

To the Graduate Council:

I am submitting herewith a dissertation written by Eve Shaw entitled "Parametrizability and small-scale geometry of certain connected fractals." I have examined the final paper copy of this dissertation for form and content and recommend that it be accepted in partial fulfillment of the requirements for the degree of Doctor of Philosophy, with a major in Mathematics.

Vyron Vellis, Major Professor

We have read this dissertation
and recommend its acceptance:

Vyron Vellis

Theodora Bourni

Matthew Badger

Joan Lind

Accepted for the Council:

Dixie L. Thompson

Vice Provost and Dean of the Graduate School

To the Graduate Council:

I am submitting herewith a dissertation written by Eve Shaw entitled "Parametrizability and small-scale geometry of certain connected fractals." I have examined the final electronic copy of this dissertation for form and content and recommend that it be accepted in partial fulfillment of the requirements for the degree of Doctor of Philosophy, with a major in Mathematics.

Vyron Vellis, Major Professor

We have read this dissertation
and recommend its acceptance:

Vyron Vellis

Theodora Bourni

Matthew Badger

Joan Lind

Accepted for the Council:

Dixie L. Thompson

Vice Provost and Dean of the Graduate School

(Original signatures are on file with official student records.)

Parametrizability and small-scale geometry of certain connected fractals

A Dissertation Presented for the
Doctor of Philosophy
Degree
The University of Tennessee, Knoxville

Eve Shaw

May 2025

© by Eve Shaw, 2025
All Rights Reserved.

*To my wonderful partner, Emily, whose love and support have kept me grounded
when the math has felt too difficult.*

*To my older sister, DOCTOR Mary Madeleine Shaw, for her solidarity and
encouragement as we both pursued our doctorates.*

*To my parents and all of my siblings, for always encouraging me to ask and pursue
difficult questions, even if the questions seem contrived, pointless, or silly.*

*And finally, to four-year-old Eve, who learned the word “mathematician” and
immediately knew how she wanted to spend at least the following 21 years of her life.*

You would be so excited.

Acknowledgements

I would like to thank my advisor Vyron Vellis, without whom I would not have had such exciting research questions, for his patience in providing feedback on my early attempts at writing mathematics, and for teaching a measure theory course which I enjoyed so much that it completely altered my career trajectory. I consider myself very lucky to have worked with him as a collaborator and to have been his student. I am endlessly grateful to him for his support, guidance, patience, and understanding.

I would like to thank my committee members Theodora Bourni, Joan Lind, and Matthew Badger for their interest in my work and commentary on my dissertation. Special thanks go to Matthew for traveling to Knoxville in order to attend my defense in person.

I would like to thank my family for their boundless encouragement and support, especially my older sister Helena for housing me so I could attend GMFB 2024 and JMM 2025 and my older brother Jerry for attending my seminar talk in Storrs to take pictures for the scrapbook. I would like also to thank my partner Emily and her family for their support, as well as my friends, in particular Jacob, Chris, and Eli, for their encouragement.

Part of the research in this dissertation was supported by the UTK Graduate Student Fellowship (Fall 2019–Spring 2023), by an NSF grant (Fall 2023), by the UTK Mathematics Summer Fellowship (Summer 2024), and by the UTK Spike Tickle STEM Endowed Fellowship (Spring 2024).

I would like to thank the professors of the UTK math department for teaching me so much extraordinary mathematics, the staff of the UTK math department, without whom travel would never happen and no one would ever graduate, and the lecturers in the UTK math department who mentored me as I taught classes during my time here. Finally, my thanks go to the math and physics department of Thomas More University, especially Professors Robert Riehemann, Wes Ryle, Jyoti Saraswat, and Harold Smith, for their encouragement and for the strong education they provided me.

Abstract

Rectifiability and parametrization of fractal sets are topics of substantial interest in geometric measure theory, including the celebrated theorem of Rademacher, Jones and Okikiolu's Analyst's Traveling Salesperson Theorem (ATST), and Remes's theorem on the parametrization of connected self-similar fractals. The first two of these, respectively, resolve fully questions related to tangents of Lipschitz curves and to Lipschitz rectifiability but leave wide open the analogous questions for Hölder curves, and the methods used by Remes have since been applied to related parametrization results for other classes of fractal sets.

First, we prove continuous and Hölder continuous parametrization results for connected fractals generated by infinite iterated function systems. In the latter portion, we investigate the tangent geometry of certain Hölder curves, proving that a certain class of randomly generated fractal sets are Hölder curves with the optimal exponent, and then proving that these curves display, at typical points, tangent geometry which is wildly more complicated than that of any self-similar curve. In our parametrization results we are able to construct these Hölder parametrizations of optimal exponent explicitly, by contrast to prior analogous parametrization results using Remes's method which must make some sacrifice here; either the parametrizations do not have optimal exponent or the proof is non-constructive. The investigation of tangent geometry to Hölder curves is a necessary first step toward obtaining a Hölder analogue of the ATST and thus in understanding Hölder rectifiability. Our results on the tangents to the randomly generated fractal sets

demonstrate that tangents to Hölder curves can be extremely complicated, and indeed the geometry which can be attained is not captured by the tangents to self-similar sets.

Table of Contents

1	Introduction	1
1.1	Overview	1
1.2	Iterated function systems	4
1.3	Tangents and rectifiability	8
1.4	Outline	10
2	Parametrizability of infinitely generated attractors	12
2.1	Disclaimer	12
2.2	Abstract	12
2.3	Introduction	13
2.4	Parametrizations of infinite IFS attractors	17
2.5	Examples of IIFS	31
3	Hölder curves with exotic tangent spaces	42
3.1	Disclaimer	42
3.2	Abstract	42
3.3	Introduction	43
3.4	Preliminaries	51
3.5	Tangents of Lipschitz curves at typical points	59
3.6	A “statistically self-similar” carpet	63
3.7	Ahlfors regularity of carpets K^η	66
3.8	Hölder parametrizations of carpets K^η	70

3.9	Random choice functions	85
3.10	Tangents of carpets K^η at typical points	89
3.11	Proof of Theorem 3.2	102
3.12	Tangents of self-similar sets	104
	Bibliography	123
	Vita	130

List of Figures

1.1	Sierpiński gasket (left) and Sierpiński carpet (right).	6
2.1	Attractor of IIFS of §2.5.1	33
2.2	First iteration of IIFS of §2.5.2.	35
2.3	First iteration of IIFS of §2.5.3.	39
3.1	First stages of sets $\mathcal{S}$ (left) and $\mathcal{C}$ (right).	46
3.2	Model 1 (left) and Model 2 (right) for $s = \log 24 / \log 6$.	50
3.3	Dendrites T^1 (left) and T^2 (right) for $n = 6$.	71
3.4	The big triangle represents all words in $\mathcal{A}_n^*$ that start with $w(\ell - N)$ and have at most $\ell + k + N$ letters. Words of the same the height have the same number of letters. The part of this set where η takes the value 2 is colored by red, and the part of the set where η takes the value 1 is colored by blue.	86

Chapter 1

Introduction

1.1 Overview

While the notion of “fractal” does not have a generally agreed upon definition, sets which are called fractals generally look geometrically complicated at a wide range of scales and positions. A natural type of question to ask when considering fractal sets is about how to quantify or bound the complexity of the set. These questions tend to come in a few different flavors, and of most interest in this dissertation are the questions involving dimension, measure, rectifiability, and parametrizability. In questions of dimension, we most commonly ask if there is a straightforward way to compute the Hausdorff dimension of a fractal, and in the related questions of measure we typically ask if the fractal has positive and finite Hausdorff measure in its dimension, and if so we ask if the fractal enjoys a particularly nice kind of scaling law, called Ahlfors regularity, in this dimension. In questions of parametrizability, we ask whether or not a connected fractal admits a continuous surjection mapping the unit interval onto the fractal, and if so can we make this parametrization particularly nice in some way. Finally, in questions of rectifiability we ask if there are any particularly nice continuous curves which contain the fractal set.

The “niceness” of the functions in parametrizability and rectifiability generally take the form of Hölder, and occasionally Lipschitz, continuity. We say that a function $f : (X, d_X) \rightarrow (Y, d_Y)$ between metric spaces is α -Hölder (continuous) for a number $\alpha > 0$ if

$$\text{Hold}_\alpha(f) := \sup_{x \neq y; x, y \in X} \frac{d_Y(f(x), f(y))}{d_X(x, y)^\alpha} < \infty.$$

If a function f is 1-Hölder then it is called Lipschitz, and $\text{Hold}_1(f)$ is denoted by $\text{Lip}(f)$. It is a well-known fact in geometric measure theory that if $f : (X, d_X) \rightarrow (Y, d_Y)$ is α -Hölder, then $\dim_{\text{H}}(f(X)) \leq \frac{1}{\alpha} \dim_{\text{H}}(X)$, where $\dim_{\text{H}}(K)$ denotes the Hausdorff dimension of any set K . This simple relationship motivates us to study parametrizability of fractals by Hölder curves. Specifically, if K is a connected fractal, then the stated relation gives us an optimum possible Hölder exponent of $(1/\dim_{\text{H}}(K))$, so we often ask if K is a $(1/\dim_{\text{H}}(K))$ -Hölder curve. Here and throughout, when we say “curve” we mean the continuous image of the closed unit interval $[0, 1]$.

1.1.1 Hausdorff dimension and measure

If (X, d) is a metric space and $E \subset X$, then for any number $s > 0$ we define the *Hausdorff s -dimensional measure* of E as

$$\mathcal{H}^s(E) := \lim_{\delta \rightarrow 0^+} \left(\inf \left\{ \sum_{i=1}^{\infty} \text{diam}(U_i)^s : E \subset \bigcup_{i=1}^{\infty} U_i, \text{diam}(U_i) \leq \delta \right\} \right).$$

Note that the limit $\lim_{\delta \rightarrow 0^+}$ could be replaced by $\sup_{\delta > 0}$ as when δ decreases, the infimum is taken over a smaller collection of possible covers of E . Thus the value $\mathcal{H}^s(E)$ is always well-defined, though it may be infinite. The Hausdorff measure is a natural generalization of the Lebesgue measure, and indeed for integer values of s , $\mathcal{H}^s$ is equal to the Lebesgue measure on $\mathbb{R}^s$ up to a multiplicative factor. For this reason, we will use the term “length” interchangeably with $\mathcal{H}^1$ -measure. Furthermore, for $E \subset X$ in any metric space, if $0 \leq \alpha < \beta$, then $\mathcal{H}^\alpha(E) < \infty$ implies that

$\mathcal{H}^\beta(E) = 0$ and similarly $\mathcal{H}^\beta(E) > 0$ implies $\mathcal{H}^\alpha(E) = \infty$. Following this observation, the *Hausdorff dimension* of the set E is defined in the following two equivalent ways

$$\dim_{\text{H}}(E) := \sup \{ \alpha \geq 0 : \mathcal{H}^\alpha(E) = \infty \} = \inf \{ \beta \geq 0 : \mathcal{H}^\beta(E) = 0 \}.$$

1.1.2 Rademacher's theorem

Lipschitz continuity is a particularly geometrically strong condition on a function, and a curve being Lipschitz (that is, being the Lipschitz image of the unit interval $[0, 1]$) implies several conditions about both the Lipschitz parametrization of the curve and the geometry of the curve itself. First among these is Rademacher's Theorem, a classical and celebrated result in geometric measure theory, which states that if $f : U \rightarrow \mathbb{R}^d$ is a Lipschitz function, where $U \subset \mathbb{R}^m$ is an open set, then f is differentiable at $\mathcal{H}^m$ -almost every point of U ; in particular if $f : [0, 1] \rightarrow \mathbb{R}^d$ is a Lipschitz map, then it is differentiable at $\mathcal{H}^1$ -almost every point of the interval $[0, 1]$. Here and throughout, for a number $s \geq 0$, $\mathcal{H}^s$ denotes the Hausdorff s -dimensional measure. For a proof of Rademacher's Theorem, see for example (Heinonen, 2001, Theorem 6.15) or (Evans and Gariepy, 2015, Theorem 3.2). Rademacher's Theorem has strong consequences for the geometry of rectifiable curves, that is curves of finite length, which are equivalent to Lipschitz curves by (Alberti and Ottolini, 2017, Theorem 4.4). Namely, for an appropriate notion of tangent set, a result of Falconer (Falconer, 1986, Corollary 3.15) states that every rectifiable curve in Euclidean space has tangent lines at $\mathcal{H}^1$ -almost every point.

1.1.3 The (analyst's) traveling salesperson problem

A classical problem in discrete optimization is the Traveling Salesperson Problem (TSP), where one asks for an algorithm to determine the most efficient path to visit every point in a given finite set of points. The analogue studied in analysis is the Analyst's Traveling Salesperson Problem (ATSP), in which we ask for a set of

conditions on a subset K of Euclidean space which guarantee that K is contained in a rectifiable curve. The version of this problem in the Euclidean plane $\mathbb{R}^2$ was resolved by Peter Jones [Jones \(1990\)](#), and the more general version in arbitrary Euclidean spaces $\mathbb{R}^d$ was resolved by Kate Okikiolu [Okikiolu \(1992\)](#), establishing the Analyst’s Traveling Salesperson Theorem (ATST). This result has since been extended to the case of Hilbert spaces by Schul [Schul \(2007b\)](#), with the proof receiving corrections and the result partially extended to the Banach space setting in a two part paper by Badger and McCurdy [Badger and McCurdy \(2023a,b\)](#).

The proofs of the ATST in these papers involve the use of beta numbers, which measure how far away the set K is from lying inside a line at various scales and locations, and they prove that in Hilbert spaces, K lies in a rectifiable curve if and only if the beta number of K is finite. The ATST fully resolves the question of Lipschitz rectifiability in Euclidean spaces and Hilbert spaces. While rectifiability by Lipschitz surfaces or by Hölder curves has seen substantial study over the years (see for example [Martín and Mattila \(1993, 2000\)](#); [Badger and Vellis \(2017\)](#); [Badger et al. \(2019\)](#); [Balogh and Züst \(2020\)](#); [Badger and Vellis \(2021\)](#); [Badger and Schul \(2023\)](#)), but an appropriate analogue to the ATST for these versions of rectifiability has not been established. In view of Rademacher’s Theorem, the ATST roughly states that if a compact set K typically has tangents which are contained in the typical tangents to Lipschitz curves, then K can be captured by a Lipschitz curve. Following this reasoning, we approach the question of Hölder rectifiability using an appropriate notion of tangent and using the tangents to connected attractors of particularly nice iterated function systems, which are a canonical family of Hölder curves.

1.2 Iterated function systems

A well-known family of fractals is the family of attractors of *iterated function systems* (IFS), and especially those attractors which are generated by similarities and satisfy the open set condition. An IFS is a finite collection $\mathcal{F}$ of Lipschitz contractions

mapping a complete metric space (X, d) to itself. We say that a map $\phi : X \rightarrow X$ is a *contraction* if $\text{Lip}(\phi) \in (0, 1)$, and we say that a function $\phi : X \rightarrow X$ is a *similarity* if for every $x, y \in X$, $d(\phi(x), \phi(y)) = \text{Lip}(\phi)d(x, y)$. An IFS $\mathcal{F}$ is said to satisfy the *open set condition* (OSC) if there exists a nonempty open set $U \subset X$ so that for every pair of distinct $\phi, \psi \in \mathcal{F}$, $\phi(U) \cap \psi(U) = \emptyset$ and $\phi(U) \subset U$. The OSC is a separation condition on $\mathcal{F}$ guaranteeing that the images of distinct maps $\phi, \psi \in \mathcal{F}$ do not overlap too much. We further define the *similarity dimension* of the IFS $\mathcal{F}$, denoted by $\text{s-dim}(\mathcal{F})$, to be the unique solution $s \geq 0$ to the equation $\sum_{\phi \in \mathcal{F}} \text{Lip}(\phi)^s = 1$.

By Hutchinson (Hutchinson, 1981, 3.1(3)(i)), if $\mathcal{F}$ is an IFS then there exists a unique nonempty compact set $K \subset X$ satisfying $K = \bigcup_{\phi \in \mathcal{F}} \phi(K)$, and we call this set K the *attractor* of $\mathcal{F}$. Attractors of IFS comprised of similarities and satisfying the OSC enjoy nice properties with respect to Hausdorff dimension and Hausdorff measure. By Hutchinson Hutchinson (1981), if K is the attractor of any IFS $\mathcal{F}$, then $\dim_{\text{H}}(K) \leq \text{s-dim}(\mathcal{F})$, and if $\mathcal{F}$ is an IFS of similarities on $\mathbb{R}^m$ satisfying the OSC then $\mathcal{H}^{\text{s-dim}(\mathcal{F})}(K) \in (0, \infty)$ and thus $\dim_{\text{H}}(K) = \text{s-dim}(\mathcal{F})$. Moreover, if K is the attractor of an IFS $\mathcal{F}$ of similarities with the OSC in Euclidean space, then K is *Ahlfors* $\dim_{\text{H}}(K)$ -*regular*, meaning that there exists a constant $C \geq 1$ such that for every $x \in K$ and each $r \in (0, \text{diam}(K))$, we have the following scaling property for the $\mathcal{H}^{\dim_{\text{H}}(K)}$ measure on K

$$\frac{1}{C}r^{\dim_{\text{H}}(K)} \leq \mathcal{H}^{\dim_{\text{H}}(K)}(B(x, r) \cap K) \leq Cr^{\dim_{\text{H}}(K)}.$$

Schief Schief (1994) proved that in Euclidean spaces, the OSC is equivalent to a stronger separation condition, called the strong open set condition, and demonstrated that this does not hold in general complete metric spaces. The system $\mathcal{F}$ is said to satisfy the *strong open set condition* (SOSC) if it satisfies the OSC via a nonempty open set U , further satisfying $U \cap K \neq \emptyset$, where K is the attractor of $\mathcal{F}$. In Figure 1.1 below are two well-known examples of attractors of IFS of similarities satisfying the OSC (and indeed the SOSC), the Sierpiński gasket and Sierpiński carpet.

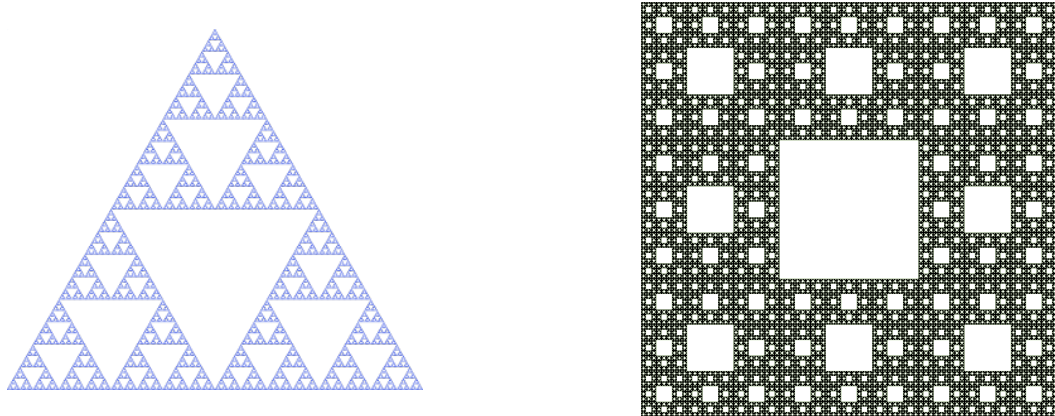


Figure 1.1: Sierpiński gasket (left) and Sierpiński carpet (right).

The parametrizability of IFS attractors has seen significant progress over the years. First, Hata [Hata \(1985\)](#) proved that if K is the connected attractor of any IFS, then K is the image of a curve. Later, Remes [Remes \(1998\)](#) showed that if K is the connected attractor of an IFS $\mathcal{F}$ of similarities with the OSC in Euclidean space, then K is a $(1/\dim_{\text{H}}(K))$ -Hölder curve. By the well-known fact that a $(1/\alpha)$ -Hölder curve has image with Hausdorff dimension at most α , this is the best possible exponent for the Hölder continuity of a parametrizing curve. Badger and Vellis [Badger and Vellis \(2021\)](#) extended Hata's result, showing that if K is a connected attractor of an IFS $\mathcal{F}$, then K is a $(1/\alpha)$ -Hölder curve for every $\alpha > \text{s-dim}(\mathcal{F})$. They also generalized Remes's result by proving that if K is the connected attractor of an IFS of similarities on any complete metric space and if $\mathcal{H}^{\dim_{\text{H}}(K)}(K) > 0$, then K is a $(1/\dim_{\text{H}}(K))$ -Hölder curve. However, as the authors note the parametrizations of optimal exponent are not constructed explicitly. Note that since the assumption that $\mathcal{F}$ is comprised of similarities and satisfies the OSC is not present in the first of these results, there is no straightforward relationship between $\dim_{\text{H}}(K)$ and $\text{s-dim}(\mathcal{F})$, in particular it is not only unknown whether or not a $(1/\text{s-dim}(\mathcal{F}))$ -Hölder parametrization can be realized in general, but it is not immediately obvious if $(1/\text{s-dim}(\mathcal{F}))$ is the best Hölder exponent we could hope for. However, they provide a class of examples of connected attractors $\Sigma \subset [0, 1]^2$ of IFS $\mathcal{F}$ for which $1 < \dim_{\text{H}}(\Sigma) < \text{s-dim}(\mathcal{F})$ and where Σ can

be parametrized by a $(1/\text{s-dim}(\mathcal{F}))$ -Hölder curve but this Hölder exponent cannot be increased.

In Chapter 2 we study similar types of questions in the context of *infinite iterated function systems* (IIFS). An IIFS is a countable collection $\mathcal{F} = \{\phi_i\}_{i \in \mathbb{N}}$ of contractions on a compact metric space (X, d) , and we define the attractor of $\mathcal{F}$ to be

$$K := \bigcup_{(i_1, i_2, \dots) \in \mathbb{N}^{\mathbb{N}}} \bigcap_{n=1}^{\infty} \phi_{i_1} \circ \dots \circ \phi_{i_n}(X).$$

Unlike in the IFS case, the attractor of an IIFS is not necessarily compact, and while the attractor K is nonempty and does satisfy $K = \bigcup_{i \in \mathbb{N}} \phi_i(K)$, it is possible that there are multiple nonempty subsets of X satisfying this property. However, if every map ϕ_i is injective and if every point $x \in X$ is contained in at most finitely many images $\phi_i(X)$, then we have that

$$K = \bigcap_{n=1}^{\infty} \bigcup_{(i_1, \dots, i_n) \in \mathbb{N}^n} \phi_{i_1} \circ \dots \circ \phi_{i_n}(X).$$

Furthermore, the expression $\sum_{i \in \mathbb{N}} \text{Lip}(\phi_i)^t$ is either infinite for all values of $t > 0$ or it is continuous and strictly decreasing on an interval (a, ∞) for some $a \geq 0$. Therefore, unlike in the IFS case, there may not be any solution to the equation $\sum_{\phi \in \mathcal{F}} \text{Lip}(\phi)^t = 1$. As a result, we must modify the definition of similarity dimension in the IIFS case, and we instead define it as

$$\text{s-dim}(\mathcal{F}) := \inf \left\{ s \geq 0 : \sum_{i \in \mathbb{N}} \text{Lip}(\phi_i)^s \leq 1 \right\}.$$

IIFS were first introduced in the papers of Mauldin and Urbański [Mauldin \(1995\)](#); [Mauldin and Urbański \(1996\)](#), and they are a subject of interest in their own right. IIFS see applications in number theory (see [Mauldin and Urbański \(1999\)](#)), where the study of continued fractions give rise to fractals like the Apollonian Gasket, which has been studied in papers such as [Jordan and Rams \(2012\)](#); [Chousionis et al. \(2024\)](#);

Vytnova and Wormell (2024). The study of IIFS has seen further applications to fields such as geometric group theory and other parts of fractal geometry, see for example Heinemann and Urbański (2002); Mauldin and Urbański (2002); Urbański and Zdunik (2002); Kesseböhmer and Zhu (2006); Mauldin et al. (2009); Seuret and Wang (2015); Rempe-Gillen and Urbański (2016); Banaji and Fraser (2023) and the contained references. In the spirit of the results mentioned at the start of this section, we prove Theorem 2.2.

Theorem. (Theorem 2.2) *Let $\mathcal{F} = \{\phi_i\}_{i \in \mathbb{N}}$ be an IIFS on a compact metric space so that the attractor K is compact, $\lim_{i \rightarrow \infty} \text{Lip}(\phi_i) = 0$, and so that there exists a curve $\gamma : [0, 1] \rightarrow K$ whose image intersects $\phi_i(K)$ for every $i \in \mathbb{N}$.*

1. *The attractor K is the image of a curve.*
2. *If γ is $(1/s)$ -Hölder for some $s \geq 1$, then for every $\alpha > \max\{s, \text{s-dim}(\mathcal{F})\}$, K is the image of a $(1/\alpha)$ -Hölder curve.*

This result comes with a longer list of hypotheses than the parametrization results about IFS attractors, but in Theorem 2.1 we prove that these assumptions are all necessary even in Euclidean space for similarity dimensions arbitrarily close to 1. We then show in Proposition 2.1 that the situation for IIFS with similarity dimension equal to 1 is much simpler, and in fact quite similar to the IFS case. Specifically, we prove that if K is the attractor of an IFS or IIFS $\mathcal{F}$ on a compact metric space so that K is a continuum and $\text{s-dim}(\mathcal{F}) = 1$, then K is isometric to a closed line segment. Here and throughout, we call a metric space K a *continuum* if K is compact and connected. If we wish to emphasize that the continuum K is not a singleton, then we will call K a *nondegenerate* continuum.

1.3 Tangents and rectifiability

Tangents to a closed set $X \subset \mathbb{R}^d$ describe the infinitesimal geometry of the set X based at a point of X . The version of tangent we use in Chapter 3 to investigate

the infinitesimal geometry of Hölder curves, and accordingly curves which are the attractors of IFS of similarities in Euclidean space with the OSC, is the notion of tangent defined in [Badger and Lewis \(2015\)](#). We say that a closed set $T \subset \mathbb{R}^d$ is a *tangent* to X at the point $x \in X$ if there exists a sequence of positive scales $r_j \rightarrow 0$ so that the blow-ups $r_j^{-1}(X - x)$ centered at x converge to T in the Attouch-Wets topology (see [Beer \(1993\)](#) for the definition of this topology); the collection of tangents to a set X at a point $x \in X$ is denoted by $\text{Tan}(X, x)$. This version of tangent is based on the notion of tangent measures, which was first defined in the celebrated 1987 paper of Preiss [Preiss \(1987\)](#), but similar ideas predate this work, for example in [Federer \(1969\)](#); see ([Mattila, 1995](#), Ch. 14) for a more thorough discussion.

A characteristic of note in the specific notion of tangent we use is that, unlike in the Pointed Gromov-Hausdorff version of tangents defined in Gromov’s work [Gromov \(1981, 1999\)](#) (see also [Burago et al. \(2001\)](#)), the tangent sets are not identified up to isometry or up to any other type of equivalence; tangents are considered to be the same precisely when they are equal as sets of points. This distinction is meaningful in [Theorem 3.1](#), which states that along a Lipschitz curve $\Gamma \subset \mathbb{R}^d$, for $\mathcal{H}^1$ -almost every point $x \in \Gamma$ there is a unique line $L \subset \mathbb{R}^d$ through the origin $\mathbf{0}$ so that $\text{Tan}(\Gamma, x) = \{L\}$. In essence, this result provides a geometric version of Rademacher’s Theorem for this particular notion of tangent, and much of the content of this result is the uniqueness of the line L , which would not be as meaningful if tangent sets were identified up to isometry.

Based on the niceness of tangents to Lipschitz curves, it was asked by Badger and Vellis whether or not every tangents to a Hölder curve “looks self-similar” in some precise sense. To this end, we prove [Proposition 3.1](#), which provides strong quantitative comparisons between the geometry of an attractor Q of an IFS of similarities satisfying the OSC in $\mathbb{R}^d$ and the geometry of its tangents. This proposition is a necessary ingredient for the main result of this chapter, which is the following theorem combining [Theorems 3.2](#) and [3.3](#). This result answers the question of Badger and Vellis about tangents to Hölder curves in the negative.

Theorem. *For all $s > 1$, there exists an integer $d \geq 2$ such that there exists a $(1/s)$ -Hölder curve $\Gamma_s \subset \mathbb{R}^d$ so that $\mathcal{H}^s(\Gamma_s) > 0$ and for $\mathcal{H}^s$ -almost every $x \in \Gamma_s$, the set $\text{Tan}(\Gamma_s, x)$ contains infinitely many topologically distinct tangent sets $\{T_1, T_2, \dots\}$ with the following property. If $Q \subset \mathbb{R}^d$ is the attractor of an IFS of similarities with the OSC and with $\dim_{\text{H}}(Q) = s$, then the set of points $z \in Q$ for which there exists a tangent $S \in \text{Tan}(Q, z)$ that is bi-Lipschitz homeomorphic to any of the T_j is an $\mathcal{H}^s$ -null set.*

We prove this by explicitly constructing Hölder curves of Hausdorff dimension close to 1 via randomly iterating different families of similarities, and we are able to use some techniques from the theory of IFS regularity and parametrizability to show that these curves are Ahlfors regular. We then use these curves to prove that the tangents to Hölder curves are not only much wilder than the tangents to Lipschitz curves, but that tangents to Hölder curves are not described by the tangents to Hölder curves arising from the well-known family of Hölder curves which are the attractors of IFS of similarities with the OSC.

1.4 Outline

The following two chapters of this dissertation are comprised of manuscripts investigating questions of measure-theoretic regularity, tangent geometry, and parametrizability for certain connected fractals which are constructed via iterates of contraction maps. While these fractals are not themselves generated by iterated function systems, they are defined in such a way that we are able to draw on many of the existing tools and ideas used to investigate the attractors of iterated function systems in many of our proofs throughout.

Chapter 2 is comprised of the manuscript ‘Parametrizability of infinitely generated attractors’ [Shaw and Vellis \(2024b\)](#), in which we investigate fractals which are generated by *infinite iterated function systems* (IIFS). After defining the appropriate notion of similarity dimension and attractor for an IIFS, we prove in Theorem 2.1 that,

even in Euclidean space, for every $\alpha > 1$ there exists a compact and connected set K generated by an IIFS with $\dim_{\text{H}}(K) < \alpha$ but so that K is not a curve. Furthermore, even if an IIFS attractor K is a curve, we still cannot guarantee that K is a Hölder curve for any exponent if $\dim_{\text{H}}(K) > 1$. Then in Theorem 2.2, we provide a set of sufficient conditions on an IIFS $\mathcal{F}$ to guarantee that its attractor K is a curve, and we further provide conditions under which the attractor is guaranteed to be a Hölder curve. In combination with Theorem 2.1 and an example in §2.5.1, we have that none of the conditions in Theorem 2.2 can be removed in general.

Chapter 3 is comprised of the manuscript ‘Hölder curves with exotic tangent spaces’ Shaw and Vellis (2024a). In it, we investigate the tangents to the attractors to IFS of similarities satisfying the OSC in Euclidean spaces and we construct Hölder curves with tangents almost everywhere which cannot be obtained from any such attractor. This chapter contains two major theorems. In Theorem 3.3, we prove that if Q is the attractor of an IFS of similarities with the OSC in Euclidean space, then at typical points of Q , every tangent is comprised of countably many similar images of Q , and satisfies a version of the OSC. We further obtain quantitative control on the sizes of these images and on the number of these images needed to cover any given bounded portion of the tangent. In Theorem 3.2, we prove that for every $s > 1$, there is a $(1/s)$ -Hölder curve Γ_s with positive $\mathcal{H}^s$ -measure in some Euclidean space so that at typical points of Γ_s , there are infinitely many tangents which are not homeomorphic to each other, and which (more importantly) are not bi-Lipschitz equivalent to *any* tangent to an attractor of an IFS of similarities satisfying the OSC in Euclidean space at typical points. This result shows in particular that typical tangents to Hölder curves are not generally described by tangents to curves which are attractors of IFS of similarities satisfying the OSC in Euclidean space.

Chapter 2

Parametrizability of infinitely generated attractors

2.1 Disclaimer

This chapter is, up to formatting and up to consolidation of the bibliography into a single bibliography for the entire dissertation, identical to the paper of the same name Shaw and Vellis (2024b). The manuscript is a collaborative work with Vyrion Vellis and was published in *Annales Fennici Mathematici* in 2024. The publication can be found under <https://afm.journal.fi/article/view/143314>.

2.2 Abstract

An infinite iterated function system (IIFS) is a countable collection of contraction maps on a compact metric space. In this paper we study the conditions under which the attractor of such a system admits a parameterization by a continuous or Hölder continuous map of the unit interval.

2.3 Introduction

Iterated function systems are among the most standard and canonical methods in mathematics of producing fractal sets. An *iterated function system* (abbrev. IFS) is a finite collection $\mathcal{F}$ of contraction maps on a complete metric space X . Hutchinson [Hutchinson \(1981\)](#) showed that for each IFS $\mathcal{F}$, there exists a unique nonempty compact set $K \subset X$ (called the *attractor* of $\mathcal{F}$) such that $K = \bigcup_{\phi \in \mathcal{F}} \phi(K)$. The *similarity dimension* of an IFS $\mathcal{F}$ is the unique solution to the equation

$$\psi_{\mathcal{F}}(t) := \sum_{\phi \in \mathcal{F}} \text{Lip}(\phi)^t = 1 \quad (2.3.1)$$

where $\text{Lip}(\phi)$ denotes the infimum of all $L > 0$ for which ϕ is L -Lipschitz. Here and for the rest of the paper we only consider nondegenerate proper contractions, that is, we always assume that $\text{Lip}(\phi) \in (0, 1)$.

The connection between the similarity dimension of an IFS $\mathcal{F}$ and the Hausdorff dimension of its attractor K was established by Hutchinson [Hutchinson \(1981\)](#) who showed that if $\mathcal{F}$ is an IFS of similarities on $\mathbb{R}^n$ satisfying the open set condition, then $\dim_H(K) = \text{s-dim}(\mathcal{F})$. Recall that a similarity in $\mathbb{R}^n$ is the composition of a dilation and an isometry. An IFS $\mathcal{F}$ on $\mathbb{R}^n$ satisfies the *open set condition* (abbrev. OSC) if there exists a nonempty open set $U \subset \mathbb{R}^n$ such that $\phi(U) \subset U$ for all $\phi \in \mathcal{F}$, and $\phi(U) \cap \phi'(U) = \emptyset$ for all distinct $\phi, \phi' \in \mathcal{F}$. Many well known fractals (such as the standard Cantor set, the Sierpiński carpet, the von Koch snowflake, etc.) are attractors of IFS of similarities on the plane with the OSC. See also [Schief \(1994, 1996\)](#); [Farkas and Fraser \(2015\)](#); [Fraser et al. \(2015\)](#) for the necessity of the OSC.

A natural question in the theory of Dynamical Systems is the regularity of an IFS attractor and whether it admits “good” parameterizations by the unit interval. Hata [Hata \(1985\)](#) showed that if the attractor K of an IFS is connected, then it is the image of a curve, that is, the image of $[0, 1]$ under a continuous map. The second named author and Badger [Badger and Vellis \(2021\)](#) improved Hata’s result

by proving that if the attractor K of an IFS $\mathcal{F}$ is connected, then it is the image of $[0, 1]$ under a $\frac{1}{\alpha}$ -Hölder continuous map for any $\alpha > \text{s-dim}(\mathcal{F})$. Under the extra assumptions that $X = \mathbb{R}^n$ and that $\mathcal{F}$ is an IFS of similarities on $\mathbb{R}^n$ satisfying the OSC, Remes [Remes \(1998\)](#) showed earlier that the attractor is the image of $[0, 1]$ under a $\frac{1}{\alpha}$ -Hölder continuous map where one can actually have $\alpha = \text{s-dim}(\mathcal{F})$. Remes' result is sharp in that there exists no $\frac{1}{\alpha}$ -Hölder parameterization if $\alpha < \text{s-dim}(\mathcal{F})$. The assumption $X = \mathbb{R}^n$ in Remes' theorem can be replaced by the assumption $\mathcal{H}^{\text{s-dim}(\mathcal{F})}(K) > 0$ [Badger and Vellis \(2021\)](#). Here and for the rest of the paper $\mathcal{H}^\alpha$ denotes the Hausdorff α -dimensional measure.

In their celebrated paper, Mauldin and Urbański [Mauldin and Urbański \(1996\)](#) (see also [Mauldin \(1995\)](#)) further extended Hutchinson's theory and introduced the notion of an *infinite iterated function system* (abbrev. IIFS); i.e., an infinite countable collection of contractions on a compact metric space X . Ever since their introduction, IIFSs have played a major role in fractal geometry, geometric group theory, and number theory; see [Mauldin and Urbański \(1999\)](#); [Heinemann and Urbański \(2002\)](#); [Mauldin and Urbański \(2002\)](#); [Urbański and Zdunik \(2002\)](#); [Kesseböhmer and Zhu \(2006\)](#); [Mauldin et al. \(2009\)](#); [Jordan and Rams \(2012\)](#); [Seuret and Wang \(2015\)](#); [Rempe-Gillen and Urbański \(2016\)](#); [Banaji and Fraser \(2023\)](#) and the references therein.

Here, unlike in most literature, we do not assume that contractions are conformal or even injective. Given an IIFS $\mathcal{F} = \{\phi_i : i \in \mathbb{N}\}$ on a compact metric space X , we define the attractor of $\mathcal{F}$ by

$$K = \bigcup_{(i_n) \subset \mathbb{N}} \bigcap_{n=1}^{\infty} \phi_{i_1} \circ \cdots \circ \phi_{i_n}(X).$$

The attractor K in the infinite setting may not be compact. Moreover, although $K = \bigcup_{\phi \in \mathcal{F}} \phi(K)$, there may exist multiple nonempty subsets of X with this property. If, additionally, each $\phi_i \in \mathcal{F}$ is injective, and if each $x \in X$ is contained in at most

finitely many $\phi_i(X)$, then

$$K = \bigcap_{n \in \mathbb{N}} \bigcup_{i_1, \dots, i_n \in \mathbb{N}} \phi_{i_1} \circ \dots \circ \phi_{i_n}(X). \quad (2.3.2)$$

In the infinite setting, the auxiliary function $\psi_{\mathcal{F}}$ in (2.3.1) is either infinite for all $t > 0$, or it is continuous and strictly decreasing on an interval (a, ∞) for some $a \geq 0$. Hence, unlike in the finite case, equation (2.3.1) may not have a solution. We define the similarity dimension of an IIFS $\mathcal{F}$ as

$$\text{s-dim}(\mathcal{F}) := \inf\{t \geq 0 : \psi_{\mathcal{F}}(t) \leq 1\}.$$

By Fatou's Lemma, the infimum above is in fact a minimum. As in the finite case, if $\mathcal{F}$ is an IIFS of similarities on $X = \overline{U} \subset \mathbb{R}^n$ where U is a bounded domain, satisfying the OSC, then the Hausdorff dimension of the attractor K is equal to $\text{s-dim}(\mathcal{F})$ (Mauldin and Urbański, 1996, Corollary 3.17).

The purpose of this paper is to study the parameterizability of IIFS attractors. In the infinite case an interesting dichotomy appears. On the one hand, in Proposition 2.1 we show that if the attractor K of an IIFS $\mathcal{F}$ is a continuum and if $\text{s-dim}(\mathcal{F}) = 1$, then K is a line segment; see also Mauldin et al. (2001) for a similar phenomenon. On the other hand, both Hata's theorem and the Badger-Vellis theorem are false when the similarity dimension is greater than 1, even if it is arbitrarily close to 1.

Theorem 2.1. *1. For each $\epsilon > 0$, there exists an IIFS of similarities on the unit square $[0, 1]^2$ having the OSC such that $\text{s-dim}(\mathcal{F}) < 1 + \epsilon$, and its attractor is a continuum but not path connected.*

2. For each $\epsilon > 0$, there exists an IIFS of similarities on the unit square $[0, 1]^2$ having the OSC such that $\text{s-dim}(\mathcal{F}) < 1 + \epsilon$, and its attractor is the image of a curve, but not the image of a Hölder curve.

While the first example may not be too surprising, the second example has the additional property that for every two points there exists a Lipschitz curve in the attractor that connects them. A common theme in both these examples is the existence of a “bad 1-skeleton” inside the attractor which is not the image of a curve (in the first case) or not the image of a Hölder curve (in the second case). If such bad structures are absent, we show that the attractor admits good parameterizations.

Theorem 2.2. *Let $\mathcal{F} = \{\phi_i\}_{i \in \mathbb{N}}$ be an IIFS on a compact metric space so that the attractor K is compact, $\lim_{i \rightarrow \infty} \text{Lip}(\phi_i) = 0$, and there exists a curve $\gamma : [0, 1] \rightarrow K$ whose image intersects $\phi_i(K)$ for all i .*

1. *The attractor K is the image of a curve.*
2. *If γ is $\frac{1}{s}$ -Hölder for some $s \geq 1$, then for each $\alpha > \max\{s, \text{s-dim}(\mathcal{F})\}$ the attractor K is the image of a $\frac{1}{\alpha}$ -Hölder curve.*

We leave it as an open question whether in the second part of the theorem one can choose $\alpha = \max\{s, \text{s-dim}(\mathcal{F})\}$; this is unknown even for IFSs. Furthermore, in the case that $\mathcal{F}$ is finite, the existence of the curve γ in both parts of Theorem 2.2 is guaranteed by (Badger and Vellis, 2021, Theorem 1.1). Finally, the condition $\lim_{i \rightarrow \infty} \text{Lip}(\phi_i) = 0$ is necessary for Theorem 2.2; see §2.5.1.

The construction of the two examples of Theorem 2.1 is given in Section 2.5 and we prove Theorem 2.2 in Section 3.8.

2.3.1 Symbolic notation

Here and for the rest of the paper, given a countable (infinite or finite) set A and an integer $n \geq 0$, we denote by A^n the set of words formed from A of length n , with the convention $A^0 = \{\varepsilon\}$ and ε is the empty word. We denote $A^* = \bigcup_{n \geq 0} A^n$ and by $A^{\mathbb{N}}$, the set of infinite words formed with letters from A . Given $w = i_1 i_2 \cdots \in A^{\mathbb{N}}$ we denote $w(n) = i_1 \cdots i_n$ the truncated sub-word of w .

Given an IFS or IIFS $\mathcal{F} = \{f_i : i \in A\}$ and a finite word $w = i_1 \cdots i_n \in A^n$, we denote the length n of w by $|w|$ and

$$f_w = f_{i_1} \circ \cdots \circ f_{i_n}.$$

Given quantities $x, y \geq 0$ and a constant $a > 0$ we write $x \lesssim_a y$ if there exists a constant C depending on at most a such that $x \leq Cy$. If C is universal, we write $x \lesssim y$. We write $x \simeq_a y$ if $x \lesssim_a y$ and $y \lesssim_a x$.

2.4 Parametrizations of infinite IFS attractors

In this section we prove Theorem 2.2. We start by proving the simple fact that if an IIFS has similarity dimension equal to 1 and the attractor is a continuum, then the attractor is a line segment. This was shown by Hutchinson for finite IFS of similarities on Euclidean spaces (Hutchinson, 1981, Remark 3.4).

