k,j} = (\delta_1^{k,j}, \delta_2^{k,j}, \dots)$, with each component defined as

$$\delta_i^{k,j} = \begin{cases} p_k - j ; & i = k \\ 0 & ; i \neq k \end{cases}$$

Proof of Claim 2. (1) will be shown using an inductive argument, with induction on k , while (2) and (3) are shown at each k -level. All three results will be obvious

for any appropriate choice of j (for a particular k), since the number of choices for j is finite (recall that G is a *bounded* Vilenkin group).

Let $k = 1$ and let $x \in G$ such that $\{x\} = \frac{1}{M_0} \equiv 1$. By definition, the first coordinate of x is a non-zero integer j between 1 and $p_1 - 1$. Thus, the placeholder is $\delta^{k,j} = (p_1 - j, 0, 0, \dots)$. Clearly, $x \dot{+} \delta^{k,j} \equiv (0, x_2, x_3, \dots) \notin G_{0,i}$ for every non-zero i , but this element *is* in the set G_{1,x_2} . Since $k = 1$, it cannot be greater than $n + 1$ for any $n \in \mathcal{N}$, so (3) is trivially satisfied. Therefore, since the choice of j is determined by the choice of x , all three hold for any appropriate j , so the claim holds for $k = 1$.

Assume now that all three conditions of the claim are satisfied for $k = m$; it remains to show that they also hold for $k = m + 1$. Since it is assumed true for $k = m$, (1) is trivially satisfied for $k = m + 1$. Let $x \in G$ such that $\{x\} = \frac{1}{M_m}$. (As above, if $x_{m+1} = 0$, then this case will fall under the next level's general case G_{m+2} , so assume that $x_{m+1} \neq 0$). Since $k = m + 1$, it is easy to see that $\delta^{k,j} = (0, \dots, 0, p_{m+1} - j, 0, \dots)$, where the non-zero entry is in the $(m+1)^{st}$ coordinate. It is easy to see, therefore, that $(x \dot{+} \delta^{k,j}) = (0, \dots, 0, 0, x_{m+2}, x_{m+3}, \dots)$ if and only if $j = x_{m+1}$. (2) is satisfied.

If $k > m + 1$, then $\delta^{k,j} = (0, \dots, 0, p_k - j, 0, \dots)$, where the $(m+1)^{st}$ entry is 0. Thus, $x \dot{+} \delta^{k,j}$ has the same first $(m+1)$ coordinates as x , implying that (3) is also satisfied (since $x_{m+1} \neq 0$). By induction on k , the claim is proven. $\square$ *Claim2.*

By Claim 2, it is clear that, for any k ,

$$\sum_{j=1}^{p_{n+1}-1} \chi_{G_{k,j}}(x \dot{+} \delta^{k,j}) = \begin{cases} 1 & : k = n + 1 \\ 0 & ; k \neq n + 1. \end{cases}$$

Thus, $\{x\}$ can be rewritten as an infinite sum.

$$\begin{aligned}\{x\} &= \frac{p_{n+1}}{x_{n+1}} \sum_{k=1}^{n+1} \frac{x_k}{M_k} \\ &= p_{n+1} \sum_{k=1}^{\infty} \frac{1}{M_k} \sum_{j=1}^{p_{n+1}-1} \chi_{G_{k,j}}(x + \delta^{k,j}).\end{aligned}$$

Examine further the effects that the place-holder $\delta^{k,j}$ has on this situation. Fix a positive integer k . For now, assume that $n < k$, and suppose that the integral is inside the sum. (It will be shown that this can be done with no problem.) Hence, $\psi_n(\delta^{k,j}) = 1$, since $\delta^{k,j} \in I_n(0)$ for all $n < k$. However, for any $x \in G_k$, $n < k$ means $\psi_n(x) = 1$. Thus, for $n < k$, (recalling that the *disjoint* union of the $G_{n,j}$'s is G_n),

$$\begin{aligned}\sum_{j=1}^{p_{n+1}-1} \int_{G_{k,j}} \psi_n(\delta^{k,j}) \overline{\psi_n(x)} dx &= \int_{G_k} \overline{\psi_n(x)} dx \\ &= \int_{G_k} 1 dx = \frac{1}{M_k}.\end{aligned}$$

If $n \geq k$, it is only necessary to recall that since $G_{k,j}$ is a closed subgroup of G for each j , the annihilator of G_k (and in turn, of $G_{k,j}$) is exactly the set of characters ψ_n where $n \geq k$ (Rudin: FA on Gps). Thus, it is clear that $\int_{G_{k,j}} \overline{\psi_n(x)} dx = 0$ for $n \geq k$, for each j . Define the parameter λ_n^k by

$$\lambda_n^k = \begin{cases} 1 & ; n < k \\ 0 & ; \text{otherwise.} \end{cases}$$

It is then clear that

$$(4) \quad \sum_{j=1}^{p_{n+1}-1} \psi_n(\delta^{k,j}) \int \chi_{G_{k,j}} \overline{\psi_n(x)} dx = \sum_{k=1}^{\infty} \left(\frac{1}{M_n} \right) \lambda_n^k.$$

Notice that (4) is a convergent series, since

$$\begin{aligned}
 \sum_{k=1}^{\infty} \left(\frac{\lambda_n^k}{M_n} \right) &= \sum_{k=n+1}^{\infty} (p_{n+1})^{-\alpha} (M_k)^{\alpha-1} \\
 &\leq \sum_{k=1}^{\infty} 2^{-\alpha} 2^{(\alpha-1)k} \\
 (5) \qquad &= 2^{-\alpha} (2^{\alpha-1} - 1)^{-1}.
 \end{aligned}$$

Define

$$(6) \qquad \gamma_n := \int K_{\alpha}(x) \overline{\psi_n(x)} dx,$$

where dx is integration with respect to the Haar measure on G . (Note: γ_n is the n^{th} coefficient of the Vilenkin-Fourier series of K_{α} .)

For a fixed x , it is clear that the sum of characteristic functions is unaffected by raising the sum to the α power (since, for fixed k , there is only one of the j 's for which the characteristic function is non-zero). Thus, it is possible to rewrite the coefficients (recalling that Haar measure dx is translation-invariant and ψ is a character) as

$$\begin{aligned}
 \gamma_n &= \int_G K_{\alpha}(x) \overline{\psi_n(x)} dx \\
 \text{by (4)} \quad &= \int_G (p_{n+1})^{-\alpha} \sum_{k=1}^{\infty} (M_k)^{\alpha} \sum_{j=1}^{p_{n+1}-1} \chi_{G_{k,j}}(x + \delta^{k,j}) \overline{\psi_n(x)} dx \\
 &= \int_G (p_{n+1})^{-\alpha} \sum_{k=1}^{\infty} (M_k)^{\alpha} \sum_{j=1}^{p_{n+1}-1} \chi_{G_{k,j}}(x) \overline{\psi_n(x)} \psi_n(\delta^{k,j}) dx.
 \end{aligned}$$

By (4) and (5), it is clear that the infinite sum above converges and is bounded above by another sum which is uniformly convergent. Thus, the Lebesgue Dominated Convergence Theorem shows that the n^{th} Vilenkin-Fourier coefficients of

$K_\alpha(x)$ are:

$$\begin{aligned}
 \gamma_n &= \int_G (p_{n+1})^{-\alpha} \sum_{k=1}^{\infty} (M_k)^\alpha \sum_{j=1}^{p_{n+1}-1} \chi_{G_{k,j}}(x) \overline{\psi_n(x)} \psi_n(\delta^{k,j}) dx \\
 &= \sum_{k=1}^{\infty} \left(\frac{M_k}{p_{n+1}} \right)^\alpha \sum_{j=1}^{p_{n+1}-1} \psi_n(\delta^{k,j}) \int_G \chi_{G_{k,j}}(x) \overline{\psi_n(x)} dx \\
 (7) \quad &= \sum_{k=n+1}^{\infty} (p_{n+1})^{-\alpha} (M_k)^{\alpha-1}.
 \end{aligned}$$

It is now possible to use a formula, (7), to explicitly calculate the n^{th} Vilenkin-Fourier coefficients for the kernel $K_\alpha(x)$. There is also an upper bound, (5), for the value of these coefficients.

Note: This is an example of where the unbounded case may be aided by a slight change in kernel. Above, it is clear that the coefficients of K_α go to zero as $n \rightarrow \infty$ (as one would expect). However, if $\frac{1}{M_n}$ were replaced by $\frac{r_{n+1}}{M_n}$ for the definition of $\{x\}$, it may be easier to get a grasp of the unbounded case. The calculation above shows that if $\sup_k \{p_k\} = \infty$, the coefficients converge even more rapidly to zero - this could prove useful, or it may be necessary to slow the convergence down with the previously noted change in kernel.

We shall now define three classes of Vilenkin series which will play a prominent role in this chapter. Let the set $\mathcal{A}$ represent the set of all Vilenkin series $S(x) = \sum_{k=0}^{\infty} a_k \psi_k(x)$ such that the averages of the M_n^{th} -partial sums of S go to 0 as $n \rightarrow \infty$ for every $x \in G$. In other words,

$$\mathcal{A} = \left\{ S(x) : \lim_{n \rightarrow \infty} \frac{S_{M_n}(x)}{M_n} = 0 \text{ for every } x \in G \right\}.$$

Define the function $[k]$ on the set of positive integers by $[k] = M_n$, where $M_n \leq k < M_{n+1}$. Represent the collection of Vilenkin series S whose coefficients satisfy

$$(8) \quad \sum_{k=0}^{\infty} |a_k|^2 [k]^{\alpha-1} < \infty$$

by $\mathcal{T}_\alpha$ for $0 < \alpha < 1$, and the collection $\mathcal{T}_\alpha^+$ as all series $S \in \mathcal{T}_\alpha$ with the additional property that there are positive integers $n_1 < n_2 < \dots$ such that $S_{M_{n_j}}(x) \geq 0$ for almost all $x \in G$. Here is how each of these classes compares with the others.

Claim 3. For each $0 < \alpha < 1$, $\mathcal{T}_\alpha^+ \subset \mathcal{T}_\alpha \subset \mathcal{A}$.

Proof of Claim 3. By definition, the first inclusion holds. For the second inclusion, notice that the Schwartz inequality implies

$$\left| \frac{1}{M_n} S_{M_n}(x) \right|^2 \leq \left(\frac{1}{M_n} \right)^2 \sum_{k=0}^{M_n-1} |a_k|^2 |\psi_k(x)|^2.$$

Recall that $(\frac{1}{M_n})^2 \leq \frac{1}{M_n}$ and $|\psi_k(x)| \equiv 1$. Replace M_n with $(M_n)^\alpha (M_n)^{1-\alpha}$. It is clear that for each $k = 0, 1, \dots, M_n - 1$, $[k] < M_n$, and thus $[k]^{\alpha-1} > (M_n)^{\alpha-1}$ (since $\alpha - 1 < 0$ for any $0 < \alpha < 1$). Consequently, it follows that

$$\begin{aligned} \left| \frac{1}{M_n} S_{M_n}(x) \right|^2 &\leq (M_n)^{-1} \sum_{k=0}^{M_n-1} |a_k|^2 |\psi_k(x)|^2 \\ &\leq (M_n)^{-\alpha} \sum_{k=0}^{M_n-1} |a_k|^2 (M_n)^{\alpha-1} \\ &\leq (M_n)^{-\alpha} \sum_{k=0}^{M_n-1} |a_k|^2 [k]^{\alpha-1}. \end{aligned}$$

This sequence of inequalities clearly shows $\mathcal{T}_\alpha \subset \mathcal{A}$ for any $0 < \alpha < 1$.

□ *Claim 3.*

Two results will now be stated without proof. The first of which comes from general probability theory, and the second of which is a trivial extension of a result of Harper (1966) to the Vilenkin case.

Lemma 4.1. Given any set $E \subset G$ of positive α -capacity, there exists a measure $\nu \in \mathcal{M}(E)$ such that $U_\nu(x) := \int_G K_\alpha(x-y) d\nu(y)$ is in $L^\infty(G)$ and $U_\nu(x) \geq W_\alpha(E) := \inf \{ \|U_\nu\|_\infty : \nu \in \mathcal{M}(E) \}$ for almost every $x \in E$.

