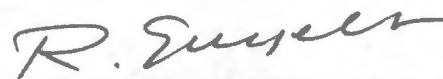


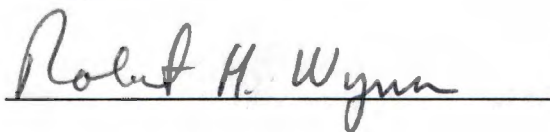
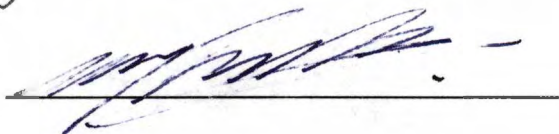
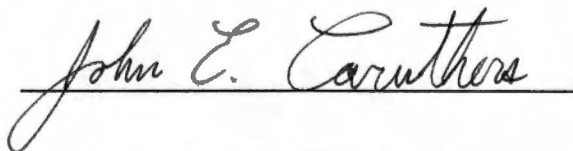
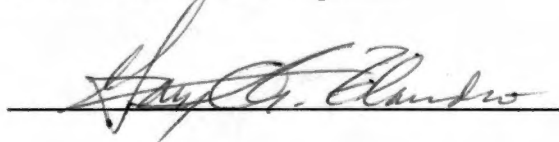
To the Graduate Council:

I am submitting herewith a dissertation written by Emma Jewell Cooley entitled "A Dynamic Finite Element Modeling Technique For Two Dimensional Structures." I have examined the final copy of this dissertation for form and content and recommend that it be accepted in partial fulfillment of the requirements for the degree of Doctor of Philosophy, with a major in Mechanical Engineering.



Remi C. Engels, Major Professor

We have read this dissertation
and recommend its acceptance:



Accepted for the Council:



Vice Provost
and Dean of the Graduate School

**A DYNAMIC FINITE ELEMENT MODELING TECHNIQUE
FOR TWO-DIMENSIONAL STRUCTURES**

**A Dissertation
Presented for the
Doctor of Philosophy
Degree
The University of Tennessee, Knoxville**

**Emma Jewell Cooley
August 2000**

DEDICATION

Several years ago, when I decided to pursue a master's degree in engineering, I met a very unique person. From the beginning, for reasons unknown to me, this person cared about my success.

After completing my master's degree, he encouraged me to pursue a doctorate. He informed me that it would be difficult, and made himself available from that moment on. After completing my course work, the dissertation was to begin. He asked to be placed on my committee, and was. At that time, I did not realize the impact of such a decision.

As the years past, many days and evenings were spent at his computer, solving problems, and extracting data from the finite element program, ANSYS. We ran many models, in an effort to obtain solutions to the dissertation problem, and to prove our theory. When the results were discouraging, I wanted to give up, but he would say, "Press On", and always gave a thumbs up.

Last year, he became very ill. Still, he continued to encourage me, in spite of everything he faced. Some how, he always found the bright side, and never complained, nor gave up. What a great teacher! What contentment, and courage!

What a blessing, and a privilege to have known such a special person. He was my mentor, cheerleader, colleague, and mostly my friend. I miss him.

"Brothers, we do not want you to be ignorant about those who fall asleep or grieve like the rest of men, who have no hope. We believe that Jesus died and rose again and so we believe that God will bring with Jesus those who have fallen asleep in him"
I Thessalonians 4:13-14

In memory of Dr. William Quinton Gurley, Sr.

ACKNOWLEDGEMENTS

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The author wishes to extend a special thanks to Dr. Robert Wynn, for accepting to serve as a committee member, due to the untimely death of Dr. William Gurley, Sr.

Finally, the author wishes to thank her family, for their patience and understanding.

ABSTRACT

The standard finite element method is used to analyze complex structures that require the solution of unsolvable partial differential equations. The solution for this method is based solely on static constraint modes (SCM's), and consider only linear displacements at the boundaries of the elements. Thus, a large number of degrees of freedom are needed for a dynamic analysis.

The purpose of this research is to develop a modeling technique, to reduce the number of degrees of freedom, needed to obtain an exact solution for the dynamic analysis of two-dimensional structures. To do so, it is necessary to account for the additional motion neglected by the standard finite element method. The dynamic finite element method accounts for this additional motion with the inclusion of interface restrained assumed modes (IRAM's), and frequencies in the mass and stiffness matrices .

Fortran programs and subroutines have been written for the dynamic analysis of the triangular and quadrilateral elements.

It is noted that the number of interface restrained assumed modes (IRAM's) needed to obtain a desired number of converged frequencies, is dependent on element size, model stiffness, and the desired percentage error.

The dynamic finite element method offers an average savings of 44%, for less than 1% error, an average savings of 54% for less than 3% error, and an average savings of 62% for less than 5 % error, when the triangular element is used for the finite element model.

The dynamic finite element method offers an average savings of 27%, for the 24-Node quadrilateral element, and an average savings of 11% for the 16-Node quadrilateral element for less than 1% error.

The dynamic finite element method has consistently used fewer degrees of freedom, than the standard finite element method, to obtain a desired number of converged frequencies. The inclusion of interface restrained assumed modes (IRAM's) and corresponding frequencies, provide an accurate model, and thus more frequencies converge.

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NOMENCLATURE

$\mathbf{R}$	Position Vector
$\mathbf{e}$	Lagrangian Elastic Displacement Vector
Φ	Assumed Modes
$\mathbf{q}$	Generalized Displacement Vector
$\mathbf{q}_I$	Interface Displacement Vector
T	Kinetic Energy
$\mathbf{M}$	Mass Matrix
U	Potential Energy
$\boldsymbol{\varepsilon}$	Strain Vector
$\boldsymbol{\tau}$	Stress Vector
$\mathbf{C}$	Material Stiffness Matrix
$\mathbf{K}$	Stiffness Matrix
$\mathbf{F}$	Generalized Force Vector
$\mathbf{u}$	Displacement Vector
ω	Circular Natural Frequency
$\overline{\mathbf{X}}_I$	Starting Iteration Vectors
E	Young's Modulus of Elasticity
ν	Possion's Ratio
A	Area of Element
ρ	Mass Density

CHAPTER 1

INTRODUCTION

1.1 Background

Structural engineers are interested in the deformation and stress state of structures subjected to applied forces and tractions. Many analytical solution techniques exist to determine these deformation and stress states in a structure. For a limited number of relatively simple configurations solutions can be found in closed form. These results are important as verification and estimation tools but are of little value in more practical situations. Most practical structures are quite complicated and require numerical solutions that are naturally approximate. The complexity of a structure is the result of irregularly shaped boundaries, variable thicknesses, and/or multiple material types. The partial differential equations governing the behavior of such structures always exist and can be shown to have a unique solution. Finding that solution or an acceptable approximation thereof is a different and more difficult matter.

Many numerical solution techniques have been developed over the years. The most popular one is the Finite Element Method (FE). The FE Method divides the structure into smaller and simpler substructures called finite elements. These finite elements are easier to model and are assembled to form a model for the entire structure or system. The resulting system equations are then solved using standard numerical solution techniques for sets of first or second order ordinary differential equations.

To find the response of a structure to a rapidly applied force system, it is common to first execute a modal analysis. The displacement response is then written as a linear combination of the computed mode shapes. For low-frequency force systems the structure will significantly respond only in its lower natural frequencies. Therefore, it is common to practice to truncate the higher frequency mode shapes thereby reducing the cost of the transient response analysis.

In general, the cost of a modal analysis is proportional to the cube of the model order (i.e., the dimension of the mass and stiffness matrices). Highly flexible structures often have many low frequency modes and are therefore quite expensive to analyze via the modal analysis approach. To reduce the computational cost of such an eigenvalue problem one can either reduce the order of the model or derive better solution algorithms.

The order of a model can be reduced at three different levels: the element level; the component level; and the system level. Order reduction at the element level is achieved by deriving element models with superior convergence characteristics. Thus, the same fidelity is obtained with fewer element degrees of freedom thereby reducing the overall order of the system. Examples of such techniques are the p - and h - versions of the FE method introduced by Ziendiewicz, Gago, and Kelly [11] for the static case. In the p -version higher order polynomial shape functions are introduced during successive refinement stages while the element size remains the same. Convergence improvement is weighed against the increase of the number of element nodes. The h -version of the finite element method is the more conventional one where the number of degrees of freedom of the element is held constant but the size is successively reduce. Peano [13] and Babusca [16] compare the two versions and discuss the advantages of the p -version over the h -version. In particular, in most cases the same accuracy is achieved with the use of fewer overall degrees of freedom. In the dynamic case Benninghof and Meirovitch [12] show that the p -version of the FE method actually blends the excellent convergence properties of the Rayleigh-Ritz method with the modeling flexibility of the standard h -version of the FE method.

Order reduction at the component level of the structure consists of truncating the modes with frequencies higher than a given cut-off frequency. This class of techniques are called component mode synthesis methods or sub-structuring methods and were originated by Hurty [14]. Complex structures are subdivided into simpler components or substructures. These substructures can then be modeled independently. In particular,

modal analysis is performed and a set of component modes are computed. The components are then reassembled by enforcing the compatibility requirements. In the process, the number of component modes are truncated according to a given cut-off frequency usually equal to twice the desired frequency content of the overall model. The resulting order reduction is usually quite significant. Hurty introduced the concept of static constraint modes thereby facilitating the component mode synthesis process. Later, Craig and Bampton [17] observed that in a formal sense no distinction between the rigid body modes and the static constraint modes is needed. This observation significantly simplifies the implementation of this class of sub-structuring techniques. The Craig-Bampton method for component mode synthesis is very powerful and is still the most popular among the many sub-structuring techniques published since Hurty. Other techniques differ mainly in the type of modes used to model the components. Benfield and Hruda [15] developed mass and stiffness loaded substructure modes, partially taking into account the dynamic effect neighboring substructures have on the substructure in question. A method that uses flexible links to interconnect free-free modes was developed by Kuhar and Stabile [33]. The concept of attachment modes was developed by Bamford [21] for constrained structures and was extended by Hintz [19] for free-free structures. Attachment modes are capable of producing the exact static deformation of a substructure subject to static loading at the interface degrees of freedom. The transformations involved leave the interface degrees of freedom untouched so that coupling can be achieved in a straightforward manner via the standard interface compatibility conditions. MacNeal [25] discovered a method to account for the stiffness contribution of the truncated modes by adding only a small number of degrees of freedom to the model. This contribution is called the residual flexibility. This idea was extended by Rubin [23] to include the second order terms of the residual flexibility.

A third option for model order reduction occurs at the system level. Guyan [27] presents a displacement transformation based on the classic static condensation technique

that relates a set of dependent degrees of freedom to a set of independent degrees of freedom. The assumption is that the degrees of freedom are related in an approximately static manner. The dependent degrees of freedom are removed from the model, resulting in a substantially smaller model. The drawback is that the mass and stiffness are redistributed in such a way as to fully populate the reduced mass and stiffness matrices. This may in the end defeat the purpose of the method because of the availability of extremely efficient solution techniques for sparsely populated matrix equations. Geradin and Thomas [24] and [31] give error bounds for the frequencies and mode shapes obtained via the Guyan reduction. Kidder [26] shows that the Guyan transformation represents the first term in a series expansion of the exact dependent-independent transformation. Flax [18] comments that for the Guyan reduction to be valid, all frequencies obtained by constraining all the independent degrees of freedom, must be larger than the frequencies of interest. Another problem with the Guyan reduction technique is the choice of the dependent degrees of freedom. Henshell and Ong [22] propose the elimination of one dependent degree of freedom at a time based on the magnitude of the associated stiffness to mass ratio.

Several order reduction techniques have been identified as useful to achieving the goal of this study. In general, useful techniques should take advantage of the sparseness and special form of the mass and stiffness matrices. They should also exploit the physical properties associated with the structural eigenvalue problem at hand. Coordinate relaxation techniques are very effective if only a few eigenvalues and vectors are needed. The Lanczos method is also effective if correctly implemented. Nour-Omid, Parlett and Taylor [29] find that the Lanczos method competes well with other methods such as the subspace iteration method. The latter method is well established and has excellent convergence properties. Jennings [30] and [32] shows that there is a connection between Guyan reduction technique and the first iteration of the subspace iteration method. Cheu, Johnson, and Craig [28] then suggest the Guyan reduction vectors as starting vectors for a

subspace iteration. Since these vectors form a first order approximation subspace of the desired dominant subspace it is expected that only a few iterations will lead to convergence.

1.2. Overview of the Present Study

In this dissertation, a dynamic finite element reduced order modeling technique pioneered by Engels [3] is extended to two-dimensional finite elements. In the next chapter, a detailed review of this technique is presented. First, the assumed modes method with static constraint modes (SCM's) and interface restrained assumed modes (IRNM's) is presented and the relationship between the SCM and the standard finite element shape functions is revealed. It is shown that the Craig-Bampton component mode synthesis technique allows for a complete representation of the element displacement vector. The standard finite element method is then shown as an approximation where all the IRNM's are truncated. Instead of relying on mesh refinement, it is proposed to retain a few of the IRNM's at the element level. This produces superior convergence at the system level with fewer generalized displacement coordinates.

Chapter 3 deals with the development of dynamic finite element models for the triangle and quadrilateral in terms of (SCM's) and interface restrained normal modes (IRNM's). Computer subroutines were written for the dynamic finite elements.

In Chapter 4, several demonstration problems are presented. Comparisons of the convergence properties of the dynamic finite element method are made with those of the standard finite element method.

The final Chapter 5, offers general conclusions and observations about the practical implementation of the new dynamic finite elements.

CHAPTER 2

REVIEW OF DYNAMIC FINITE ELEMENT METHOD

2.1 Introduction

In this chapter a concise review is given of the dynamic finite element modeling method. This method was first presented in reference 3 and further developed in reference 25. There are essentially two parts to the method. The first part consists of derivation of the element equations of motion. This is done for a general finite element. The second part of the method consists of the assembly process for the elements. Again this is done for generic elements so that the method is valid for all types of finite elements. Some examples of element types will be given as well as a summary of all elements developed thus far. The next section presents the assumed modes method which underlies all subsequent developments.

2.2 Assumed Modes Method

The structural component E in figure 1 is shown in its deformed state. The component is assumed to undergo small displacements about a stable equilibrium position. The inertial position vector $\mathbf{R}$ of point P is

$$\mathbf{R} = \mathbf{R}_o + \mathbf{r} + \mathbf{e} \quad (2.2.1)$$

where $\mathbf{r}$ is the position vector of point A relative to the local reference frame. The point P coincides with A in the undeformed state. The vector $\mathbf{e}$ is the Lagrangian elastic displacement vector measured relative to the local reference frame.

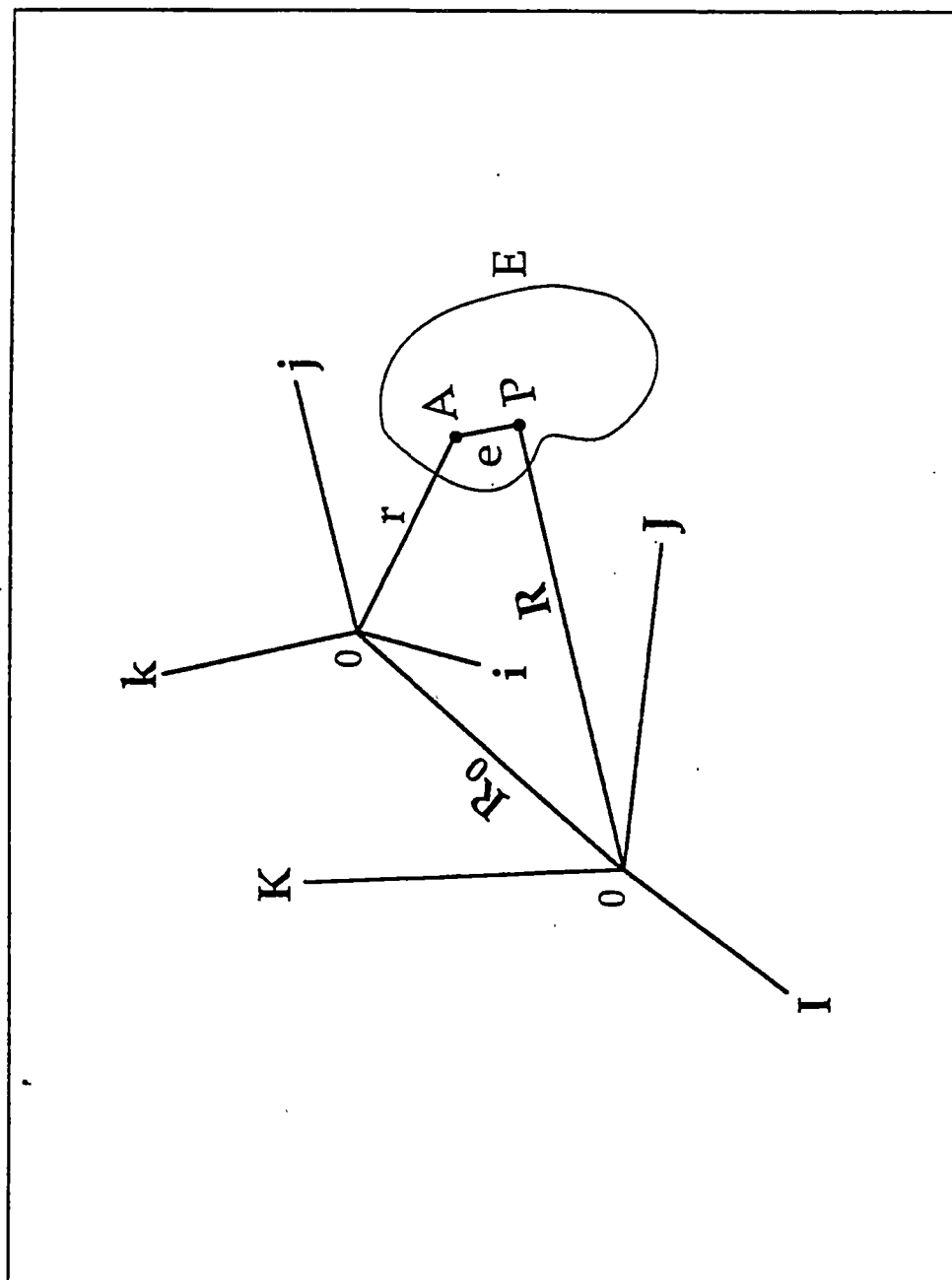


Figure 1. Structural Element and Reference Frames :

2.2.1 Assumed Modes

The first step in the assumed modes method [24] is the approximation of the elastic displacement vector $\mathbf{e}(x, y, z, t) = [u \ v \ w]^T$ by a linear combination of assumed modes $\Phi_i(x, y, z)$ ($i = 1, \dots, n$)

$$\mathbf{e} = \sum_{i=1}^n \Phi_i(x, y, z) q_i(t) \triangleq \Phi \mathbf{q} \quad (2.2.2)$$

The time dependent coefficients $\mathbf{q}(t)$ are called generalized coordinates.