Proposition 2.1. *Let K be the attractor of an infinite or finite IFS $\mathcal{F}$ on a compact space. If K is a continuum and if $\text{s-dim}(\mathcal{F}) = 1$, then K is isometric to a closed line segment.*

Proof. Write $\mathcal{F} = \{\phi_i : i \in A\}$. We first claim that $\mathcal{H}^1(K) \leq \text{diam } K$. Fix $\delta > 0$ and let $n \in \mathbb{N}$ such that $(\text{diam } K) \text{Lip}(\phi_w) < \delta$ for all $w \in \mathbb{N}^n$. Since $\psi_{\mathcal{F}}(1) \leq 1$,

$$\begin{aligned} \mathcal{H}_\delta^1(K) &\leq \sum_{w \in A^n} \text{diam } \phi_w(K) \leq \text{diam } K \sum_{i_1, \dots, i_n \in A} \text{Lip } \phi_{i_1} \cdots \text{Lip } \phi_{i_n} \\ &= \text{diam } K (\psi_{\mathcal{F}}(1))^n \leq \text{diam } K \end{aligned}$$

and the claim follows by taking $\delta \rightarrow 0$. Hence, K is the Lipschitz image of $[0, 1]$ (Alberti and Ottolini, 2017, Theorem 4.4). Fix $p, q \in K$ such that $d(p, q) = \text{diam } K$ and fix an arc $\gamma \subset K$ with endpoints p, q . Note that

$$\text{diam } K = \mathcal{H}^1(K) \geq \mathcal{H}^1(\gamma) \geq \text{diam } \gamma = \text{diam } K.$$

Therefore, $K = \gamma$. For each $x, y \in \gamma$ denote by $\gamma(x, y)$ the subarc of γ with endpoints x, y .

Let $x, y \in \gamma$ such that x is between p and y . Then,

$$\begin{aligned} \text{diam } \gamma &\leq d(p, x) + d(x, y) + d(y, q) \\ &\leq \mathcal{H}^1(\gamma(p, x)) + \mathcal{H}^1(\gamma(x, y)) + \mathcal{H}^1(\gamma(y, q)) \\ &= \mathcal{H}^1(\gamma) \\ &= \text{diam } \gamma. \end{aligned}$$

Therefore, $\mathcal{H}^1(\gamma(x, y)) = d(x, y)$ for all $x, y \in \gamma$, which yields that K is isometric to the line segment $[0, \text{diam } K]$. $\square$

The rest of Section 3.8 is devoted to the proof of Theorem 2.2. Henceforth, we assume that we have an IIFS $\mathcal{F} = \{\phi_1, \phi_2, \dots\}$ on a compact space X so that K is compact, $\lim_{i \rightarrow \infty} \text{Lip}(\phi_i) = 0$, and there exists a curve $\gamma : [0, 1] \rightarrow K$ whose image intersect each set $\phi_i(K)$. We make some standard reductions.

First, if K is a point, then the claim of the theorem is trivial. Therefore, we may assume that K is nondegenerate and, rescaling the metric, we may also assume that $\text{diam}(K) = 1$. Second, since $\lim_{i \rightarrow \infty} \text{Lip}(\phi_i) = 0$, we may assume that

$$\text{Lip}(\phi_1) = \max_{i \in \mathbb{N}} \text{Lip}(\phi_i).$$

Third, by traversing the image of γ backwards if necessary, we may assume that $\gamma(0) = \gamma(1)$. For any point p in the image of γ , by reparameterizing γ , we may assume that $\gamma(0) = \gamma(1) = p$. Moreover, for any p, q in the image of γ , by reparameterizing γ , we may assume that there exists $[a, b] \subset [0, 1]$ such that $\gamma(a) = p$ and $\gamma(b) = q$.

Here and for the rest of this section, given $w = i_1 i_2 \dots i_n \in \mathbb{N}^*$, we denote

$$L_w = \text{Lip}(\phi_{i_1}) \text{Lip}(\phi_{i_2}) \dots \text{Lip}(\phi_{i_n})$$

with the convention $L_\varepsilon = 1$. Note that in general, $L_w \geq \text{Lip}(\phi_w)$.

2.4.1 Path connectedness of IIFS attractors

The first step in the proof of Theorem 2.2 is the following lemma which shows that K is pathwise connected.

Lemma 2.2.1. *Let $w \in \mathbb{N}^*$ and $x, y \in \phi_w(K)$. There exists a continuous map $f : [0, 1] \rightarrow \phi_w(K)$ such that $f(0) = x$ and $f(1) = y$. If γ is $\frac{1}{s}$ -Hölder for some $s > \text{s-dim}(\mathcal{F})$ and with Hölder constant H_0 , then f can be chosen to be $\frac{1}{s}$ -Hölder with Hölder constant $H \lesssim_{\psi_{\mathcal{F}}(s), s, H_0, L_1} L_w, \cdot$.*

Proof. Clearly, we may assume that $x \neq y$. Moreover, it suffices to assume that $w = \varepsilon$, and that there is no $j \in \mathbb{N}$ so that $x, y \in \phi_j(K)$, as otherwise we could pass to the longest common word. By the Kuratowski embedding theorem, we may further assume that K is a subset of ℓ_∞ .

We construct a sequence of continuous maps $(f_n : [0, 1] \rightarrow \ell_\infty)_{n \geq 0}$, sequences of finite collections of closed nondegenerate intervals $(\mathcal{B}_n)_{n \geq 0}$, $(\mathcal{E}_n)_{n \geq 0}$ in $[0, 1]$, and an injection $\mathbf{w} : \bigcup_{n \geq 0} \mathcal{E}_n \rightarrow \mathbb{N}^*$ with the following properties.

- (P1) For each $n \in \mathbb{N}$, intervals in $\mathcal{B}_n \cup \mathcal{E}_n$ intersect only at endpoints and the union of all these intervals is $[0, 1]$.
- (P2) For each $J \in \mathcal{E}_n$ there exists unique $S \in \mathcal{B}_{n+1}$ such that $S \subset J$. Conversely, for each $S \in \mathcal{B}_{n+1} \setminus \mathcal{B}_n$ there exists unique $J_S \in \mathcal{E}_n$ such that $S \subset J_S$.
- (P3) For any $n \geq 0$ and any $J \in \mathcal{E}_{n+1}$, there exists unique $J' \in \mathcal{E}_n$ such that $J \subset J'$. Moreover, there exists $u \in \mathbb{N}^*$ such that $|u| > 0$ and $\mathbf{w}(J) = \mathbf{w}(J')u$.
- (P4) For each $n \geq 0$ and $J \in \mathcal{E}_n$, there exist distinct $i, j \in \mathbb{N}$ such that $f_n|_J$ is a linear map mapping the left endpoint in $\phi_{\mathbf{w}(J)_i}(K)$ and the right endpoint in $\phi_{\mathbf{w}(J)_j}(K)$.

(P5) For each $n \geq 1$, and each $S \in \mathcal{B}_n$ there exists a closed nondegenerate interval I and a linear map $\zeta : S \rightarrow I$ such that

$$f_{n+1}|_S = f_n|_S = \phi_{\mathbf{w}(J_S)} \circ (\gamma|_I) \circ \zeta.$$

(P6) For each $n \geq 0$, $f_n(0) = x$ and $f_n(1) = y$.

(P7) For each $n \geq 0$, and each $J \in \mathcal{E}_n$, $\|f_n - f_{n+1}\|_{J,\infty} \leq 2L_{\mathbf{w}(J)}$.

Before the construction of $(f_n)_n$, $(\mathcal{B}_n)_n$, $(\mathcal{E}_n)_n$, and $\mathbf{w}$ we remark that (P3) and a simple induction yield that

(P8) For all $n \in \mathbb{N}$ and all $J \in \mathcal{E}_n$, $|\mathbf{w}(J)| \geq n$.

The construction is done in an inductive fashion. Define $\mathcal{E}_0 = \{[0, 1]\}$, $\mathcal{B}_0 = \emptyset$, $\mathbf{w}([0, 1]) = \varepsilon$, and $f_0 : [0, 1] \rightarrow \ell_\infty$ to be the linear map with $f_0(0) = x$ and $f_0(1) = y$. Property (P4) is immediate while the rest of the properties are vacuous.

Assume now that for some integer $n \geq 0$ we have defined a continuous map $f_n : [0, 1] \rightarrow \ell_\infty$, collections $\mathcal{B}_n$, $\mathcal{E}_n$ in $[0, 1]$, and an injection $\mathbf{w} : \bigcup_{k=0}^n \mathcal{E}_k \rightarrow \mathbb{N}^*$ with properties (P1)–(P7). The new collections of intervals will be

$$\mathcal{B}_{n+1} = \mathcal{B}_n \cup \bigcup_{J \in \mathcal{E}_n} \mathcal{B}_{n+1}(J), \quad \mathcal{E}_{n+1} = \bigcup_{J \in \mathcal{E}_n} \mathcal{E}_{n+1}(J).$$

If $S \in \mathcal{B}_n$, then we set $f_{n+1}|_S = f_n|_S$.

Fix now $J \in \mathcal{E}_n$ and write $J = [t, s]$. By (P3) there exist distinct $i, j \in \mathbb{N}$ such that $|\mathbf{w}(J)| \geq n$, $f_n(t) \in \phi_{\mathbf{w}(J)i}(K)$, and $f_n(s) \in \phi_{\mathbf{w}(J)j}(K)$. There also exists an interval $I = [a, b] \subset [0, 1]$ such that $\gamma(a) \in \phi_i(K)$ and $\gamma(b) \in \phi_j(K)$. We consider three possible cases.

Case I: Suppose that $\phi_{\mathbf{w}(J)} \circ \gamma(a) = f_n(t)$ and $\phi_{\mathbf{w}(J)} \circ \gamma(b) = f_n(s)$. Set $\mathcal{B}_{n+1}(J) = \{J\}$, $\mathcal{E}_{n+1}(J) = \emptyset$, and $f_{n+1}|_J = \phi_{\mathbf{w}(J)} \circ (\gamma|_I) \circ \zeta$ where $\zeta : J \rightarrow I$ is the orientation preserving linear map.

Case II: Suppose that $\phi_{\mathbf{w}(J)} \circ \gamma(a) \neq f_n(t)$ and $\phi_{\mathbf{w}(J)} \circ \gamma(b) = f_n(s)$. Let $z \in (t, s)$ and let $u \in \mathbb{N}^*$ be the shortest word such that there exist distinct $i, j \in \mathbb{N}$ with $\phi_{\mathbf{w}(J)} \circ \gamma(a) \in \phi_{\mathbf{w}(J)ui}(K)$ and $f_n(t) \in \phi_{\mathbf{w}(J)uj}(K)$. By (P4) we have that $|u| \geq 1$. Set $\mathcal{E}_{n+1}(J) = \{[t, z]\}$, $\mathcal{B}_{n+1}(J) = \{[z, s]\}$, and $\mathbf{w}([t, z]) = \mathbf{w}(J)u$. Define f_{n+1} on J continuously so that $f_{n+1}|[z, s]$ is as in Case I, and $f_{n+1}|[t, z]$ is linear with $f_{n+1}(t) = f_n(t)$. We work similarly if $\phi_{\mathbf{w}(J)} \circ \gamma(a) = f_n(t)$ and $\phi_{\mathbf{w}(J)} \circ \gamma(b) \neq f_n(s)$.

Case III: Suppose that $\phi_{\mathbf{w}(J)} \circ \gamma(a) \neq f_n(t)$ and $\phi_{\mathbf{w}(J)} \circ \gamma(b) \neq f_n(s)$. Let $t < z < w < s$ and set $\mathcal{E}_{n+1}(J) = \{[t, z], [w, s]\}$, $\mathcal{B}_{n+1}(J) = \{[z, w]\}$. For the definitions of $\mathbf{w}([t, z])$, $\mathbf{w}([w, s])$, and $f_{n+1}|J$ we work as in Cases I, II.

Properties (P1)–(P7) are easy to verify. Since $\mathbf{w}$ is injective on $\bigcup_{k=0}^n \mathcal{E}_k$, then by (P3) and the fact that $\mathbf{w}$ is injective on $\mathcal{E}_{n+1}(J)$ for each $J \in \mathcal{E}_n$ we have that $\mathbf{w}$ is injective on $\bigcup_{k=0}^{n+1} \mathcal{E}_k$. Finally, continuity of f_{n+1} follows from the facts that f_{n+1} is the same as f_n outside of intervals in $\mathcal{E}_n$, that $f_{n+1}|J$ is continuous for all $J \in \mathcal{E}_n$, and that $f_{n+1}|\partial J = f_n|\partial J$ for all $J \in \mathcal{E}_n$. This completes the inductive construction.

From (P5) and (P7) we have that $(f_n)_{n \in \mathbb{N}}$ converges to a continuous map $f : [0, 1] \rightarrow \ell_\infty$. Fix $J \in \bigcup_{n \geq 0} \mathcal{E}_n$. By (P5) we have that for all $S \in \bigcup_{n \geq 0} \mathcal{B}_n$ with $S \subset J$, $f(S) \subset \phi_{\mathbf{w}(J)}(K)$. By this fact, by (P3), and by (P4) we have that for all $m \geq n$,

$$\sup_{t \in J} \text{dist}(f_m(t), \phi_{\mathbf{w}(J)}(K)) \leq \max_{\substack{J' \subset J \\ J' \in \mathcal{E}_m}} 2L_{\mathbf{w}(J')} \leq 2L_{\mathbf{w}(J)}L_1^{m-n}.$$

Since K is closed, it follows that $f(J) \subset \phi_{\mathbf{w}(J)}(K)$. This proves the first part of the lemma.

Assume now that γ is $\frac{1}{s}$ -Hölder for some $s > \text{s-dim}(\mathcal{F})$ and with Hölder constant H_0 . Then, $\psi_{\mathcal{F}}(s) \in (0, 1)$. Set $\mathcal{E} = \bigcup_{n \geq 0} \mathcal{E}_n$ and $\mathcal{B} = \bigcup_{n \geq 0} \mathcal{B}_n$. For each $J \in \mathcal{E}$, set

$$\mathcal{M}(J) := \sum_{\substack{J' \in \mathcal{E} \\ J' \subset J}} (L_{\mathbf{w}(J')})^s.$$

By injectivity of $\mathbf{w}$ and (P3) we have that for each $J \in \mathcal{E}$

$$(L_{\mathbf{w}(J)})^s \leq \mathcal{M}(J) \leq \sum_{w \in \mathbb{N}^*} (L_{\mathbf{w}(J)w})^s = (L_{\mathbf{w}(J)})^s \sum_{n=0}^{\infty} \left(\sum_{i \in \mathbb{N}} L_i^s \right)^n = \frac{(L_{\mathbf{w}(J)})^s}{1 - \psi_{\mathcal{F}}(s)}.$$

The only difference in the construction in this case is that we require that for all $S \in \mathcal{B}$ and all $J \in \mathcal{E}$

$$|S| = (L_{\mathbf{w}(J_S)})^s / \mathcal{M}([0, 1]) \quad \text{and} \quad |J| = \mathcal{M}(J) / \mathcal{M}([0, 1]).$$

To see why this is possible, note that if $J \in \mathcal{E}_n$ and $\mathcal{B}_{n+1}(J) = \{S\}$ for some $n \geq 0$, then

$$|J| = \frac{(L_{\mathbf{w}(J)})^s}{\mathcal{M}([0, 1])} + \sum_{\substack{J' \in \mathcal{E} \\ J' \subsetneq J}} \frac{(L_{\mathbf{w}(J')})^s}{\mathcal{M}([0, 1])} = |S| + \sum_{J' \in \mathcal{E}_{n+1}(J)} |J'|.$$

We claim that the map f defined above is $\frac{1}{s}$ -Hölder continuous with Hölder constant depending only on $\psi_{\mathcal{F}}(s)$, L_1 , s , and H_0 . To show the claim, fix $p, q \in [0, 1]$. Clearly, we may assume that $p \neq q$. There exist $n \in \mathbb{N}$ and $J \in \mathcal{E}_n$ so that $p, q \in J$ and n is maximal. Proving the claim falls to a case study.

Case 1. Suppose that $p \in J_1$ and $q \in J_2$ where $J_1, J_2 \in \mathcal{E}_{n+1}(J)$ are distinct. On the one hand

$$|p - q| \geq |S| \gtrsim_{s, \psi_{\mathcal{F}}(s)} (L_{\mathbf{w}(J)})^s$$

while on the other hand, by (P5) and (P7), $f(p), f(q) \in \phi_{\mathbf{w}(J)}(K)$, so

$$d(f(p), f(q)) \leq L_{\mathbf{w}(J)}.$$

Case 2. Suppose that $p, q \in S$ where $S \in \mathcal{B}_{n+1}(J)$. By (P5),

$$d(f(p), f(q)) \leq L_{\mathbf{w}(J)} |S|^{-1/s} H_0 |p - q|^{1/s} \lesssim_{s, \psi_{\mathcal{F}}(s), H_0} |p - q|^{1/s}.$$

Case 3. Suppose that $p \in J_1 \setminus S$ and $q \in S \setminus J_1$ where $J_1 \in \mathcal{E}_{n+1}(J)$ and $S \in \mathcal{B}_{n+1}(J)$. There exists integer $m \geq n+1$ and $J' \in \mathcal{E}_m$ such that $p \in J' \subset J_1$, $J' \cap S \neq \emptyset$, and if $S' \in \mathcal{B}_{m+1}(J')$, then S' separates p from q . Let z be the unique point in $J' \cap S$. By Case 1 for p, z and Case 2 for z, q

$$d(f(p), f(q)) \leq d(f(p), f(z)) + d(f(z), f(q)) \lesssim_{L_1, s, \psi_{\mathcal{F}}(s), H_0} |p - q|^{1/s}$$

which completes the proof of the claim. $\square$

2.4.2 Parameterizations of IIFS attractors

The second step in the proof of Theorem 2.2 is the following lemma that allows us to reparametrize γ so that preimages of $\phi_i(K)$ have nonempty interior.

Lemma 2.2.2. *Let p be in the image of γ . There exists a map $\Gamma : [0, 1] \rightarrow K$ and a collection of nondegenerate closed intervals $\{\mathcal{I}_n\}_{n \in \mathbb{N}}$ with disjoint interiors such that Γ has the same image as γ , satisfies $\Gamma(0) = \Gamma(1) = p$, and for each $n \in \mathbb{N}$, $\mathcal{I}_n \subset \Gamma^{-1}(\phi_n(K))$. Moreover, if γ is $\frac{1}{s}$ -Hölder with constant H_0 , and if $(a_n) \in \ell^1$ is a sequence of positive numbers, then Γ is $\frac{1}{s}$ -Hölder with constant $H \leq 2^{1/s} H_0 (1 + \|(a_n)\|_1)^{1/s}$ and for each $n \in \mathbb{N}$, $|\mathcal{I}_n| = a_n (1 + \|(a_n)\|_1)^{-1}$.*

Proof. We may assume that $\gamma(0) = \gamma(1) = p$. For each $n \in \mathbb{N}$ fix a point $x_n \in \gamma^{-1}(\phi_n(K)) \subset [0, 1]$. It is possible that for some $n \neq m$ we have $x_n = x_m$. Let $\{p_k\}_{k \in B}$ be an enumeration of the set $\{x_n\}_{n \in \mathbb{N}}$ where B is either a finite set, or $\mathbb{N}$. For each $k \in B$, define $A_k = \{n \in \mathbb{N} : x_n = p_k\}$.

For the first claim, fix a decreasing sequence $(b_n)_{n \in \mathbb{N}}$ of positive numbers that converges to 0. Identify $\mathbb{R}^3$ with $\mathbb{C} \times \mathbb{R}$ and define the set

$$E = (\{0\} \times [0, 1]) \cup \bigcup_{k \in B} \bigcup_{n \in A_k} (\{te^{2\pi i/n} : t \in [0, b_n]\} \times \{p_k\}) \subset \mathbb{R}^3.$$

Since $b_n \rightarrow 0$, it is easy to see that E is closed. Moreover, there exists a continuous increasing map $\eta : [0, 1] \rightarrow [0, 1]$ such that $\eta(0) = 0$,

$$\max_{n \in A_k} b_n \leq \eta \left(\min_{j \in \{1, \dots, k-1\}} |p_j - p_k| \right) \quad \text{for all } k \in B$$

and

$$b_n \leq \eta \left(\min_{m \in \{1, \dots, n-1\}} |e^{2\pi i/m} - e^{2\pi i/n}| \right) \quad \text{for all } n \in \mathbb{N}.$$

We claim that there exists a continuous $\omega : [0, 1] \rightarrow [0, 1]$ with $\omega(0) = 0$ such that for all $x, y \in E$, there exists a curve $\sigma : [0, 1] \rightarrow E$ such that $\sigma(0) = x$, $\sigma(1) = y$ and the diameter of its image is at most $\omega(|x - y|)$. The proof of the claim is a simple case study.

If $x, y \in \{0\} \times [0, 1]$, then use the line segment $[x, y]$.

If $x = (te^{2\pi i/n}, p_k)$ and $y \in \{0\} \times [0, 1]$ for some $t \in [0, b_n]$, $n \in A_k$ and $k \in B$, then use the union of line segments $[x, (0, p_k)] \cup [(0, p_k), y]$.

If $x = (t_1 e^{2\pi i/n}, p_k)$ and $y = (t_2 e^{2\pi i/n}, p_k)$ for some distinct $t_1, t_2 \in [0, b_n]$, $n \in A_k$ and $k \in B$, then use the line segment $[x, y]$.

If $x = (t_1 e^{2\pi i/n}, p_k)$ and $y = (t_2 e^{2\pi i/m}, p_k)$ for some $t_1, t_2 \in [0, b_n]$, $n, m \in A_k$ with $m < n$, and $k \in B$, then use the union of line segments $[x, (0, p_k)] \cup [(0, p_k), y]$. Note that

$$\begin{aligned} |x - y| &\gtrsim |t_1 - t_2| + \min\{t_1, t_2\} |e^{2\pi i/m} - e^{2\pi i/n}| \\ &\geq |t_1 - t_2| + \min\{t_1, t_2\} \eta^{-1}(\min\{t_1, t_2\}) \end{aligned}$$

while

$$\text{diam}(\sigma([0, 1])) \leq t_1 + t_2 \leq |t_1 - t_2| + \min\{t_1, t_2\}.$$

Finally, suppose that $x = (t_1 e^{2\pi i/m}, p_k)$ and $y = (t_2 e^{2\pi i/n}, p_j)$, for some distinct $k, j \in B$, $m \in A_k$, $n \in A_j$, $t_1 \in [0, b_m]$, and $t_2 \in [0, b_n]$. Then,

$$|x - y| \gtrsim |p_k - p_j| + |t_1 - t_2|$$

and if σ is the union of line segments $[x, (0, p_k)] \cup [(0, p_k), (0, p_j)] \cup [(0, p_j), y]$,

$$\begin{aligned} \text{diam}(\sigma([0, 1])) &\leq t_1 + t_2 + |p_k - p_j| \leq |t_1 - t_2| + \min\{t_1, t_2\} + |p_k - p_j| \\ &\lesssim |x - y| + \eta(|p_k - p_j|) \end{aligned}$$

and the proof of the claim is complete.

Thus, E is connected and locally connected, and by the Hahn-Mazurkiewicz Theorem ([Hocking and Young, 1961](#), Theorem 3.30), there exists continuous surjection $g : [0, 1] \rightarrow E$. Note that for each $n \in \mathbb{N}$, the preimage

$$g^{-1}(\{te^{2\pi i/n} : t \in [0, b_n]\} \times \{x_n\})$$

contains a nondegenerate closed interval. Define now $\tilde{\gamma} : E \rightarrow K$ by $\tilde{\gamma}|_{\{0\} \times [0, 1]} = \gamma$ and for each $n \in \mathbb{N}$ and $t \in [0, b_n]$, $\tilde{\gamma}(te^{2\pi i/n}, x_n) = \gamma(x_n)$. Then $\tilde{\gamma}$ is continuous and $\Gamma := \tilde{\gamma} \circ g$ satisfies the conclusions of the lemma.

For the second part of the lemma, assume that γ is $\frac{1}{s}$ -Hölder with constant H_0 , and assume that $(a_n) \in \ell^1$ is a sequence of positive numbers. Define E as above replacing b_n by a_n . Then,

$$\mathcal{H}^1(E) = 1 + \|(a_n)\|_1 < \infty.$$

Therefore, by ([Alberti and Ottolini, 2017](#), Theorem 4.4), there exists a Lipschitz surjection $g : [0, 1] \rightarrow E$ with constant speed equal to $2\mathcal{H}^1(E)$. Thus, for each $n \in \mathbb{N}$,

$g^{-1}(\{te^{2\pi i/n} : t \in [0, a_n]\} \times \{x_n\})$ contains a closed subinterval of length

$$(1 + \|(a_n)\|_1)^{-1}a_n.$$

Define $\tilde{\gamma}$ as above and note that $\tilde{\gamma}$ is $\frac{1}{s}$ -Hölder with constant H_0 . Setting $\Gamma := \tilde{\gamma} \circ g$, we have for all $x, y \in [0, 1]$

$$d(\Gamma(x), \Gamma(y)) \leq H_0|g(x) - g(y)|^{1/s} \leq 2^{1/s}H_0(1 + \|(a_n)\|_1)^{1/s}|x - y|^{1/s}. \quad \square$$

We are now ready to prove Theorem 2.2.

Proof of Theorem 2.2. Fix $p_0 \in K$. We construct a sequence of continuous maps $(f_n : [0, 1] \rightarrow K)_{n \geq 0}$, sequences of collections of closed nondegenerate intervals $(\mathcal{N}_n)_{n \geq 0}$, $(\mathcal{J}_n)_{n \geq 0}$ in $[0, 1]$, and a bijection $\mathbf{w} : \bigcup_{n \geq 0} \mathcal{N}_n \rightarrow \mathbb{N}^*$ with the following properties.

- (P1) For each $n \geq 0$, intervals in $\mathcal{N}_n \cup \mathcal{J}_n$ intersect at most on endpoints.
- (P2) For each $n \geq 0$, each $I \in \mathcal{N}_n$ and each $i \in \mathbb{N}$, there exists $I' \in \mathcal{N}_{n+1}$ contained in the interior of I such that $\mathbf{w}(I') = \mathbf{w}(I)i$. Conversely, for each $n \geq 0$ and each $I' \in \mathcal{N}_{n+1}$, there exists $I \in \mathcal{N}_n$ and $i \in \mathbb{N}$ such that I' is contained in the interior of I and $\mathbf{w}(I') = \mathbf{w}(I)i$.
- (P3) For each $n \geq 0$ and each $I \in \mathcal{N}_n$, there exist exactly two intervals $J, J' \in \mathcal{J}_{n+1}$ contained in I . Conversely, for each $n \geq 0$ and each $J \in \mathcal{J}_{n+1} \setminus \mathcal{J}_n$, there exists unique interval $I_J \in \mathcal{N}_n$ such that $J \subset I_J$.
- (P4) If $n \geq 1$, $I \in \mathcal{N}_{n-1}$ and $J, J' \in \mathcal{J}_n$ are contained in I , then there exists an orientation reversing linear map $\zeta_J : J' \rightarrow J$, such that $f_n|_{J'} = (f_n|_J) \circ \zeta_J$. Moreover, $f_n|_J = g \circ \zeta$ where $g : [0, 1] \rightarrow \phi_{\mathbf{w}(I)}(K)$ is the map from Lemma 2.2.1 and $\zeta : J \rightarrow [0, 1]$ is an increasing linear map.
- (P5) For each $n \geq 0$ and each $I \in \mathcal{N}_n$, $f_n|_I$ is constant and its image is in $\phi_{\mathbf{w}(I)}(K)$. Moreover, $f_n(I) \subset f_{n+1}(I) \subset \phi_{\mathbf{w}(I)}(K)$.

(P6) For each $n \in \mathbb{N}$, if x is not in the interior of some $I \in \mathcal{N}_n$, then $f_{n+1}(x) = f_n(x)$.

(P7) For each $n \in \mathbb{N}$, $f_n(0) = f_n(1) = p_0$.

For the construction, apply Lemma 2.2.2 and obtain a map $\Gamma : [0, 1] \rightarrow K$ and a collection $\{\mathcal{I}_i : i \in \mathbb{N}\}$ of closed nondegenerate intervals in $[0, 1]$ such that for each $i \in \mathbb{N}$, $\Gamma|_{\mathcal{I}_i}$ is constant and its image is in $\phi_i(K)$. The proof of the claim is done in an inductive fashion.

For $n = 0$, let $\mathcal{N}_0 = \{[0, 1]\}$ and $\mathcal{J}_0 = \emptyset$, let $f_0 : [0, 1] \rightarrow K$ be the constant map p_0 , and let $\mathbf{w}([0, 1]) = \varepsilon$. Properties (P1), (P5) and (P7) are trivial while the rest of them are vacuous.

Assume now that for some $n \geq 0$ we have defined a continuous $f_n : [0, 1] \rightarrow K$, collections of intervals $\mathcal{N}_n, \mathcal{J}_n$, and a bijection $\mathbf{w} : \mathcal{N}_n \rightarrow \mathbb{N}^n$ satisfying assumptions (P1)–(P7). The new collections of intervals will be

$$\mathcal{J}_{n+1} = \mathcal{J}_n \cup \bigcup_{I \in \mathcal{N}_n} \mathcal{J}_{n+1}(I), \quad \mathcal{N}_{n+1} = \bigcup_{I \in \mathcal{N}_n} \mathcal{N}_{n+1}(I).$$

We set

$$f_{n+1}|_{[0, 1] \setminus \bigcup \mathcal{N}_n} = f_n|_{[0, 1] \setminus \bigcup \mathcal{N}_n}.$$

Fix now $I \in \mathcal{N}_n$. By (P5), there exists $i \in \mathbb{N}$ such that $f_n(I) \in \phi_{\mathbf{w}(I)i}(K)$. Reparameterizing Γ , we may assume that $\Gamma(0) \in \phi_j(K)$ and 0 is the left endpoint of $\mathcal{I}_j$. Write $I = [a, b]$, and let $a < a_1 < a_2 < b$. Let $\xi_I : [a_1, a_2] \rightarrow [0, 1]$ be an increasing linear map and let $\zeta_I : [a, a_1] \rightarrow [0, 1]$ and $\zeta'_I : [a_2, b] \rightarrow [0, 1]$ be increasing linear maps. Set

$$\mathcal{N}_{n+1}(I) = \{\xi_I^{-1}(\mathcal{I}_j) : j \in \mathbb{N}\} \quad \text{and} \quad \mathcal{J}_{n+1}(I) = \{[a, a_1], [a_2, b]\}$$

and for each $j \in \mathbb{N}$ define $\mathbf{w}(\xi_I^{-1}(\mathcal{I}_j)) = \mathbf{w}(I)j$.

Let $g : [0, 1] \rightarrow \phi_{\mathbf{w}(I)}(K)$ be the map given from Lemma 2.2.1 that connects $f_n(I)$ to $\phi_{\mathbf{w}(I)}(\Gamma(\mathcal{I}_j))$ and define

1. $f_{n+1}|[a, a_1] = g \circ \zeta_I$,
2. $f_{n+1}|[a_1, a_2] = \phi_{\mathbf{w}(I)} \circ \Gamma \circ \xi_I$,
3. $f_{n+1}|[a_2, b] = g \circ h \circ \zeta'_I$ where $h : [0, 1] \rightarrow [0, 1]$ with $h(x) = 1 - x$.

Properties (P1)–(P6) are clear from design and the properties of Γ . Note that for all $I \in \mathcal{N}_n$ the function $\mathbf{w} : \mathcal{N}_{n+1}(I) \rightarrow \{\mathbf{w}(I)i : i \in \mathbb{N}\}$ is bijective. Therefore, $\mathbf{w} : \mathcal{N}_{n+1} \rightarrow \mathbb{N}^{n+1}$ is a bijection. Finally, since $0, 1$ are not contained in the interior of any $I \in \mathcal{N}_n$, by (P6), $f_{n+1}(0) = f_{n+1}(1) = p_0$.

It remains to prove continuity of f_{n+1} . Fix $x \in [0, 1]$. We only show continuity of f_{n+1} at x from the right. To this end, fix a sequence $x_m \subset (x, 1]$ that converges to x and consider the following three cases.

Case 1. Suppose that for all m sufficiently large, $x_m \in [0, 1] \setminus \overline{\bigcup \mathcal{N}_n}$. Then $f_{n+1}(x_m)$ converges to $f_{n+1}(x)$ by (P6) and continuity of f_n .

Case 2. Suppose that for all m sufficiently large, $x_m \in I$ for some $I \in \mathcal{N}_n$. Then, $f_{n+1}(x_m)$ converges to $f_{n+1}(x)$ by design.

Case 3. Suppose that for all m sufficiently large, there exists $I_m \in \mathcal{N}_n$ such that $x_m \in I_m$ and that the collection $\{I_m\}_m$ is infinite. Fixing $\epsilon > 0$, there exists $i_0 \in \mathbb{N}$ such that for every $i \geq i_0$, $L_i < \epsilon/2$. Since the collection $\{\mathbf{w}(I_m)\}_m$ is infinite, there exists $N \in \mathbb{N}$ such that for every $m \geq N$, some character of the word $\mathbf{w}(I_m)$ is larger than i_0 . It follows that $L_{\mathbf{w}(I_m)} < \epsilon/2$ for every $m \geq N$. By continuity of f_n , we may further assume that for every $m \geq N$, $d(f_n(x), f_n(x_m)) < \epsilon/2$. By (P5), for every $m \geq N$,

$$d(f_{n+1}(x), f_{n+1}(x_m)) \leq d(f_n(x), f_n(x_m)) + d(f_n(x_m), f_{n+1}(x_m)) < \epsilon.$$

This completes the induction and the proof of (P1)–(P7).

By (P5) and (P6),

$$\|f_{n+1} - f_n\|_\infty \leq \sup_{I \in \mathcal{N}_n} \|f_n - f_{n+1}\|_{I, \infty} \leq \sup_{I \in \mathcal{N}_n} L_{\mathbf{w}(I)} \leq L_1^n$$

so the maps f_n converge uniformly to a continuous map $f : [0, 1] \rightarrow K$. By (P5) and the bijectivity of $\mathbf{w}$, we have $f([0, 1]) \cap \phi_w(K) \neq \emptyset$ for all $w \in \mathbb{N}^*$. Therefore, for all $x \in K$ and $n \in \mathbb{N}$,

$$\text{dist}(x, f([0, 1])) \leq \inf_{w \in \mathbb{N}^n} \text{diam } \phi_w(K) \leq L_1^n.$$

Hence, $K \subset f([0, 1])$ and it follows that $f([0, 1]) = K$. This proves the first part of Theorem 2.2.

For the second part of the theorem, assume that γ is $\frac{1}{s}$ -Hölder for some $s > \text{s-dim}(\mathcal{F})$ and with Hölder constant H_0 . Define for all $w \in \mathbb{N}^*$

$$M_w := 3 \sum_{u \in \mathbb{N}^*} (L_{wu})^s.$$

Working as in the proof of Lemma 2.2.1, we have that for all $w \in \mathbb{N}^*$,

$$M_w = 3 (L_w)^s (1 - \psi_{\mathcal{F}}(s))^{-1}.$$

For each $i \in \mathbb{N}$ set $a_i = M_i > 0$ and note that $\|(a_i)\|_1 = M_\varepsilon \lesssim_{s, \psi_{\mathcal{F}}(s)} 1$. We apply on each stage of the construction, the second part of Lemma 2.2.2 with $a_i = M_i$ and we may assume that Γ is $\frac{1}{s}$ -Hölder with constant $H \lesssim_{s, \psi_{\mathcal{F}}(s), L_1} H_0$.

The other change in the construction, is that we require that if $n \geq 0$, $I \in \mathcal{N}_n$, and $J \in \mathcal{J}_{n+1}(I)$, then

$$|I| = M_\varepsilon^{-1} M_{\mathbf{w}(I)} \quad \text{and} \quad |J| = M_\varepsilon^{-1} (L_{\mathbf{w}(I)})^s.$$

To see why this is possible, fix $n \geq 0$ and $I \in \mathcal{N}_n$. By Lemma 2.2.2

$$\begin{aligned}
|I| &= M_\varepsilon^{-1} M_{\mathbf{w}(I)} \\
&= 2M_\varepsilon^{-1} (L_{\mathbf{w}(I)})^s + M_\varepsilon^{-1} (L_{\mathbf{w}(I)})^s + \sum_{i \in \mathbb{N}} M_\varepsilon^{-1} M_{\mathbf{w}(I)i} \\
&= \sum_{J \in \mathcal{J}_{n+1}(I)} |J| + \left| I \setminus \bigcup \mathcal{J}_{n+1}(I) \setminus \bigcup \mathcal{N}_{n+1}(I) \right| + \sum_{J \in \mathcal{N}_{n+1}(I)} |J|.
\end{aligned}$$

We claim that the limit f of the maps f_n is $\frac{1}{s}$ -Hölder continuous. To this end, fix distinct $p, q \in [0, 1]$ and let $n \geq 0$ be the maximal integer such that there is some $I \in \mathcal{N}_n$ with $p, q \in I$. Denote by J, J' the two elements in $\mathcal{J}_{n+1}(I)$, by I' the closure of $I \setminus (J \cup J')$, and by $\mathcal{B}_I$ the closure of the set of points in I which are not contained in any interval in $\mathcal{N}_{n+1}(I) \cup \mathcal{J}_{n+1}(I)$. The proof of the claim falls to a case study.

Case 1. Suppose that $p, q \in J$ or $p, q \in J'$. By (P6) and Lemma 2.2.1,

$$\begin{aligned}
d(f(p), f(q)) &= d(g \circ \zeta_I(p), g \circ \zeta_I(q)) \\
&\lesssim_{\psi_{\mathcal{F}(s)}, L_1, s, H_0} L_{\mathbf{w}(I)} |J|^{-1/s} |p - q|^{1/s} \\
&\lesssim_{\psi_{\mathcal{F}(s)}, s, L_1, H_0} |p - q|^{1/s}.
\end{aligned}$$

Case 2. Suppose that p and q are separated by the interior of one of I', J, J' . Then $|p - q| \gtrsim_{s, \psi_{\mathcal{F}(s)}} (L_{\mathbf{w}(I)})^s$ while, by (P5), $f(I) \subset \phi_{\mathbf{w}(I)}(K)$ and

$$d(f(p), f(q)) \leq L_{\mathbf{w}(I)}.$$

Case 3. Suppose that $p, q \in \mathcal{B}_I$. By (P6) and design of f_{n+1} , $f|_{\mathcal{B}_I} = f_{n+1}|_{\mathcal{B}_I} = \phi_{\mathbf{w}(I)} \circ \Gamma \circ \xi_I$. Therefore, by Lemma 2.2.2 we have

$$d(f(p), f(q)) \lesssim_{H_0, s, \psi_{\mathcal{F}(s)}} L_{\mathbf{w}(I)} |I'|^{-1/s} |p - q|^{1/s} \lesssim_{H_0, \psi_{\mathcal{F}(s)}, s} |p - q|^{1/s}.$$

Case 4. Suppose that $p \in I_1$ and $q \in I_2$ where $I_1, I_2 \in \mathcal{N}_{n+1}(I)$. By maximality of n , we have $I_1 \neq I_2$. Let $a \in I_1$ and $b \in I_2$ such that $|a - b| = \text{dist}(I_1, I_2)$. Then

the pair p, a satisfy either Case 1 or Case 2, with I replaced by I_1 . Similarly for q, b . Moreover, $a, b \in \mathcal{B}_I$, and hence satisfy Case 3. Therefore, by triangle inequality,

$$d(f(p), f(q)) \lesssim_{\psi_{\mathcal{F}}(s), s, H_0, L_1} |p - q|^{1/s}.$$

Case 5. Suppose that $p \in I_1$ for some $I_1 \in \mathcal{N}_{n+1}(I)$ and $q \in \mathcal{B}_I$. Let $a \in I_1$ be such that $|a - q| = \text{dist}(I_1, q)$. Note that points a, p satisfy one of Case 1 or Case 2 (with I replaced by I_1), while points a, q satisfy Case 3. Therefore, by the triangle inequality,

$$d(f(p), f(q)) \lesssim_{\psi_{\mathcal{F}}(s), s, H_0, L_1} |p - q|^{1/s}.$$

Case 6. Suppose that $p \in J \cup J'$ (say J) and $q \in I'$. Let $a \in J$ such that $|a - q| = \text{dist}(J, q)$. Note that points a, p satisfy Case 1, while points a, q satisfy Case 3 or Case 5. Therefore, by the triangle inequality,

$$d(f(p), f(q)) \lesssim_{\psi_{\mathcal{F}}(s), s, H_0, L_1} |p - q|^{1/s}. \quad \square$$

2.5 Examples of IIFS

In this section we provide three examples of IIFS. In §2.5.1 we show that the condition $\lim_{n \rightarrow \infty} \text{Lip}(\phi_n) = 0$ is necessary in Theorem 2.2, in §2.5.2 we prove Theorem 2.1(1), and in §2.5.3 we prove Theorem 2.1(2).

2.5.1 An IIFS without vanishing Lipschitz norms

For this example we use complex coordinates. For each $n \in \mathbb{N}$ define a contraction $\phi_n : \overline{\mathbb{B}^2} \rightarrow \overline{\mathbb{B}^2}$ on the closed unit disk $\overline{\mathbb{B}^2}$ by

$$\phi_n(z) = e^{\frac{2\pi i}{n}} \frac{1}{2}(\text{Re}(z) + 1).$$

Note that $\text{Lip}(\phi_n) = 1/2$ for all $n \in \mathbb{N}$, and let $\mathcal{F} = \{\phi_n : n \in \mathbb{N}\}$.

Set $G = \{te^{\frac{2\pi i}{n}} : n \in \mathbb{N}, t \in [0, 1]\}$; see Figure 2.1 below for the first generation of images of the system used in this section. Note that the figure shows only the images of finitely many maps in the family, the images in fact accumulate to the real interval $[0, 1]$.

We claim that G is the attractor K of the IIFS $\mathcal{F}$. Note that for all $n \in \mathbb{N}$ we have $\phi_n(\overline{\mathbb{B}^2}) = \phi_n(G)$. Therefore, $\phi_w(\overline{\mathbb{B}^2}) = \phi_w(G)$ for all $w \in \mathbb{N}^*$. Moreover, it is easy to see that

$$\bigcup_{n \in \mathbb{N}} \phi_n(\overline{\mathbb{B}^2}) = G,$$

which yields that $K \subset G$. For the opposite inclusion, fix $x \in G$. There exists $n_1 \in \mathbb{N}$ such that $x \in \phi_{n_1}(\overline{\mathbb{B}^2})$. Assume now that for some $m \in \mathbb{N}$ we have defined a word $w \in \mathbb{N}^m$ such that $x \in \phi_w(\overline{\mathbb{B}^2}) = \phi_n(G)$. Since $G = \bigcup_{n \in \mathbb{N}} \phi_n(\overline{\mathbb{B}^2})$, there exists $n_{m+1} \in \mathbb{N}$ such that $x \in \phi_{wn_{m+1}}(\overline{\mathbb{B}^2})$. It follows that there exists an infinite word $w = n_1 n_2 \cdots \in \mathbb{N}^{\mathbb{N}}$ such that $x \in \bigcap_{m \in \mathbb{N}} \phi_{n_1 \cdots n_m}(\overline{\mathbb{B}^2})$ which yields that $G \subset K$.

To complete the example, note that there exists a curve $\gamma : [0, 1] \rightarrow K$ (namely the constant curve with image the origin) whose image intersects every $\phi_n(K)$, and the attractor K is a continuum but it is not locally connected.