Lemma 4.2. Let $E_1 \supset E_2 \supset \dots$ be a nested sequence of closed subsets of G such that

$$E = \bigcap_{n=1}^{\infty} E_n$$

is of α -capacity zero. Then, $\lim_{n \rightarrow \infty} W_{\alpha}(E_n) = \infty$.

The statement of the first lemma of this section which shall be proven is an extension of Lemma 3 of Wade (1974), which limits the types of Vilenkin series on which it is necessary to concentrate. In more concrete terms, given any Vilenkin series S , and an appropriate interval in $[0, 1)$, a Vilenkin series T can be found which is equivalent to the original series on the given interval, but is zero elsewhere, thus making calculations easier by using T instead of S . Included in the proof of this lemma is a calculation which generalizes a result of Shneider (1949; p.285).

Lemma 4.3. If $S(x) = \sum_{k=0}^{\infty} a_k \psi_k(x) \in \mathcal{A}$ and c, d are M_n -adic rational numbers in $[0, 1)$, then there exists a Vilenkin series $T \in \mathcal{A}$, $N \in \mathcal{N}$ such that

$$T_{M_n}(x) = \begin{cases} S_{M_n}(x) & ; \text{ for all } x \in [c, d) \\ 0 & ; \text{ for all } x \notin [c, d) \end{cases}$$

for all $n > N$.

Proof of 4.3. It is first necessary to show the existence of appropriate Vilenkin polynomials.

Claim 4. There exists a Vilenkin polynomial $P(x) = \sum_{k=0}^M \beta_k \psi_k(x)$ such that

$$(9) \quad P(x) = \begin{cases} 1 ; & \text{ if } x \in [c, d) \\ 0 ; & \text{ otherwise.} \end{cases}$$

Proof of Claim 4. Because c and d are M_n -adic rationals, there exists a positive integer M so that $\frac{j}{M_M} = c$ and $\frac{j+1}{M_M} = d$ for some $j = 0, 1, \dots, p_M - 1$. Define

$\eta_k := \psi_k(t)$, where $t \in [c, d)$. Clearly, this implies that $\sum_{k=0}^M \eta_k \psi_k(x)$ is nothing more than the M^{th} Vilenkin-Dirichlet kernel. Thus, by the Vilenkin extension of Paley's Lemma, it is clear that if $M = M_n$ for some n ,

$$\sum_{k=0}^M \eta_k \psi_k(x) = M_n$$

on $[c, d)$ and zero elsewhere. Let $M = M_n$ for some n and define the coefficients $\beta_k = \frac{\eta_k}{M_n}$. Clearly, $P(x) = \sum_{k=0}^M \beta_k \psi_k(x)$ satisfies (9). $\square$ *Claim4.*

Hence, let $P(x)$ be any Vilenkin polynomial satisfying (9). For any two integers j, k , define the operation $k \circ j$ by $\psi_{k \circ j} := \psi_k \overline{\psi_j}$. Define $\gamma_k := \sum_{j=0}^M \beta_j a_{k \circ j}$ and $T(x) := \sum_{k=0}^{\infty} \gamma_k \psi_k(x)$. The following calculation shows that this is the right choice for T .

$$\begin{aligned} T_{M_n}(x) &= \sum_{k=0}^{M_n-1} \gamma_k \psi_k(x) \\ &= \sum_{k=0}^{M_n-1} \left(\sum_{j=0}^M \beta_j a_{k \circ j} \right) \psi_k(x) \\ &= \sum_{j=0}^M \sum_{k=0}^{M_n-1} \beta_j a_{k \circ j} \psi_k(x). \end{aligned}$$

Recall that $\psi_j(x) \overline{\psi_j(x)} = |\psi_j(x)|^2 \equiv 1$, and continue:

$$\begin{aligned} &= \sum_{j=0}^M \sum_{k=0}^{M_n-1} \beta_j a_{k \circ j} \psi_k(x) \psi_j(x) \overline{\psi_j(x)} \\ &= \sum_{j=0}^M \beta_j \psi_j(x) \sum_{k=0}^{M_n-1} a_{k \circ j} \psi_{k \circ j}(x) \\ &= P(x) S_{M_n}(x) \end{aligned}$$

for all $x \in G$, as long as n is such that $M_n > M$. Clearly, by the choice of P , the series T satisfies the conclusion. $\square$ *Lemma4.3.*

The next lemma is an extension of Lemma 4 in Wade (1974) to bounded Vilenkin groups, and shows for a given interval I with M_n -adic endpoints, the M_n^{th} partial sums of a series S eventually **equal** zero on I .

Lemma 4.4. *Let $F \subset [0, 1]$ be closed and $S = \sum_{k=0}^{\infty} a_k \psi_k$ be in $\mathcal{A}$. Suppose $S_{M_n}(x) \rightarrow 0$ for almost every $x \in [0, 1] \setminus F$ and $\limsup_{n \rightarrow \infty} |S_{M_n}| < \infty$ for all but (at most) countably many $x \in [0, 1] \setminus F$. Then, for any interval $(c, d) \subset [0, 1] \setminus F$, where c and d are M_n -adic rationals, there exists an $N \in \mathcal{N}$ such that for every $n \geq N$, $S_{M_n}(x) = 0$ for all $x \in (c, d)$.*

Proof of 4.4. Let S and $(c, d) \subset [0, 1] \setminus F$ be as above. Then, by Lemma 3, we know there exists a series $T \in \mathcal{A}$ and an $N \in \mathcal{N}$ such that

$$T_{M_n} = \begin{cases} S_{M_n} & ; \text{ if } x \in (c, d) \\ 0 & ; \text{ otherwise} \end{cases}$$

for all $n > N$. By hypothesis, it is clear that $T_{M_n} \rightarrow 0$ a.e. as $n \rightarrow \infty$ (since $(c, d) \subset [0, 1] \setminus F$, and $T_{M_n} = 0$ off (c, d) for all $n > N$). This part of the hypothesis also guarantees that $\limsup_{n \rightarrow \infty} |T_{M_n}(x)| < \infty$ off a countable set in $[0, 1] \setminus F$. Thus, a theorem of Grubb (1987) which is an extension of the main theorem of Wade (1969), implies T is the zero series. Since the M_n^{th} partial sums of S and T coincide on (c, d) for all $n > N$, the lemma is proven. $\square$ *Lemma 4.4.*

The main theorem of this section is finally ready to be proven. It characterizes all sets of α -capacity zero as sets of uniqueness for T_α^+ .

Theorem 4.5. *Let α be a number between 0 and 1. Then, a necessary and sufficient condition for a closed set $E \subset [0, 1]$ to be a set of uniqueness for T_α^+ is that E be of α -capacity zero.*

Proof. Let E be a closed subset of $[0, 1]$.

Necessity. Suppose that E is not of α -capacity zero. Then, Lemma 1 implies the

existence of a measure $\nu \in \mathcal{M}(E)$ such that $U_\nu \in L^\infty(G)$; hence, by definition, $W_\alpha^\nu(E) = \|U_\nu\|_\infty < \infty$. Let $S \sim \sum_{k=0}^{\infty} \widehat{\nu}(k)\psi_k$ be the Vilenkin-Fourier-Stieltjes series of ν . Clearly, since $1 = \|\nu\| = \widehat{\nu}(0)$, S is not the zero series. Recalling that $S_{M_n} = D_{M_n} * \nu$ for all n , it is easy to see $\lim_{n \rightarrow \infty} S_{M_n} = 0$ for all $x \notin E$, since ν is supported on E . Thus, it remains only to show that $S \in \mathcal{T}_\alpha^+$. [This will prove E is *not* a set of uniqueness for $\mathcal{T}_\alpha^+$, a contradiction to the hypothesis.]

Lemma 4.6. *Let $0 < \alpha < 1$ and $E \subset [0, 1]$ be closed. For $\nu \in \mathcal{M}(E)$, there exists a constant $\tilde{B}$ (depending on α and $\{p_n\}$) such that*

$$\sum_{k=0}^{\infty} |\widehat{\nu}(k)|^2 [k]^{\alpha-1} \leq \tilde{B} \cdot W_\alpha^\nu(E)$$

where $\widehat{\nu}(k)$ is the k^{th} Vilenkin-Fourier-Stieltjes coefficient of ν .

Proof of 4.6. Clearly, it is only necessary to deal with the case in which $W_\alpha^\nu(E) < \infty$. Let γ_k represent the V-F coefficients of the kernel $K_\alpha(x)$. From the general theory of Fourier transforms (see Rudin) it is known that

$$(10) \quad \sum_{k=0}^{\infty} |\widehat{\nu}(k)|^2 \gamma_k = \int_G U_\nu(x) d\nu(x).$$

Let $B := \sup_n p_n$, where $\{p_n\}$ is the *bounded* sequence of positive primes which generates the Vilenkin group G . Then, by definition of $[k]$,

$$(11) \quad B^{-\alpha} \sum_{k=0}^{\infty} |\widehat{\nu}(k)|^2 [k]^{\alpha-1} = B^{-\alpha} \sum_{j=0}^{\infty} \sum_{k=M_j}^{M_{j+1}-1} |\widehat{\nu}(j)|^2 (M_j)^{\alpha-1}$$

where $M_j \leq k < M_{j+1}$. Since $B \geq p_{k+1}$ for all k , the representation for the coefficients of the kernel $K_\alpha(x)$ yields

$$(12) \quad \begin{aligned} \gamma_k &= \sum_{j=k+1}^{\infty} (p_{k+1})^{-\alpha} (M_j)^{\alpha-1} \\ &\geq \sum_{j=k+1}^{\infty} B^{-\alpha} (M_j)^{\alpha-1}. \end{aligned}$$

Putting (10), (11), and (12) together shows

$$\begin{aligned}
B^{-\alpha} \sum_{k=0}^{\infty} |\widehat{\nu}(k)|^2 [k]^{\alpha-1} &\leq B^{-\alpha} \sum_{j=0}^{\infty} |\widehat{\nu}(j)|^2 \sum_{k=j}^{\infty} (M_j)^{\alpha-1} \\
&\leq \sum_{j=0}^{\infty} |\widehat{\nu}(j)|^2 \gamma_j \\
&= \int_G U_{\nu}(x) d\nu(x) \\
&\leq W_{\alpha}^{\nu}(E). \quad \square \text{ Lemma 4.6.}
\end{aligned}$$

It is clear that $S \in \mathcal{A}$. Since $S_{M_n} = D_{M_n} * \nu$ for all n (see above), it is also clear that $S_{M_n} \geq 0$ for every n , since $D_{M_n} \geq 0$ and ν is nonnegative. Therefore, S is a non-zero series in $\mathcal{T}_{\alpha}^{+}$ which is zero off E , contradicting the fact that E is a set of uniqueness for $\mathcal{T}_{\alpha}^{+}$. Hence, E must be of α -capacity zero.

Sufficiency. Let E be of α -capacity zero. Suppose $S \sim \sum a_k \psi_k$ is a Vilenkin series in $\mathcal{T}_{\alpha}^{+}$ such that

$$\lim_{n \rightarrow \infty} S_{M_n}(x) = 0 \quad \text{for all } x \notin E.$$

It remains to show that S is the zero series.

Lemma 4.7. Any series $T \in \mathcal{T}_{\alpha}^{+}$ such that

$$\lim_{n \rightarrow \infty} T_{M_n}(x) = 0$$

for every x off a set of α -capacity zero has constant term equal zero.

Proof of 4.7. Let $\lambda : G \rightarrow [0, 1]$ be Fine's map for Vilenkin groups.

$$\text{i.e.} \quad \lambda(x_0, x_1, \dots) = \sum_{k=0}^{\infty} \frac{x_k}{M_{k+1}}.$$

It is known that λ is continuous and 1-1 (when using the appropriate choice of x_i 's for M_n -adic rationals) and $[0, 1]$ is compact, so since E is a closed set in G ,

$\lambda(E)$ is closed in $[0, 1]$. Thus, let $\{I_k\}$ be a sequence of open intervals in $[0, 1]$ with M_n -adic endpoints such that $\bigcup_k I_k = [0, 1] \setminus \lambda(E)$. Define a nested sequence of closed sets $E_1 \supset E_2 \supset \dots$ in G by

$$E_N := G \setminus \lambda^{-1}\left(\bigcup_{k=1}^N I_k\right).$$

This will allow exploration of the series S on both the group G and the interval $[0, 1]$.

