To the Graduate Council:

I am submitting herewith a thesis written by Robert E. Bodenheimer, Jr. entitled "Elliptical Bounds, Robustness, and Performance in Control System Design." I have examined the final copy of this thesis for form and content and recommend that it be accepted in partial fulfillment of the requirements for the degree of Master of Science, with a major in Electrical Engineering.

J. Douglas Birdwell

J. Douglas Birdwell, Major Professor

We have read this thesis
and recommend its acceptance:

J. S. Fowler

Kenneth Stephenson

Accepted for the Council:

Cunningham

Vice Provost
and Dean of The Graduate School

## STATEMENT OF PERMISSION TO USE

In presenting this thesis in partial fulfillment of the requirements for a Master's degree at The University of Tennessee, Knoxville, I agree that the Library shall make it available to borrowers under rules of the Library. Brief quotations from this thesis are allowable without special permission, provided that accurate acknowledgment of the source is made.

Requests for permission for extensive quotation from or reproduction of this thesis in whole or in parts may be granted by the copyright holder.

Signature Robert E. Bodenheimer Jr.

Date DECEMBER 1987

# ELLIPTICAL BOUNDS, ROBUSTNESS, AND PERFORMANCE IN CONTROL SYSTEM DESIGN

A Thesis

Presented for the

Master of Science

Degree

The University of Tennessee, Knoxville

Robert E. Bodenheimer, Jr.

June 1988

# ACKNOWLEDGMENTS

The author thanks everyone who helped in the research and preparation of this thesis. I especially thank my major professor, Dr. Birdwell, for the privilege of working with him, as well as the advice and support he has given throughout my graduate career. I extend thanks to Drs. Lawler and Stephenson for serving on my thesis committee and for teaching me so much throughout my association with them. Hal Beck and my father proofread this thesis, and their criticisms were invaluable in contributing to the finished product.

Finally, thanks to my parents and family for their support. Portions of the research were completed under sponsorship of the Oak Ridge National Laboratory under subcontract 41B-07685C, project authorization S89, whose support the author gratefully acknowledges.

# ABSTRACT

Generalizations of the standard singular-value robust stability and nominal performance criteria are proven. The robustness test allows the inclusion of information about the structure of uncertainty, and the nominal performance result allows independent constraints on each input channel. These tests provide a natural characterization of sensitivity and complementary sensitivity weighting matrices for sensitivity minimization problems. An $H^{\infty}$ optimization problem is presented which provides a method of synthesizing controllers satisfying these requirements based on the solution of the general distance problem.

# TABLE OF CONTENTS

# LIST OF FIGURES

# NOTATION

| | |
|---|---|
| $\mathbb{R}$ | The scalar field of real numbers. |
| $\mathbb{C}$ | The scalar field of complex numbers. |
| $\mathbb{R}^{m\times n}$ | The vector space of all $m \times n$ matrices with elements from $\mathbb{R}$. $\mathbb{R}^{m\times 1}$ will customarily be denoted as $\mathbb{R}^m$. |
| $\mathbb{C}^{m\times n}$ | The vector space of all $m \times n$ matrices with elements from $\mathbb{C}$. $\mathbb{C}^{n\times 1}$ will customarily be denoted as $\mathbb{C}^m$. |
| $j\mathbb{R}$ | The imaginary axis (also the $j\omega$-axis). |
| $j\omega$ | A point on the imaginary axis, where $\omega \in \mathbb{R}$. |
| $x$ | A vector in $\mathbb{R}^n$ or $\mathbb{C}^n$. |
| $\mathbf{A}$ | A matrix in $\mathbb{R}^{m\times n}$ or $\mathbb{C}^{m\times n}$. |
| $\mathbf{A}^H$ | Complex conjugate transpose of the matrix $\mathbf{A}$. |
| $\sigma_i(\mathbf{A})$ | The $i^{\mathrm{th}}$ singular value of the matrix $\mathbf{A}$. |
| $\sigma_{\mathbf{max}}(\mathbf{A})$ | The maximum singular value of the matrix $\mathbf{A}$. |
| $\sigma_{\mathbf{min}}(\mathbf{A})$ | The minimum singular value of the matrix $\mathbf{A}$. |
| $\lambda_i(\mathbf{A})$ | the $i^{\mathrm{th}}$ eigenvalue of the square matrix $\mathbf{A}$. |
| $\mathrm{Tr}[\mathbf{A}]$ | The trace of the square matrix $\mathbf{A}$. |
| $\mathbf{A} > 0$ | The square matrix $\mathbf{A}$ is positive definite. |
| $\mathbf{A} \geq 0$ | The square matrix $\mathbf{A}$ is positive semi-definite. |
| $\mathrm{Re}\, s$ | The real part of the complex number $s$. |
| ORHP | The open right-half of the complex plane, i.e., $\{s \in \mathbb{C}\ \vert \mathrm{Re}\, s > 0\}$. |
| CRHP | The closed right-half of the complex plane, i.e., $\{s \in \mathbb{C}\ \vert \mathrm{Re}\, s \geq 0\}$. We will also denote the CHRP as $\mathbb{C}^+$. |

$L^2_{m\times n}(j\mathbb{R})$ The Hilbert space of square-integrable matrix valued functions of finite norm on $j\mathbb{R}$ taking values in $\mathbb{C}^{m\times n}$ with the inner product

$$\langle X,Y\rangle \equiv \frac{1}{2\pi}\int_{-\infty}^{\infty} \mathrm{Tr}\left[X(j\omega)^H Y(j\omega)\right] d\omega$$

for $X$, $Y \in L^2(j\mathbb{R})$. The domain $j\mathbb{R}$ and the subscript $m \times n$ are often omitted when there is no chance of ambiguity.

$H^2_{m\times n}$ The (Hardy) space of functions in $L^2_{m\times n}(j\mathbb{R})$ taking values in $\mathbb{C}^{m\times n}$ and analytic in the ORHP with

$$\sup_{\sigma>0}\int_{-\infty}^{\infty} \mathrm{Tr}\left[X(\sigma+j\omega)^H X(\sigma+j\omega)\right] d\omega < \infty.$$

The subscript $m \times n$ is often omitted when there is no chance of ambiguity.

$\|G\|_2$ $L^2$ or $H^2$ norm

$$\equiv \left[\frac{1}{2\pi}\int_{-\infty}^{\infty} \mathrm{Tr}\left[X(j\omega)^H X(j\omega)\right]\right]^{\frac{1}{2}} d\omega$$

if $X \in L^2(j\mathbb{R})$ or $H^2(j\mathbb{R})$.

$L^\infty_{m\times n}(j\mathbb{R})$ Banach space of essentially bounded $m\times n$ functions on $j\mathbb{R}$ taking values in $\mathbb{C}^{m\times n}$. The domain $j\mathbb{R}$ and the subscript $m\times n$ is often omitted when there is no chance of ambiguity.

$H^\infty_{m\times n}$ The space of functions in $L^\infty_{m\times n}(j\mathbb{R})$ with bounded analytic continuation to the ORHP. The subscript $m \times n$ is often omitted when there is no chance of ambiguity.

$\|G\|_\infty$ $L^\infty$ or $H^\infty$ norm $\equiv \operatorname*{ess\,sup}_{\omega\in\mathbb{R}} \sigma_{\max}[X(j\omega)]$, if $X \in L^\infty(j\mathbb{R})$ or $H^\infty(j\mathbb{R})$.

| | |
|---|---|
| $\mu(\mathbf{A})$ | The structured singular value of $\mathbf{A}$. |
| $\mathcal{R}(\mathbf{A})$ | The range of the linear transformation $\mathbf{A}$. |
| $\mathbf{RH}_\infty$ | The space of rational polynomial matrices with real coefficients which are proper and stable. |
| $\mathbf{A}^T$ | The transpose of the matrix $\mathbf{A}$. |

# CHAPTER 1

# INTRODUCTION

## 1.1 Motivation

The objective of the control engineer is to cause the output of a physical plant with feedback around it to behave in a specified manner. Feedback is used because there is uncertainty, usually in the form of modelling errors, noise and disturbance signals, and nonlinear dynamics [64,43]. The fundamental problem the control engineer faces is to achieve desired behavior (performance) in the face of these uncertainties. Additionally, the potential for instability arises when feedback is present around a plant. Stability is essentially just another type of performance, but is so fundamental that it is usually treated separately from other types of performance. The ability of the plant to perform in the presence of uncertainties is called its *robustness*. The system is *robustly stable* if it is stable in the presence of uncertainties; it will *perform robustly* if other performance bounds for the system are met in the presence of uncertainties.

If the system for which a controller is being designed is complex, then causing the plant to behave as desired is not trivial. Usually, it is advantageous to have some procedure to follow which indicates how to achieve these desirable characteristics in the system. This procedure is called a *design methodology*. Examples of modern design methodologies are linear quadratic synthesis with loop transfer recovery (LQG/LTR) [26], $H^{\infty}$-optimization [41,77], $\mu$-synthesis [20], quantitative feedback theory (QFT) [43] and $\ell^1$-optimization [14]. These are examples of multi-input multioutput (MIMO) methodologies, i.e., the system has more than one input and more than one output: the F100 turbofan engine is a good example of such a system [70]. Multivariable controls are used because in complex systems the interaction between variables can be so severe that acceptable performance is unobtainable with single loop controllers. Single-input single-output (SISO) systems are much simpler than MIMO systems, and their interest to researchers is usually only as a stepping stone to the development of MIMO techniques. A difficulty of control theory is that while MIMO techniques generally apply (trivially) to SISO systems, the converse is not true: SISO techniques applied indiscriminately to MIMO systems can lead to catastrophic results. In particular, it can be shown [20] that for SISO systems, nominal performance and robust stability imply robust performance; however, simple examples show that nominal performance and robust stability do not imply robust performance for MIMO systems, and robust performance is more difficult to guarantee.

## 1.2 Problem Description

The mathematical model of a system does not truly represent a real physical system; as mentioned before, uncertainty is always present. A control system designer attempts to model the uncertainty as lying in a class of *perturbations* to the system. These perturbations are also mathematical models; hopefully, the class of modelled perturbations includes the set of true perturbations that the system may encounter. When this occurs, the class of perturbations is said to be conservative. An example of a conservative class of perturbations is the class of norm-bounded perturbations. Researchers in control theory try to create robustness tests and design methodologies which reduce this conservatism, because there is often a trade-off between stability and performance in a control system: A system stable against perturbations which cannot actually occur often exhibits degraded performance. This is commonly seen when the class of perturbations is norm-bounded. Norm-bounded perturbations are spherical; they constrain only the magnitude of the perturbation and place no constraint on its trajectory.

This thesis addresses that problem by devising a robust stability criteria which is elliptical, placing both magnitude and trajectory constraints on the class of allowable perturbations. An analogous performance test is presented which allows a nominal performance specification based on the trajectory and magnitude of inputs and disturbances. These tests provide a natural characterization of weighting matrices for use in sensitivity minimization design methods which have gained popularity in recent years. A controller synthesis method based on sensitivity minimization which uses the robust stability and nominal performance tests is presented. This synthesis method can be approximately solved by computing a solution to a particular type of minimization problem.

This thesis is organized as follows: Chapter 2 reviews concepts from control theory and mathematics of fundamental importance to this work, Chapter 3 presents the results of the research effort, and Chapter 4 discusses the work and presents suggestions for future research.

# CHAPTER 2

# REVIEW OF FUNDAMENTAL MATHEMATICS AND CONTROL THEORY

Excellent general references for this chapter are [38,69,74,35]. Note that a polynomial matrix $M(s)$ is often written simply as $M$ for convenience. This notation, while perhaps overly terse, has gained acceptance in the literature.

## 2.1 The Singular Value Decomposition

The singular value decomposition (SVD) is one of the most important tools in numerical linear algebra and linear systems theory. The following theorem defines the SVD:

**Theorem 1.** *Let* $\mathbf{A} \in \mathbb{C}^{m \times n}$*. Then there exist unitary matrices*

$$U = [u_1, \cdots, u_m] \in \mathbb{C}^{m \times m}$$

*and*

$$V = [v_1, \cdots, v_n] \in \mathbb{C}^{n \times n}$$

*such that*

$$\mathbf{A} = U \begin{pmatrix} S & 0 \\ 0 & 0 \end{pmatrix} V^H$$

*where* $S = \mathrm{diag}(\sigma_1, \sigma_2, \cdots, \sigma_r)$ *with*

$$\sigma_1 \geq \cdots \geq \sigma_r > 0.$$

The vectors $u_i$ and $v_i$ are the left singular vectors and right singular vectors of $\mathbf{A}$, respectively; the $\sigma_i$ with $\sigma_{r+1} = \cdots = \sigma_p = 0$, $p = \min(m,n)$, are the singular values of $\mathbf{A}$. Another way of writing the SVD is as

$$\mathbf{A} = \sum_{i=1}^{r} \sigma_i u_i v_i^H.$$

Note that $\sigma_{\max}(\mathbf{A}) = \sigma_1(\mathbf{A})$ and $\sigma_{\min}(\mathbf{A}) = \sigma_p(\mathbf{A})$. The following is a list of some important and well-known properties of the SVD:

1. $\sigma_i(\mathbf{A}) = \sqrt{\lambda_i(\mathbf{A}^H\mathbf{A})}$.

2. $\sigma_{\max}(\mathbf{A}) = \|A\|_2$.

3. $\det(\mathbf{A}) = 0$ iff $\sigma_{\min}(\mathbf{A}) = 0$.

4. $\sigma_{\min}(\mathbf{A}) = \frac{1}{\sigma_{\max}(\mathbf{A}^{-1})}$, if $\mathbf{A}$ is nonsingular.

5. $\mathbf{A}$ nonsingular and $\|\mathbf{E}\|_2 < \sigma_{\min}(\mathbf{A})$ implies $\mathbf{A} + \mathbf{E}$ nonsingular.

The reader desiring more information than presented here, particularly for a proof of the theorem, should consult [38] and [47] and the references therein.

## 2.2 Norms and Spaces

The approach to systems we take is an *input-output* approach; that is, we view a system as an operator which takes a frequency response from the input space and transforms it into a frequency response in the output space. The idea that frequency repsonses are functions in a Hardy space was introduced to the systems community by Helton [41], who applied it to electronics problems, and Zames [77], who applied it to weighted sensitivity minimization problems. We assume the reader is familiar with the Lebesgue spaces $L^p$, and note that all norms used are such that $\|I\| = 1$.