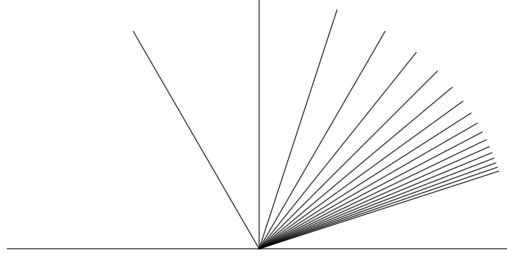


Figure 2.1: Attractor of IFS of §2.5.1

2.5.2 An IFS where the attractor is a continuum but not path connected

Fix $s > 1$ and fix $M \in \mathbb{N}$ such that $M > \max \left\{ 4^{\frac{1}{s-1}}, 7 \right\}$. For each $n \in \mathbb{N}$ let

$$a_n = \frac{1 + M^n(2M + 1)^{1-n}}{M + 1} \quad \text{and} \quad b_n = M^{-n}(a_n - a_{n+1} - M^{-n}).$$

Note that $a_n \in (0, 1]$ for each n , that a_n is strictly decreasing, that $a_1 = 1$, and that for all $n \in \mathbb{N}$

$$a_n - a_{n+1} - M^{-n} = \left(\frac{M}{2M + 1} \right)^n - M^{-n} > M^{-n}.$$

The construction of the IFS $\mathcal{F}$ is done in three subfamilies

$$\mathcal{F} = \{\phi_{n,i}\}_{n,i} \cup \{\tau_{n,i}\}_{n,i} \cup \{\sigma_n\}_n.$$

First, for $n \in \mathbb{N}$ and $i \in \{1, \dots, M^n\}$, define $\phi_{n,i} : [0, 1]^2 \rightarrow [0, 1]^2$ by

$$\phi_{n,i}((x, y)) = (a_n - M^{-n}, M^{-n}(i - 1)) + M^{-n}(x, y).$$

Second, for $n \in \mathbb{N}$ and $i \in \{1, \dots, M^n\}$, define $\tau_{n,i} : [0, 1]^2 \rightarrow [0, 1]^2$ by

$$\begin{aligned}\tau_{n,i}((x, y)) &= (a_n - M^{-n} - (i-1)b_n, 0) + b_n(x, y) \quad \text{if } n \text{ is even} \\ \tau_{n,i}((x, y)) &= (a_n - M^{-n} - (i-1)b_n, 1 - b_n) + b_n(x, y) \quad \text{if } n \text{ is odd.}\end{aligned}$$

Third, for $i \in \{1, \dots, M+1\}$, define $\sigma_i : [0, 1]^2 \rightarrow [0, 1]^2$ by

$$\sigma_i((x, y)) = (0, \frac{i-1}{M+1}) + \frac{1}{M+1}(x, y).$$

See Figure 2.2 below for the first generation of images of the system used in this section. Note that this figure shows only finitely many of the first generation images, and the images in fact accumulate to the boxes along the left edge of the square.

Using the definition of M we now compute

$$\begin{aligned}\psi_{\mathcal{F}}(s) &= \sum_{n=1}^{\infty} \sum_{i=1}^{M^n} \text{Lip}(\phi_{n,i})^s + \sum_{n=1}^{\infty} \sum_{i=1}^{M^n} \text{Lip}(\tau_{n,i})^s + \sum_{n=1}^{M+1} \text{Lip}(\sigma_n)^s \\ &= \sum_{n=1}^{\infty} M^{(1-s)n} + \sum_{n=1}^{\infty} M^n b_n^s + (M+1)^{1-s} \\ &\leq \frac{2}{M^{s-1} - 1} + \frac{1}{(M+1)^{s-1}} \\ &< 1.\end{aligned}$$

Therefore, the similarity dimension of $\mathcal{F}$ is no more than s . We are now ready to prove the first part of Theorem 2.1.

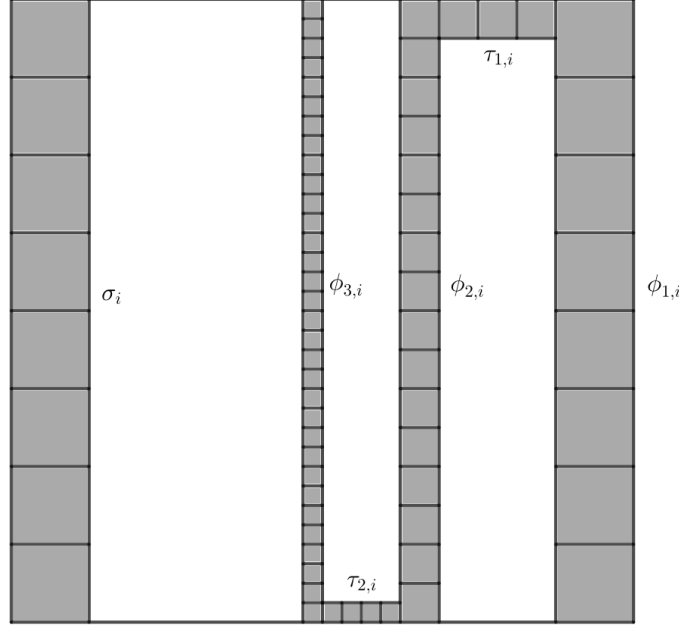


Figure 2.2: First iteration of IIFS of §2.5.2.

Proof of Theorem 2.1(1). Define $K_0 = [0, 1]^2$ and for each $m \in \mathbb{N}$ define $K_m = \bigcup_{f \in \mathcal{F}} f(K_{m-1})$. By (2.3.2), $(K_m)_{m \in \mathbb{N}}$ is a nested family of sets with $K = \bigcap_{m \geq 0} K_m$. Thus in order to show that K is a continuum, it suffices to show that each K_m is a continuum.

To show compactness, we proceed by induction. The base case $m = 0$ is trivial. Suppose now that K_m is compact. Let (p_k) be a sequence in K_{m+1} converging to some $p = (x, y) \in \mathbb{R}^2$. If $x < \frac{1}{2^{M-1}}$, then for all k large enough, $p_k \in \bigcup_i \sigma_i(K_m)$ which is compact. If $x > \frac{1}{2^{M-1}}$ then there exists a finite set $\mathcal{F}' \subset \mathcal{F}$ such that for all k large enough, p_k is contained in $\bigcup_{f \in \mathcal{F}'} f(K_m)$ which is compact. Finally, if $x = \frac{1}{2^{M-1}}$, then $p \in \bigcup_i \sigma_i(K_m)$. In either case, $p \in K_{m+1}$.

Connectedness is also shown inductively. The case $m = 0$ is trivial. Suppose now that for some $m \geq 0$, the set K_m is connected. Let

$$L = \bigcup_{i=1}^{2M-1} \sigma_i(K_m), \quad R = K_{m+1} \setminus L = \bigcup_{n,i} \phi_{n,i}(K_m) \cup \bigcup_{n,i} \tau_{n,i}(K_m).$$

Note that L is connected since for every $i \in \{1, \dots, 2M-2\}$

$$\sigma_{i+1}((1, 0)) \in \sigma_i(K_m) \cap \sigma_{i+1}(K_m)$$

and each $\sigma_i(K_m)$ is connected by the inductive hypothesis. We also claim that R is connected. This follows from the fact that sets $\tau_{n,i}(K_m)$ and $\phi_{n,i}(K_m)$ are connected and from the fact that for all $n \in \mathbb{N}$

$$\begin{aligned}\phi_{n,i+1}((1, 0)) &\in \phi_{n,i}(K_m) \cap \phi_{n,i+1}(K_m) \\ \tau_{n,i+1}((1, 0)) &\in \tau_{n,i}(K_m) \cap \tau_{n,i+1}(K_m) \\ \tau_{n,M^n}((0, 0)) &\in \tau_{n,M^n}(K_m) \cap \phi_{n+1,1}(K_m) \\ \phi_{n,M^n}((0, 0)) &\in \tau_{n,1}(K_m) \cap \phi_{n,M^n}(K_m).\end{aligned}$$

Therefore if we were to have some partition $K_{m+1} = A \cup B$ by disjoint nonempty open sets A, B , we must have that $A = L$ or $B = L$. However, R is not a closed set, as it is disjoint from $\{\frac{1}{2M-1}\} \times [0, 1]$ but its sequential closure contains $(\frac{1}{2M-1}, 0)$. Thus, K_{m+1} is connected.

To finish the proof, we show that there is no path in K_1 connecting $p = (\frac{1}{2M-1}, 0)$ to $q = (1, 0)$. Since both of these points are in K , and $K \subset K_1$ the latter implies that K is not path connected. To this end, assume for a contradiction that $f = (f_1, f_2) : [0, 1] \rightarrow K_1$ is a continuous map with $f(0) = p$, $f(1) = q$. By design of K_1 , for any $n \in \mathbb{N}$

1. for any $t \in [0, 1]$ with

$$\frac{1}{2M-1} + 2^{-2n} < f_1(t) < \frac{1}{2M-1} + 2^{-2n+1} - M^{-2n}$$

we have that $f_2(t) \leq (2M)^{-2n}$ and

2. for any $t \in [0, 1]$ with

$$\frac{1}{2M-1} + 2^{-2n-1} < f_1(t) < \frac{1}{2M-1} + 2^{-2n} - M^{-2n-1}$$

we have that $f_2(t) \geq 1 - (2M)^{-2n-1}$.

It follows now that f_2 (and consequently f) is not continuous at $t = 0$. $\square$

2.5.3 An IIFS where the attractor is the image of a curve but not the image of a Hölder curve

Fix $s > 1$ and fix an even integer $M \geq \max\{10, 7^{\frac{1}{s-1}}\}$. For each $n \in \mathbb{N}$ let

$$a_n = \frac{\log(2)}{M^n \log(n+1)} \quad \text{and} \quad b_n = \frac{4 + 2M(2^{-1} - M^{-1})^n}{M + 2}.$$

A simple calculation shows that for all $n \in \mathbb{N}$, $b_{n+1} < b_n - 2a_n < b_n$. Additionally, for each $n \in \mathbb{N}$ let N_n be an integer such that

$$2a_{n+1}^{-1}(b_n - a_n - b_{n+1}) \leq N_n \leq 4a_{n+1}^{-1}(b_n - a_n - b_{n+1}).$$

The construction of the IIFS $\mathcal{F}$ is done in three subfamilies

$$\mathcal{F} = \{\phi_{n,i}\}_{n,i} \cup \{\tau_{n,i}\}_{n,i} \cup \{\sigma\}.$$

First, for each $n \in \mathbb{N}$ and $i \in \{1, \dots, M^n\}$ define $\phi_{n,i} : [0, 1]^2 \rightarrow [0, 1]^2$ by

$$\phi_{n,i}((x, y)) = (b_n - a_n, (i-1)a_n) + a_n(x, y).$$

Second, for $n \in \mathbb{N}$ and $i \in \{1, \dots, N_n\}$ define $\tau_{n,i} : [0, 1]^2 \rightarrow [0, 1]^2$ by

$$\tau_{n,i}((x, y)) = \left(b_{n+1} + (i-1)\frac{b_n - a_n - b_{n+1}}{N_n}, 0\right) + \frac{b_n - a_n - b_{n+1}}{N_n}(x, y).$$

Third, we define $\sigma : [0, 1]^2 \rightarrow [0, 1]^2$ such that

$$\sigma((x, y)) = \frac{4}{M+2}(x, y).$$

See Figure 2.3 below for the first generation of images of the system used in this section. Note that this figure shows only finitely many of the first generation images, and the images in fact accumulate to the box in the bottom left corner of the square.

Using the definition of M we compute

$$\begin{aligned} \psi_{\mathcal{F}}(s) &= \sum_{n=1}^{\infty} \sum_{i=1}^{M^n} (\text{Lip}(\phi_{n,i}))^s + \sum_{n=1}^{\infty} \sum_{i=1}^{N_n} (\text{Lip}(\tau_{n,i}))^s + (\text{Lip}(\sigma))^s. \\ &= \sum_{n=1}^{\infty} M^n a_n^s + \sum_{n=1}^{\infty} N_n \left(\frac{b_n - a_n - b_{n+1}}{N_n} \right)^s + \left(\frac{4}{M+2} \right)^s \\ &\leq \frac{M^{1-s}}{1 - M^{1-s}} + \frac{M^{1-s}}{1 - M^{1-s}} + \frac{4^s}{M^s} \\ &< 1. \end{aligned}$$

Therefore, the similarity dimension of $\mathcal{F}$ is at most s .

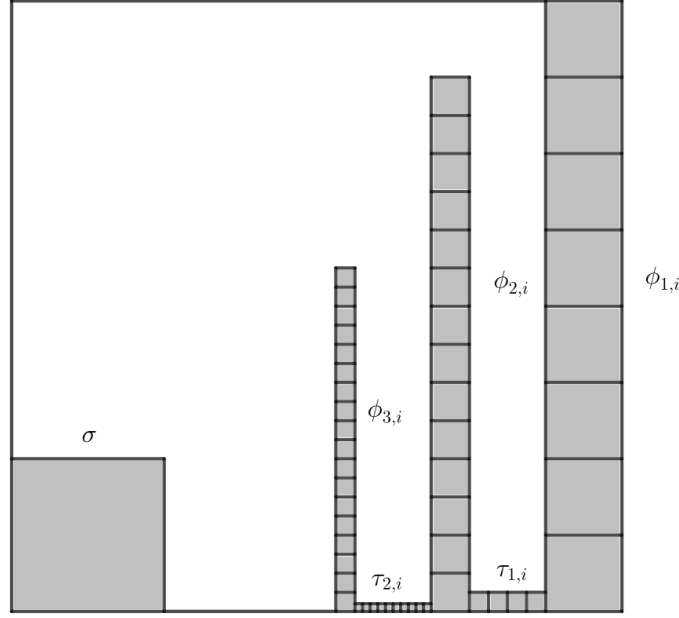


Figure 2.3: First iteration of IIFS of §2.5.3.

Lemma 2.2.3. *The attractor K of the IIFS $\mathcal{F}$ is the image of a curve.*

Proof. Define L_b, L_r to be the bottom and left, respectively, edges of the unit square $[0, 1]^2$. Define also $K_0 = [0, 1]^2$, and for each $m \in \mathbb{N}$ define $K_m = \bigcup_{f \in \mathcal{F}} f(K_{m-1})$.

We claim that for every $f \in \mathcal{F}$, $f(L_b \cup L_r) \subset f(K)$. Assuming the claim, we note that the set

$$E = L_b \cup \bigcup_{n,i} \phi_{n,i}(L_r)$$

is contained in K and, working as in the proof of Lemma 2.2.2, it follows that E is the image of a curve. Now, by Theorem 2.2(1), it follows that K is the image of a curve.

To prove the claim, we show that $L_b \cup L_r \subset K_m$ for every $m \in \mathbb{N}$. By a simple inductive argument, this immediately implies that $f(L_b \cup L_r) \subset f(K)$ for all $f \in \mathcal{F}$. The proof is by induction on m . Clearly $L_b \cup L_r \subset K_0$. Assume now that $L_b \cup L_r \subset K_m$

for some integer $m \geq 0$. First,

$$L_r = \bigcup_{i=1}^M \left(\{1\} \times \left[\frac{i-1}{M}, \frac{i}{M} \right] \right) = \bigcup_{i=1}^M \phi_{1,i}(L_r) \subset K_{m+1}.$$

Second,

$$\begin{aligned} L_b &= \bigcup_{n=1}^{\infty} ([b_n - a_n, b_n] \times \{0\}) \cup \bigcup_{n=1}^{\infty} ([b_{n+1}, b_n - a_n] \times \{0\}) \cup ([0, \frac{4}{M+2}] \times \{0\}) \\ &= \bigcup_{n=1}^{\infty} \phi_{n,1}(L_b) \cup \bigcup_{n,i} \tau_{n,i}(L_b) \cup \sigma(L_b) \\ &\subset \bigcup_{f \in \mathcal{F}} f(K_m). \end{aligned} \quad \square$$

Proof of Theorem 2.1(2). We show that the attractor K of the IIFS $\mathcal{F}$ is not the image of a Hölder curve. Assume for a contradiction that there exists $\alpha \geq 1$, $H > 0$ and a surjection $f : [0, 1] \rightarrow K$ such that for all $x, y \in [0, 1]$

$$|f(x) - f(y)|^\alpha \leq H|x - y|.$$

For each $n \in \mathbb{N}$ let

$$A_n = \bigcup_{i=1}^{M^n} \phi_{n,i}(K).$$

Note that the height of each “tower” A_n is equal to $\log(2)/\log(n+1)$.

Recall that M was chosen even and for each $n \in \mathbb{N}$ define $p_n = \phi_{n,M^n}((1, 1))$ and $q_n = \phi_{n, \frac{1}{2}M^n}((1, 1))$. Following the proof of Lemma 2.2.3, for each $n \in \mathbb{N}$, the vertical segment $[q_n, p_n] \subset A_n$. Moreover, for any $n \in \mathbb{N}$, p_n is the point of A_n with the highest y -coordinate, and the y -coordinate of q_n is $\frac{1}{2} \log(2)/\log(n+1)$. Setting

$$B_n = \bigcup_{i=\frac{1}{2}M^n+1}^{M^n} \phi_{n,i}(K),$$

we have $B_n \cap \overline{K \setminus B_n} = \{q_n\}$. Given that B_n is connected, there exists for each $n \in \mathbb{N}$ an interval $I_n \subset [0, 1]$ such that

$$\{p_n, q_n\} \subset f(I_n) \subset B_n.$$

It follows that the intervals $I_1, I_2, \dots$ are mutually disjoint and

$$1 \geq \sum_{n=1}^{\infty} \text{diam } I_n \geq \sum_{n=1}^{\infty} H^{-1} |p_n - q_n|^\alpha = \frac{(\log 2)^\alpha}{2^\alpha H} \sum_{n=1}^{\infty} \frac{1}{(\log(n+1))^\alpha}.$$

However, the latter series diverges and we reach a contradiction. □

Chapter 3

Hölder curves with exotic tangent spaces

3.1 Disclaimer

This chapter is, up to formatting and up to consolidation of the bibliography into a single bibliography for the entire dissertation, identical to the manuscript of the same name [Shaw and Vellis \(2024a\)](#). This manuscript is a collaborative work with Vyrion Vellis and the preprint is available at <https://arxiv.org/abs/2409.13662>.

3.2 Abstract

An important implication of Rademacher’s Differentiation Theorem is that every Lipschitz curve Γ infinitesimally looks like a line at almost all of its points in the sense that at $\mathcal{H}^1$ -almost every point of Γ , the only tangent to Γ is a straight line through the origin. In this article, we show that, in contrast, the infinitesimal structure of Hölder curves can be much more extreme. First we show that for every $s > 1$ there exists a $(1/s)$ -Hölder curve Γ_s in a Euclidean space with $\mathcal{H}^s(\Gamma_s) > 0$ such that $\mathcal{H}^s$ -almost every point of Γ_s admits infinitely many topologically distinct tangents. Second, we

study the tangents of self-similar connected sets (which are canonical examples of Hölder curves) and prove that the curves Γ_s have the additional property that $\mathcal{H}^s$ -almost every point of Γ_s admits infinitely many homeomorphically distinct tangents to Γ_s which are not admitted as (not even bi-Lipschitz to) tangents to any self-similar set at typical points.

3.3 Introduction

Rademacher's Theorem, one of the most important theorems in geometric measure theory, states that every Lipschitz function defined on $[0, 1]$ is differentiable at $\mathcal{H}^1$ -almost every point of $[0, 1]$. It is natural to ask whether a similar result may hold for more general functions. Calderon [Calderón \(1951\)](#) extended Rademacher's Theorem by proving that every function in the Sobolev class $\mathcal{W}^{1,p}$ with $p > 1$ is $\mathcal{H}^1$ -almost everywhere differentiable. However, any further generalization would be futile as for each $s > 1$ there exists a Weierstrass function on $[0, 1]$ which is $\frac{1}{s}$ -Hölder but nowhere differentiable [Zygmund \(2002\)](#).

A major application of Rademacher's Theorem is towards the understanding of the “infinitesimal structure” of Lipschitz curves (i.e. Lipschitz images of $[0, 1]$). To state this application, let us first define the notion of tangents. Following [Badger and Lewis \(2015\)](#), given a closed set $X \subset \mathbb{R}^n$ and a point $x \in X$, we say that a closed set T is a *tangent of X at x* if there exists a sequence of positive scales r_j that go to zero such that the blow-up sets $r_j^{-1}(X - x)$ converge to the set T in the Attouch-Wets topology; see §3.4.4 for the precise definition. Other notions of metric space convergences which produce similar tangents are known in the literature; see [Gromov \(1981, 1999\)](#); [David and Semmes \(1997\)](#). Another well-known notion of infinitesimal structure in geometric measure theory is that of the *tangent cone* ([Federer, 1969](#), 3.1.21). The tangent cone of X at x is the union of all tangents of X at x , which means that some local information is lost. We denote by $\text{Tan}(X, x)$ the collection of

all tangents of X at x . It is well-known that $\text{Tan}(X, x)$ is nonempty and that if T is in $\text{Tan}(X, x)$, then λT is in $\text{Tan}(X, x)$ for every $\lambda > 0$.

By Rademacher's Theorem and by a theorem of Besicovitch [Besicovitch \(1944\)](#) (see also [\(Falconer, 1986, Corollary 3.15\)](#)), every Lipschitz curve is infinitesimally a line at $\mathcal{H}^1$ -almost every point. More precisely, the following is true, and we provide a proof in [Section 3.5](#).

Theorem 3.1. *If $f : [0, 1] \rightarrow \mathbb{R}^N$ is Lipschitz, then for $\mathcal{H}^1$ -a.e. $x \in f([0, 1])$, there exists a straight line $L \subset \mathbb{R}^N$ through the origin such that $\text{Tan}(f([0, 1]), x) = \{L\}$.*

More generally, if X is the Lipschitz (or even p -Sobolev for $p > n$) image of $[0, 1]^n$ into some $\mathbb{R}^m$ with $m > n$, then for $\mathcal{H}^n$ -a.e. $x \in X$, the tangent space of X at x contains exactly one element which is an n -plane [Badger et al. \(2024\)](#). Here and for the rest of this paper, given $\alpha > 0$, we denote by $\mathcal{H}^\alpha(X)$ the Hausdorff α -dimensional measure of a metric space X .

It is worthwhile to note that while typical points of a Lipschitz curve have a simple tangent space, exceptional points may exhibit extreme behaviors. In particular, there exists a Lipschitz curve $\Gamma \subset \mathbb{R}^2$ (or in any Euclidean space of dimension at least 2) and a point $x_0 \in \Gamma$ such that $\text{Tan}(\Gamma, x_0)$ contains *every possible tangent*; see [Appendix 3.12.2](#) for a precise statement and the proof.

The infinitesimal structure of Lipschitz curves plays an important role in the classification of 1-rectifiable sets by Jones [Jones \(1990\)](#) and Okikiolu [Okikiolu \(1992\)](#). A feature of the proofs is the use of Jones's *beta numbers*, which roughly measure how well a given set E can be approximated by lines at a given scale and location, and showing that if these values are small enough at all scales and locations then E can be captured in a rectifiable curve. Roughly speaking, a set E is contained in a Lipschitz curve if and only if at almost all points, tangent spaces are lines in a strong quantitative way. For a more complete discussion of the history of the Analyst's Traveling Salesperson Theorem, see [Schul \(2007a\)](#) and the citations therein.

Recent years have seen great interest in obtaining a Hölder version of the Analyst’s Traveling Salesperson Theorem, that is, in characterizing all sets which are contained in a Hölder curve; see for example [Martín and Mattila \(1993\)](#); [Remes \(1998\)](#); [Martín and Mattila \(2000\)](#); [Badger and Vellis \(2017\)](#); [Badger et al. \(2019\)](#); [Balogh and Züst \(2020\)](#); [Badger and Vellis \(2021\)](#); [Badger and Schul \(2023\)](#); [Shaw and Vellis \(2024b\)](#). This notion of “Hölder rectifiability” is greatly motivated by the fact that many fractals in analysis on metric spaces admit a Hölder parameterization but not a Lipschitz one. For example, if the attractor of an iterated function system is connected (e.g. the von Koch curve, the Sierpiński gasket, the Sierpiński carpet), then it admits a Hölder parameterization by $[0, 1]$; see [Remes \(1998\)](#); [Badger and Vellis \(2021\)](#).

In lieu of the Lipschitz rectifiability results discussed above, a crucial step towards a comprehensive theory of Hölder rectifiable sets would be to understand the tangent spaces of Hölder curves at typical points. Unfortunately, unlike the Lipschitz case, it would be naive to expect that given $s > 1$ there exists one single set T_s which is the tangent of all $\frac{1}{s}$ -Hölder curves at typical points. For example, consider the two self-similar sets $\mathcal{S}$ and $\mathcal{C}$ constructed as follows. Set $S_0 = T$ where T is an isosceles triangle with side-lengths $1, 5^{-1/4}, 5^{-1/4}$, and for each $n \in \mathbb{N}$ set $S_n = \phi_1(S_{n-1}) \cup \phi_2(S_{n-1})$ where ϕ_1, ϕ_2 are the two similarities in the left image of Figure 3.1. The sets S_n converge to a snowflake-like curve $\mathcal{S} = \bigcap_{n \in \mathbb{N}} S_n$. Similarly, set $C_0 = [0, 1]^2$ and for each $n \in \mathbb{N}$ set $C_n = \bigcup_{i=1}^{16} \psi_i(C_{n-1})$ where $\psi_1, \dots, \psi_{16}$ are the similarities in the right image of Figure 3.1. The sets C_n converge to a carpet-like set $\mathcal{C} = \bigcap_{n \in \mathbb{N}} C_n$. Both $\mathcal{S}$ and $\mathcal{C}$ are connected self-similar sets which have Hausdorff dimension equal to $s = \log 16 / \log 5$ and positive $\mathcal{H}^s$ measure, and by a theorem of Remes [Remes \(1998\)](#), both are $\frac{1}{s}$ -Hölder curves. However, the tangents of $\mathcal{S}$ at $\mathcal{H}^s$ -a.e. point are infinite snowflakes while the tangents of $\mathcal{C}$ at $\mathcal{H}^s$ -a.e. point are infinite carpets; hence they are topologically different.

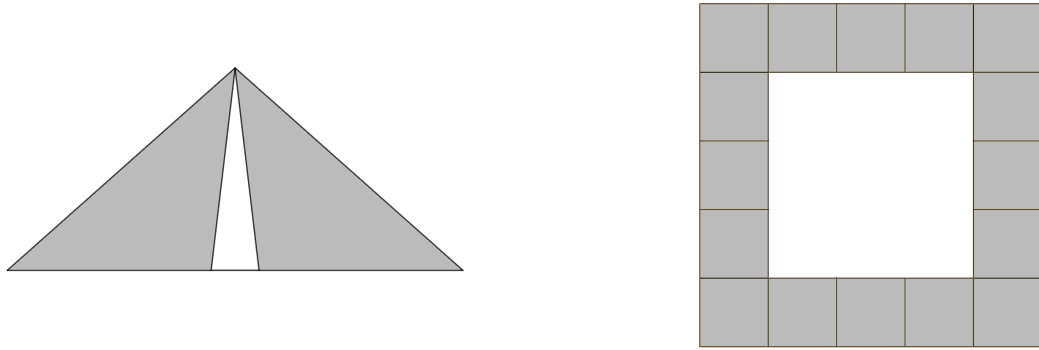


Figure 3.1: First stages of sets $\mathcal{S}$ (left) and $\mathcal{C}$ (right).

The previous two examples lead us to ask whether there are infinitely many Hölder curves $\Gamma_1, \Gamma_2, \dots$ and infinitely many sets $T_1, T_2, \dots$ such that T_i are pairwise topologically different and each T_i is a tangent of Γ_i at typical points. In our main result, we show that not only is this true but it is even worse: this situation can arise in one single Hölder curve. Furthermore, the Hölder exponent of such a curve can be chosen arbitrarily close to 1.

Theorem 3.2. *For each $s > 1$, there exists a $\frac{1}{s}$ -Hölder curve Γ_s in some Euclidean space such that $\mathcal{H}^s(\Gamma_s) > 0$ and for $\mathcal{H}^s$ -almost every $x \in \Gamma_s$, the space $\text{Tan}(\Gamma_s, x)$ contains infinitely many topologically different tangents.*

We note that little is currently known about the geometry of tangents at typical points. For example, are tangents at typical points connected? If so, are they Hölder images of $\mathbb{R}$? Theorem 3.1 answers the previous two questions in the affirmative for Lipschitz curves.

3.3.1 Tangents of self-similar sets

Perhaps the richest source of Hölder curves with positive and finite Hausdorff measure is that of self-similar sets. Recall that an *iterated function system* (abbrv. IFS) of similarities on $\mathbb{R}^n$ is a finite collection $\mathcal{F}$ of contracting similarities of $\mathbb{R}^n$. We say that a set K is the *attractor* of an IFS $\mathcal{F}$ if $K = \bigcup_{\phi \in \mathcal{F}} \phi(K)$. An IFS $\mathcal{F}$ satisfies the

open set condition (abbv. OSC) if there exists a nonempty open set $U \subset \mathbb{R}^n$ such that for all distinct $\phi, \phi' \in \mathcal{F}$ we have $\phi(U) \subset U$ and $\phi(U) \cap \phi'(U) = \emptyset$. In this paper, we say that a set Q is *self-similar* if it is the attractor of an IFS of similarities with the OSC. A well-known theorem of Hutchinson [Hutchinson \(1981\)](#) states that every self-similar set has finite and positive $\mathcal{H}^s$ -measure for some $s > 0$.

While there are examples of Hölder curves with positive and finite Hausdorff measure which are not self-similar sets (e.g. a bi-Lipschitz embedding of the snowflaked space $([0, 1], |\cdot|^\epsilon)$ [Bonk and Heinonen \(2004\)](#); [Wu \(2015\)](#); [Romney and Vellis \(2017\)](#)), these examples are all bi-Lipschitz equivalent to self-similar sets. Therefore, it is natural to ask if Hölder curves have tangents possessing some form of self-similarity at typical points. Note that tangents of Lipschitz curves at typical points are straight lines which do possess local self-similarity. This discussion motivates two questions. First, can we classify tangents of (connected) self-similar sets? Second, are tangents of general Hölder curves always bi-Lipschitz equivalent to tangents of connected self-similar sets?

In pursuit of the first of question, we investigate the tangents of self-similar sets and we show that if Q is a self-similar set in a Euclidean space, then at $\mathcal{H}^{\dim_H(Q)}$ -almost every point x of Q , every tangent is locally made up of “big pieces of Q ”.

Proposition 3.1. *Let $\{\phi_i : \mathbb{R}^N \rightarrow \mathbb{R}^N\}_{i=1}^m$ be an IFS of similarities satisfying the OSC for some open set U with $U \cap Q \neq \emptyset$, where Q is the attractor, and let $s = \dim_H Q$. Then there exists $c_0 \in \mathbb{N}$ depending only on the Lipschitz norms of $\phi_1, \dots, \phi_m$ such that for $\mathcal{H}^s$ -almost every $x \in Q$, every tangent $T \in \text{Tan}(Q, x)$, and every $R > 0$, there exist similarities $\{f_i^R : Q \rightarrow T\}_{i=1}^{M_R}$ with $M_R \leq c_0$ with Lipschitz norms in $[c_0^{-1}R, R]$ such that*

$$\overline{B}(\mathbf{0}, R) \cap T \subset \bigcup_{j=1}^{M_R} f_j^R(Q) \subset \overline{B}(\mathbf{0}, 2R) \cap T,$$

and for all distinct $i, j \in \{1, \dots, M_R\}$, $f_i^R(U) \cap f_j^R(U) = \emptyset$ and $f_i^R(U \cap Q)$ is an open subset of T .

Here and throughout we denote by $\mathbf{0}$ the origin in $\mathbb{R}^N$. For the special case of self-similar *sponges* (including self-similar carpets such as the Sierpiński carpet), we show in Section 3.12 that at typical points of such a sponge, each tangent is *locally self-similar*, which means roughly that every open ball contains a similar copy of every other open ball as an open subset.

Regarding the second question, following the construction in Theorem 3.2, we show in Section 3.12 that the curves Γ_s from Theorem 3.2, infinitesimally, do not resemble self-similar sets.

Theorem 3.3. *For each $s > 1$, at $\mathcal{H}^s$ -a.e. $x \in \Gamma_s$ there exists $T \in \text{Tan}(\Gamma_s, x)$ with the following property. If $Q \subset \mathbb{R}^n$ is a self-similar set with $\dim_{\text{H}} Q = s$, then the set of points $z \in Q$ for which T is bi-Lipschitz homeomorphic to an element of $\text{Tan}(Q, z)$ is $\mathcal{H}^s$ -null.*

To the best of our knowledge, the curves of Theorem 3.3 are the first examples of Hölder curves that $\mathcal{H}^s$ -almost everywhere possess tangents which are not realizable as a tangent of a self-similar set at $\mathcal{H}^s$ -almost any point. However, it is worth noting that the Hölder curves of Theorem 3.3 have the property that at typical points, *some* tangent does arise as a tangent of a self-similar set at typical points. In light of this, we leave open a weakened version of the second question as a conjecture, originally due to Matthew Badger and the second named author in 2019.

Question 3.4 (Badger, Vellis). *If Γ is a Hölder curve in Euclidean space, then is it true that at typical points, there is some tangent to Γ which is bi-Lipschitz to a tangent of a connected self-similar set at typical points?*

3.3.2 Outline of the proof of Theorem 3.2

To simplify the exposition, we first describe the construction of Γ_s in the special case that $s = \log 24 / \log 6$. The set Γ_s is constructed by applying two sets of iterated function systems on $\mathbb{R}^2$ in a somewhat random fashion. We start with the unit

square $\mathcal{S}^0$. Assume now that for some $k \geq 0$ we have defined $\mathcal{S}^k$ which is the union of 24^k closed squares with disjoint interiors $\mathcal{S}_{i_1 \dots i_k}^k$, corresponding to strings $i_1 \dots i_k \in \{1, \dots, 24\}^k$ of length k in 24 letters. Each such square is replaced by a copy of Model 1 or a copy of Model 2 in Figure 3.2, rescaled by 6^{-k} . After replacing each square, we obtain $\mathcal{S}^{k+1}$. We have that $\mathcal{S}^0 \supset \mathcal{S}^1 \supset \dots$ and the set Γ_s is the Hausdorff limit of these sets. The choice between Model 1 or Model 2 at every stage of the construction is encoded by a *choice function* η which is defined on all finite words made from characters $\{1, \dots, 24\}$ and takes values in $\{1, 2\}$. See Section 3.6 for the rigorous construction. The important detail is that Model 1 contains one *local cut-point* (i.e. a point x which if removed makes a neighborhood of x disconnected) while Model 2 has no such point.

In Section 3.7 we show that, no matter what the choice function η is, the resulting “limit set” Γ_s will satisfy $0 < \mathcal{H}^s(\Gamma_s) < \infty$. Then, in Section 3.8, drawing inspiration from the techniques in parametrization results of attractors of iterated function systems (e.g. in Hata (1985) and Badger and Vellis (2021)), for each choice function η , we construct Lipschitz curves which approximate Γ_s and converge to a $\frac{1}{s}$ -Hölder parameterization of Γ_s .

Note that if a choice function takes only one value (say 1), then the resulting set Γ_s is self-similar, and in fact, it is a self-similar *sponge* which, as we prove, implies that it has very nice tangent spaces; see §3.12.1. Thus in order to obtain a set where at typical points, the tangent space contains infinitely many topologically different elements which cannot be obtained from a self-similar set, the choice function necessarily must exhibit some form of randomness.

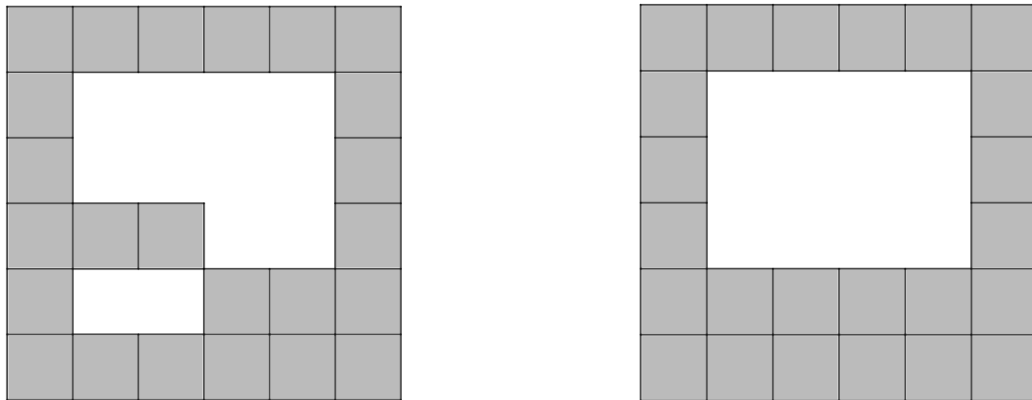


Figure 3.2: Model 1 (left) and Model 2 (right) for $s = \log 24 / \log 6$.

In Section 3.9, using a measure-theoretic argument on the set of choice functions, we show that almost every choice function (in terms of a suitable probability measure) yields a continuum Γ_s which “sees” both models in arbitrarily small neighborhoods at $\mathcal{H}^s$ -almost every point. With such a choice function in hand, we show in Section 3.10 that at $\mathcal{H}^s$ -almost every $x \in \Gamma_s$, and for each $k \in \mathbb{N}$, there exists a tangent $T \in \text{Tan}(\Gamma_s, x)$ that contains exactly $\frac{1}{23}(24^k - 1)$ many *local cut-points* (see §3.4.5). Therefore, there are infinitely many topologically different tangents.

A construction similar to the one described above works for all values s of the form $\alpha_n = \frac{\log(5n-6)}{\log(n)}$ where $n \geq 4$ is an even number. To obtain Theorem 3.2 for an arbitrary $s > 1$, we first choose $n \geq 4$ even such that $s > \alpha_n$. Then in Section 3.11 we appropriately “snowflake” the curve Γ_{α_n} , apply Assouad’s embedding theorem Assouad (1983) to bi-Lipschitz embed the snowflaked Γ_{α_n} into some Euclidean space, and use a weak-tangent-snowflaking argument to show that the embedded image (which has dimension s) satisfies the conclusions of Theorem 3.2.

Finally, in Section 3.12 we prove that these tangents cannot be obtained as tangents of self-similar sets at generic points.

3.4 Preliminaries

3.4.1 Dendrites

A *dendrite* is a Peano continuum (that is, compact, connected, and locally connected at every point) which contains no simple closed curves. The *leaves* of a dendrite X are exactly those points $x \in X$ such that $X \setminus \{x\}$ is connected.

Lemma 3.4.1. *Let T_1, T_2 be two dendrites in $\mathbb{R}^n$ that intersect on a point. Then $T_1 \cup T_2$ is a dendrite.*

Proof. Recall that a metric space is a dendrite if and only if any two distinct points can be separated by a third point (Nadler, 1992, Theorem X.10.2). Suppose that $T_1 \cap T_2 = \{x_0\}$. Fix distinct $x, y \in T_1 \cup T_2$. If $x, y \in T_1$, since T_1 is a dendrite, there exists a point $z \in T_1$ that separates x and y . Similarly if $x, y \in T_2$. If $x \in T_1 \setminus T_2$ and $y \in T_2 \setminus T_1$, then x_0 separates x and y . $\square$

3.4.2 Words

For each even integer $n \geq 4$ we denote

$$\mathcal{A}_n = \{1, \dots, 5n - 6\}.$$

Given n as above and an integer $m \geq 0$, let $\mathcal{A}_n^m$ be the set of words of length m formed by characters in $\mathcal{A}_n$, with the convention that $\mathcal{A}_n^0 = \{\varepsilon\}$, where ε is the empty word. Define $\mathcal{A}_n^* := \bigcup_{m=0}^{\infty} \mathcal{A}_n^m$ and $\mathcal{A}_n^{\mathbb{N}}$ be the set of infinitely countable words.

Given a word $w \in \mathcal{A}_n^*$, we denote by $|w|$ the length of w . Also, given $w = i_1 \cdots i_m \in \mathcal{A}_n^m$ and $j \leq m$, we write $w(j) = i_1 \cdots i_j$, and similarly given an integer $j \geq 0$ and an infinite word $\tau = i_1 i_2 \cdots \in \mathcal{A}_n^{\mathbb{N}}$ we write $\tau(j) = i_1 i_2 \cdots i_j$. For any $w \in \mathcal{A}_n^*$, define the *cylinder set* $\mathcal{A}_{n,w}^{\mathbb{N}} := \{\tau \in \mathcal{A}_n^{\mathbb{N}} : \tau(m) = w\}$, that is, the set of infinite words which agree with w for the first m characters. Similarly, for $j, m \in \mathbb{N}$ with $m \geq j$ and for each $w \in \mathcal{A}_n^j$, we define $\mathcal{A}_{n,w}^m := \{v \in \mathcal{A}_n^m : v(j) = w\}$.

Denote by Σ_n the σ -algebra generated by the cylinders $\mathcal{A}_{n,w}^{\mathbb{N}}$ where $w \in \mathcal{A}_n^*$. Then there exists a unique probability measure $\nu_n : \Sigma_n \rightarrow [0, 1]$ such that $\nu_n(\mathcal{A}_{n,w}^{\mathbb{N}}) = (5n - 6)^{-|w|}$ for all $w \in \mathcal{A}_n^*$; see for example (Stroock, 1993, §3.1).

3.4.3 Combinatorial graphs and trees

A *combinatorial graph* is a pair $G = (V, E)$ of a finite or countable vertex set V and an edge set

$$E \subset \{\{v, v'\} : v, v' \in V \text{ and } v \neq v'\}.$$

If $\{v, v'\} \in E$, we say that the vertices v and v' are *adjacent* in G .

If $V \subset \mathbb{R}^n$ for some $n \in \mathbb{N}$, then we define the *image of G* to be the set

$$\text{Im}(G) := \bigcup_{\{v, v'\} \in E} [v, v']$$

where $[v, v']$ denotes the line segment from v to v' . Recall that if $v \in V$ is a vertex, then the *valence* of v in G is the number of vertices $u \in V \setminus \{v\}$ so that $\{u, v\} \in E$.

A *simple path* in G is a set $\gamma = \{\{v_1, v_2\}, \dots, \{v_{n-1}, v_n\}\} \subset E$ such that for all distinct $i, j \in \{1, \dots, n\}$ we have $v_i \neq v_j$; in this case we say that γ joins v_1, v_n . A graph $T = (V, E)$ is a *combinatorial tree* if for any distinct v, v' there exists a unique simple path γ that joins v with v' .

3.4.4 Tangents

If A and B are nonempty subsets of a Euclidean space $\mathbb{R}^N$, then we define the *excess of A over B* as

$$\text{excess}(A, B) = \sup_{a \in A} \inf_{b \in B} |a - b|;$$

see (Beer, 1993, §3.1), Badger and Lewis (2015), and (Badger et al., 2017, Appendix A). In the next remark, we list six important properties of the excess which will be used heavily throughout this article. The first and fifth properties follow

straightforwardly from the definition of excess while the other four are given in (Badger and Lewis, 2015, Section 2).

Remark 3.5. *The excess satisfies the following six properties.*

1. Translation invariance. *For nonempty sets $A, B \subset \mathbb{R}^N$ and any point $x \in \mathbb{R}^N$,*

$$\text{excess}(A, B) = \text{excess}(A + x, B + x).$$

2. Triangle inequality. *For nonempty sets $A, B, C \subset \mathbb{R}^N$,*

$$\text{excess}(A, C) \leq \text{excess}(A, B) + \text{excess}(B, C).$$

3. Containment. *For nonempty sets $A, B \subset \mathbb{R}^N$, $\text{excess}(A, B) = 0$ if, and only if, $A \subset \overline{B}$.*

4. Monotonicity. *If $A \subset A'$, $B' \subset B$ are all nonempty subsets of $\mathbb{R}^N$, then*

$$\text{excess}(A, B) \leq \text{excess}(A', B').$$

5. Subadditivity. *If $A, B, C \subset \mathbb{R}^N$ are all nonempty, then*

$$\text{excess}(A \cup B, C) \leq \text{excess}(A, C) + \text{excess}(B, C).$$

6. Closure. *If $A, B \subset \mathbb{R}^N$ are nonempty, then*

$$\text{excess}(A, B) = \text{excess}(\overline{A}, \overline{B}).$$

Let $\mathfrak{C}(\mathbb{R}^N)$ be the collection of nonempty closed subsets of $\mathbb{R}^N$, and let $\mathfrak{C}(\mathbb{R}^N; \mathbf{0})$ be the collection of nonempty closed subsets of $\mathbb{R}^N$ containing the origin $\mathbf{0}$. We consider both of these spaces equipped with the *Attouch-Wets* topology, which is defined in (Beer, 1993, Definition 3.1.2).