As a series on $[0, 1]$, $S_{M_n}(x) \rightarrow 0$ as $n \rightarrow \infty$ for every $x \notin \lambda(E)$ by hypothesis. Thus, for each k , Lemma 4.4 implies

$$(13) \quad S_{M_n}(x) \equiv 0 \quad \text{on } I_k \text{ for every } n > \Gamma(k)$$

where $\Gamma(k)$ is some positive integer. However, since G is of bounded Vilenkin type, there exists a number $\Gamma \in \mathcal{N}$, independent of k , so that (13) holds for every $n > \Gamma$. Thus, $S_{M_n}(x) = 0$ for all $x \in \bigcup I_k$, which in turn implies that as a series on the group, $S_{M_n} \equiv 0$ off E_N , for large enough n .

Let $\{n_j\}$ be a sequence of positive integers such that $n_1 < n_2 < \dots$ and $S_{M_{n_j}}(x) \geq 0$. Such numbers exist since $S \in \mathcal{T}_\alpha^+$. Since $a_0 = \int_G S_{M_{n_j}}(x) dx$, it is clear that $a_0 \geq 0$. Also, by general probability theory (Kahne and Salem 1963, see Lemma 6.2), there is a $\nu \in \mathcal{M}(E)$ such that $U_\nu(x) \geq W_\alpha(E_N)$ a.e. $[\nu]$ on E_N . Therefore,

$$\begin{aligned} |a_0| &= \int_G S_{M_{n_j}}(x) dx \\ &\leq \int_G S_{M_{n_j}}(x) \frac{U_\nu(x)}{W_\alpha(E_N)} \\ &= \frac{1}{W_\alpha(E_N)} \int_G \int_G S_{M_{n_j}}(x) K_\alpha(x \cdot y) d\nu(y) dx \end{aligned}$$

by the definition of $U_\nu(x)$.

Since all of the above integrands are non-negative, Fubini's Theorem applies. This, along with the definition of the $M_{n_j}^{th}$ partial sums of S show:

$$|a_0| \leq \frac{1}{W_\alpha(E_N)} \int_G \sum_{k=0}^{M_{n_j}-1} a_k \int_G K_\alpha(x-y) \psi_k(x) dx d\nu(y).$$

Recall that dx is integration with respect to the Haar measure on G , and since Haar measure is translation invariant, a change of variables, along with the character properties of ψ yields

$$\begin{aligned} |a_0| &\leq \frac{1}{W_\alpha(E_N)} \int_G \sum_{k=0}^{M_{n_j}-1} a_k \int_G K_\alpha(x) \psi_k(x+y) dx d\nu(y) \\ &= \frac{1}{W_\alpha(E_N)} \int_G \sum_{k=0}^{M_{n_j}-1} a_k \psi_k(y) \int_G K_\alpha(x) \psi_k(x) dx d\nu(y) \\ &= \frac{1}{W_\alpha(E_N)} \int_G \sum_{k=0}^{M_{n_j}-1} a_k \psi_k(y) \gamma_k d\nu(y) \end{aligned}$$

where γ_k is the k^{th} V-F coefficient of the kernel K_α .

From the discussion at the beginning of this chapter about the properties of K_α and its coefficients, there exists a constant C , depending only on α , which is a bound for the value of the γ_k 's. This, along with the fact that S is continuous on G (and thus integrable) yields

$$\begin{aligned} |a_0| &\leq \frac{1}{W_\alpha(E_N)} \int_G \sum_{k=0}^{M_{n_j}-1} a_k \psi_k(y) \gamma_k d\nu(y) \\ &\leq \frac{C}{W_\alpha(E_N)} \int_G \sum_{k=0}^{\infty} a_k \psi_k(y) d\nu(y) \\ &= \frac{C}{W_\alpha(E_N)} \int_G S(y) d\nu(y) \\ &\leq \frac{\tilde{C}}{W_\alpha(E_N)}. \end{aligned}$$

Clearly, $E = \bigcap_N E_N$, so by Lemma 4.2, E of α -capacity zero implies that

$$\lim_{N \rightarrow \infty} W_\alpha(E_N) = \infty.$$

Thus, $a_0 = 0$ $\square$ Lemma 4.7.

Theorem 4.5 will now be proven by induction. Assume that $a_k = 0$ for all $k = 0, 1, \dots, M_n - 1$. It remains to show that $a_k = 0$ for all $k = M_n, \dots, M_{n+1} - 1$. This will complete the proof.

Choose a number $\ell \in [M_n, M_{n+1})$ and define the polynomial

$$P_\ell(x) := D_{M_{n+1}}\left(\frac{\ell}{M_{n+1}} + x\right)$$

where $D_n(x)$ is defined by $D_n(x) = \sum_{j=0}^{M_n-1} \psi_j(x)$. Define $\beta_j^\ell := \psi_j(\frac{\ell}{M_{n+1}})$; the character properties of ψ yield

$$P_\ell(x) = \sum_{j=0}^{M_{n+1}-1} \psi_j\left(\frac{\ell}{M_{n+1}}\right) \psi_j(x) = \sum_{j=0}^{M_{n+1}-1} \beta_j^\ell \psi_j(x).$$

Define the matrix A to be

$$[\beta_j^\ell : j, \ell = M_n, \dots, M_{n+1} - 1].$$

It is clear that A is nonsingular. Define $\gamma_k := \sum_{j=0}^{M_{n+1}-1} \beta_j^\ell a_{k \circ j}$ and then define the Vilenkin series $T(x) := \sum_{k=0}^{\infty} \gamma_k \psi_k(x)$. Following the steps of the proof of Lemma 4.3, it is easy to see that $T \in \mathcal{T}_a$ and that there exists an $N \in \mathcal{N}$ such that $T_{M_n}(x) = P_\ell(x) S_{M_n}(x)$ for all $x \in G$, when $n > N$. Thus, by the previous discussion about the series S , it is clear that $T_{M_n} \rightarrow 0$ off E and that $T_{M_n}(x) \geq 0$ a.e. (for all j). Lemma 4.7, therefore, shows that $\gamma_0 = 0$.

Because $\gamma_0 = 0$ and (by assumption) $a_k = 0$ for all $k = 0, 1, \dots, M_n - 1$, it can be shown that

$$\begin{aligned} 0 = \gamma_0 &= \sum_{k=0}^{M_{n+1}-1} \beta_j^\ell a_{k \circ j} \\ &= 0 + \sum_{k=M_n}^{M_{n+1}-1} \beta_j^\ell a_j. \end{aligned}$$

This, in turn, is true for every integer $\ell \in [M_n, M_{n+1})$. Hence:

$$A \begin{bmatrix} a_{M_n} \\ \vdots \\ a_{M_{n+1}-1} \end{bmatrix} = 0.$$

Since A is nonsingular, it is clear that $a_k = 0$ for $k = M_n, \dots, M_{n+1} - 1$. By induction, $a_k = 0$ for every k , and thus the series S must be the zero series. Hence, E is a set of uniqueness for $\mathcal{T}_\alpha^+$. $\square$

CHAPTER 5 : THE COLLECTION $\mathcal{M}_\epsilon$

This chapter contains two results about sets of uniqueness for a more general collection of Vilenkin series than was considered in the previous chapter. For the proper choice of ϵ (see below), our collection actually is equivalent to the collection $\mathcal{T}_\alpha$ from Chapter 4. This will be shown in the discussion prior to the proof of Theorem 5.2. Thus, the work done here is a partial generalization of Theorem 4.5, and is inspired by work done on the dyadic group (Yoneda (1983)). Quasi-measures will be used as a method of working in the group setting analytically. Both results in this section apply to *bounded* multiplicative Vilenkin groups, and give conditions for a subset of the group to be a set of uniqueness for the collection of Vilenkin series

$$\mathcal{M}_\epsilon = \left\{ S(x) = \sum_{k=0}^{\infty} a_k \psi_k(x) : \sum_{k=0}^{\infty} \epsilon_k |a_k|^2 < \infty \right\}$$

where the ϵ_k 's are a sequence of positive numbers satisfying

$$(1) \quad \epsilon_0 = 1, \epsilon_k \searrow 0, \text{ and } \liminf_{n \rightarrow \infty} \frac{\epsilon_{M_{n+1}}}{\epsilon_{M_n}} = \theta > 0.$$

The first result reinforces the theme that the "dispersal" of a set throughout the group is as important as the measure of that set in determining its uniqueness properties.

Theorem 5.1. *Let $\{n_k\}$ be any unbounded sequence of integers, and define the sequence $\{N_k\}$ recursively by $N_0 = 0$ and $N_k = N_{k-1} + n_k$ for all k . Let E be a subset of the bounded Vilenkin group G generated by the sequence $\{p_n\}$ which satisfies*

$$(2) \quad I(jp_{N_k+1} + \alpha_{j,k}, M_{N_k+1}) \cap E = \emptyset$$

for all but one of the values $\alpha_{j,k} = 0, 1, \dots, p_{N_k+1} - 1$. E must satisfy (2) for every integer k and for every $j = 0, 1, \dots, M_{N_k} - 1$. Then, E is a set of uniqueness for the collection $\mathcal{M}_\epsilon$ if and only if the sequence $\{\epsilon_k\}$ satisfies

$$(3) \quad \sum_{k=0}^{\infty} (p_{N_k-k+2} \cdots p_{N_k}) \epsilon_{M_{N_k}} = \infty.$$

Proof. Let S be any Vilenkin series and let m_S be its associated quasi-measure (see Proposition 2.4). Because of Lemma 2.6 and the liminf property of the sequence ϵ_k , the following relationships hold:

$$\sum_{k=M_n}^{M_{n+1}-1} \epsilon_k |a_k|^2 \leq M_n \epsilon_{M_n} \sum_{j=0}^{M_n-1} |\Delta m_S(I(j, M_n))|^2$$

and

$$\begin{aligned} \sum_{k=M_n}^{M_{n+1}-1} \epsilon_k |a_k|^2 &\geq M_n \epsilon_{M_{n+1}} \sum_{j=0}^{M_n-1} |\Delta m_S(I(j, M_n))|^2 \\ &= \theta \epsilon_{M_n} \sum_{j=0}^{M_n-1} |\Delta m_S(I(j, M_n))|^2, \end{aligned}$$

where Δm_S is the unitary difference from Definition 2.5. It is easy to see that the growth condition on the coefficients of series in $\mathcal{M}_\epsilon$ is satisfied if and only if the new sum introduced above is finite. In other words,

$$\sum_{k=0}^{\infty} \epsilon_k |a_k|^2 < \infty \quad \text{if and only if}$$

$$(4) \quad \sum_{n=0}^{\infty} \epsilon_{M_n} M_n \sum_{j=0}^{M_n-1} |\Delta m_S(I(j, M_n))|^2 < \infty.$$

We are now prepared to prove Theorem 5.1.

First, suppose that E is a set of uniqueness for $\mathcal{M}_\epsilon$ but (3) is finite. It suffices to show the existence of a quasi-measure which corresponds to a non-zero Vilenkin series in the collection $\mathcal{M}_\epsilon$ with M_n^{th} partial sums converging to zero off E . This will be a contradiction to the hypothesis that E is a set of uniqueness. Define the set function m^* recursively as follows:

For all positive integers $n \leq N_1$, define

$$\begin{aligned} m^*(I(j, M_n)) &= \frac{1}{M_n} \quad ; \text{ if } I(j, M_{N_1+1}) \cap E \neq \emptyset \\ m^*(I(j, M_{N_1+1})) &= \frac{1}{M_{N_1}} \quad ; \text{ if } I(j, M_{N_1+1}) \cap E \neq \emptyset, \end{aligned}$$

with both being equal to zero in the case of an empty intersection with the set E . Continuing in this manner, define the function m^* so that, for any integer n such that $N_k < n \leq N_{k+1}$

$$m^*(I(j, M_n)) = \frac{1}{M_{n-k}} \quad ; \text{ if } I(j, M_n) \cap E \neq \emptyset,$$

and zero otherwise.