Assume $f : \text{ORHP} \to \mathbb{C}$ is analytic. For $1 \le p < \infty$, $f \in H^p(\mathbb{C}^+)$ iff

$$\sup_{\xi \ge 0} \left[ \frac{1}{2\pi} \int_{-\infty}^{\infty} |f(\xi + j\omega)|^p \, d\omega \right]^{\frac{1}{p}} = \|f\|_p < \infty.$$

Also, $f \in H^\infty$ iff

$$\sup_{\xi \ge 0} \sup_{\omega} |f(\xi + j\omega)| = \|f\|_\infty < \infty.$$

It can be shown that if $f \in H^p$, then the function $\phi : \omega \to f(j\omega)$ belongs to $L^p(j\mathbb{R})$, and that

$$\|f\|_p = \left[ \frac{1}{2\pi} \int_{-\infty}^{\infty} |f(j\omega)|^p \, d\omega \right]^{\frac{1}{p}} \quad \text{if } p \in [1, \infty),$$

$$\|f\|_\infty = \text{ess} \sup_{\omega} |f(j\omega)|.$$

The results are extended to cover vector and matrix-valued functions as follows: Let $G(s) : \mathbb{C} \to \mathbb{C}^n$. $G \in L^2_n$ iff $G$ has finite norm on $j\mathbb{R}$ in the sense of the inner product

$$\langle F, G \rangle = \frac{1}{2\pi} \int_{-\infty}^{\infty} F^H(j\omega) G(j\omega) d\omega.$$

$G \in H^2_n$ iff $G \in L^2_n$, $G$ is analytic in the ORHP, and

$$\sup_{\xi > 0} \frac{1}{2\pi} \int_{-\infty}^{\infty} G^H(\xi + j\omega) G(\xi + j\omega) \, d\omega < \infty.$$

Let $G(s) : \mathbb{C} \to \mathbb{C}^{m \times n}$. $G \in L^2_{m \times n}$ iff $G$ has finite norm on $j\mathbb{R}$ in the sense of the inner product

$$\langle F, G \rangle = \frac{1}{2\pi} \int_{-\infty}^{\infty} \text{Tr} \left[ F^H(j\omega) G(j\omega) \right] d\omega.$$

$G \in H^2_{m \times n}$ iff $G \in L^2_{m \times n}$, $G$ is analytic in the ORHP, and

$$\sup_{\xi > 0} \frac{1}{2\pi} \int_{-\infty}^{\infty} \text{Tr} \left[ G^H(\xi + j\omega) G(\xi + j\omega) \right] d\omega < \infty.$$

Let $F(s) : \mathbb{C} \to \mathbb{C}^{m \times n}$. $F \in L^{\infty}_{m \times n}$ iff

$$\|F\|_{\infty} = \sup_{\omega \in \mathbb{R}} \|F(j\omega)\|_2 = \sup_{\omega \in \mathbb{R}} \sigma_{\max}(F(j\omega));$$

$F \in H^{\infty}_{m \times n}$ iff $F \in L^{\infty}_{m \times n}$ and $F$ has a bounded analytic continuation to the ORHP.

The 2-norm was introduced in Section 2.1. An important property of the 2-norm is *unitary invariance*, i.e., for all unitary $\mathbf{Q}$ and $\mathbf{Z}$ we have

$$\|\mathbf{QAZ}\|_2 = \|\mathbf{A}\|_2 .$$

This idea will be generalized later in this thesis.

Functions such as $f$ and $g$ are viewed as frequency responses, i.e., inputs and outputs, which the system acts on. Matrix-valued functions such as $F$ and $G$ are the mathematical constructions representing the systems themselves. Whether we choose to view $F$ as in $H^2$ or $H^{\infty}$ is problem specific. Attaching an $\mathbf{R}$ as a prefix to a space, as in $\mathbf{RH}_{\infty}$, means we are restricting attention to functions in the space which are real and rational.

See Zames [78,79,77] for a more detailed presentation of the material here from a control viewpoint; the books by Willems [73] and Vidyasagar [72] are also good references, although Vidyasagar's book is not as rigorous as one might hope.

### 2.3 Definiteness and Ellipsoids

Recall that a matrix $\mathbf{A} \in \mathbb{C}^{n \times n}$ is positive definite iff $x^H \mathbf{A} x > 0 \, \forall x \in \mathbb{C}^n$, $x \neq 0$, and positive semi-definite iff $x^H \mathbf{A} x \geq 0 \, \forall x \in \mathbb{C}^n$. A matrix is also positive definite iff all its eigenvalues are positive real and positive semi-definite iff all its eigenvalues are non-negative real. Let $\mathbf{A}$ have rank $r$; if $\mathbf{A}$ is positive semi-definite (possibly definite), then the vectors satisfying the equation $x^H Ax = 1$ determine an $r$-dimensional hyperellipsoid in $\mathbb{C}^n$. Given two positive semi-definite matrices $\mathbf{A}$ and $\mathbf{B}$ the hyperellipsoid generated by $\mathbf{A}$ "fits inside" the hyperellipsoid generated by $\mathbf{B}$ iff $\mathbf{B}-\mathbf{A}$ is also positive semidefinite. We make this idea more precise with the following lemmas:

**Lemma 1.** *Let $A : \mathcal{X} \to \mathcal{Y}$ and $B : \mathcal{X} \to \mathcal{Y}$ be linear transformations from the vector space $\mathcal{X}$ to the vector space $\mathcal{Y}$. Then $\|Ax\|_2 < \|Bx\|_2$ $\forall x \in \mathcal{X}$ iff $B^H B - A^H A$ is positive definite.*

PROOF. $\|Ax\|_2 < \|Bx\|_2$ $\forall x \in \mathcal{X}$, $x \neq 0$, iff $\|Ax\|_2^2 < \|Bx\|_2^2$ $\forall x \in \mathcal{X}$. By definition, $\|Ax\|_2^2 = x^H A^H Ax$ and $\|Bx\|_2^2 = x^H B^H Bx$, so $\|Ax\|_2 < \|Bx\|_2 \forall x \in \mathcal{X}$, $x \neq 0$, iff $x^H A^H Ax < x^H B^H Bx$ $\forall x \in \mathcal{X}$, $x \neq 0$, iff $x^H \left(B^H B - A^H A\right) x > 0$ $\forall x \in \mathcal{X}$, $x \neq 0$, iff $B^H B - A^H A$ is positive definite. Q.E.D.

**Lemma 2.** *Let $A : \mathcal{X} \to \mathcal{Y}$ and $B : \mathcal{X} \to \mathcal{Y}$ be linear transformations from the vector space $\mathcal{X}$ to the vector space $\mathcal{Y}$. Then $\|Ax\|_2 \leq \|Bx\|_2$ $\forall x \in \mathcal{X}$ iff $B^H B - A^H A$ is positive semi-definite.*

PROOF. Identical to Lemma 1 with "$<$" and "$>$" replaced with "$\leq$" and "$\geq$," respectively. Q.E.D.

Note that $\mathbf{A}^H\mathbf{A}$, $\mathbf{B}^H\mathbf{B}$, and $\mathbf{A}^H\mathbf{A} - \mathbf{B}^H\mathbf{B}$ are all Hermitian.

### 2.4 State-Space Representation

Given a linear time-invariant (LTI) system with $n$ states, $m$ inputs and $p$ outputs, the dynamics of the *nominal* model of the plant obey the equations

$$\dot{x}(t) = \mathbf{A}x(t) + \mathbf{B}u(t), \tag{2.1}$$

$$y(t) = \mathbf{C}x(t) + \mathbf{D}u(t), \tag{2.2}$$

where $x(t)$ is the plant state, $u(t)$ is the plant input, $y(t)$ is the plant output, $\mathbf{A} \in \mathbb{R}^{n \times n}$, $\mathbf{B} \in \mathbb{R}^{n \times m}$, $\mathbf{C} \in \mathbb{R}^{p \times n}$, and $\mathbf{D} \in \mathbb{R}^{p \times m}$. The above is called the *state-space* representation of the system. The system pair $(\mathbf{A}, \mathbf{B})$ is *controllable* iff $[\mathbf{B}, \mathbf{AB}, \ldots, \mathbf{A}^{n-1}\mathbf{B}]$ has rank $n$. The matrix $\mathbf{A}$ is *stable* iff all its eigenvalues have negative real parts, and the system $(\mathbf{A}, \mathbf{B})$ is *stabilizable* iff there exists an $m \times n$ matrix $\mathbf{F}$ such that $\mathbf{A} + \mathbf{BF}$ is stable. The system pair $(\mathbf{C}, \mathbf{A})$ is *observable* iff $(\mathbf{A}^T, \mathbf{C}^T)$ is controllable; it is *detectable* iff $(\mathbf{A}^T, \mathbf{C}^T)$ is stabilizable. Finally, the system realization $(\mathbf{A}, \mathbf{B}, \mathbf{C})$ is *minimal* iff $(\mathbf{A}, \mathbf{B})$ is controllable and $(\mathbf{C}, \mathbf{A})$ is observable.

### 2.5 Transfer Functions

Related to the LTI system (2.1) and (2.2) is the convolution

$$Y(t) = (G_0 * u)(t),$$

with

$$G_0(t) = \begin{cases} \mathbf{C}e^{\mathbf{A}t}\mathbf{B} + \mathbf{D}\delta(t) & \text{if } t \geq 0, \\ 0 & \text{if } t < 0, \end{cases}$$

where we assume the system is relaxed at $t = 0$; the usual practice is to then transform from the state-space representation to the transfer function frequency domain representation using the Laplace transform. The validity of transforming the system in this manner is a deep philosophical issue in control theory, but any difficulties with it are far beyond the scope of this thesis. The system is represented in the frequency domain by

$$y(s) = G_0(s)u(s), \tag{2.3}$$

where the nominal transfer function $G_0(s)$ is a $p \times m$ matrix of rational functions of $s$ with real coefficients (*real-rational*), given by

$$G_0(s) = \mathbf{C}(s\mathbf{I} - \mathbf{A})^{-1}\mathbf{B} + \mathbf{D}. \tag{2.4}$$

Note that this is an input-output or black box description of the system. For any fixed value of $s$, $G_0(s)$ is a linear transformation from the space of input vectors, $\mathbb{C}^m$, to the space of output vectors $\mathbb{C}^p$.

The transfer function $G_0(s)$ is stable if it has no poles in the CRHP; proper if $\lim_{\omega\to\infty} \|G_0(j\omega)\|$ exists and is finite; and strictly proper if $\lim_{\omega\to\infty} G_0(j\omega) = 0$. Since $G_0(s)$ is the transform of the system in state space form, it is proper (possible strictly proper). Proper transfer functions are almost always required in practice, since high-frequency noise is amplified in a non-proper system, resulting in poor performance and possible stability problems.

A transfer function $G_0(s)$ is *inner* if $G_0^T(-s)G_0(s) = I$ for every $s \in \mathbb{C}$, and *co-inner* if $G_0(-s)G_0^T(s) = I$ for every $s \in \mathbb{C}$. Inner transfer functions are important because the $\infty$-norm is invariant under multiplication by square inner matrices, analogous to the unitary invariance of the 2-norm, mentioned in Section 2.2.

## 2.6 Compensators

A standard closed-loop control system is shown in Figure 2.1. Here $G$ is the plant, $K$ is the compensator, $u$ is the input to the plant, $r$ is the exogenous input, $e$ is the error response, and $y$ is the output of the plant corrupted by a disturbance $d$ not subject to control. The signals are all assumed to be multivariable, and $G$ and $K$ are finite-dimensional LTI. It is easily verified that the configuration has the following

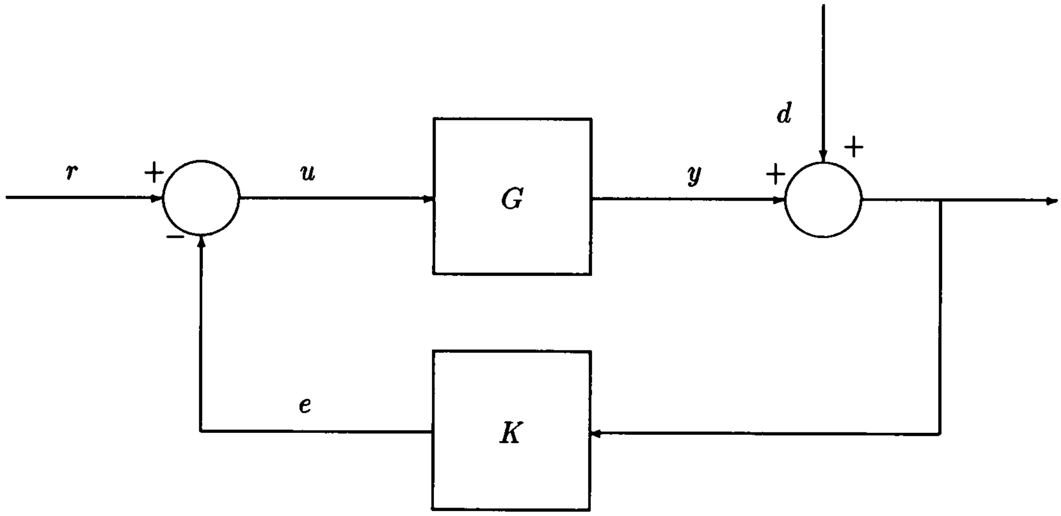


Figure 2.1: Standard Closed-Loop System

input-output properties:

$$
\begin{aligned}
y &= G(I+KG)^{-1}r - GK(I+GK)^{-1}d, && (2.5)\\
e &= KG(I+KG)^{-1}r + K(I+GK)^{-1}d, && (2.6)\\
u &= (I+KG)^{-1}r - (I+KG)^{-1}Kd. && (2.7)
\end{aligned}
$$

where we have dropped the $s$ dependency.