Lemma 3.5.1 ((Beer, 1993, Chapter 3)). *There exists a metrizable topology, called the Attouch-Wets topology, on $\mathfrak{C}(\mathbb{R}^N)$ in which a sequence of sets $(X_i)_{i=1}^\infty \subset \mathfrak{C}(\mathbb{R}^N)$ converges to a set $X \in \mathfrak{C}(\mathbb{R}^N)$ if and only if for every $r > 0$,*

$$\lim_{m \rightarrow \infty} \text{excess}(X_m \cap \overline{B}(\mathbf{0}, r), X) = 0 \quad \text{and} \quad \lim_{m \rightarrow \infty} \text{excess}(X \cap \overline{B}(\mathbf{0}, r), X_m) = 0.$$

Moreover, the subcollection $\mathfrak{C}(\mathbb{R}^N; \mathbf{0})$ is sequentially compact; that is, for any sequence $(X_i)_{i=1}^\infty \subset \mathfrak{C}(\mathbb{R}^N; \mathbf{0})$ there exists a subsequence $(X_{i_j})_{j=1}^\infty$ and a set $X \in \mathfrak{C}(\mathbb{R}^N; \mathbf{0})$ such that $(X_{i_j})_{j=1}^\infty$ converges to X .

In the following lemma, we record a useful property of Attouch-Wets convergence in $\mathfrak{C}(\mathbb{R}^N; \mathbf{0})$, which roughly says that sequences in $\mathfrak{C}(\mathbb{R}^N; \mathbf{0})$ that converge in the Attouch-Wets topology satisfy a type of Cauchy condition with respect to excess.

Lemma 3.5.2. *If $(A_j)_{j \in \mathbb{N}} \subset \mathfrak{C}(\mathbb{R}^N; \mathbf{0})$ is a sequence converging to a set $A \in \mathfrak{C}(\mathbb{R}^N; \mathbf{0})$ with respect to the Attouch-Wets topology, then for every $r > 0$ and each $\epsilon > 0$, there exists an integer $j_0 \in \mathbb{N}$ so that for every pair of integers $j_1, j_2 \geq j_0$, $\text{excess}(A_{j_1} \cap \overline{B}(\mathbf{0}, r), A_{j_2}) < \epsilon$.*

Proof. Let $\epsilon > 0$ and let $r > 0$. Since $A_j \rightarrow A$ in the Attouch-Wets topology as $j \rightarrow \infty$, there exists an integer $j_0 \in \mathbb{N}$ so that for every integer $j \geq j_0$

$$\text{excess}(A_j \cap \overline{B}(\mathbf{0}, r), A) < \min\{\epsilon/2, r\} \quad \text{and} \quad \text{excess}(A \cap \overline{B}(\mathbf{0}, 2r), A_j) < \min\{\epsilon/2, r\}.$$

Then for every pair of integers $j_1, j_2 \geq j_0$, by the triangle inequality for excess (see Remark 3.5),

$$\text{excess}(A_{j_1} \cap \overline{B}(\mathbf{0}, r), A_{j_2}) \leq \text{excess}(A_{j_1} \cap \overline{B}(\mathbf{0}, r), A \cap \overline{B}(\mathbf{0}, 2r)) + \text{excess}(A \cap \overline{B}(\mathbf{0}, 2r), A_{j_2}).$$

Now since $j_1 \geq j_0$, $\text{excess}(A_{j_1} \cap \overline{B}(\mathbf{0}, r), A) < \min\{\epsilon/2, r\}$, so for every $x \in A_{j_1} \cap \overline{B}(\mathbf{0}, r)$ there exists a point $y \in A$ so that $|x - y| < \min\{\epsilon/2, r\}$. Then it must hold that such a point y is contained in $\overline{B}(\mathbf{0}, 2r) \cap A$, and therefore $\text{excess}(A_{j_1} \cap \overline{B}(\mathbf{0}, r), A \cap \overline{B}(\mathbf{0}, 2r)) <$

$\epsilon/2$. Additionally, since $j_2 \geq j_0$, we have also that $\text{excess}(A \cap \overline{B}(\mathbf{0}, 2r), A_{j_2}) < \epsilon/2$, and this completes the proof. $\square$

Let $X \in \mathfrak{C}(\mathbb{R}^N)$ and let $x \in X$. Following [Badger and Lewis \(2015\)](#), we say that a set $T \in \mathfrak{C}(\mathbb{R}^N; \mathbf{0})$ is a *tangent set of X at x* if there exists a sequence of scales $(r_m)_{m \in \mathbb{N}} > 0$ such that $r_m \rightarrow 0$ and $\frac{X-x}{r_m} \rightarrow T$ with respect to the Attouch-Wets topology. We denote by $\text{Tan}(X, x)$ the set of all tangent sets to X at x .

For the next lemma, denote by $\mathfrak{C}_U(\mathbb{R}^N; \mathbf{0})$ the collection of all sets $X \in \mathfrak{C}(\mathbb{R}^N; \mathbf{0})$ such that every component of X is unbounded.

Lemma 3.5.3. *If $X \subset \mathbb{R}^N$ is a nondegenerate continuum and $x \in X$, then $\text{Tan}(X, x) \subset \mathfrak{C}_U(\mathbb{R}^N; \mathbf{0})$.*

Proof. Fix $T \in \text{Tan}(X, x)$. First, we prove that the component of T which contains $\mathbf{0}$ is unbounded. To this end, let $(r_j) > 0$ be a sequence of scales converging to 0 so that $(r_j)^{-1}(X - x) \rightarrow T$ as $j \rightarrow \infty$ in the Attouch-Wets topology. Given $j \in \mathbb{N}$, define the similarity $g_j : X \rightarrow \mathbb{R}^N$ by $g_j(y) = (r_j)^{-1}(y - x)$, and for $R > 0$ let $Q_{j,R}$ be the component of $g_j(X) \cap \overline{B}(\mathbf{0}, R)$ containing $\mathbf{0}$. Since X is connected and nondegenerate, for each $R > 0$ there exists a $j_R \in \mathbb{N}$ so that $Q_{j,R} \cap \partial B(\mathbf{0}, R) \neq \emptyset$ for every integer $j \geq j_R$. Note that for every pair of scales $R_2 > R_1 > 0$ and for each $j \in \mathbb{N}$, $Q_{j,R_1} \subset Q_{j,R_2}$. By compactness of $\mathfrak{C}(\mathbb{R}^N; \mathbf{0})$, for each $R > 0$ the sequence $(Q_{j,R})_{j \in \mathbb{N}}$ has an Attouch-Wets subsequential limit K_R , and for $R_2 > R_1 > 0$ we have $K_{R_1} \subset K_{R_2}$. Additionally, since $g_j(X)$ converges to T in the Attouch-Wets topology, we have that $K_R \subset T$, $\mathbf{0} \in K_R$, and $K_R \cap \partial B(\mathbf{0}, R) \neq \emptyset$ for every $R > 0$. Therefore, $\bigcup_{m \in \mathbb{N}} K_m \subset T$ and the former union is unbounded, connected, and contains $\mathbf{0}$, so the component of T containing $\mathbf{0}$ is unbounded, thus T has an unbounded component.

Assume now for a contradiction that T has a bounded component K . Let L be the component of $T' = T \cup \{\infty\}$ in $\mathbb{S}^N$ that contains ∞ . Then clearly K, L are distinct connected components of T' . Regarding K and L as quasi-components of T' , one can choose two disjoint closed sets $Y, Z \subset T'$ satisfying $K \subset Y$, $L \subset Z$, and $T' = Y \cup Z$.

Then both Y and Z are closed in $\mathbb{S}^N$, and thus by normality there exists an open set U such that

$$Y \subset U \subset \overline{U} \subset \mathbb{S}^N \setminus Z.$$

Since $\infty \notin \overline{U}$, we have that $\overline{U}$ is a compact subset of $\mathbb{R}^N$. Set $d = \text{dist}(Z, Y)$.

Fix $R > 0$ large enough that $Y \subset \overline{B}(\mathbf{0}, R)$ and $Z \cap \overline{B}(\mathbf{0}, R) \neq \emptyset$. By definition of Attouch-Wets convergence, there exists a $j_0 \in \mathbb{N}$ so that for every integer $j \geq j_0$,

$$\text{excess}(r_j^{-1}(X - x) \cap \overline{B}(\mathbf{0}, R), T) + \text{excess}(T \cap \overline{B}(\mathbf{0}, R), r_j^{-1}(X - x)) < d/3.$$

Let $y, z \in r_j^{-1}(X - x)$ so that $\text{dist}(y, Y) < d/3$ and $\text{dist}(z, Z) < d/3$. Since X is connected, there is another point $p \in r_j^{-1}(X - x)$ so that $p \in \overline{B}(\mathbf{0}, R)$ and $\text{dist}(p, Y) = d/3$; thus $\text{dist}(p, Z) \geq 2d/3$. However, it follows that $\text{dist}(p, T) \geq d/3$, which is a contradiction. $\square$

Remark 3.6. *Note that since $\mathfrak{C}(\mathbb{R}^N; \mathbf{0})$ is sequentially compact, we have that if $X \in \mathfrak{C}(\mathbb{R}^N)$ and if $x_0 \in X$, then the sequence of sets $(n(X - x_0))_{n \in \mathbb{N}}$ has a subsequential limit in $\mathfrak{C}_U(\mathbb{R}^N; \mathbf{0})$. In particular, such a limit must be a tangent of X at x_0 , so the set of tangents to a nonempty closed set at a point contained in the set is always nonempty. More simply, every nonempty closed set has tangents at every point (and every tangent contains the origin $\mathbf{0}$).*

3.4.5 Local cut-points

Recall if $X \subset \mathbb{R}^N$, then a point $x \in X$ is called a *cut-point* if $C \setminus \{x\}$ is not connected, where $C \subset X$ is the component of X containing x . For a nondegenerate closed set $X \subset \mathbb{R}^N$ in a Euclidean space, following Whyburn [Whyburn \(1935\)](#), we say that a point $x_0 \in X$ is a *local cut-point* of X if there exists some $r > 0$ such that $C_X(x_0, r) \setminus \{x_0\}$ is not connected, where $C_X(x_0, r)$ is the component of $\overline{B}(x_0, r) \cap X$ containing x_0 . That is, x_0 is a local cut-point if x_0 is a cut-point in sufficiently small neighborhoods of itself.

Lemma 3.6.1. *Let $X \subset \mathbb{R}^N$ and $Y \subset \mathbb{R}^M$ be nondegenerate closed subsets of Euclidean spaces and let $f : X \rightarrow Y$ be a homeomorphism. If p is a local cut-point of X , then $f(p)$ is a local cut-point of Y .*

Proof. Let p be a local cut-point of X and let $r > 0$ such that $C_X(p, r) \setminus \{p\}$ is disconnected. As a matter of notation, throughout this proof when we write $B_X(p, r)$ we mean $B(p, r) \cap X$, and similarly when we write $B_Y(f(p), r)$ we mean $B(f(p), r) \cap Y$. Let $\epsilon > 0$ so that $\overline{B}_Y(f(p), \epsilon) \subset f(\overline{B}_X(p, r))$, and let $\delta \in (0, r)$ such that

$$f(\overline{B}_X(p, \delta)) \subset \overline{B}_Y(f(p), \epsilon) \subset f(\overline{B}_X(p, r)).$$

Then we have that

$$f(C_X(p, \delta)) \subset C_Y(f(p), \epsilon) \subset f(C_X(p, r)).$$

Let $A, B \subset C_X(p, r) \setminus \{p\}$ be a disjoint pair of nonempty closed subsets of $C_X(p, r) \setminus \{p\}$ so that $A \cup B = C_X(p, r) \setminus \{p\}$. Note that $A \cup \{p\}$ and $B \cup \{p\}$ are both closed sets in X . We claim that $A \cap C_X(p, \delta) \neq \emptyset$.

First, we show that $A \cup \{p\}$ is connected. If $A \cup \{p\}$ is not connected, then let C, D be a disjoint pair of nonempty closed subsets of $A \cup \{p\}$ with $C \cup D = A \cup \{p\}$. Then since $A \cup \{p\}$ is closed in X , we have that C and D are both closed in X . Without loss of generality, assume that $p \in C$. Then the pair $B \cup C$ and D is a disjoint pair of nonempty closed subsets of $C_X(p, r)$ with $(B \cup C) \cup D = C_X(p, r)$, which is a contradiction, so $A \cup \{p\}$ is connected.

Let C_A be the component of $(A \cup \{p\}) \cap \overline{B}_X(p, \delta)$ which contains p . Then $C_A = A \cup \{p\}$ or $C_A \cap \partial B_X(p, \delta) \neq \emptyset$, and in either case we have that $C_A \neq \{p\}$. Furthermore, since C_A is connected, we have that $C_A \setminus \{p\} \subset C_X(p, \delta) \setminus \{p\}$, so since $C_A \setminus \{p\} \subset A$, we have that $A \cap C_X(p, \delta) \neq \emptyset$. Similarly, $B \cap C_X(p, \delta) \neq \emptyset$.

Finally, we have

$$C_Y(f(p), \epsilon) \setminus \{f(p)\} \subset f(A) \cup f(B),$$

therefore

$$C_Y(f(p), \epsilon) \setminus \{f(p)\} = (f(A) \cap C_Y(f(p), \epsilon)) \cup (f(B) \cap C_Y(f(p), \epsilon)).$$

Furthermore, the sets on the right hand side form a disjoint pair of nonempty closed subsets of $C_Y(f(p), \epsilon) \setminus \{f(p)\}$ with $(f(A) \cap C_Y(f(p), \epsilon)) \cup (f(B) \cap C_Y(f(p), \epsilon)) = C_Y(f(p), \epsilon) \setminus \{f(p)\}$, so $f(p)$ is a local cut-point of Y . $\square$

3.4.6 Self-similarity

If a function $f : \mathbb{R}^N \rightarrow \mathbb{R}^M$ is Lipschitz continuous, then we denote by $\text{Lip}(f)$ (the *Lipschitz norm* of f) the smallest $L \geq 0$ so that for every pair of points $x, y \in \mathbb{R}^N$, $|f(x) - f(y)| \leq L|x - y|$. If $\text{Lip}(f) < 1$, then f is a *contraction*. A map $f : \mathbb{R}^N \rightarrow \mathbb{R}^M$ is *affine* if there exists a linear map $L : \mathbb{R}^N \rightarrow \mathbb{R}^M$ so that for every $x \in \mathbb{R}^N$, $f(x) = L(x) + f(\mathbf{0})$. A map $f : \mathbb{R}^N \rightarrow \mathbb{R}^N$ is called a *similarity* if there exists $\lambda > 0$ such that for every pair of points $x, y \in \mathbb{R}^N$, $|f(x) - f(y)| = \lambda|x - y|$. Every similarity is a composition of an orthogonal transformation, a scalar multiplication, and a translation; in particular similarity maps are affine. If $f : \mathbb{R}^N \rightarrow \mathbb{R}^N$ is a rotation-free and reflection-free similarity, then for every $x \in \mathbb{R}^N$, $f(x) = \text{Lip}(f)x + f(\mathbf{0})$.

Remark 3.7. A map $f : \mathbb{R}^N \rightarrow \mathbb{R}^M$ is affine if and only if there exists a linear transformation $L : \mathbb{R}^N \rightarrow \mathbb{R}^M$ so that for every $x, y \in \mathbb{R}^N$, $f(x) - f(y) = L(x - y)$. In particular, if $f : \mathbb{R}^N \rightarrow \mathbb{R}^N$ is a rotation-free and reflection-free similarity, then for every $x, y \in \mathbb{R}^N$, $f(x) - f(y) = \text{Lip}(f)(x - y)$.

An *iterated function system* (IFS for short) is a finite collection $\mathcal{F}$ of contractions on $\mathbb{R}^N$. By a theorem of Hutchinson [Hutchinson \(1981\)](#), for each IFS $\mathcal{F}$ there exists a unique nonempty compact set K (called the *attractor* of $\mathcal{F}$) such that $K = \bigcup_{\phi \in \mathcal{F}} \phi(K)$.

We say that an IFS $\mathcal{F}$ satisfies the *open set condition* (OSC for short) if there exists a nonempty open set $U \subset \mathbb{R}^N$ such that for any distinct $\phi, \psi \in \mathcal{F}$,

$$\phi(U) \subset U, \quad \phi(U) \cap \psi(U) = \emptyset.$$

By a theorem of Schief (Schief, 1994, Theorem 2.2), the OSC is equivalent to the *strong open set condition* (SOSC): if K is the attractor of $\mathcal{F}$, then there exists a nonempty open set $U \subset \mathbb{R}^N$ for which the OSC holds so that $U \cap K \neq \emptyset$.

It is well-known that if K is the attractor of an IFS $\mathcal{F}$ of similarities with the OSC, then the Hausdorff dimension, the Minkowski dimension, and the Assouad dimension are all equal to s , where s is the unique solution of the equation

$$\sum_{\phi \in \mathcal{F}} (\text{Lip}(\phi))^s = 1.$$

Moreover, $0 < \mathcal{H}^{\dim_{\text{H}}(K)}(K) < \infty$ where $\dim_{\text{H}}(K)$ is the Hausdorff dimension of K Hutchinson (1981).

Henceforth, we say that a compact set $K \subset \mathbb{R}^N$ is *self-similar* if there exists an IFS $\mathcal{F}$ of similarities with the OSC such that K is the attractor of $\mathcal{F}$.

3.5 Tangents of Lipschitz curves at typical points

The goal of this section is to prove Proposition 3.1. The proof is based on two results. The first is Rademacher's Theorem.

Lemma 3.7.1 (Rademacher's Theorem). *If $f : \mathbb{R}^N \rightarrow \mathbb{R}^M$ is a locally Lipschitz continuous function, then f is differentiable $\mathcal{L}^N$ -a.e., where $\mathcal{L}^N$ is the N -dimensional Lebesgue measure on $\mathbb{R}^N$.*

For a proof see for example (Evans and Gariepy, 2015, Theorem 3.2). The second ingredient is a result of Falconer which roughly says that Lipschitz curves look flat

around typical points. Following (Falconer, 1986, §3.2), for a point $x_0 \in \mathbb{R}^N$, for a line $L \subset \mathbb{R}^N$ through the origin, and for a number $\alpha > 0$, define

$$C(x_0, L, \alpha) := \{x \in \mathbb{R}^N : d(x, L + x_0) < \alpha|x - x_0|\}$$

to be the open cone centered at x_0 in the direction of L with aperture α . Given a closed set $K \subset \mathbb{R}^N$, we say that a point $x_0 \in K$ is *flat in K* if there exists a line $L \subset \mathbb{R}^N$ through the origin such that for every $\alpha > 0$, there exists $r_\alpha > 0$ with $B(x_0, r_\alpha) \cap K \subset C(x_0, L, \alpha)$.

Lemma 3.7.2. *If $X \subset \mathbb{R}^N$ is a continuum with $\mathcal{H}^1(X) < \infty$, then $\mathcal{H}^1$ -a.e. $x \in X$ is flat in X .*

Proof. The claim is trivially true if $\mathcal{H}^1(X) = 0$, so we may assume that $\mathcal{H}^1(X) > 0$.

By (Falconer, 1986, Corollary 3.15), for $\mathcal{H}^1$ -almost every point $x \in X$, there exists a unique line L passing through $\mathbf{0}$ such that for every $\alpha > 0$,

$$\lim_{r \rightarrow 0} \frac{\mathcal{H}^1((X \cap \overline{B}(x, r)) \setminus C(x, L, \alpha))}{r} = 0. \quad (3.5.1)$$

Let $x_0 \in X$ be such a point, let L be the line given above.

Assume for a contradiction that there exists $\alpha \in (0, 1)$, there exists a sequence of positive scales $(r_j)_{j \in \mathbb{N}}$ going to zero, and there exists a sequence $(y_j)_{j \in \mathbb{N}}$ of points in X such that

$$y_j \in (\overline{B}(x_0, r_j) \cap X) \setminus C(x_0, L, \alpha).$$

Since X is a continuum, the component of $(\overline{B}(x_0, \frac{3}{2}|y_j - x_0|) \cap X) \setminus C(x_0, L, \alpha/2)$ which contains y_j must intersect at least one of $\partial B(x_0, \frac{3}{2}|y_j - x_0|)$ and $\overline{C(x_0, L, \alpha/2)} \cap$

$\overline{B}(x_0, \frac{3}{2}|y_j - x_0|)$. Therefore,

$$\begin{aligned} \mathcal{H}^1 \left((\overline{B}(x_0, \tfrac{3}{2}|y_j - x_0|) \cap X) \setminus C(x_0, L, \alpha/2) \right) \\ \geq \min \left\{ \text{dist} \left(y_j, \partial B(x_0, \tfrac{3}{2}|y_j - x_0|) \right), \text{dist} \left(y_j, \overline{C(x_0, L, \alpha/2)} \cap \overline{B}(x_0, \tfrac{3}{2}|y_j - x_0|) \right) \right\} \\ \geq \min \left\{ \tfrac{1}{2}|y_j - x_0|, \text{dist} \left(y_j, \overline{C(x_0, L, \alpha/2)} \cap \overline{B}(x_0, \tfrac{3}{2}|y_j - x_0|) \right) \right\}. \end{aligned}$$

To estimate the second distance, fix $z \in \overline{C(x_0, L, \alpha/2)} \cap \overline{B}(x_0, \frac{3}{2}|y_j - x_0|)$. Since $y_j \notin C(x_0, L, \alpha)$, we have $\text{dist}(y_j, L + x_0) \geq \alpha|y_j - x_0|$. Moreover, $\text{dist}(z, L + x_0) \leq (\alpha/2)|z - x_0|$. Fix a point $p \in L + x_0$ so that $|z - p| \leq \frac{\alpha}{2}|z - x_0|$, so $|z - p| \leq \frac{3}{4}\alpha|y_j - x_0|$. Note that $|y_j - p| \geq \alpha|y_j - x_0|$. Thus, $|y_j - z| \geq \frac{\alpha}{4}|y_j - x_0|$. Hence, for each $j \in \mathbb{N}$,

$$\frac{\mathcal{H}^1((\overline{B}(x_0, \frac{3}{2}|y_j - x_0|) \cap X) \setminus C(x_0, L, \alpha/2))}{\frac{3}{2}|y_j - x_0|} \geq \frac{\alpha}{6}$$

which contradicts (3.5.1) as $j \rightarrow \infty$ and $\frac{3}{2}|y_j - x_0| \rightarrow 0$. $\square$

We are now ready to prove Proposition 3.1.

Proof of Proposition 3.1. Let $f : [0, 1] \rightarrow \mathbb{R}^N$ be Lipschitz and let $X = f([0, 1])$. By (Alberti and Ottolini, 2017, Theorem 4.4), there exists $g : [0, 1] \rightarrow X$, an essentially two-to-one Lipschitz parametrization with constant speed equal to $2\mathcal{H}^1(X)$. By Rademacher's Theorem, for $\mathcal{H}^1$ -a.e. $x \in X$, there exists $t \in g^{-1}(\{x\})$ so that $g'(t)$ exists. Furthermore, by Lemma 3.7.2, $\mathcal{H}^1$ -a.e. point $x \in X$ is flat in X . Let $x_0 \in X$ be such that x_0 is flat in X and there exists $t \in g^{-1}(\{x_0\})$ such that $g'(t)$ exists. Let $T \in \text{Tan}(X, x_0)$, and let $L \subset \mathbb{R}^N$ be the line through the origin in the direction of $g'(t)$.

First, we show that $T \subset L$. To this end, let $r_j \rightarrow 0$ be a sequence of positive scales so that $\frac{1}{r_j}(X - x_0) \rightarrow T$ in the Attouch-Wets topology. Fix $\epsilon > 0$ and $R > 0$. By Lemma 3.7.2, there is $j_0 \in \mathbb{N}$, such that for every $j \geq j_0$,

$$B(x_0, r_j(R + \epsilon)) \cap X \subset C(x_0, L, \frac{\epsilon}{2(R + \epsilon)})$$

and

$$\text{excess}(T \cap \overline{B}(\mathbf{0}, R), \frac{1}{r_j}(X - x_0)) < \epsilon/2.$$

Note that

$$\text{excess}(T \cap \overline{B}(\mathbf{0}, R), (\frac{1}{r_j}(X - x_0)) \cap \overline{B}(\mathbf{0}, R + \epsilon)) < \epsilon/2$$

as well. Then by the triangle inequality for excess from Remark 3.5 and since x_0 is flat in X ,

$$\begin{aligned} & \text{excess}(T \cap \overline{B}(\mathbf{0}, R), L) \\ & \leq \text{excess}(T \cap \overline{B}(\mathbf{0}, R), \frac{1}{r_j}(X - x_0) \cap \overline{B}(\mathbf{0}, R + \epsilon)) + \text{excess}(\frac{1}{r_j}(X - x_0) \cap \overline{B}(\mathbf{0}, R + \epsilon), L) \\ & < \epsilon. \end{aligned}$$

Since the latter is true for all $\epsilon > 0$ and $R > 0$, it follows from Remark 3.5 that $T \subset L$.

Next, we show that $L \subset T$. Fix $\epsilon > 0$ and $R > 0$. There exists $h > 0$ so that for every $s \in (-hR, hR)$,

$$\frac{1}{h}|g(t + s) - x_0 - g'(t)s| < \epsilon/2.$$

Moreover, there exists $j_0 \in \mathbb{N}$ so that for every $j \geq j_0$ and every $s \in (-r_jR, r_jR)$ we get

$$\frac{1}{r_j}|g(t + s) - x_0 - g'(t)s| < \epsilon.$$

Fix $y \in L \cap \overline{B}(\mathbf{0}, \frac{R}{2\mathcal{H}^1(X)})$. For each $j \geq j_0$ there exists $s_j \in (-r_jR, r_jR)$ such that $y = g'(t)\frac{s_j}{r_j}$. Hence,

$$\left| \frac{g(t + s_j) - x_0}{r_j} - y \right| = \left| \frac{g(t + s_j) - x_0}{r_j} - g'(t)\frac{s_j}{r_j} \right| < \epsilon.$$

Noting that $\frac{g(t+s)-x_0}{r_j} \in \frac{1}{r_j}(X - x_0)$, we obtain

$$\text{excess}(L \cap \overline{B}(\mathbf{0}, (2\mathcal{H}^1(X))^{-1}R), r_j^{-1}(X - x_0)) < \epsilon.$$

Therefore, for every $R > 0$ and every $\epsilon > 0$, there exists some $j_0 \in \mathbb{N}$ such that for all $j \geq j_0$,

$$\text{excess}(L \cap \overline{B}(\mathbf{0}, R), r_j^{-1}(X - x_0)) < \epsilon.$$

Proceeding to the conclusion as in the previous paragraph, we have that $L \subset T$, and therefore $L = T$ as desired. $\square$

3.6 A “statistically self-similar” carpet

Fix for the rest of this section an even integer $n \geq 4$. Recall the alphabets $\mathcal{A}_n$ and associated word spaces from §3.4.2. Divide the unit square $[0, 1]^2$ into n^2 -many closed squares of side-length $1/n$ that have disjoint interiors and let $S_1, \dots, S_{4n-4}$ be those squares that intersect with $\partial[0, 1]^2$.

We define two iterated function systems

$$\mathcal{F}_n^1 = \{\psi_{n,j}^1 : [0, 1]^2 \rightarrow [0, 1]^2\}_{j \in \mathcal{A}_n} \quad \text{and} \quad \mathcal{F}_n^2 = \{\psi_{n,j}^2 : [0, 1]^2 \rightarrow [0, 1]^2\}_{j \in \mathcal{A}_n}$$

as follows.

1. For $j \in \{1, \dots, 4n - 4\}$, $\psi_{n,j}^1 = \psi_{n,j}^2$ and $\psi_{n,j}^1$ is the unique composition of a translation and a dilation that maps $[0, 1]^2$ onto S_j .
2. For $j \in \{4n - 3, \dots, 4n + \frac{n}{2} - 4\}$ define

$$\psi_{n,j}^1(x) = \psi_{n,j}^2(x) = \frac{1}{n}x + \left(\frac{1}{2} + \frac{j-1}{n}, \frac{1}{n}\right).$$

3. For $j \in \{4n + \frac{n}{2} - 3, 5n - 6\}$ define

$$\psi_{n,j}^1(x) = \frac{1}{n}x + \left(\frac{j-4n+\frac{n}{2}+3}{n}, \frac{2}{n}\right), \quad \psi_{n,j}^2(x) = \frac{1}{n}x + \left(\frac{j-4n+\frac{n}{2}+3}{n}, \frac{1}{n}\right).$$

See Figure 3.2 for the first iterations of $\mathcal{F}_n^1$ and $\mathcal{F}_n^2$ in the case $n = 6$.

For $j \in \{1, 2\}$ we let $\psi_{n,\varepsilon}^j$ be the identity map and for $w = i_1 \dots i_m \in \mathcal{A}_n^m$ with $m \geq 1$ we let

$$\psi_{n,w}^j := \psi_{n,i_1}^j \circ \dots \circ \psi_{n,i_m}^j.$$

We set

$$\mathcal{C}_n = \{\eta : \mathcal{A}_n^* \rightarrow \{1, 2\}\}$$

and each function $\eta \in \mathcal{C}_n$ will henceforth be called a *choice function*.

Fix now a choice function $\eta \in \mathcal{C}_n$. Set ϕ_ε^η to be the identity on $[0, 1]^2$, and for any $w = i_1 \dots i_m \in \mathcal{A}_n^m$ with $m \geq 1$, define

$$\phi_w^\eta := \psi_{n,i_1}^{\eta(\varepsilon)} \circ \dots \circ \psi_{n,i_m}^{\eta(i_1 \dots i_{m-1})}.$$

Define now the “attractor” associated with the choice function η to be

$$K^\eta := \bigcap_{m=0}^{\infty} \bigcup_{w \in \mathcal{A}_n^m} \phi_w^\eta([0, 1]^2).$$

We list a couple of elementary facts about the set K^η .

Lemma 3.7.3. *For every $\eta \in \mathcal{C}_n$, the set K^η is compact.*

Proof. This is immediate noting that each set $\bigcup_{w \in \mathcal{A}_n^m} \phi_w^\eta([0, 1]^2)$ is compact as a finite union of compact sets, and that K^η is the countable intersection of compact sets. $\square$

Given $w \in \mathcal{A}_n^*$, we define

$$K_w^\eta := \phi_w^\eta([0, 1]^2) \cap K^\eta.$$

The following version of the open set condition is satisfied.

Lemma 3.7.4. *For every $\eta \in \mathcal{C}_n$, every $m \in \mathbb{N}$, and every pair of distinct words $w, v \in \mathcal{A}_n^m$,*

$$\phi_w^\eta((0, 1)^2) \cap \phi_v^\eta((0, 1)^2) = \emptyset \quad \text{and} \quad \phi_w^\eta((0, 1)^2) \subset (0, 1)^2.$$

Proof. We proceed by induction on m . For $m = 1$, if $i, j \in \mathcal{A}_n$ are distinct characters, then by definition we have that $\phi_j^\eta([0, 1]^2)$ and $\phi_i^\eta([0, 1]^2)$ are distinct squares of side length $1/n$ with disjoint interiors contained in $[0, 1]^2$. Thus, $\phi_i^\eta((0, 1)^2) \subset (0, 1)^2$ and $\phi_i^\eta((0, 1)^2) \cap \phi_j^\eta((0, 1)^2) = \emptyset$.

Fix now an $m \in \mathbb{N}$ and assume that for all distinct words $v, w \in \mathcal{A}_n^m$, we have $\phi_w^\eta((0, 1)^2) \cap \phi_v^\eta((0, 1)^2) = \emptyset$ and $\phi_w^\eta((0, 1)^2) \subset (0, 1)^2$. Let $u, u' \in \mathcal{A}_n^{m+1}$ be distinct words. Then by the base case we have $\phi_u^\eta((0, 1)^2) \subset \phi_{u(m)}^\eta((0, 1)^2)$, which is contained in $(0, 1)^2$ by the induction hypothesis.

The remainder of the proof falls to a case study.

Case 1. If $u(m) \neq u'(m)$, then we have that $\phi_u^\eta((0, 1)^2) \subset \phi_{u(m)}^\eta((0, 1)^2)$ and $\phi_{u'}^\eta((0, 1)^2) \subset \phi_{u'(m)}^\eta((0, 1)^2)$ by the base case. Then, since $u(m) \neq u'(m)$, we have by the induction hypothesis that $\phi_{u(m)}^\eta((0, 1)^2) \cap \phi_{u'(m)}^\eta((0, 1)^2) = \emptyset$, so $\phi_u^\eta((0, 1)^2) \cap \phi_{u'}^\eta((0, 1)^2) = \emptyset$.

Case 2. If $u(m) = u'(m)$, then there exist distinct characters $i, j \in \mathcal{A}_n$ so that $u = u(m)j$ and $u' = u'(m)i$. By definition of ϕ_u^η we then have

$$\phi_u^\eta((0, 1)^2) = \phi_{u(m)}^\eta \circ \psi_j^{\eta(u(m))}((0, 1)^2)$$

and

$$\phi_{u'}^\eta((0, 1)^2) = \phi_{u(m)}^\eta \circ \psi_i^{\eta(u(m))}((0, 1)^2).$$

Thus we can see that

$$\phi_{u'}^\eta((0, 1)^2) \cap \phi_u^\eta((0, 1)^2) = \phi_{u(m)}^\eta(\psi_j^{\eta(u(m))}((0, 1)^2) \cap \psi_i^{\eta(u(m))}((0, 1)^2)),$$

and the latter set is empty by the base case. □

Lemma 3.7.5. *Let $\eta \in \mathcal{C}_n$, $w, v \in \mathcal{A}_n^m$, $x \in K_w^\eta$, and $y \in K_v^\eta$. If $|x - y| > 2\sqrt{2} \cdot n^{-m}$, then $K_w^\eta \cap K_v^\eta = \emptyset$. If $|x - y| < n^{-m}$, then $K_w^\eta \cap K_v^\eta \neq \emptyset$.*

Proof. Let $X_0 = [0, 1]^2$, and for $m \in \mathbb{N}$, define $X_m = \bigcup_{w \in \mathcal{A}_n^m} \phi_w^\eta([0, 1]^2)$. Recall that for $j \in \{1, \dots, 4n-4\}$ we have $\psi_{n,j}^1 = \psi_{n,j}^2$. Therefore, for every integer $m \geq 0$, we have that

$$\bigcup_{j=1}^{4n-4} \psi_{n,j}^1(X_m) \subset X_{m+1}$$

and

$$\partial[0, 1]^2 \subset \bigcup_{j=1}^{4n-4} \psi_{n,j}^1(\partial[0, 1]^2).$$

Thus, since $\partial[0, 1]^2 \subset X_0$, we have that $\partial[0, 1]^2 \subset X_m$ for every integer $m \geq 0$, and so since $K^\eta = \bigcap_{m=1}^\infty X_m$, we get $\partial[0, 1]^2 \subset K^\eta$. Similarly, for every $w \in \mathcal{A}_n^*$ we have that $\phi_w^\eta(\partial[0, 1]^2) \subset K_w^\eta$. Let $m \geq 1$ and let $v, w \in \mathcal{A}_n^m$. If $K_v^\eta \cap K_w^\eta \neq \emptyset$, then $\text{diam}(K_v^\eta \cup K_w^\eta) \leq 2\sqrt{2}n^{-m}$. Furthermore, if $K_v^\eta \cap K_w^\eta = \emptyset$, then $\phi_v^\eta(\partial[0, 1]^2) \cap \phi_w^\eta(\partial[0, 1]^2) = \emptyset$, so $\text{dist}(\phi_v^\eta([0, 1]^2), \phi_w^\eta([0, 1]^2)) \geq n^{-m}$, and the result follows as $K_v^\eta \subset \phi_v^\eta([0, 1]^2)$. $\square$

3.7 Ahlfors regularity of carpets K^η

Fix for the rest of this section an even integer $n \geq 4$ and set $\alpha_n := \frac{\log(5n-6)}{\log(n)}$. We show that for every choice function $\eta \in \mathcal{C}_n$, we have $0 < \mathcal{H}^{\alpha_n}(K^\eta) < \infty$. In fact, we show the following stronger statement.

Proposition 3.2. *There exists $C_n > 1$ such that for every $\eta \in \mathcal{C}_n$, every $x \in K^\eta$, and every $r \in (0, 1/n)$ we have*

$$C_n^{-1}r^{\alpha_n} \leq \mathcal{H}^{\alpha_n}(K^\eta \cap B(x, r)) \leq C_n r^{\alpha_n}.$$

For a choice function $\eta \in \mathcal{C}_n$ define $\pi_\eta : \mathcal{A}_n^\mathbb{N} \rightarrow K^\eta$ by

$$\pi_\eta(\tau) := \lim_{m \rightarrow \infty} \phi_{\tau(m)}^\eta(\mathbf{0}). \quad (3.7.1)$$

We start with an elementary topological fact.

Lemma 3.7.6. *The σ -algebra generated by the collection $\{K_w^\eta : w \in \mathcal{A}_n^*\}$ is equal to the Borel σ -algebra on K^η .*

Proof. Fix $\eta \in \mathcal{C}_n$. Let Σ denote the σ -algebra generated by the collection $\{K_w^\eta : w \in \mathcal{A}_n^*\}$, and let $\mathcal{B}(K^\eta)$ be the Borel σ -algebra on K^η . That $\Sigma \subset \mathcal{B}(K^\eta)$ is clear, as each K_w^η is a Borel set. For the reverse inclusion, we show that for any $x \in K^\eta$ and any $r > 0$, $B(x, r) \cap K^\eta \in \Sigma$. To see this, let

$$W(x, r) = \{w \in \mathcal{A}_n^* : K_w^\eta \subset B(x, r)\}.$$

Note that $W(x, r)$ is a countable set since $\mathcal{A}_n^*$ is countable, and so $\bigcup_{w \in W(x, r)} K_w^\eta \in \Sigma$. Furthermore, for each point $y \in B(x, r) \cap K^\eta$ there is a word $\tau \in \mathcal{A}_n^\mathbb{N}$ with $\pi_\eta(\tau) = y$. Then since $B(x, r) \cap K^\eta$ is an open subset of K^η and $\lim_{j \rightarrow \infty} \text{diam}(K_{\tau(j)}^\eta) = 0$, there is some $m \in \mathbb{N}$ large enough that $\tau(m) \in W(x, r)$. Since $y \in K_{\tau(m)}^\eta$, $\bigcup_{w \in W(x, r)} K_w^\eta = B(x, r) \cap K^\eta$, so we have that $\Sigma = \mathcal{B}(K^\eta)$. $\square$

Recall from §3.4.2 the σ -algebra Σ_n on $\mathcal{A}_n^\mathbb{N}$ and the probability measure $\nu_n : \Sigma_n \rightarrow [0, 1]$. We say a word $\tau \in \mathcal{A}_n^\mathbb{N}$ is an *injective word* if for every $\eta \in \mathcal{C}_n$, we have $\pi_\eta^{-1}(\{\pi_\eta(\tau)\}) = \{\tau\}$. That is, τ is called an injective word if it always uniquely defines a point in K^η .

Lemma 3.7.7. *The set of non-injective words $\tau \in \mathcal{A}_n^\mathbb{N}$ is a ν_n -null set.*

Proof. Fix $\eta \in \mathcal{C}_n$. Recall that if $j \in \mathcal{A}_n$ satisfies $j > 4n - 4$, then

$$(\psi_j^1([0, 1]^2) \cup \psi_j^2([0, 1]^2)) \cap \partial[0, 1]^2 = \emptyset.$$

Thus if $\tau = i_1 i_2 \cdots \in \mathcal{A}_n^\mathbb{N}$ and $i_{m+1} > 4n - 4$ for some $m \in \mathbb{N}$, then $\pi_\eta(\tau) \notin \bigcup_{v \in \mathcal{A}_n^m} \phi_v^\eta(\partial[0, 1]^2)$. Furthermore, if $\tau, \zeta \in \mathcal{A}_n^\mathbb{N}$ are distinct words with $\pi_\eta(\tau) = x = \pi_\eta(\zeta)$, then there exists some $N \in \mathbb{N}$ such that $\tau(N) \neq \zeta(N)$. Thus for every integer $m \geq N$, we have $\tau(m) \neq \zeta(m)$ and $x \in \phi_{\tau(m)}^\eta([0, 1]^2) \cap \phi_{\zeta(m)}^\eta([0, 1]^2)$. By Lemma 3.7.4,

this implies that $x \in \phi_{\tau(m)}^\eta(\partial[0, 1]^2) \cap \phi_{\zeta(m)}^\eta(\partial[0, 1]^2)$, so

$$\{x \in K^\eta : \pi_\eta^{-1}(\{x\}) \text{ is not a singleton}\} \subset \bigcup_{m=1}^{\infty} \bigcup_{v \in \mathcal{A}_n^m} \phi_v^\eta(\partial[0, 1]^2).$$

Moreover, for every $m \in \mathbb{N}$,

$$\bigcup_{v \in \mathcal{A}_n^m} \phi_v^\eta(\partial[0, 1]^2) \subset \bigcup_{v \in \mathcal{A}_n^{m+1}} \phi_v^\eta(\partial[0, 1]^2).$$

Thus if a character $j \in \{4n-3, \dots, 5n-6\}$ appears infinitely often in $\tau \in \mathcal{A}_n^\mathbb{N}$, then τ is injective; so the set of non-injective words is contained in the set of words for which every character $j \in \{4n-3, \dots, 5n-6\}$ appears only finitely often. To see that this set is ν_n -null, for each $m \in \mathbb{N}$ let $U_m := \{\tau = i_1 i_2 \dots \in \mathcal{A}_n^\mathbb{N} : i_m > 4n-4\}$. We have $\nu_n(U_m) = \frac{n-2}{5n-6}$ for each m , and if $\{U_{m_1}, \dots, U_{m_j}\}$ is a collection of these sets, then

$$\bigcap_{i=1}^j U_{m_i} = \{\tau = i_1 i_2 \dots \in \mathcal{A}_n^\mathbb{N} : i_{m_i} > 4n-4 \text{ for each } i \in \{1, \dots, j\}\}$$

and this intersection has $\nu_n(\bigcap_{i=1}^j U_{m_i}) = (\frac{n-2}{5n-6})^j$. Thus, $V_m := \bigcap_{j=m}^{\infty} U_j$ has $\nu_n(V_m) = 0$ for every $m \in \mathbb{N}$. Noting further that the set F of words for which every character $j \in \{4n-3, \dots, 5n-6\}$ appears only finitely often is contained in $\bigcup_{m=0}^{\infty} V_m$, we have that F is ν_n -null. $\square$

We are now ready to prove Proposition 3.2.

Proof of Proposition 3.2. Fix $\eta \in \mathcal{C}_n$. We claim that for some $C_n > 1$ the measure $\pi_\eta \# \nu_n$ defined on $\mathcal{B}(K^\eta)$ by Lemma 3.7.6 satisfies

$$C_n^{-1} r^{\alpha_n} \leq \pi_\eta \# \nu_n(B(x, r) \cap K^\eta) \leq C_n r^{\alpha_n} \quad (3.7.2)$$

for all $x \in K^\eta$ and $r \in (0, 1/n)$. Assuming (3.7.2), by (Mackay and Tyson, 2010, §1.4.3), we have that $\mathcal{H}^{\alpha_n} \llcorner K^\eta$ satisfies (3.7.2) (perhaps with a different constant

C_n), since the measure $\pi_\eta \# \nu_n$ has the same null sets as the restricted Hausdorff measure $\mathcal{H}^{\alpha_n} \llcorner K^\eta$.