The assumed modes must be admissible functions, i.e., they must be complete, linearly independent, sufficiently differentiable (at least to an order equal to that of the highest derivative in the element strain energy), and satisfy at least the geometric boundary conditions of the element E .

The second step in the method involves the choice of the assumed modes.

2.2.2 Static Constraint Modes and Interface Restrained Assumed Modes

The physical interface of the element E is defined as the collection of all points, curves, and surfaces the element has in common with its neighbors. If this interface can be suitably modeled, all interface displacement coordinates can be collected into a single interface displacement vector $\mathbf{q}_I(t)$. This vector $\mathbf{q}_I(t)$ is chosen as a partition of $\mathbf{q}$

$$\mathbf{q} = \begin{Bmatrix} \mathbf{q}_I \\ \bar{\mathbf{q}} \end{Bmatrix} \quad (2.2.3)$$

where $\bar{\mathbf{q}}$ represents the remaining $(n - I)$ generalized coordinates.

Applying the continuous assumed modes method as shown by Engles [24], $\mathbf{e}$ is written as the sum of two separate vectors $\mathbf{e}_I$ and $\bar{\mathbf{e}}$

$$\mathbf{e} = \mathbf{e}_I + \bar{\mathbf{e}} \quad (2.2.4)$$

where $\mathbf{e}_I$ is the quasi-static displacement due to the nodal displacement $\mathbf{q}_I$ and is expressed as a linear combination of the assumed modes Φ_I

$$\mathbf{e}_I = \Phi_I \mathbf{q}_I \quad (2.2.5)$$

The assumed modes, i.e., the columns of Φ_I , are chosen as static constraint modes (SCM's). SCM's are designed to accurately capture the static deformation of the element E caused by the physical interface displacements q_{Ij} ($j = 1, \dots, I$). I is the number of interface degrees of freedom.

The static constraint mode Φ_{Ij} (i.e., the j -th column in Φ_I) corresponding to the displacement coordinate q_{Ij} is defined as the static deformation of the element E for $q_{Ij}=1$ and $q_{Ii}=0$ for all $i \neq j$.

The second part $\bar{\mathbf{e}}$ in equation (2.2.4) represents the remainder of the total displacement vector $\mathbf{e}$. It is that part of $\mathbf{e}$ which is measured relative to $\mathbf{e}_I$ by an absolute observer. Clearly, $\bar{\mathbf{e}}$ vanishes at the interface and therefore can be expressed as a linear combination of assumed modes $\bar{\Phi}_i$ that are restrained at the q_{Ij} ($j = 1, \dots, I$) coordinates.

The interface restrained assumed mode (IRAM) $\bar{\Phi}_i$ corresponding to the generalized coordinate $\bar{q}_i$ is defined as an admissible deformation pattern of the element E with $\mathbf{q}_I=0$. Collecting all the $\bar{\Phi}_i$ as the columns of the matrix $\bar{\Phi}$ leads to

$$\bar{e} = \bar{\Phi} \bar{q} \quad (2.2.6)$$

The vector $\bar{q}$ represents a set of generalized coordinates to be determined as part of the solution. One example of IRAM's are the normal modes of E restrained at the q_I coordinates. It should be stressed that although interface restrained normal modes (IRNM's) have unique advantages, they are only one of many possible sets. In fact, IRAM's need only be admissible functions that vanish at the q_I coordinates.

Substituting equations (2.2.5) and (2.2.6) into equation (2.2.4), the displacement vector e becomes

$$e = \begin{bmatrix} \Phi_I & \bar{\Phi} \end{bmatrix} \begin{Bmatrix} q_I \\ \bar{q} \end{Bmatrix} \quad (2.2.7)$$

It is important to realize that the representation of e in equation (2.2.7) is complete in the sense that any degree of accuracy is theoretically possible as long as enough IRAM $\bar{\Phi}_i$ are retained.

2.2.3 Shape Functions

In the standard FE method, the variation of the elastic displacement vector e over the element is represented by interpolation or shape functions. These shaped functions are identical to the SCM's Φ_i [25]. The FE method neglects the displacement $\bar{e}$ completely. Also, the total element mass is redistributed over the interface degrees of freedom. This redistribution via the shape functions is known as the consistent mass matrix approach to FE modeling. The resulting error in the response vector is negligible only when the displacement vector $\bar{e}$ is small. This condition is fulfilled when the element deforms in a nearly static manner. This is accomplished by refining the FE mesh thereby producing

stiffer elements.

Another way to ensure convergence to a desired model fidelity is suggested by equation (2.2.7) and leads to the so called dynamic finite element modeling technique [25]. Instead of completely neglecting the displacement vector $\bar{\mathbf{e}}$, a limited number of coordinates $\bar{q}_1$ is retained thereby improving the mass distribution of the element E .

2.3. Element Mass and Stiffness Matrices

The mass and stiffness matrix for the generic element E is now derived in terms of the generalized displacement vector $\mathbf{q}$ given by equation (2.2.3). This represents the third step in the assumed modes method.

The inertial position vector $\mathbf{R}$ is given as

$$\mathbf{R} = \mathbf{R}_o + \mathbf{r} + \Phi \mathbf{q} \quad (2.3.1)$$

where equation (2.2.2) is substituted into equation (2.2.1). Because in this type of application the vector $\mathbf{r}$ and the matrix Φ are time independent, the inertial velocity vector $\dot{\mathbf{R}}$ is

$$\dot{\mathbf{R}} = \Phi \dot{\mathbf{q}} \quad (2.3.2)$$

and the kinetic energy T becomes

$$T = \frac{1}{2} \int_E \dot{\mathbf{R}}^T \dot{\mathbf{R}} dm = \frac{1}{2} \dot{\mathbf{q}}^T \left(\int_E \Phi^T \Phi dm \right) \dot{\mathbf{q}} \quad (2.3.3)$$

Thus, the element mass matrix takes on the form

$$M = \int_E \Phi^T \Phi \, dm \quad (2.3.4)$$

The final form of M is obtained by substituting $\Phi = [\Phi_I \quad \bar{\Phi}]$ from equation (2.2.7) into equation (2.3.4) and writing

$$M = \begin{bmatrix} M_{II} & M_{IN} \\ M_{NI} & M_{NN} \end{bmatrix} \quad (2.3.5)$$

where

$$\begin{aligned} M_{II} &= \int_E \Phi_I^T \Phi_I \, dm \\ M_{IN} &= \int_E \Phi_I^T \bar{\Phi} \, dm \\ M_{NN} &= \int_E \bar{\Phi}^T \bar{\Phi} \, dm \end{aligned} \quad (2.3.6)$$

Note that if the interface restrained assumed modes are taken to be orthogonal to the mass distribution, then M_{NN} becomes diagonal.

The element potential energy U is by definition

$$U = \frac{1}{2} \int_E \epsilon^T \tau \, dV \quad (2.3.7)$$

where ϵ and τ are the 6 x 1 strain and stress vectors for the element E respectively. For a linear material

$$\tau = C \epsilon \quad (2.3.8)$$

in which C is termed the material stiffness matrix. Also, ϵ is related to e as follows,

$$\epsilon = B e \quad (2.3.9)$$

where the elements of B are the appropriate partial derivatives $\partial/\partial x$, $\partial/\partial y$, and $\partial/\partial z$.

Combining the equations (2.3.7), (2.3.8), (2.3.9) and (2.2.2) yields

$$U = \frac{1}{2} q^T K q \quad (2.3.10)$$

with

$$K = \int_E (B\Phi)^T C (B\Phi) dV \quad (2.3.11)$$

substituting $\Phi = [\Phi_I \quad \bar{\Phi}]$ from equation (2.2.7) into equation (2.3.11) leads to the following expression for the stiffness matrix

$$K = \begin{bmatrix} K_{II} & 0 \\ 0 & K_{NN} \end{bmatrix} \quad (2.3.12)$$

with

$$\begin{aligned} K_{II} &= \int_E (B\Phi_I)^T C (B\Phi_I) dV \\ K_{NN} &= \int_E (B\bar{\Phi})^T C (B\bar{\Phi}) dV \end{aligned} \quad (2.3.13)$$

If $\bar{\Phi}$ is chosen to be orthogonal to the stiffness distribution, then K_{NN} becomes diagonal.

It is important to note that the IN -partition K_{IN} in K vanishes, i.e

$$K_{IN} = \int_E (B\Phi_I)^T C (B\bar{\Phi}) dV \quad (2.3.14)$$

Physically, this means that no stiffness coupling exists between $\mathbf{q}_I$ and $\bar{\mathbf{q}}$, and mathematically, it indicates stiffness orthogonality between the SCM's and the IRAM's.

A proof of equation (2.3.14) can be found in reference [24].

Lagrange's equations of motion for an undamped generic element can finally be written as

$$M\ddot{\mathbf{q}} + K\mathbf{q} = \mathbf{F} \quad (2.3.15)$$

where M and K are given by equations (2.3.5) and (2.3.12).

The load vector $\mathbf{F}$ corresponding to the distributed for $\mathbf{f}(x, y, z, t)$ can be obtained from the virtual work expression

$$\delta W = \int_E \mathbf{f}^T \delta \mathbf{e} dV \triangleq \mathbf{F} \delta \mathbf{q} \quad (2.3.16)$$

Substituting equation (2.2.2) into equation (2.3.16) yields

$$\mathbf{F} = \int_E \Phi^T \mathbf{f} dV$$

(2.3.17)

Furthermore using equation (2.2.7) in equation (2.3.17) leads to

$$\mathbf{F} \triangleq \begin{Bmatrix} \mathbf{F}_I \\ \overline{\mathbf{F}} \end{Bmatrix} = \int_E \begin{bmatrix} \Phi_I^T \\ \overline{\Phi}^T \end{bmatrix} \mathbf{f} dV$$

(2.3.18)

Note that in the case where only interface forces are applied, $\mathbf{F} = \mathbf{0}$ because $\overline{\Phi}$ vanishes at the interface.

2.4 Discrete Version

Continuous ICM's and IRNM's are only available in a few mainly one-dimensional cases such as the Axial Bar and the Bernoulli-Euler Beam. For the majority of elements (particularly the two- and three-dimensional elements), the boundaries and the interior of the element need to be discretized yielding a finite number of degrees of freedom. In the discrete case the equivalent of equation (2.3.7) becomes

$$\mathbf{u} \triangleq \begin{Bmatrix} \mathbf{u}_I \\ \mathbf{u}_N \end{Bmatrix} = \begin{bmatrix} \mathbf{I} & \mathbf{0} \\ \mathbf{S} & \overline{\Phi} \end{bmatrix} \begin{Bmatrix} \mathbf{u}_I \\ \overline{\mathbf{q}} \end{Bmatrix}$$

(2.4.1)

where $\mathbf{u}$ is the discrete displacement vector of the element, $\mathbf{u}_I$ the discrete interface displacement vector, $\mathbf{u}_N$ the discrete non-interface displacement vector, and $\overline{\mathbf{q}}$ the modal coordinate vector corresponding to the discrete IRNM's. Interface means the boundary of the element while non-interface means the interior of the element. The matrix $\mathbf{S}$ has the

non-interface part of discrete ICM's as its columns while $\bar{\Phi}$ has the non-interface part of the discrete IRNM's as its columns.

Assuming a discrete mass and stiffness matrix M and K were formed relative to the displacement vector $\mathbf{u}$ we can write

$$M = \begin{bmatrix} M_{II} & M_{IN} \\ M_{NI} & M_{NN} \end{bmatrix}, \quad K = \begin{bmatrix} K_{II} & K_{IN} \\ K_{NI} & K_{NN} \end{bmatrix} \quad \leftrightarrow \quad \mathbf{u} = \begin{Bmatrix} \mathbf{u}_I \\ \mathbf{u}_N \end{Bmatrix} \quad (2.4.2)$$

where $\mathbf{u}_N$ is the non-interface displacement vector. The kinetic and strain energies are then

$$T = \frac{1}{2} \dot{\mathbf{u}}^T M \dot{\mathbf{u}}, \quad U = \frac{1}{2} \mathbf{u}^T K \mathbf{u} \quad (2.4.3)$$

The S matrix, by definition, is obtained from solving the lower partition of

$$\begin{bmatrix} K_{II} & K_{IN} \\ K_{NI} & K_{NN} \end{bmatrix} \begin{bmatrix} I \\ S \end{bmatrix} = \begin{bmatrix} R \\ 0 \end{bmatrix} \quad (2.4.4)$$

where R are the reactions necessary to keep the interface degrees of freedom at their zero value. Note also that no forces are applied at the non-interface degrees of freedom. To compute S , we need not apply any forces, because the ICM's are the same whether or not forces are applied. In fact, we could make the absence of forces at the non-interface degrees of freedom part of the definition of the ICM's. Thus,

$$S = -K_{NN}^{-1} K_{NI} \quad (2.4.5)$$

The desired form M^* and K^* of M and K is obtained by substituting equations (2.4.5), (2.4.1) and (2.4.2) into equation (2.4.3)

$$M^* = \begin{bmatrix} M^*_{II} & M^*_{IN} \\ M^*_{NI} & M^*_{NN} \end{bmatrix}, \quad K^* = \begin{bmatrix} K^*_{II} & 0 \\ 0 & K^*_{NN} \end{bmatrix}, \quad \leftrightarrow \quad \begin{Bmatrix} u_I \\ \bar{q} \end{Bmatrix} \quad (2.4.6)$$

with

$$\begin{aligned} Y &\triangleq K^{-1}_{NN} K_{NI}, \quad X \triangleq M_{NI} - M_{NN} Y, \\ M^*_{II} &\triangleq M_{II} - M_{IN} Y - Y^T X, \quad M^*_{NI} \triangleq \bar{\Phi}^T X, \quad M^*_{NN} \triangleq I \\ K^*_{II} &\triangleq K_{II} - K_{IN} Y, \quad K^*_{NN} \triangleq \bar{\omega}^2 \end{aligned} \quad (2.4.7)$$

where we used the properties

$$\bar{\Phi}^T M_{NN} \bar{\Phi} = I, \quad \bar{\Phi}^T K_{NN} \bar{\Phi} = \bar{\omega}^2 \quad (2.4.8)$$

which result from the non-interface eigenvalue problem solution, i.e., the IRNM eigenvalue problem

$$\bar{\omega}^2 M_{NN} \bar{\Phi} = K_{NN} \bar{\Phi} \quad (2.4.9)$$

where $\bar{\omega}$ is an interface restrained frequency and $\bar{\Phi}$ is an interface restrained normal mode (IRNM).

Note that the II - partition in equation (2.4.6) represents a static or Guyan reduction of the non-interface degrees of freedom into the interface degrees of freedom. It can then be seen that the NN -partition serves as a correction to this Guyan reduction. The frequency

matrix $\bar{\omega}^2$ is then truncated according to a preset frequency $\bar{\omega}_p^2$.

2.5 Element Assembly Process

The element assembly process and its computer implementation is quite similar to that of the standard FE method. In particular, it will be shown that the structural component assembled from two elements is again an element formally the same as any other single element.

Following reference [24], consider the elements E_k and E_l in figure 2. Element E_k has one interface and element E_l has two distinct interfaces. In order to enforce displacement compatibility, it is necessary to transform the local interface vector $\mathbf{q}_1^k$ of element E_k into a global vector $\mathbf{Q}_1^k$ using a matrix L_k containing the constant direction cosines of the local system axes relative to the global reference frame

$$\mathbf{q}_1^k = L_k \mathbf{Q}_1^k \quad (2.5.1)$$

Similarly for element E_l

$$\mathbf{q}_1^l = \begin{Bmatrix} \mathbf{q}_{11}^l \\ \mathbf{q}_{12}^l \end{Bmatrix} = \begin{bmatrix} L_{11} & 0 \\ 0 & L_{12} \end{bmatrix} \begin{Bmatrix} \mathbf{Q}_{11}^l \\ \mathbf{Q}_{12}^l \end{Bmatrix} = L_l \mathbf{Q}_1^l \quad (2.5.2)$$

No need exists to transform the vectors $\bar{\mathbf{q}}_k$ and $\bar{\mathbf{q}}_l$ because elements only connect through their interfaces.