Define the *return ratio* at the $u$ node, $L_u(s)$, as

$$L_u(s) \equiv K(s)G(s),$$

and the return ratio at the $y$ node, $L_y(s)$, as

$$L_y(s) \equiv G(s)K(s).$$

The $u$ and $y$ node return ratios are also called the input and output return ratios, respectively. The following definitions apply to both $L_u(s)$ and $L_y(s)$, and when there is no ambiguity, we drop the subscript. We define the *return difference* matrix transfer function as

$$I + L(s),$$

and the *inverse return difference* matrix transfer function as

$$I + L^{-1}(s).$$

The *sensitivity* function $S(s)$ is defined as

$$S(s) \equiv (I + L(s))^{-1},$$

and the *complementary sensitivity* is defined as

$$T(s) \equiv (I + L(s))^{-1}L(s) = L(s)(I + L(s))^{-1} = (I + L^{-1}(s))^{-1}.$$

Then equations (2.5)-(2.7) become

$$y = GS_u(s)r - T_y(s)d, \tag{2.8}$$

$$e = T_u(s)r + KS_y(s)d, \tag{2.9}$$

$$u = S_u(s)r - S_u(s)Kd. \tag{2.10}$$

If the system is to be stable, $S$ and $T$ must be stable at both the $y$ and $u$ nodes.

The sensitivity and complementary sensitivity satisfy the identity

$$S(s) + T(s) = I. \tag{2.11}$$

This identity represents the fundamental reason trading-off various design constraints is necessary; $S$ and $T$ cannot be made small at the same time. If it is desired that the output $y$ or error $e$ be small with respect to $r$, the sensitivity $S_y$ should be made small. Usually, $T$ should be made small at high frequencies; this is because small $T$ contributes to the ability of the system to remain stable in the presence of large modelling errors, time delays, nonlinearities, and other effects which are normally significant only at high frequencies. These errors will eventually become large as frequency increases; therefore a well-designed control system's transfer function exhibits strictly proper behavior [20]. $S$ should be made small at low frequencies; modelling accuracy and linearity are normally high at low frequencies, and performance requirements such as small tracking error are stringent. The problem often occurs at frequencies where $S \approx T$, the *crossover* region. In the sequel, we will usually be referring to bounds at the input of the plant; at the input of the plant, the input sensitivity and complementary sensitivity are

$$\begin{aligned} S_u(S) &= (I + KG)^{-1}, \\ T_u(s) &= KG(I + KG)^{-1}. \end{aligned}$$

In those situations we will also omit the subscript; the fact that we are referring to the input will be clear from the context.

The crossover *frequency* of the $i^{\text{th}}$ singular value is defined as the frequency $\omega_{c_i}$ where

$$\sigma_i(L(j\omega_{c_i})) = 1; \tag{2.12}$$

in particular, $\omega_{c_{\max}}$ and $\omega_{c_{\min}}$ are the crossover frequencies of the maximum and minimum singular values, respectively. From a strictly mathematical point of view, there is no guarantee that $\sigma_i(L(j\omega))$ equals one at a unique frequency, but a well-designed control system should exhibit this property. If the crossover *gap*, the region where performance requirements are small, modelling accuracy high, time delays insignificant, and nonlinearities small, is large, then there is no difficulty with following that strategy. Unfortunately, this is usually not the case, and $S$ and $T$ can exhibit bad behavior by not crossing over in all directions at the same frequency. The procedure for handling these methods is still very much *ad hoc* and on a case-by-case basis.

$L$ is unknown. We know the nominal plant $G_0$, and thus nominal values of the return ratio $L_0(s)$, sensitivity $S_0$, and complementary sensitivity $T_0$, but the true plant is some perturbed version of $G_0$. One way of representing this model uncertainty is with a multiplicative perturbation of the form

$$I + \Delta(s);$$

the physical plant is assumed to be correctly modelled by

$$L(s) = L_0(s)(I + \Delta(s)) \tag{2.13}$$

for a perturbation $\Delta(s)$. $\Delta(s)$ is unknown, but we may be able to restrict it to a certain class. Our objective then would be to guarantee stability and performance for all perturbations in this class. The test for stability we will use is that given in Safonov, Laub, and Hartmann [63] for asymptotic stability:

**Lemma 3.** *The system of equations (2.5)-(2.7) is (asymptotically) stable if and only if there does not exist $s_0$ with $Re\, s_0 \geq 0$ and nonzero $x_1 \in \mathbb{C}^n$ such that*

$$x_1 = -\Delta(s_0)T_0(s_0)x_1 \tag{2.14}$$

*where $T_0$ is the nominal complementary sensitivity, and $\Delta$ is a multiplicative perturbation to the plant.*

Of course, another way of stating Lemma 3 is that the closed-loop system is unstable iff $\exists s_0 \in \mathbb{C}^+$ and $\exists x_1 \in \mathbb{C}^n$ such that equation (2.14) holds. In particular, since all these functions are linear, if equation (2.14) holds for some $x_1 \neq 0 \in \mathbb{C}^n$, it holds for some $x_1 \in \mathbb{C}^n$ with $\|x_1\|_2 = 1$.

## 2.7 Overview of LQG/LTR

In this section we present an overview of the LQG/LTR design methodology; this presentation is based upon [26,63,52,6,11,40,3,5]. We first note that the computations needed in LQG/LTR can be done effectively and quickly, and there is little work left to be done [10,9,7,13].

We start with the LTI system described by equations (2.1)-(2.4). Assume the feed-through matrix $\mathbf{D}$ is zero, and that the transfer function represented by equation (2.4) is minimum-phase, i.e., has no zeros in ORHP. Additionally, assume $(\mathbf{A}, \mathbf{B})$ is stabilizable, and that $\mathbf{B}$ has full rank (see [26,5] for details).

LQG/LTR yields a closed-loop system of the structure shown in Figure 2.2. A filter or a regulator is designed so that $L(s)$ satisfies constraints on the sensitivity, complementary sensitivity, or other performance bounds. After this, a modified regulator (if the filter was designed first) or a filter (if the regulator was designed first) is designed to "recover" the return ratio of a point internal to the compensator which has desirable robustness properties. Performance bounds and sensitivities can be specified at either the input or the output of the plant in Figure 2.2; in general, bounds cannot be specified at both the input and the output except in special cases [49]. If the bounds are specified with respect to the plant input, the regulator is designed first and then the filter; if the bounds are specified at the output of the plant, the filter is designed first and then the regulator. The design methods are duals of each other, therefore without loss of generality, only bounds at the plant input are considered in the sequel. Note that the closed-loop system has a structure different from that of the system in Figure 2.1 on page 11; this does not effect the use of the input sensitivity and complementary

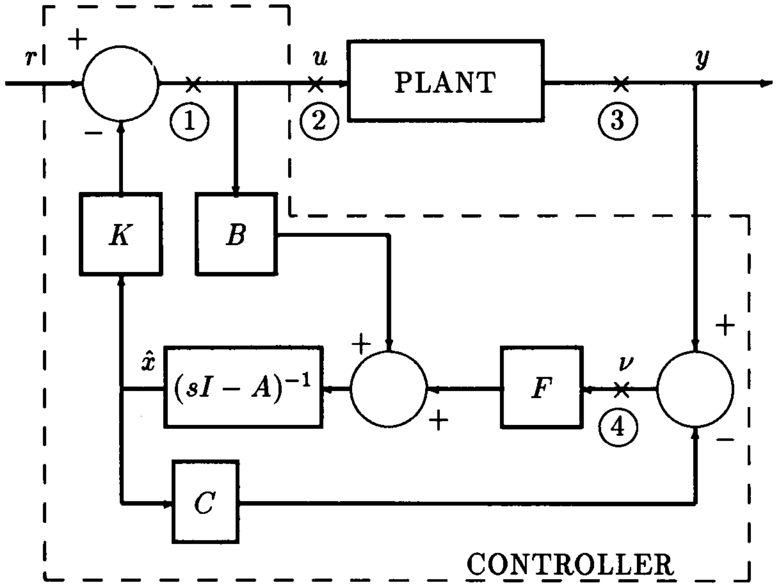


Figure 2.2: LQG/LTR Controller

sensitivity in specifying the bounds.

The nominal return ratio at the node 1 in Figure 2.2 is

$$L_1(s) = \mathbf{K}(sI - \mathbf{A})^{-1}\mathbf{B}, \tag{2.15}$$

the standard linear quadratic regulator (LQR) return ratio, which has well-known robustness properties [6]. The matrix $\mathbf{K}$ is called the regulator gain matrix. For performance bounds at the input of the plant, the return ratio of interest is at the $u$ node,

$$L_2(s) = \mathbf{K}\left[(sI - \mathbf{A})^{-1} + \mathbf{BK} + \mathbf{FC}\right]^{-1}\mathbf{FC}(sI - \mathbf{A})^{-1}\mathbf{B}, \tag{2.16}$$

where $\mathbf{F}$ is the filter gain matrix. This return ratio has, unfortunately, no guaranteed robustness properties [21]. When the plant is minimum-phase, however, we can pick $\mathbf{F}$ so that $L_2 \approx L_1$, where the measure of equality is the singular values of the matrices as a function of frequency. The method of making this approximation is elaborated on in the sequel.

For robustness, the physical plant is assumed to be correctly modelled by equation (2.13) on page 14. The regulator in Figure 2.2 is designed as a standard LQR. Since we are designing for bounds at the input, we ignore the output and consider only the state dynamics given by equation (2.1) on page 8. The optimal LQR is designed by finding the optimal closed-loop controller which minimizes the cost function $J$, where

$$J = \int_0^\infty \left( \underline{x}^{\mathrm{T}} \mathbf{Q} \underline{x} + \underline{u}^{\mathrm{T}} \mathbf{R} \underline{u} \right) dt,$$

and $\mathbf{Q} \in \mathbb{R}^{n \times n}$, $\mathbf{Q} \geq 0$, is the state weighting matrix and $\mathbf{R} \in \mathbb{R}^{m \times m}$, $\mathbf{R} > 0$, is the control input weighting matrix. Note that the preceding optimality is with respect to the minimization of $J$; this fact is not used in the LQG/LTR design methodology.

In order that certain guaranteed margins exist the control weighting matrix $\mathbf{R}$ must be diagonal [52]. The usual choice is to choose $\mathbf{R} = \rho \mathbf{I}$, where $\rho$ is a positive scalar. Also, $\mathbf{Q}$ is positive semi-definite and of the form $\mathbf{Q} = \mathbf{H}^{\mathrm{T}}\mathbf{H}$ for some matrix $\mathbf{H}$. With these assumptions on $\mathbf{Q}$ and $\mathbf{R}$ it can be shown [26] that the singular values of the return difference satisfy the inequality

$$\sigma_i(I + L(j\omega)) \approx \sqrt{1 + \frac{1}{\rho}\sigma_i^2 \left[ \mathbf{H} \left( j\omega \mathbf{I} - \mathbf{A} \right)^{-1} \mathbf{B} \right]}. \tag{2.17}$$

If the return ratio $L(j\omega)$ is large at a frequency $\omega$ ($\sigma_{\min}(L(j\omega)) \gg 1$), the above becomes [26]

$$\sigma_i(I + L(j\omega)) \approx \frac{1}{\sqrt{\rho}}\sigma_i \left[ \mathbf{H} \left( j\omega \mathbf{I} - \mathbf{A} \right)^{-1} \mathbf{B} \right]. \tag{2.18}$$

The above equation is the fundamental relation used to select $\rho$ and $\mathbf{H}$.

Doyle and Stein [26] recommend selecting $\mathbf{H}$ to force as many of the singular values of $I + L(j\omega)$ as possible to be equal (*balancing*); Birdwell and Laub [11] have devised a method of balancing the singular values at an arbitrary frequency $\omega_b$. However, no method for determining a suitable value for $\omega_b$ is currently available. Moreover, the balancing transform $\mathbf{H}$ for a particular $\omega_b$ is not unique, and there is currently no method for selecting an appropriate $\mathbf{H}$ to exploit this non-uniqueness.

The gain matrix $\mathbf{K}$ is computed by first solving the algebraic Riccati equation (ARE) in $\mathbf{X}$

$$\mathbf{XA} + \mathbf{A}^T\mathbf{X} - \mathbf{XBR}^{-1}\mathbf{B}^T\mathbf{X} + \mathbf{Q} = 0$$

and setting $\mathbf{K}$ equal to

$$\mathbf{K} = \frac{1}{\rho}\mathbf{B}^T\mathbf{X}.$$

In the above we have assumed that $(\mathbf{A}, \mathbf{H})$ is detectable.

To design a standard LQG compensator, a state estimator must also be designed. Unfortunately, as mentioned previously, the usual Kalman filter design has no guaranteed robustness properties [21]. This is reasonable since the LQG cost has no factor for including robustness. The Kalman filter design must be modified slightly so that we can asymptotically approximate (or "recover") the good robustness properties of the LQR, by making $L_2 \approx L_1$. In this role, the estimator serves as an approximate right inverse to the nominal plant, $G_0(s)$ [26]. To do this, we use a design parameter $q > 0$, and solve the following ARE for $\mathbf{X}$:

$$\mathbf{AX} + \mathbf{XA}^T - \mathbf{XC}^T\mathbf{CX} + q^2\mathbf{BB}^T = 0.$$

The filter gain matrix $\mathbf{F}$ is given by

$$\mathbf{F} = \mathbf{XC}^T.$$

By choosing $q$ sufficiently large we can cause the return ratio given by equation (2.16) to again satisfy the original performance bounds, since

$$\lim_{q\to\infty} L_2(q,s) = L_1(s),$$

where we have explicitly noted the dependence of $L_2$ on $q$ [26].

Robust stability can be guaranteed by the LQG/LTR method provided the multiplicative perturbation satisfies a frequency bound $b_H(\omega)$, in that

$$\sigma_{\max}(\Delta(j\omega)) < b_H(\omega) \quad \forall\omega, \tag{2.19}$$

and the closed-loop inverse return difference matrix satisfies

$$\sigma_{\min}(I + L^{-1}(j\omega)) > b_H(\omega) \quad \forall\omega. \tag{2.20}$$

This can be derived from the Small Gain Theorem and the Loop Transformation theorem [65,73], which states that the loop will be stable if

$$\left\| \Delta(j\omega)L(j\omega)(I + L(j\omega))^{-1} \right\|_2 < 1 \; \forall\omega. \tag{2.21}$$

Nominal performance can be guaranteed if the return difference matrix satisfies

$$\sigma_{\min}(I + L(j\omega)) \geq b_L(\omega) \quad \forall\omega, \tag{2.22}$$

where $b_L(j\omega)$ is the minimum sensitivity to command following error signals at the input as a function of frequency. These properties are achievable for some $q > 0$ if $L_2$ satisfies equations 2.20 and 2.22. Note that $b_H(\omega)$ and $b_L(\omega)$ are scalar functions of frequency.