For any $w \in \mathcal{A}_n^*$, $\mathcal{A}_{n,w}^\mathbb{N} \subset \pi_\eta^{-1}(K_w^\eta) \subset \mathcal{N}_n \cup \mathcal{A}_{n,w}^\mathbb{N}$, where $\mathcal{N}_n$ denotes the set of non-injective words in $\mathcal{A}_n^\mathbb{N}$. Hence by Lemma 3.7.7, for any $w \in \mathcal{A}_n^*$ we have

$$\pi_\eta \# \nu_n(K_w^\eta) = (5n - 6)^{-|w|} = n^{-|w|\alpha_n}.$$

Fix for the rest of the proof a point $x \in K^\eta$, a radius $r \in (0, 1/n)$, and a word $\tau \in \mathcal{A}_n^\mathbb{N}$ such that $\pi_\eta(\tau) = x$.

For the upper bound of (3.7.2), let $m \in \mathbb{N}$ satisfy

$$n^{-m}\sqrt{2} > 2r \geq n^{-m-1}\sqrt{2}.$$

Let $\{v_1, \dots, v_j\} \subset \mathcal{A}_n^m$ be those words with $K_{v_i}^\eta \cap K_{\tau(m)}^\eta \neq \emptyset$. By Lemma 3.7.4, $j \leq 9$ and by Lemma 3.7.5, we have that $B(x, n^{-m}) \cap K^\eta \subset \bigcup_{i=1}^j K_{v_i}^\eta$. We have also that $n^{-m} \geq r$, so

$$B(x, r) \cap K^\eta \subset B(x, n^{-m}) \cap K^\eta \subset \bigcup_{i=1}^j K_{v_i}^\eta,$$

and therefore

$$\pi_\eta \# \nu_n(B(x, r) \cap K^\eta) \leq \sum_{i=1}^j \pi_\eta \# \nu_n(K_{v_i}^\eta) \leq 9\pi_\eta \# \nu_n(K_{\tau(m)}^\eta) = 9n^{-m\alpha_n} \leq 9 \left(\frac{8}{\sqrt{2}}\right)^{\alpha_n} r^{\alpha_n}.$$

For the lower bound of (3.7.2), let $m \in \mathbb{N}$ satisfy

$$n^{-m}\sqrt{2} < \frac{r}{2} \leq n^{-m+1}\sqrt{2}.$$

Since $r < 1/n$, we have that m is at least 1. If $y \in K_{\tau(m)}^\eta$, then

$$|x - y| \leq n^{-m}\sqrt{2} < \frac{r}{2},$$

so $K_{\tau(m)}^\eta \subset B(x, r) \cap K^\eta$. Thus

$$\pi_\eta \# \nu_n(B(x, r) \cap K^\eta) \geq \pi_\eta \# \nu_n(K_{\tau(m)}^\eta) = n^{-m\alpha_n} \geq (8\sqrt{2})^{-\alpha_n} r^{\alpha_n}. \quad \square$$

3.8 Hölder parametrizations of carpets K^η

In this section, we show that sets K^η defined in Section 3.6 are Hölder curves. Recall the numbers $\alpha_n := \frac{\log(5n-6)}{\log(n)}$ from Section 3.7.

Proposition 3.3. *For any even integer $n \geq 4$ and any $\eta \in \mathcal{C}_n$, there exists a $(1/\alpha_n)$ -Hölder continuous surjection $f : [0, 1] \rightarrow K^\eta$.*

For the rest of this section we fix an even integer $n \geq 4$ and a choice function $\eta \in \mathcal{C}_n$. The dependence of sets and functions on n and η in this section is omitted.

Define the set $T \subset [0, 1]^2$ by

$$T := \left(\partial[0, 1 - \frac{1}{n}]^2 \setminus \left(\{1 - \frac{1}{n}\} \times [1 - \frac{2}{n}, 1 - \frac{1}{n}] \right) \right) \cup \left(\{\frac{1}{2}\} \times [0, \frac{1}{n}] \right) \cup \left([\frac{1}{2}, 1 - \frac{2}{n}] \times \{\frac{1}{n}\} \right)$$

and the sets

$$T^1 := T \cup \left([0, \frac{1}{2} - \frac{1}{n}] \times \{\frac{2}{n}\} \right) \quad \text{and} \quad T^2 := T \cup \left([0, \frac{1}{2} - \frac{1}{n}] \times \{\frac{1}{n}\} \right).$$

See Figure 3.3 for $n = 6$. Some elementary properties of the sets T^1, T^2 are given in the next lemma and its proof is left to the reader.

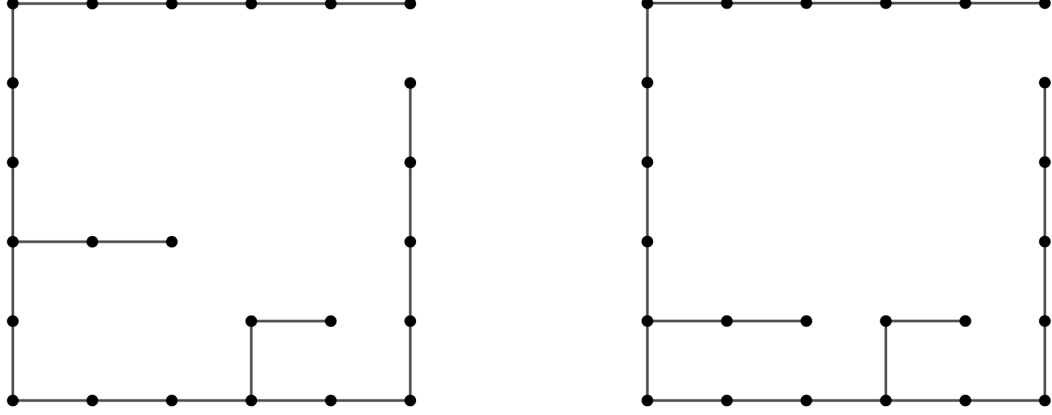


Figure 3.3: Dendrites T^1 (left) and T^2 (right) for $n = 6$.

Lemma 3.7.8. *Both T^1 and T^2 are dendrites and are contained in $[0, 1]^2$.*

For each $i \in \{1, 2\}$ define combinatorial graphs $G^i = (V^i, E^i)$ by

$$V^i := \{\psi_j^i(\mathbf{0}) : j \in \mathcal{A}_n\} \quad \text{and} \quad E^i := \{\{p, q\} : p, q \in V^i, [p, q] \subset T^i, |q - p| = 1/n\}.$$

Next, we define a sequence of sets $(T_m)_{m \geq 0}$ by $T_0 = \{\mathbf{0}\}$, and for any integer $m \geq 0$,

$$T_{m+1} = T_m \cup \left(\bigcup_{w \in \mathcal{A}_n^m} \phi_w^\eta(T^{\eta(w)}) \right).$$

Define also for $m \geq 0$ the combinatorial graph $G_m = (V_m, E_m)$ via

$$V_m = \{\phi_w^\eta(\mathbf{0}) : w \in \mathcal{A}_n^m\} \quad \text{and} \quad E_m = \{\{p, q\} : p, q \in V_m, [p, q] \subset T_m, |p - q| = n^{-m}\}.$$

Lemma 3.7.9. *The sequence of sets $(T_m)_{m=0}^\infty$ is a nested sequence of dendrites contained in K^η . Moreover, for every $m \in \mathbb{N}$, a vertex $u \in V_m$ is a leaf of T_m if and only if u has no edges emanating to the right or upward in T_m .*

Proof. By the arguments in the proof of Lemma 3.7.5, for each $w \in \mathcal{A}_n^*$ we have that $\phi_w^\eta(\partial[0, 1]^2) \subset K^\eta$. Therefore,

$$T_m \subset \bigcup_{w \in \mathcal{A}_n^m} \phi_w^\eta(\partial[0, 1]^2) \subset K^\eta.$$

Next, we show that each T_m is a dendrite by induction on m . The set $T_0 = \{\mathbf{0}\}$ is clearly a dendrite. Now assume that T_m is a dendrite for some $m \geq 0$. We make three observations. First, by Lemma 3.7.8, $\phi_w^\eta(T^{\eta(w)})$ is a dendrite for any $w \in \mathcal{A}_n^{m+1}$. Second, by Lemma 3.7.4 and by Lemma 3.7.8, if $v, w \in \mathcal{A}_n^{m+1}$ are distinct, then

$$\phi_w^\eta(T^{\eta(w)}) \cap \phi_v^\eta(T^{\eta(v)}) \subset \phi_w^\eta([0, 1]^2) \cap \phi_v^\eta([0, 1]^2) = \emptyset.$$

Third, for any $w \in \mathcal{A}_n^{m+1}$ we have that $\phi_w^\eta(T^{\eta(w)}) \cap T_{m+1}$ is either $\phi_w^\eta(\{0\} \times [0, 1 - \frac{1}{n}])$, or $\phi_w^\eta([0, 1 - \frac{1}{n}] \times \{0\})$, or the union of the latter two sets. In either case, $\phi_w^\eta(T^{\eta(w)}) \cap T_{m+1}$ is an arc. Write now $T_{m+1} = T_m \cup \bigcup_{k=1}^N X_k$ where $X_1, \dots, X_N$ are the components of the closure of $T_{m+1} \setminus T_m$. Note the sets X_k are pairwise disjoint, by the first and second observation each X_k is a dendrite, and by the first and third observation, $X_i \cap T_m$ is a point. Applying Lemma 3.4.1, we conclude that T_{m+1} is a dendrite.

Finally, we prove the claim about the leaves of T_m by induction on m . For $m = 1$, we note that the leaves of T^1 are exactly the points $(1 - \frac{1}{n}, 1 - \frac{1}{n})$, $(1 - \frac{1}{n}, 1 - \frac{2}{n})$, $(\frac{2}{n}, \frac{2}{n})$, $(1 - \frac{2}{n}, \frac{1}{n})$ for which it is easy to check the claim. Similarly for T^2 .

Suppose now that the claim holds for some $m \in \mathbb{N}$ and let $u \in V_{m+1}$ be a vertex. Then there is a unique word $w \in \mathcal{A}_n^m$ so that $u \in \phi_w^\eta(T^{\eta(w)})$, and u is a leaf in T_{m+1} if and only if $(\phi_w^\eta)^{-1}(u)$ is a leaf in $T^{\eta(w)}$, which is a leaf if and only if it has no edges in $T^{\eta(w)}$ emanating to the right or upward. This holds if and only if u has no edges emanating upward or to the right in T_{m+1} . $\square$

Note that for each integer $m \geq 0$,

$$T_m = \text{Im}(G_m) = V_m \cup \left(\bigcup_{\{p,q\} \in E_m} (p,q) \right),$$

so when we refer to a “vertex” of T_m we mean a point $p \in V_m$, and when we refer to an “edge (p,q) of T_m ” we mean the open segment (p,q) in $\mathbb{R}^2$ associated with the edge $\{p,q\} \in E_m$. Set

$$e_b = (0,1) \times \{0\} \quad \text{and} \quad e_l = \{0\} \times (0,1),$$

the bottom and left, respectively, open edges of $(0,1)^2$. Note that if $e \subset T_m$ is an edge in T_m , then e is parallel to either the x -axis or to the y -axis, as this is true for T^1 and T^1 and the maps ψ_j^i are all rotation-free and reflection-free similarities.

Lemma 3.7.10. *For each $m \in \mathbb{N}$, if (p,q) is an edge of T_m , then there exists $w \in \mathcal{A}_n^m$ such that either $(p,q) = \phi_w^\eta(e_b)$ or $(p,q) = \phi_w^\eta(e_l)$.*

Proof. We proceed by induction on m . For $m = 1$, the result follows by simple inspection, as either $T_1 = T^1$ or $T_1 = T^2$. Now assume the result holds for some $m \in \mathbb{N}$. Let $(p,q) \subset T_{m+1}$ be an edge. Then there exist words $u, v \in \mathcal{A}_n^{m+1}$ so that $p = \phi_u^\eta(\mathbf{0})$ and $q = \phi_v^\eta(\mathbf{0})$. If there is a word $w \in \mathcal{A}_n^m$ so that $(p,q) \subset \phi_w^\eta(T^{\eta(w)})$, then there is some edge $e \subset T^{\eta(w)}$ so that $(p,q) = \phi_w^\eta(e)$, and the result follows from the $m = 1$ case and from the definition of ϕ_w^η .

If there is no such word w , then we have that $(p,q) \subset T_m$. Since the only nondegenerate line segments contained in T_m are contained in its edges, there is some edge $e \subset T_m$ so that $(p,q) \subset e$, and since (p,q) is not a subset of $\phi_v^\eta(T^{\eta(v)})$ for any $v \in \mathcal{A}_n^m$, we have that $[p,q]$ is not a subset of $\phi_v^\eta([0,1]^2)$ for any $v \in \mathcal{A}_n^m$. Then by the induction hypothesis there is a word $w \in \mathcal{A}_n^m$ so that $e = \phi_w^\eta(e_b)$ or $e = \phi_w^\eta(e_l)$. Note that the length of (p,q) is n^{-m-1} , that $[p,q] \subset \phi_w^\eta([0,1]^2)$, and that $(p,q) \subset \phi_w^\eta([0,1]^2)$. Then either p or q is $\phi_w^\eta((0,1))$ or $\phi_w^\eta((1,0))$ since the edge e is

equal to either $\phi_w^\eta(e_b)$ or to $\phi_w^\eta(e_l)$. Then there is a $j \in \mathcal{A}_n$ so that the other of p and q is equal to $\phi_{wj}^\eta(\mathbf{0})$, so $\phi_{wj}^\eta(e_b) = (p, q)$ or $\phi_{wj}^\eta(e_l) = (p, q)$. $\square$

In the next lemma, which is the crux of the proof of Proposition 3.3, we construct intermediate parametrizations $f_m : [0, 1] \rightarrow T_m$.

Lemma 3.7.11. *There exist a sequence of piecewise linear continuous surjections $(f_m : [0, 1] \rightarrow T_m)_{m \geq 0}$, a sequence $(\mathcal{N}_m)_{m \geq 0}$ of families of nondegenerate closed intervals contained in $[0, 1]$, a sequence $(\mathcal{E}_m)_{m \geq 0}$ of families of open intervals in $[0, 1]$, another sequence $(\mathcal{F}_m)_{m \geq 0}$ of families of nondegenerate closed intervals in $[0, 1]$, and a sequence of functions $(\mathbf{w}_m : \mathcal{N}_m \rightarrow \mathcal{A}_n^m)_{m \geq 0}$ satisfying the following properties.*

- (P1) *For each $m \geq 0$, $\mathbf{w}_m$ is a bijection between $\mathcal{N}_m$ and $\mathcal{A}_n^m$.*
- (P2) *For each $m \geq 0$, the families $\mathcal{E}_m, \mathcal{N}_m, \mathcal{F}_m$ are pairwise disjoint and the elements of $\mathcal{E}_m \cup \mathcal{N}_m \cup \mathcal{F}_m$ are pairwise disjoint. Moreover, $[0, 1] = \bigcup (\mathcal{E}_m \cup \mathcal{N}_m \cup \mathcal{F}_m)$.*
- (P3) *For each $m \geq 0$ and $J \in \mathcal{E}_m$, there exists $w \in \mathcal{A}_n^m$ such that $f_m|J$ is a linear bijection onto either $\phi_w^\eta(e_b)$ or $\phi_w^\eta(e_l)$, and there exists a unique interval $J' \in \mathcal{E}_m \setminus \{J\}$, so that $f_m|J' = (f_m|J) \circ \zeta_{J'}$ where $\zeta_{J'} : J' \rightarrow J$ is the unique linear orientation reversing map. Furthermore, $f_{m+1}(J) \subset \phi_w^\eta([0, 1]^2)$. Conversely, if (p, q) is an edge of T_m , then there exist exactly two intervals $J, J' \in \mathcal{E}_m$ such that $f_m(J) = f_m(J') = (p, q)$.*
- (P4) *For each $m \geq 0$ and $I \in \mathcal{N}_m$, $f_m|I$ is constant equal to $\phi_{\mathbf{w}_m(I)}^\eta(\mathbf{0})$. Conversely, $f_m(\bigcup \mathcal{N}_m) = V_m$. Moreover, $f_{m+1}(I) \subset \phi_{\mathbf{w}_m(I)}^\eta([0, 1]^2)$.*
- (P5) *For each $m \geq 0$ and $I \in \mathcal{F}_m$, $f_m|I$ is constant and $f_m(I) \in V_m$. Furthermore, $f_{m+1}|I = f_m|I$.*
- (P6) *For each $m \geq 0$ and $J \in \mathcal{E}_{m+1} \cup \mathcal{N}_{m+1}$, there exists $I \in \mathcal{E}_m \cup \mathcal{N}_m$ such that $J \subset I$. Moreover, if $J \in \mathcal{N}_{m+1}$ and $I \in \mathcal{N}_m$ with $J \subset I$, then $\mathbf{w}_m(I) = \mathbf{w}_{m+1}(J)(m)$.*

(P7) For each $m \geq 0$ and $J \in \mathcal{E}_m$, there exists $I \in \mathcal{N}_{m+1}$ such that $I \subset J$ and $f_{m+1}(I)$ is a leaf in T_{m+1} .

Proof. We start by defining two pairs of preliminaries maps.

First, fix for each $i \in \{1, 2\}$ a continuous surjection $\tau_i : [0, 1] \rightarrow T^i$ with the following properties.

1. The map τ_i is a 2-to-1 tour of the edges of T^i and is linear along edges with $\tau_i(0) = \tau_i(1) = \mathbf{0}$.
2. For every vertex $v \in V^i$ other than $\mathbf{0}$, the preimage $\tau_i^{-1}(\{v\})$ has a number of components equal to the valence of the vertex v and each component is a nondegenerate closed interval.
3. The preimage $\tau_i^{-1}(\{\mathbf{0}\})$ is made up of three disjoint nondegenerate closed intervals.
4. There exists t_i in the component of $\tau_i^{-1}(\{\mathbf{0}\})$ which does not contain 0 or 1 such that $\tau_i^{-1}(\{(0, 1 - 1/n)\})$ lies to the left of t_i and $\tau_i^{-1}(\{(1 - 1/n, 0)\})$ lies to the right of t_i .

Next, for each $i \in \{1, 2\}$, define the combinatorial tree $\tilde{G}^i = (\tilde{V}^i, \tilde{E}^i)$ via

$$\tilde{V}^i := V^i \cup \{(1, 0), (0, 1)\}, \quad \tilde{E}^i = E^i \cup \{(1 - 1/n, 0), (1, 0)\}, \{(0, 1 - 1/n), (0, 1)\},$$

and let $\tilde{T}^i := \text{Im}(\tilde{G}^i)$. Fix for each $i \in \{1, 2\}$ a continuous surjection $\tilde{\tau}_i : [0, 1] \rightarrow \tilde{T}^i$ with the following properties.

1. The map $\tilde{\tau}_i$ is a 2-to-1 tour of the edges of $\tilde{T}^i$ and is linear along edges with $\tilde{\tau}_i(0) = \tilde{\tau}_i(1) = \mathbf{0}$.
2. For every vertex $v \in \tilde{V}^i$ other than $\mathbf{0}$, $(0, 1)$, and $(1, 0)$, the preimage $\tilde{\tau}_i^{-1}(\{v\})$ has a number of components equal to the valence of the vertex v and each component is a nondegenerate closed interval.

3. The preimage $\tilde{\tau}_i^{-1}(\{\mathbf{0}\})$ is made up of three disjoint nondegenerate closed intervals.
4. The preimages $\tilde{\tau}_i^{-1}(\{(1, 0)\})$ and $\tilde{\tau}_i^{-1}(\{(0, 1)\})$ are both singletons, denoted by $s_{i,U}$ and $s_{i,L}$, respectively.
5. There exists $\tilde{t}_i$ in the component of $\tilde{\tau}_i^{-1}(\{\mathbf{0}\})$ which does not contain 0 or 1 such that $s_{i,U}$ lies to the left of $\tilde{t}_i$ and $s_{i,L}$ lies to the right of $\tilde{t}_i$.
6. There are disjoint nondegenerate closed intervals $I_{i,1}, I_{i,2}, I_{i,3}, I_{i,4} \subset [0, 1]$, denoted by $I_{i,j} = [a_{i,j}, b_{i,j}]$, $j = 1, \dots, 4$ equal to the preimages of the four leaves of T^i , so that

$$0 < a_{i,1} < s_{i,U} < a_{i,2} < \tilde{t}_i < a_{i,3} < s_{i,L} < a_{i,4} < b_{i,4} < 1.$$

Let T be one of the trees T^i or $\tilde{T}^i$ above, let V and E be the corresponding vertex and edge sets, respectively, and let $\tau : [0, 1] \rightarrow T$ be the map defined above corresponding to T .

- Define $\mathcal{E}(\tau)$ to be the set of components of preimages of open edges in E .
- If $\tau \in \{\tau_1, \tau_2\}$, then for each $v \in V$ choose a nondegenerate component I_v of the preimage $\tau^{-1}(\{v\})$ and let $\mathcal{N}(\tau) = \{I_v : v \in V\}$.
- If $\tau \in \{\tilde{\tau}_1, \tilde{\tau}_2\}$, then for a vertex $v \in V$ that is not a leaf, choose a nondegenerate component I_v of the preimage $\tau^{-1}(\{v\})$, and for each vertex $v \in V$ that is a leaf other than $(1, 0)$ or $(0, 1)$, let $I_v = I_{i,j}$ where $j \in \{1, 2, 3, 4\}$ is chosen so that $\tau(I_{i,j}) = v$. We then define $\mathcal{N}(\tau) = \{I_v : v \in V \setminus \{(1, 0), (0, 1)\}\}$.
- Define $\mathcal{F}(\tau)$ to be the set of nondegenerate components of preimages under τ of vertices in T which are not already in $\mathcal{N}(\tau)$.

The construction of maps f_m and families $\mathcal{E}_m, \mathcal{F}_m, \mathcal{N}_m$ is done in an inductive fashion.

Set $f_0 : [0, 1] \rightarrow T_0$ to be the constant map $f_0 \equiv \mathbf{0}$, set $\mathcal{N}_0 = \{[0, 1]\}$, $\mathcal{E}_0 = \emptyset$, $\mathcal{F}_0 = \emptyset$, and set $\mathbf{w}_0([0, 1]) = \varepsilon$. Properties (P1)–(P7) are all either clear or vacuous for $m = 0$.

Assume now that for some $m \geq 0$ we have constructed the map f_m , families $\mathcal{E}_m$, $\mathcal{F}_m$, and $\mathcal{N}_m$, and bijection $\mathbf{w}_m : \mathcal{N}_m \rightarrow \mathcal{A}_n^m$ satisfying (P1)–(P7). For elements $I \in \mathcal{E}_m \cup \mathcal{N}_m$, we will define the collections $\mathcal{E}_{m+1}(I)$, $\mathcal{N}_{m+1}(I)$, and $\mathcal{F}_{m+1}(I)$. We then set

$$\begin{aligned}\mathcal{E}_{m+1} &= \bigcup_{I \in \mathcal{E}_m \cup \mathcal{N}_m} \mathcal{E}_{m+1}(I), \\ \mathcal{N}_{m+1} &= \bigcup_{I \in \mathcal{E}_m \cup \mathcal{N}_m} \mathcal{N}_{m+1}(I), \\ \mathcal{F}_{m+1} &= \mathcal{F}_m \cup \bigcup_{I \in \mathcal{E}_m \cup \mathcal{N}_m} \mathcal{F}_{m+1}(I).\end{aligned}$$

Intervals in $\mathcal{F}_m$. If $I \in \mathcal{F}_m$, then set $f_{m+1}|I = f_m$.

Intervals in $\mathcal{E}_m$. Let $I \in \mathcal{E}_m$ and write $I = (a, b)$. By (P3) there exists $I' = (a', b') \in \mathcal{E}_m \setminus \{I\}$ such that $f_m|I' = (f_m|I) \circ \zeta_{I'}$ where $\zeta_{I'} : I' \rightarrow I$ is the unique linear orientation reversing map and there exists a word $v \in \mathcal{A}_n^m$ so that $f_m(I) = \phi_v^\eta(e_b)$ or $f_m(I) = \phi_v^\eta(e_l)$. If $f_m(I) = \phi_v^\eta(e_l)$, then, without loss of generality, we may assume that the y -coordinate of f_m is increasing inside I and decreasing inside I' . Let $\tilde{\theta}_I : I \rightarrow (0, s_{\eta(v), U})$ be linear, bijective, and increasing, and let $\tilde{\xi}_{I'} : I' \rightarrow (s_{\eta(v), U}, \tilde{t}_{\eta(v)})$ be linear, bijective, and decreasing. We define

$$f_{m+1}|I = \phi_v^\eta \circ \tilde{\tau}_{\eta(v)} \circ \tilde{\theta}_I \quad \text{and} \quad f_{m+1}|I' = \phi_v^\eta \circ \tilde{\tau}_{\eta(v)} \circ \tilde{\xi}_{I'}.$$

Define $\mathcal{E}_{m+1}(I) = \{\tilde{\theta}_I^{-1}(J) : J \in \mathcal{E}(\tilde{\tau}_{\eta(v)})\}$ and $\mathcal{E}_{m+1}(I') = \{\tilde{\xi}_{I'}^{-1}(J) : J \in \mathcal{E}(\tilde{\tau}_{\eta(v)})\}$. We define $\mathcal{N}_{m+1}(I)$, $\mathcal{N}_{m+1}(I')$, $\mathcal{F}_{m+1}(I)$, and $\mathcal{F}_{m+1}(I')$ in a similar manner. If $f_m(I) = \phi_v^\eta(e_b)$, then we proceed in a similar manner, interchanging $s_{\eta(v), U}$ with $s_{\eta(v), L}$ above.

If $J \in \mathcal{N}_{m+1}(I)$, then there is a character $j \in \mathcal{A}_n$ so that $f_{m+1}(J) = \phi_{vj}^\eta(\mathbf{0})$, and we define $\mathbf{w}_{m+1}(I) = vj$. Note that if $J, Q \in \mathcal{N}_{m+1}(I) \cup \mathcal{N}_{m+1}(I')$ are distinct, then $\mathbf{w}_{m+1}(J) \neq \mathbf{w}_{m+1}(Q)$ as $\tilde{\theta}_I(J) \neq \tilde{\theta}_I(Q)$ are distinct intervals in, for example, $\mathcal{N}(\tilde{\tau}_{\eta(v)})$ ($\tilde{\theta}_I$ may be $\tilde{\xi}_{I'}$ instead).

Intervals in $\mathcal{N}_m$. Let $I \in \mathcal{N}_m$ and write $I = [a, b] \subset [0, 1]$. We consider four cases.

Case 1. If the vertex $f_m(I)$ in T_m has an edge in T_m emanating to the right and no edge emanating upward, then define $\zeta_I : I \rightarrow [0, t_{\eta(\mathbf{w}_m(I))}]$ to be the unique linear increasing bijection, and let

$$f_{m+1}|I = \phi_{\mathbf{w}_m(I)}^\eta \circ \tau_{\eta(\mathbf{w}_m(I))} \circ \zeta_I.$$

Define $\mathcal{F}_{m+1}(I), \mathcal{E}_{m+1}(I), \mathcal{N}_{m+1}(I)$ in a manner similar to those for $I \in \mathcal{E}_m$ above.

Case 2. If the vertex $f_m(I)$ in V_m has an edge in T_m emanating upward and no edge emanating to the right, then define $\xi_I : I \rightarrow [t_{\eta(\mathbf{w}_m(I))}, 1]$ to be the unique increasing linear bijection, and let

$$f_{m+1}|I = \phi_{\mathbf{w}_m(I)}^\eta \circ \tau_{\eta(\mathbf{w}_m(I))} \circ \xi_I.$$

Define $\mathcal{F}_{m+1}(I), \mathcal{E}_{m+1}(I), \mathcal{N}_{m+1}(I)$ in a manner similar to those for $I \in \mathcal{E}_m$ above.

Case 3. If the vertex $f_m(I)$ is a leaf in T_m , then let $\sigma_I : I \rightarrow [0, 1]$ be the unique increasing linear bijection, and let

$$f_{m+1}|I = \phi_{\mathbf{w}_m(I)}^\eta \circ \tau_{\eta(\mathbf{w}_m(I))} \circ \sigma_I.$$

Define $\mathcal{F}_{m+1}(I), \mathcal{E}_{m+1}(I), \mathcal{N}_{m+1}(I)$ in a manner similar to those for $I \in \mathcal{E}_m$ above.

Case 4. If the vertex $f_m(I)$ has an edge in T_m emanating upward and another edge emanating to the right, then simply set $f_{m+1}|I = f_m|I$ and let $\mathcal{N}_{m+1}(I) = \{I\}$, $\mathcal{E}_{m+1}(I) = \emptyset$, and $\mathcal{F}_{m+1}(I) = \emptyset$.

By definition of $\tau_{\eta(\mathbf{w}_m(I))}$, for any $Q \in \mathcal{N}_{m+1}(I)$, there is a character $j \in \mathcal{A}_n$ so that $f_{m+1}(Q) = \{\phi_{\mathbf{w}_m(I)j}^\eta(\mathbf{0})\}$, and we set $\mathbf{w}_{m+1}(Q) = \mathbf{w}_m(I)j$.

We now prove (P1)–(P7) for f_{m+1} , $\mathcal{E}_{m+1}$, $\mathcal{N}_{m+1}$, $\mathcal{F}_{m+1}$, and $\mathbf{w}_{m+1}$.

We start with (P2). Since (P2) holds for $\mathcal{E}_m$, $\mathcal{N}_m$, and $\mathcal{F}_m$, and since intervals in $\mathcal{F}_{m+1}$ are not partitioned, it suffices to show that for every $I \in \mathcal{E}_m \cup \mathcal{N}_m$, the families $\mathcal{E}_{m+1}(I)$, $\mathcal{N}_{m+1}(I)$, and $\mathcal{F}_{m+1}(I)$ are pairwise disjoint with disjoint elements covering I . If $I \in \mathcal{E}_m$, then this holds by definitions of $\mathcal{E}(\tilde{\tau}_i)$, $\mathcal{N}(\tilde{\tau}_i)$, and $\mathcal{F}(\tilde{\tau}_i)$. If $I \in \mathcal{N}_m$, then this follows from the definitions of $\mathcal{E}(\tau_i)$, $\mathcal{N}(\tau_i)$, and $\mathcal{F}(\tau_i)$.

The first part of (P3) follows from the analogous properties defining $\tilde{\tau}_i$. The converse part of (P3) follows from (P3) and (P6) at m and by the definitions of $\tilde{\tau}_i$ and τ_i .

The first part of (P4) follows from design of $\mathcal{N}(\tau_i)$ and $\mathcal{N}(\tilde{\tau}_i)$ along with (P1) and (P4) at m . The converse part follows from (P1), (P3), (P4), and (P6) at m along with design of τ_i and $\tilde{\tau}_i$.

Property (P5) follows immediately from the definitions of $\mathcal{F}(\tau_i)$ and $\mathcal{F}(\tilde{\tau}_i)$.

The first part of (P6) follows from (P2) at m , as for every $I \in \mathcal{E}_m \cup \mathcal{N}_m$ and every $J \in \mathcal{E}_{m+1}(I) \cup \mathcal{N}_{m+1}(I)$, $J \subset I$, while every $J \in \mathcal{E}_{m+1} \cup \mathcal{N}_{m+1}$ is in $\mathcal{E}_{m+1}(I) \cup \mathcal{N}_{m+1}(I)$ for some $I \in \mathcal{E}_m \cup \mathcal{N}_m$. The converse part of (P6) is immediate from the definition of $\mathbf{w}_{m+1}(J)$ for $J \in \mathcal{N}_{m+1}(I)$ with $I \in \mathcal{N}_m$.

Property (P7) follows from properties (4), (5), and (6) in the definition of $\tilde{\tau}_i$.

Finally, we verify (P1). We first show that $\mathbf{w}_{m+1}$ is injective. Let $I, J \in \mathcal{N}_{m+1}$ be distinct intervals. Then by (P6), let $\tilde{I}, \tilde{J} \in \mathcal{E}_m \cup \mathcal{N}_m$ so that $I \in \mathcal{N}_{m+1}(\tilde{I})$, $J \in \mathcal{N}_{m+1}(\tilde{J})$. If $\tilde{I} = \tilde{J}$ are the same element of $\mathcal{E}_m$, then $\mathbf{w}_{m+1}(I) \neq \mathbf{w}_{m+1}(J)$ by definitions of $\mathcal{N}(\tilde{\tau}_i)$. If $\tilde{I} = \tilde{J}$ are the same element of $\mathcal{N}_m$, then by (P6) there are characters $i, j \in \mathcal{A}_n$ with $\mathbf{w}_{m+1}(I) = \mathbf{w}_m(\tilde{I})i$, $\mathbf{w}_{m+1}(J) = \mathbf{w}_m(\tilde{I})j$. Then by the definitions of $\mathcal{N}(\tau_i)$, we have that $i \neq j$. If $\tilde{I} \neq \tilde{J}$ are distinct intervals, both in $\mathcal{N}_m$, then by (P6) at $m+1$ and (P1) at m , we have that $\mathbf{w}_{m+1}(I)(m) \neq \mathbf{w}_{m+1}(J)(m)$, so $\mathbf{w}_{m+1}(I) \neq \mathbf{w}_{m+1}(J)$. If $\tilde{I} \neq \tilde{J}$ with $\tilde{J} \in \mathcal{E}_m$ and $\tilde{I} \in \mathcal{N}_m$, then by definitions of τ_i

and $\tilde{\tau}_i$, $f_{m+1}(\tilde{I}) \cap f_{m+1}(\tilde{J}) = \emptyset$, so by (P3) and (P4), $\mathbf{w}_{m+1}(I) \neq \mathbf{w}_{m+1}(J)$. Similarly $\mathbf{w}_{m+1}(I) \neq \mathbf{w}_{m+1}(J)$ if $\tilde{I} \neq \tilde{J}$ with $\tilde{J} \in \mathcal{E}_m$ and $\tilde{I} \in \mathcal{E}_m \setminus \{\tilde{J}, \tilde{J}'\}$. If $\tilde{J} \in \mathcal{E}_m$ and $\tilde{I} = \tilde{J}'$, then $\mathbf{w}_{m+1}(I) \neq \mathbf{w}_{m+1}(J)$ by definition of $\mathcal{N}(\tilde{\tau}_i)$.

To conclude the proof, we show that $\mathbf{w}_{m+1} : \mathcal{N}_{m+1} \rightarrow \mathcal{A}_n^{m+1}$ is surjective. Let $u \in \mathcal{A}_n^{m+1}$ be a word. By (P1) at m , there exists a unique interval $I \in \mathcal{N}_m$ so that $\mathbf{w}_m(I) = u(m)$. If $f_m(I)$ is a leaf in T_m , then by definition of $\mathcal{N}(\tau_i)$, for each $j \in \mathcal{A}_n$ there is an interval $J \in \mathcal{N}_{m+1}(I)$ so that $\mathbf{w}_{m+1}(J) = \mathbf{w}_{m+1}(I)j$, in particular one of them has $\mathbf{w}_{m+1}(J) = u$. If $f_m(I)$ is not a leaf in T_m , then the result follows by definitions of $\mathcal{N}(\tilde{\tau}_i)$ and $\mathcal{N}(\tau_i)$ as well as (P3) at m . $\square$

Throughout the remainder of this section, when we write any of (P1)–(P7) we mean the appropriate property (P1)–(P7) from Lemma 3.7.11.

Remark 3.8. *It is possible that $\mathcal{N}_{m+1} \cap \mathcal{N}_m \neq \emptyset$ for some $m \in \mathbb{N}$. Define a map $\mathbf{w} : \bigcup_{m \geq 0} \mathcal{N}_m \rightarrow \mathcal{A}_n^*$ as follows. Set $\mathbf{w}([0, 1]) = \varepsilon$. If $m \in \mathbb{N}$ and $I \in \mathcal{N}_m \setminus \mathcal{N}_{m-1}$, set $\mathbf{w}(I) = \mathbf{w}_m(I)$. If $m \in \mathbb{N}$ and $I \in \mathcal{N}_m \cap \mathcal{N}_{m-1}$, set $\mathbf{w}(I) = \mathbf{w}_{m-k}(I)$, where $k \in \mathbb{N}$ is the greatest integer so that $I \in \mathcal{N}_{m-k}$. Note that $\mathbf{w}$ is injective and $V_m = \{\phi_{\mathbf{w}(I)}^\eta(\mathbf{0}) : I \in \mathcal{N}_m\}$ for every $m \in \mathbb{N}$.*

Corollary 3.8.1. *The maps $(f_m)_{m \geq 0}$ of Lemma 3.7.11 converge uniformly to a continuous surjection $f : [0, 1] \rightarrow K^\eta$.*

Proof. Since $T_m \subset K^\eta$, the Hausdorff distance

$$\text{dist}_H(T_m, K^\eta) \leq \text{dist}_H(V_m, K^\eta) \leq \max_{w \in \mathcal{A}_n^m} \text{diam } K_w^\eta \leq n^{-m} \sqrt{2}.$$

Therefore, $\overline{\bigcup_{m \in \mathbb{N}} T_m} = K^\eta$, as K^η is compact.

Next we claim that for every $m \in \mathbb{N}$,

$$\|f_m - f_{m+1}\|_\infty \leq n^{-m} \sqrt{2}. \quad (3.8.1)$$

Fix $m \in \mathbb{N}$ and $x \in [0, 1]$. By (P1), there exists $I \in \mathcal{N}_m \cup \mathcal{E}_m \cup \mathcal{F}_m$ so that $x \in I$. If $I \in \mathcal{F}_m$, then by (P5) we have that $f_m(x) = f_{m+1}(x)$. If $I \in \mathcal{N}_m$, then by (P4) we have that $f_m(x), f_{m+1}(x) \in \phi_{\mathbf{w}_m(I)}^\eta([0, 1]^2)$. Finally, if $I \in \mathcal{E}_m$, then by (P3) we have that there is some word $v \in \mathcal{A}_n^m$ such that $f_m(x), f_{m+1}(x) \in \phi_v^\eta([0, 1]^2)$. In any case, we have the claimed estimate.

Hence, the functions f_m converge uniformly to a continuous map $f : [0, 1] \rightarrow [0, 1]^2$. Furthermore, because the dendrites $T_m \subset K^\eta$ are nested and $f_m : [0, 1] \rightarrow T_m$ are surjective, we have that the uniform limit f is a continuous parametrization of K^η , as desired. $\square$

Lemma 3.8.1. *For each $m \in \mathbb{N}$ and every pair of distinct intervals $I, J \in \mathcal{E}_m \cup \mathcal{N}_m$, we have that $\mathcal{H}^{\alpha_n}(f(I) \cap f(J)) = 0$.*

Before proving the lemma we recall the well-known definition of porosity. Given a metric space X , we say that a set $Y \subset X$ is *porous* in X if there exists $c > 1$ such that for any $r < \text{diam } X$ and any $x \in X$, there exists $y \in B(x, r)$ such that $B(y, c^{-1}r) \subset B(x, r) \setminus Y$.

Proof of Lemma 3.8.1. We claim that $f(I) \cap f(J)$ is porous in K^η . Assuming the claim, since K^η is Ahlfors α_n -regular by Proposition 3.2, it follows from (Bandt et al., 2006, Lemma 3.12) that the Hausdorff dimension (in fact, the Assouad dimension) of $f(I) \cap f(J)$ is less than α_n which gives the lemma.

To prove porosity, note first that by (P3), (P4), and (P6) there exists a word $v \in \mathcal{A}_n^m$ such that $f(I) \cap f(J) \subset K_v^\eta$. Fix $x \in K^\eta$ and $r \in (0, n^{-m}]$. There exists an integer $k_0 \in \mathbb{N}$ such that $n^{-k_0} \leq r < n^{-k_0+1}$, and so there exists $w \in \mathcal{A}_n^{k_0+1}$ with $K_w^\eta \subset \overline{B}(x, r) \cap K^\eta$.

We now prove that there are characters $i, j \in \mathcal{A}_n$ such that $f(I) \cap f(J) \cap K_{wji}^\eta = \emptyset$. This falls to a case study.

Case 1. Suppose that $I, J \in \mathcal{N}_m$. Then $f(I) \cap \phi_{\mathbf{w}_m(J)}^\eta((0, 1)^2) = \emptyset$ and $f(J) \cap \phi_{\mathbf{w}_m(I)}^\eta((0, 1)^2) = \emptyset$ by Lemma 3.7.4, (P4), and the fact that $\mathbf{w}_m(I), \mathbf{w}_m(J)$ are distinct words of length m . Moreover, at least one of $\phi_w^\eta((0, 1)^2) \cap \phi_{\mathbf{w}_m(I)}^\eta([0, 1]^2)$

and $\phi_w^\eta((0,1)^2) \cap \phi_{\mathbf{w}_m(J)}^\eta([0,1]^2)$ is empty. Since there are $i, j \in \mathcal{A}_n$ for which $K_{wji}^\eta \subset \phi_w^\eta((0,1)^2)$, we have $f(I) \cap f(J) \cap K_{wji}^\eta = \emptyset$.

Case 2. Suppose that at least one of I, J is in $\mathcal{E}_m$. By (P3), (P4), and (P6), there exist words $u, v \in \mathcal{A}_n^m$ such that $f(J) \subset K_u^\eta$ and $f(I) \subset K_v^\eta$.

If $u \neq v$, then we proceed as in Case 1 with u playing the role of $\mathbf{w}_m(J)$ and v playing the role of $\mathbf{w}_m(I)$. If $f(I)$ or $f(J)$ is a singleton, then the existence of such characters $i, j \in \mathcal{A}_n$ is clear.

Therefore, we may assume for the rest of Case 2 that $u = v$ and that neither of $f(I), f(J)$ is a singleton. If $f_{k_0+1}(J) \cap K_w^\eta = \emptyset$, then $f(J) \cap \phi_w^\eta((0,1)^2) = \emptyset$ by Lemma 3.7.4 and by (P3), (P4), and (P5), so the existence of such characters $i, j \in \mathcal{A}_n$ is clear. Similarly if $f_{k_0+1}(I) \cap K_w^\eta = \emptyset$. Now assume that $f_{k_0+1}(I) \cap K_w^\eta \neq \emptyset$ and $f_{k_0+1}(J) \cap K_w^\eta \neq \emptyset$. Then there are distinct leaves $p_1, p_2 \in T_{k_0+2}$ with $p_1, p_2 \in K_w^\eta$ and distinct intervals $Q_1, Q_2 \in \mathcal{N}_{k_0+2}$ such that $Q_1 \subset J$ and $Q_2 \subset I$ satisfy $f_{k_0+2}|_{Q_1} \equiv p_1$ and $f_{k_0+2}|_{Q_2} \equiv p_2$. Then $f([0,1] \setminus Q_1) \cap \phi_{\mathbf{w}_{k_0+2}(Q_1)}^\eta((0,1)^2) = \emptyset$ and $\mathbf{w}(Q_1)(k_0+1) = w$, so there is a pair of characters $i, j \in \mathcal{A}_n$ so that $wj = \mathbf{w}_{k_0+2}(Q_1)$ and $K_{wji}^\eta \subset \phi_{w_j}^\eta((0,1)^2)$. Since I, J are distinct, and thus disjoint, we have that $f(I) \cap \phi_{w_j}^\eta((0,1)^2) = \emptyset$, concluding Case 2.

Additionally, there exists a point $y \in K_{wji}^\eta$ with $\overline{B}(y, n^{-k_0-5}) \cap K^\eta \subset K_{wji}^\eta$. Thus, for every $r \in (0, n^{-m}]$ and each $x \in K^\eta$, there exists $y \in K^\eta$ such that

$$\overline{B}(y, n^{-5}r) \cap K^\eta \subset K^\eta \cap (\overline{B}(x, r) \setminus (f(I) \cap f(J))). \quad \square$$

For the rest of this section, f is the uniform limit of the maps f_m from Corollary 3.8.1, ν_n is the probability measure from §3.4.2, and $\pi : \mathcal{A}_n^\mathbb{N} \rightarrow K^\eta$ is the map from (3.7.1).