The first step in the assembly process is to require displacement compatibility

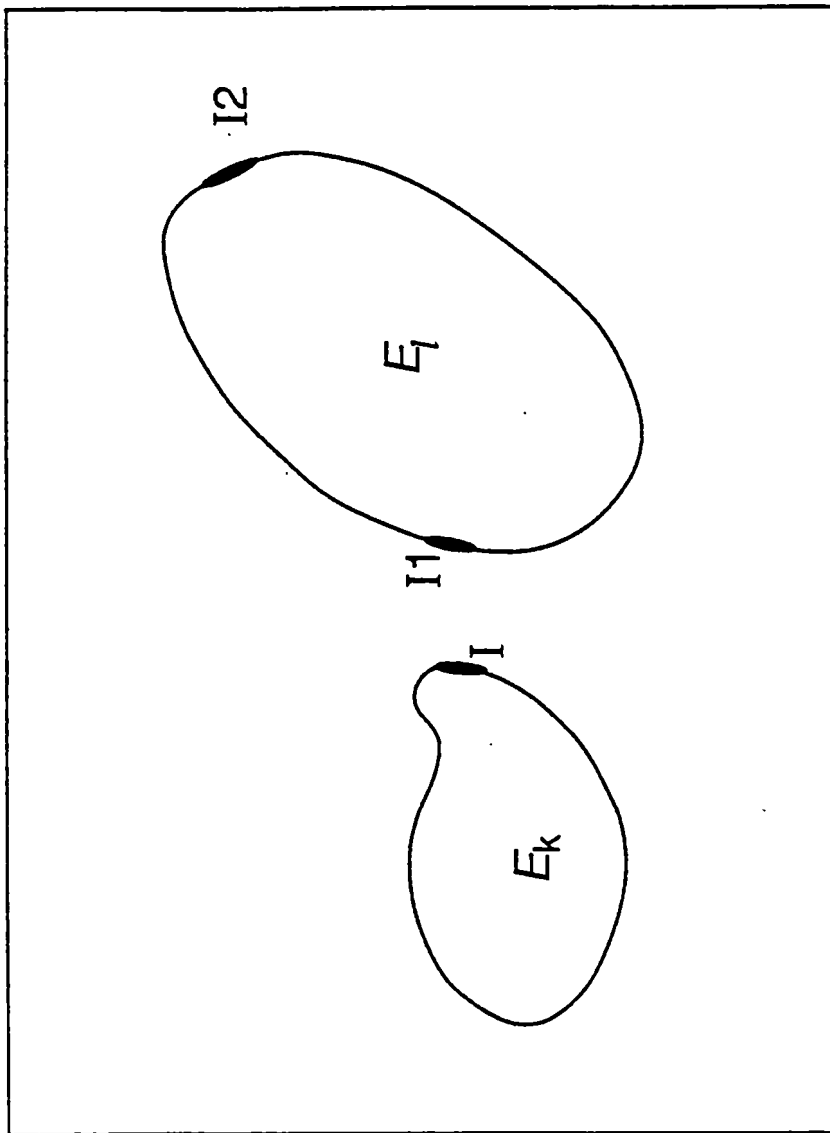


Figure 2. Assembly of Two Elements

$$\mathbf{Q}_I^k = \mathbf{Q}_{II}^l$$

(2.5.3)

The displacement vector for the assembled kl - element becomes

$$\mathbf{q} = \begin{Bmatrix} \mathbf{Q}_{I2}^l \\ \mathbf{Q}_I^k \\ \mathbf{q}_k \\ \mathbf{q}_l \end{Bmatrix}$$

(2.5.4)

The corresponding stiffness and mass matrices are

$$K = \begin{bmatrix} k_{22}^l & k_{12}^{lT} & 0 & 0 \\ k_{12}^l & k_{11}^l + k_{11}^k & 0 & 0 \\ 0 & 0 & \omega_k^2 & 0 \\ 0 & 0 & 0 & \omega_l^2 \end{bmatrix}, \quad M = \begin{bmatrix} m_{22}^l & m_{12}^{lT} & 0 & m_{23}^l \\ m_{12}^l & m_{11}^l + m_{11}^k & m_{12}^k & m_{13}^l \\ 0 & m_{12}^{kT} & I & 0 \\ m_{23}^{lT} & m_{13}^{lT} & 0 & I \end{bmatrix}$$

(2.5.5)

where

$$m_{11}^k = L_k^T M_{II}^k L_k, \quad m_{12}^k = L_k^T M_{IN}^k, \quad k_{11}^k = L_k^T K_{II}^k L_k,$$

(2.5.6)

and

$$\begin{bmatrix} m_{11}^l & m_{12}^l \\ m_{12}^{lT} & m_{22}^l \end{bmatrix} = L_l^T M_{II}^l L_l, \quad \begin{bmatrix} m_{13}^l \\ m_{23}^l \end{bmatrix} = L_l^T M_{IN}^l, \quad \begin{bmatrix} k_{11}^l & k_{12}^l \\ k_{12}^{lT} & k_{22}^l \end{bmatrix} = L_l^T K_{II}^l L_l$$

(2.5.7)

The second step in the assembly process is to write the interface vector $\mathbf{Q}_I^k$ in terms of a linear combination of static constraint modes Φ_I and interface restrained normal modes

$\bar{\Phi}$ similar to equation (2.2.7)

$$\mathcal{Q}_I^k = \begin{bmatrix} \Phi_I & \bar{\Phi} \end{bmatrix} \begin{Bmatrix} \mathcal{Q}_{I2}^I \\ \bar{q}_I \end{Bmatrix} \quad (2.5.8)$$

where

$$\Phi_I = -(k_{11}^k + k_{11}^I)^{-1} k_{12}^I \quad (2.5.9)$$

and the $\bar{\Phi}$ are obtained by solving the interface eigenvalue problem

$$\omega_I^2 [m_{11}^k + m_{11}^I] \bar{\Phi} = [k_{11}^k + k_{11}^I] \bar{\Phi} \quad (2.5.10)$$

If necessary, the modes in $\bar{\Phi}$ can be truncated according to the preset system cut-off frequency. Using the transformation

$$\begin{Bmatrix} \mathcal{Q}_{I2}^I \\ \mathcal{Q}_I^k \\ \bar{q}_k \\ \bar{q}_I \end{Bmatrix} = \begin{bmatrix} I & 0 & 0 & 0 \\ \Phi_I & \bar{\Phi} & 0 & 0 \\ 0 & 0 & I & 0 \\ 0 & 0 & 0 & I \end{bmatrix} \begin{Bmatrix} \mathcal{Q}_{I2}^I \\ \bar{q}_I \\ \bar{q}_k \\ \bar{q}_I \end{Bmatrix} \quad (2.5.11)$$

the stiffness and mass matrices in equation (2.5.5) becomes

$$K_1 = \begin{bmatrix} k_{11} & 0 & 0 & 0 \\ 0 & \omega_I^2 & 0 & 0 \\ 0 & 0 & \omega_k^2 & 0 \\ 0 & 0 & 0 & \omega_l^2 \end{bmatrix}, \quad M_1 = \begin{bmatrix} m_{11} & m_{12} & m_{13} & m_{14} \\ m_{12}^T & I & m_{23} & m_{24} \\ m_{13}^T & m_{23}^T & I & 0 \\ m_{14}^T & m_{24}^T & 0 & I \end{bmatrix} \quad (2.5.12)$$

with

$$\begin{aligned} m_{11} &= m_{22}^l + m_{12}^{lT} \Phi_I + \Phi_I^T m_{12}^l + \Phi_I^T (m_{11}^k + m_{11}^l) \Phi_I \\ m_{12} &= [m_{12}^{lT} + \Phi_I^T (m_{11}^k + m_{11}^l)] \bar{\Phi} \\ m_{13} &= \Phi_I^T m_{12}^k, \quad m_{14} = m_{23}^l + \Phi_I^T m_{13}^l \\ m_{23} &= \bar{\Phi}^T m_{12}^k, \quad m_{24} = \bar{\Phi}^T m_{13}^l, \quad k_{11} = k_{22}^l + k_{12}^{lT} \Phi_I \end{aligned} \quad (2.5.13)$$

The decomposition of vector Q^k_I in equation (2.5.8) allows for stiffness decoupling between Q^k_{I2} and Q^k_I in equation (2.5.12). The transformations in equation (2.5.11) will allow for frequency and mode shape truncation on the kl - element level without compromising the accuracy of the quasi-static deformation pattern of the kl - element.

The last step in the assembly process is to solve for the interface restrained modes of the kl - element

$$\omega_{kl}^2 = \begin{bmatrix} I & m_{23} & m_{24} \\ m_{23}^T & I & 0 \\ m_{24}^T & 0 & I \end{bmatrix} \bar{\Phi}_{kl} = \begin{bmatrix} \omega_I^2 & 0 & 0 \\ 0 & \omega_k^2 & 0 \\ 0 & 0 & \omega_l^2 \end{bmatrix} \bar{\Phi}_{kl} \quad (2.5.14)$$

Again the modes in $\bar{\Phi}_{kl}$ can be truncated according to the system cut-off frequency.

Using the modal transformation

$$\begin{Bmatrix} \mathbf{Q}_{I2}^I \\ \overline{\mathbf{q}}_I \\ \overline{\mathbf{q}}_k \\ \overline{\mathbf{q}}_l \end{Bmatrix} = \begin{bmatrix} I & 0 \\ 0 & \overline{\Phi}_{kl} \end{bmatrix} \begin{Bmatrix} \mathbf{Q}_{I2}^I \\ \overline{\mathbf{q}}_{kl} \end{Bmatrix}$$

(2.5.15).

the transformed mass and stiffness matrices in equation (2.5.12) become

$$M_2 = \begin{bmatrix} M_{II} & M_{IN} \\ M_{IN}^T & I \end{bmatrix}, \quad K_2 = \begin{bmatrix} K_{II} & 0 \\ 0 & \omega_{kl}^2 \end{bmatrix}$$

(2.5.16)

where

$$M_{II} = m_{11}, \quad M_{IN} = [m_{12} \ m_{13} \ m_{14}] \overline{\Phi}_{kl}, \quad K_{II} = k_{11} \quad (2.5.17)$$

Comparing the mass and stiffness matrices in equation. (2.5.16) and the mass and stiffness matrices for a generic element in equations (2.3.5) and (2.3.12) shows that the assembled kl - element can now be considered as a new single element.

The same assembly procedure can be used when adding the next element E_m to the E_{kl} element. The resulting element E_{klm} , again has the minimum number of degrees of freedom necessary to obtain the desired accuracy in the system frequencies and mode shapes of interest. Also, any element chosen by the user can be taken off-line, in order to construct a minimum order model using the same procedure. For example, a model for the bay element in figure 3 may be constructed by assembling sequentially the axial bars

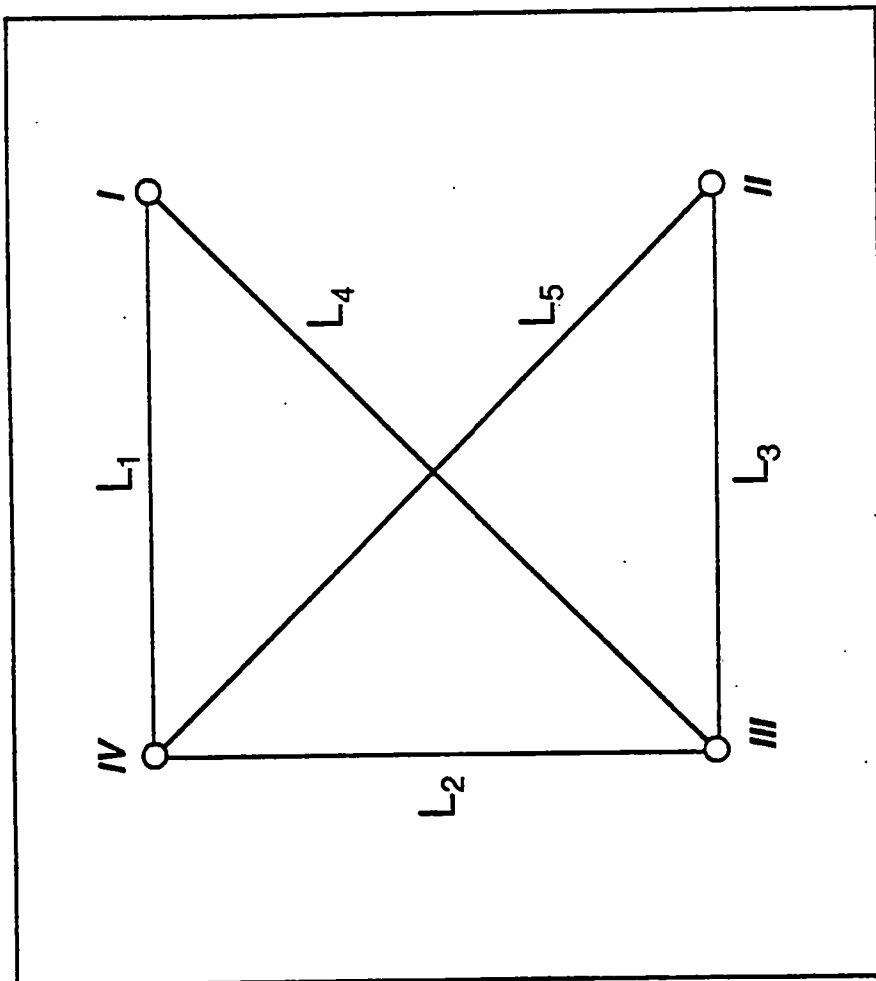


Figure 3. Single Bay of Planar Truss

and applying modal truncation at every step. The result is a minimum order model for the entire bay element. Furthermore, if this element is used to build up an entire structure, then the same bay model can be used over again. In fact, multiple bay elements can be constructed from single bay elements, using the same assembly process. The present technique is quite suitable to take advantage of spatial periodicity wherever it occurs in the system. This idea will be applied to the standard two-dimensional finite elements.

Re-analysis of structures is often required in order to improve the accuracy of previously obtained solutions. Because the proposed approach is hierarchical in nature, more refined models can be easily derived from the existing lower order models. Extra interface restrained assumed modes can be added according to a newly defined cut-off frequency, without disturbing the old grid. In addition, when subspace iteration is used, old intermediate eigenvalue problem solutions can be used as starting vector sets for the new iteration process.

The proposed assembly process allows for the replacement of a large system eigenvalue problem by a series of relatively small eigenvalue problems and is well suited for use with micro computers. Since the cost of an eigenvalue problem roughly increases with the cube of its dimension, a significant cost reduction can be achieved. It is important to note that even when the elements are small enough so that no interface restrained assumed modes are necessary, the sequential nature of the system eigenvalue problem solution still allows for substantial cost reduction. In the following sections, a cost-effective algorithm for the eigenvalue problem in equation (2.5.14) will be developed.

2.6 System Eigenvalue Problem

The ultimate objective of the analysis process is the determination of a set of low frequency system modes. As was indicated in the previous section, the proposed assembly process allows for the replacement of a large system eigenvalue problem by a series of relatively small and inexpensive eigenvalue problems of the type shown in equation (2.5.14).

Of the multitude of existing algorithms, it was judged that the subspace iteration technique would be the most suitable in the present context. Subspace iteration uses Ritz analysis to perform simultaneous inverse iteration on the starting vectors in order to extract the lowest eigenvalue and corresponding eigenvector approximations. These approximations can be further improved by iteration. The basic objective of the subspace iteration method is to solve the following generalized eigenvalue problem of order n

$$\lambda M\Phi = K\Phi \quad (2.6.1)$$

for the lowest p ($\leq n$) eigenvalues and eigenvectors. Collecting the p eigenvectors as the columns of Φ and the corresponding eigenvalues as the diagonal entries of matrix Λ , a consequence of equation (2.6.1) is that

$$\Phi^T K \Phi = \Lambda, \quad \Phi^T M \Phi = I \quad (2.6.2)$$

where the eigenvectors are mass normalized.

The underlying philosophy of the subspace iteration solution method is that the eigenvectors Φ , form a mass orthonormal basis for the p -dimensional dominant subspace E_∞ of the matrices K and M . The iteration with p linearly independent vectors can therefore be regarded as an iteration with a subspace. The starting iteration vectors, $\bar{X}_i$

span subspace E_1 and the iteration continues until E_k spans E_∞ to a sufficient degree of accuracy. The following algorithm is set up

For $k = 1, 2, \dots$, iterate from E_k to E_{k+1} :

$$KX_{k+1} = M\overline{X}_k \quad (2.6.3)$$

Find the projections of the matrix operators K and M onto E_{k+1} :

$$K_{k+1} = X_{k+1}^T K X_{k+1}, \quad M_{k+1} = X_{k+1}^T M X_{k+1} \quad (2.6.4)$$

Solve for the eigenvalues and eigenvectors of the projected operators:

$$K_{k+1} Z_{k+1} = M_{k+1} Z_{k+1} \overline{\Lambda}_{k+1} \quad (2.6.5)$$

Find an improved approximation to the eigenvectors:

$$\overline{X}_{k+1} = X_{k+1} Z_{k+1} \quad (2.6.6)$$

Provided that the vectors in $\overline{X}_1$ are not orthogonal to one of the eigenvectors, then

$\overline{X}_{k+1} \rightarrow A$ and $\overline{X}_{k+1} \rightarrow \Phi$ as $k \rightarrow \infty$. Bathe and Wilson [31] indicate that the number of starting vectors in $\overline{X}_1$ is chosen as $q = \min [2p, p+8]$ rather than p , in order to achieve faster convergence.