Additionally, note that by adding integrators to the input or output of the plant we can augment the dynamics of the plant (and thus change them). This may be necessary to insure performance requirements such as, for example, zero steady-state error. Also it may be desirable to augment the plant to insure the system has uniform type as $\omega \to 0$.

Augmenting the plant extraneously, however, leads to computational difficulties with the controller's numerical calculations; thus we prefer to keep the order of augmentation at a minimum.

In summary, the LQG/LTR design methodology reduces the design procedure to the selection of five parameters: scalar gains $\rho$ and $q$, a balancing frequency $\omega_b$, a balancing transform $\mathbf{H}$, and the order of the augmenting dynamics to apply. The design method above assumed that the real process noise was zero, but a non-zero process noise merely adds details to the method. Without more information about a specific application this is as simple as the design methodology currently is.

The above methodology can be interpreted as the solution to a weighted $H^2$ optimization problem [68], but this interpretation has severe problems, as we now show. Stein and Athans describe a procedure [68, page 112], "to define $H^2$-weightings which achieve essentially arbitrary shapes for sensitivity functions in the first step." The procedure described, however, is extremely restrictive in its requirements and does not, in general, allow arbitrary loopshaping. In the following we restrict attention to the case where the filter is designed first; the case where the regulator is designed first is completely analogous.

We modify our LTI system slightly to make it conform to the standard LQG setup

$$\begin{aligned} \dot{x}(t) &= \mathbf{A}x(t) + \mathbf{B}u(t) + \mathbf{L}\xi(t), \\ y(t) &= \mathbf{C}x(t) + \mu I \eta(t) \end{aligned}$$

where everything is as before, and $\xi(t)$ and $\eta(t)$ are fictitious independent white noise processes with unit intensity, and $\mathbf{L} \in \mathbb{R}^{n \times m}$ is the (designer-specified) noise input matrix. The transfer function is the same and we are requiring that $G(s)$ be strictly proper in addition to minimum-phase and finite-dimensional.

Given a strictly proper finite dimensional weight $\mathbf{W}(s)$, Stein and Athans require it to satisfy, for some choice of $\mathbf{L}$ and a positive scalar $\mu$,

$$\frac{\mathbf{C}(s\mathbf{I}-\mathbf{A})^{-1}\mathbf{L}}{\mu} = \mathbf{W}(s). \tag{2.23}$$

This implies that

$$\mathcal{R}(\mathbf{W}(s)) \subseteq \mathcal{R}(\mathbf{C}(s\mathbf{I}-\mathbf{A})^{-1}), \tag{2.24}$$

where $\mathcal{R}(\mathbf{M})$ denotes the range space of the operator $\mathbf{M}$. In general, each column of $\mathbf{W}(s)$ is restricted to be a linear combination of the columns of $\mathbf{C}(s\mathbf{I}-\mathbf{A})^{-1}$ which span at most an $n$-dimensional subspace of $L^{\infty}$. In particular, it may not be possible to select a matrix $\mathbf{L}$ which satisfies equation (2.23) at $s = 0$; this difficulty was the motivation for the work described by Birdwell and Laub [11].

As an example of the restrictiveness of this requirement, consider

$$\mathbf{A} = \begin{bmatrix} -1 & 0 \\ 0 & -2 \end{bmatrix} \quad \mathbf{C} = \begin{bmatrix} 1 & 1 \end{bmatrix} \quad \mathbf{L} = \begin{bmatrix} l_1 \\ l_2 \end{bmatrix}$$

$$\mathbf{W}(s) = \frac{1}{(s+3)^3},$$

for scalars $l_1$ and $l_2$, with $\mu$ absorbed into them. Note that $\mathbf{G}(s)$ is stable, strictly proper, and minimum phase. Then, any matrix $\mathbf{W}(s)$ which satisfies equation (2.23) must be of the form

$$\mathbf{W}(s) = \frac{l_1}{s+1} + \frac{l_2}{s+2} = \frac{(l_1+l_2)s + (2l_1+l_2)}{(s+1)(s+2)}, \tag{2.25}$$

which our chosen $\mathbf{W}(s)$ is not. The choice of a specific weighting matrix is determined by the particular system for which the design is being done, but there is usually some flexibility in its exact specification. A reasonable question to ask is whether the method proposed by Stein and Athans can yield a good approximation of $\mathbf{W}(s)$. By equation (2.25), the answer is no, since any weighting matrix allowable by Stein and Athans' method can roll off at high frequencies by at most −40dB/decade, while the particular weight above rolls off by −60dB/decade.

Weights in the form given by equation (2.25) comprise a very limited subset of strictly proper, finite dimensional, and stable weights. Athans has replied, in a private communication, that the article is intended to imply that the $\mathbf{C}$ and $\mathbf{A}$ matrices are to be augmented specifically to realize the weighting function $\mathbf{W}$; instead of just picking an $\mathbf{L}$ and $\mu$ they would change $\mathbf{C}$ and $\mathbf{A}$ so that they *can* pick an $\mathbf{L}$ and $\mu$. If there is a method for doing that, they do not provide it. Furthermore, a subject of ongoing research is the issue of whether the plant can be augmented to achieve an arbitrary loopshape. The difficulty with augmenting dynamics to the plant is that the outputs of the plant are affected, so that while the plant may remain stabilizable from $\xi(t)$ and $u(t)$, and remain detectable from $y(t)$ and $z(t)$ (thus guaranteeing the existence of an LQG solution), it may cease to be controllable from $\xi(t)$ and $u(t)$, and observable from $y(t)$ and $z(t)$, resulting in the system having arbitrarily bad robustness properties. Thus, viewing LQG/LTR as a weighted $H^2$ design method in the manner of Stein and Athans does not yield a viable design technique for general systems, except for those in which the input has a fixed power spectrum, and the output is bounded in power, or in which the input is a stochastic process of known power spectral density, and the variance of the output is the performance criterion [24].

LQG/LTR does produce a controller which is the same order as the system; this is an advantage, albeit a somewhat limited one. Unfortunately, even when not viewed as an $H^2$-optimization, the methodology has several disadvantages, the chief of which are the conservatism of its designs and its inability to handle disturbances at more than one point in the loop. Additionally, the compensator represents a partial inversion of the plant, and plant-inverting controllers can exhibit extremely bad robust performance properties if they are ill-conditioned [44,20]. These drawbacks have lead several researchers to abandon it altogether as a design methodology.

## 2.8 $H^\infty$ Optimal Control

A more recent design methodology currently receiving much attention is $H^\infty$ optimal control. This design methodology, first expounded in Zames [77], treats transfer functions as functions in the Hardy space $H^\infty$, discussed earlier, and, given two weighting matrices, minimizes the plant sensitivity function with respect to the $\infty$-norm. There are several reasons for treating sensitivity functions with the $\infty$-norm rather than the 2-norm LQG/LTR essentially employs. Foremost among these is the observation that the 2-norm treats disturbances as having a fixed power spectrum or known covariance properties, whereas, from a control viewpoint, it is perhaps more appropriate to treat disturbances as belonging to a prescribed set, e.g., having a bounded power spectrum. Also, robustness analysis seems more easily addressed in the $\infty$-norm than in the 2-norm. The presentation here is based upon results in [35,59,22] and more loosely upon those in [81,42,36,55,46]. Also, we will be considering real-rational matrix transfer functions, but all the computations detailed below can be done in a state-space framework [22,24].

The plant transfer function, $P(s)$, will be denoted by

$$P(s) = \left[\begin{array}{c|c} P_{11} & P_{12} \\ \hline P_{21} & P_{22} \end{array}\right](s) = \left[\begin{array}{c|cc} \mathbf{A} & \mathbf{B}_1 & \mathbf{B}_2 \\ \hline \mathbf{C}_1 & \mathbf{D}_{11} & \mathbf{D}_{12} \\ \mathbf{C}_2 & \mathbf{D}_{21} & \mathbf{D}_{22} \end{array}\right]$$

and means

$$P_{ij}(s) = \mathbf{C}_i(s\mathbf{I} - \mathbf{A})^{-1}\mathbf{B}_j + \mathbf{D}_{ij}.$$

The relationship between the frequency-domain polynomial matrix transfer function and its state space matrices is given by

$$G(s) = \left[\begin{array}{c|c} \mathbf{A} & \mathbf{B} \\ \hline \mathbf{C} & \mathbf{D} \end{array}\right] = \langle \mathbf{A}, \mathbf{B}, \mathbf{C}, \mathbf{D} \rangle .$$

The general feedback configuration we consider is shown in Figure 2.3. The equations describing this diagram are

$$\begin{bmatrix} y_1 \\ y_2 \end{bmatrix}(s) = P(s)\begin{bmatrix} u_1 \\ u_2 \end{bmatrix}(s) \equiv \left[\begin{array}{c|c} P_{11} & P_{12} \\ \hline P_{21} & P_{22} \end{array}\right](s)\begin{bmatrix} u_1 \\ u_2 \end{bmatrix}(s),$$

$$y_1 = Zu_1 = \mathsf{LFT}(P, K)u_1,$$

$$u_2 = K(s)y_2,$$

where the plant $P(s)$ is partitioned into $P_{11}$, $P_{12}$, $P_{21}$, and $P_{22}$ to be consistent with the dimensions of the input and output vectors $u_1$, $u_2$, $y_1$, and $y_2$. Note that

$$Z = \mathsf{LFT}(P, K) \equiv P_{11} + P_{12}K(I - P_{22}K)^{-1}P_{21},$$

and $\mathsf{LFT}(P, K)$ is called the linear fractional trasformation of $P$ and $K$. If $u_1$ is a prefiltered command and $y_1$ is the tracking error, then $Z$ is the sensitivity function $S$

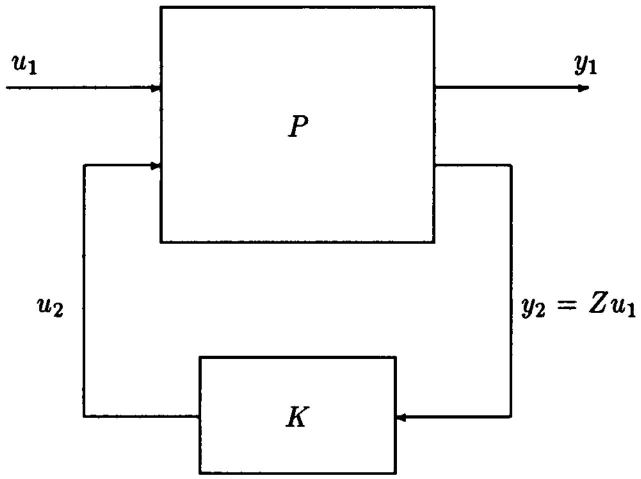


Figure 2.3: Basic System Block Diagram

defined earlier. We assume $P$ is stabilizable from $u_2$ ( $(\mathbf{A}, \mathbf{B}_2)$ is stabilizable) and is detectable from $y_2$ ( $(\mathbf{C}_2, \mathbf{A})$ is detectable).

In $H^\infty$-optimal control, the *standard problem* is to find a proper compensator $K$ such that $\|\mathsf{LFT}(P, K)\|_\infty$ is minimized under the constraint that $K$ stabilize $P$. It can be shown that, under the above assumptions, $K$ stabilizes $P$ iff $K$ stabilizes $P_{22}$ [22,35].

We now parameterize the set of all stabilizing compensators $K$. Introduce left and right coprime factorizations (respectively) of $P_{22}$,

$$P_{22} = M_l^{-1} N_l = N_r M_r^{-1}, \tag{2.26}$$

with the Bezout identity

$$\left[\begin{array}{c|c} V_r & U_r \\ \hline -N_l & M_l \end{array}\right] \left[\begin{array}{c|c} M_r & -U_l \\ \hline N_l & V_l \end{array}\right] = \left[\begin{array}{c|c} I & 0 \\ \hline 0 & I \end{array}\right],$$

for some $U_r$, $V_r$, $U_l$, and $V_l$. Then the compensators giving internal stability to $P_{22}$, hence to $P$, are given by

$$\begin{aligned} K &= -(U_l + M_r Q)(V_l - N_r Q)^{-1} \\ &= -(V_r - QN_l)^{-1}(U_r + QM_l), \end{aligned} \tag{2.27}$$

where $Q \in \mathbf{RH}_\infty$ is arbitrary. Using equations (2.26), (2.27), and the Bezout identity, a little algebra shows

$$K(I - P_{22}K)^{-1} = -(U_l + M_r Q)M_l,$$

hence we have

$$\mathsf{LFT}(P, K) = (P_{11} - P_{12}U_l M_l P_{21}) + (P_{12}M_r)Q(M_l P_{21}) = T_{11} + T_{12}QT_{21}.$$

Since $Q \in \mathbf{RH}_\infty$ implies that $\mathsf{LFT}(P, K)$ is stable, we must have $T_{11}$, $T_{12}$, and $T_{21}$ stable, even though $P$ is not necessarily stable. Thus all stabilizing compensators can be parameterized as an affine function of $Q$. This parameterization is called the *$Q$-parameterization.*

In view of the $Q$-parameterization then, an equivalent to the standard problem is the *model matching problem*: find $Q \in \mathbf{RH}_\infty$ such that

$$\min_{Q \in \mathbf{RH}_\infty} \|T_{11} + T_{12}QT_{21}\|_\infty$$

is achieved (or approximated arbitrarily closely).

Now, if we require that $P_{12}$ and $P_{21}$ have no multivariable zeros on $j\mathbb{R}$, the resulting $T_{12}$ is inner and the resulting $T_{21}$ is co-inner. Then $T_\perp$ and $\tilde{T}_\perp$ can be found so that the matrices

$$D_{12} = \begin{bmatrix} T_{12} & T_\perp \end{bmatrix} \text{ and } D_{21} = \begin{bmatrix} T_{21} \\ \tilde{T}_\perp \end{bmatrix}$$

are square and inner [12,22]. Additionally, we can reparameterize the $Q$-parameterization to yield

$$\left\| T_{11} - D_{12}\hat{Q}D_{21} \right\|_\infty = \left\| \begin{bmatrix} R_{11} - \hat{Q} & R_{12} \\ R_{21} & R_{22} \end{bmatrix} \right\|_\infty ,$$

where

$$R = \begin{bmatrix} R_{11} & R_{12} \\ R_{21} & R_{22} \end{bmatrix} = \begin{bmatrix} -T_{12}^H \\ T_\perp^H \end{bmatrix} T_{11} \begin{bmatrix} T_{21}^H & \tilde{T}_\perp^H \end{bmatrix} .$$

This is the *general distance problem* (GDP). Minimizing this quantity over $Q \in H^\infty$ yields the optimal $H^\infty$ controller. A corresponding GDP can be formulated for $H^2$, but is relatively trivial as $L^2$ is a Hilbert space, and orthogonal projection can be used. The GDP in $H^\infty$ is much more complicated, and its reliable and efficient solution is currently a research topic of substantial interest, since an iterative technique is presently the only method of solving it except in certain special cases [12]. Some cases of the GDP result in Hankel-norm approximation problems which can usually be solved [35,37]. At this point we also see why $H^\infty$ controllers have a high order: the order of $K$ is equal to the original order of $P$ plus the order of $Q$. Since the order of $Q$ is not restricted, the order of the controller can be high, although an exciting rumour is circulating among the control community that the actual minimum to this problem has order $n - 1$ (one less than the order of the system).