It is clear by the hypothesis on the set E that the set function m^* satisfies the finite additivity property, except possibly when the set function is changed at the $M_{N_k}^{th}$ level. However, the hypothesis on E says that for any interval $I(j, M_{N_k})$ that has non-empty intersection with E , there is **exactly** one interval, $I(jp_{N_k+1} + \alpha_{j,k}, M_{N_k+1})$, that has non-empty intersection with E . Thus,

$$\begin{aligned} \frac{1}{M_{N_k-k+1}} &= m^*(I(j, M_{N_k})) \\ &= \sum_{\alpha=0}^{p_{N_k+1}-1} m^*(I(jp_{N_k+1} + \alpha, M_{N_k+1})) \\ &= m^*(I(jp_{N_k+1} + \alpha_{j,k}, M_{N_k+1})) \\ &= \frac{1}{M_{(N_k+1)-k}}. \end{aligned}$$

m^* is therefore a quasi-measure. There are a couple of other facts about m^* that will also be needed.

Claim. The quasi-measure m^* has three properties.

- (a) $|\Delta m^*(I(j, M_{N_k}))| = \frac{1}{M_{N_k-k+1}}$; if $I(j, M_{N_k}) \cap E \neq \emptyset$
- (b) $|\Delta m^*(I(j, M_n))| = 0$; for all $n \neq N_k$ for some k
- (c) $m^* \equiv 0$; off E .

Proof of Claim. Both (a) and (b) are simple calculations using the definition of m^* , along with the fact that the p_k^{th} roots of unity sum to zero.

(a) For every integer k , if $I(j, M_{N_k}) \cap E \neq \emptyset$, the definition of Δm^* yields

$$\begin{aligned}
 |\Delta m^*(I(j, M_{N_k}))| &= \left| \sum_{\alpha=0}^{p_{N_k+1}-1} e^{\alpha(\frac{2\pi i}{p_{N_k+1}})} m^*(I(jp_{N_k+1} + \alpha, M_{N_k+1})) \right| \\
 (\text{by defn. of } m^*) &= \left| e^{\alpha_{j,k}(\frac{2\pi i}{p_{N_k+1}})} \frac{1}{M_{N_k+1-k}} \right| \\
 &= \frac{1}{M_{N_k+1-k}}.
 \end{aligned}$$

(b) If $I(j, M_n) \cap E = \emptyset$, then (b) is trivially satisfied. Assume there is a nonempty intersection. If integer $n \neq N_i$ for all i , there exists an integer k such that $N_k < n < N_{k+1}$. By definition of m^* ,

$$|\Delta m^*(I(j, M_n))| = \left| \sum_{\alpha=0}^{p_{n+1}-1} e^{\alpha(\frac{2\pi i}{p_{n+1}})} m^*(I(jp_{n+1} + \alpha, M_{n+1})) \right|.$$

Note: Non-empty intersection of the interval $I(j, M_n)$ with the set E implies that each subinterval $I(jp_{n+1} + \alpha, M_{n+1})$ of $I(j, M_n)$ must also have non-empty intersection with E , or else the additivity property of m^* would be violated.

Hence, the above equality can be rewritten as:

$$\begin{aligned}
 |\Delta m^*(I(j, M_n))| &= \left| \sum_{\alpha=0}^{p_{n+1}-1} e^{\alpha(\frac{2\pi i}{p_{n+1}})} \left(\frac{1}{M_{n-k}} \right) \right| \\
 &= \frac{1}{M_{n-k}} \left| \sum_{\alpha=0}^{p_{n+1}-1} e^{\alpha(\frac{2\pi i}{p_{n+1}})} \right| \\
 &= 0, \quad \text{since the sum of all } p_{n+1}^{th} \text{ roots of unity is zero.}
 \end{aligned}$$

(c) By definition of the quasi-measure m^* , $m^*(I(j, M_n)) = 0$ if the interval does not intersect E . $\square$ *Claim.*

Using the Claim, comparisons can be made with the summing condition on the ϵ_k 's. The first equality below is clear by part (b) of the Claim.

$$\begin{aligned} \sum_{n=0}^{\infty} \epsilon_{M_n} M_n \sum_{j=0}^{M_n-1} |\Delta m_S(I(j, M_n))|^2 &= \sum_{k=0}^{\infty} \epsilon_{M_{N_k}} M_{N_k} \sum_{j=0}^{M_{N_k}-1} |\Delta m_S(I(j, M_{N_k}))|^2 \\ &= \sum_{k=0}^{\infty} \epsilon_{M_{N_k}} M_{N_k} \sum_{I(j, M_{N_k}) \cap E \neq \emptyset} |\Delta m_S(I(j, M_{N_k}))|^2 \\ &\text{(by (a) of the Claim)} = \sum_{k=0}^{\infty} \epsilon_{M_{N_k}} M_{N_k} \sum_{I(j, M_{N_k}) \cap E \neq \emptyset} \left(\frac{1}{M_{N_k+1-k}} \right)^2. \end{aligned}$$

It is an obvious fact that $I(j p_{N_k} + \alpha, M_{N_k}) \cap E = \emptyset$ implies $I(j p_{N_k} p_{N_k+1} \cdots p_n + \beta, M_n) \cap E = \emptyset$ for any M_n interval contained in $I(j p_{N_k+1} + \alpha, M_{N_k+1})$, and for all $n \geq N_k$. Together with the hypothesis on the set E , this fact yields that there are at most $M_{N_k-(k-1)}$ intervals $I(j, M_{N_k})$ which have non-empty intersection with E . Therefore, the above sum reduces even further.

$$\begin{aligned} (5) \quad \sum_{k=0}^{\infty} \epsilon_{M_{N_k}} M_{N_k} \sum_{I(j, M_{N_k}) \cap E \neq \emptyset} \left(\frac{1}{M_{N_k+1-k}} \right)^2 &\leq \sum_{k=0}^{\infty} \epsilon_{M_{N_k}} M_{N_k} M_{N_k-k+1} \left(\frac{1}{M_{N_k-k+1}} \right)^2 \\ &= \sum_{k=0}^{\infty} (p_{N_k-k+2} \cdots p_{N_k}) \epsilon_{M_{N_k}}. \end{aligned}$$

Since this last series was assumed to be finite-valued, relationship (4) shows that the Vilenkin series $S^*(x)$ associated with the quasi-measure m^* satisfies the criterion to be a member of the collection $\mathcal{M}_\epsilon$. Furthermore, it is clear by the definition of m^* that S^* is supported on E , and thus $S^*(x) = 0$ for any x not in E . It is easy to see

$$\begin{aligned} S^*(0) &= \int_G \psi_0(x) dm^*(x) \\ &= \int_G 1 dm^*(x) \\ &= m^*(G) \equiv 1. \end{aligned}$$

Hence, S^* is not the zero series. This is a contradiction to the hypothesis that E is a set of uniqueness for $\mathcal{M}_\epsilon$ and thus $\sum_{k=0}^{\infty} \epsilon_k p_{N_k-k+2} \cdots p_{N_k} = \infty$.

Conversely, assume that $\sum_{k=0}^{\infty} (p_{N_k-k+2} \cdots p_{N_k}) \epsilon_{M_{N_k}} = \infty$. It remains to show E is a set of uniqueness for $\mathcal{M}_\epsilon$. It suffices to show, by the definition of set of uniqueness, that if there exists a non-zero Vilenkin series $S(x)$ whose M_n^{th} -partial sums converge pointwise to zero everywhere off E , then S is not a member of the collection $\mathcal{M}_\epsilon$.

Let $S(x) = \sum_{k=0}^{\infty} a_k \psi_k(x)$ be any non-zero Vilenkin series with M_n^{th} -partial sums converging to zero off the set E . Proposition 2.4 gives the existence a quasi-measure m_S associated with S which is supported on the set E and is normalized so that $m_S(G) = 1$. Using a direct inequality argument, similar to the one found in the first direction of this proof, along with the triangle inequality and Hölder's inequality, it will be possible to obtain an inequality which bounds the summing condition of the ϵ_k 's.

$$\begin{aligned} |m(G)| &= \left| \sum_{I(j, M_{N_k}) \cap E \neq \emptyset} m_S(I(j, M_{N_k})) \right| \\ (\text{by finite additivity}) &= \left| \sum_{I(j, M_{N_k}) \cap E \neq \emptyset} \sum_{\alpha=0}^{p_{N_k+1}-1} m_S(I(j p_{N_k+1} + \alpha, M_{N_k+1})) \right|. \end{aligned}$$

Recall that the set E satisfies (2), and thus the above sum can be reduced considerably.

$$\begin{aligned} &= \left| \sum_{I(j, M_{N_k}) \cap E \neq \emptyset} m_S(I(j p_{N_k+1} + \alpha_{j,k}, M_{N_k+1})) \right| \\ (\text{by the triangle inequality}) &\leq \sum_{I(j, M_{N_k}) \cap E \neq \emptyset} |m_S(I(j p_{N_k+1} + \alpha_{j,k}, M_{N_k+1}))|. \end{aligned}$$

Of course, $\left| e^{\alpha_{j,k} (\frac{2\pi i}{p_{N_k+1}})} \right| = 1$ for any value of $\alpha_{j,k}$, so nothing changes by includ-

ing it in the sum.

$$\begin{aligned}
(\text{continued from above}) &= \sum_{I(j, M_{N_k}) \cap E \neq \emptyset} |e^{\alpha_{j,k}(\frac{2\pi i}{p_{N_k+1}})} m_S(I(j p_{N_k+1} + \alpha_{j,k}, M_{N_k+1}))| \\
(\text{by hypothesis on } E) &= \sum_{I(j, M_{N_k}) \cap E \neq \emptyset} |\Delta m_S(I(j, M_{N_k}))| \\
(\text{by Hölder's inequality}) &\leq \left(\sum_{I(j, M_{N_k}) \cap E \neq \emptyset} |\Delta m_S(I(j, M_{N_k}))|^2 \right)^{\frac{1}{2}} \left(\sum_{I(j, M_{N_k}) \cap E \neq \emptyset} 1 \right)^{\frac{1}{2}} \\
(\text{by (5)}) &\leq \left(\sum_{j=0}^{M_{N_k}-1} |\Delta m_S(I(j, M_{N_k}))|^2 \right)^{\frac{1}{2}} \left(\frac{1}{M_{N_k}-k+1} \right)^{\frac{1}{2}}.
\end{aligned}$$

Thus, it is clear that

$$\sum_{j=0}^{M_{N_k}-1} |\Delta m_S(I(j, M_{N_k}))|^2 \geq \frac{1}{M_{N_k}-k+1},$$

which in turn implies

$$(6) \quad M_{N_k} \sum_{j=0}^{M_{N_k}-1} |\Delta m_S(I(j, M_{N_k}))|^2 \geq (p_{N_k-k+2} \cdots p_{N_k}).$$

However, since the ϵ_k 's decrease monotonically to zero, we can write

$$\begin{aligned}
\sum_{k=0}^{\infty} \epsilon_k |a_k|^2 &\geq \sum_{k=0}^{\infty} \epsilon_{M_{N_k+1}} \sum_{n=M_{N_k}}^{M_{N_k+1}-1} |a_n|^2 \\
&= \sum_{k=0}^{\infty} \epsilon_{M_{N_k}} \frac{\epsilon_{M_{N_k+1}}}{\epsilon_{M_{N_k}}} \sum_{n=M_{N_k}}^{M_{N_k+1}-1} |a_n|^2 \\
(\text{by lim inf property of the } \epsilon_k \text{'s}) &\geq \theta \sum_{k=0}^{\infty} \epsilon_{M_{N_k}} \sum_{n=M_{N_k}}^{M_{N_k+1}-1} |a_n|^2 \\
(\text{by Lemma 2.6}) &= \sum_{k=0}^{\infty} \epsilon_{M_{N_k}} M_{N_k} \sum_{j=0}^{M_{N_k}-1} |\Delta m_S(I(j, N_k))|^2 \\
(\text{by (6)}) &\geq \sum_{k=0}^{\infty} (p_{N_k-k+2} \cdots p_{N_k}) \epsilon_{M_{N_k}} \\
&= \infty \quad (\text{by assumption}).
\end{aligned}$$

Thus, any non-zero series S which is zero off E is not in the collection $\mathcal{M}_\epsilon$, implying that E is a set of uniqueness for this collection. $\square$.