Lemma 3.8.2. *For every $m \in \mathbb{N}$ and every $J \in \mathcal{E}_m$, $\pi_\eta \# \nu_n(f(J)) \geq (5n-6)^{-m-1}$.*

Proof. Note that by (P7), there is some $I \in \mathcal{N}_{m+1}$ so that $f_{m+1}(I)$ is a leaf in T_{m+1} and $I \subset J$. Then since $f_{m+1}(I) = \phi_{\mathbf{w}(I)}^\eta(\mathbf{0})$ is a leaf in T_{m+1} , we have that

$$f_{m+1}([0, 1] \setminus I) \cap \phi_{\mathbf{w}(I)}^\eta(T^{\eta(\mathbf{w}(I))}) = \emptyset.$$

We claim that $f_k([0, 1] \setminus I) \cap \phi_{\mathbf{w}(I)}^\eta((0, 1)^2) = \emptyset$ for every integer $k \geq m+1$. To this end, fix an integer $k \geq m+1$ and let $t \in [0, 1] \setminus I$. By Lemma 3.7.4, it is sufficient to show that there exists a word $u \in \mathcal{A}_n^{m+1}$ different from $\mathbf{w}(I)$ such that $f_k(t) \in \phi_u^\eta([0, 1]^2)$. By (P2), there exists a unique interval $Q \in (\mathcal{E}_{m+1} \cup \mathcal{F}_{m+1} \cup \mathcal{N}_{m+1}) \setminus \{I\}$ with $t \in Q$.

If $Q \in \mathcal{F}_{m+1}$, then by (P5) there exists a word $u \in \mathcal{A}_n^{m+1}$ so that

$$f_k(t) = f_{m+1}(t) = \phi_u^\eta(\mathbf{0}).$$

Then since

$$f_{m+1}([0, 1] \setminus I) \cap \phi_{\mathbf{w}(I)}^\eta(T^{\eta(\mathbf{w}(I))}) = \emptyset,$$

we have that $u \neq \mathbf{w}(I)$.

If $Q \in \mathcal{E}_{m+1}$, then by (P3) and Lemma 3.7.10, there exists a word $u \in \mathcal{A}_n^{m+1}$ so that either $f_{m+1}(Q) = \phi_u^\eta(e_b)$ or $f_{m+1}(Q) = \phi_u^\eta(e_l)$. In either case, by (P3), (P4), and (P5) we have that $f_k(Q) \subset \phi_u^\eta([0, 1]^2)$. Note that $\phi_{\mathbf{w}(I)}^\eta(e_l)$ emanates from $\phi_{\mathbf{w}(I)}^\eta(\mathbf{0})$ upward and $\phi_{\mathbf{w}(I)}^\eta(e_b)$ emanates from $\phi_{\mathbf{w}(I)}^\eta(\mathbf{0})$ to the right. Since $\phi_{\mathbf{w}(I)}^\eta(\mathbf{0})$ is a leaf in T_{m+1} , by Lemma 3.7.9 we must have that $u \neq \mathbf{w}(I)$.

Finally, if $Q \in \mathcal{N}_{m+1} \setminus \{I\}$, then by (P1), $\mathbf{w}_{m+1}(Q) \neq \mathbf{w}(I)$. By (P3), (P4), and (P5), $f_k(Q) \subset \phi_{\mathbf{w}_{m+1}(Q)}^\eta([0, 1]^2)$. Therefore, $f_k(t) \notin \phi_{\mathbf{w}(I)}^\eta((0, 1)^2)$, completing the proof of the claim.

Thus, if $J \in \mathcal{E}_m$, then there is an interval $I \in \mathcal{N}_{m+1}$ with $I \subset J$ and this I satisfies

$$f([0, 1] \setminus I) \cap \phi_{\mathbf{w}(I)}^\eta((0, 1)^2) = \emptyset,$$

so $\phi_{\mathbf{w}(I)}^\eta((0,1)^2) \cap K^\eta \subset f(I)$. Since I is compact, we have $K_{\mathbf{w}(I)}^\eta \subset f(I)$, so $K_{\mathbf{w}(I)}^\eta \subset f(J)$ and $|\mathbf{w}(I)| = m+1$. Therefore, $\pi_\eta \# \nu_n(f(J)) \geq (5n-6)^{-m-1}$. $\square$

We are now ready to prove Proposition 3.3.

Proof of Proposition 3.3. Let $f_m, \mathcal{E}_m, \mathcal{N}_m, \mathcal{F}_m$, and $\mathbf{w}$ be as in Lemma 3.7.11 and Remark 3.8. There exists a non-decreasing surjection $\zeta : [0,1] \rightarrow [0,1]$ so that

$$|\zeta(J)| = \pi_\eta \# \nu_n(f(J)), \quad \text{for every integer } m \geq 0 \text{ and each } J \in \mathcal{E}_m \cup \mathcal{N}_m \cup \mathcal{F}_m.$$

Let $\mathcal{F}_\infty = \bigcup_{m \geq 0} \mathcal{F}_m$ and let $\mathcal{L}_\infty$ be the collection of intervals $I \in \bigcup_{m \geq 0} \mathcal{N}_m$ so that $\phi_{\mathbf{w}(I)}^\eta(\mathbf{0})$ has an edge emanating upward and an edge emanating to the right in $T_{|\mathbf{w}(I)|}$. Note that $\mathcal{L}_\infty = \bigcup_{m \geq 0} \{I \in \mathcal{N}_m : |\mathbf{w}(I)| < m\}$. By (P7) and by design of the families $\mathcal{F}_m, \mathcal{E}_m$, and $\mathcal{N}_m$ we have that the restriction of ζ on $[0,1] \setminus \bigcup(\mathcal{F}_\infty \cup \mathcal{L}_\infty)$ is strictly increasing and injective.

For any interval $J \in \mathcal{F}_\infty \cup \mathcal{L}_\infty$, let p_J be the unique element of $\zeta(J)$ and let q_J be the unique element of $f(J)$. Define the function $F : [0,1] \rightarrow K^\eta$ by

$$F(t) = \begin{cases} f \circ \zeta^{-1}(t), & \text{if } t \in \zeta([0,1] \setminus \bigcup(\mathcal{F}_\infty \cup \mathcal{L}_\infty)), \\ q_J, & \text{if } t \in \zeta(J) \text{ where } J \in \mathcal{F}_\infty \cup \mathcal{L}_\infty. \end{cases}$$

To complete the proof, it remains to show that F is $(1/\alpha_n)$ -Hölder continuous. To this end, fix distinct points $x, y \in [0,1]$ and assume that $x < y$, and let $m \geq 0$ be the maximal integer so that there exists an interval $J \in \mathcal{E}_m \cup \mathcal{N}_m$ with $x, y \in \overline{\zeta(J)}$. The remainder of the proof falls to a case study.

Case 1. If there is an interval $I \in \mathcal{E}_{m+1}(J)$ so that x and y are separated by $\zeta(I)$, then by (P3) and (P4), $|F(x) - F(y)| \leq n^{-m} \sqrt{2}$ and $|x - y| \geq (5n-6)^{-m-2}$, so $|F(x) - F(y)| \leq n^2 \sqrt{2} |x - y|^{\alpha_n}$.

Case 2. If there is an interval $I \in \mathcal{N}_{m+1}(J) \setminus \mathcal{L}_\infty$ so that $\zeta(I)$ separates x and y , then proceeding similarly as in Case 1. we obtain $|F(x) - F(y)| \leq n^3 \sqrt{2} |x - y|^{\alpha_n}$.

Case 3. If neither Case 1 nor Case 2 holds, then there exist intervals $I_1, I_2 \in \mathcal{E}_{m+1}(J) \cup (\mathcal{N}_{m+1}(J) \setminus \mathcal{L}_\infty)$ so that $x \in \overline{\zeta(I_1)}$, $y \in \overline{\zeta(I_2)}$. Then the intersection $\overline{\zeta(I_1)} \cap \overline{\zeta(I_2)} = \{p\}$ for some p , so we may apply Case 1 to the pairs x, p and p, y to obtain $|F(x) - F(y)| \leq n^3 \sqrt{2} |x - y|^{\alpha_n}$. $\square$

3.9 Random choice functions

Fix an even integer $n \geq 4$. Note that the set $\mathcal{C}_n$ of choice functions $\eta : \mathcal{A}_n^* \rightarrow \{1, 2\}$ can be identified with $\{1, 2\}^{\mathcal{A}_n^*}$ which is a countable product of finite sets. Then by (Stroock, 1993, §3.1), there exists a probability measure μ_n on $\mathcal{C}_n$ such that for any $w \in \mathcal{A}_n^*$

$$\mu_n(\{\eta \in \mathcal{C}_n : \eta(w) = 1\}) = \mu_n(\{\eta \in \mathcal{C}_n : \eta(w) = 2\}) = \frac{1}{2}.$$

We say that a choice function $\eta \in \mathcal{C}_n$ is a *random choice function* if ν_n -a.e. word $w \in \mathcal{A}_n^{\mathbb{N}}$ satisfies the following property: for each $N \in \mathbb{N}$ and every integer $k \geq 0$, there exists $\ell \in \mathbb{N}$ depending on w, k, N such that

(R1) the restrictions

$$\eta \left| \bigcup_{j=0}^{k-1} \mathcal{A}_{n, w(\ell)}^{\ell+j} \right. = 1 \quad \text{and} \quad \eta \left| \bigcup_{j=0}^{2N+k-1} \mathcal{A}_{n, w(\ell-N)}^{\ell-N+j} \setminus \bigcup_{j=0}^{k-1} \mathcal{A}_{n, w(\ell)}^{\ell+j} \right. = 2,$$

(R2) the character $i_{\ell-N+1} > 4n - 4$.

For completeness, if $k = 0$, then the union $\bigcup_{j=0}^{k-1} \mathcal{A}_{n, w(\ell)}^{\ell+j}$ is taken over an empty set thus we regard the union itself to be empty, and so we consider only the second restriction of (R1) in this case. See Figure 3.4 for a schematic representation of property (R1).

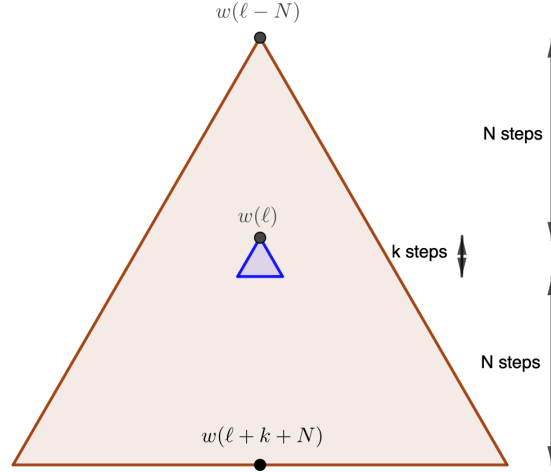


Figure 3.4: The big triangle represents all words in $\mathcal{A}_n^*$ that start with $w(\ell - N)$ and have at most $\ell + k + N$ letters. Words of the same the height have the same number of letters. The part of this set where η takes the value 2 is colored by red, and the part of the set where η takes the value 1 is colored by blue.

To simplify our notation, for a word $w \in \mathcal{A}_n^{\mathbb{N}}$, a function $\eta \in \mathcal{C}_n$, and two integers $N \geq 1, k \geq 0$, we say (η, w, N, k) satisfies (R1), (R2) if there exists $\ell \in \mathbb{N}$ for which (R1) and (R2) hold. For a word $w \in \mathcal{A}_n^{\mathbb{N}}$ and a function $\eta \in \mathcal{C}_n$, we say that (η, w) satisfies (R1) and (R2) if for every pair of integers $N \geq 1, k \geq 0$, we have that (η, w, N, k) satisfies (R1) and (R2). Thus, a choice function η is random if for ν_n -a.e. $w \in \mathcal{A}_n^{\mathbb{N}}$, (η, w) satisfies (R1) and (R2).

The goal of this section is to show that random choice functions exist. In fact, we prove the following stronger statement.

Proposition 3.4. *For every even integer $n \geq 4$, μ_n -a.e. $\eta \in \mathcal{C}_n$ is random.*

Since $\pi_\eta \# \nu_n$ and $\mathcal{H}^{\alpha_n} \sqsubset K^\eta$ have the same null sets in K^η for every $\eta \in \mathcal{C}_n$, Proposition 3.4 yields immediately the following corollary.

Corollary 3.8.2. *For every even integer $n \geq 4$, there exists $\eta \in \mathcal{C}_n$ such that $\mathcal{H}^{\alpha_n}$ -almost every point $x \in K^\eta$ has $\pi_\eta^{-1}(\{x\}) = \{w\}$ and (η, w) satisfies (R1) and (R2).*

Before proving Proposition 3.4, we establish some notation. For $w \in \mathcal{A}_n^{\mathbb{N}}$ define

$$D^w := \{\eta \in \mathcal{C}_n : (\eta, w) \text{ satisfies (R1) and (R2)}\}$$

and for $\eta \in \mathcal{C}_n$ define

$$C_\eta := \{w \in \mathcal{A}_n^\mathbb{N} : (\eta, w) \text{ satisfies (R1) and (R2)}\}.$$

Define also

$$\Delta_n := \{(\eta, w) \in \mathcal{C}_n \times \mathcal{A}_n^\mathbb{N} : w \in C_\eta\} = \{(\eta, w) \in \mathcal{C}_n \times \mathcal{A}_n^\mathbb{N} : \eta \in D^w\}.$$

Proof of Proposition 3.4. Using the notation above, we claim that $\mu_n(D^w) = 1$ for ν_n -almost every word $w \in \mathcal{A}_n^\mathbb{N}$. Assuming the claim, by Fubini's theorem we obtain

$$\int_{\mathcal{C}_n} \nu_n(C_\eta) d\mu_n(\eta) = (\mu_n \times \nu_n)(\Delta_n) = \int_{\mathcal{A}_n^\mathbb{N}} \mu_n(D^w) d\nu_n(w) = 1.$$

Therefore, $\nu_n(C_\eta) = 1$ for μ_n -a.e. $\eta \in \mathcal{C}_n$ which implies that μ_n -a.e. $\eta \in \mathcal{C}_n$ is random.

For the proof of the claim, recall that

$$\nu_n(\{i_1 i_2 \cdots \in \mathcal{A}_n^\mathbb{N} : i_j > 4n - 4 \text{ infinitely often}\}) = 1,$$

so it suffices to show that the claim holds for every word w in the above set.

Fix $w = i_1 i_2 \cdots \in \mathcal{A}_n^\mathbb{N}$ such that $i_j > 4n - 4$ infinitely often. Fix also integers $N \geq 1$ and $k \geq 0$. We construct a sequence $(M_m)_{m \in \mathbb{N}} \subset \mathbb{N}$ in an inductive fashion. Let $M_1 \in \mathbb{N}$ be such that $i_{M_1+1} > 4n - 4$. Assuming that we have defined M_m for some $m \in \mathbb{N}$, let $M_{m+1} \in \mathbb{N}$ such that $M_{m+1} > M_m + 2N + k + 1$ and $i_{M_{m+1}+1} > 4n - 4$.

For each $m \in \mathbb{N}$ let $E_{m,w}$ be the set of all choice functions $\eta \in \mathcal{C}_n$ such that (η, w, N, k) satisfies (R1) and (R2) with $\ell = M_m + N$. Since $i_{M_{m+1}+1} > 4n - 4$ for each $m \in \mathbb{N}$, (R2) is satisfied for $\ell = M_m + N$. For each $m \in \mathbb{N}$, let

$$B_m = \bigcup_{j=0}^{2N+k-1} \mathcal{A}_{n,w(M_m)}^{M_m+j},$$

that is, the set of all finite words u starting with $w(M_m)$ and having length at most $M_m + 2N + k - 1$. Define also for each $m \in \mathbb{N}$ the function $\xi_m : B_m \rightarrow \{1, 2\}$ via

$$\xi_m \left| \bigcup_{j=0}^{k-1} \mathcal{A}_{n,w(M_m)}^{M_m+j} \right| = 1 \quad \text{and} \quad \xi_m \left| B_m \setminus \bigcup_{j=0}^{k-1} \mathcal{A}_{n,w(M_m)}^{M_m+j} \right| = 2.$$

Then $E_{m,w} = \{\eta \in \mathcal{C}_n : \eta|_{B_m} = \xi_m\}$ and $|B_m| = \frac{(5n-6)^{2N+k-1}}{5n-7}$ for every $m \in \mathbb{N}$. By uniformity of the measure μ_n , we have that for every $m \in \mathbb{N}$, $\mu_n(E_{m,w}) = 2^{-|B_m|}$, in particular each set $E_{m,w}$ has the same measure independent of m , and that measure is positive.

Next we claim that the sets $(E_{m,w})_{m \in \mathbb{N}}$ are independent. To see this, let $\{E_{m_1,w}, \dots, E_{m_j,w}\}$ be a finite collection of the sets $(E_{m,w})_{m \in \mathbb{N}}$, and note that

$$\bigcap_{i=1}^j E_{m_i,w} = \{\eta \in \mathcal{C}_n : \eta|_{B_{m_i}} = \xi_{m_i} \text{ for } i \in \{1, \dots, j\}\}.$$

Since the sets $\{B_m\}_{m \in \mathbb{N}}$ are pairwise disjoint as $M_{m+1} - M_m > 2N + k + 1$ for each $m \in \mathbb{N}$, we have that $|\bigcup_{i=1}^j B_{m_i}| = \sum_{i=1}^j |B_{m_i}|$ and $\mu_n(\bigcap_{i=1}^j E_{m_i,w}) = 2^{-|\bigcup_{i=1}^j B_{m_i}|}$. Thus,

$$\mu_n(\bigcap_{i=1}^j E_{m_i,w}) = \prod_{i=1}^j \mu_n(E_{m_i,w}),$$

so the sets $(E_{m,w})_{m \in \mathbb{N}}$ are independent.

Furthermore, each set $E_{m,w}$ has the same positive measure, so

$$\sum_{m=1}^{\infty} \mu_n(E_{m,w}) = \infty.$$

Thus, by the second Borel-Cantelli Lemma,

$$\mu_n(\bigcap_{j=1}^{\infty} \bigcup_{m=j}^{\infty} E_{m,w}) = 1,$$

or equivalently, for μ_n -almost every choice function $\eta \in \mathcal{C}_n$, (η, w, N, k) satisfies (R1) and (R2) infinitely often. In particular, for every pair of integers $N \geq 1, k \geq 0$ we have that for μ_n -almost every $\eta \in \mathcal{C}_n$, (η, w, N, k) satisfies (R1) and (R2). Since there are countably many choices of N and k , we have immediately that $\mu_n(D^w) = 1$ for ν_n -almost every word $w \in \mathcal{A}_n^{\mathbb{N}}$. $\square$

3.10 Tangents of carpets K^η at typical points

Fix for the rest of this section an even integer $n \geq 4$. Recall the definitions of choice functions $\eta \in \mathcal{C}_n$ and carpets K^η from Section 3.6, the number $\alpha_n = \frac{\log(5n-6)}{\log(n)}$ from Section 3.7, and the definition of random choice functions from Section 3.9.

In this section we prove the following result about tangents of carpets K^η at typical points when η is random.

Proposition 3.5. *If $\eta \in \mathcal{C}_n$ is a random choice function, then for $\mathcal{H}^{\alpha_n}$ -a.e. $x \in K^\eta$ there exist $T_{n,0}, T_{n,1}, T_{n,2}, \dots \in \text{Tan}(K^\eta, x)$ such that $T_{n,k}$ has exactly $\frac{1}{5n-7}((5n-6)^k - 1)$ many cut points.*

In §3.10.1 we study the local cut-points of a certain class of “almost self-similar” carpets and in §3.10.2, we relate the tangents of K^η with these carpets.

3.10.1 Local cut-points in a class of carpets

Define, for an integer $k \geq 0$, the set

$$K^{n,k} := \bigcup_{w \in \mathcal{A}_n^k} \psi_{n,w}^1 \left(\bigcap_{m=1}^{\infty} \bigcup_{v \in \mathcal{A}_n^m} \psi_{n,v}^2([0, 1]^2) \right) \quad (3.10.1)$$

where maps $\psi_{n,v}^i$ and ϕ_v^η for $v \in \mathcal{A}_n^*$ and $i \in \{1, 2\}$ are as in Section 3.6.

Roughly speaking, $K^{n,k}$ is obtained by first iterating k times the maps from the system $\mathcal{F}_n^1$, and then applying these iterates to the self-similar attractor of the IFS of similarities $\mathcal{F}_n^2$ (recall the systems $\mathcal{F}_n^1, \mathcal{F}_n^2$ from Section 3.6).

The next lemma is the main result of this subsection.

Lemma 3.8.3. *For each integer $k \geq 0$, $K^{n,k}$ has exactly $\frac{1}{5n-7}((5n-6)^k - 1)$ many local cut-points.*

The proof of Lemma 3.8.3 is by induction on k . The base case $k = 0$ is given in Lemma 3.9.3. In the proof, we make use of the following definition.

Let $\{f_j(x) = Lx + b_j\}_{j=1}^l$ with $b_j \in \mathbb{R}^2$ be similarities on $\mathbb{R}^2$ where $L > 0$ and $l \geq 2$ such that

1. if $i, j \in \{1, \dots, l\}$ are distinct, then, $f_i((0, 1)^2) \cap f_j((0, 1)^2) = \emptyset$, and
2. $\bigcap_{i=1}^l f_i([0, 1]^2) \neq \emptyset$.

We say that $f_1, \dots, f_l$ *meet at edges of* $[0, 1]^2$ if for every $i \in \{1, \dots, l\}$, there exists $j \in \{1, \dots, l\} \setminus \{i\}$ so that $f_i([0, 1]^2) \cap f_j([0, 1]^2)$ is an edge of $f_i([0, 1]^2)$. Note that necessarily $l \in \{2, 3, 4\}$.

Remark 3.9. *Recall from the proof of Lemma 3.7.5 that for any $k \in \mathbb{N}$,*

$$\begin{aligned} \bigcup_{j \in \mathcal{A}_n} \psi_{n,j}^2(\partial[0, 1]^2) &\subset K^{n,0} \subset \bigcup_{j \in \mathcal{A}_n} \psi_{n,j}^2([0, 1]^2) \\ \bigcup_{j \in \mathcal{A}_n} \psi_{n,j}^1(\partial[0, 1]^2) &\subset K^{n,k} \subset \bigcup_{j \in \mathcal{A}_n} \psi_{n,j}^1([0, 1]^2). \end{aligned}$$

Note also that if there exist distinct $i, j \in \mathcal{A}_n$ and $x \in [0, 1]^2$ with $x \in \psi_{n,j}^2([0, 1]^2) \cap \psi_{n,i}^2([0, 1]^2)$, then the maps in $\{\psi_{n,l}^2 : x \in \psi_{n,l}^2([0, 1]^2), l \in \mathcal{A}_n\}$ meet at edges of $[0, 1]^2$. Note further that if there exist distinct $i, j \in \mathcal{A}_n$ and $x \in [0, 1]^2$ with $x \in \psi_{n,j}^2([0, 1]^2) \cap \psi_{n,i}^1([0, 1]^2)$, then either $x = (1/2, 2/n)$ or the functions in the set $\{\psi_{n,l}^1 : x \in \psi_{n,l}^1([0, 1]^2), l \in \mathcal{A}_n\}$ meet at edges of $[0, 1]^2$. See for example Figure 3.2 for the case $n = 6$.

Lemma 3.9.1. *Let $\eta \in \mathcal{C}_n$, let $m \in \mathbb{N}$, let $u, v, w \in \mathcal{A}_n^m$ be distinct, and let*

$$x \in \phi_u^\eta([0, 1]^2) \cap \phi_v^\eta([0, 1]^2) \cap \phi_w^\eta([0, 1]^2).$$

Then the maps $\{\phi_\beta^\eta : x \in \phi_\beta^\eta([0, 1]^2), \beta \in \mathcal{A}_n^m\}$ meet at edges of $[0, 1]^2$.

Proof. Recall that for all $\beta \in \mathcal{A}_n^m$, $\phi_\beta^\eta(x) = n^{-m}y + \phi_\beta^\eta(\mathbf{0})$ with $\phi_\beta^\eta(\mathbf{0}) \in \{0, \frac{1}{n^m}, \dots, \frac{n^m-1}{n^m}\}^2$.

Let $E = \{\beta \in \mathcal{A}_n^m : x \in \phi_\beta^\eta([0, 1]^2)\}$ which, by assumption, contains at least three words. Clearly the intersection $\bigcap_{\beta \in E} \phi_\beta^\eta([0, 1]^2)$ is nonempty as it contains x .

Suppose that $a, b \in E$ are distinct words so that ϕ_a^η, ϕ_b^η do not meet at edges of $[0, 1]^2$. Then $\phi_a^\eta(\mathbf{0}) - \phi_b^\eta(\mathbf{0}) \in \{-n^{-m}, n^{-m}\}^2$. Since $\text{card}(E) \geq 3$, there exists $c \in E \setminus \{a, b\}$ with

$$\phi_a^\eta(\mathbf{0}) - \phi_c^\eta(\mathbf{0}) \in \{-n^{-m}, 0, n^{-m}\}^2 \quad \text{and} \quad \phi_b^\eta(\mathbf{0}) - \phi_c^\eta(\mathbf{0}) \in \{-n^{-m}, 0, n^{-m}\}^2.$$

If, for example, $\phi_a^\eta(\mathbf{0}) - \phi_c^\eta(\mathbf{0}) \in \{-n^{-m}, n^{-m}\}^2$, then

$$\phi_c^\eta(\mathbf{0}) - \phi_b^\eta(\mathbf{0}) = (\phi_c^\eta(\mathbf{0}) - \phi_a^\eta(\mathbf{0})) + (\phi_a^\eta(\mathbf{0}) - \phi_b^\eta(\mathbf{0})) \in \{-2n^{-m}, 0, 2n^{-m}\}^2 \cap \{-n^{-m}, 0, n^{-m}\}^2.$$

But then $\phi_c^\eta(\mathbf{0}) = \phi_b^\eta(\mathbf{0})$, and this is a contradiction. Hence, $\phi_a^\eta(\mathbf{0}) - \phi_c^\eta(\mathbf{0}) \notin \{-n^{-m}, n^{-m}\}^2$, and by a simple calculation we get $\phi_a^\eta([0, 1]^2) \cap \phi_c^\eta([0, 1]^2)$ is equal to an edge of $\phi_a^\eta([0, 1]^2)$ containing x , and similarly for $\phi_b^\eta([0, 1]^2) \cap \phi_c^\eta([0, 1]^2)$. From this we may conclude that the maps in $\{\phi_\beta^\eta : \beta \in E\}$ meet at edges of $[0, 1]^2$. $\square$

Lemma 3.9.2. *Let $m \in \mathbb{N}$, let $u, v \in \mathcal{A}_n^m$ be distinct, and let $x \in K^{n,0}$ with $x \in \psi_{n,u}^2(K^{n,0}) \cap \psi_{n,v}^2(K^{n,0})$. Then the maps in $\{\psi_{n,w}^2 : x \in \psi_{n,w}^2(K^{n,0}), w \in \mathcal{A}_n^m\}$ meet at edges of $[0, 1]^2$.*

Proof. The proof is by induction on m . The base case $m = 1$ holds by Remark 3.9.

Assume now that the claim holds for some $m \in \mathbb{N}$. Let $x \in K^{n,0}$ and $u, v \in \mathcal{A}_n^{m+1}$ distinct so that $x \in \psi_{n,u}^2(K^{n,0}) \cap \psi_{n,v}^2(K^{n,0})$. By Lemma 3.9.1, we may assume that $\{w \in \mathcal{A}_n^{m+1} : x \in \psi_{n,w}^2(K^{n,0})\} = \{u, v\}$. We prove that $\psi_{n,u}^2(K^{n,0}) \cap \psi_{n,v}^2(K^{n,0})$ is an edge of $\psi_{n,u}^2([0, 1]^2)$.

Let $u = a_1 \dots a_{m+1}$, $v = b_1 \dots b_{m+1}$. If $a_1 = b_1$, then applying $(\psi_{n,a_1}^2)^{-1}$ returns us to the claim at m , so by the inductive hypothesis we may assume that $a_1 \neq b_1$. Next,

if x is not a vertex of $\psi_{n,u}^2([0, 1]^2)$, then by Lemma 3.11.1 the claim holds, so we may further assume that x is a vertex of $\psi_{n,u}^2([0, 1]^2)$ and of $\psi_{n,v}^2([0, 1]^2)$. If there exists a word $w \in \mathcal{A}_n^m \setminus \{u(m), v(m)\}$ so that $x \in \psi_{n,w}^2(K^{n,0})$, then there exists a $j \in \mathcal{A}_n$ with $x \in \psi_{n,u}^2(K^{n,0}) \cap \psi_{n,v}^2(K^{n,0}) \cap \psi_{n,wj}^2(K^{n,0})$, and this is a contradiction.

Therefore, we may further assume that $\{w \in \mathcal{A}_n^m : x \in \psi_{n,w}^2(K^{n,0})\} = \{v(m), u(m)\}$. Then by the inductive hypothesis, $\psi_{n,u(m)}^2(K^{n,0}) \cap \psi_{n,v(m)}^2(K^{n,0})$ is an edge of $\psi_{n,u(m)}^2([0, 1]^2)$. The remainder of the proof is a case study on which edge of $\psi_{n,u(m)}^2([0, 1]^2)$ this intersection is equal to, but these cases are all essentially identical so we show only one of them and leave the remaining three to the reader. Assume that

$$\psi_{n,u(m)}^2(K^{n,0}) \cap \psi_{n,v(m)}^2(K^{n,0}) = \psi_{n,u(m)}^2([0, 1] \times \{0\}),$$

and as the maps $\psi_{n,u(m)}^2$ and $\psi_{n,v(m)}^2$ are both rotation-free and reflection-free similarity maps with Lipschitz norms n^{-m} , by Remark 3.7 this intersection must also be equal to the edge $\psi_{n,v(m)}^2([0, 1] \times \{1\})$. Then

$$\psi_{n,u(m)}^2(\mathbf{0}) - \psi_{n,v(m)}^2(\mathbf{0}) = (0, n^{-m}), \quad \psi_{n,a_{m+1}}^2(\mathbf{0}) = (\frac{k}{n}, 0), \quad \psi_{n,b_{m+1}}^2(\mathbf{0}) = (\frac{k \pm 1}{n}, 1)$$

with $k, k \pm 1 \in \{0, 1, \dots, n-1\}$. Then there is some $j \in \mathcal{A}_n$ so that $\psi_{n,j}^2(\mathbf{0}) = (\frac{k}{n}, 1)$, thus by a straightforward calculation we obtain $x \in \psi_{n,v(m)j}^2(K^{n,0})$. As $a_1 \neq b_1$ and necessarily $j \neq b_{m+1}$, $v(m)j \in \mathcal{A}_n^{m+1} \setminus \{v, u\}$, and this contradicts our assumption that $\{w \in \mathcal{A}_n^{m+1} : x \in \psi_{n,w}^2(K^{n,0})\} = \{u, v\}$, concluding the proof. $\square$

The next lemma is the base case for the proof of Lemma 3.8.3.

Lemma 3.9.3. *The set $K^{n,0}$ does not contain local cut-points.*

Proof. It suffices to show that $K^{n,0}$ contains no cut-points. Assuming this to be true, the proof of the lemma follows from (Dai et al., 2023, Theorem 1.6), as Lemma 3.9.1 and Lemma 3.9.2 alongside the fact that $K^{n,0}$ is a self-similar set with no cut-points are sufficient to imply the hypotheses of this theorem.

To prove that $K^{n,0}$ contains no cut-points, fix $y \in K^{n,0}$ and distinct $p, q \in K^{n,0} \setminus \{y\}$. Let $m \in \mathbb{N}$ so that there exist distinct $u, v, w \in \mathcal{A}_n^m$ for which $\psi_{n,u}^2(K^{n,0})$, $\psi_{n,v}^2(K^{n,0})$, and $\psi_{n,w}^2(K^{n,0})$ are pairwise disjoint with $p \in \psi_{n,v}^2(K^{n,0})$, $q \in \psi_{n,u}^2(K^{n,0})$, and $y \in \psi_{n,w}^2(K^{n,0})$. Then by Lemma 3.9.2 and by (Badger and Vellis, 2021, Lemma 3.1), there exist distinct $u_1, \dots, u_k \in \mathcal{A}_n^m$ so that $p \in \psi_{n,u_1}^2(K^{n,0})$, $q \in \psi_{n,u_k}^2(K^{n,0})$, and for every $j \in \{1, \dots, k-1\}$, $\psi_{n,u_j}^2(K^{n,0}) \cap \psi_{n,u_{j+1}}^2(K^{n,0})$ contains at least two points. Now let $z_0, \dots, z_k \in K^{n,0}$ be distinct points so that $z_0 = p$, $z_k = q$, and $z_j \in (\psi_{n,u_j}^2(K^{n,0}) \cap \psi_{n,u_{j+1}}^2(K^{n,0})) \setminus \{y\}$ for every $j \in \{1, \dots, k-1\}$. Then since $\partial[0, 1]^2 \subset K^{n,0}$ by the proof of Lemma 3.7.5, for every $j \in \{1, \dots, k-2\}$ there is a continuous function $\gamma_j : [0, 1] \rightarrow \psi_{n,u_{j+1}}^2(\partial[0, 1]^2) \setminus \{y\}$ so that $\gamma_j(0) = z_j$, $\gamma_j(1) = z_{j+1}$ and necessarily $\gamma_j([0, 1]) \subset K^{n,0} \setminus \{y\}$. By (Badger and Vellis, 2021, Proposition 3.2), there exist continuous $\gamma_0 : [0, 1] \rightarrow \psi_{n,u_1}^2(K^{n,0})$ with $\gamma_0(0) = p$, $\gamma_0(1) = z_1$ and continuous $\gamma_{k-1} : [0, 1] \rightarrow \psi_{n,u_k}^2(K^{n,0})$ with $\gamma_{k-1}(0) = z_{k-1}$, $\gamma_{k-1}(1) = q$. Concatenating these curves yields a path inside of $K^{n,0}$ connecting p to q and avoiding y , so y cannot be a cut-point for $K^{n,0}$. $\square$

As a straightforward consequence, we obtain the following corollary.

Corollary 3.9.1. *If $f_1, \dots, f_j : \mathbb{R}^2 \rightarrow \mathbb{R}^2$ with $j \in \{2, 3, 4\}$ is a collection of rotation-free and reflection-free similarities with a common Lipschitz norm which meet at edges of $[0, 1]^2$, then $\bigcup_{i=1}^j f_i(K^{n,0})$ does not contain local cut-points.*

Proof. We leave most of this proof to the reader. A proof follows from the observation that, up to re-scaling and translating, there are only 7 ways for such a collection of maps to meet at edges of $[0, 1]^2$, and all of these arrangements are present in $\{\psi_{n,j}^2 : j \in \mathcal{A}_n\}$. $\square$

We are now finally ready to prove Lemma 3.8.3.

Proof of Lemma 3.8.3. Denote by N_k the number of local cut-points in $K^{n,k}$. By Lemma 3.9.3 we have $N_0 = 0$. It remains to show that $N_k = (5n - 6)N_{k-1} + 1$ for every $k \in \mathbb{N}$.

By definition of sets $K^{n,k}$,

$$\begin{aligned}
K^{n,k} &= \bigcup_{v \in \mathcal{A}_n^k} \psi_{n,v}^1 \left(\bigcap_{m \in \mathbb{N}} \bigcup_{w \in \mathcal{A}_n^m} \psi_{n,w}^2([0, 1]^2) \right) \\
&= \bigcup_{j \in \mathcal{A}_n} \psi_{n,j}^1 \left(\bigcup_{v \in \mathcal{A}_n^{k-1}} \psi_{n,v}^1 \left(\bigcap_{m \in \mathbb{N}} \bigcup_{w \in \mathcal{A}_n^m} \psi_{n,w}^2([0, 1]^2) \right) \right) \\
&= \bigcup_{j \in \mathcal{A}_n} \psi_{n,j}^1(K^{n,k-1}).
\end{aligned}$$

By Lemma 3.7.4, each $\psi_{n,j}^1(K^{n,k-1} \cap (0, 1)^2)$ contributes N_{k-1} many local cut-points, and there is one additional local cut-point at $(1/2, 2/n)$. Then by the last part of Remark 3.9, every point of $K^{n,k} \setminus (\bigcup_{j \in \mathcal{A}_n} \psi_{n,j}^1(K^{n,k-1} \cap (0, 1)^2))$ other than $(1/2, 2/n)$ is either contained in exactly one image $\psi_{n,v}^1(K^{n,0})$ for some $v \in \mathcal{A}_n^k$ or in an intersection of at least two such images of maps which meet at edges of $[0, 1]^2$, so there are no other local cut-points of $K^{n,k}$ by Corollary 3.9.1. Therefore, $N_k = (5n - 6)N_{k-1} + 1$. $\square$

3.10.2 Proof of Proposition 3.5

Fix for the rest of this subsection an integer $k \geq 0$, a random choice function $\eta \in \mathcal{C}_n$, and an injective word

$$w = i_1 i_2 \cdots \in \mathcal{A}_n^{\mathbb{N}}$$

such that (η, w) satisfy (R1) and (R2) from the definition of a random choice function (see the paragraph before Lemma 3.7.7 for the definition of injective words). Set $x = \pi_\eta(w)$. Given $N \in \mathbb{N}$, let $\ell_N \in \mathbb{N}$ be the integer given in properties (R1) and (R2). For $N \in \mathbb{N}$, define

$$\begin{aligned}
X_N &:= n^{\ell_N}(K^\eta - x), \\
y_N &:= (\phi_{w(\ell_N)}^\eta)^{-1} \circ \phi_{w(\ell_N + N + k - 1)}^\eta(\mathbf{0}), \\
x_N &:= n^{\ell_N}(x - \phi_{w(\ell_N)}^\eta(\mathbf{0})).
\end{aligned}$$

Intuitively, x_N are the bottom-left corners of the blow-ups of pieces of K^η that contain x , while the points $(y_N)_{N \in \mathbb{N}}$ form a sequence of points in $[0, 1]^2$ which approximate the points x_N . Additionally, the sets X_N are a sequence of blow-ups of K^η centered at the point x which we use to find the desired tangents in $\text{Tan}(K^\eta, x)$.

By (R2), $i_{\ell_N - N + 1} > (4n - 4)$, so $\psi_{n, i_{\ell_N - N + 1}}^{\eta(w(\ell_N - N))}([0, 1]^2)$ does not intersect $\partial[0, 1]^2$. Thus, for every $N \geq 3$ we get

$$\text{dist} \left(K_{w(\ell_N)}^\eta, K^\eta \setminus K_{w(\ell_N - N)}^\eta \right) \geq n^{-\ell_N - 1 + N}. \quad (3.10.2)$$

The general strategy of the proof of Proposition 3.5 is to first establish estimates for the Hausdorff distance between X_N and translates of $K^{n,k}$ in a region near $[0, 1]^2$, then establish estimates for the Hausdorff distance between the rest of X_N and some blown-up and translated image of $K^{n,0}$ out to some large ball containing the origin. Then, exploiting the triangle inequality for excess (as in Remark 3.5), we show that some subsequence of X_N converges to a tangent set for which all local cut-points (see §3.4.5) lie inside a unit square near the origin, for which outside of this square the tangent “looks like” a tangent of $K^{n,0}$, and for which inside this unit square the tangent has exactly $\frac{1}{5n-7}((5n-6)^k - 1)$ many local cut-points.

Lemma 3.9.4. *For each $N \in \mathbb{N}$, $|y_N - x_N| \leq n^{-N-k+3}$.*

Proof. Since $x = \pi_\eta(w)$, the point x_N is in the set

$$\begin{aligned} n^{\ell_N} (\phi_{w(\ell_N + N + k - 1)}^\eta([0, 1]^2) - \phi_{w(\ell_N)}^\eta(\mathbf{0})) \\ = n^{\ell_N} \left(\phi_{w(\ell_N)}^\eta \circ (\phi_{w(\ell_N)}^\eta)^{-1} \circ \phi_{w(\ell_N + N + k - 1)}^\eta([0, 1]^2) - \phi_{w(\ell_N)}^\eta(\mathbf{0}) \right). \end{aligned}$$

Now by Remark 3.7, the latter set is equal to $(\phi_{w(\ell_N)}^\eta)^{-1} \circ \phi_{w(\ell_N + N + k - 1)}^\eta([0, 1]^2)$, which has diameter no more than $n^{-N-k+2}\sqrt{2}$ and contains y_N , so $|y_N - x_N| \leq n^{-N-k+3}$. $\square$

In the following lemma, we show that the parts of the sets X_N inside of some unit square containing the origin become close to translates of $K^{n,k}$ in terms of Hausdorff distance.

Lemma 3.9.5. *For each $N \in \mathbb{N}$, the Hausdorff distance*

$$\text{dist}_H(n^{\ell_N}(K_{w(\ell_N)}^\eta - x), K^{n,k} - y_N) \leq n^{-N-k+4}.$$

Proof. Recall that if $A, B \subset \mathbb{R}^M$ are compact sets, then Hausdorff distance is given by

$$\text{dist}_H(A, B) := \max \left\{ \sup_{x \in A} \inf_{y \in B} |x - y|, \sup_{y \in B} \inf_{x \in A} |y - x| \right\},$$

and recall from §3.4.4 that excess is given by $\text{excess}(A, B) := \sup_{x \in A} \inf_{y \in B} |x - y|$.

Thus to prove the lemma, it suffices to show

$$\text{excess}(n^{\ell_N}(K_{w(\ell_N)}^\eta - x), K^{n,k} - y_N) \leq n^{-N-k+4},$$

$$\text{excess}(K^{n,k} - y_N, n^{\ell_N}(K_{w(\ell_N)}^\eta - x)) \leq n^{-N-k+4}.$$

For the first inequality, by Lemma 3.9.4 and by the triangle inequality for excess as in Remark 3.5, it is enough to show that

$$\text{excess}(n^{\ell_N}(K_{w(\ell_N)}^\eta - x), K^{n,k} - x_N) \leq n^{-N-k+3}.$$

Note that since excess is translation invariant (see Remark 3.5), this excess is equal to

$$\text{excess}(n^{\ell_N}(K_{w(\ell_N)}^\eta - \phi_{w(\ell_N)}^\eta(\mathbf{0})), K^{n,k}),$$

which is at most $n^{-N-k+2}\sqrt{2}$ by (R1) and by definition of $K^{n,k}$. Via a similar argument, we obtain the other inequality

$$\text{excess}(K^{n,k} - y_N, n^{\ell_N}(K_{w(\ell_N)}^\eta - x)) \leq n^{-N-k+4}. \quad \square$$

Now define

$$F_N := n^{\ell_N}(K^{n,0} - \phi_{w(\ell_N)}^\eta(\mathbf{0})) - y_N,$$

$$Y_N := (F_N \setminus ([0, 1]^2 - y_N)) \cup (K^{n,k} - y_N).$$

The sets F_N are simply blown-up shifts of the self-similar set $K^{n,0}$. The sets Y_N are constructed by removing the copy of $K^{n,0}$ in F_N which contains the origin, then replacing it by a shift of $K^{n,k}$. Before continuing, note that for $k = 0$, $F_N = Y_N$.

Lemma 3.9.6. *For every $r > 0$,*

$$\lim_{N \rightarrow \infty} \text{excess}(Y_N \cap \overline{B}(\mathbf{0}, r), X_N) = 0 \quad \text{and} \quad \lim_{N \rightarrow \infty} \text{excess}(X_N \cap \overline{B}(\mathbf{0}, r), Y_N) = 0.$$

Proof. Fix $r > 0$. By (3.10.2) there exists $N_0 \in \mathbb{N}$ such that for every integer $N \geq N_0$,

$$\text{dist}(K_{w(\ell_N)}^\eta, K^\eta \setminus K_{w(\ell_N - N)}^\eta) \geq n^{-\ell_N} 2r.$$

By (R2), $\text{dist}(\partial[0, 1]^2, (\phi_{w(\ell_N - N)}^\eta)^{-1} \circ \phi_{w(\ell_N)}^\eta(\mathbf{0})) \geq n^{-1}$. Since $y_N \in [0, 1]^2$,

$$\overline{B}(\mathbf{0}, r) \subset n^N([0, 1]^2 - (\phi_{w(\ell_N - N)}^\eta)^{-1} \circ \phi_{w(\ell_N)}^\eta(\mathbf{0})) - y_N.$$

Fix $y \in Y_N \cap \overline{B}(\mathbf{0}, r)$. If $y \in [0, 1]^2 - y_N$, then by Lemma 3.9.5 there exists some $z \in X_N$ so that $|y - z| \leq n^{-N-k+4}$.