In this section we have given an extremely brief overview of $H^\infty$-optimal synthesis, ignoring finer points and most of the computational details. For more detail, the report by Doyle and Chu [24] is particularly good; also, see the other references given in this section.

## 2.9 $\mu$-synthesis

This is the last design methodology to be reviewed; $\mu$-synthesis is wholly the work of Doyle [17,18,24]. This design methodology is also the newest and can be considered still in its infancy. The advantage this design methodology has over others is that it can guarantee robust performance as well as robust stability, and thus the technique has great potential. The difficulty with it at present is there are no good methods for doing the numerical calculations involved, although work is progressing in this area. Hence $\mu$-synthesis is currently limited as a practical design methodology. Computing $\mu$ for various existing controllers also provides a method for analyzing various aspects of system performance [20,19]. The basic configuration which it deals with is a slight modification of Figure 2.3 and is shown in Figure 2.4. Doyle often points out [17,27,19,23] that "any interconnection of inputs and outputs with components and perturbations may be rearranged into this form." For the purposes of synthesis, the $\Delta$ block in Figure 2.4 can be absorbed into the plant, resulting in Figure 2.3; for the purposes of analysis, the controller $K$ can be absorbed into the plant, resulting in Figure 2.5, where $P_K$ is the plant with controller absorbed. For the synthesis model, the transfer function is

$$e = \mathsf{LFT}(P, K)v,$$

while for the analysis model, the transfer function is

$$e = \mathsf{LFT}(P_K, \Delta)v.$$

Doyle's $\mu$ is a generalization of singular values and the spectral norm [17]. The $\Delta$

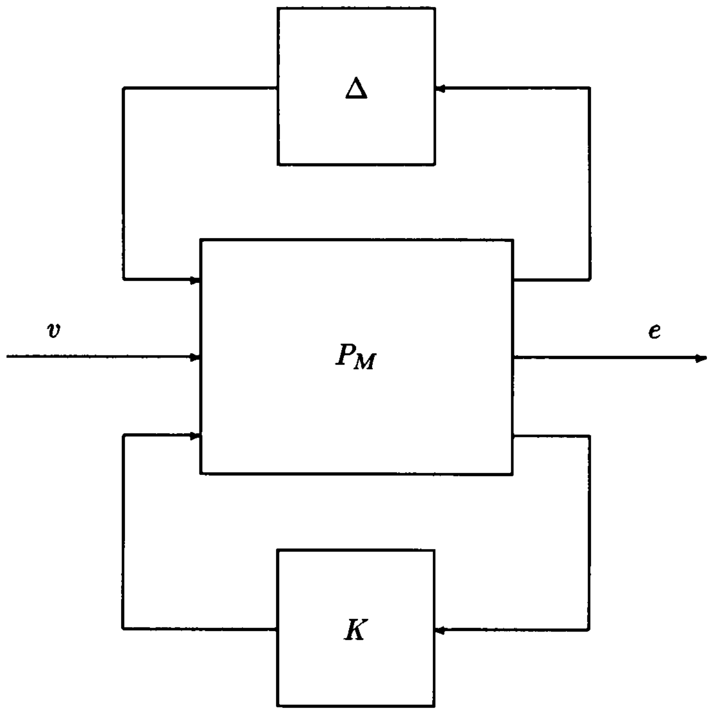


Figure 2.4: General Model

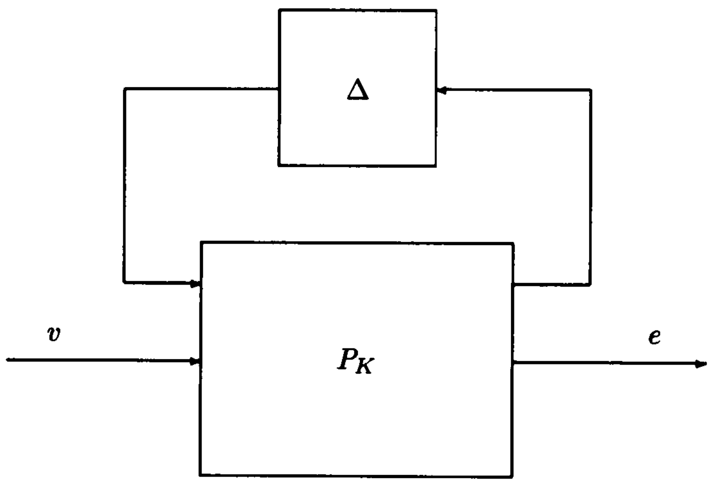


Figure 2.5: Analysis Model

in Figure 2.4 is a a member of a set such as

$$\boldsymbol{\Delta} = \left\{ \operatorname{diag}(\delta_1, \ldots, \delta_n, \Delta_1, \ldots, \Delta_m) \mid \delta_i \in \mathbb{R},\ \Delta_j \in \mathbb{C}^{k_j \times k_j} \right\},$$

or its bounded subset

$$\boldsymbol{B\Delta} = \{\Delta \in \boldsymbol{\Delta} \mid \sigma_{\max}(\Delta) < 1\}.$$

Each $\Delta_j$ above represents a perturbation occurring for one component in one part of the loop, and each $\delta_i$ represents, for instance, a parameter variation in one component of the system. For the set $\Delta$ above, $\mu$ is defined as the the positive real function which satisfies the property

$$\det(I - M\Delta) \neq 0\ \forall \Delta \in \boldsymbol{\Delta},\ \sigma_{\max}(\Delta) < \gamma,\ \text{iff}\ \gamma\mu(M) \leq 1. \tag{2.28}$$

Noting that $\mu(M)$ depends on $\boldsymbol{\Delta}$, if $\mu(M) \neq 0$ then it can be shown that

$$\mu(M) = \left( \min_{\Delta \in \boldsymbol{\Delta}} \{\sigma_{\max}(\Delta) \mid \det(I - M\Delta) = 0\} \right)^{-1}. \tag{2.29}$$

Doyle himself points out that equation 2.29 does not provide a very useful characterization of $\mu$ [23]. Reasonable methods of approximating it exist [20]. In terms of design, the robust performance requirement is obtained by adding a block $\Delta_{m+1}$ to $\boldsymbol{\Delta}$ which translates the performance specification into a "stability" requirement. The flexibility of $\mu$ allows this to be done, and is the real benefit of using $\mu$, but the exact method of doing this has not yet been made clear in the literature. When this is done, robust performance and stability are *guaranteed* if a stabilizing compensator can be found such that the condition

$$\|\mathsf{LFT}(P_K, \Delta)\|_\infty < 1 \text{ and } \mathsf{LFT}(P_K, \Delta) \text{ stable } \forall \Delta \in \boldsymbol{\Delta},$$

or, equivalently,

$$\|\mu(P_K)\|_\infty \leq 1,$$

holds for the structure shown in Figure 2.5. Doyle has *ad hoc* techniques for which he can approximately carry out the computations, but, as mentioned before, no effective way of doing them exists yet [20].

## 2.10 Problem Formulation

The problem this thesis addresses can now be stated. Recent efforts in control theory try to characterize the structure of the types of uncertainty faced in the real world. Motivating these efforts is dissatisfaction with the linear quadratic design methodologies developed in the 1970's, particularly LQG/LTR, although it is the best of all those methods. The dissatisfaction exists somewhat because the robust stability and nominal performance given by equations (2.19), (2.20), and (2.22) often lead to excessive conservatism in the design, but they are *not* conservative if all the perturbations which they bound can in fact occur. This means, in effect, that some of the perturbations which satisfy the bounds of equations (2.19)-(2.22) are not physically possible. The structured singular value of Section 2.9 has perhaps the greatest potential in this direction, but cannot yet be considered effective. Other attempts to deal with this problem have been in the area of loopshaping, or minimizing weighted sensitivity functions to obtain desirable singular value characteristics. The selection of weights is important, but the attributes of the weights are often overlooked in trying to do the minimization problem. Equations (2.19)-(2.22) constrain only the magnitude of the perturbations as a function of frequency, and thus place *spherical* requirements on them. This thesis generalizes these particular performance and stability requirements into *elliptical* constraints as a function of frequency, that is, constrain perturbations by both magnitude and direction as a function of frequency. Using the Cauchy-Schwartz inequality in proceeding from equation (2.21) to (2.20) on page 19 translated the bound into a spherical constraint as a scalar function of frequency, forcing the return difference and inverse return difference to satisfy a spherical bound in $\mathbb{C}^n$ or $\mathbb{C}^m$ as function of frequency, thus losing all directional information. To maintain directional information, we must work directly with equation (2.21), but still provide a useful bound. This can be accomplished by finding *matrices* $B_H(j\omega)$ and $B_L(j\omega)$ from constraints on nominal performance and robust stability such that

$$\left\| B_H L(I+L)^{-1} \right\|_\infty \leq 1, \tag{2.30}$$

and

$$\left\| (I+L)^{-1} B_L^{-1} \right\|_\infty \leq 1. \tag{2.31}$$

Inequalities such as these characterize nominal perfomance and robust stability. As an illustration of these bounds, we shall show how to translate these constraints into a GDP for the purpose of synthesizing a controller to meet them, and show how to translate the LQG/LTR controller into a form suitable for $\mu$-analysis.

### 2.11 Survey of Related Work

Stability is of fundamental importance in control. $H^\infty$ and $H^2$ minimization methods start with the parameterization of all stabilizing controllers [24], as we did in Section 2.8. This parameterization is due to Youla *et al.*, [75,76], who applied it to Wiener-Hopf filters, which are essentially the frequency domain analogs of LQG regulators. Since that basic work many researchers have redone and perhaps slightly improved the method: a few of these efforts are [15,57,28,54]. All are algebraic and frequency-domain efforts; Doyle [22] was the first to do the computations in state-space, thus making them practical for large and complex systems.

These parameterizations are used in the minimization of weighted sensitivity functions or loopshaping designs. These minimizations are done based on mathematical results by Sarason [66] and Adamjan, Arov, and Krein [1]. Weighted $H^\infty$-sensitivity minimization is not new: The fundamental work in this area is that of Zames [77]. By minimizing a sensitivity function, Zames is attempting to design a controller of optimal robustness, and works dealing with strict sensitivity minimization tend to ignore other performance objectives. Since that paper, the results have been significantly extended to various systems: SISO [81,46,48], MIMO [36], decentralized [80], delay systems [32], distributed plants [31,30], and time-varying periodic systems [29], to name a few. Mixed sensitivity, that is, minimizing both sensitivity and complementary sensitivity over frequency bands have also been studied [45,71,33,34]. As these bear close resemblance to aspects of this paper, we will return to them later. $H^\infty$ techniques have also been applied to positivity and sector constraints [64,59], and computer code written to solve these problems.

The fundamental papers on LQG/LTR are Doyle and Stein [26], Lehtomaki *et al.* [52], and Safonov, Laub, and Hartmann [63], although the last two are more concerned with general issues of MIMO performance and robustness. LQG/LTR is adaptable to computer implementation [5,7,9], and an expert system has been written with it as the design methodology [10,13,4,8]. LQG/LTR has been used to design many real-world controllers. Unfortunately, as pointed out previously, it suffers several limitations. The one we will focus on here is that of (un)structured uncertainty.

LQG/LTR deals with *unstructured* uncertainty. This means that all perturbations of a given magnitude or less are allowed to perturb the plant, and LQG/LTR designs a controller stable against them all, often leading to excessively conservative results, as pointed out by Doyle [17]. He developed the various $\mu$-based results to explicitly deal with this problem, as well as the added problem of simultaneous perturbations at different points in the loop.

Structure in the uncertainty has also been considered by Lehtomaki *et al.* [51,50]. These results, while different from the ones this thesis pursues, have the same motivation, that of characterizing the types of perturbations that can destabilize a plant. Using the multivariable Nyquist criterion, he considers various small matrices $\mathbf{E}$ (perturbations) which can destabilize a complex matrix $\tilde{A}$. The complex matrix $\tilde{A}$ is the system matrix $sI - \mathbf{A} + \mathbf{BC}$, and it is destabilized if $\tilde{A} + \mathbf{E}$ is singular, so that the characteristic polynomial is zero. The matrices capable of accomplishing this are then written in terms of the basis of orthogonal matrices of $\mathbb{C}^{p \times m}$ given by $u_i v_j^H$, where $u_i$ and $v_j$ are the singular vectors of $\tilde{A}$ as a function of frequency. The characterization given by Lehtomaki is different from the one to be given here in that his approach deals with different representations of uncertainty perturbing the plant, i.e., other than the simple multiplicative perturbations we consider, rather than with the structure of the perturbations as a function of frequency. The approach is interesting, but does not seem to offer insight into the basic robustness issues.

Many other researchers have noted the need to take directional information into account when considering weights or uncertainty. Doyle has commented upon it in [23], as well as Morari [20], who considers the "ease" of driving a system at various trajectories. Safonov has independently noted the usefulness of definite matrices in weighting and robustness studies [58]. What distinguishes this thesis from this work and others, is that the above works all assume weighting matrices $W_i$, $1 \leq i \leq 4$, and proceed from there; this thesis gives a characterization, in terms of system performance and robustness of what weighting matrices to select for a particular control design.