The second result of this chapter gives two separate conditions on a set E , each of which is sufficient for a set E to be a set of uniqueness for the collection $\mathcal{M}_\epsilon$, given that the set E is closed and has Haar measure zero.

Theorem 5.2. *Let E be a closed subset of the bounded Vilenkin group G having Haar measure zero. Suppose E satisfies either*

$$(i) \quad \sum_{n=0}^{\infty} \frac{\epsilon_{M_n} - \epsilon_{M_{n+1}}}{|E_n|} = \infty,$$

where the series $\{\epsilon_k\}$ of course, satisfies (1).

or

$$(ii) \quad \limsup_{n \rightarrow \infty} \frac{\epsilon_{M_n}}{|E_n|} = \infty,$$

where $E_n := \bigcup_{x \in E} I_{M_n}(x)$ and $|E_n|$ represents the Haar measure of E_n . Then, E is a set of uniqueness for the collection of Vilenkin series $\mathcal{M}_\epsilon$.

This theorem gives a sufficient condition for a set E to be a set of uniqueness for $\mathcal{M}_\epsilon$ which clearly depends on the sequence $\{\epsilon_k\}$, as one might expect. Thus, Theorem 5.2 generalizes Theorem 4.5 by showing that a closed subset E of G of α -capacity zero is a set of uniqueness for T_α . This can be seen easily because if E is of α -capacity zero, then E is of α -Hausdorff measure zero (Kahane and Salem, "Ensembles parfaits et series trig"). This implies that E satisfies (7) below. If $0 < \alpha < 1$, define the sequence ϵ^* by setting $\epsilon_k^* := \frac{1}{k^{1-\alpha}}$. It is easy to see that $\epsilon_{M_n}^* = (\frac{1}{M_n})^{1-\alpha}$, and that condition (ii) of Theorem 5.2 is satisfied, implying E is a set of uniqueness for $\mathcal{M}_\epsilon^* \equiv T_\alpha$.

Proof of 5.2. Let E be a closed subset of G having Haar measure zero. Define a sequence of non-negative integers $\{\gamma_n\}$ by

$$\gamma_n := \text{the number of } I(j, M_n) \text{ such that } I(j, M_n) \cap E \neq \emptyset.$$

It is clear the sequence $\frac{\gamma_n}{M_n}$ must go to zero as n gets large since $|E| = 0$. (Recall that the total number of intervals $I(j, M_n)$ is M_n .) Because of this, it is clear that

$$(7) \quad \begin{aligned} \lim_{n \rightarrow \infty} |E_n| &= \lim_{n \rightarrow \infty} \sum_{I(j, M_n) \cap E \neq \emptyset} |I(j, M_n)| \\ &= \lim_{n \rightarrow \infty} \frac{\gamma_n}{M_n} = 0. \end{aligned}$$

Suppose $S(x) = \sum_{k=0}^{\infty} a_k \psi_k(x)$ is a non-zero Vilenkin series such that S is zero off the set E . Let m_S be the associated quasi-measure from Proposition 2.4. As in the proof of Theorem 5.1, it suffices to show that any S satisfying this criterion cannot belong to the collection $\mathcal{M}_\epsilon$ if either (i) or (ii) are satisfied.

Let N be any positive integer. It is clear that both the liminf condition on the ϵ_k 's and the fact that they monotonically decrease to zero yield the first inequality below.

$$\begin{aligned} \sum_{k=0}^{M_N-1} \epsilon_k |a_k|^2 &\geq \theta \sum_{n=0}^N \epsilon_{M_n} \sum_{k=M_n}^{M_{n+1}-1} |a_k|^2 \\ &= \theta \sum_{n=0}^N \epsilon_{M_n} \left(\sum_{k=0}^{M_{n+1}-1} |a_k|^2 - \sum_{k=0}^{M_n-1} |a_k|^2 \right) \\ &= \theta \left(\sum_{n=0}^{N-1} \epsilon_{M_n} \sum_{k=0}^{M_{n+1}-1} |a_k|^2 + \epsilon_{M_N} \sum_{k=0}^{M_{N+1}-1} |a_k|^2 - \sum_{n=1}^N \epsilon_{M_n} \sum_{k=0}^{M_n-1} |a_k|^2 - \epsilon_{M_0} |a_0|^2 \right) \\ &= \theta \left(\sum_{n=0}^{N-1} (\epsilon_{M_n} - \epsilon_{M_{n+1}}) \sum_{k=0}^{M_{n+1}-1} |a_k|^2 + \epsilon_{M_N} \sum_{k=0}^{M_{N+1}-1} |a_k|^2 - \epsilon_1 |a_0|^2 \right), \end{aligned}$$

i.e.,

$$(8) \quad \sum_{k=0}^{M_N-1} \epsilon_k |a_k|^2 \geq \theta \left(\sum_{n=0}^{N-1} (\epsilon_{M_n} - \epsilon_{M_{n+1}}) \sum_{k=0}^{M_{n+1}-1} |a_k|^2 + \epsilon_{M_N} \sum_{k=0}^{M_{N+1}-1} |a_k|^2 - \epsilon_1 |a_0|^2 \right).$$

This is one of relationships needed to show $S \notin \mathcal{M}_\epsilon$. Another relationship similar to (8) can be found in much the same way as was done in the second half of the

proof of Theorem 5.1. The first step in the computation below is nothing more than the triangle inequality.

$$\begin{aligned}
|m_S(G)| &\leq \sum_{j=0}^{M_n-1} |m_S(I(j, M_n))| \\
(\text{by Hölder's inequality}) &\leq \left(\sum_{j=0}^{M_n-1} |m_S(I(j, M_n))|^2 \right)^{\frac{1}{2}} \left(\sum_{I(j, M_n) \cap E \neq \emptyset} 1 \right)^{\frac{1}{2}} \\
(\text{by Lemma 2.6}) &= \left(\frac{1}{M_n} \sum_{k=M_n}^{M_{n+1}-1} |a_k|^2 \right)^{\frac{1}{2}} \left(\sum_{I(j, M_n) \cap E \neq \emptyset} 1 \right)^{\frac{1}{2}} \\
(\text{by the defn of the } \gamma_n) &= \left(\frac{1}{M_n} \sum_{k=M_n}^{M_{n+1}-1} |a_k|^2 \right)^{\frac{1}{2}} (\gamma_n)^{\frac{1}{2}} \\
&\leq \left(\frac{1}{M_n} \sum_{k=0}^{M_{n+1}-1} |a_k|^2 \right)^{\frac{1}{2}} (\gamma_n)^{\frac{1}{2}} \\
&= \left(\sum_{k=0}^{M_{n+1}-1} |a_k|^2 \right)^{\frac{1}{2}} \left(\frac{\gamma_n}{M_n} \right)^{\frac{1}{2}} \\
(\text{as in (7)}) &= \left(\sum_{k=0}^{M_{n+1}-1} |a_k|^2 \right)^{\frac{1}{2}} (|E_n|)^{\frac{1}{2}}.
\end{aligned}$$

Since $m_S(G) = 1$, it follows that

$$(9) \quad \frac{1}{|E_n|} \leq \sum_{k=0}^{M_{n+1}-1} |a_k|^2$$

for all $n = 1, 2, \dots$

Using (8) and (9), and remembering that $a_0 = 1$, it is easy to see that

$$\begin{aligned}
\sum_{k=0}^{M_N-1} \epsilon_k |a_k|^2 &\geq \theta \left(\sum_{n=0}^{N-1} (\epsilon_{M_n} - \epsilon_{M_{n+1}}) \sum_{k=0}^{M_{n+1}-1} |a_k|^2 + \epsilon_{M_N} \sum_{k=0}^{M_{N+1}-1} |a_k|^2 - \epsilon_1 \right) \\
(10) \quad &\geq \theta \left(\sum_{n=0}^{N-1} \frac{\epsilon_{M_n} - \epsilon_{M_{n+1}}}{|E_n|} + \frac{\epsilon_{M_N}}{|E_N|} - \epsilon_1 \right).
\end{aligned}$$

Hence, if either (i) or (ii) hold, then (10) implies $\sum_{k=0}^{\infty} \epsilon_k |a_k|^2 = \infty$. Therefore, S is not in the collection $\mathcal{M}_\epsilon$, which implies E is a set of uniqueness for $\mathcal{M}_\epsilon$. $\square$

CHAPTER 6 : THE COLLECTION $\mathcal{U}_\epsilon^+$

The past two chapters have dealt with sets of α -capacity zero, and shown them to be sets of uniqueness for small collections of Vilenkin series. This chapter will bring many of the same ideas into play, but in a more general setting. Beginning with the definition of a generalized capacity, it is shown that a subset E of a bounded Vilenkin group G is a set of uniqueness for a collection of Vilenkin series containing the collection $\mathcal{T}_G^+$ from Chapter 4 if and only if E is of capacity zero. The generalized capacity introduced below is adapted from Yoneda (1988).

Let $\bar{\epsilon} = \{\epsilon_n\}$ be a sequence of numbers such that:

$$(1) \quad 0 < \epsilon_n < 1 \text{ and } \{\epsilon_n\} \searrow 0 \quad \text{monotonically as } n \rightarrow \infty.$$

Define the function α from the group G to the complex numbers as

$$\alpha(x) = 1 + \sum_{n=0}^{\infty} \epsilon_n \sum_{k=M_n}^{M_{n+1}-1} \psi_k(x).$$

This function is well-defined since G is of bounded type, and

$$(2) \quad \sum_{k=M_n}^{M_{n+1}-1} \psi_k(x) = \begin{cases} (p_{n+1} - 1)M_n & \text{if } x \in I_{M_{n+1}}(0) \\ -M_n & \text{if } x \in I_{M_n}(0) \setminus I_{M_{n+1}}(0) \\ 0 & \text{otherwise} \end{cases}$$

for every $n = 0, 1, \dots$. Because the above sum is equal to zero off $I(0, M_n)$ for each n , it is clear that the sum in $\alpha(x)$ converges uniformly on $G \setminus I(0, M_n)$ for any n . It is also clear that $\alpha(x)$ is well-defined and integrable. There are other important properties which the function $\alpha(x)$ possesses.

Claim. $\alpha(x)$ satisfies:

- i. $\int_0^1 \alpha(x) dx = 1.$
- ii. $\alpha(x)$ is constant on each $I(1, M_n),$
- iii. $\alpha(x) = \sum_{n=0}^{\infty} \alpha_n \chi_{I(0, M_n)}(x)$
for some sequence α_n of non-negative integers such that $\sum_{n=0}^{\infty} \frac{\alpha_n}{M_n} = 1.$
- iv. $|\alpha(x)| \leq |\alpha(y)|$ if $x > y.$

Proof of Claim. (i) By definition,

$$\int_0^1 \alpha(x) dx = 1 + \int_G \sum_{n=0}^{\infty} \epsilon_n \sum_{k=M_n}^{M_{n+1}-1} \psi_k(x) dx$$

where dx will represent Haar measure on G and Lebesgue measure on the interval. Clearly, all that remains to be shown is that the above integral over G is zero. Note that Fatou's Lemma implies

$$\begin{aligned} \int_G \sum_{n=0}^{\infty} \epsilon_n \sum_{k=M_n}^{M_{n+1}-1} \psi_k(x) dx &= \int_G \lim_{N \rightarrow \infty} \sum_{n=0}^N \epsilon_n \sum_{k=M_n}^{M_{n+1}-1} \psi_k(x) dx \\ &\leq \liminf_{N \rightarrow \infty} \sum_{n=0}^N \epsilon_n \int_G \sum_{k=M_n}^{M_{n+1}-1} \psi_k(x) dx. \end{aligned}$$

As has been mentioned in previous sections, the preset oscillation of the values of the characters of G means that $\int_G \psi_k(x) dx = 0$ for every $k \neq 0$. Therefore, the sum over n only gets a non-zero contribution when $k = 0$. However, since $M_0 = 1$, k is never equal to zero in the character sum, and thus, the entire integral must be equal to zero. (i) is proven.