Next, a typical intermediate eigenvalue problem, equation. (2.6.1) is written in a more convenient form

$$\lambda \begin{bmatrix} I & m \\ m^T & I \end{bmatrix} \Phi = \begin{bmatrix} \mu & 0 \\ 0 & \Omega \end{bmatrix} \Phi \quad (2.6.7)$$

The efficiency of the subspace iteration method depends on (i) the sparseness of M and K and (ii) the starting vectors. From equation (2.6.7) it is seen that the mass and stiffness matrices are indeed quite sparse. In particular, the stiffness matrix is diagonal, and therefore the solution becomes trivial. In addition, the form of the mass matrix is such that further simplifications can be easily implemented.

Furthermore, it is possible to derive an excellent set of starting vectors at relatively little cost. To this end, the q lowest diagonal elements in the stiffness matrix in equation (2.6.7) are identified. For each of these values, the following simple eigenvalue problem

$$\theta \begin{bmatrix} I & m_i \\ m_i^T & I \end{bmatrix} \begin{Bmatrix} \xi_0 \\ q \end{Bmatrix} = \begin{bmatrix} \mu_i & 0 \\ 0 & \Omega \end{bmatrix} \begin{Bmatrix} \xi_0 \\ q \end{Bmatrix} \quad (2.6.8)$$

is solved for the eigenvalue and corresponding eigenvector which is closest to μ_i , where μ_i is one of the chosen q diagonal elements. It is shown in reference 28 that the resulting starting set $\bar{X}_1$ is in general, already a very good approximation of the desired dominant subspace of problem (2.6.7).

Next, in order to solve the eigenvalue problem in equation (2.6.8) for the eigenvalue θ_c closest to μ_i and for the corresponding eigenvalue problem $[\xi_{0c} \ q_c^T]^T$, consider the lower partition of equation (2.6.8) and solve for q

$$q = \theta [\Omega - \theta I]^{-1} m_i^T \xi_0 \quad (2.6.9)$$

The characteristic equation for this problem can then be obtained by substituting equation (2.6.9) into the top partition of equation (2.6.8)

$$\theta - \mu_i + \theta^2 \mathbf{m}_i [\Omega - \theta I]^{-1} \mathbf{m}_i^T = 0 \quad (2.6.10)$$

This equation is solved for the desired root via the Newton-Raphson iteration method

$$\theta_{r+1} = \theta_r - f(\theta_r) \left[\frac{df(\theta_r)}{d\theta} \right]^{-1} \quad (2.6.11)$$

in which r is the iteration index and

$$\begin{aligned} f(\theta_r) &= \theta_r - \mu_i + \theta_r^2 \sum_{j=1}^{n-q} \frac{\mathbf{m}_{ij}^2}{\Omega_j - \theta_r} \\ \frac{df(\theta_r)}{d\theta} &= 1 + 2\theta_r \sum_{j=1}^{n-q} \frac{\mathbf{m}_{ij}^2}{\Omega_j - \theta_r} + \theta_r^2 \sum_{j=1}^{n-q} \frac{\mathbf{m}_{ij}^2}{(\Omega_j - \theta_r)^2} \end{aligned} \quad (2.6.12)$$

For $r = 0$, θ_0 can be taken as μ_i . Because the Newton-Raphson method has quadratic convergence and only a good approximation of the desired root is needed, no more than two iterations are needed in equation (2.6.12). In addition, because the difference $(\Omega_j - \theta_r)$ becomes large with increasing j , all sums in equation (2.6.12) can usually be truncated. In practice, the cost to compute the desired root in equation (2.6.11) is negligible. The value of ξ_0 is obtained by requiring the eigenvector corresponding to θ_0 to be mass orthogonal, i.e.

$$\begin{Bmatrix} \xi_{0c} \\ \mathbf{q}_c \end{Bmatrix}^T \begin{bmatrix} 1 & \mathbf{m}_i \\ \mathbf{m}_i^T & I \end{bmatrix} \begin{Bmatrix} \xi_{0c} \\ \mathbf{q}_c \end{Bmatrix} = 1 \quad (2.6.13)$$

Substituting equation (2.6.9) into equation (2.6.13), ξ_{0c} is determined as

$$\xi_{0c} = \left[\frac{df(\theta_c)}{d\theta} \right]^{-1/2} \quad (2.6.14)$$

where θ_c is the converged desired root of equation. (2.6.11). Note that $df/d\theta$ is always positive and the vector $\mathbf{q}_c$ can be computed using equations (2.6.14) and (2.6.9). Finally, the i -th column of the starting subspace is taken as

$$\overline{X}_{1_i} = \begin{Bmatrix} \xi_{0c} \mathbf{e}_i \\ \mathbf{q}_c \end{Bmatrix}, \quad i = 1, 2, \dots, q \quad (2.6.15)$$

where $\mathbf{e}_i = [0 \ 0 \ \dots \ 0 \ 1 \ 0 \ \dots \ 0]^T$ and $\mathbf{q}_c$ can be computed from equation (2.6.9)

In the next section we illustrate a simple example of a dynamic finite element and briefly discuss the other already existing dynamic finite element.

2.7 Axial Bar

As an illustration of a simple dynamic finite element consider the axial bar shown in figure 4. The bar is assumed to undergo vibrations along its own axis and as a rigid body can only move along the same axis. This is the simplest element type and is included as a simple illustration of the construction of the element equations. Following references [25] and [15], the interface displacements are defined as the end displacements, i.e.

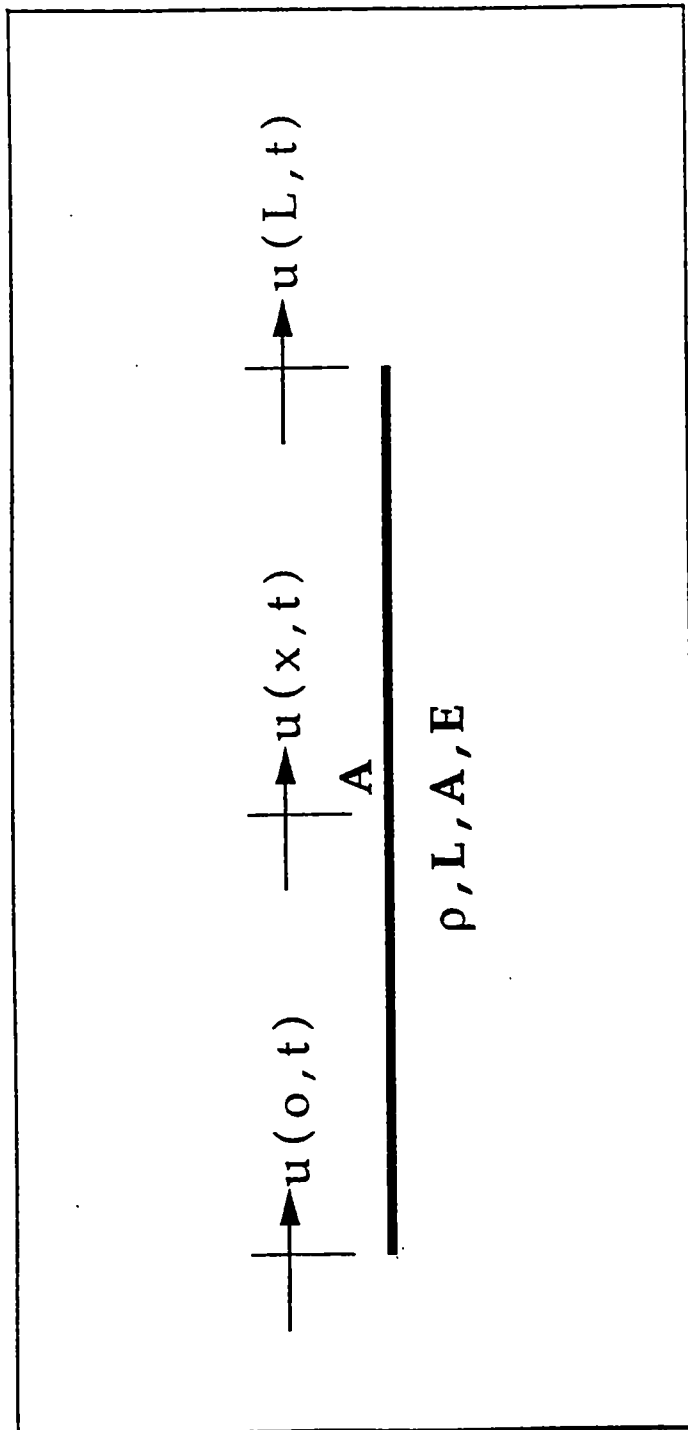


Figure 4. Axial Bar (Line Element)

$$\mathbf{q}_I = \begin{bmatrix} q_1 & q_2 \end{bmatrix}^T = \begin{bmatrix} u(0,t) & u(L,t) \end{bmatrix}^T \quad (2.7.1)$$

The non-interface degrees of freedom are $u(x, t)$ with $x = 0, L$

The static constraint modes are computed from the following static equilibrium equation

$$EA \frac{\partial^2 u}{\partial x^2} = 0 \quad (2.7.2)$$

where E is Young's modulus and A is the cross-sectional area. The appropriate boundary conditions for the static constraint modes are $u(0, t) = 1$, $u(0, t) = 0$ and $u(L, t) = 0$, $u(L, t) = 1$, respectively. The Φ_I matrix then becomes

$$\Phi_I = \begin{bmatrix} 1 & -\frac{x}{L} & \frac{x}{L} \end{bmatrix} \quad (2.7.3)$$

where L is the length of the bar.

The interface restrained normal modes and the corresponding natural frequencies are found by solving the eigenvalue problem given by the partial differential equation

$$\rho A \frac{\partial^2 u}{\partial t^2} - EA \frac{\partial^2 u}{\partial x^2} = 0 \quad (2.7.4)$$

subject to the clamped-clamped boundary conditions, i.e., $u(0, t) = 0$ and $u(L, t) = 0$ and where ρ represents the mass density of the bar. The interface restrained normal modes are found as

$$\bar{\Phi}_j(x) = \sqrt{\frac{2}{\rho AL}} \sin j \frac{\pi x}{L}, \quad j=1,2,\dots,\infty \quad (2.7.5)$$

According to equation (2.2.7), the displacement vector, $\mathbf{u}$ can be written as

$$\mathbf{u} = \begin{bmatrix} \Phi_I & \bar{\Phi} \end{bmatrix} \begin{Bmatrix} \dot{\mathbf{q}}_I \\ \bar{\dot{\mathbf{q}}} \end{Bmatrix} \quad (2.7.6)$$

The kinetic energy T of the axial bar element is

$$T = \frac{1}{2} \int_0^L \rho A u^2 dx \quad (2.7.7)$$

where the dot ($\cdot$) denotes the partial derivative with respect to time t . Substituting equations (2.7.3), (2.7.5) and (2.7.6) into equation (2.7.7) yields

$$T = \frac{1}{2} \begin{Bmatrix} \dot{\mathbf{q}}_I \\ \bar{\dot{\mathbf{q}}} \end{Bmatrix}^T \begin{bmatrix} M_{II} & M_{IN} \\ M_{IN}^T & M_{NN} \end{bmatrix} \begin{Bmatrix} \dot{\mathbf{q}}_I \\ \bar{\dot{\mathbf{q}}} \end{Bmatrix} \quad (2.7.8)$$

where the mass matrix partitions are evaluated as

$$\begin{aligned} M_{II} &= \int_0^L \rho A \Phi_I^T \Phi_I dx = \frac{\rho AL}{6} \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} \\ M_{NN} &= \int_0^L \rho A \bar{\Phi}_I^T \bar{\Phi}_I dx = I \\ M_{IN} &= \int_0^L \rho A \Phi_I^T \bar{\Phi}_I dx = \frac{\sqrt{2\rho AL}}{\pi} \begin{bmatrix} 1 & \frac{1}{2} & \dots & \frac{1}{j} & \dots \\ 1 & -\frac{1}{2} & \dots & \frac{(-1)^{j+1}}{j} & \dots \end{bmatrix} \end{aligned} \quad (2.7.9)$$

Note that M_{II} is the standard consistent finite element mass matrix for an axial bar. The matrix in equation (2.7.9) represents the mass coupling between q_I and $\bar{q}$ which is neglected in the standard finite element method. Because of the orthonormality of the interface restrained normal modes, M_{NN} becomes a unit matrix.

The element strain energy U is

$$U = \frac{1}{2} \int_0^L E A u'^2 dx \quad (2.7.10)$$

where the $()'$ indicates the partial derivative with respect to x . This time, from equation (2.7.6) it follows that

$$u' = \begin{bmatrix} \Phi_I' & \bar{\Phi}' \end{bmatrix} \begin{Bmatrix} q_I \\ q \end{Bmatrix} \quad (2.7.11)$$

Substituting equation (2.7.11) into equations (2.7.10, 2.7.12), the stiffness matrix K becomes

$$K = \begin{bmatrix} K_{II} & 0 \\ 0 & K_{NN} \end{bmatrix} \quad (2.7.12)$$

where

$$\begin{aligned}
K_{II} &= \int_0^L EA \Phi_I'^T \Phi_I' dx = \frac{EA}{L} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \\
K_{NN} &= \int_0^L EA \bar{\Phi}'^T \bar{\Phi}' dx = \text{Diag}(\omega_1^2, \omega_2^2, \dots, \infty) \\
\omega_j^2 &= j^2 \pi^2 \frac{E}{\rho L^2}, \quad j = 1, 2, \dots, \infty
\end{aligned}
\tag{2.7.13}$$

Here the ω_j are the clamped-clamped circular natural frequencies of the bar. Note that K_{II} in equation (2.7.13) is the standard finite element stiffness matrix for the axial bar and that as expected

$$K_{IN} = \int_0^L EA \Phi_I'^T \bar{\Phi}' dx = 0$$

(2.7.14)

CHAPTER 3

TWO-DIMENSIONAL DYNAMIC FINITE ELEMENTS

3.1 Introduction

In this chapter we will develop the equations of motion for the most commonly used two-dimensional finite elements: the triangle and the quadrilateral.

One difficulty in forming two-dimensional dynamic finite elements is that the boundaries of such elements are generally collections of curves, and not points as was the case for the one-dimensional elements. In the case of the triangle and the quadrilateral, the boundaries are straight lines. As a practical matter the boundaries must be discretized. This discretization process brings with it the difficulty that the ICM's and the IRNM's are not readily available. The computation of the ICM's and the IRNM's may become expensive if not carefully crafted. The approach taken to overcome these difficulties is now described in broad terms.

The IRNM's for a typical element will be computed off-line in an optimum manner. The element is subdivided into smaller sub-elements and the fixed-fixed modes computed. The detail of this procedure will show the opportunities for substantial cost savings. The number of ICM's for the element will follow from the imposed degree of accuracy, i.e., the number of nodes on the boundary of the element and the types of shape functions used to determine the IRNM's.

In the next section, the standard constant strain triangular element is discussed. This is necessary as an introduction to the understanding of the development associated with the generic triangular element.

3.2 Constant Strain Triangular Element

Consider the triangular element with corner nodes 1, 2, and 3 as shown in figure 5.

For reasons to be explained later the corner nodes must be ordered counter-clockwise.

The general relations between the cartesian coordinates (x,y) and the area coordinates

(L_1, L_2, L_3) are

$$\begin{aligned} x &= \sum_{k=1}^3 L_k x_k \\ y &= \sum_{k=1}^3 L_k y_k \\ 1 &= \sum_{k=1}^3 L_k \end{aligned} \quad (3.2.1)$$

and

$$L_k = \frac{a_k + b_k x + c_k y}{2\Delta}, \quad k = 1, 2, 3 \quad (3.2.2)$$

with

$$\begin{aligned} \Delta &= \frac{1}{2} \text{Det} \begin{vmatrix} 1 & x_1 & y_1 \\ 1 & x_2 & y_2 \\ 1 & x_3 & y_3 \end{vmatrix} = \text{Area (123)} \\ a_1 &= x_2 y_3 - x_3 y_2, \quad a_2 = x_3 y_1 - x_1 y_3, \quad a_3 = x_1 y_2 - x_2 y_1 \\ b_1 &= y_2 - y_3, \quad b_2 = y_3 - y_1, \quad b_3 = y_1 - y_2 \\ c_1 &= x_3 - x_2, \quad c_2 = x_1 - x_3, \quad c_3 = x_2 - x_1 \end{aligned} \quad (3.2.3)$$

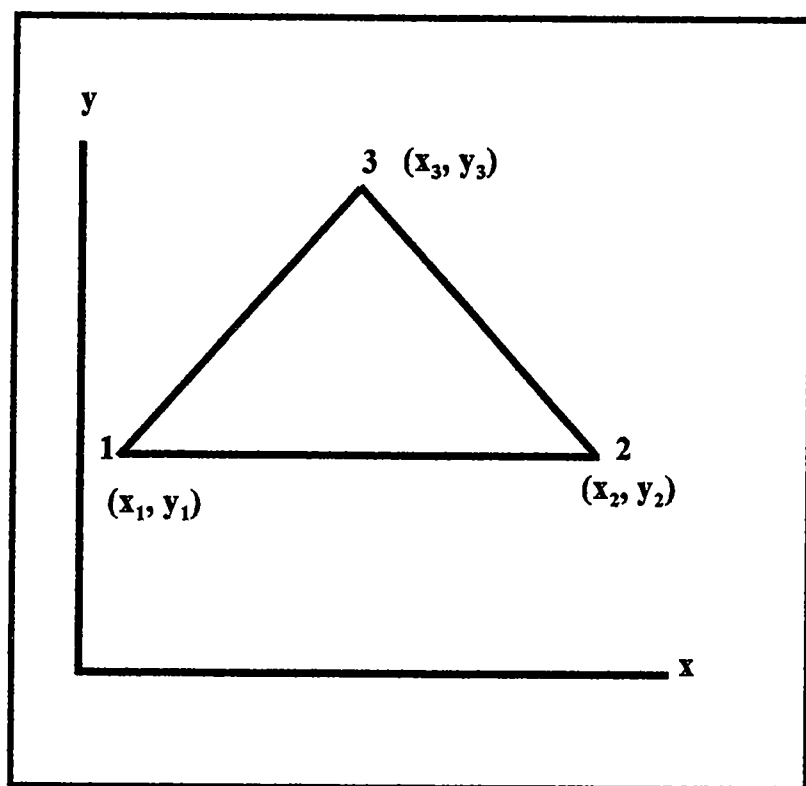


Figure 5. Triangular Element

The coordinates (x_i, y_i) ($i = 1, 2, 3$) are the cartesian coordinates of the corner nodes 1, 2, and 3.