Robustness and performance results derived in this thesis are also translated into a mixed sensitivity problem, mentioned above. The mixed sensitivity function that Foo and Postlethwaite [33,34] study is almost identical to the one we consider, having the form

$$\left\| \begin{bmatrix} W_1 S \\ W_2 T \end{bmatrix} \right\|_{\infty}.$$

They propose an actual method of computing the minimum to their problem, although it is based in the frequency domain and therefore probably inapplicable for realistic design problems. Foo and Postlethwaite also consider the structure of perturbations in their paper, basing the discussion upon the characterization of Lehtomaki above, and try to introduce other optimization constraints to the problem. This paper is parallel and very closely related to that of this thesis, but fundamentally different. Verma and Jonckheere [71] discuss mixed sensitivity optimization for SISO systems in relation to Helton's broadband matching problem [42], while Jonckheere and Verma [45] discuss mixed sensitivity in relation to putting the problem into the GDP form. The results in Jonckheere and Verma's paper ultimately lead to the GDP developments given in Chu, Doyle and Lee [12].

Finally, many papers have noted the connection between design using $H^{\infty}$, $H^2$, and LQG [23,22,39,67,68]. This is in part due to the nice interpretation that can be given to LQG in terms of an $H^2$-optimization problem. Sideris [67] considers LQG/LTR in the $H^{\infty}$ context, but only as a method of synthesizing an approximately optimal $H^{\infty}$

controller for the SISO case. Packard, in [56], has commented upon the $H^\infty$ nature of LQG/LTR, but this thesis is the first work to make the connection clear.

# CHAPTER 3

# ROBUSTNESS, PERFORMANCE, AND THE GDP

The major results of this thesis are presented here. In the first section, a slight generalization of the unstructured uncertainty robustness condition is given; in the second, a nominal performance theorem analagous to the robustness condition is proved; and in the final section, it is shown how to translate these problems into a GDP for the purpose of synthesizing a controller meeting them.

## 3.1 Robust Stability

As mentioned earlier, *robust stability* is the ability of a control system to remain stable in the presence of uncertainty, which is present because the mathematical models a control engineer uses are not exact. In the SISO case, robustness properties are characterized by gain and phase margins. For MIMO systems, such simple measures are inadequate. The basic work on robustness in MIMO systems was done by Safonov and Athans [64,60,62,61], and by Doyle [16]. The following is a slight generalization of the well-known results presented by Doyle.

**Theorem 2.** *Let $B_H \in H^{\infty}_{p\times p}$ be a matrix transfer function with no poles on $j\mathbb{R}$, and which is full rank on $j\mathbb{R}$. Let $\mathcal{D}_{B_H}$ be the class of all stable perturbations $\Delta$ such that $B_H^H(j\omega)B_H(j\omega) - \Delta^H(j\omega)\Delta(j\omega)$ is positive definite $\forall\omega \in \mathbb{R}$. Assume the nominal complementary sensitivity $T_0 = L_0(I + L_0)^{-1}$ is stable. Then the closed-loop system is* *stable with respect to all perturbations in $\mathcal{D}_{\mathcal{B}_{\mathcal{H}}}$ if*

$$\|B_H T_0\|_\infty \leq 1. \tag{3.1}$$

PROOF. For all $x \in \mathbb{C}^p$ with $\|x\|_2 = 1$ and almost all $\omega \in \mathbb{R}$ we have

$$\begin{aligned}
\|x\|_2 &= 1 \geq \|B_H T_0\|_\infty \geq \|B_H(j\omega)T_0(j\omega)\|_2 \\
&\geq \|B_H(j\omega)T_0(j\omega)\|_2 \, \|x\|_2 \\
&\geq \|B_H(j\omega)T_0(j\omega)x\|_2 \\
&> \|\Delta(j\omega)T_0(j\omega)x\|_2 .
\end{aligned}$$

where the last inequality is due to Lemma 1 on page 8.

By the Cauchy Residue Theorem [2], and, since $\Delta(s)T_0(s)$ is analytic in the CRHP, we have the following $\forall s_0 \in \mathbb{C}^+$,

$$x = \frac{1}{2\pi j}\int_{-\infty}^{\infty} \left.\frac{x}{s - s_0}\right|_{s=j\omega} d\omega$$

$$\Delta(s_0)T_0(s_0)x = \frac{1}{2\pi j}\int_{-\infty}^{\infty} \left.\frac{\Delta(s)T_0(s)x}{s - s_0}\right|_{s=j\omega} d\omega,$$

hence,

$$\begin{aligned}
\|\Delta(s_0)T(s_0)x\|_2 &= \frac{1}{2\pi j}\int_{-\infty}^{\infty} \left.\frac{\|\Delta(s)T_0(s)x\|_2}{s - s_0}\right|_{s=j\omega} d\omega \qquad (3.2)\\
&< \frac{1}{2\pi j}\int_{-\infty}^{\infty} \left.\frac{\|x\|_2}{s - s_0}\right|_{s=j\omega} d\omega \\
&= \|x\|_2 ,
\end{aligned}$$

where the equality in equation (3.2) comes from the fact that the norm of an analytic function is an analytic function. Thus,

$$\|x\|_2 > \|\Delta(s_0)T_0(s_0)x\|_2 .$$

Since this holds for all $x$ with $\|x\|_2 = 1$ it holds for all $x \neq 0 \in \mathbb{C}^p$. Therefore, by Lemma 3 on page 14, the system is stable with respect to all perturbations in $\mathcal{D}_{B_H}$. Q.E.D.

Recall the LQG/LTR robust stability condition is that for all stable perturbations $\Delta(s)$ satisfying

$$\sigma_{\max}(\Delta(j\omega)) = \|\Delta(j\omega)\|_2 < b_H(w)\, \forall\omega \in \mathbb{R},$$

robust stability is guaranteed iff

$$\sigma_{\min}(I + L_0^{-1}(j\omega)) > b_H(\omega)\, \forall\omega \in \mathbb{R}.$$

Let $\mathcal{D} = \{\Delta : \|\Delta\|_2 < b_H(\omega)\, \forall\omega\}$. For all $\Delta \in \mathcal{D}$ and $\forall x \in \mathbb{C}^m$

$$\|\Delta x\|_2 \leq \|\Delta\|_2\, \|x\|_2 < b_H(\omega)\, \|x\|_2 = \|b_H(\omega)x\|_2 = \|b_H(\omega)Ix\|_2\, \forall\omega \in \mathbb{R},$$

and, by Lemma 1, this is true iff

$$b_H^2(\omega)I - \Delta^H(j\omega)\Delta(j\omega)$$

is positive definite $\forall\omega \in \mathbb{R}$. And, if $b_H(\omega) < \sigma_{\min}(I + L_0^{-1}(j\omega))$ then

$$b_H(\omega) < \frac{1}{\left\|(I + L_0^{-1}(j\omega))^{-1}\right\|_2} = \frac{1}{\|T(j\omega)\|_2}\, \forall\omega \in \mathbb{R}.$$

Hence,

$$b_H(\omega)\, \|T_0(j\omega)\|_2 = \|(b_H(\omega)I)T_0(j\omega)\| < 1,$$

which implies

$$\|(b_H I)T_0\|_\infty \leq 1.$$

Thus the LQG/LTR robust stability condition is a special case of Theorem 2.

The above result considers the structure of the multiplicative perturbation as well as its magnitude. Though not as powerful a tool as $\mu$-analysis, it does not suffer the computational drawbacks from which $\mu$-analysis suffers. The result also gives a characterization of the structure of perturbations and thus insight into the nature of destabilizing perturbations. One drawback of the theorem, shared by singular value methods but not by $\mu$, is that the perturbations are constrained to be stable. Parameter variations in components, such as those caused by aging, can result in poles migrating across $j\mathbb{R}$. No solution to this problem short of $\mu$ is apparent.

## 3.2 Nominal Performance

This discussion considers performance criteria at the input of the plant only; that is, we are only concerned about controlling the response of the input $u$ in Figure 2.1 on page 11 to commands $r$. From equation (2.7), it is clear that the error response is controlled by placing constraints on the input sensitivity $S$. Given that, our performance criterion is: *restrict the the magnitude of the input response* u *as a function* p($\omega$,r) *of frequency, input trajectory, and input magnitude.* This definition was motivated by the work of Zames [78,79]. We require the function $p$ to be linear so we can deal with it in the frequency domain. This is an obvious approximation which is clearly violated in the physical world. There are always nonlinear saturation effects such as slewing or "clipping" in a physical device, which can change both the magnitude and the trajectory of the response (see, for example, [25]). We can only require linearity over a limited operating range, but we will pretend that saturation effects do not exist for the purpose of this discussion. Then the function $p$ can be a matrix transfer function $B_L$, and the nominal performance condition is to have $u$ satisfy, for all $r$,

$$\|u\|_2 \leq \|B_L r\|_2 .$$

**Theorem 3.** *Let $B_L \in H^{\infty}_{m\times m}$ be a matrix transfer function which has full rank on $j\mathbb{R}$ and which adequately characterizes the desired error response. Then the closed-loop system satisfies the nominal performance criterion if and only if*

$$\left\|(I+L)^{-1}B_L^{-1}\right\|_{\infty} \leq 1. \tag{3.3}$$

PROOF. Immediately from Lemma 2 on page 8, we have that the nominal performance constraint is satisfied if and only if

$$B_L^H(j\omega)B_L(j\omega) - (I + L_0(j\omega))^{-H}(I + L_0(j\omega))^{-1}$$

is positive semi-definite $\forall\omega \in \mathbb{R}$, if and only if

$$y^H\left(B_L^H(j\omega)B_L(j\omega) - (I + L_0(j\omega))^{-H}(I + L_0(j\omega))^{-1}\right)y \geq 0 \ \ \forall\omega \in \mathbb{R}\ \forall y \in \mathbb{C}^m. \tag{3.4}$$

Since $B_L$ is full-rank it is surjective, so $\forall y \in \mathbb{C}^m$ there exists $x \in \mathbb{C}^m$ such that

$$y = B_L^{-1}x.$$

Substituting into equation (3.4) then gives

$$(B_L(j\omega)^{-1}x)^H\left(B_L^H(j\omega)B_L(j\omega) - (I + L_0(j\omega))^{-H}(I + L_0(j\omega))^{-1}\right)(B_L(j\omega)^{-1}x) \geq 0,$$

for all $\omega \in \mathbb{R}$ and $\forall x \in \mathbb{C}^m$, or,

$$x^H\left(I - B_L^{-H}(j\omega)(I + L_0(j\omega))^{-H}(I + L_0(j\omega))^{-1}B_L^{-1}(j\omega)\right)x \geq 0 \ \ \forall\omega \in \mathbb{R}\ \forall x \in \mathbb{C}^m,$$

which is true if and only if

$$\left\|(I + L_0(j\omega))^{-1}B_L^{-1}(j\omega)\right\|_2 \leq 1 \ \ \forall\omega \in \mathbb{R}.$$

This is equivalent to equation (3.3) and therefore by Lemma 2 on page 8 the theorem is true. Q.E.D.

Recall that the LQG/LTR constraint is to attenuate the error or disturbance equally in all directions, i.e., requiring equation (2.22) on page 19 to hold. From the properties of singular values, equation (2.22) becomes,

$$\frac{1}{\|(I + L_0(j\omega))^{-1}\|_2} \geq b_L(j\omega),$$

where we have changed the function $b_L : \mathbb{R} \to \mathbb{R}^+$ to $b_L : j\mathbb{R} \to \mathbb{R}^+$. Then, this becomes

$$\left\|(I + L_0(j\omega))^{-1}(\frac{1}{b_L(j\omega)I})^{-1}\right\|_2 \leq 1 \; \forall\omega,$$

which is equivalent to

$$\left\|(I + L)^{-1}(\frac{1}{b_L}I)^{-1}\right\|_\infty \leq 1.$$

Hence, we see that the LQG/LTR nominal performance criterion is a special case of Theorem 3.

### 3.3 Translation to a GDP

Recall that any interconnection of inputs, blocks, and perturbations can be restructured to fit into the form of Figure 2.4 on page 29. We now give an example of this type of translation by putting the robustness and performance requirements into a block diagram and translating them into a GDP; a controller could then be synthesized by solving the GDP. This is not difficult; its purpose is to show how LQG/LTR fits into the framework of current efforts in control theory. This emphasizes the essential relationship between $H^\infty$ performance specifications and LQG/LTR, and how LQG/LTR in its proper interpretation is just one way of meeting those specifications; $H^\infty$-optimization is another.

Let $W_b$ and $W_u$ be scalar high-pass filter and low-pass filters, respectively, with idealized characteristics shown in Figure 3.1. Both have their cut-off frequency at the desired crossover frequency $\omega_c$. We assume that the transition from the low frequency uncertainties to the high frequency modelling errors is well-defined. As mentioned previously, this is not usually the case in practice, but crossover characteristics are so specific to particular problems that it is difficult to make generalizations. The idea of using these filters is to zero the robust stability information at frequencies below $\omega_c$ and zero the performance bounds at frequencies above it. By incorporating these weights into the synthesis, we force the compensator to meet the performance bounds at low frequencies and the robustness bounds at high frequencies. Note that we drop the $s$ dependency of everything in this section, and that performance and robustness criteria are specified with respect to the input of the plant.