(ii) This clearly follows from (2), as $I(0, M_n)$ and $I(1, M_n)$ are disjoint for each n .

(iii) This is a simple calculation. Let $\{\alpha_n\}$ be any sequence of non-negative integers satisfying $\sum_{n=0}^{\infty} \frac{\alpha_n}{M_n} = 1$. Because the series $\sum_{n=0}^{\infty} \epsilon_n \sum_{k=M_n}^{M_{n+1}-1} \psi_k(x)$ is uniformly convergent off $I_{M_n}(0)$ for all n , the integral and infinite sum are interchangeable in the following calculation.

$$\begin{aligned} \int_G \sum_{k=0}^{\infty} \alpha_n \chi_{I_{M_n}(0)}(x) dx &= \sum_{k=0}^{\infty} \int_G \alpha_n \chi_{I_{M_n}(0)}(x) dx \\ &= \sum_{k=0}^{\infty} \int_{I_{M_n}(0)} \alpha_n dx \\ &= \sum_{k=0}^{\infty} \alpha_n \frac{1}{M_n} \\ &\equiv 1 \end{aligned}$$

by definition of the $\{\alpha_n\}$. But part (i) of the claim implies

$$\int_G \alpha(x \dot{+} t) dx = \int_G \sum_{k=0}^{\infty} \alpha_n(x) \chi_{I_{M_n}(0)}(x) dx \quad \text{a.e.}$$

with respect to the Haar measure on G . Recalling that the characters of G are continuous on each interval $I_{M_n}(x)$, this implies that (iii) holds everywhere on G .

(iv) This clearly follows from (iii). $\square$ Claim.

Because of the claim, it is possible to define the generalized *potential function* U_S , where $S(x) = \sum_{k=0}^{\infty} a_k \psi_k(x)$ is a Vilenkin series concentrated on E with associated quasi-measure m_S , from Proposition 2.4.

$$U_S := \int_0^1 \alpha(x \dot{+} t) dm_S(t).$$

Part (iii) of the claim implies that the above integral is actually

$$\begin{aligned} U_S &= \int_0^1 \sum_{n=0}^{\infty} \alpha_n \chi_{I(0, M_n)}(x \dot{+} t) dm_S(t) \\ (3) \quad &= \sum_{n=0}^{\infty} \alpha_n m_S(I_{M_n}(x)). \end{aligned}$$

Hence, define the *energy integral* of S by

$$(4) \quad I_S := \int_0^1 U_S(x) dm_S(x).$$

Lemma 2.6, part (iii) of the Claim and (3) show I_S can be rewritten as

$$\begin{aligned} I_S &= \sum_{n=0}^{\infty} \alpha_n \sum_{j=0}^{M_n-1} |m_S(I(j, M_n))|^2 \\ &= \sum_{n=0}^{\infty} \sum_{k=M_n}^{M_{n+1}-1} |a_k|^2. \end{aligned}$$

Let $C(E)$ be the number which satisfies

$$\frac{1}{C(E)} = \inf \left\{ \|U_S\|_{\infty} : S(x) = \sum_{k=0}^{\infty} a_k \psi_k(x) \text{ and } S \equiv 0 \text{ off } E \right\}.$$

Define the ϵ -capacity of a set E as the number $C(E)$. The following theorem uses this definition of capacity to characterize sets of uniqueness for a collection of Vilenkin series which contains the collection $\mathcal{T}_{\alpha}^+$ from Chapter 4.

Theorem 6.1. *A closed set E in G is a set of uniqueness for the collection of Vilenkin series $\mathcal{U}_{\epsilon}^+$ if and only if the ϵ -capacity of E is zero. The collection $\mathcal{U}_{\epsilon}^+$ is given by*

$$\mathcal{U}_{\epsilon}^+ = \left\{ S(x) = \sum_{k=0}^{\infty} a_k \psi_k(x) : \sum_{n=0}^{\infty} \epsilon_n \sum_{k=M_n}^{M_{n+1}-1} |a_k|^2 < \infty \text{ and } S(x) \geq 0 \text{ a.e.} \right\}$$

where the ϵ_k 's satisfy (1).

The proof of this theorem follows directly from seven lemmas, four of which are routine extensions of Lemmas 4.1 through 4.4 to the notion of ϵ -capacity, and three others which are stated below, these three following directly from results of Kahane and Salem (1963, p. 31-37).

Lemma 6.2. *Let E be a closed subset of G . Then*

(a) $C(E) > 0$ if and only if there exists a Vilenkin series S concentrated on E such that $I_S < \infty$

and

$$(b) \quad I(E) := \frac{1}{C(E)} = \inf \{I_S : S \equiv 0 \text{ off } E\}$$

Lemma 6.3. *If $I(E) < \infty$, there exists a unique quasi-measure m_S^* such that the associated Vilenkin series S^* is zero off E , and $I_S^* = I(E)$.*

Lemma 6.4. *A nonnegative quasi-measure m_S associated with the Vilenkin series S that satisfies $I_S < \infty$ is such that, for any closed subset E of G ,*

$$m_S(\{x \in E : U_S^*(x) < I(E)\}) = \lim_{n \rightarrow \infty} m_S(\cup I_{M_n}(x) : x \in E, U_S^*(x) < I(E)) = 0.$$

Now proceed with the proof of Theorem 6.1.

Proof. Necessity. Assume that the set E is not of ϵ -capacity zero. Then, by Lemmas 4.1 and 6.3, and Proposition 2.4, there exists a quasi-measure m_S^* supported on E (with associated Vilenkin series $S^* \sim \sum a_k \psi_k$) satisfying $\|U_{S^*}\|_\infty < \infty$ and $I_{S^*} = I(E)$, such that S^* is not the zero series (since $U_{S^*} > 0$ a.e. on E). Since m_S^* is supported on E , $S^*(x) \equiv 0$ off the set E . Moreover, since $I(E) = I_{S^*} < \infty$, it is clear by (4) that

$$\begin{aligned} \infty > I_{S^*} &= \sum_{n=0}^{\infty} \alpha_n \sum_{j=0}^{M_n-1} |m_S^*(I(j, M_n))|^2 \\ &= 1 + \sum_{n=0}^{\infty} \epsilon_n \sum_{M_n}^{M_{n+1}-1} |a_k|^2. \end{aligned}$$

Part (iii) of the Claim and (3) imply $S(x) \geq 0$ a.e. since U_S is. Therefore, S^* is a non-zero Vilenkin series in the collection $\mathcal{U}_\epsilon^+$ which is zero off E , contradicting the hypothesis that E is a set of uniqueness for $\mathcal{U}_\epsilon^+$.

Sufficiency. Let E be a closed set of ϵ -capacity zero. Let $S(x) = \sum_{k=0}^{\infty} a_k \psi_k(x)$ be a Vilenkin series in the collection $\mathcal{U}_\epsilon^+$ which is zero off E . We must show that S is the zero series. From the same argument found at the end of Chapter 4, it suffices to show $a_0 = 0$.

Define $E_N = \cup \{I_N(x) : x \in E\}$ for every positive integer N . Since E is closed, it is easy to see that $E = \cap_N E_N$. Because of this, and the fact that S is zero off the set E , there exists a positive integer n_0 such that

$$(6) \quad \sum_{k=0}^{M_n-1} a_k \psi_k(x) = 0 \quad \text{off } E_N$$

for all $n > n_0$. This $n_0(N)$ exists for any N .

By the definition of $\mathcal{U}_\epsilon^+$, $a_0 \geq 0$. But by definition of the quasi-measure m_S , it is clear that

$$\begin{aligned} a_0 &= |a_0| = \int_G \psi_0(x) dm_S(x) \\ &= \int_G 1 dm_S(x) \\ &= \sum_{j=0}^{M_n-1} \int_{I_{M_n,j}} 1 dm_S(x), \end{aligned}$$

where $I_{M_n,j}$ is the interval $I_{M_n}(x)$, with $x = \frac{j}{M_n}$ for each $j = 0, 1, \dots, M_n - 1$. The above holds for any positive integer n . The definition of quasi-measure yields

$$m_S(I_{M_n}(x)) = \frac{1}{M_n} \sum_{k=0}^{M_n-1} a_k \psi_k(x),$$

but because the series $S(x)$ is zero off the set E , the above representation of a_0 can be further analyzed. Let N be a positive integer. Then,

$$\begin{aligned} \sum_{k=0}^{M_n-1} \sum_{j=0}^{M_n-1} \frac{1}{M_n} \int_{I_{M_n,j}} a_k \psi_k(x) &= \frac{1}{M_n} \int_E \sum_{k=0}^{M_n-1} a_k \psi_k(x) dx \\ (\text{by (6)}) &= \frac{1}{M_n} \int_{E_N} \sum_{k=0}^{M_n-1} a_k \psi_k(x) dx \end{aligned}$$

for any $n > n_0(N)$.

By Lemma 4.1, there exists a Vilenkin series S^* which is zero off E , and an associated quasi-measure m_S^* , such that for any N ,

$$(7) \quad U_{S^*}(x) \geq W_\epsilon(E_N) := \inf \{ \|U_S\|_\infty : S \text{ is a Vilenkin series concentrated on } E_N \}$$

almost everywhere $[dx]$ on E_N . Thus,

$$\begin{aligned} |a_0| &= \frac{1}{M_n} \int_{E_N} \sum_{k=0}^{M_n-1} a_k \psi_k(x) dx \\ (\text{by (7)}) &\leq \frac{1}{M_n} \int_{E_N} \frac{U_{S^*}(x)}{W_\epsilon(E_N)} \sum_{k=0}^{M_n-1} a_k \psi_k(x) dx \\ &= \frac{1}{M_n W_\epsilon(E_N)} \int_{E_N} \sum_{k=0}^{M_n-1} a_k \psi_k(x) U_{S^*}(x) dx \\ (\text{by defn. of q.msr.}) &= \frac{1}{M_n W_\epsilon(E_N)} \int_{E_N} M_n m_S(I_{M_n}(x)) U_{S^*}(x) dx \\ (\text{by } m_S \text{ finite}) &\leq \frac{1}{W_\epsilon(E_N)} \int_{E_N} C U_{S^*}(x) dx \\ (\text{by defn. of } U_{S^*}) &= \frac{C}{W_\epsilon(E_N)} \int_{E_N} \int_G \alpha(x+t) dm_{S^*}(x) dx \\ (\text{by Fubini}) &= \frac{C}{W_\epsilon(E_N)} \int_G \left(\int_{E_N} \alpha(x+t) dx \right) dm_{S^*}(x) \\ (\text{by trans.invariance of } dx) &= \frac{C}{W_\epsilon(E_N)} \int_G \left(\int_{E_N} \alpha(x) dx \right) dm_{S^*}(x) \\ (\text{by Claim. part } i \text{ and } \alpha \geq 0) &\leq \frac{C}{W_\epsilon(E_N)} \int_G 1 dm_{S^*}(x) \\ (\text{by } m_{S^*}(G) \equiv 1) &= \frac{C}{W_\epsilon(E_N)} \end{aligned}$$

for all N .

Note: the constant C does **not** depend on either n or N because the quasi-measure m_S is finite-valued on G .

By Lemma 4.2, it is clear that

$$\lim_{N \rightarrow \infty} \frac{C}{W_\epsilon(E_N)} = 0,$$

implying $a_0 = 0$. Following the inductive argument at the end of Chapter 4 will show E is a set of uniqueness for $\mathcal{U}_\epsilon^+$. $\square$

CHAPTER 7 : THE COLLECTION $\mathcal{B}$

In this chapter sets of uniqueness are examined with respect to a smaller collection of Vilenkin series. It will be shown that any subset of the group G is a set of uniqueness for this collection as long as it intersects the intervals $I(j, M_n)$ in a certain pattern. It will also be shown that there is a set of multiplicity of Haar measure zero for this collection of Vilenkin series, showing that the growth condition on the coefficients of series in this collection is not too strong. Finally, a remarkable fact (for sets of uniqueness) is proven about this particular collection of series. As has been mentioned, a set of uniqueness is not only dependent upon the *thinness* of the set (in a measure-theoretic sense), but also on the “dispersal” and relative location of the set throughout the interval. However, the last result of this section shows the existence of a measure-preserving transform from the group G onto itself, under which the uniqueness property of any set E in the group is preserved.