Assume now that $y \in Y_N \cap \overline{B}(\mathbf{0}, r) \setminus ([0, 1]^2 - y_N)$. Let

$$Q = \bigcup_{v \in \mathcal{A}_n^{2N+k-1}} \psi_{n,v}^2([0, 1]^2) \quad \text{and} \quad P = \bigcup_{v \in \mathcal{A}_n^{2N+k-1}} \psi_{n,v}^2(\partial[0, 1]^2).$$

By (R2) and since $\phi_v^\eta(\partial[0, 1]^2) \subset K^\eta$ for every finite word $v \in \mathcal{A}_n^*$ by the proof of Lemma 3.7.5, we have that

$$n^{\ell_N}((\phi_{w(\ell_N - N)}^\eta(P) - x) \setminus (\phi_{w(\ell_N)}^\eta([0, 1]^2) - x)) \subset X_N.$$

Since the maps $\psi_{n,j}^i$ are all similarities with Lipschitz norm n^{-1} , by Remark 3.7 we have

$$\begin{aligned} n^{\ell_N}(\phi_{w(\ell_N-N)}^\eta(P) - x) &= n^{\ell_N}(\phi_{w(\ell_N-N)}^\eta(P) - \phi_{w(\ell_N-N)}^\eta \circ (\phi_{w(\ell_N-N)}^\eta)^{-1}(x)) \\ &= n^N(P - (\phi_{w(\ell_N-N)}^\eta)^{-1}(x)). \end{aligned}$$

Similarly, we obtain $n^{\ell_N}(\phi_{w(\ell_N)}^\eta([0, 1]^2) - x) = [0, 1]^2 - (\phi_{w(\ell_N)}^\eta)^{-1}(x)$ and $x_N = (\phi_{w(\ell_N)}^\eta)^{-1}(x)$. Furthermore, $(\phi_{w(\ell_N-N)}^\eta)^{-1}(x) = (\phi_{w(\ell_N-N)}^\eta)^{-1} \circ \phi_{w(\ell_N)}^\eta(x_N)$, and so

$$(n^N(P - (\phi_{w(\ell_N-N)}^\eta)^{-1} \circ \phi_{w(\ell_N)}^\eta(\mathbf{0})) - x_N) \setminus ([0, 1]^2 - x_N) \subset X_N.$$

Since $y \in Y_N \cap \overline{B}(\mathbf{0}, r) \setminus ([0, 1]^2 - y_N)$, by calculations similar to those above we obtain

$$y \in (n^N(Q - (\phi_{w(\ell_N-N)}^\eta)^{-1} \circ \phi_{w(\ell_N)}^\eta(\mathbf{0})) - y_N) \setminus ([0, 1]^2 - y_N).$$

Therefore, there exists $u \in \mathcal{A}_n^{2N+k-1}$ so that

$$y \in (n^N(\psi_{n,u}^2([0, 1]^2) - (\phi_{w(\ell_N-N)}^\eta)^{-1} \circ \phi_{w(\ell_N)}^\eta(\mathbf{0})) - y_N) \setminus ([0, 1]^2 - y_N),$$

which implies that there is a point

$$p \in (n^N(\psi_{n,u}^2(\partial[0, 1]^2) - (\phi_{w(\ell_N-N)}^\eta)^{-1} \circ \phi_{w(\ell_N)}^\eta(\mathbf{0})) - x_N) \setminus ([0, 1]^2 - x_N),$$

hence $p \in X_N$. Then by the triangle inequality and Lemma 3.9.4 we have

$$|p - y| \leq |x_N - y_N| + |(p + x_N) - (y + y_N)| \leq |x_N - y_N| + n^{-N-k+1}\sqrt{2} \leq n^{-N-k+4}$$

since $(p + x_N), (y + y_N) \in n^N\psi_{n,u}^2([0, 1]^2)$, and the latter set has diameter $n^{-N-k+1}\sqrt{2}$.

The other limit can be proven in the same way, yielding the result. $\square$

In the following two lemmas, we prove that the sequence $(Y_N)_{N \in \mathbb{N}}$ has a subsequence converging in the Attouch-Wets topology to a set with $\frac{1}{5n-7}((5n-6)^k - 1)$ local cut-points and that the closeness of excesses from Lemma 3.9.6 implies that this set is also the Attouch-Wets limit of a subsequence of $(X_N)_{N \in \mathbb{N}}$, thus it is a tangent of K^η at x .

Lemma 3.9.7. *The sequence $(Y_N)_{N \in \mathbb{N}}$ has a subsequence converging with respect to the Attouch-Wets topology to a set $L_k \in \mathfrak{C}(\mathbb{R}^2; \mathbf{0})$ that has exactly $\frac{1}{5n-7}((5n-6)^k - 1)$ many local cut-points.*

Proof. Let $r > 3$, and recall from the proof of Lemma 3.9.6 that there exists a minimal $N_{0,r} \in \mathbb{N}$ so that for every integer $N \geq N_{0,r}$,

$$(Y_N + y_N) \cap \overline{B}(\mathbf{0}, r) = \overline{B}(\mathbf{0}, r) \cap (n^N(K^{n,0} - (\phi_{w(\ell_N-N)}^\eta)^{-1} \circ \phi_{w(\ell_N)}^\eta)(\mathbf{0})) \setminus [0, 1]^2 \cup K^{n,k}.$$

Furthermore, $N_{0,r}$ is nondecreasing in r . By (R1), for each $N \in \mathbb{N}$ there exists a word $v_N \in \mathcal{A}_n^N$ so that $(\phi_{w(\ell_N-N)}^\eta)^{-1} \circ \phi_{w(\ell_N)}^\eta = \psi_{n,v_N}^2$. Then by Remark 3.7 we obtain

$$[0, 1]^2 \cap n^N(K^{n,0} - \psi_{n,v_N}^2(\mathbf{0})) = n^N(\psi_{n,v_N}^2(K^{n,0}) - \psi_{n,v_N}^2(\mathbf{0})) = K^{n,0}.$$

Thus,

$$(Y_N + y_N) \cap \overline{B}(\mathbf{0}, r) = \overline{B}(\mathbf{0}, r) \cap (n^N(((K^{n,0} \setminus \psi_{n,v_N}^2([0, 1]^2)) \cup (\psi_{n,v_N}^2(K^{n,k}))) - \psi_{n,v_N}^2(\mathbf{0}))).$$

Since $\mathfrak{C}(\mathbb{R}^2; \mathbf{0})$ is sequentially compact in the Attouch-Wets topology, the sequence of sets $(Y_N + y_N)_{N \in \mathbb{N}}$ has a subsequence $(Y_{N_j} + y_{N_j})_{j \in \mathbb{N}}$ converging to a limit L in the Attouch-Wets topology and so that $(y_{N_j})_{j \in \mathbb{N}}$ converges to a point $y_0 \in [0, 1]^2$ as $j \rightarrow \infty$.

For $j \in \mathbb{N}$, let

$$D_{r,j} := \{(m_1, m_2) \in \mathbb{Z}^2 \setminus \{\mathbf{0}\} : [m_1, m_1 + 1] \times [m_2, m_2 + 1] \subset \overline{B}(\mathbf{0}, r), \\ \text{and there exists } v \in \mathcal{A}_n^{N_j} \text{ with } (m_1, m_2) = n^{N_j}(\psi_{n,v}^2(\mathbf{0}) - \psi_{n,v_{N_j}}^2(\mathbf{0}))\}.$$

If $j \in \mathbb{N}$ so that $N_j \geq N_{0,r+2}$, then

$$\bigcup_{(m_1, m_2) \in D_{r,j}} ((m_1, m_2) + K^{n,0}) \cup K^{n,k} \subset (Y_{N_j} + y_{N_j}) \cap \overline{B}(\mathbf{0}, r) \\ \subset \bigcup_{(m_1, m_2) \in D_{r+2,j}} ((m_1, m_2) + K^{n,0}) \cup K^{n,k}$$

and $D_{r,j} \subset D_{r+2,j}$. Then since for every $N \in \mathbb{N}$, every $v \in \mathcal{A}_n^N$ has $n^N(\psi_{n,v}^2(\mathbf{0}) - \psi_{n,v_N}^2(\mathbf{0})) \in \mathbb{Z}^2$, and since $(Y_{N_j} + y_{N_j})_{j \in \mathbb{N}}$ converges, there exists $j_{0,r} \in \mathbb{N}$ so that for every integer $j \geq j_{0,r}$, $D_{r,j} = D_{r,j_{0,r}}$. Indeed, if $D_{r,j} \neq D_{r,j+1}$ then

$$\text{excess}((Y_{N_j} + y_{N_j}) \cap \overline{B}(\mathbf{0}, r), (Y_{N_{j+1}} + y_{N_{j+1}})) \geq n^{-1}$$

or vice-versa, which cannot happen infinitely often by Lemma 3.5.2. Consequently, for every $r > 3$ and every integer $j \geq j_{0,r+4}$, we have that $(Y_{N_j} + y_{N_j}) \cap \overline{B}(\mathbf{0}, r) = \overline{B}(\mathbf{0}, r) \cap L$, so by Lemma 3.8.3, Corollary 3.9.1, and Remark 3.9, L has exactly $\frac{1}{5n-7}((5n-6)^k - 1)$ many local cut-points inside $[0, 1]^2$ and no other local cut-points. Finally, $L_k = L - y_0$ is the desired limit. $\square$

Remark 3.10. *The tangent L_k of Lemma 3.9.7 is a countable union $\bigcup_{j \in \mathbb{N}} S_j(K^{n,0})$ of rescaled copies of $K^{n,0}$, and if $i, j \in \mathbb{N}$ are distinct, then $S_i((0, 1)^2) \cap S_j((0, 1)^2) = \emptyset$. Since $\dim_H(K^{n,0}) = \alpha_n$ and $\mathcal{H}^{\alpha_n}(K^{n,0}) > 0$, by countable stability of Hausdorff dimension we have that $\dim_H(L_k) = \alpha_n$ and $\mathcal{H}^{\alpha_n}(L_k) > 0$.*

The next lemma completes the proof of Proposition 3.5 by showing that the closeness of the sequences Y_{N_j} and X_{N_j} from Lemma 3.9.6 implies that they share a common Attouch-Wets limit.

Lemma 3.10.1. *Let $(N_j)_j \subset \mathbb{N}$ and $L_k \in \mathfrak{C}(\mathbb{R}^2; \mathbf{0})$ be the subsequence and limit set from Lemma 3.9.7, respectively. Then $(X_{N_j})_{j \in \mathbb{N}}$ converges to L_k with respect to the Attouch-Wets topology.*

Proof. Since $\mathbb{R}^2$ has the Heine-Borel property, in order to show that $\lim_{j \rightarrow \infty} X_{N_j} = L_k$ with respect to the Attouch-Wets topology, by (Beer, 1993, Lemma 2.1.2) and (Beer, 1993, Lemma 3.1.4) it is sufficient to verify the following two conditions.

- (i) If $V \subset \mathbb{R}^2$ is an open set such that $V \cap L_k \neq \emptyset$, then there exists $j_0 \in \mathbb{N}$ such that for every integer $j \geq j_0$, $X_{N_j} \cap V \neq \emptyset$ as well.
- (ii) If $\delta > 0$ and $z \in \mathbb{R}^2$ such that $B(z, \delta) \cap L_k = \emptyset$, then for every $\epsilon \in (0, \delta)$ there exists $j_0 \in \mathbb{N}$ such that for every integer $j \geq j_0$, $B(z, \epsilon) \cap X_{N_j} = \emptyset$ as well.

To verify (i), fix such an open set $V \subset \mathbb{R}^2$. Then there exist $z \in V$ and $\epsilon > 0$ such that $B(z, \epsilon) \subset V$ and $B(z, \epsilon/2) \cap L_k \neq \emptyset$. Assume for the sake of contradiction that (i) fails to hold for this V , and thus $X_{N_j} \cap B(z, \epsilon) = \emptyset$ infinitely often. Now by Lemma 3.9.6 and since $\lim_{j \rightarrow \infty} Y_{N_j} = L_k$, there exists $j_0 \in \mathbb{N}$ so that for every integer $j \geq j_0$, $\text{excess}(Y_{N_j} \cap \overline{B}(\mathbf{0}, |z| + \epsilon), X_{N_j}) < \epsilon/2$ and $B(z, \epsilon/2) \cap Y_{N_j} \neq \emptyset$. By our assumption, there exists an integer $j \geq j_0$ such that $X_{N_j} \cap B(z, \epsilon) = \emptyset$ and there exists a point $y \in Y_{N_j} \cap B(z, \epsilon/2)$. Then we have

$$\text{excess}(Y_{N_j} \cap \overline{B}(\mathbf{0}, |z| + \epsilon), X_{N_j}) \geq \text{dist}(y, X_{N_j}) \geq \epsilon/2,$$

which is a contradiction.

Now to verify (ii), fix $\delta > 0$ and $z \in \mathbb{R}^2$ so that $B(z, \delta) \cap L_k = \emptyset$ and let $\epsilon \in (0, \delta)$. Since $Y_{N_j} \rightarrow L_k$ as $j \rightarrow \infty$ with respect to the Attouch-Wets topology, there is a $j_0 \in \mathbb{N}$ so that for every integer $j \geq j_0$, $Y_{N_j} \cap B(z, \frac{\epsilon+\delta}{2}) = \emptyset$. Furthermore by Lemma 3.9.6, choosing j_0 large enough, we may assume that for every integer $j \geq j_0$, $\text{excess}(X_{N_j} \cap \overline{B}(\mathbf{0}, |z| + \delta), Y_{N_j}) < \frac{\delta-\epsilon}{2}$. Then for every integer $j \geq j_0$ and for each $p \in X_{N_j} \cap B(\mathbf{0}, |z| + \delta)$, $\text{dist}(p, Y_{N_j}) < \frac{\delta-\epsilon}{2}$. Since $Y_{N_j} \cap B(z, \frac{\epsilon+\delta}{2}) = \emptyset$, if $p \in B(z, \epsilon)$,

then $\text{dist}(p, Y_{N_j}) \geq \frac{\delta-\epsilon}{2}$ and $p \in B(\mathbf{0}, |z| + \delta)$. Therefore for each integer $j \geq j_0$, $X_{N_j} \cap B(z, \epsilon) = \emptyset$, as desired. $\square$

3.11 Proof of Theorem 3.2

The proof of Theorem 3.2 in the special case that $s = \alpha_n := \frac{\log(5n-6)}{\log(n)}$ follows from several of the propositions we have established so far. For the general case, we require the following lemma which shows that bi-Hölder equivalence is hereditary. See Li (2021) for a quasimetric version of this lemma.

Lemma 3.10.2. *Let $f : X \rightarrow Y$ be a $(1/t)$ -bi-Hölder homeomorphism between two closed sets $X \subset \mathbb{R}^d$ and $Y \subset \mathbb{R}^m$ for some $t > 0$. Then for every point $x_0 \in X$ and every tangent $T \in \text{Tan}(X, x_0)$, there exist a tangent $T' \in \text{Tan}(Y, f(x_0))$ and a $(1/t)$ -bi-Hölder homeomorphism $g : T \rightarrow T'$.*

Proof. Assume first that $t \geq 1$. Let $x_0 \in X$, let $T \in \text{Tan}(X, x_0)$, and let $r_j \rightarrow 0$ be a sequence of scales so that $r_j^{-1}(X - x_0) \rightarrow T$ as $j \rightarrow \infty$ in the Attouch-Wets topology. For each $j \in \mathbb{N}$, define the function

$$f_j : r_j^{-1}(X - x_0) \rightarrow r_j^{-1/t}(Y - f(x_0)) \quad \text{with} \quad f_j(z) = r_j^{-1/t}(f(r_j z + x_0) - f(x_0)).$$

It is easy to see that each f_j is a $(1/t)$ -bi-Hölder homeomorphism with a bi-Hölder coefficient H independent of j . By McShane's extension theorem (McShane, 1934, Corollary 1), each f_j extends to a $(1/t)$ -Hölder map $F_j : \mathbb{R}^d \rightarrow \mathbb{R}^m$ with the Hölder coefficient equal to H . Passing to a subsequence, by the Arzelà-Ascoli theorem, we may assume that $\{F_j\}$ converges locally uniformly to a $(1/t)$ -Hölder map $F : \mathbb{R}^d \rightarrow \mathbb{R}^m$.

Since the maps $F_j|_T$ are $(1/t)$ -bi-Hölder with the same bi-Hölder coefficients, $F|_T : T \rightarrow F(T)$ is a $(1/t)$ -bi-Hölder homeomorphism. We claim that $r_j^{-1/t}(Y -$

$f(x_0)) \rightarrow F(T)$ as $j \rightarrow \infty$ in the Attouch-Wets topology. First note that

$$r_j^{-1/t}(Y - f(x_0)) = F_j(r_j^{-1}(X - x_0)).$$

Let $R > 0$, let $\epsilon > 0$, and let $j_0 \in \mathbb{N}$ so that for every integer $j \geq j_0$,

$$\begin{aligned} \|F_j - F\|_{\infty, \overline{B}(\mathbf{0}, R)} &< \epsilon \\ \text{excess}(\overline{B}(\mathbf{0}, R) \cap r_j^{-1}(X - x_0), T) &< \epsilon \\ \text{excess}(\overline{B}(\mathbf{0}, R) \cap T, r_j^{-1}(X - x_0)) &< \epsilon. \end{aligned}$$

For an integer $j \geq j_0$, given $z \in \overline{B}(\mathbf{0}, R) \cap r_j^{-1}(X - x_0)$ there exists $y \in T$ with $|z - y| < \epsilon$, so

$$|F_j(z) - F(y)| \leq |F_j(y) - F_j(z)| + |F_j(z) - F(z)| < H\epsilon^{1/t} + \epsilon.$$

Therefore,

$$\lim_{j \rightarrow \infty} \text{excess}(\overline{B}(\mathbf{0}, HR^{1/t}) \cap F_j(r_j^{-1}(X - x_0)), F(T)) = 0.$$

The other excess estimate follows via a similar calculation, so $r_j^{-1/t}(Y - f(x_0)) \rightarrow F(T)$ as $j \rightarrow \infty$ in the Attouch-Wets topology. This concludes the case $t \geq 1$.

Now let $t \in (0, 1)$. Then $f^{-1} : Y \rightarrow X$ is a t -bi-Hölder homeomorphism. Fix $x_0 \in X$, let $T \in \text{Tan}(X, x_0)$, and let $r_j \rightarrow 0$ be a sequence of scales so that $r_j^{-1}(X - x_0) \rightarrow T$ as $j \rightarrow \infty$ in the Attouch-Wets topology. Then $r_j^t \rightarrow 0$ is a sequence of scales, and there is a convergent subsequence $r_{j_k}^{-t}(Y - f(x_0)) \rightarrow S$ as $k \rightarrow \infty$ in the Attouch-Wets topology. As in the case $t \geq 1$, we have $g_{j_k} : r_{j_k}^{-t}(Y - f(x_0)) \rightarrow r_{j_k}^{-1}(X - x_0)$, a sequence of t -bi-Hölder homeomorphisms with uniform coefficients which extend to t -Hölder functions $G_{j_k} : \mathbb{R}^m \rightarrow \mathbb{R}^d$, and the maps G_{j_k} converge locally uniformly to a t -Hölder map $G : \mathbb{R}^m \rightarrow \mathbb{R}^d$. Then

$$G_{j_k}(r_{j_k}^{-t}(Y - f(x_0))) \rightarrow G(S)$$

as $j \rightarrow \infty$ in the Attouch-Wets topology, and since $G_{j_k}(r_{j_k}^{-t}(Y - f(x_0))) = r_{j_k}^{-1}(X - x_0)$ converges, it must converge to T . Therefore $G(S) = T$, and by locally uniform convergence $G|_S$ is a t -bi-Hölder homeomorphism onto T . Finally, $G^{-1} : T \rightarrow S$ is a $(1/t)$ -bi-Hölder homeomorphism, as desired. $\square$

We can now prove Theorem 3.2.

Proof of Theorem 3.2. Fix $s > 1$. Let $n \geq 4$ be an even integer such that $\alpha_n \in (1, s)$ and let $\eta \in \mathcal{C}_n$ be a random function. By Proposition 3.3 there exists a $\frac{1}{\alpha_n}$ -Hölder surjection $f : [0, 1] \rightarrow K^\eta$ and by Proposition 3.2, $\mathcal{H}^{\alpha_n}(K^\eta) > 0$. Finally, by Proposition 3.5, for $\mathcal{H}^{\alpha_n}$ -a.e. point $x \in K^\eta$ and for all integers $k \geq 0$, there exists $T_{n,k} \in \text{Tan}(K^\eta, x)$ such that $T_{n,k}$ has exactly $\frac{1}{5n-7}((5n-6)^k - 1)$ many cut points. By Lemma 3.6.1, it follows that tangents $T_{n,0}, T_{n,1}, \dots$ are pairwise topologically distinct.

By Assouad's embedding theorem Assouad (1983), there exists $N \in \mathbb{N}$ and a bi-Lipschitz embedding of the metric space $(K^\eta, |\cdot|^{\alpha_n/s})$ into $\mathbb{R}^N$ and denote by Γ the embedded image. This mapping produces a (α_n/s) -bi-Hölder homeomorphism $g : K^\eta \rightarrow \Gamma$. Therefore, there exists some $C > 1$ such that for every Borel $B \subset K^\eta$,

$$C^{-1}\mathcal{H}^{\alpha_n}(B) \leq \mathcal{H}^s(g(B)) \leq C\mathcal{H}^{\alpha_n}(B),$$

so $\mathcal{H}^s(\Gamma) > 0$. Moreover, the map $g \circ f : [0, 1] \rightarrow \Gamma$ is a $(1/s)$ -Hölder surjection. By Lemma 3.10.2, we have that for every point $x \in K^\eta$ and every tangent $T \in \text{Tan}(K^\eta, x)$ there exists a tangent $T' \in \text{Tan}(\Gamma, g(x))$ so that T and T' are (α_n/s) -bi-Hölder equivalent; in particular, T and T' are homeomorphic to each other. Therefore, $\text{Tan}(\Gamma, x)$ contains infinitely many topologically distinct elements. $\square$

3.12 Tangents of self-similar sets

In this section we prove Proposition 3.1 and Theorem 3.3. We start with the proof of Proposition 3.1.

Proof of Proposition 3.1. Without loss of generality, assume that $\text{diam } Q = 1$ and that $0 < \text{Lip}(\phi_1) \leq \dots \leq \text{Lip}(\phi_m) < 1$. By (Schief, 1994, Theorem 2.2) and by (Fan and Lau, 1999, Theorem 2.1, Lemma 2.5), we may assume that $\mathcal{H}^s(Q \cap U) = \mathcal{H}^s(Q)$. In particular, $Q \cap U \neq \emptyset$.

Fix $x \in Q \cap U$, $S \in \text{Tan}(Q, x)$, and $R > 0$. Let $r_j \rightarrow 0$ be a decreasing sequence of positive scales so that $r_1 < 1/R$ and the sets $D_j := (r_j)^{-1}(Q - x)$ converge to S in the Attouch-Wets topology as $j \rightarrow \infty$. Define for each $j \in \mathbb{N}$ the similarity $g_j : \mathbb{R}^N \rightarrow \mathbb{R}^N$, given by $g_j(y) = (r_j)^{-1}(y - x)$, and note that $g_j(Q) = D_j$. Define the alphabet $A = \{1, \dots, m\}$ and for a word $w \in A^*$, define $L_w := \text{Lip}(\phi_w)$. Following (Badger and Vellis, 2021, §2.3), for any $\delta \in (0, 1)$ define

$$A^*(\delta) := \{i_1 \dots i_n \in A^* : n \geq 1 \text{ and } L_{i_1 i_2 \dots i_n} < \delta \leq L_{i_1 i_2 \dots i_{n-1}}\}$$

and $A^*(1) := \{\varepsilon\}$, the set containing only the empty word.

Note that if $w \in A^*(r_j R)$, then

$$L_1 r_j R \leq L_w < r_j R. \quad (3.12.1)$$

For each $j \in \mathbb{N}$ define

$$W_j^R := \{w \in A^* : \phi_w(Q) \cap \overline{B}(x, r_j R) \neq \emptyset\} \quad \text{and} \quad C_j^R := W_j^R \cap A^*(r_j R).$$

Note that for each j , $\overline{B}(x, r_j R) \cap Q \subset \bigcup_{w \in C_j^R} \phi_w(Q)$. Additionally, by the OSC, if $v, w \in A^*$ are distinct words so that $\phi_w(U) \cap \phi_v(U) \neq \emptyset$, then there exists $u \in A^* \setminus \{\varepsilon\}$ so that either $w = vu$ or $v = wu$, which further implies that $\frac{L_w}{L_v} \leq L_1$ or $\frac{L_v}{L_w} \leq L_1$. Moreover, if $j \in \mathbb{N}$ and $w, v \in C_j^R$, then $\frac{L_w}{L_v} > L_1$ (or vice-versa); thus if $w, v \in C_j^R$ are distinct, $\phi_w(U) \cap \phi_v(U) = \emptyset$. Roughly speaking, each set $\{\phi_w : w \in C_j^R\}$ respects the “disjoint images” part of the open set condition for U . Observe that for each $j \in \mathbb{N}$ and each $w \in C_j^R$ by (3.12.1), the fact that $\text{diam}(Q) = 1$ and the fact that

$\phi_w(Q) \cap \overline{B}(x, r_j R) \neq \emptyset$, we have $\phi_w(Q) \subset \overline{B}(x, 2r_j R) \cap Q$. Hence, for each $j \in \mathbb{N}$,

$$\overline{B}(x, r_j R) \cap Q \subset \bigcup_{w \in C_j^R} \phi_w(Q) \subset \overline{B}(x, 2r_j R) \cap Q. \quad (3.12.2)$$

Next, since Q is self-similar, it follows that $\mathcal{H}^s(Q) > 0$ (Hutchinson, 1981, §5.3) and, in fact, Q is Ahlfors s -regular (Badger and Vellis, 2021, Lemma 2.4). That is, there exists $c_1 \geq 1$ so that for every $y \in Q$ and $r \in (0, 1)$,

$$(c_1)^{-1} r^s \leq \mathcal{H}^s(\overline{B}(y, r) \cap Q) \leq c_1 r^s.$$

Moreover, by (Hutchinson, 1981, Theorem 5.3.1), for every $w \in A^*$ we have $\mathcal{H}^s(\phi_w(Q)) = L_w^s$.

We claim that $\text{card}(C_j^R) \leq (2/L_1)^s c_1$ for all $j \in \mathbb{N}$. To this end, fix $j \in \mathbb{N}$. By (3.12.2),

$$\text{card}(C_j^R) (L_1 r_j R)^s \leq \sum_{w \in C_j^R} \mathcal{H}^s(\phi_w(Q)) = \mathcal{H}^s\left(\bigcup_{w \in C_j^R} \phi_w(Q)\right) \leq \mathcal{H}^s(\overline{B}(x, 2r_j R) \cap Q) \leq (2R)^s c_1 (r_j)^s,$$

where the equality follows from the fact that sets $\phi_w(U)$ for $w \in C_j^R$ are mutually disjoint. The claimed inequality follows.

By the preceding claim, the sequence $(\text{card}(C_j^R))_{j \in \mathbb{N}}$ is a bounded sequence of positive integers, so passing to a subsequence, we may assume that it is constant. That is, there exists an integer $M_R \in [1, (2/L_1)^s c_1]$ such that $\text{card}(C_j^R) = M_R$ for all $j \in \mathbb{N}$. We write $C_j^R = \{w_{1,j}^R, \dots, w_{M_R,j}^R\}$. Note that for every $j \in \mathbb{N}$ and $i \in \{1, \dots, M_R\}$ the map $g_j \circ \phi_{w_{i,j}^R} : \mathbb{R}^N \rightarrow \mathbb{R}^N$ is a similarity with $L_1 R \leq \text{Lip}(g_j \circ \phi_{w_{i,j}^R}) \leq R$. Moreover,

$$D_j \cap \overline{B}(\mathbf{0}, R) \subset \bigcup_{i=1}^{M_R} (g_j \circ \phi_{w_{i,j}^R}(Q)) \subset D_j \cap \overline{B}(\mathbf{0}, 2R),$$

and if $i_1, i_2 \in \{1, \dots, M_R\}$ are distinct, then $g_j \circ \phi_{w_{i_1,j}^R}(U) \cap g_j \circ \phi_{w_{i_2,j}^R}(U) = \emptyset$.

Since the similarities $g_j \circ \phi_{w_{i,j}}^R$ have uniformly bounded Lipschitz norms and are pointwise uniformly bounded, by the Arzelà-Ascoli theorem, and passing to a subsequence, we may assume that for each $i \in \{1, \dots, M_R\}$, $g_j \circ \phi_{w_{i,j}}^R \rightarrow f_i^R$ locally uniformly as $j \rightarrow \infty$. Since $D_j \rightarrow S$ as $j \rightarrow \infty$ in the Attouch-Wets topology, for each $i \in \{1, \dots, M_R\}$, $f_i^R(Q) \subset S$. Furthermore by the properties of the maps $g_j \circ \phi_{w_{i,j}}^R$ and by local uniform convergence, we obtain analogous properties for the f_i^R . Precisely, for each $i \in \{1, \dots, M_R\}$ the map $f_i^R : \mathbb{R}^N \rightarrow \mathbb{R}^N$ is a similarity with $L_1 R \leq \text{Lip}(f_i^R) \leq R$. Moreover,

$$\overline{B}(\mathbf{0}, R) \cap S \subset \bigcup_{i=1}^{M_R} f_i^R(Q) \subset \overline{B}(\mathbf{0}, 2R) \cap S$$

and if $i, j \in \{1, \dots, M_R\}$ are distinct, then $f_i^R(U) \cap f_j^R(U) = \emptyset$.

To complete the proof, we show that for every $i \in \{1, \dots, M_R\}$, $f_i^R(U \cap Q)$ is an open subset of S . Since f_i^R is a similarity, $f_i^R(U)$ is an open subset of $\mathbb{R}^N$, which implies that $f_i^R(U) \cap S$ is an open subset of S . By properties of functions f_i^R above we have that

$$f_i^R(U \cap Q) \cap B(\mathbf{0}, R) = f_i^R(U) \cap S \cap B(\mathbf{0}, R).$$

Furthermore, if $\{f_1^{2R}, \dots, f_{M_{2R}}^{2R}\}$ is constructed in a similar manner (replacing R by $2R$), then for every $i \in \{1, \dots, M_R\}$ there exists unique $j \in \{1, \dots, M_{2R}\}$ so that $f_i^R(Q) \subset f_j^{2R}(Q)$. Indeed, the sets C_j^R and C_j^{2R} satisfy $\bigcup_{v \in C_j^R} \phi_v(Q) \subset \bigcup_{u \in C_j^{2R}} \phi_u(Q)$, so for each $v \in C_j^R$, there exists a unique $u \in C_j^{2R}$ so that $\phi_v(U \cap Q) \subset \phi_u(U \cap Q)$. Then it must hold that $f_i^R(U \cap Q)$ is an open subset of $f_j^{2R}(U \cap Q) \cap B(\mathbf{0}, 2R)$ which is an open subset of $S \cap B(\mathbf{0}, 2R)$. Thus, $f_i^R(U \cap Q)$ is an open subset of S . $\square$

The following remark follows easily from the proof of Proposition 3.1.

Remark 3.11. *Let Q and S be as in Proposition 3.1. Then the Hausdorff dimensions of Q and of S are equal and they satisfy $\mathcal{H}^{\dim_H(Q)}(S) > 0$, $\mathcal{H}^{\dim_H(Q)}(Q) \in (0, \infty)$.*

3.12.1 Self-similar sponges

We say that a set $K \subset [0, 1]^N$ is a *self-similar sponge* if K is the attractor of a system

$$\{S_i(y) = k^{-1}y + p_i : [0, 1]^N \rightarrow [0, 1]^N\}_{i=1}^m, \quad (3.12.3)$$

where $k \in \{2, 3, \dots\}$, and points $\{p_1, \dots, p_m\}$ are mutually distinct and contained in the set $\{0, \frac{1}{k}, \dots, \frac{k-1}{k}\}^N$. It is easy to see that the system $\{S_i\}_{i=1}^m$ satisfies the OSC with $U = (0, 1)^N$.

We say that a subset $C \subset [0, 1]^N$ is a *face* of $[0, 1]^N$ if C is of the form $C = I_1 \times \dots \times I_N$, where each I_j is either equal to $[0, 1]$, to $\{0\}$, or to $\{1\}$. Additionally, we call $C = I_1 \times \dots \times I_N$ an M -face of $[0, 1]^N$ if exactly M -many of the $I_1, \dots, I_N$ are nondegenerate.

Lemma 3.11.1. *Let $\{S_1, \dots, S_m\}$ be a system of similarities as in (3.12.3). If $i_1, \dots, i_l \in \{1, \dots, m\}$ are distinct indices such that $\bigcap_{j=1}^l S_{i_j}([0, 1]^N) \neq \emptyset$, then this intersection is equal to $S_{i_1}(C)$, where C is a face of $[0, 1]^N$.*

Proof. For $z \in \mathbb{R}^N$ and $m \in \{1, \dots, N\}$, denote by $z(m)$ the m -th coordinate of z .

We begin by proving the result in the case $l = 2$. For simplicity, assume that $i_1 = 1$ and $i_2 = 2$. Since $S_1([0, 1]^N) \cap S_2([0, 1]^N) \neq \emptyset$ and since $S_1((0, 1)^N) \cap S_2((0, 1)^N) = \emptyset$, we have that

$$e := S_1(\mathbf{0}) - S_2(\mathbf{0}) \in \{-k^{-1}, 0, k^{-1}\}^N.$$

Write $e = (e(1), \dots, e(N))$. For $m \in \{1, \dots, N\}$, set $I_m = [0, 1]$ if $e(m) = 0$, $I_m = \{0\}$ if $e(m) > 0$, and $I_m = \{1\}$ if $e(m) < 0$. We claim that

$$S_1([0, 1]^N) \cap S_2([0, 1]^N) = S_1(I_1 \times \dots \times I_N).$$

To see this, fix $z_1, z_2 \in [0, 1]^N$ such that $S_1(z_1) = S_2(z_2)$. Then $z_2 - z_1 = ke$ and there are three cases to consider for $m \in \{1, \dots, N\}$. If $e(m) = 0$, then $z_2(m) - z_1(m) = 0$ which means that $z_1(m)$ can take any value in $[0, 1]$. If $e(m) > 0$, then $e(m) = k^{-1}$,

and $z_2(m) - z_1(m) = 1$ which implies that $z_1(m) = 0$. If $e(m) < 0$, then $e(m) = -k^{-1}$, and $z_2(m) - z_1(m) = -1$ which implies that $z_1(m) = 1$. In either of the three cases, $z_1 \in I_1 \times \cdots \times I_N$. This proves the claim and since $I_1 \times \cdots \times I_N$ is a face of $[0, 1]^N$, the lemma for $l = 2$.

For the case $l \geq 3$, we have that

$$\bigcap_{j=1}^l S_{i_j}([0, 1]^N) = \bigcap_{j=2}^l (S_{i_1}([0, 1]^N) \cap S_{i_j}([0, 1]^N)).$$

By the previous case, each $S_{i_1}([0, 1]^N) \cap S_{i_j}([0, 1]^N) = S_{i_1}(C_j)$, where C_j is a face of $[0, 1]^N$. The nonempty intersection of faces of $[0, 1]^N$ is also a face of $[0, 1]^N$, and the result follows. $\square$

We say that a set $F \subset \mathbb{R}^N$ is *locally self-similar* if for every pair of points $x, y \in F$ and every pair of scales $\rho_1, \rho_2 > 0$, there exists a similarity $g : \mathbb{R}^N \rightarrow \mathbb{R}^N$ such that $g(B(x, \rho_1) \cap F)$ is a subset of $B(y, \rho_2) \cap F$ and is open relative to F . Note that if F is locally self-similar, then it cannot have a positive and finite number of local cut-points.

Lemma 3.11.2. *Let K be a self-similar sponge. Then for every $x \in (0, 1)^N \cap K$, every tangent $T \in \text{Tan}(K, x)$ is locally self-similar.*

Proof. Suppose that K is the attractor of $\{S_i\}_{i=1}^m$ with $S_i(y) = k^{-1}y + p_i$ where k and $\{p_1, \dots, p_m\}$ are as in (3.12.3). If $m = 1$, then the result is trivial, so we may assume that $m \geq 2$. Set $A = \{1, \dots, m\}$. Note that for each $n \in \mathbb{N}$, $v \in A^n$, and $i \in A$,

$$S_{iv}(y) = k^{-n-1}y + S_{iv}(\mathbf{0}) = k^{-n-1}y + k^{-1}S_v(\mathbf{0}) + S_i(\mathbf{0})$$

and that $S_{iv}(\mathbf{0}) \in \{0, \frac{1}{k^{n+1}}, \dots, 1 - \frac{1}{k^{n+1}}\}^N$. Thus by a simple inductive argument, for every $n \in \mathbb{N}$ and every $w \in A^n$, $S_w(y) = k^{-n}y + S_w(\mathbf{0})$, where $S_w(\mathbf{0}) \in \{0, \frac{1}{k^n}, \dots, \frac{k^n-1}{k^n}\}^N$.

If $(0, 1)^N \cap K = \emptyset$, then the result is trivial, so assume $(0, 1)^N \cap K \neq \emptyset$. Fix $x \in (0, 1)^N \cap K$, $T \in \text{Tan}(K, x)$, $p, q \in T$, $\rho_1, \rho_2 > 0$, and a sequence of positive scales $r_j \rightarrow 0$ so that the sets $X_j := (r_j)^{-1}(K - x)$ converge to T in the Attouch-Wets topology as $j \rightarrow \infty$.

We first show that there exist a set $U \subset K \cap (0, 1)^N$ open relative to K , and a similarity map $G : U \rightarrow B(p, \rho_1) \cap T$ which is surjective. For $j \in \mathbb{N}$, define $g_j : K \rightarrow X_j$ by $g_j(y) = (r_j)^{-1}(y - x)$, and set $B_j := B(r_j p + x, \rho_1 r_j) \cap K$. Since $X_j \rightarrow T$ as $j \rightarrow \infty$ in the Attouch-Wets topology, we have

$$\lim_{j \rightarrow \infty} \text{excess}(B(p, \rho_1/2) \cap T, g_j(B_j)) = 0,$$

so passing to a subsequence, we may assume that $B_j \neq \emptyset$ for all $j \in \mathbb{N}$. Additionally, since $\lim_{j \rightarrow \infty} \text{diam}(B_j) = 0$, since $x \in K \cap (0, 1)^N$, and since $\lim_{j \rightarrow \infty} \text{dist}(x, B_j) = 0$, again passing to a subsequence, we may assume that $B_j \subset K \cap (0, 1)^N$ for all $j \in \mathbb{N}$.

For each $j \in \mathbb{N}$, let $n_j \in \mathbb{N}$ such that

$$2\rho_1 r_j < k^{-n_j} \leq 2\rho_1 k r_j$$

and define

$$W_j := \{v \in A^{n_j} : S_v([0, 1]^N) \cap B_j \neq \emptyset\}.$$

The sets W_j are nonempty and a simple computation shows that

$$B_j \subset \bigcup_{v \in W_j} S_v(K) \subset K \cap B(r_j p + x, 4k\sqrt{N}\rho_1 r_j).$$

Thus we may assume that for all $j \in \mathbb{N}$, $\bigcup_{v \in W_j} S_v(K) \subset K \cap (0, 1)^N$. Furthermore, by the OSC, for every pair of distinct words $w, v \in W_j$, $S_v((0, 1)^N) \cap S_w((0, 1)^N) = \emptyset$. Therefore, by the doubling property of $\mathbb{R}^N$, there exists $M' \in \mathbb{N}$ such that $\text{card}(W_j) \leq M'$ for all $j \in \mathbb{N}$.

We claim that $\bigcap_{w \in W_j} S_w([0, 1]^N) \neq \emptyset$ for every $j \in \mathbb{N}$. We first show the claim for two words only. Fix distinct $v, w \in W_j$ and let $z_v, z_w \in [0, 1]^N$ so that

$$|S_v(z_v) - (r_j p + x)| < \rho_1 r_j \quad \text{and} \quad |S_w(z_w) - (r_j p + x)| < \rho_1 r_j,$$

so by the triangle inequality we obtain

$$\text{dist}(S_v([0, 1]^N), S_w([0, 1]^N)) < 2\rho_1 r_j < k^{-n_j}.$$

Therefore, $S_w([0, 1]^N) \cap S_v([0, 1]^N) \neq \emptyset$ and the claim in this special case holds.

For $l \in \{1, \dots, N\}$ and $w \in W_j$, denote by $S_w(\mathbf{0})^{(l)}$ the l -th coordinate of $S_w(\mathbf{0})$. Note that if $w, v \in W_j$ are distinct and $S_v(\mathbf{0})^{(l)} - S_w(\mathbf{0})^{(l)} > 0$ for some $l \in \{1, \dots, N\}$, then $S_v(\mathbf{0})^{(l)} - S_w(\mathbf{0})^{(l)} = k^{-n_j}$. This further implies that if $u \in W_j \setminus \{w, v\}$, then $S_v(\mathbf{0})^{(l)} - S_u(\mathbf{0})^{(l)} \geq 0$.

Fix now a word $v \in W_j$. For each $w \in W_j \setminus \{v\}$ and $l \in \{1, 2, \dots, N\}$, let

$$I_w^l := \begin{cases} [0, 1], & \text{if } S_v(\mathbf{0})^{(l)} - S_w(\mathbf{0})^{(l)} = 0 \\ \{0\}, & \text{if } S_v(\mathbf{0})^{(l)} - S_w(\mathbf{0})^{(l)} = k^{-n_j} \\ \{1\}, & \text{if } S_v(\mathbf{0})^{(l)} - S_w(\mathbf{0})^{(l)} = -k^{-n_j}. \end{cases}$$

For each $l \in \{1, \dots, N\}$, if there exists $w \in W_j \setminus \{v\}$ with $I_w^l = \{0\}$, then for every $u \in W_j \setminus \{v\}$ either $I_u^l = \{0\}$ or $I_u^l = [0, 1]$, and similarly in the case that $I_w^l = \{1\}$. In particular, $I_w^l \cap I_u^l \neq \emptyset$ for every pair of words $w, u \in W_j \setminus \{v\}$. Therefore $\bigcap_{w \in W_j \setminus \{v\}} (I_w^1 \times \dots \times I_w^N) \neq \emptyset$. By Lemma 3.11.1, $S_v(I_w^1 \times \dots \times I_w^N) = S_v([0, 1]^N) \cap S_w([0, 1]^N)$ for each $w \in W_j \setminus \{v\}$, and by the special case this intersection is nonempty. Thus we obtain $\bigcap_{w \in W_j} S_w([0, 1]^N) \neq \emptyset$, proving the claim.