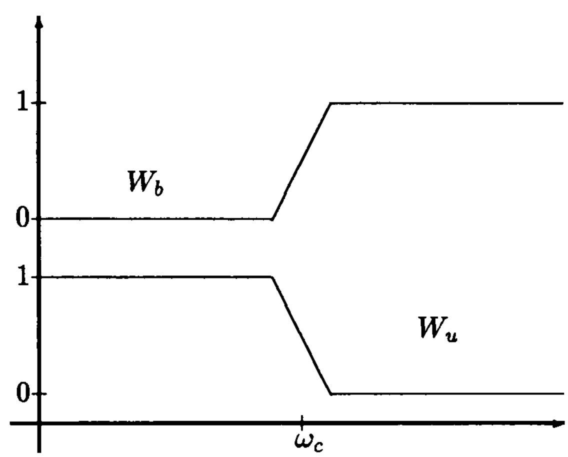


Figure 3.1: Scalar Weights

The synthesis block diagram is shown in Figure 3.2. Here $G$ is the plant, $B_L^{-1}$ and $B_H$ are as described in Theorems 2 and 3, $K$ is the compensator to be designed, $r$ represents all exogenous inputs, $k$ represents $r$ after filtering, $u_1$ is the input to the true plant, $u$ is the weighted input, $y$ is the true output of the system, $w$ is the feedback, $z$ represents the robustness constraints, and $v = [u \;\; z]^T$ is the quantity to be minimized. Note that $B_L^{-1}$, $B_H$, $W_b$, $W_u$, and the structure shown in Figure 3.2 are neither real filters nor the real physical structure of the system; they are control theory "tricks" for mathematically dealing with behavior which a physical system has. Let $X = [B_L^{-1}(1 - W_b) + W_b]$; note that $u_1 = k - w$, so that

$$\begin{aligned}
k &= [B_L^{-1}(1 - W_b) + W_b]r = Xr, \\
u &= W_u u_1 = W_u(k - w) = W_u k - W_u w = W_u X r - W_u w, \\
y &= G u_1 = G(k - w) = Gk - Gw = GXr - Gw, \\
z &= B_H W_b(k - u_1) = B_H W_b(k - k + w) = B_H W_b w, \\
v &= \begin{bmatrix} u \\ z \end{bmatrix} = \begin{bmatrix} \begin{bmatrix} W_u X \\ 0 \end{bmatrix} & \begin{bmatrix} -W_u \\ B_H W_B \end{bmatrix} \end{bmatrix} \begin{bmatrix} r \\ w \end{bmatrix}.
\end{aligned}$$

Then

$$\begin{bmatrix} v \\ y \end{bmatrix} = \begin{bmatrix} \begin{bmatrix} W_u \\ 0 \end{bmatrix} X & \begin{bmatrix} -W_u \\ B_H W_b \end{bmatrix} \\ GX & -G \end{bmatrix} \begin{bmatrix} r \\ w \end{bmatrix},$$

$$P = \left[\begin{array}{c|c} P_{11} & P_{12} \\ \hline P_{21} & P_{22} \end{array}\right] = \left[\begin{array}{c|c} \begin{bmatrix} W_u \\ 0 \end{bmatrix} X & \begin{bmatrix} -W_u \\ B_H W_b \end{bmatrix} \\ \hline GX & -G \end{array}\right]$$

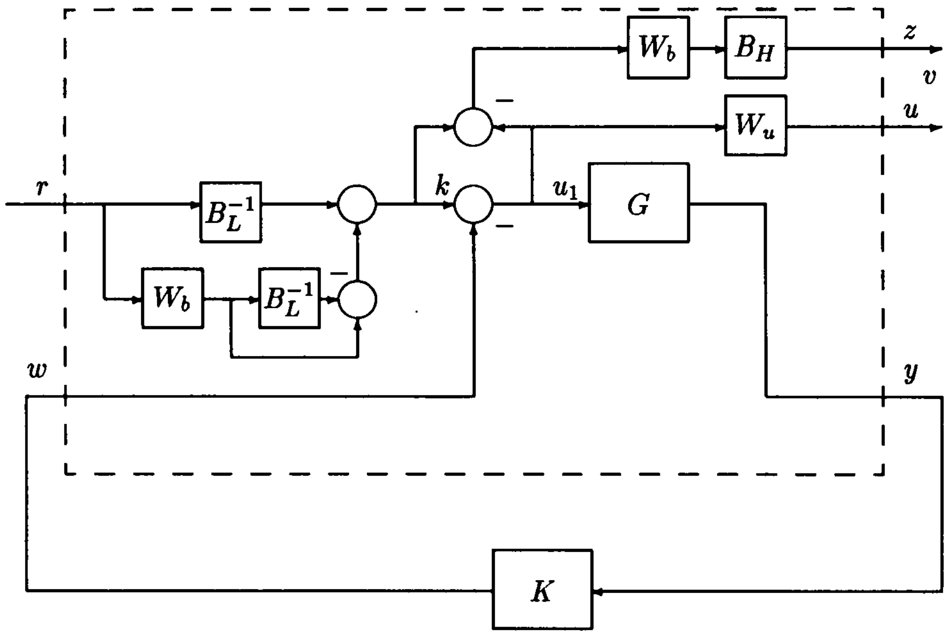


Figure 3.2: General Distance Problem Diagram

Recall that the GDP minimizes the linear fractional transformation from $r$ to $v$, i.e.,

$$\begin{aligned}
v &= \mathsf{LFT}(P,K)r = [P_{11} + P_{12}K(I - P_{22}K)^{-1}P_{21}]r \qquad (3.5)\\
&= \begin{bmatrix} W_u \\ 0 \end{bmatrix} Xr + \begin{bmatrix} -W_u \\ B_H W_b \end{bmatrix} K(I+KG)^{-1}GXr \\
&= \begin{bmatrix} W_u \\ 0 \end{bmatrix} Xr + \begin{bmatrix} -W_u \\ B_H W_b \end{bmatrix} KG(I+KG)^{-1}Xr \\
&= \begin{bmatrix} W_u Xr \\ 0 \end{bmatrix} + \begin{bmatrix} -W_u KG(I+KG)^{-1}Xr \\ B_H W_b KG(I+KG)^{-1}Xr \end{bmatrix} \\
&= \begin{bmatrix} W_u(I - KG(I+KG)^{-1}Xr \\ B_H W_b KG(I+KG)^{-1}Xr \end{bmatrix} \\
&= \begin{bmatrix} W_u(I+KG)^{-1}Xr \\ B_H W_b KG(I+KG)^{-1}Xr \end{bmatrix} \\
&= \begin{bmatrix} W_u(I+KG)^{-1}(B_L^{-1}(1-W_b)+W_b) \\ B_H W_b KG(I+KG)^{-1}(B_L^{-1}(1-W_b)+W_b) \end{bmatrix} r. \qquad (3.6)
\end{aligned}$$

For $\omega \ll \omega_c$, $W_b \approx 0$ and $W_u \approx 1$ so equation 3.6 becomes

$$v = \begin{bmatrix} (I+KG)^{-1}B_L^{-1} \\ 0 \end{bmatrix} r,$$

which is the input sensitivity, the quantity to minimize to meet the nominal performance requirement. At $\omega \gg \omega_c$, $W_b \approx 1$ and $W_u \approx 0$ so equation 3.6 becomes

$$v = \begin{bmatrix} 0 \\ B_H KG(I+KG)^{-1} \end{bmatrix} r,$$

which is input complementary sensitivity, the quantity to minimize to meet the robust performance requirement. There is a trade-off at $\omega \approx \omega_c$, which is to be expected.

In the above, the matrix $P$ is the starting point for the GDP minimization. To meet the requirements for a GDP, $P$ must satisfy the following requirements [24]:

1. $P_{ij}$ must be rational. This is clearly the case for the $P$ above.

2. $P_{12}$ and $P_{21}$ must have no zeroes on $j\mathbb{R}$. $B_L$ and $B_H$ were required to be full rank, therefore they will not have multivariable zeroes on $j\mathbb{R}$; hence $X$ will not have multivariable zeroes on $j\mathbb{R}$, and $W_u$ and $B_H W_b$ will not be zero simultaneously. A limitation of the method is that $G$ may have multivariable zeros on $j\mathbb{R}$, and thus will have to be checked.

If the above conditions are met, that is, if $G$ has no multivariable zeroes on $j\mathbb{R}$, then $P$ can be approximately solved using the methods outlined in [24].

The $H^\infty$-performance of an existing MIMO controller can be gauged using the above formulation; we note that simpler techniques exist when the controller is SISO [35,53].

# CHAPTER 4

# CONCLUSIONS

This thesis has generalized the stability and performance criteria of the LQG/LTR design methodology and shown how they are related to $H^\infty$-optimization methods. In truth, calling the performance criteria of equations 2.20 and 2.22 LQG/LTR criteria is somewhat misleading since they apply to any control system, not just those designed by the LQG/LTR methodology. A simple method was given for translating these criteria into an optimization problem which can be approximately solved by existing techniques and which provides a method for gauging the $H^\infty$ performance of an existing controller. The generalized criterion for robust stability, Theorem 2, provides for including information about the trajectories of disturbances in addition to their magnitudes. Moreover, it assumes no correlation between disturbances in different channels of the output, an assumption implicit in spherical criteria of the form $b(s)I$. The primary limitation of this approach would seem to be that all the information about uncertainty must be related to one point in the loop. The criterion on nominal performance, Theorem 3, gives necessary and sufficient conditions for independent attenuation of each input channel's error response, as well as attenuation of cross-feed between input channels. This type of criterion is much more useful than a spherical bound which only gives one general attenuation factor for all channels of the error response and provides no limit on cross-feed (since all the cross-feed entries of the weight are zero). Also, these two theorems provide guidelines for incorporating information about the trajectories and magnitudes of inputs into weights, for any design methodology using them, be it LQG/LTR or $\mu$-synthesis. Note that although all results were formulated in terms of performance bounds and perturbations occurring at the input of the plant, equivalent formulations exist for the dual case of performance bounds and perturbations at the output of the plant, but not at both.

Moreover, performance specifications for an LQG/LTR design can be translated into constraints for an $H^\infty$ optimization problem with the results given in this thesis. While it does provide a way of translating specifications into a form suitable for $\mu$-synthesis also, the desirability of doing this is arguable, since $\mu$-synthesis is a much more powerful technique, and therefore a re-evaluation of performance specifications seems warranted if this technique is to be used.

This thesis does have several limitations. The original motivation for the work was an attempt to reduce the conservatism of LQG/LTR designs by adding structure to the type of uncertainty the robustness and performance criteria characterize. The $H^\infty$ nature of the generalizations then led to the formulation of the problem as an $H^\infty$-optimization, and the issue of synthesizing an LQG/LTR controller with the resulting tests, as well as a reduction of conservatism in the design, was not addressed. Some effort was expended in this direction, but noteworthy results were not obtained, and this remains a possible direction for future research efforts, although reasons for abandoning LQG/LTR as a design methodology will be given below.

The subject of robust performance has not been addressed, and its importance in control system design should not be underestimated. If the system is ill-conditioned, then its nominal performance will be drastically different from its true performance, for even a small perturbation to the plant, and the effort that went into the compensator design would essentially be wasted. Robust performance is generally intractable short of employing powerful techniques such as $\mu$-synthesis; however, the $B_L$ matrix developed in this thesis would provide a performance block for $\mu$-synthesis.

Obtaining robust performance in a design represents a reasonable motivation for abandoning LQG/LTR for $H^\infty$-optimality. LQG/LTR attempts to invert the plant, and recent results [20] on plant inversion indicate that it is definitely not a technique to employ if robust performance is a consideration. The reason for this is that plant inversion is just pole-zero cancellation, and since the poles and zeros of the plant are not known exactly the sensitivity of the plant to small perturbations can be arbitrarily large. While results such as Theorem 2 guarantee that inexact pole-zero cancellation will not destabilized the plant, no guarantees can be made about the performance, except for an upper bound, as mentioned previously. While $H^\infty$ designs do not guarantee robust performance either, they are not exacerbating the situation with a plant inversion. An excellent topic for future research is to attempt to characterized the performance robustness of $H^\infty$ designs analogous to the results for plant inversion given in [20].

Another issue this thesis did not address was the possibility of using a similarity transform of $B_H$ and $B_L$ as weights in order to scale the inputs and outputs into a diagonal or spherical form. This appears to be common among practicing control engineers employing the techniques discussed in this thesis. Using similarity transforms in this manner has great appeal since it makes some of the details conceptually clearer. The similarity transforms involved, however, would not be unitary in general, and non-unitary similarity transforms suffer from numerical difficulties. Also, in light of the results presented in this thesis, the question remains: Why employ similarity transforms at all? But it must be conceded that this remains an open question.

Additionally, there are more general topics of importance remaining which this thesis did not address. Efficiently computing a solution to the GDP would represent a major contribution, as well as finally resolving the issue of the order of the $H^{\infty}$-optimal controller. Computational accuracy and efficiency of algorithms relating to $\mu$-analysis and synthesis have barely been addressed in the literature, and much work needs to be done in this area.

In summary, this thesis has provided a method for incorporating directional information into norm tests for robust stability and nominal performance. The issue of robust performance and the desirability of using similarity transformations on the plant instead were not discussed and remain topics for future research.

# BIBLIOGRAPHY

## BIBLIOGRAPHY


[1] V. M. Adamjan, D. Z. Arov, and M. G. Kreĭn, "Infinite hankel block matrices and related extension problems," *American Mathematical Society Translations*, vol. 111, no. 2, pp. 133–156, 1978.

[2] L. V. Ahlfors, *Complex Analysis.* New York: McGraw-Hill, third ed., 1979.

[3] M. Athans, "A tutorial on the LQG/LTR method," in *Proceedings of the American Control Conference*, pp. 1289–1296, 1986.

[4] J. D. Birdwell, G. Chang, P. Hansen, L. Hong, J. Lai, and G. V. Murphy, "Teaching with the CASCADE computer-aided control system design environment," in *Proceedings of the Nineteenth Southeastern Symposium on System Theory*, pp. 115–120, IEEE, 1987.

[5] J. D. Birdwell, R. Cockett, R. W. Heller, R. W. Rochelle, A. J. Laub, M. T. Athans, R. Strunce, S. A. Bly, C. J. Herget, M. T. Heath, and J. P. Stovall, "Issues in the design of a computer-aided systems and control analysis and design environment," Tech. Rep. ORNL/TM-9038, Oak Ridge National Laboratory, Oak Ridge, TN 37831, Aug. 1984.

[6] J. D. Birdwell, "Linear multivariable control system design," 1984. Course notes, University of Tennessee.

[7] J. D. Birdwell, R. E. Bodenheimer, Jr., and A. J. Laub, "The cascade library user's guide," Tech. Rep., The University of Tennessee, Knoxville, TN 37996-2100, 1987.

[8] J. D. Birdwell and J. R. B. Cockett, "Experiences in the design and use of an expert system for computer-aided control system design," in *Applications of Artificial Intelligence V, Proceedings of SPIE*, pp. 468–473, SPIE, 1987.

[9] J. D. Birdwell, J. R. B. Cockett, R. E. Bodenheimer, Jr., and G. Chang, "The cascade tools and knowledge base," Tech. Rep., The University of Tennessee, Knoxville, TN 37996-2100, 1987.

[10] J. D. Birdwell, J. R. B. Cockett, and A. J. Laub, "The cascade user's guide," Tech. Rep., The University of Tennessee, Knoxville, TN 37996-2100, 1987.

[11] J. D. Birdwell and A. J. Laub, "Balanced singular values for LQG/LTR design," *International Journal of Control*, vol. 45, pp. 939–950, Feb. 1987.