Next, let the element be a constant-strain triangle. The shape functions (N_1, N_2, N_3) then are simply the area coordinates (L_1, L_2, L_3) of the element themselves

$$N_1 = L_1, \quad N_2 = L_2, \quad N_3 = L_3 \quad (3.2.4)$$

The displacement field within the element becomes

$$u = \sum_{k=1}^3 N_k u_k, \quad v = \sum_{k=1}^3 N_k v_k \quad (3.2.5)$$

or using equations (3.2.4) and (3.2.5)

$$u = (u_1 - u_3)L_1 + (u_2 - u_3)L_2 + u_3, \quad v = (v_1 - v_3)L_1 + (v_2 - v_3)L_2 + v_3 \quad (3.2.6)$$

or, writing equation (3.2.5) in matrix form

$$\mathbf{u} = \mathbf{N} \mathbf{u}_c \quad (3.2.7)$$

with

$$\mathbf{u} \triangleq \begin{Bmatrix} u \\ v \end{Bmatrix}, \quad \mathbf{N} \triangleq \begin{bmatrix} N_1 & 0 & N_2 & 0 & N_3 & 0 \\ 0 & N_1 & 0 & N_2 & 0 & N_3 \end{bmatrix}, \quad \mathbf{u}_c \triangleq \begin{Bmatrix} u_1 \\ v_1 \\ u_2 \\ v_2 \\ u_3 \\ v_3 \end{Bmatrix} \quad (3.2.8)$$

From equations (3.2.1) and (3.2.4) it also follows that

$$x = \sum_{k=1}^3 N_k x_k, \quad y = \sum_{k=1}^3 N_k y_k \quad (3.2.9)$$

so that the displacements as well as the coordinates are interpolated within the element using the same shape functions. Thus, the present representation is an isoparametric one. From equations (3.2.1), (3.2.4), and (3.2.9) it follows that

$$x = x_{13}L_1 + x_{23}L_2 + x_3, \quad y = y_{13}L_1 + y_{23}L_2 + y_3 \quad (3.2.10)$$

where

$$x_{ij} \triangleq x_i - x_j, \quad y_{ij} \triangleq y_i - y_j \quad (3.2.11)$$

Next, the chain-rule gives

$$\begin{Bmatrix} \frac{\partial u}{\partial L_1} \\ \frac{\partial u}{\partial L_2} \end{Bmatrix} = J \begin{Bmatrix} \frac{\partial u}{\partial x} \\ \frac{\partial u}{\partial y} \end{Bmatrix} \quad (3.2.12)$$

where

$$J = \begin{bmatrix} \frac{\partial x}{\partial L_1} & \frac{\partial y}{\partial L_1} \\ \frac{\partial x}{\partial L_2} & \frac{\partial y}{\partial L_2} \end{bmatrix} = \begin{bmatrix} x_{13} & y_{13} \\ x_{23} & y_{23} \end{bmatrix} \quad (3.2.13)$$

is the Jacobian matrix. Solving equation (3.2.12) yields

$$\begin{Bmatrix} \frac{\partial u}{\partial x} \\ \frac{\partial u}{\partial y} \end{Bmatrix} = J^{-1} \begin{Bmatrix} \frac{\partial u}{\partial L_1} \\ \frac{\partial u}{\partial L_2} \end{Bmatrix}, \quad J^{-1} = \frac{1}{\text{Det } J} \begin{bmatrix} y_{23} & -y_{13} \\ -x_{23} & x_{13} \end{bmatrix}, \quad \text{Det } J = x_{13}y_{23} - x_{23}y_{13} \quad (3.2.14)$$

For counter-clockwise ordering of the corners of the triangle we have that the $\text{Det } J$ is positive and equal to two times the area of the triangle

$$\text{Det } J = 2 \text{Area } (1, 2, 3) \quad (3.2.15)$$

From equation (3.2.14) it follows that

$$\begin{Bmatrix} \frac{\partial u}{\partial x} \\ \frac{\partial u}{\partial y} \end{Bmatrix} = \frac{1}{\text{Det } J} \begin{Bmatrix} y_{23} \frac{\partial u}{\partial L_1} - y_{13} \frac{\partial u}{\partial L_2} \\ -x_{23} \frac{\partial u}{\partial L_1} + x_{13} \frac{\partial u}{\partial L_2} \end{Bmatrix}, \quad \begin{Bmatrix} \frac{\partial v}{\partial x} \\ \frac{\partial v}{\partial y} \end{Bmatrix} = \frac{1}{\text{Det } J} \begin{Bmatrix} y_{23} \frac{\partial v}{\partial L_1} - y_{13} \frac{\partial v}{\partial L_2} \\ -x_{23} \frac{\partial v}{\partial L_1} + x_{13} \frac{\partial v}{\partial L_2} \end{Bmatrix} \quad (3.2.16)$$

where the corresponding equation for the v -displacement component is also listed. By definition, the strain-displacement relations are

$$\epsilon \triangleq \begin{Bmatrix} \frac{\partial u}{\partial x} \\ \frac{\partial v}{\partial y} \\ \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} \end{Bmatrix} \quad (3.2.17)$$

Taking into account the previous equations, the strain vector ϵ becomes

$$\epsilon = B \mathbf{u}_c$$

(3.2.18)

with

$$B = \frac{1}{\text{Det } J} \begin{bmatrix} y_{23} & 0 & y_{31} & 0 & y_{12} & 0 \\ 0 & x_{32} & 0 & x_{13} & 0 & x_{21} \\ x_{32} & y_{23} & x_{13} & y_{31} & x_{21} & y_{12} \end{bmatrix}$$

(3.2.19)

Next, the strain energy is

$$U_{(1,2,3)} = \frac{1}{2} \int_{(1,2,3)} \epsilon^T D \epsilon t dA$$

(3.2.20)

where D is the material matrix

$$D = \frac{E}{1-\nu^2} \begin{bmatrix} 1 & \nu & 0 \\ \nu & 1 & 0 \\ 0 & 0 & \frac{1-\nu}{2} \end{bmatrix}, \quad D = \frac{E}{(1+\nu)(1-2\nu)} \begin{bmatrix} 1-\nu & \nu & 0 \\ \nu & 1-\nu & 0 \\ 0 & 0 & \frac{1-2\nu}{2} \end{bmatrix}$$

(3.2.21)

for plane stress and plane strain, respectively. t is the thickness of the plate and the integral is taken over the area of the element (1,2,3). Using equation (3.2.20) and the fact that D and B are constant it follows that

$$U_{(1,2,3)} = \frac{1}{2} \mathbf{u}_c^T k \mathbf{u}_c, \quad k = t A B^T D B$$

(3.2.22)

where A is the area of the (1,2,3) triangle. Explicitly, for the plane stress case we have

$$k = \begin{bmatrix} k_{11} & k_{12} & k_{13} & k_{14} & k_{15} & k_{16} \\ k_{12} & k_{22} & k_{23} & k_{24} & k_{25} & k_{26} \\ k_{13} & k_{23} & k_{33} & k_{34} & k_{35} & k_{36} \\ k_{14} & k_{24} & k_{34} & k_{44} & k_{45} & k_{46} \\ k_{15} & k_{25} & k_{35} & k_{45} & k_{55} & k_{56} \\ k_{16} & k_{26} & k_{36} & k_{46} & k_{56} & k_{66} \end{bmatrix} \quad (3.2.23)$$

To find the k_{ij} we let

$$C \triangleq R^T R, \text{ with } R \triangleq \begin{bmatrix} y_{23} & x_{23} & y_{31} & x_{31} & y_{12} & x_{12} \end{bmatrix} \\ \alpha \triangleq \frac{1-\nu}{2}, \quad \beta \triangleq \frac{1+\nu}{2} \quad (3.2.24)$$

Because C is symmetric, equation (3.2.24) represents an operations count of 21 flops (i.e., floating point operations). With the elements of C defined as c_{ij} , the elements of k are written as

$$\begin{aligned} k_{11} &= c_{11} + \alpha c_{22}, \quad k_{22} = c_{22} + \alpha c_{11}, \quad k_{33} = c_{33} + \alpha c_{44}, \quad k_{44} = c_{44} + \alpha c_{33} \\ k_{55} &= c_{55} + \alpha c_{66}, \quad k_{66} = c_{66} + \alpha c_{55}, \quad k_{12} = \beta c_{12}, \quad k_{34} = \beta c_{34}, \quad k_{56} = \beta c_{56} \\ k_{13} &= c_{13} + \alpha c_{24}, \quad k_{24} = c_{24} + \alpha c_{13}, \quad k_{15} = c_{15} + \alpha c_{26}, \quad k_{26} = c_{26} + \alpha c_{15} \\ k_{35} &= c_{35} + \alpha c_{46}, \quad k_{46} = c_{46} + \alpha c_{35}, \quad k_{14} = -\nu c_{14} - \alpha c_{23}, \quad k_{23} = -\nu c_{23} + \alpha c_{14} \\ k_{16} &= -\nu c_{16} - \alpha c_{25}, \quad k_{25} = -\nu c_{25} + \alpha c_{16}, \quad k_{36} = -\nu c_{36} - \alpha c_{45}, \quad k_{45} = -\nu c_{45} + \alpha c_{36} \end{aligned} \quad (3.2.25)$$

which represent an additional 27 flops. Note that k is, of course, symmetric and that it depends on the coordinates of the corner nodes of the (1,2,3) triangle.

Next, the kinetic energy T is

$$T_{(1,2,3)} = \frac{1}{2} \int_{(1,2,3)} \dot{\mathbf{u}}^T \dot{\mathbf{u}} \rho dV = \frac{1}{2} \int_{(1,2,3)} \dot{\mathbf{u}}_c^T N^T N \dot{\mathbf{u}}_c \rho t dA = \frac{1}{2} \rho t \dot{\mathbf{u}}_c^T \int_{(1,2,3)} N^T N dA \dot{\mathbf{u}}_c \quad (3.2.26)$$

so that the mass matrix m of the element (1,2,3) becomes

$$m = \rho t \int_{(1,2,3)} N^T N dA \quad (3.2.27)$$

or, explicitly

$$m = \frac{\rho t A}{12} \begin{bmatrix} 2 & 0 & 1 & 0 & 1 & 0 \\ 0 & 2 & 0 & 1 & 0 & 1 \\ 1 & 0 & 2 & 0 & 1 & 0 \\ 0 & 1 & 0 & 2 & 0 & 1 \\ 1 & 0 & 1 & 0 & 2 & 0 \\ 0 & 1 & 0 & 1 & 0 & 2 \end{bmatrix} \quad (3.2.28)$$

This concludes the description of the procedure to find the element properties of the standard constant-strain triangle.

3.3 Generic Triangular Element

3.3.1 Element Properties of Sub-Element

Figure 6 shows a triangle (not necessarily constant strain) subdivided into triangular sub-elements. Each side of the (1,2,3) triangle has $N+1$ nodes, where N is equal to the chosen number of subdivisions per side. For example, for $N=1$ the (1,2,3) triangle is reproduced and for $N=2$ a quadratic element is obtained. Thus, in general there are N^2 congruent triangular sub-elements. These sub-elements will be taken as constant-strain triangles.

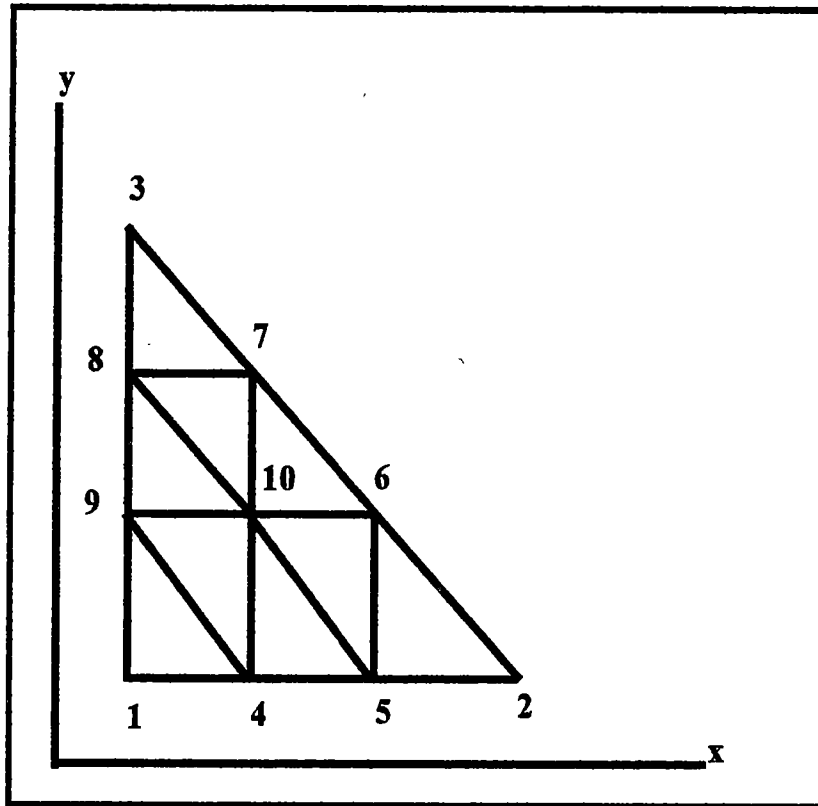


Figure 6. Triangular Sub-Element

The first problem is to find the consistent mass and stiffness matrix of one of the sub-elements.

There are two types of sub-elements as seen relative to the global coordinate system (x,y) . Elements (1,4,9), (4,5,10), etc. are the one type and elements (10,9,4), (6,10,5), etc. are the other type. Clearly, all members of one type have the same mass and stiffness matrix given by equations (3.2.22, 3.2.27) and (3.2.28). We will now explore the relationship between the mass and stiffness matrices of the two types.

Samples of the two types of triangular elements are shown in figure 6. The mass and stiffness matrices of element (1,4,9) are given by equations (3.2.22, 3.2.27) and (3.2.28)

$$m = \frac{\rho t A}{12} \begin{bmatrix} 2 & 0 & 1 & 0 & 1 & 0 \\ 0 & 2 & 0 & 1 & 0 & 1 \\ 1 & 0 & 2 & 0 & 1 & 0 \\ 0 & 1 & 0 & 2 & 0 & 1 \\ 1 & 0 & 1 & 0 & 2 & 0 \\ 0 & 1 & 0 & 1 & 0 & 2 \end{bmatrix} \quad (3.3.1)$$

and

$$k = t A B^T D B \quad (3.3.2)$$

with

$$B = \frac{1}{\text{Det } J} \begin{bmatrix} y_{49} & 0 & y_{91} & 0 & y_{14} & 0 \\ 0 & x_{94} & 0 & x_{19} & 0 & x_{41} \\ x_{94} & y_{49} & x_{19} & y_{91} & x_{41} & y_{14} \end{bmatrix}, \quad \text{Det } J = x_{19}y_{49} - x_{49}y_{19} \quad (3.3.3)$$

Note that A is the area of the triangle (1,4,9) and that

$$\text{Det } J = 2A \quad (3.3.4)$$

Next, it is important to order the nodes of both element types in a counter-clockwise manner. This will ensure that $\text{Det } J$ will be positive. Following the same procedure as in Section 3.2 the mass and stiffness matrices of an element of the second type such as (10,9,4) are

$$m^* = \frac{\rho t A^*}{12} \begin{bmatrix} 2 & 0 & 1 & 0 & 1 & 0 \\ 0 & 2 & 0 & 1 & 0 & 1 \\ 1 & 0 & 2 & 0 & 1 & 0 \\ 0 & 1 & 0 & 2 & 0 & 1 \\ 1 & 0 & 1 & 0 & 2 & 0 \\ 0 & 1 & 0 & 1 & 0 & 2 \end{bmatrix} \quad (3.3.5)$$

and

$$k^* = t A^* B^{*T} D B^* \quad (3.3.6)$$

with

$$B^* = \frac{1}{\text{Det } J^*} \begin{bmatrix} y_{94} & 0 & y_{4,10} & 0 & y_{10,9} & 0 \\ 0 & x_{49} & 0 & x_{10,4} & 0 & x_{9,10} \\ x_{49} & y_{94} & x_{10,4} & y_{4,10} & x_{9,10} & y_{10,9} \end{bmatrix}, \quad \text{Det } J^* = x_{10,4} y_{94} - x_{94} y_{10,4} \quad (3.3.7)$$

where

$$x_{ij} \triangleq x_i - x_j, \quad y_{ij} \triangleq y_i - y_j \quad (3.3.8)$$

Now, note that $A^* = A$, $\text{Det } J^* = \text{Det } J = 2A$ and that the elements of B^* are equal but opposite of those of B . Taking into account equations (3.3.1-3.3.8) yields the fact that

$$m^* = m, \quad k^* = k \quad (3.3.9)$$

Therefore, if the corners of the sub-elements are ordered as stated, we have that the mass and stiffness matrices of all the sub-elements are identical.