Theorem 7.1. *Let G be a Vilenkin group of bounded type and $\{N_k\}$ be any unbounded, strictly increasing sequence of integers. If $E \subseteq G$ satisfies*

$$E \cap I(jp_{N_k+1} + \alpha_{j,k}, M_{N_k+1}) \neq \emptyset$$

for only one $\alpha_{j,k} = 0, 1, \dots, p_{N_k+1} - 1$ (i.e. $E \cap I(jp_{N_k+1} + \gamma, M_{N_k+1}) = \emptyset$ for all $\gamma \neq \alpha_{j,k}$), then E is a set of uniqueness for the collection

$$\mathcal{B} = \left\{ S(x) = \sum_{k=1}^{\infty} a_k \psi_k(x) : \sum_{k=M_n}^{M_{n+1}-1} |a_k|^2 = o(1) \text{ as } n \rightarrow \infty \right\}.$$

The set $\mathcal{B}$ has some basis in the classical literature (of the dyadic case). Define Γ_1 to be the set of all Walsh series whose coefficients a_n go to zero as $n \rightarrow \infty$. It has been shown that E is a set of uniqueness for Γ_1 if and only if E is a U-set. Also, if the set Γ_2 is defined to be the set of all Walsh series whose coefficients are ℓ^2 summable, then it is known that E is a set of uniqueness for Γ_2 if and only if $|E| = 0$. Both of these results can be easily extended to the bounded Vilenkin case, and thus the inclusions $\Gamma_2 \subset \mathcal{B} \subset \Gamma_1$ hold. From this, the trivial observations can be made that any set of positive measure is a set of multiplicity for $\mathcal{B}$ and any set of uniqueness for Γ_1 is also one for $\mathcal{B}$.

Proof of Theorem 7.1. Let m_S be a quasi-measure associated with some bounded Vilenkin character series $S(x) = \sum_{k=1}^{\infty} a_k \psi_k(x)$ such that $m_S(G) = 1$ and $m_S(I) = 0$ for all intervals $I(j, M_n)$ such that $I \cap E = \emptyset$ (i.e. m_S is a quasi-measure such that the associated S is not the zero series, but is zero off the set E). Thus, by definition of Δm_S ,

$$\sum_{j=0}^{M_{N_k}-1} |\Delta m_S(I(j, M_{N_k}))|^2 = \sum_{j=0}^{M_{N_k}-1} \left| \sum_{\ell=0}^{p_{N_k+1}-1} e^{\frac{2\pi i \ell}{p_{N_k+1}}} m_S(I(j p_{N_k+1} + \ell, M_{N_k+1})) \right|^2$$

Now, by the hypothesis on the set E , the above sum can be simplified considerably.

$$\begin{aligned} &= \sum_{j=0}^{M_{N_k}-1} \left| e^{\frac{2\pi i \alpha_{j,k}}{p_{N_k+1}}} m_S(I(j p_{N_k+1} + \alpha_{j,k}, M_{N_k+1})) \right|^2 \\ &\left(\text{by } \left| e^{\frac{2\pi i \alpha_{j,k}}{p_{N_k+1}}} \right| = 1 \right) = \sum_{j=0}^{M_{N_k}-1} |m_S(I(j p_{N_k+1} + \alpha_{j,k}, M_{N_k+1}))|^2 \\ &(\text{by hypothesis again}) = \sum_{j=0}^{M_{N_k}-1} \left| \sum_{\ell=0}^{p_{N_k+1}-1} m_S(I(j p_{N_k+1} + \ell, M_{N_k+1})) \right|^2 \\ &(\text{by } m_S \text{ a quasi-measure}) = \sum_{j=0}^{M_{N_k}-1} |m_S(I(j, M_{N_k}))|^2. \end{aligned}$$

Using the fact that $|z|^2 = z\bar{z}$, the calculation continues:

$$\begin{aligned}
&= \sum_{j=0}^{M_{N_k}-1} m_S(I(j, M_{N_k})) \overline{m_S(I(j, M_{N_k}))} \\
&\text{(by definition of } m_S) = \sum_{j=0}^{M_{N_k}-1} \left(\frac{1}{M_{N_k}} \right)^2 \sum_{\ell=0}^{M_{N_k}-1} a_\ell \psi_\ell(j/M_{N_k}) \sum_{m=0}^{M_{N_k}-1} \overline{a_m \psi_m(j/M_{N_k})} \\
&= \left(\frac{1}{M_{N_k}} \right)^2 \sum_{j=0}^{M_{N_k}-1} \sum_{\ell=0}^{M_{N_k}-1} \sum_{m=0}^{M_{N_k}-1} a_\ell \overline{a_m} \psi_\ell(j/M_{N_k}) \overline{\psi_m(j/M_{N_k})} \\
&= \left(\frac{1}{M_{N_k}} \right)^2 \sum_{\ell=0}^{M_{N_k}-1} \sum_{m=0}^{M_{N_k}-1} a_\ell \overline{a_m} \sum_{j=0}^{M_{N_k}-1} \psi_\ell(j/M_{N_k}) \overline{\psi_m(j/M_{N_k})} \\
&\text{(by } \psi_\ell \perp \bar{\psi}_m) = \left(\frac{1}{M_{N_k}} \right)^2 \sum_{\ell=0}^{M_{N_k}-1} |a_\ell|^2 |\psi_\ell(j/M_{N_k})|^2 M_{N_k} \\
&\text{(by } |\psi_\ell| \equiv 1) = \frac{1}{M_{N_k}} \sum_{\ell=0}^{M_{N_k}-1} |a_\ell|^2 \\
&= \frac{1}{M_{N_k}} \sum_{\ell=0}^{M_{N_k}-1} \left| \int_G \psi_\ell(x) dm_S(x) \right|^2.
\end{aligned}$$

Recalling that $\int_G \psi_0(x) dm_S(x) = 1$, it is easy to see

$$\frac{1}{M_{N_k}} \sum_{\ell=0}^{M_{N_k}-1} \left| \int_G \psi_\ell(x) dm_S(x) \right|^2 \geq \frac{1}{M_{N_k}}.$$

Thus, by Lemma 2.6:

$$\begin{aligned}
\sum_{i=M_{N_k}}^{M_{N_k+1}-1} |a_i|^2 &= M_{N_k} \sum_{j=0}^{M_{N_k}-1} |\Delta m_S(I(j, M_{N_k}))|^2 \\
&\geq M_{N_k} \left(\frac{1}{M_{N_k}} \right) = 1.
\end{aligned}$$

Hence, if any $S \equiv 0$ on E and $S \neq 0$, then $S \notin \mathcal{B}$, implying E is a set of uniqueness for $\mathcal{B}$. $\square$

It is now clear that even rather thick sets E of measure zero are sets of uniqueness for this small class of Vilenkin series. Is this a trivial case, therefore?

In other words, is every set of measure zero a set of uniqueness for $\mathcal{B}$? The following result shows that the answer to this question is no.

Theorem 7.2. *There exists a set of multiplicity of measure zero for $\mathcal{B}$.*

As it turns out, the set of multiplicity which shall be constructed is a variation of Cantor's middle p_n th set, where the deletion of the appropriate middle section is not done at every stage, but only at specified stages. (Another difference is that this construction removes half-open intervals instead of open intervals). Below is the definition of the set of multiplicity in the general bounded Vilenkin setting.

Let $\{N_k\}$ be the sequence of integers defined by $N_k = 1 + 2 + \cdots + k$. Define the set of multiplicity by $E = \bigcap_k E_k$, where the E_k are recursively defined as follows:

$$E_1 = \left[0, \frac{[p_1]}{p_1}\right) \cup \left[\frac{[p_1]}{p_1} + 1, 1\right)$$

$$E_k = E_{k-1} \setminus \bigcup_{j=0}^{(M_{N_{k-1}})-1} \left[\frac{jp_{N_k} + [\frac{p_{N_k}}{2}]}{M_{N_k}}, \frac{jp_{N_k} + [\frac{p_{N_k}}{2}] + 1}{M_{N_k}} \right)$$

where $[\frac{p_k}{2}] =$ the greatest integer less than or equal to $\frac{p_k}{2}$, unless $p_k = 2$ for some k . The definition above necessitates that we define $[1] = 0$. Since the $\{p_k\}$ are prime, this is the only possible problem. It is easy to calculate the measure of each set E_k .

$$|E_1| = 1 - \frac{1}{p_1} = \frac{p_1 - 1}{p_1}.$$

$$|E_2| = \frac{p_1 - 1}{p_1} - \left(\sum_{j=0}^{(M_{N_2-1})-1} \frac{1}{M_{N_2}} \right) + \frac{p_2}{M_3}$$

$$= \frac{(p_1 - 1)p_2p_3 - p_1p_2 + p_2}{M_3} = \frac{(p_1 - 1)p_2(p_3 - 1)}{M_3}$$

$$\begin{aligned}
&= \frac{(p_1 - 1)(p_3 - 1)}{p_1 p_3} \\
&\vdots \\
|E_k| &= \dots = \frac{(p_{N_1} - 1)(p_{N_2} - 1) \cdots (p_{N_k} - 1)}{p_{N_1} p_{N_2} \cdots p_{N_k}}.
\end{aligned}$$

Since the sequence $\{p_n\}$ is bounded above, it is clear that

$$\begin{aligned}
|E_k| &= \frac{(p_{N_1} - 1)(p_{N_2} - 1) \cdots (p_{N_k} - 1)}{p_{N_1} p_{N_2} \cdots p_{N_k}} \\
&\leq \max \left\{ \frac{p_{N_j} - 1}{p_{N_j}} \right\}^k \\
&\leq c^k
\end{aligned}$$

where c is some constant strictly less than 1. Therefore, $|E_k| \rightarrow 0$ as $k \rightarrow \infty$, implying that $|E| = 0$. It remains to show that E is a set of multiplicity for the collection $\mathcal{B}$.

Each E_k is such that

$$|E_k| = \frac{(p_{N_1} - 1)(p_{N_2} - 1) \cdots (p_{N_k} - 1)}{p_{N_1} p_{N_2} \cdots p_{N_k}}$$

and each interval $I_{M_{N_k}}(x)$ taken away from the set E_{k-1} in the construction of the set E_k has measure $\frac{1}{M_{N_k}}$. Thus, it is possible to find the number of intervals $I_{M_{N_k}}$ removed at each stage with a simple calculation.

$$\begin{aligned}
|E_k| - |E_{k-1}| &= \frac{(p_{N_1} - 1)(p_{N_2} - 1) \cdots (p_{N_k} - 1)}{p_{N_1} p_{N_2} \cdots p_{N_k}} - \frac{(p_{N_1} - 1)(p_{N_2} - 1) \cdots (p_{N_{k-1}} - 1)}{p_{N_1} p_{N_2} \cdots p_{N_{k-1}}} \\
&= \frac{(p_{N_1} - 1)(p_{N_2} - 1) \cdots (p_{N_k} - 1) - (p_{N_1} - 1)(p_{N_2} - 1) \cdots (p_{N_{k-1}} - 1)(p_{N_k})}{p_{N_1} p_{N_2} \cdots p_{N_k}} \\
&= \frac{(p_{N_1} - 1)(p_{N_2} - 1) \cdots (p_{N_{k-1}} - 1)(p_{N_k} - 1 - p_{N_k})}{p_{N_1} p_{N_2} \cdots p_{N_k}} \\
&= \frac{(p_{N_1} - 1)(p_{N_2} - 1) \cdots (p_{N_{k-1}} - 1)}{p_{N_1} p_{N_2} \cdots p_{N_k}} (-1).
\end{aligned}$$

Let $\eta(N_k)$ represent the number of intervals $I_{M_{N_k}}$ taken away from E_{k-1} to construct E_k . Then, $\eta(N_k)$ satisfies

$$\frac{\eta(N_k)}{M_{N_k}} = \frac{(p_{N_1} - 1)(p_{N_2} - 1) \cdots (p_{N_{k-1}} - 1)}{p_{N_1} p_{N_2} \cdots p_{N_k}}.$$

Define the set function m^* on the intervals of G by

$$m^*(I(j, M_n)) = \begin{cases} \frac{1}{\eta(N_k)} ; & \text{if } I(j, M_n) \cap E \neq \emptyset \\ 0 ; & \text{otherwise.} \end{cases}$$

Clearly, by definition of $\eta(N_k)$, m^* is finitely additive, and is therefore a quasi-measure which is zero off the set E . Let $S^*(x) = \sum_{k=0}^{\infty} a_k c_k(x)$ be the Vilenkin series associated with m^* . It is clear by the definition of m^* that S is zero off E , but is not the zero series. Thus, it remains to show that $S \in \mathcal{B}$.