Since the sequence $(\text{card}(W_j))_j$ is bounded, passing to a subsequence, we may assume that $\text{card}(W_j) = M$ for all $j \in \mathbb{N}$ for some $M \in \mathbb{N}$. Write $W_j = \{v_1^j, \dots, v_M^j\}$ and note that for every $i \in \{1, \dots, M\}$ and $j \in \mathbb{N}$, $g_j \circ S_{v_i^j} : K \rightarrow B(p, 4k\sqrt{N}\rho_1) \cap X_j$

is a similarity with $\text{Lip}(g_j \circ S_{v_i^j}) \in (2\rho_1, 2\rho_1 k]$. By the Arzelà-Ascoli theorem and passing to a subsequence, we may assume that for every $i \in \{1, \dots, M\}$, the maps $g_j \circ S_{v_i^j}$ converge uniformly to a similarity $f_i : K \rightarrow T$ as $j \rightarrow \infty$. Moreover, for some $L \in (2\rho_1, 2\rho_1 k]$, we have $\text{Lip}(f_i) = L$ for all $i \in \{1, 2, \dots, M\}$. Additionally,

$$B(p, \rho_1) \cap T \subset \bigcup_{i=1}^M f_i(K),$$

and by the OSC, if $i, j \in \{1, \dots, M\}$ are distinct, then $f_i((0, 1)^N) \cap f_j((0, 1)^N) = \emptyset$.

By Lemma 3.11.1, for every $j \in \mathbb{N}$ there exists a face C_j of $[0, 1]^N$ so that $\bigcap_{i=1}^M S_{v_i^j}([0, 1]^N) = S_{v_1^j}(C_j)$. Thus, for every $j \in \mathbb{N}$ there exist distinct $e_{j,1}, \dots, e_{j,M} \in \{0, 1\}^N$ so that $S_{v_i^j}(e_{j,i}) = S_{v_1^j}(e_{j,1})$ for every $i \in \{1, \dots, M\}$. Passing to a subsequence, we may assume that there exist distinct $e_1, \dots, e_M \in \{0, 1\}^N$ so that $e_{j,i} = e_i$ for every $j \in \mathbb{N}$ and $i \in \{1, \dots, M\}$. Then $f_i(e_i) = f_1(e_1)$ and $S_{v_i^j}(e_i) = S_{v_1^j}(e_1)$ for every $j \in \mathbb{N}$ and $i \in \{1, \dots, M\}$. Therefore,

$$\begin{aligned} f_i \circ S_{v_i^j}^{-1}(y) &= L(S_{v_i^j}^{-1}(y) - e_i) + f_i(e_i) = Lk^{-n_j}(y - S_{v_i^j}(e_i)) = L(S_{v_1^j}^{-1}(y) - e_1) + f_1(e_1) \\ &= f_1 \circ S_{v_1^j}^{-1}(y). \end{aligned} \tag{3.12.4}$$

Define now $G = f_1 \circ (S_{v_1^1})^{-1}$. By (3.12.4), G is a similarity satisfying

$$B(p, \rho_1) \cap T \subset G \left(\bigcup_{i=1}^M S_{v_i^1}(K) \right) \subset T.$$

Further, since $\bigcup_{i=1}^M S_{v_i^1}(K) \subset K \cap (0, 1)^N$, G maps the open subset $G^{-1}(B(p, \rho_1) \cap T)$ of $K \cap (0, 1)^N$ onto $B(p, \rho_1) \cap T$.

By Proposition 3.1, there exists a similarity $f : \mathbb{R}^N \rightarrow \mathbb{R}^N$ such that $q \in f(K) \subset T$ and so that $f((0, 1)^N \cap K) \cap B(q, \rho_2)$ is an open subset of T . Let $z \in f((0, 1)^N \cap K) \cap B(q, \rho_2)$ and let $w \in A^*$ so that $f^{-1}(z) \in S_w(K) \subset (0, 1)^N \cap K$. Then $f \circ S_w$ is a

similarity so that $f \circ S_w((0, 1)^N \cap K)$ is an open subset of $f((0, 1)^N \cap K) \cap B(q, \rho_2)$, and therefore $f \circ S_w((0, 1)^N \cap K)$ is an open subset of $T \cap B(q, \rho_2)$. Therefore,

$$(f \circ S_w) \circ G^{-1} : B(p, \rho_1) \cap T \rightarrow B(q, \rho_2) \cap T$$

is a similarity so that $(f \circ S_w) \circ G^{-1}(B(p, \rho_1) \cap T)$ is an open subset of $T \cap B(q, \rho_2)$. $\square$

Remark 3.12. Lemma 3.11.2 can be easily extended to show that at typical points of a self-similar sponge, only locally self-similar tangents are admitted, even if the sponge does not intersect $(0, 1)^N$. To argue this, we would need only make the simple observation that in such a case, the sponge would be contained in an M -face of $[0, 1]^N$ for some $M < N$ for which the sponge intersects the interior of that face. Then we can repeat the argument in the lower dimension M .

3.12.2 Proof of Theorem 3.3

Let $s > 1$, and given an even integer $n \geq 4$ recall numbers α_n from Section 3.7, sets $K^{n,0}$ from (3.10.1), and tangents $\{T_{n,k}\}_{k \in \mathbb{N}}$ of Proposition 3.5. Fix an even integer $n \geq 4$ so that $\alpha_n < s$ and a random choice function $\eta \in \mathcal{C}_n$. Let $I : K^\eta \rightarrow \Gamma_s$ be the snowflake map composed with the bi-Lipschitz embedding given in Section 3.11. Recall that I is an (α_n/s) -bi-Hölder homeomorphism onto its image Γ_s , and $\Gamma_s \subset \mathbb{R}^d$ for some $d \in \mathbb{N}$. Furthermore, Γ_s is a $(1/s)$ -Hölder curve with $\mathcal{H}^s(\Gamma_s) \in (0, \infty)$. Fix a point $x_0 \in K^\eta$ so that for every $k \in \mathbb{N}$, $T_{n,k} \in \text{Tan}(K^\eta, x_0)$, and recall by Lemma 3.10.2 for every $k \in \mathbb{N}$ there exists a tangent $T_k^* \in \text{Tan}(\Gamma_s, I(x_0))$ and an (α_n/s) -bi-Hölder homeomorphism $I_k^* : T_{n,k} \rightarrow T_k^*$ induced by the map I . The next proposition completes the proof of Theorem 3.3.

Proposition 3.6. Let $k \in \mathbb{N}$, $\{S_i : \mathbb{R}^{d'} \rightarrow \mathbb{R}^{d'}\}_{i=1}^m$ be an IFS of similarities satisfying the OSC for some open set $U \subset \mathbb{R}^{d'}$, and let Q be the attractor. Further assume $Q \cap U \neq \emptyset$. Let Q_k be the set of $z \in Q \cap U$ such that T_k^* is bi-Lipschitz equivalent to some $T \in \text{Tan}(Q, z)$. If $Q_k \neq \emptyset$, then $\dim_H(Q) = s$ and for $\mathcal{H}^s$ -almost every $y \in Q$,

each $S \in \text{Tan}(Q, y)$ is bi-Hölder equivalent to a locally self-similar subset of $\mathbb{R}^2$. In particular, $\mathcal{H}^s(Q_k) = 0$.

Proof. First, if $Q_k \neq \emptyset$, then by Remark 3.11 $\dim_{\text{H}}(Q) = s$. Let $z \in Q_k$, let $T \in \text{Tan}(Q, z)$ be bi-Lipschitz homeomorphic to T_k^* for some $k \in \mathbb{N}$, and let $g : T_k^* \rightarrow T$ be a bi-Lipschitz homeomorphism. Then by Remark 3.10, there exists a countable collection of similarities $\{\phi_j : \mathbb{R}^2 \rightarrow \mathbb{R}^2\}_{j \in \mathbb{N}}$ so that $T_{n,k} = \bigcup_{j \in \mathbb{N}} \phi_j(K^{n,0})$ and so that for every pair of distinct $i, j \in \mathbb{N}$, $\phi_i((0, 1)^2) \cap \phi_j((0, 1)^2) = \emptyset$. Then for each $j \in \mathbb{N}$, the composition $(g \circ I_k^* \circ \phi_j) : K^{n,0} \rightarrow T$ is an (α_n/s) -bi-Hölder homeomorphism onto its image, further satisfying $(g \circ I_k^* \circ \phi_j)(K^{n,0} \cap (0, 1)^2)$ is an open subset of T . Additionally, by Lemma 3.1 and its proof, for every $R > 0$ there is an integer $M_R \in \mathbb{N}$ and a collection of similarities $f_1, \dots, f_{M_R} : \mathbb{R}^{d'} \rightarrow \mathbb{R}^{d'}$ such that

$$\overline{B}(\mathbf{0}, R) \cap T \subset \bigcup_{j=1}^{M_R} f_j(Q) \subset \overline{B}(\mathbf{0}, 2R) \cap T$$

and so that for every pair of distinct $i, j \in \{1, \dots, M_R\}$, $f_i(U) \cap f_j(U) = \emptyset$ and $f_i(U \cap Q)$ is an open subset of T .

Now let $i \in \mathbb{N}$ and $j \in \{1, \dots, M_R\}$ so that

$$V := (g \circ I_k^* \circ \phi_i)(K^{n,0} \cap (0, 1)^2) \cap f_j(Q \cap U) \neq \emptyset,$$

and note that V is an open subset of T . Then $f_j^{-1}(V)$ is an open subset of $Q \cap U$ and the composition

$$(f_j^{-1} \circ g \circ I_k^* \circ \phi_i) : (\phi_i^{-1} \circ (I_k^*)^{-1} \circ g^{-1})(V) \rightarrow f_j^{-1}(V)$$

is an (α_n/s) -bi-Hölder homeomorphism. Recall the similarities $\psi_{n,1}^2, \dots, \psi_{n,5n-6}^2 : \mathbb{R}^2 \rightarrow \mathbb{R}^2$ from Section 3.6, the similarities generating the self-similar sponge $K^{n,0}$. Recall that $K^{n,0} \cap (0, 1)^2 \neq \emptyset$ and recall from the definition of self-similar sponge that the system $\{\psi_{n,1}^2, \dots, \psi_{n,5n-6}^2\}$ satisfies the OSC via the open set $(0, 1)^2$. Then there is a

word $u \in \mathcal{A}_n^*$ so that

$$\psi_{n,u}^2(K^{n,0}) \subset (\phi_i^{-1} \circ (I_k^*)^{-1} \circ g^{-1})(V)$$

and so that the composition

$$(f_j^{-1} \circ g \circ I_k^* \circ \phi_i) : \psi_{n,u}^2(K^{n,0}) \rightarrow Q \cap U$$

is an (α_n/s) -bi-Hölder homeomorphism onto its image. Let now

$$D := (f_j^{-1} \circ g \circ I_k^* \circ \phi_i)(\psi_{n,u}^2(K^{n,0} \cap (0,1)^2)).$$

Note that D is a nonempty open subset of $Q \cap U$ and for every word $w \in \{1, \dots, m\}^*$, $S_w(D)$ is a nonempty open subset of $Q \cap U$ as well. Therefore, letting $B := \bigcup_{w \in \{1, \dots, m\}^*} S_w(D)$, we have that B is a nonempty open subset of $Q \cap U$, further satisfying for every pair of distinct $s, t \in \{1, \dots, m\}$, $S_t(B) \subset B$ and $S_t(B) \cap S_s(B) = \emptyset$. Then since B is an open subset of $Q \cap U$, there exists an open set $B_0 \subset \mathbb{R}^{d'}$ so that $B_0 \cap Q = B$ and so that $B_0 \subset U$. Let $B' := \bigcup_{w \in \{1, \dots, m\}^*} S_w(B_0)$. Then it is easy to verify that the OSC is obtained by the open set B' for the system of maps $\{S_1, \dots, S_m\}$ and $B' \cap Q = B$ is nonempty, therefore by (Fan and Lau, 1999, Theorem 2.1, Lemma 2.5) $\mathcal{H}^s \llcorner Q = \mathcal{H}^s \llcorner B$. We claim that for every point $y \in B$ and every tangent $S \in \text{Tan}(Q, y)$, there is a point $\tilde{y} \in K^{n,0} \cap (0,1)^2$ and some $\tilde{S} \in \text{Tan}(K^{n,0}, \tilde{y})$ so that $\tilde{S}$ is (α_n/s) -bi-Hölder homeomorphic to S . Assuming the claim, we can conclude the proof by recalling that by Lemma 3.11.2, the set $\tilde{S}$ is locally self-similar, in particular has either 0 or infinitely-many local cut-points, so by Lemma 3.6.1 the tangent S cannot be bi-Lipschitz homeomorphic to (or homeomorphic to at all) any set T_k^* for any $k \in \mathbb{N}$.

All that remains is to prove the claim, and to this end let $y \in B$ and let $S \in \text{Tan}(Q, y)$. Since B is an open subset of Q , $S \in \text{Tan}(B, y)$, so there exists a word

$w \in \{1, \dots, m\}^*$ with $S \in \text{Tan}(S_w(D), y)$. Note that the composition

$$(S_w \circ f_j^{-1} \circ g \circ I_k^* \circ \phi_i)^{-1} : S_w(D) \rightarrow \psi_{n,u}^2(K^{n,0} \cap (0,1)^2)$$

is an (α_n/s) -bi-Hölder homeomorphism, so by Lemma [3.10.2](#) the tangents S and $\tilde{S}$ are (α_n/s) -bi-Hölder homeomorphic. $\square$

Appendix: A Lipschitz curve and a point with extreme tangent space

Recall that $\mathfrak{C}_U(\mathbb{R}^N; \mathbf{0})$ denotes the collection of all sets $X \in \mathfrak{C}(\mathbb{R}^N; \mathbf{0})$ (i.e. closed sets in $\mathbb{R}^N$ that contain the origin $\mathbf{0}$) such that every component of X is unbounded. In Lemma 3.5.3 we showed that for every non-constant Lipschitz curve in $\mathbb{R}^N$, the tangent space at every point is a subset of $\mathfrak{C}_U(\mathbb{R}^N; \mathbf{0})$. In this appendix we prove that the tangent space can in fact be all of $\mathfrak{C}_U(\mathbb{R}^N; \mathbf{0})$.

Theorem 3.13. *For each $N \in \{2, 3, \dots\}$ there exists a Lipschitz curve $f : [0, 1] \rightarrow \mathbb{R}^N$ such that $f(0) = \mathbf{0}$ and $\text{Tan}(f([0, 1]), \mathbf{0}) = \mathfrak{C}_U(\mathbb{R}^N; \mathbf{0})$.*

Fix, for the rest of the appendix, an integer $N \geq 2$. For each $k \in \mathbb{N}$ denote by $\mathcal{G}_k$ the collection of all connected graphs $G = (V, E)$ such that

1. $\{\mathbf{0}\} \cup \{2^{-k}\mathbf{m} : \mathbf{m} \in \partial[-2^k, 2^k]^N \cap \mathbb{Z}^N\} \subset V \subset \{2^{-k}\mathbf{m} : \mathbf{m} \in [-2^k, 2^k]^N \cap \mathbb{Z}^N\}$,
2. for $v, v' \in V$ we have $\{v, v'\} \in E$, if and only if $|v - v'| = 2^{-k}$.

We can enumerate $\bigcup_{k \in \mathbb{N}} \mathcal{G}_k = \{G_1, G_2, \dots\}$ and for $n \in \mathbb{N}$, we define $k_n \in \mathbb{N}$ to be the unique positive integer with $G_n \in \mathcal{G}_{k_n}$. Note that for all $n \in \mathbb{N}$ we have $\mathcal{H}^1(\text{Im}(G_n)) \leq (2^{k_n+1} + 1)^N$.

Let $(r_n)_{n \geq 0}$ be a decreasing sequence of positive numbers such that $r_0 = 1$ and for each $n \in \mathbb{N}$

$$r_n \leq \min \left\{ \frac{1}{2^{n+1} \mathcal{H}^1(\text{Im}(G_n))}, \frac{r_{n-1}}{2^{n+k_{n+1}+1}} \right\}.$$

For each $n \in \mathbb{N}$ let $\phi_n : \mathbb{R}^N \rightarrow \mathbb{R}^N$ be the map $\phi_n(x) = r_n x$, and let

$$H_n = \phi_n(\text{Im}(G_n)) \setminus ((-r_{n+1}, r_{n+1})^N).$$

Since $\text{Im}(G_n)$ is a connected set and it contains $\mathbf{0}$, the set $H_n \cap [-r_{n+1}, r_{n+1}]^N$ is nonempty; in fact, every point therein is the center of an $(N - 1)$ -face of the cube

$[-r_{n+1}, r_{n+1}]^N$. Therefore,

$$H_n \cap [-r_{n+1}, r_{n+1}]^N = H_n \cap \partial[-r_{n+1}, r_{n+1}]^N \subset H_{n+1} \cap \partial[-r_{n+1}, r_{n+1}]^N \subset \phi_{n+1}(\text{Im}(G_{n+1})).$$

Define

$$H = \{\mathbf{0}\} \cup \bigcup_{n=1}^{\infty} H_n.$$

We show below that there exists a Lipschitz surjection $f : [0, 1] \rightarrow H$ with $f(0) = \mathbf{0}$ and that $\text{Tan}(H, \mathbf{0}) = \mathfrak{C}_U(\mathbb{R}^N; \mathbf{0})$. We split the proof in several steps.

Lemma 3.13.1. *There exists a Lipschitz surjection $f : [0, 1] \rightarrow H$ with $f(0) = \mathbf{0}$.*

Proof. First note that H is the countable union of compact sets H_n which converge in Hausdorff distance to $\{\mathbf{0}\}$ which is contained in H . Therefore, H is compact.

Now we claim that every point of H can be connected to $\mathbf{0}$. To see that, fix $n \in \mathbb{N}$ and $p \in H_n$. Since $\text{Im}(G_n)$ is connected, there exists $p_1 \in H_n \cap H_{n+1}$ and a path γ_1 in H_n that joins p with p_1 . Assume that for some $k \in \mathbb{N}$ we have defined a point $p_k \in H_{n+k} \cap H_{n+k+1}$. By connectedness of $\text{Im}(G_{n+k+1})$, there exists $p_{k+1} \in H_{n+k+1} \cap H_{n+k+2}$ and a path γ_{k+2} in H_{n+k+1} that connects p_k to p_{k+1} . The concatenation of paths $\gamma_1, \gamma_2, \dots$ produces a path that connects p to $\mathbf{0}$. Therefore, H is connected.

Finally, by the choice of scales r_n we get

$$\mathcal{H}^1(H) = \sum_{n=1}^{\infty} \mathcal{H}^1(H_n) \leq \sum_{n=1}^{\infty} r_n \mathcal{H}^1(\text{Im}(G_n)) \leq 1.$$

Therefore, H is a continuum with $\mathcal{H}^1(H) \leq 1$ and by (Alberti and Ottolini, 2017, Theorem 4.4) there exists a Lipschitz surjection $f : [0, 1] \rightarrow H$ such that $\text{Lip}(f) \leq 2$ and $f(0) = \mathbf{0}$. $\square$

It remains to show that $\text{Tan}(H, \mathbf{0}) = \mathfrak{C}_U(\mathbb{R}^N; \mathbf{0})$. To this end, fix for the rest of this section a set $T \in \mathfrak{C}_U(\mathbb{R}^N; \mathbf{0})$.

For each $j \in \mathbb{N}$ set

$$\begin{aligned} Q_j &= \{2^{-j}\mathbf{m} : \mathbf{m} \in [1 - 4^j, 4^j - 1]^N \cap \mathbb{Z}^N\}, \\ W_j &= \{v \in Q_j : \text{dist}(v, T \cap [2^{-j} - 2^j, 2^j - 2^{-j}]^N) \leq 2^{2-j}\sqrt{N}\}, \\ X_j &= \bigcup_{\substack{v, v' \in W_j \\ |v-v'| \leq 2^{-j}}} [v, v']. \end{aligned}$$

Since $\mathbf{0} \in T$ we have that $\mathbf{0} \in W_j$.

Lemma 3.13.2. *Every component of X_j intersects the boundary $\partial[2^{-j} - 2^j, 2^j - 2^{-j}]^N$.*

Proof. Let B be a component of X_j . Fix a vertex $v \in B \cap V_j$ and note that there is a point $p \in T \cap [2^{-j} - 2^j, 2^j - 2^{-j}]^N$ such that $|v - p| \leq 2^{2-j}\sqrt{N}$. Let $K \subset T$ be the component of $T \cap [2^{-j} - 2^j, 2^j - 2^{-j}]^N$ which contains p . Since every component of T is unbounded, $K \cap \partial[2^{-j} - 2^j, 2^j - 2^{-j}]^N \neq \emptyset$. Now for any point $q \in K$, let $Z_q = \{w \in W_j : |w - q| \leq 2^{2-j}\sqrt{N}\}$ and let

$$Y_q = \bigcup_{\substack{w, w' \in Z_q \\ |w-w'| \leq 2^{-j}}} [w, w'].$$

We claim that Y_q is connected. To this end, let $z_0 \in Z_q$ be a point in Z_q with minimal distance to q , and let $z \in Z_q$ be any other vertex. We now construct a chain of points $z_0, z_1, \dots, z_M \in Z_q$ such that $z_M = z$, and so that for every $i \in \{0, 1, \dots, M-1\}$, $[z_i, z_{i+1}] \subset Y_q$. We proceed by induction, noting that the base vertex z_0 has already been established. Fix some integer $i \geq 0$ and assume we have already some point $z_i \in Z_q$, furthermore satisfying that if $i \geq 1$ then $[z_{i-1}, z_i] \subset Y_q$. Let $z_i = (z_i^1, \dots, z_i^N)$. If $z_i = z$, then we terminate the process. If not, then let $z = (z^1, \dots, z^N)$ and let $m \in \{1, \dots, N\}$ be the first index such that $z_i^m \neq z^m$. If $z_i^m < z^m$, then let $z_{i+1}^m = z_i^m + 2^{-j}$, and if instead $z_i^m > z^m$, then let $z_{i+1}^m = z_i^m - 2^{-j}$. For each $\ell \in \{1, \dots, N\}$ other than $\ell = m$, let $z_{i+1}^\ell = z_i^\ell$, and let $z_{i+1} = (z_{i+1}^1, \dots, z_{i+1}^N)$. Note that since $|z_0 - q|$ is minimal, we have that for every $\ell \in \{1, \dots, N\}$, the difference

in coordinates $|z_0^\ell - q^\ell|$ is also minimal among points in Z_q . Furthermore, by the construction of z_{i+1} , we have that for every $\ell \in \{1, \dots, N\}$, z_{i+1}^ℓ is between z_0^ℓ and z^ℓ , possibly equal to one of these. Then $|z_i - z_{i+1}| = 2^{-j}$ and $|z_{i+1} - q| \leq |z - q| \leq 2^{2-j}\sqrt{N}$, so $z_{i+1} \in Z_q$ and $[z_i, z_{i+1}] \subset Y_q$. This process terminates after finitely many steps, completing the construction, and so we have that Y_q is connected.

Note that for points $q, u \in K$ which have $|q - u| \leq 2^{-j}$, we have that $Y_q \cap Y_u \neq \emptyset$. Now let $q \in K \cap \partial[2^{-j} - 2^j, 2^j - 2^{-j}]^N$, and since K is connected there exists finite chain of points $\{p_1, \dots, p_M\} \subset K$ so that $p_1 = p$, $p_M = q$, and for every $i \in \{1, \dots, M-1\}$, $|p_i - p_{i+1}| \leq 2^{-j}$. Then $\bigcup_{i=1}^M Y_{p_i} \subset X_j$ is a connected set containing v and intersecting with $\partial[2^{-j} - 2^j, 2^j - 2^{-j}]^N$, so it is contained in B , and therefore $B \cap \partial[2^{-j} - 2^j, 2^j - 2^{-j}]^N \neq \emptyset$. $\square$

Lemma 3.13.3. *For each $j \in \mathbb{N}$ there exists $n_j \in \mathbb{N}$ such that $G_{n_j} \in \mathcal{G}_{2j}$ and that X_j is a re-scaled copy of $\text{Im}(G_{n_j}) \cap [-1 + 4^{-j}, 1 - 4^{-j}]^N$.*

Before proceeding to the proof, note that while every $\text{Im}(G_{n_j}) \subset [-1, 1]^N$ is connected, intersecting with $[-1 + 4^{-j}, 1 - 4^{-j}]^N$ in effect removes the boundary, which may disconnect the image. Essentially, this lemma states that every X_j extends to some $\text{Im}(G_{n_j})$ (up to re-scaling) by attaching the components of X_j along the boundary $\partial[-1, 1]^N$.

Proof of Lemma 3.13.3. If we define

$$\begin{aligned} V^j &= 4^{-j} \left(2^j W_j \cup (\partial[-4^j, 4^j]^N \cap \mathbb{Z}^N) \right), \\ E^j &= \{ \{v, v'\} : v, v' \in V^j, |v - v'| = 4^{-j} \}, \end{aligned}$$

then (V^j, E^j) is connected by Lemma 3.13.2, so it is equal to $G_{n_j} \in \mathcal{G}_{2j}$ for some $n_j \in \mathbb{N}$. Furthermore, $\text{Im}(G_{n_j}) \cap [-1 + 4^{-j}, 1 - 4^{-j}]^N = 2^{-j} X_j$, as desired. $\square$

It follows that

$$\begin{aligned}\text{excess}(T \cap [2^{-j} - 2^j, 2^j - 2^{-j}]^N, 2^j \text{Im}(G_{n_j})) &\leq 2^{2-j} \sqrt{N} \\ \text{excess}(2^j \text{Im}(G_{n_j}) \cap [2^{-j} - 2^j, 2^j - 2^{-j}]^N, T) &\leq 2^{2-j} (\sqrt{N} + 1/4).\end{aligned}$$

This implies that for every $R > 0$,

$$\lim_{j \rightarrow \infty} \text{excess}(T \cap \overline{B}(\mathbf{0}, R), 2^j \text{Im}(G_{n_j})) = 0, \quad \lim_{j \rightarrow \infty} \text{excess}(2^j \text{Im}(G_{n_j}) \cap \overline{B}(\mathbf{0}, R), T) = 0. \quad (3.12.5)$$

Proof of Theorem 3.13. Let $\rho_j = 2^j(r_{n_j})^{-1} > 0$. We claim that $\lim_{j \rightarrow \infty} \rho_j H = T$, with respect to the Attouch-Wets topology.

Recall that $k_{n_j} = 2j$ for each $j \in \mathbb{N}$, so $r_{n_j+1}\rho_j \leq 2^{-n_j-j+1}$ and $\rho_j r_{n_j} = 2^j$. Hence, for every $j \in \mathbb{N}$,

$$(\rho_j H_{n_j}) \setminus (2^{-n_j-j+1}[-1, 1]^N) = (2^j \text{Im}(G_{n_j})) \setminus (2^{-n_j-j+1}[-1, 1]^N).$$

Fix $R > 0$, and let $j_0 \in \mathbb{N}$ such that $2^{j_0-1} \geq R$. For every integer $j \geq j_0$, we have

$$\begin{aligned}\rho_j H \cap \overline{B}(\mathbf{0}, R) &\subset ((\rho_j H_{n_j}) \cup (2^{-n_j-j+1}[-1, 1]^N)) \cap \overline{B}(\mathbf{0}, R) \\ &= (2^j \text{Im}(G_{n_j})) \cup (2^{-n_j-j+1}[-1, 1]^N) \cap \overline{B}(\mathbf{0}, R).\end{aligned}$$

Thus by design of G_{n_j} and by the monotonicity and subadditivity properties of excess (see Remark 3.5), we have

$$\begin{aligned}\text{excess}(\rho_j H \cap \overline{B}(\mathbf{0}, R), T) &\leq \text{excess}((2^j \text{Im}(G_{n_j})) \cup (2^{-n_j-j+1}[-1, 1]^N) \cap \overline{B}(\mathbf{0}, R), T) \\ &\leq \text{excess}(2^j \text{Im}(G_{n_j}) \cap \overline{B}(\mathbf{0}, R), T) + \text{excess}(2^{-n_j-j+1}[-1, 1]^N, T).\end{aligned}$$

As $j \rightarrow \infty$, by (3.12.5) the first term in the latter sum approaches 0. Furthermore, since $\mathbf{0} \in T$ and as $j \rightarrow \infty$ $\text{diam}(2^{-n_j-j+1}[-1, 1]^N)$ approaches 0, as $j \rightarrow \infty$ the second term in the latter sum approaches 0 as well.

We now show that $\lim_{j \rightarrow \infty} \text{excess}(T \cap \overline{B}(\mathbf{0}, R), \rho_j H) = 0$. Note that

$$(\rho_j H_{n_j}) \setminus (\rho_j r_{n_j+1}[-1, 1]^N) \subset \rho_j H,$$

so by monotonicity and triangle inequality for excess (as in Remark 3.5), we have

$$\begin{aligned} \text{excess}(T \cap \overline{B}(\mathbf{0}, R), \rho_j H) &\leq \text{excess}(T \cap \overline{B}(\mathbf{0}, R), (\rho_j H_{n_j}) \setminus (\rho_j r_{n_j+1}[-1, 1]^N)) \\ &= \text{excess}(T \cap \overline{B}(\mathbf{0}, R), (2^j \text{Im}(G_{n_j})) \setminus (\rho_j r_{n_j+1}[-1, 1]^N)) \\ &\leq \text{excess}(T \cap \overline{B}(\mathbf{0}, R), 2^j \text{Im}(G_{n_j})) \\ &\quad + \text{excess}(2^j \text{Im}(G_{n_j}), (2^j \text{Im}(G_{n_j})) \setminus (\rho_j r_{n_j+1}[-1, 1]^N)). \end{aligned}$$

As $j \rightarrow \infty$, the first term of the latter sum approaches 0 by (3.12.5) and by design of the sets G_{n_j} . In the second term of the latter sum, the sets differ only inside $\rho_j r_{n_j+1}[-1, 1]^N$, so the excess is at most $\text{diam}(\rho_j r_{n_j+1}[-1, 1]^N) \leq 2\rho_j r_{n_j+1}\sqrt{N}$, which approaches 0 as $j \rightarrow \infty$. This completes the proof, by definition of convergence $\lim_{j \rightarrow \infty} \rho_j H$ in the Attouch-Wets topology. $\square$

Bibliography

- Alberti, G. and Ottolini, M. (2017). On the structure of continua with finite length and Golab’s semicontinuity theorem. *Nonlinear Anal.*, 153:35–55. [3](#), [17](#), [25](#), [61](#), [118](#)
- Assouad, P. (1983). Plongements lipschitziens dans $\mathbf{R}^n$. *Bull. Soc. Math. France*, 111(4):429–448. [50](#), [104](#)
- Badger, M., Engelstein, M., and Toro, T. (2017). Structure of sets which are well approximated by zero sets of harmonic polynomials. *Anal. PDE*, 10(6):1455–1495. [52](#)
- Badger, M., Krandel, J., and Vellis, V. (2024). Gromov-Hausdorff tangents of rectifiable metric spaces. *in preparation*. [44](#)
- Badger, M. and Lewis, S. (2015). Local set approximation: Mattila-Vuorinen type sets, Reifenberg type sets, and tangent sets. *Forum Math. Sigma*, 3:Paper No. e24, 63. [9](#), [43](#), [52](#), [53](#), [55](#)
- Badger, M. and McCurdy, S. (2023a). Subsets of rectifiable curves in Banach spaces I: Sharp exponents in traveling salesman theorems. *Illinois J. Math.*, 67(2):203–274. [4](#)
- Badger, M. and McCurdy, S. (2023b). Subsets of rectifiable curves in Banach spaces II: Universal estimates for almost flat arcs. *Illinois J. Math.*, 67(2):275–331. [4](#)

- Badger, M., Naples, L., and Vellis, V. (2019). Hölder curves and parameterizations in the analyst’s traveling salesman theorem. *Adv. Math.*, 349:564–647. [4](#), [45](#)
- Badger, M. and Schul, R. (2023). Square packings and rectifiable doubling measures. [4](#), [45](#)
- Badger, M. and Vellis, V. (2017). Geometry of measures in real dimensions via hölder parameterizations. *The Journal of Geometric Analysis*, 29:1153 – 1192. [4](#), [45](#)
- Badger, M. and Vellis, V. (2021). Hölder parameterization of iterated function systems and a self-affine phenomenon. *Anal. Geom. Metr. Spaces*, 9(1):90–119. [4](#), [6](#), [13](#), [14](#), [16](#), [45](#), [49](#), [93](#), [105](#), [106](#)
- Balogh, Z. M. and Züst, R. (2020). Box-counting by Hölder’s traveling salesman. *Arch. Math. (Basel)*, 114(5):561–572. [4](#), [45](#)
- Banaji, A. and Fraser, J. M. (2023). Intermediate dimensions of infinitely generated attractors. *Trans. Amer. Math. Soc.*, 376(4):2449–2479. [8](#), [14](#)
- Bandt, C., Hung, N. V., and Rao, H. (2006). On the open set condition for self-similar fractals. *Proc. Amer. Math. Soc.*, 134(5):1369–1374. [81](#)
- Beer, G. (1993). *Topologies on closed and closed convex sets*, volume 268 of *Mathematics and its Applications*. Kluwer Academic Publishers Group, Dordrecht. [9](#), [52](#), [53](#), [54](#), [101](#)
- Besicovitch, A. S. (1944). On the existence of tangents to rectifiable curves. *J. London Math. Soc.*, 19:205–207. [44](#)
- Bonk, M. and Heinonen, J. (2004). Smooth quasiregular mappings with branching. *Publ. Math. Inst. Hautes Études Sci.*, (100):153–170. [47](#)
- Burago, D., Burago, Y., and Ivanov, S. (2001). *A course in metric geometry*. Graduate studies in mathematics ; v. 33. American Mathematical Society, Providence, Rhode Island. [9](#)

- Calderón, A. P. (1951). On the differentiability of absolutely continuous functions. *Rivista Mat. Univ. Parma*, 2:203–213. [43](#)
- Chousionis, V., Leykekhman, D., Urbański, M., and Wendt, E. (2024). Rigorous hausdorff dimension estimates for conformal fractals. [arXiv:2408.06330](#). [7](#)
- Dai, X.-R., Luo, J., Ruan, H.-J., Wang, Y., and Xiao, J.-C. (2023). Connectedness and local cut points of generalized Sierpinski carpets. *Asian J. Math.*, 27(4):529–570. [92](#)
- David, G. and Semmes, S. (1997). *Fractured fractals and broken dreams*, volume 7 of *Oxford Lecture Series in Mathematics and its Applications*. The Clarendon Press, Oxford University Press, New York. Self-similar geometry through metric and measure. [43](#)
- Evans, L. C. and Gariepy, R. F. (2015). *Measure theory and fine properties of functions*. Textbooks in Mathematics. CRC Press, Boca Raton, FL, revised edition. [3](#), [59](#)
- Falconer, K. J. (1986). *The geometry of fractal sets*, volume 85 of *Cambridge Tracts in Mathematics*. Cambridge University Press, Cambridge. [3](#), [44](#), [60](#)
- Fan, A. H. and Lau, K.-S. (1999). Iterated function system and Ruelle operator. *J. Math. Anal. Appl.*, 231(2):319–344. [105](#), [115](#)
- Farkas, A. and Fraser, J. M. (2015). On the equality of Hausdorff measure and Hausdorff content. *J. Fractal Geom.*, 2(4):403–429. [13](#)
- Federer, H. (1969). *Geometric measure theory*, volume Band 153 of *Die Grundlehren der mathematischen Wissenschaften*. Springer-Verlag New York, Inc., New York. [9](#), [43](#)
- Fraser, J. M., Henderson, A. M., Olson, E. J., and Robinson, J. C. (2015). On the Assouad dimension of self-similar sets with overlaps. *Adv. Math.*, 273:188–214. [13](#)

- Gromov, M. (1981). Groups of polynomial growth and expanding maps. *Inst. Hautes Études Sci. Publ. Math.*, (53):53–73. [9](#), [43](#)
- Gromov, M. (1999). *Metric structures for Riemannian and non-Riemannian spaces*, volume 152 of *Progress in Mathematics*. Birkhäuser Boston, Inc., Boston, MA. Based on the 1981 French original [MR0682063 (85e:53051)], With appendices by M. Katz, P. Pansu and S. Semmes, Translated from the French by Sean Michael Bates. [9](#), [43](#)
- Hata, M. (1985). On the structure of self-similar sets. *Japan J. Appl. Math.*, 2(2):381–414. [6](#), [13](#), [49](#)
- Heinemann, S.-M. and Urbański, M. (2002). Hausdorff dimension estimates for infinite conformal IFSs. *Nonlinearity*, 15(3):727–734. [8](#), [14](#)
- Heinonen, J. (2001). *Lectures on analysis on metric spaces*. Universitext. Springer-Verlag, New York. [3](#)
- Hocking, J. G. and Young, G. S. (1961). *Topology*. Addison-Wesley Publishing Co., Inc., Reading, Mass.-London. [25](#)
- Hutchinson, J. E. (1981). Fractals and self-similarity. *Indiana Univ. Math. J.*, 30(5):713–747. [5](#), [13](#), [17](#), [47](#), [58](#), [59](#), [106](#)
- Jones, P. W. (1990). Rectifiable sets and the traveling salesman problem. *Invent. Math.*, 102(1):1–15. [4](#), [44](#)
- Jordan, T. and Rams, M. (2012). Increasing digit subsystems of infinite iterated function systems. *Proc. Amer. Math. Soc.*, 140(4):1267–1279. [7](#), [14](#)
- Kesseböhmer, M. and Zhu, S. (2006). Dimension sets for infinite IFSs: the Texan conjecture. *J. Number Theory*, 116(1):230–246. [8](#), [14](#)
- Li, W. (2021). Quasisymmetric embeddability of weak tangents. *Ann. Fenn. Math.*, 46(2):909–944. [102](#)

- Mackay, J. M. and Tyson, J. T. (2010). *Conformal dimension*, volume 54 of *University Lecture Series*. American Mathematical Society, Providence, RI. Theory and application. [68](#)
- Martín, M. A. and Mattila, P. (1993). Hausdorff measures, Hölder continuous maps and self-similar fractals. *Math. Proc. Cambridge Philos. Soc.*, 114(1):37–42. [4](#), [45](#)
- Martín, M. A. and Mattila, P. (2000). On the parametrization of self-similar and other fractal sets. *Proc. Amer. Math. Soc.*, 128(9):2641–2648. [4](#), [45](#)
- Mattila, P. (1995). *Geometry of sets and measures in Euclidean spaces*, volume 44 of *Cambridge Studies in Advanced Mathematics*. Cambridge University Press, Cambridge. Fractals and rectifiability. [9](#)
- Mauldin, R. D. (1995). Infinite iterated function systems: theory and applications. In *Fractal geometry and stochastics (Finsterbergen, 1994)*, volume 37 of *Progr. Probab.*, pages 91–110. Birkhäuser, Basel. [7](#), [14](#)
- Mauldin, R. D., Mayer, V., and Urbański, M. (2001). Rigidity of connected limit sets of conformal IFSs. *Michigan Math. J.*, 49(3):451–458. [15](#)
- Mauldin, R. D., Szarek, T., and Urbański, M. (2009). Graph directed Markov systems on Hilbert spaces. *Math. Proc. Cambridge Philos. Soc.*, 147(2):455–488. [8](#), [14](#)
- Mauldin, R. D. and Urbański, M. (1996). Dimensions and measures in infinite iterated function systems. *Proc. London Math. Soc. (3)*, 73(1):105–154. [7](#), [14](#), [15](#)
- Mauldin, R. D. and Urbański, M. (1999). Conformal iterated function systems with applications to the geometry of continued fractions. *Trans. Amer. Math. Soc.*, 351(12):4995–5025. [7](#), [14](#)
- Mauldin, R. D. and Urbański, M. (2002). Fractal measures for parabolic IFS. *Adv. Math.*, 168(2):225–253. [8](#), [14](#)

- McShane, E. J. (1934). Extension of range of functions. *Bull. Amer. Math. Soc.*, 40(12):837–842. [102](#)
- Nadler, Jr., S. B. (1992). *Continuum theory*, volume 158 of *Monographs and Textbooks in Pure and Applied Mathematics*. Marcel Dekker, Inc., New York. An introduction. [51](#)
- Okikiolu, K. (1992). Characterization of subsets of rectifiable curves in $\mathbf{R}^n$. *J. London Math. Soc. (2)*, 46(2):336–348. [4](#), [44](#)
- Preiss, D. (1987). Geometry of measures in $\mathbf{R}^n$: distribution, rectifiability, and densities. *Ann. of Math. (2)*, 125(3):537–643. [9](#)
- Remes, M. (1998). Hölder parametrizations of self-similar sets. *Ann. Acad. Sci. Fenn. Math. Diss.*, (112):68. [6](#), [14](#), [45](#)
- Rempe-Gillen, L. and Urbański, M. (2016). Non-autonomous conformal iterated function systems and Moran-set constructions. *Trans. Amer. Math. Soc.*, 368(3):1979–2017. [8](#), [14](#)
- Romney, M. and Vellis, V. (2017). Bi-Lipschitz embedding of the generalized Grushin plane into Euclidean spaces. *Math. Res. Lett.*, 24(4):1177–1203. [47](#)
- Schief, A. (1994). Separation properties for self-similar sets. *Proc. Amer. Math. Soc.*, 122(1):111–115. [5](#), [13](#), [59](#), [105](#)
- Schief, A. (1996). Self-similar sets in complete metric spaces. *Proc. Amer. Math. Soc.*, 124(2):481–490. [13](#)
- Schul, R. (2007a). Analyst’s traveling salesman theorems. A survey. In *In the tradition of Ahlfors-Bers. IV*, volume 432 of *Contemp. Math.*, pages 209–220. Amer. Math. Soc., Providence, RI. [44](#)
- Schul, R. (2007b). Subsets of rectifiable curves in Hilbert space—the analyst’s TSP. *J. Anal. Math.*, 103:331–375. [4](#)

- Seuret, S. and Wang, B.-W. (2015). Quantitative recurrence properties in conformal iterated function systems. *Adv. Math.*, 280:472–505. [8](#), [14](#)
- Shaw, E. and Vellis, V. (2024a). Hölder curves with exotic tangent spaces. arXiv:2409.13662. [11](#), [42](#)
- Shaw, E. and Vellis, V. (2024b). Parametrizability of infinitely generated attractors. *Ann. Fenn. Math.*, 49(1):81–97. [10](#), [12](#), [45](#)
- Stroock, D. W. (1993). *Probability theory, an analytic view*. Cambridge University Press, Cambridge. [52](#), [85](#)
- Urbański, M. and Zdunik, A. (2002). Hausdorff dimension of harmonic measure for self-conformal sets. *Adv. Math.*, 171(1):1–58. [8](#), [14](#)
- Vytnova, P. and Wormell, C. (2024). Hausdorff dimension of the apollonian gasket. arXiv:2406.04922. [8](#)
- Whyburn, G. T. (1935). Concerning continua of finite degree and local separating points. *American Journal of Mathematics*, 57(1):11–16. [56](#)
- Wu, J.-M. (2015). Bilipschitz embedding of Grushin plane in $\mathbb{R}^3$. *Ann. Sc. Norm. Super. Pisa Cl. Sci. (5)*, 14(2):633–644. [47](#)
- Zygmund, A. (2002). *Trigonometric series. Vol. I, II*. Cambridge Mathematical Library. Cambridge University Press, Cambridge, third edition. With a foreword by Robert A. Fefferman. [43](#)

Vita

Eve Shaw was born and raised in Cincinnati, Ohio. After high school, she studied at Thomas More University, where she earned her Bachelor of Science in Physics and Bachelor of Arts in Mathematics, *Summa cum Laude*, in 2019. During this time, she worked as an undergraduate math and physics tutor and collaborated with Professors Robert Riehemann and Harold Smith to write her senior projects.

After completing her undergraduate education, in 2019 Eve began study as a doctoral student in the mathematics department at the University of Tennessee, Knoxville, where she obtained her Master of Science in Mathematics in 2021. She then began research under the advisement of Professor Vyron Vellis. At the time this dissertation is submitted, Eve is expected to graduate from the University of Tennessee, Knoxville with a Doctorate in Mathematics.

Eve's research interests revolve primarily around Geometric Measure Theory, Fractal Geometry, and Measure Theory more generally. She has accepted an offer from the University of Illinois, Urbana-Champaign as a J.L. Doob Research Assistant Professor, where she plans to spend the next three years.