3.3.2 Relationship Between Sub-Element and Generic Element

The next objective is to relate the elements of m and k to the given coordinates (x_i, y_i) ($i = 1, 2, 3$) of the given generic triangle (1,2,3). To this end, consider the generic sub-element in figure 7. The mass and stiffness matrix for this element (e) are

$$m = \frac{\rho t A^e}{12} \begin{bmatrix} 2 & 0 & 1 & 0 & 1 & 0 \\ 0 & 2 & 0 & 1 & 0 & 1 \\ 1 & 0 & 2 & 0 & 1 & 0 \\ 0 & 1 & 0 & 2 & 0 & 1 \\ 1 & 0 & 1 & 0 & 2 & 0 \\ 0 & 1 & 0 & 1 & 0 & 2 \end{bmatrix} \quad (3.3.10)$$

and

$$k = t A^e B^{eT} D B^e \quad (3.3.11)$$

with

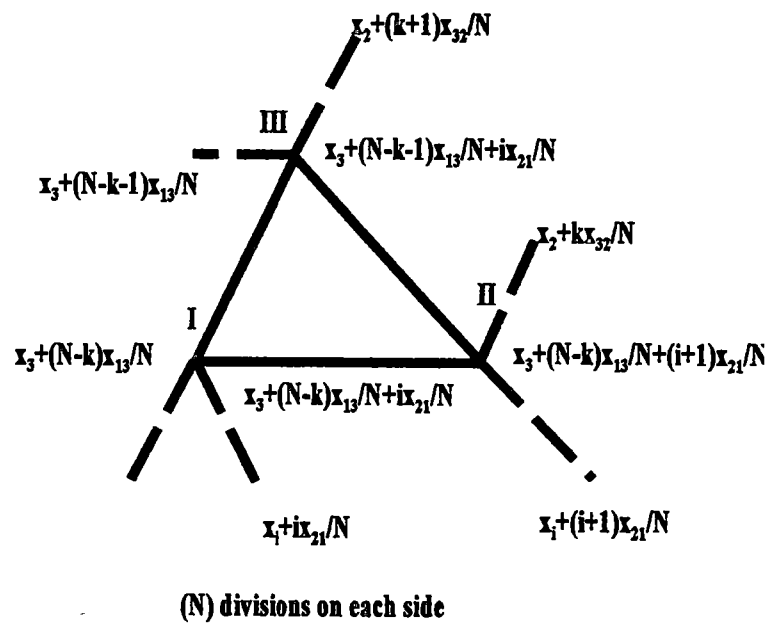


Figure 7. Generic Triangular Element

$$B^e = \frac{1}{\text{Det } J^e} \begin{bmatrix} y_{II,III} & 0 & y_{III,I} & 0 & y_{I,II} & 0 \\ 0 & x_{III,II} & 0 & x_{I,III} & 0 & x_{II,I} \\ x_{III,II} & y_{II,III} & x_{I,III} & y_{III,I} & x_{II,I} & y_{I,II} \end{bmatrix}, \quad \text{Det } J^e = x_{I,III}y_{II,III} - x_{II,III}y_{I,III} \quad (3.3.12)$$

where A^e is the area of the sub-element and $\text{Det } J^e = 2A^e$.

Because there are N^2 sub-elements in the triangle (1,2,3) it follows that

$$A^e = \frac{A}{N^2} = \frac{\text{Det } J}{2N^2}, \quad \text{Det } J^e = 2A^e = \frac{\text{Det } J}{N^2} \quad (3.3.13)$$

where A is the area of triangle (1,2,3) and $\text{Det } J$ is the determinant of the Jacobian matrix for triangle (1,2,3). In addition, from figure 7 it can be seen that, for example,

$$x_{I,II} = x_3 + (N-k)\frac{x_{13}}{N} + i\frac{x_{21}}{N} - x_3 - (N-k)\frac{x_{13}}{N} - (i+1)\frac{x_{21}}{N} = -\frac{x_{21}}{N} = \frac{x_{12}}{N} \quad (3.3.14)$$

where $x_{I,II} \triangleq x_I - x_{II}$. In general then

$$x_{IJ} \triangleq x_I - x_J = \frac{x_{ij}}{N}, \quad y_{IJ} \triangleq y_I - y_J = \frac{y_{ij}}{N} \quad (3.3.15)$$

So, from equation (3.3.12) it follows that

$$B^e = NB, \quad \text{Det } J^e = \frac{\text{Det } J}{N^2} \quad (3.3.16)$$

Then, from equations (3.3.10-3.3.16) we have

$$m = \frac{\rho t A}{12 N^2} \begin{bmatrix} 2 & 0 & 1 & 0 & 1 & 0 \\ 0 & 2 & 0 & 1 & 0 & 1 \\ 1 & 0 & 2 & 0 & 1 & 0 \\ 0 & 1 & 0 & 2 & 0 & 1 \\ 1 & 0 & 1 & 0 & 2 & 0 \\ 0 & 1 & 0 & 1 & 0 & 2 \end{bmatrix} = \frac{M}{N^2}, \quad k = t A B^T D B = K \quad (3.3.17)$$

with M and K the mass and stiffness matrix of element (1,2,3) and

$$B = \frac{1}{\text{Det } J} \begin{bmatrix} y_{23} & 0 & y_{31} & 0 & y_{12} & 0 \\ 0 & x_{32} & 0 & x_{13} & 0 & x_{21} \\ x_{32} & y_{23} & x_{13} & y_{31} & x_{21} & y_{12} \end{bmatrix}, \quad \text{Det } J = x_{13}y_{23} - x_{23}y_{13} \quad (3.3.18)$$

Note that we assumed the mass density ρ , the plate thickness t , Young's Modulus E , and Poisson's ratio to be constant throughout the element (1,2,3).

3.3.3. Model for Generic Triangle

At this point it is necessary to distinguish between the interface degrees of freedom and the non-interface degrees of freedom. The user has a choice. For the generic element in figure 6, for example, one possibility is to consider nodes 1 through 9 as interface nodes and node 10 as a non-interface node. On the other hand, it is also possible to take nodes 1, 2, and 3 as interface nodes while taking nodes 4 through 10 as non-interface nodes. In either case, the next step is to assemble all the sub-elements to form the overall generic triangle. This is done according to the procedure outlined in Section 2.5. In the next section several practical assemblies of sub-elements are explicitly described.

3.4 Mass and Stiffness Matrix for $N=1$ and $N=2$

For $N = 1$, one sub-element is the same as the overall generic triangle. Equation (3.3.17) shows that $m = M$ and $k = K$ where from (3.3.1)

$$M = \frac{\rho t A}{12} \begin{bmatrix} 2 & 0 & 1 & 0 & 1 & 0 \\ 0 & 2 & 0 & 1 & 0 & 1 \\ 1 & 0 & 2 & 0 & 1 & 0 \\ 0 & 1 & 0 & 2 & 0 & 1 \\ 1 & 0 & 1 & 0 & 2 & 0 \\ 0 & 1 & 0 & 1 & 0 & 2 \end{bmatrix} \quad (3.4.1)$$

and from equation (3.3.2)

$$K = t A B^T D B \quad (3.4.2)$$

with

$$B = \frac{1}{\text{Det } J} \begin{bmatrix} y_{23} & 0 & y_{31} & 0 & y_{12} & 0 \\ 0 & x_{32} & 0 & x_{13} & 0 & x_{21} \\ x_{32} & y_{23} & x_{13} & y_{31} & x_{21} & y_{12} \end{bmatrix} \quad (3.4.3)$$

and

$$x_{ij} \triangleq x_i - x_j, \quad y_{ij} \triangleq y_i - y_j \quad (3.4.4)$$

and from equation (3.3.18)

$$\text{Det } J = x_{13}y_{23} - x_{23}y_{13}$$

(3.4.5)

and where D is the material matrix

$$D = \frac{E}{1-\nu^2} \begin{bmatrix} 1 & \nu & 0 \\ \nu & 1 & 0 \\ 0 & 0 & \frac{1-\nu}{2} \end{bmatrix}, \quad D = \frac{E}{(1+\nu)(1-2\nu)} \begin{bmatrix} 1-\nu & \nu & 0 \\ \nu & 1-\nu & 0 \\ 0 & 0 & \frac{1-2\nu}{2} \end{bmatrix} \quad (3.4.6)$$

for plane stress and plane strain, respectively. ρ is the mass density, t is the thickness of the triangle, A is the area of the triangle, E is Young's modulus, and ν is Poisson's ratio.

For $N = 2$, we have, again from Eq. (3.3.18) that

$$m = \frac{1}{4}M, \quad k = K \quad (3.4.7)$$

with M and K given in Eqs. (3.4.1) and (3.4.2). To obtain the mass and stiffness matrix of the overall triangle (1,2,3) as shown in figure 8, we must assemble sub-elements (1,4,6), (5,6,4), (4,2,5), and (6,5,3). The mass matrix becomes

$$M_{(1,2,3)} = \frac{\rho t A}{48} \begin{bmatrix} 2 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 2 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 2 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 2 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 2 & 0 & 0 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 2 & 0 & 0 & 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 & 0 & 0 & 6 & 0 & 2 & 0 & 2 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 & 0 & 6 & 0 & 2 & 0 & 2 \\ 0 & 0 & 1 & 0 & 1 & 0 & 2 & 0 & 6 & 0 & 2 & 0 \\ 0 & 0 & 0 & 1 & 0 & 1 & 0 & 2 & 0 & 6 & 0 & 2 \\ 1 & 0 & 0 & 0 & 1 & 0 & 2 & 0 & 2 & 0 & 6 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 & 0 & 2 & 0 & 2 & 0 & 6 \end{bmatrix}$$

(3.4.8)

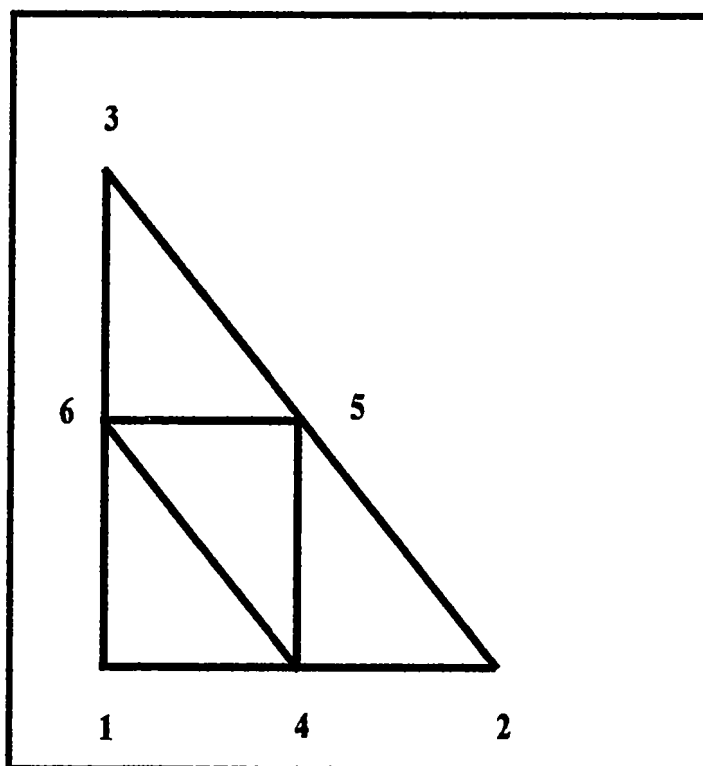


Figure 8. Generic Triangle for $N=2$

It is important to note that this matrix does not depend on the coordinates of the nodes of the triangle. This matrix therefore represents a one time computation and can be wired into the program. The stiffness matrix $K_{(1,2,3)}$ is found from assembling four k matrices. The matrix $K_{(1,2,3)}$ does depend on the coordinates of the corner nodes 1, 2, and 3.

Both cases $N = 1$ and $N = 2$ have no interior nodes in the ordinary sense. The case $N = 2$ can however be considered as having nodes 4, 5, and 6 as interior nodes and nodes 1, 2, and 3 as interface nodes. In that case the matrices M^* and K^* of equation (2.4.6) must be computed.

3.5 Four- Node Quadrilateral Element

Consider the quadrilateral element with the corner nodes 1, 2, 3, and 4 in counter clockwise order, as show in figure 9 such that the edges are bounded by +1 and -1.

The general relations between the cartesian coordinates (x, y) and the shape functions (N_1, N_2, N_3, N_4) are

$$\begin{aligned} x &= \sum_{m=1}^4 N_m x_m \\ y &= \sum_{m=1}^4 N_m y_m \\ 1 &= \sum_{m=1}^4 N_m \end{aligned} \tag{3.5.1}$$

where the shape functions are expressed in the natural $(s-t)$ coordinate system.

$$\begin{aligned} N_1 &= \frac{1}{4} (1-s) (1-t) \\ N_2 &= \frac{1}{4} (1+s) (1-t) \\ N_3 &= \frac{1}{4} (1+s) (1+t) \\ N_4 &= \frac{1}{4} (1-s) (1+t) \end{aligned} \tag{3.5.2}$$

The shape functions are used to map the square $(s-t)$ coordinate system onto a

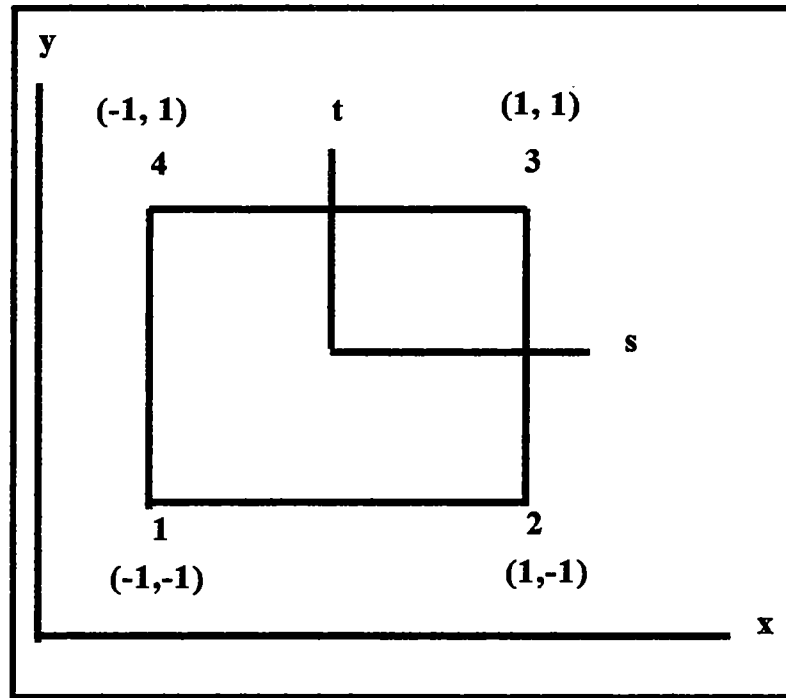


Figure 9. Quadrilateral Element (x - y) (s - t) Coordinate System

quadrilateral in the $(x-y)$ coordinate system for any arbitrarily shaped quadrilateral. The displacement field within the element becomes

$$u = \sum_{m=1}^4 N_m u_m, \quad v = \sum_{m=1}^4 N_m v_m \quad (3.5.3)$$

In matrix form

$$\mathbf{u} = N \mathbf{u}_c \quad (3.5.4)$$

with

$$\mathbf{u} \triangleq \begin{Bmatrix} u \\ v \end{Bmatrix}, \quad N \triangleq \begin{bmatrix} N_1 & 0 & N_2 & 0 & N_3 & 0 & N_4 \\ 0 & N_1 & 0 & N_2 & 0 & N_3 & 0 \end{bmatrix}, \quad \mathbf{u}_c \triangleq \begin{Bmatrix} u_1 \\ v_1 \\ u_2 \\ v_2 \\ u_3 \\ v_3 \\ u_4 \\ v_4 \end{Bmatrix} \quad (3.5.5)$$

In order to construct the stiffness matrix the strains, which are defined in terms of the derivatives of the displacement with respect to the (x) and (y) coordinates must be determined. Since the displacements are functions of s and t , the chain rule is needed for differentiation.