[12] C. Chu, J. C. Doyle, and E. B. Lee, "The general distance problem in $H_\infty$ optimal control theory," *International Journal of Control*, vol. 44, no. 2, pp. 565–596, 1986.

[13] J. R. B. Cockett, S. Heard, A. Chen, G. Chang, and J. D. Birdwell, "The decide expert system shell," Tech. Rep., The University of Tennessee, Knoxville, TN 37996-2100, 1987.

[14] M. A. Dahleh and J. B. Pearson, Jr., "$l^1$-optimal feedback controllers for MIMO discrete-time systems," *IEEE Transactions on Automatic Control*, vol. AC-32, pp. 314–322, Apr. 1987.

[15] C. A. Desoer, R. Liu, J. Murray, and R. Saeks, "Feedback system design: the fractional representation approach to analysis and synthesis," *IEEE Transactions on Automatic Control*, vol. AC-25, pp. 399–412, June 1980.

[16] J. C. Doyle, "Robustness of multiloop linear feedback systems," in *Proceedings of the 1978 Conference on Decision and Control*, pp. 12–18, IEEE, 1978.

[17] J. Doyle, "Analysis of feedback systems with structured uncertainty," *IEE Proceedings, Part D*, vol. 129, pp. 242–250, Nov. 1982.

[18] J. Doyle, K. Lenz, and A. Packard, "Design examples using $\mu$-synthesis: space shuttle lateral axis FCS during reentry," 1986. Honeywell Systems and Research Center.

[19] J. Doyle and A. Packard, "Uncertain multivariable systems from a state space perspective," in *Proceedings of the American Control Conference*, pp. 2147–2152, IEEE, 1987.

[20] J. Doyle, G. Stein, M. Morari, S. Boyd, R. Smith, and D. Enns, "Case studies in robust control," 1987. ACC Tutorial Workshop Notes.

[21] J. C. Doyle, "Guaranteed margins for LQG regulators," *IEEE Transactions on Automatic Control*, vol. AC-23, pp. 756–757, Aug. 1978.

[22] J. C. Doyle, "Lecture notes on advances in multivariable control," 1984. ONR/-Honeywell Workshop, Minneapolis, MN.

[23] J. C. Doyle, "Synthesis of robust controllers and filters," in *Proceedings of the $22^{nd}$ Conference on Decision and Control*, pp. 109–114, IEEE, 1983.

[24] J. C. Doyle and C. Chu, "Robust control of multivariable and large scale systems," Final Report, Honeywell Systems and Research Center, March 1986.

[25] J. C. Doyle, R. S. Smith, and D. F. Enns, "Control of plants with input saturation nonlinearities," in *Proceedings of the American Control Conference*, pp. 1034–1039, IEEE, 1987.

[26] J. C. Doyle and G. Stein, "Multivariable feedback design: concepts for a classical/modern synthesis," *IEEE Transactions on Automatic Control*, vol. AC-26, pp. 4–16, Feb. 1981.

[27] J. C. Doyle, J. E. Wall, and G. Stein, "Performance and robustness analysis for structured uncertainty," in *Proceedings of the 21st Conference on Decision and Control*, pp. 629–636, IEEE, 1982.

[28] A. Feintuch and B. A. Francis, "Uniformly optimal control of linear time-varying systems," *Systems & Control Letters*, vol. 5, pp. 67–71, Oct. 1984.

[29] A. Feintuch, P. Khargonekar, and A. Tannenbaum, "On the sensitivity minimization problem for linear time-varying periodic systems," *SIAM Journal of Control and Optimization*, vol. 24, pp. 164–176, Sep. 1986.

[30] A. Feintuch and A. Tannenbaum, "Gain optimization for distributed plants," *Systems & Control Letters*, vol. 6, pp. 295–301, Jan. 1986.

[31] C. Foias, A. Tannenbaum, and G. Zames, "Sensitivity minimization for arbitrary SISO distributed plants," *Systems & Control Letters*, vol. 8, pp. 189–195, 1987.

[32] C. Foias, A. Tannenbaum, and G. Zames, "Weighted sensitivity minimization for delay systems," in *Proceedings of the 24th Conference on Decision and Control*, pp. 244–249, IEEE, 1985.

[33] Y. K. Foo and I. Postlethwaite, "An $H^\infty$-minimax approach to the design of robust control systems," *Systems & Control Letters*, vol. 5, pp. 81–88, Nov. 1984.

[34] Y. K. Foo and I. Postlethwaite, "An $H^\infty$-minimax approach to the design of robust control systems, part II: all solutions, all-pass form solutions, and the 'best' solution," *Systems & Control Letters*, vol. 7, pp. 261–268, July 1986.

[35] B. A. Francis, *A Course in $H_\infty$ Control Theory.* Vol. 88 of *Lecture Notes in Control and Information Sciences*, Berlin: Springer-Verlag, 1987.

[36] B. A. Francis, J. W. Helton, and G. Zames, "$H^\infty$-optimal feedback controllers for linear multivariable systems," *IEEE Transactions on Automatic Control*, vol. AC-29, pp. 888–900, Oct. 1984.

[37] K. Glover, "All optimal Hankel-norm approximations of linear multivariable systems and their $L^\infty$-error bounds," *International Journal of Control*, vol. 39, no. 6, pp. 1115–1193, 1984.

✓ [38] G. H. Golub and C. F. Van Loan, *Matrix Computations.* Baltimore: The John Hopkins University Press, 1983.

[39] M. J. Grimble, "Optimal $H_\infty$ robustness and the relationship to LQG design problems," *International Journal of Control*, vol. 43, no. 2, pp. 351–372, 1986.

[40] P. Hansen and J. D. Birdwell, "Robust LQG/LTR control of robotic manipulators," in *Proceedings IEEE Southeastcon '87*, pp. 437–441, IEEE, 1987.

[41] J. W. Helton, "Operator theory and broadband matching," in *Proceedings of the Allerton Conference on Circuits and System Theory*, pp. 91–97, IEEE, 1976.

[42] J. W. Helton, "Worst case analysis in the frequency domain: the $H^\infty$ approach to control," *IEEE Transactions on Automatic Control*, vol. AC-30, pp. 1154–1170, Dec. 1985.

[43] I. Horowitz, "Quantitative feedback theory," *IEE Proceedings, Part D*, vol. 129, pp. 215–226, Nov. 1982.

[44] R. D. Johnston and G. W. Barton, "Control system performance robustness," *International Journal of Control*, vol. 45, no. 2, pp. 641–648, 1987.

[45] E. Jonckheere and M. Verma, "A spectral characterization of $H^\infty$-optimal feedback performance and its efficient computation," *Systems & Control Letters*, vol. 8, pp. 13–22, 1986.

[46] H. Kimura, "Robust stabilizability for a class of transfer functions," *IEEE Transactions on Automatic Control*, vol. AC-29, pp. 788–793, Sep. 1984.

[47] V. C. Klema and A. J. Laub, "The singular value decomposition: its computation and some applications," *IEEE Transactions on Automatic Control*, vol. AC-25, pp. 164–176, Apr. 1980.

[48] H. Kwakernaak, "A polynomial approach to minimax frequency domain optimization of multivariable feedback systems," *International Journal of Control*, vol. 44, no. 1, pp. 117–156, 1986.

[49] T. Lee and M. Chen, "Robustness recovery of LQG-based multivariable control designs," *International Journal of Control*, vol. 45, pp. 1131–1136, Apr. 1987.

[50] N. A. Lehtomaki, D. A. Castanon, B. C. Levy, G. Stein, N. R. Sandell, Jr., and M. Athans, "Robustness tests utilizing the structure of modelling error," in *Proceedings of the $20^{th}$ Conference on Decision and Control*, pp. 1173–1190, IEEE, 1981.

[51] N. A. Lehtomaki, D. A. Castanon, B. C. Levy, G. Stein, N. R. Sandell, Jr., and M. Athans, "Robustness and modeling error characterization," *IEEE Transactions on Automatic Control*, vol. AC-29, pp. 212–220, March 1984.

[52] N. A. Lehtomaki, N. R. Sandell, Jr., and M. Athans, "Robustness results in linear-quadratic Gaussian based multivariable control designs," *IEEE Transactions on Automatic Control*, vol. AC-26, pp. 75–93, Feb. 1981.

[53] K. E. Lenz, P. P. Khargonekar, and J. C. Doyle, "When is classical loop shaping $H^\infty$-optimal?," in *Proceedings of the American Control Conference*, pp. 617–621, IEEE, 1987.

[54] C. N. Nett, "Algebraic aspects of linear controller system stability," *IEEE Transactions on Automatic Control*, vol. AC-31, pp. 941–949, Oct. 1986.

[55] S. D. O'Young and B. A. Francis, "Optimal performance and robust stabilization," *Automatica*, vol. 22, no. 2, pp. 171–183, 1986.

[56] A. Packard and J. C. Doyle, "Robust control with an $H_2$ performance perspective," in *Proceedings of the American Control Conference*, pp. 2141–2146, IEEE, 1987.

[57] R. Saeks and J. Murray, "Fractional representation, algebraic geometry, and the simultaneous stabilization problem," *IEEE Transactions on Automatic Control*, vol. AC-27, pp. 895–903, Aug. 1982.

[58] M. G. Safonov, "Stability margins of diagonally perturbed multivariable feedback systems," *IEE Proceedings, Part D*, vol. 129, pp. 251–156, Nov. 1982.

[59] M. G. Safonov, E. A. Jonckheere, M. Verma, and D. J. N. Limebeer, "Synthesis of positive real multivariable feedback systems," *International Journal of Control*, vol. 45, pp. 817–842, Feb. 1987.

[60] M. G. Safonov and M. Athans, "Gain and phase margins for multiloop LQG regulators," *IEEE Transactions on Automatic Control*, vol. AC-22, pp. 173–179, Apr. 1977.

[61] M. G. Safonov and M. Athans, "A multiloop generalization of the circle criterion for stability margin analysis," *IEEE Transactions on Automatic Control*, vol. AC-26, pp. 415–422, Apr. 1981.

[62] M. G. Safonov and M. Athans, "Robustness and computational aspects nonlinear stochastic estimators and regulators," *IEEE Transactions on Automatic Control*, vol. AC-23, pp. 717–725, Aug. 1978.

[63] M. G. Safonov, A. J. Laub, and G. L. Hartmann, "Feedback properties of multivariable systems: the role and use of the return difference matrix," *IEEE Transactions on Automatic Control*, vol. AC-26, pp. 47–65, Feb. 1981.

[64] M. G. Safonov, *Stability and Robustness of Multivariable Feedback Systems. MIT Press series in signal processing, optimization, and control*, Cambridge, MA: The MIT Press, 1980.

[65] N. R. Sandell, Jr., "Robust stability of systems with application to singular perturbations," *Automatica*, vol. 15, pp. 467–470, July 1979.

[66] D. Sarason, "Generalized interpolation in $H^\infty$," *Transactions of the American Mathematical Society*, vol. 127, pp. 179–203, 1967.

[67] A. Sideris and D. Kavranoglu, "$H_\infty$ optimal control as weighted Wiener-Hopf optimization," in *Proceedings of the $26^{th}$ Conference on Decision and Control*, IEEE, 1987. To appear.

[68] G. Stein and M. Athans, "The LQG/LTR procedure for multivariable feedback control design," *IEEE Transactions on Automatic Control*, vol. AC-32, pp. 105–114, Feb. 1987.

[69] G. Strang, *Linear Algebra and Its Applications.* New York: Academic Press, 1980.

[70] J. Van de Vegte, "Classical design, with application to a $3 \times 3$ turbofan engine model," *International Journal of Control*, vol. 45, pp. 1–16, Jan. 1987.

[71] M. Verma and E. Jonckheere, "$L^\infty$-compensation with mixed sensitivity as a broadband matching problem," *Systems & Control Letters*, vol. 4, pp. 125–129, May 1984.

[72] M. Vidyasagar, *Control System Synthesis: A Factorization Approach.* Cambridge, Massachusetts: The MIT Press, 1985.

[73] J. C. Willems, *The Analysis of Feedback Systems.* Cambridge, MA: The MIT Press, 1971.

[74] W. M. Wonham, *Linear Multivariable Control.* New York: Springer-Verlag, third ed., 1985.

[75] D. C. Youla, J. J. Bongiorno, Jr., and H. A. Jabr, "Modern Wiener-Hopf design of optimal controllers part i: the single-input-output case," *IEEE Transactions on Automatic Control*, vol. AC-21, pp. 3–13, Feb. 1976.

[76] D. C. Youla, J. J. Bongiorno, Jr., and H. A. Jabr, "Modern Wiener-Hopf design of optimal controllers part ii: the multivariable case," *IEEE Transactions on Automatic Control*, vol. AC-21, pp. 319–338, June 1976.

[77] G. Zames, "Feedback and optimal sensitivity: model reference transformations, multiplicative seminorms, and approximate inverses," *IEEE Transactions on Automatic Control*, vol. AC-26, pp. 301–320, Apr. 1981.

[78] G. Zames, "On the input-output stability of time-varying nonlinear feedback systems part i: conditions derived using concepts of loop gain, conicity, and positivity," *IEEE Transactions on Automatic Control*, vol. AC-11, pp. 228–238, Apr. 1966.

[79] G. Zames, "On the input-output stability of time-varying nonlinear feedback systems–part ii: conditions involving circles in the frequency plane and sector nonlinearities," *IEEE Transactions on Automatic Control*, vol. AC-11, pp. 465–476, July 1966.

[80] G. Zames and D. Bensoussan, "Multivariable feedback, sensitivity, and decentralized control," *IEEE Transactions on Automatic Control*, vol. AC-28, pp. 1030–1035, Nov. 1983.

[81] G. Zames and B. A. Francis, "Feedback, minimax sensitivity, and optimal robustness," *IEEE Transactions on Automatic Control*, vol. AC-28, pp. 585–601, May 1983.

# VITA

Robert Edward Bodenheimer, Jr., was born in Deerfield, Illinois, on September 24, 1963. He has lived in Knoxville, Tennessee, for most of his life, graduating from Bearden High School in June 1982. He started taking classes at the University of Tennessee in June 1979, and became a full-time student there in September 1982. He graduated with highest honors in August 1986 obtaining a Bachelor of Science in Electrical Engineering and a Bachelor of Arts in Mathematics. Thereupon, he enrolled as a graduate student and began study toward a Master's Degree, awarded in December 1987. The author plans to study for a doctorate at the California Institute of Technology.