For any positive integer n , there exists a k such that $N_k \leq k < N_{k+1}$. Thus, Lemma 2.6 shows

$$\begin{aligned} \sum_{k=M_n}^{M_{n+1}-1} |a_k|^2 &= M_n \sum_{j=0}^{M_n-1} |\Delta m^*(I(j, M_n))|^2 \\ (\text{by defn. of } \Delta m^*) &= M_n \sum_{j=0}^{M_n-1} \left| \sum_{\alpha=0}^{p_{n+1}-1} e\left(\frac{2\pi i \alpha}{p_{n+1}}\right) m^*(I(j p_{n+1} + \alpha, M_{n+1})) \right|^2. \end{aligned}$$

Because m^* is finitely-additive, it is possible to rewrite each $m^*(I(j p_{n+1} + \alpha, M_{n+1}))$ as a sum of $m^*(I(\ell, M_{n+2}))$ over appropriate ℓ , and then as a sum of $m^*(I(m, M_{n+3}))$ over appropriate m , and so forth, until the intervals $I_{M_{k+1}}(x)$ are reached. Let the parameter $\gamma := \gamma(k, n)$ represent the number of times the above process is carried out before reaching the $I_{M_{k+1}}(x)$ 'level'. In other words,

$\gamma = N_{k+1} - n$. Thus,

$$\begin{aligned}
& \sum_{k=M_n}^{M_{n+1}-1} |a_k| \\
&= M_n \sum_{j=0}^{M_n-1} \left| \sum_{\alpha=0}^{p_{n+1}-1} e^{\left(\frac{2\pi i \alpha}{p_{n+1}}\right)} m^*(I(jp_{n+1} + \alpha, M_{n+1})) \right|^2 \\
&= M_n \sum_{j=0}^{M_n-1} \left| \sum_{\alpha=0}^{p_{n+1}-1} e^{\left(\frac{2\pi i \alpha}{p_{n+1}}\right)} \sum_{\ell=0}^{(p_{n+1} \cdots p_{N_{k+1}})-1} m^*(I(j(p_n \cdots p_{N_{k+1}}) + \ell, M_{N_{k+1}})) \right|^2 \\
&= M_n \sum_{j=0}^{M_n-1} \left| \sum_{\alpha=0}^{p_{n+1}-1} e^{\left(\frac{2\pi i \alpha}{p_{n+1}}\right)} \frac{p_{n+1} \cdots p_{N_{k+1}}}{\eta(N_{k+1})} \right|^2 \\
&= M_n \sum_{j=0}^{M_n-1} \left(\frac{p_n \cdots p_{N_{k+1}}}{\eta(N_{k+1})} \right)^2 \left| \sum_{\alpha=0}^{p_{n+1}-1} e^{\left(\frac{2\pi i \alpha}{p_{n+1}}\right)} \right|^2 \\
&= 0,
\end{aligned}$$

since the sum of the p_{n+1}^{th} roots of unity is zero. Therefore, the coefficients of the series S^* satisfies the growth condition of the collection $\mathcal{B}$. Hence, the set E is a set of multiplicity of $\mathcal{B}$ of Haar measure zero. $\square$.

The final result of this section shows the existence of a measure-preserving transform which takes sets of uniqueness for $\mathcal{B}$ to sets of uniqueness for $\mathcal{B}$.

Theorem 7.3. *Let G be a Vilenkin group of bounded type. Let φ be any measure preserving transform on G defined as follows:*

For $x = (t_1, t_2, \dots) \in G$, define the function $\varphi(x) := (\epsilon_1(t_1), \epsilon_2(t_2), \dots)$, where $\epsilon_k(t_k)$ is equal to one of the numbers $0, 1, \dots, p_k - 1$ for each k and $\epsilon_k(t_k + s) = \epsilon_k(t_k) + s\alpha_k$, for $s = 1, 2, \dots, p_n - 1$ where α_k is relatively prime to p_k . (This last hypothesis ensures φ is onto.) Then:

E is a set of uniqueness for $\mathcal{B}$ if and only if $\varphi(E) = \{\varphi(x) : x \in E\}$ is a set of uniqueness for $\mathcal{B}$.

Note: φ defined in this manner actually represents a class of measure preserving functions, each of which will satisfy the theorem.

Proof. Assume E is an set of multiplicity for $\mathcal{B}$. Then, by Proposition 2.4, any $S \in \mathcal{B}$ has a corresponding quasi-measure m_S such that $m_S(G) = 1$ and $m_S(I) = 0$ for all $I(j, M_n)$ such that $I \cap E = \emptyset$.

Claim. φ is a one to one function on G which takes M_n -adic intervals to M_n -adic intervals.

Proof of Claim. First recall that the group G is totally disconnected. Thus, by the definition of φ , it is clearly one to one on the entire bounded Vilenkin group G . Also, if $x \in I(j, M_n)$, then $\sum_{k=1}^n \frac{t_k}{M_k} = \frac{j}{M_n}$. Thus, it is clear that φ will take each interval $I(j, M_n)$ to some other interval $I(j', M_n)$, as each element of $I(j, M_n)$ has the same first n -elements. $\square$ *Claim.*

Let m_{S^*} be the quasi-measure defined by $m_{S^*}(I) = m_S(\varphi^{-1}(I))$, where $\varphi^{-1}(I) = \{y : \varphi(y) \in I\}$. (Note: m_{S^*} is clearly a quasi-measure, since φ is measure-preserving and φ is one to one). By the Claim, $\{\varphi(x) : x \in I(j, M_n)\} = I(j', M_n)$ for some $j' = 0, 1, \dots, M_n - 1$ and for every $n \in \mathcal{N}$. In fact, more structure on the intervals is preserved by φ than the Claim states. Not only does φ map M_n -adic intervals to one another (as in the Claim), but because of the additive properties of quasi-measures, φ actually takes each sub-interval $I(\ell, M_{n+1})$ of $I(j, M_n)$ to a unique sub-interval of the $I(j', M_n)$ for every sub-interval of $I(j, M_n)$. (Here, of course, $I(j', M_n) = \varphi(I(j, M_n))$.) Thus, it is clear that for all $j = 0, 1, \dots, M_n - 1$:

$$\varphi(I(jp_{n+1} + \ell, M_{n+1})) = I(j'p_{n+1} + \eta_\ell, M_{n+1})$$

where $\eta_\ell = 0, 1, \dots, p_n - 1$ and $\eta_\ell \neq \eta_{\ell'}$ for all $\ell \neq \ell'$. Now, by the definition of Δm_S^* ,

$$|\Delta m_S^*(I(j, M_n))| = \left| \sum_{\ell=0}^{p_{n+1}-1} e^{\frac{2\pi i \ell}{p_{n+1}}} m_S^*(I(jp_{n+1} + \ell, M_{n+1})) \right|.$$

Because of the definition of S^* and the fact that φ^{-1} takes the sub-intervals of one M_n -adic interval to the sub-intervals of another M_n -adic interval, continue the calculation to see:

$$\begin{aligned} \left| \sum_{\ell=0}^{p_{n+1}-1} e^{\frac{2\pi i \ell}{p_{n+1}}} m_S^*(I(j p_{n+1} + \ell, M_{n+1})) \right| &= \left| \sum_{\ell=0}^{p_{n+1}-1} e^{\frac{2\pi i \ell}{p_{n+1}}} m_S(\varphi^{-1}(I(j p_{n+1} + \ell, M_{n+1}))) \right| \\ &= \left| \sum_{\ell=0}^{p_{n+1}-1} e^{\frac{2\pi i \ell}{p_{n+1}}} m_S(I(j' p_{n+1} + \eta_\ell, M_{n+1})) \right|. \end{aligned}$$

Because of the definition of φ , intervals are taken to intervals and sub-intervals to sub-intervals; however, additional sub-interval structure is preserved. In fact, they are rearranged in a particular order. For example, if $\varphi(I(j, M_n)) = I(j', M_n)$, then $\varphi(I(j p_{n+1} + \ell, M_{n+1})) = I(j' p_{n+1} + (\ell + s \alpha_n), M_{n+1})$. In other words, the parameter $\eta_\ell = \ell + s \alpha_n$. This is clear by the definition of the coordinate-functions ϵ_k . Because of this, continue to manipulate the above sum.

$$\begin{aligned} &\left| \sum_{\ell=0}^{p_{n+1}-1} e^{\frac{2\pi i \ell}{p_{n+1}}} m_S(I(j' p_{n+1} + \eta_\ell, M_{n+1})) \right| \\ &= \left| \sum_{\ell=0}^{p_{n+1}-1} e^{\frac{2\pi i (\ell - \eta_\ell)}{p_{n+1}}} e^{\frac{2\pi i \eta_\ell}{p_{n+1}}} m_S(I(j' p_{n+1} + \eta_\ell, M_{n+1})) \right| \\ &= \left| \sum_{\ell=0}^{p_{n+1}-1} e^{\frac{2\pi i s \alpha_n}{p_{n+1}}} e^{\frac{2\pi i \eta_\ell}{p_{n+1}}} m_S(I(j' p_{n+1} + \eta_\ell, M_{n+1})) \right| \\ &= \left| e^{\frac{2\pi i s \alpha_n}{p_{n+1}}} \right| \left| \sum_{\ell=0}^{p_{n+1}-1} e^{\frac{2\pi i \eta_\ell}{p_{n+1}}} m_S(I(j' p_{n+1} + \eta_\ell, M_{n+1})) \right| \end{aligned}$$

(of course, $\ell - \eta_\ell$ is done modulo p_{n+1}).

Since $\left| e^{\frac{2\pi i s \alpha_n}{p_{n+1}}} \right| = 1$ for all n , it is clear that for any M_n -adic interval $I(j, M_n)$ such that $\varphi(I(j, M_n)) = I(j', M_n)$,

$$|\Delta m_S^*(I(j, M_n))| = |\Delta m_S(I(j', M_n))|.$$

By Lemma 2.6:

$$\sum_{i=M_n}^{M_{n+1}-1} |a_i^*|^2 = M_n \sum_{j=0}^{M_n-1} |\Delta m_S^*(I(j, M_n))|^2$$

and

$$M_n \sum_{j'=0}^{M_n-1} |\Delta m_S(I(j'), M_n)|^2 = \sum_{i=M_n}^{M_{n+1}-1} |a_i|^2.$$

Hence, it is clear that $\sum_{i=M_n}^{M_{n+1}-1} |a_i^*|^2 = o(1)$ as $k \rightarrow \infty$. (It is also clear that $m_S^*(G) = 0$ and $m_S^*(I) = 0$ for all $I \cap \varphi(E) = \emptyset$.) Thus, $S^* \in \mathcal{B}$ is associated with a non-zero quasi-measure m_S^* which is zero off of $\varphi(E)$; hence, $\varphi(E)$ is a set of multiplicity for $\mathcal{B}$. Clearly, if φ is measure-preserving, then φ^{-1} is too, so that if $\varphi(E)$ is a set of multiplicity for $\mathcal{B}$, then $\varphi^{-1}(\varphi(E)) = E$ is a set of multiplicity for $\mathcal{B}$. Since every subset of G is either a set of multiplicity or a set of uniqueness, the above argument also shows that E is a set of uniqueness for $\mathcal{B}$ if and only if $\varphi(E)$ is. $\square$

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