A function $z = z(x, y)$ may be considered an implicit function of s and t as

$z = z[x(s, t), y(s, t)]$. Using the chain rule of differentiation

$$\begin{aligned}\frac{\partial z}{\partial s} &= \frac{\partial z}{\partial x} \frac{\partial x}{\partial s} + \frac{\partial z}{\partial y} \frac{\partial y}{\partial s} \\ \frac{\partial z}{\partial t} &= \frac{\partial z}{\partial x} \frac{\partial x}{\partial t} + \frac{\partial z}{\partial y} \frac{\partial y}{\partial t}\end{aligned}\tag{3.5.6}$$

or

$$\begin{Bmatrix} \frac{\partial z}{\partial s} \\ \frac{\partial z}{\partial t} \end{Bmatrix} = J \begin{Bmatrix} \frac{\partial x}{\partial s} \\ \frac{\partial x}{\partial t} \end{Bmatrix}\tag{3.5.7}$$

where

$$J = \begin{bmatrix} \frac{\partial x}{\partial s} & \frac{\partial y}{\partial s} \\ \frac{\partial x}{\partial t} & \frac{\partial y}{\partial t} \end{bmatrix}\tag{3.5.8}$$

Differentiating and substituting equations (3.5.1) and (3.5.2)

$$J = \frac{1}{4} \begin{bmatrix} -(1-t)x_1 + (1-t)x_2 + (1+t)x_3 - (1+t)x_4 & -(1-t)y_1 + (1-t)y_2 + (1+t)y_3 - (1+t)y_4 \\ -(1-s)x_1 - (1+s)x_2 + (1+s)x_3 + (1-s)x_4 & -(1-s)y_1 - (1+s)y_2 + (1+s)y_3 + (1-s)y_4 \end{bmatrix}\tag{3.5.9}$$

or

$$J = \begin{bmatrix} J_{11} & J_{12} \\ J_{21} & J_{22} \end{bmatrix}\tag{3.5.10}$$

Inverting equation (3.5.7)

$$\begin{Bmatrix} \frac{\partial z}{\partial x} \\ \frac{\partial z}{\partial y} \end{Bmatrix} = J^{-1} \begin{Bmatrix} \frac{\partial z}{\partial s} \\ \frac{\partial z}{\partial t} \end{Bmatrix} \quad (3.5.11)$$

or

$$\begin{Bmatrix} \frac{\partial z}{\partial x} \\ \frac{\partial z}{\partial y} \end{Bmatrix} = \frac{1}{\det J} \begin{Bmatrix} J_{22} & -J_{12} \\ -J_{21} & J_{11} \end{Bmatrix} \begin{Bmatrix} \frac{\partial z}{\partial s} \\ \frac{\partial z}{\partial t} \end{Bmatrix} \quad (3.5.12)$$

where

$$\det J = (J_{11})(J_{22}) - (J_{21})(J_{12}) \quad (3.5.13)$$

Next, the strain energy is given as

$$U = \frac{1}{2} \int_v \sigma^T \epsilon dV \quad (3.5.14)$$

or

$$U = \sum_e t e \frac{1}{2} \int_e \sigma^T \epsilon dA \quad (3.5.15)$$

where

$$\epsilon \triangleq \begin{Bmatrix} \frac{\partial u}{\partial x} \\ \frac{\partial v}{\partial y} \\ \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} \end{Bmatrix} \quad (3.5.16)$$

Noting that z may represent the components of displacement, temperature, and so forth, consider $z = u$. Then equation (3.5.12) becomes

$$\begin{Bmatrix} \frac{\partial u}{\partial x} \\ \frac{\partial u}{\partial y} \end{Bmatrix} = \frac{1}{\det J} \begin{Bmatrix} J_{22} & -J_{12} \\ -J_{21} & J_{11} \end{Bmatrix} \begin{Bmatrix} \frac{\partial u}{\partial s} \\ \frac{\partial u}{\partial t} \end{Bmatrix} \quad (3.5.17)$$

Similarly, consider $z = v$

$$\begin{Bmatrix} \frac{\partial v}{\partial x} \\ \frac{\partial v}{\partial y} \end{Bmatrix} = \frac{1}{\det J} \begin{Bmatrix} J_{22} & -J_{12} \\ -J_{21} & J_{11} \end{Bmatrix} \begin{Bmatrix} \frac{\partial v}{\partial s} \\ \frac{\partial v}{\partial t} \end{Bmatrix} \quad (3.5.18)$$

Combining equations (3.5.16), (3.5.17), and (3.5.18)

$$\epsilon = A \begin{Bmatrix} \frac{\partial u}{\partial s} \\ \frac{\partial u}{\partial t} \\ \frac{\partial v}{\partial s} \\ \frac{\partial v}{\partial t} \end{Bmatrix} \quad (3.5.19)$$

where

$$A = \frac{1}{\det J} \begin{bmatrix} J_{22} & -J_{21} & 0 & 0 \\ 0 & 0 & -J_{21} & J_{11} \\ -J_{21} & J_{11} & J_{22} & -J_{12} \end{bmatrix} \quad (3.5.20)$$

Differentiating equation (3.5.3)

$$\begin{Bmatrix} \frac{\partial u}{\partial s} \\ \frac{\partial u}{\partial t} \\ \frac{\partial v}{\partial s} \\ \frac{\partial v}{\partial t} \end{Bmatrix} = G \mathbf{u}_c \quad (3.5.21)$$

where

$$G = \frac{1}{4} \begin{bmatrix} -(1-t) & 0 & (1-t) & 0 & (1+t) & 0 & -(1+t) & 0 \\ -(1-s) & 0 & -(1+s) & 0 & (1+s) & 0 & (1-s) & 0 \\ 0 & -(1-t) & 0 & (1-t) & 0 & (1+t) & 0 & -(1+t) \\ 0 & -(1-s) & 0 & -(1+s) & 0 & (1+s) & 0 & (1-s) \end{bmatrix} \quad (3.5.22)$$

Combining equations (3.5.19) and (3.5.21), strain becomes

$$\epsilon = A G \mathbf{u}_c \quad (3.5.23)$$

or

$$\epsilon = B \mathbf{u}_c \quad (3.5.24)$$

where

$$B = AG \quad (3.5.25)$$

Stress for the element is given by

$$\sigma = DBu_c \quad (3.5.26)$$

where D is the material matrix

$$D = \frac{E}{1-\nu^2} \begin{bmatrix} 1 & \nu & 0 \\ \nu & 1 & 0 \\ 0 & 0 & \frac{1-\nu}{2} \end{bmatrix}, \quad D = \frac{E}{(1+\nu)(1-2\nu)} \begin{bmatrix} 1-\nu & \nu & 0 \\ \nu & 1-\nu & 0 \\ 0 & 0 & \frac{1-2\nu}{2} \end{bmatrix} \quad (3.5.27)$$

for plane stress and plain strain respectively.

Substituting equations (3.5.24) and (3.5.26) for strain and stress, the strain energy equation (3.5.15) becomes

$$U = \sum_e \frac{1}{2} u_c^T \left[t^e \int_{-1}^1 \int_{-1}^1 B^T DB \det J ds dt \right] u \quad (3.5.28)$$

or

$$U = \sum_e \frac{1}{2} u_c^T k^e u_c \quad (3.5.29)$$

where

$$k = t^e \int_{-1}^1 \int_{-1}^1 B^T DB \det J ds dt \quad (3.5.30)$$

The mass is given as

$$m = \rho t^e \int_{-1}^1 \int_{-1}^1 N^T N \det J ds dt \quad (3.5.31)$$

3.6 Generic Four Node Quadrilateral Element

3.6.1 Element Properties of Sub-Element

Figure 10 shows a four node quadrilateral of arbitrary shape, subdivided into four node quadrilateral sub-elements. Each side of the (1,2,3,4) quadrilateral has $N+1$ nodes, where N is equal to the number of sub-divisions per side. For example, for $N=1$, the (1,2,3,4) quadrilateral is reproduced. For $N=2$, a quadratic element is obtained. Thus, in general there are N^2 congruent quadrilateral sub-elements. These sub-elements will be taken as four-node quadrilaterals. See figure 10.

3.6.2 Relationship Between Sub-Elements and Generic Element

Recall, the mass and stiffness matrix of any generic quadrilateral element is given by equations (3.5.30) and (3.5.31) reprinted here.

$$m = \rho t^e \int_{-1}^1 \int_{-1}^1 N^T N \det J ds dt \quad (3.6.1)$$

and

$$k = t^e \int_{-1}^1 \int_{-1}^1 B^T D B \det J ds dt \quad (3.6.2)$$

with

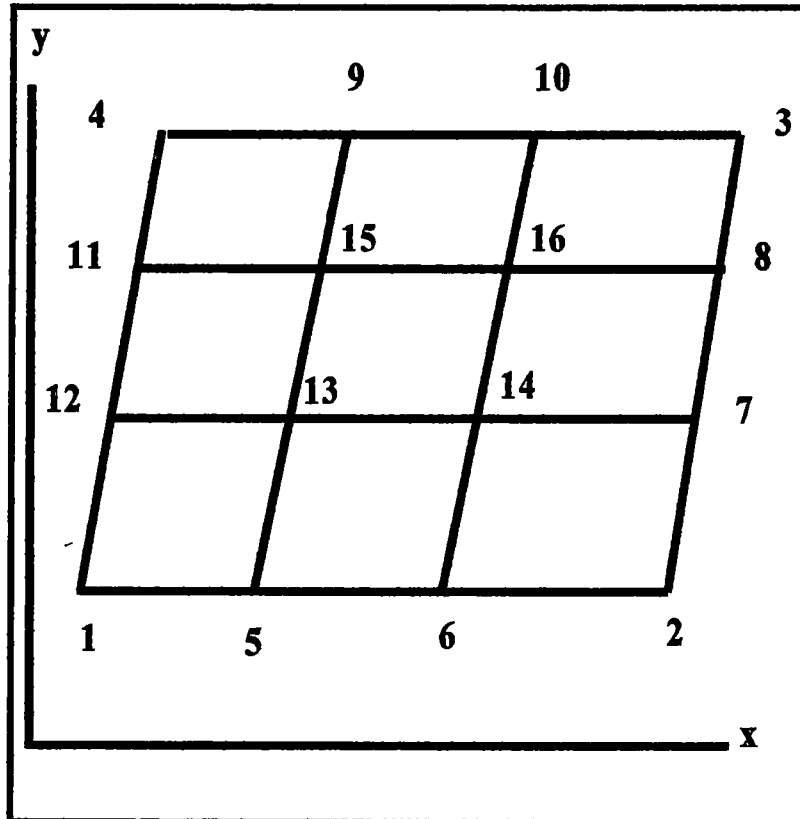


Figure 10. Quadrilateral Sub-Element

$$B = AG$$

(3.6.3)

where

$$A = \frac{1}{\det J} \begin{bmatrix} J_{22} & -J_{21} & 0 & 0 \\ 0 & 0 & -J_{21} & J_{11} \\ -J_{21} & J_{11} & J_{22} & -J_{12} \end{bmatrix}$$

(3.6.4)

and

$$G = \frac{1}{4} \begin{bmatrix} -(1-t) & 0 & (1-t) & 0 & (1+t) & 0 & -(1+t) & 0 \\ -(1-s) & 0 & -(1+s) & 0 & (1+s) & 0 & (1-s) & 0 \\ 0 & -(1-t) & 0 & (1-t) & 0 & (1+t) & 0 & -(1+t) \\ 0 & -(1-s) & 0 & -(1+s) & 0 & (1+s) & 0 & (1-s) \end{bmatrix}$$

(3.6.5)

The next objective is to relate the elements of m and k of the generic four-node quadrilateral element to those of the sub-element. Consider the generic sub-element figure 11. The mass and stiffness matrices for the element (e) is given as

$$m^e = \rho t^e \int_{-1}^1 \int_{-1}^1 N^T N \det J^e ds dt$$

(3.6.6)

and

$$k^e = t^e \int_{-1}^1 \int_{-1}^1 B^{eT} D B^e \det J^e ds dt$$

(3.6.7)

where

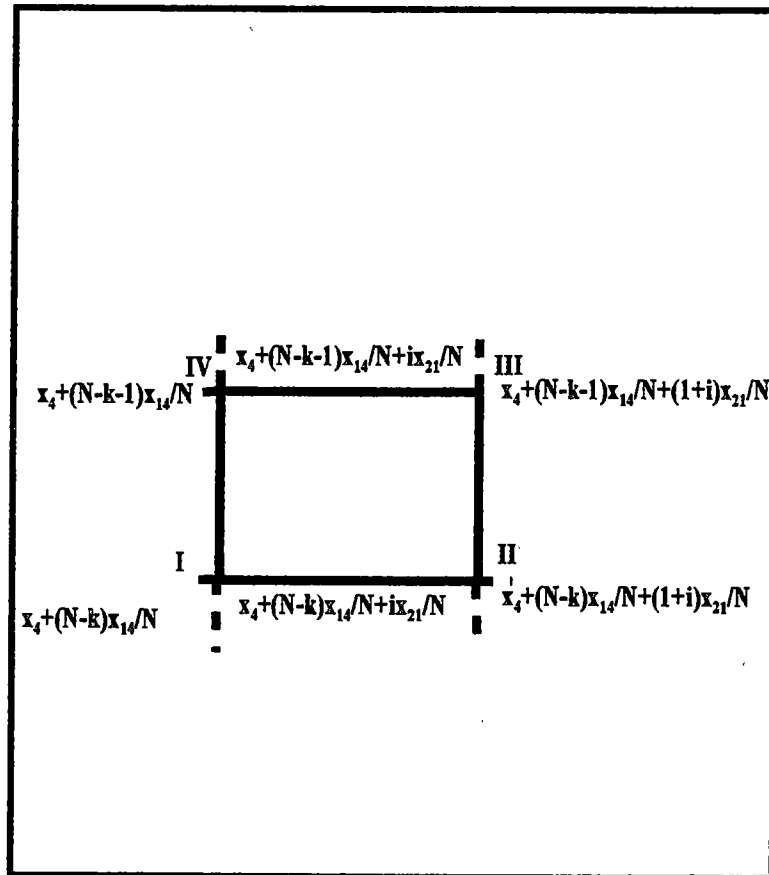


Figure 11. Generic Quadrilateral Element

$$J^e = \begin{bmatrix} J_{11}^e & J_{12}^e \\ J_{21}^e & J_{22}^e \end{bmatrix} \quad (3.6.8)$$

and

$$J_{11}^e = -(1-t)x_I + (1-t)x_{II} + (1+t)x_{III} - (1+t)x_{IV} \quad (3.6.9)$$

Combining like terms,

$$J_{11}^e = (1-t)(-x_I + x_{II}) + (1+t)(x_{III} - x_{IV}) \quad (3.6.10)$$

From figure 11

$$-x_I + x_{II} = -(x_4 + (N-k)\left(\frac{x_{14}}{N}\right) + i\frac{x_{21}}{N}) + (x_4 + (N-k)\left(\frac{x_{14}}{N}\right) + (1+i)\frac{x_{21}}{N}) \quad (3.6.11)$$

Thus

$$-x_I + x_{II} = \frac{x_{21}}{N} = \frac{(x_2 - x_1)}{N} \quad (3.6.12)$$

Also from figure 11 it is seen that

$$x_{III} - x_{IV} = (x_4 + (N-k-1)\left(\frac{x_{14}}{N}\right) + (1+i)\frac{x_{21}}{N}) - (x_4 + (N-k-1)\left(\frac{x_{14}}{N}\right) + i\frac{x_{21}}{N}) \quad (3.6.13)$$

Thus,

$$x_{III} - x_{IV} = \frac{x_{21}}{N} = \frac{(x_2 - x_1)}{N} \quad (3.6.14)$$

Continuing in the same manner,

$$J_{21}^e = -(1-s)x_I - (1+s)x_{II} + (1+s)x_{III} + (1-s)x_{IV} \quad (3.6.15)$$

Combining like terms,

$$J_{21}^e = (1-s)(-x_I + x_{IV}) + (1+s)(-x_{II} + x_{III}) \quad (3.6.16)$$

From figure 11

$$-x_I + x_{IV} = -(x_4 + (N-k)\left(\frac{x_{14}}{N}\right) + i\frac{x_{21}}{N}) + (x_4 + (N-k-1)\left(\frac{x_{14}}{N}\right) + i\frac{x_{21}}{N}) \quad (3.6.17)$$

Thus,

$$-x_I + x_{IV} = -\frac{x_{14}}{N} = \frac{(x_4 - x_1)}{N} \quad (3.6.18)$$

Also, from figure 11 it is seen that

$$-x_{II} + x_{III} = -(x_4 + (N-k)\left(\frac{x_{14}}{N}\right) + (1+i)\frac{x_{21}}{N}) + (x_4 + (N-k-1)\left(\frac{x_{14}}{N}\right) + (1+i)\frac{x_{21}}{N}) \quad (3.6.19)$$

$$-x_{II} + x_{III} = -\frac{x_{14}}{N} = \frac{(x_4 - x_1)}{N} \quad (3.6.20)$$

$$-y_{II} + y_{III} = -\frac{y_{14}}{N} = \frac{(y_4 - y_1)}{N} \quad (3.6.21)$$

Thus

$$y_{III} - y_{IV} = \frac{y_{21}}{N} = \frac{(y_2 - y_1)}{N} \quad (3.6.22)$$

$$-y_I + y_{IV} = -\frac{y_{14}}{N} = \frac{(y_4 - y_1)}{N} \quad (3.6.23)$$

$$-y_I + y_{II} = \frac{y_{21}}{N} = \frac{(y_2 - y_1)}{N} \quad (3.6.24)$$

Equation (3.6.10) reprinted here

$$J_{11}^e = (1-t)(-x_I + x_{II}) + (1+t)(x_{III} - x_{IV}) \quad (3.6.25)$$

becomes

$$J_{11}^e = (1-t)\frac{x_2 - x_1}{N} + (1+t)\frac{x_2 - x_1}{N} \quad (3.6.26)$$

Equation (3.6.16) reprinted here

$$J_{21}^e = (1-s)(-x_I + x_{IV}) + (1+s)(-x_{II} + x_{III}) \quad (3.6.27)$$

becomes

$$J_{21}^e = (1-s)\frac{x_4 - x_1}{N} + (1+s)\frac{x_4 - x_1}{N} \quad (3.6.28)$$

Similarly,

$$J_{12}^e = (1-t)(-y_I + y_{II}) + (1+t)(y_{III} - y_{IV}) \quad (3.6.29)$$

becomes

$$J_{12}^e = (1-t) \frac{y_2 - y_1}{N} + (1+t) \frac{y_2 - y_1}{N} \quad (3.6.30)$$

also

$$J_{22}^e = (1-s)(-y_I + y_{IV}) + (1+s)(-y_{II} + y_{III}) \quad (3.6.31)$$

becomes

$$J_{22}^e = (1-s) \frac{y_4 - y_1}{N} + (1+s) \frac{y_4 - y_1}{N} \quad (3.6.32)$$

Finally J^e becomes

$$J^e = \frac{1}{4N} \begin{bmatrix} (1-t)(x_2 - x_1) + (1+t)(x_2 - x_1) & (1-t)(y_2 - y_1) + (1+t)(y_2 - y_1) \\ (1-s)(x_4 - x_1) + (1+s)(x_4 - x_1) & (1-s)(y_4 - y_1) + (1+s)(y_4 - y_1) \end{bmatrix} \quad (3.6.33)$$

Recall equation (3.5.9) rewritten here

$$J = \frac{1}{4} \begin{bmatrix} (1-t)(x_2 - x_1) + (1+t)(x_3 - x_4) & (1+t)(y_3 - y_4) + (1-t)(y_2 - y_1) \\ (1-s)(x_4 - x_1) + (1+s)(x_3 - x_2) & (1-s)(y_4 - y_1) + (1+s)(y_3 - y_2) \end{bmatrix} \quad (3.6.34)$$

But

$$\begin{aligned} (x_2 - x_1) &= (x_3 - x_4) \\ (-x_1 + x_4) &= (-x_2 + x_3) \\ (y_2 - y_1) &= (y_3 - y_4) \\ (-y_1 + y_4) &= (-y_2 + y_3) \end{aligned} \quad (3.6.35)$$

Therefore the values of the determinants are the same with the exception of the factor N in equation (3.6.33). Thus

$$J^e = \frac{J}{N}$$

(3.6.36)

Since J_{22} , J_{12} , J_{11} , and J_{21} are the same for J and J^e

$$A^e = \frac{1}{\det J^e} \begin{bmatrix} J_{22} & -J_{21} & 0 & 0 \\ 0 & 0 & -J_{21} & J_{11} \\ -J_{21} & J_{11} & J_{22} & -J_{12} \end{bmatrix}$$

(3.6.37)

Then

$$A^e = \frac{1}{\frac{\det J}{N}} \begin{bmatrix} J_{22} & -J_{21} & 0 & 0 \\ 0 & 0 & -J_{21} & J_{11} \\ -J_{21} & J_{11} & J_{22} & -J_{12} \end{bmatrix}$$

(3.6.38)

or

$$A^e = \frac{N}{\det J} \begin{bmatrix} J_{22} & -J_{21} & 0 & 0 \\ 0 & 0 & -J_{21} & J_{11} \\ -J_{21} & J_{11} & J_{22} & -J_{12} \end{bmatrix}$$

(3.6.39)

Thus

$$A^e = NA$$

(3.6.40)

Recall the sub-element stiffness given by equation (3.6.7) and reprinted here

$$k^e = t^e \int_{-1}^1 \int_{-1}^1 B^{eT} D B^e \det J^e ds dt$$

(3.6.41)

where

$$B^e = A^e G, \quad J^e = \frac{J}{N} \quad (3.6.42)$$

Substituting equation (3.6.42) into equation (3.6.41)

$$k^e = t^e \int_{-1}^1 \int_{-1}^1 A^{eT} G^T D A^e G \frac{\det J}{N} ds dt \quad (3.6.43)$$

Substituting equation (3.6.40) into equation (3.6.43), k^e becomes

$$k^e = t^e \int_{-1}^1 \int_{-1}^1 N A^T G^T D N A G \frac{\det J}{N} ds dt \quad (3.6.44)$$

Simplifying

$$k^e = N t^e \int_{-1}^1 \int_{-1}^1 B^T D B \det J ds dt \quad (3.6.45)$$

Thus

$$k^e = N k \quad (3.6.46)$$

Recall the sub-element mass given by equation (3.6.6) and reprinted here

$$m^e = \rho t^e \int_{-1}^1 \int_{-1}^1 N^T N \det J^e ds dt \quad (3.6.47)$$

Substituting the equation for J^e from equation (3.6.42)

$$m^e = \rho t^e \int_{-1}^1 \int_{-1}^1 N^T N \frac{\det J}{N} ds dt \quad (3.6.48)$$

Simplifying

$$m^e = \frac{1}{N} \rho t^e \int_{-1}^1 \int_{-1}^1 N^T N \det J ds dt \quad (3.6.49)$$

Thus

$$m^e = \frac{1}{N} m \quad (3.6.50)$$

3.6.3 Model for Generic Quadrilateral

At this point it is necessary to distinguish between the interface degrees of freedom and the non-interface degrees of freedom. For the generic quadrilateral element in figure 10, for example, one possibility is to consider nodes 1 through 12 as interface nodes and nodes 13 through 16 as non-interface nodes. The next step is to assemble all the sub-elements to form the overall generic quadrilateral. This is done according to the procedure outlined in Section 2.5.

CHAPTER 4

DEMONSTRATION EXAMPLES

4.1 Plate

4.1.1 Free-Free Plate

To demonstrate the capability of the dynamic finite element method, the natural frequencies of a (2 x 1) plate are investigated. The properties of the plate are: $E = 3 \times 10^7 \text{ lb/in}^2$, $\nu = 0.28$, thickness=0.1, and $\rho = 7.34 \times 10^{-4} \text{ lb/in}^3$. Since the solution is not exact, an approximation of the exact natural frequencies were obtain using the COSMOS, and ANSYS finite element programs.

4.1.2 Convergence of Mode Shapes

Since there is no difference between the computation of the Dynamic Finite Element Method and the Standard Finite Element Method mode shape, the results are identical. The first eight modes are illustrated in figures 12-14.

4.1.3 Dynamic Finite Element Method (Triangular Element)

Fortran programs were written using the dynamic finite element method developed in sections 3.2 and 3.3 for the triangular element.

As a first example, the dynamic finite element model, shown in figure 15, is compared to the standard finite element model. Graphs were generated using data from the standard finite element model, and from the dynamic finite element model, for the 18-Node Triangular Element. See figures 16-18. The graphs were used to obtain the number of degrees of freedom, needed for the convergence of a desired number of frequencies.

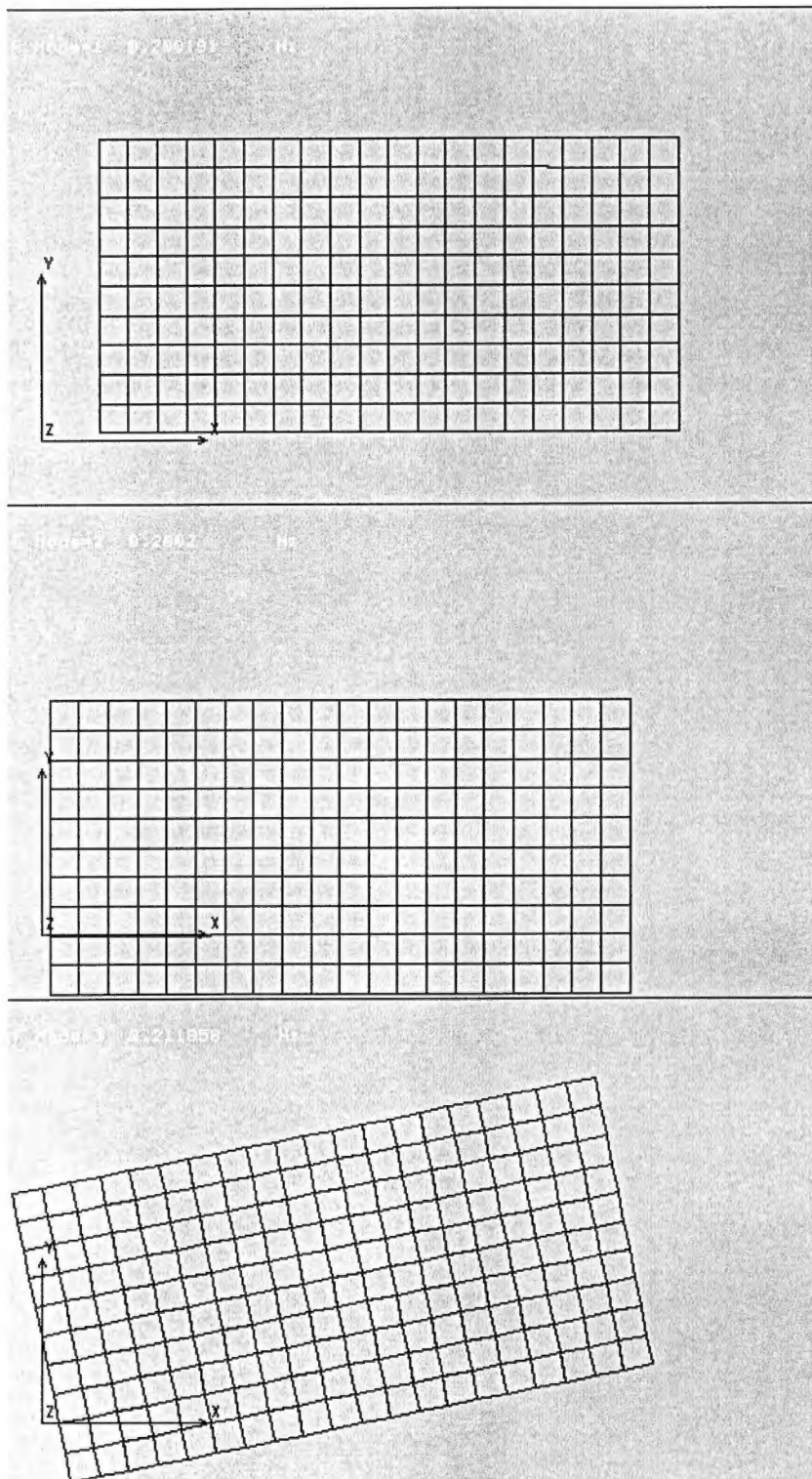


Figure 12. Illustration of First Three Rigid Body Mode Shapes

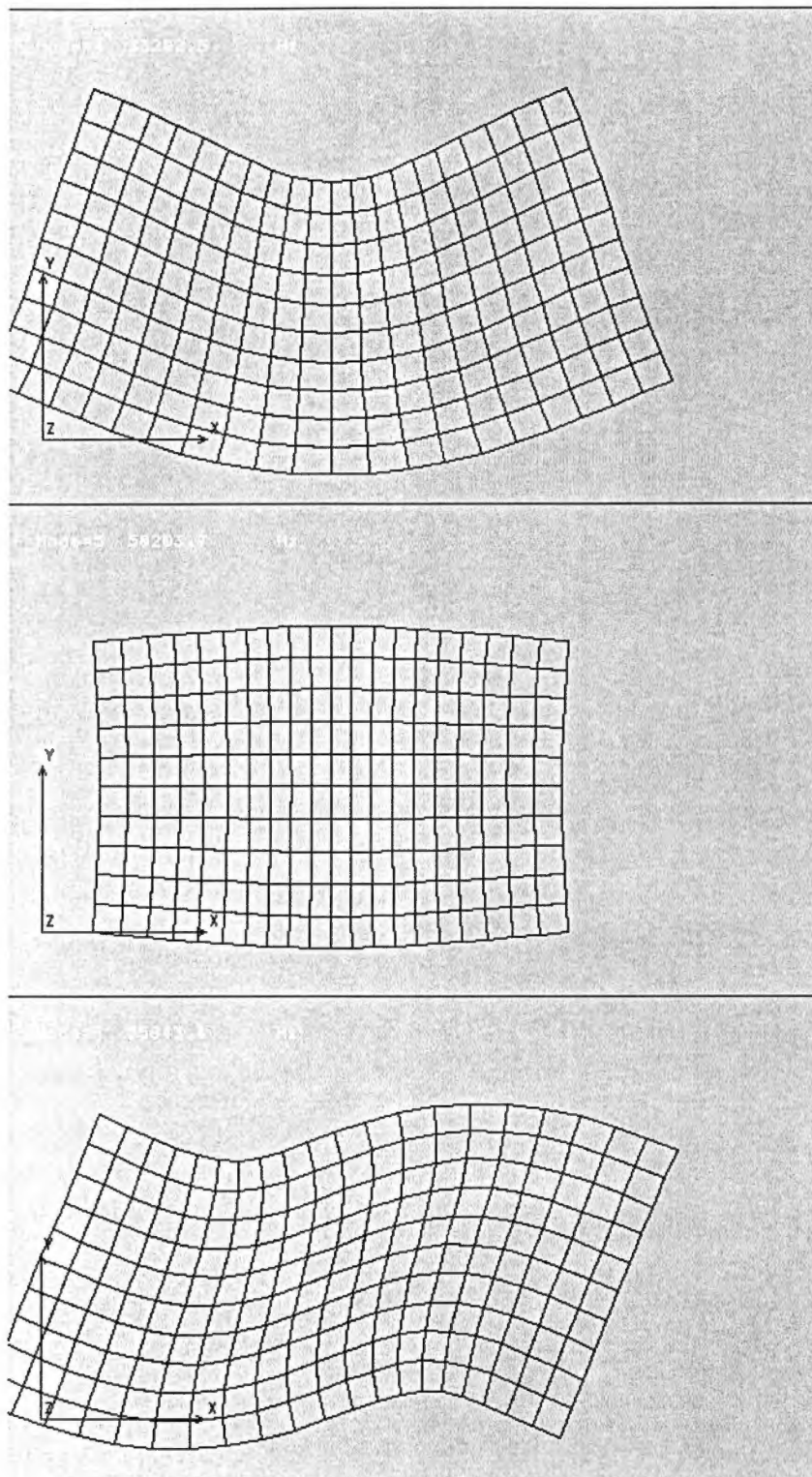


Figure 13. Illustration of Next Three Mode Shapes

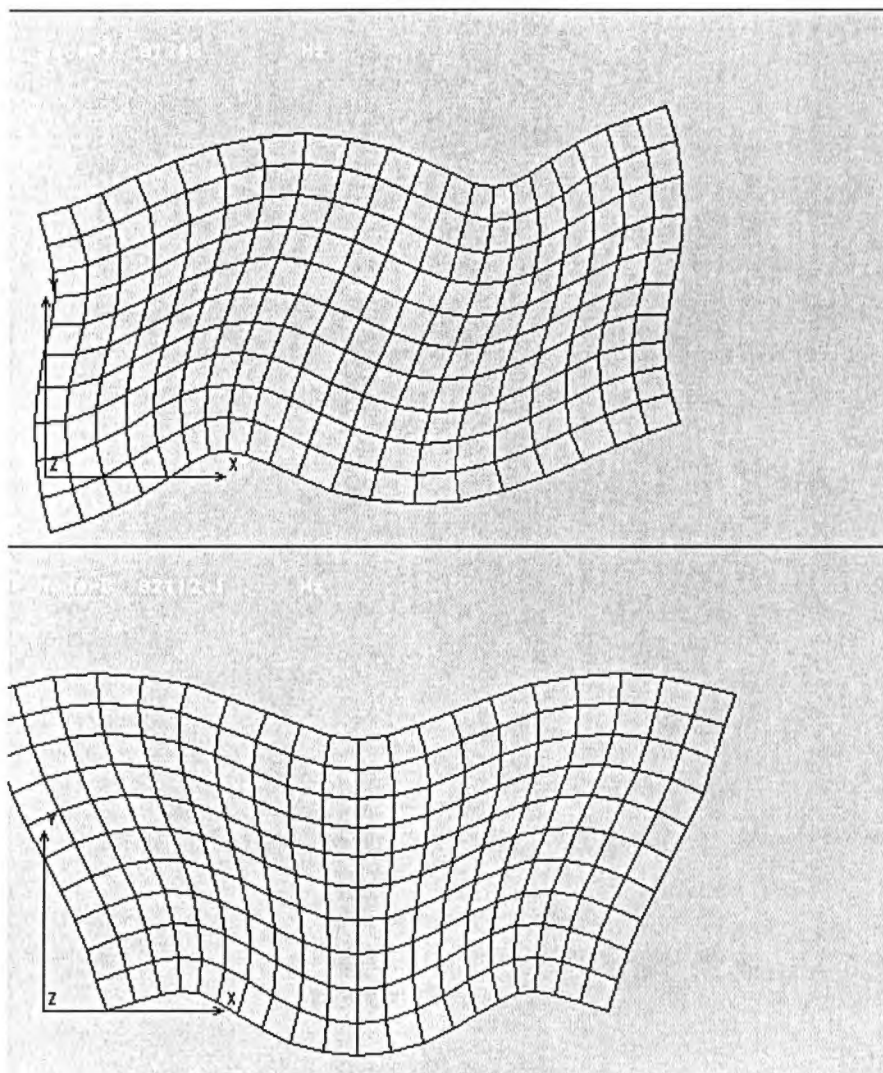


Figure 14. Illustration of Next Two Mode Shapes

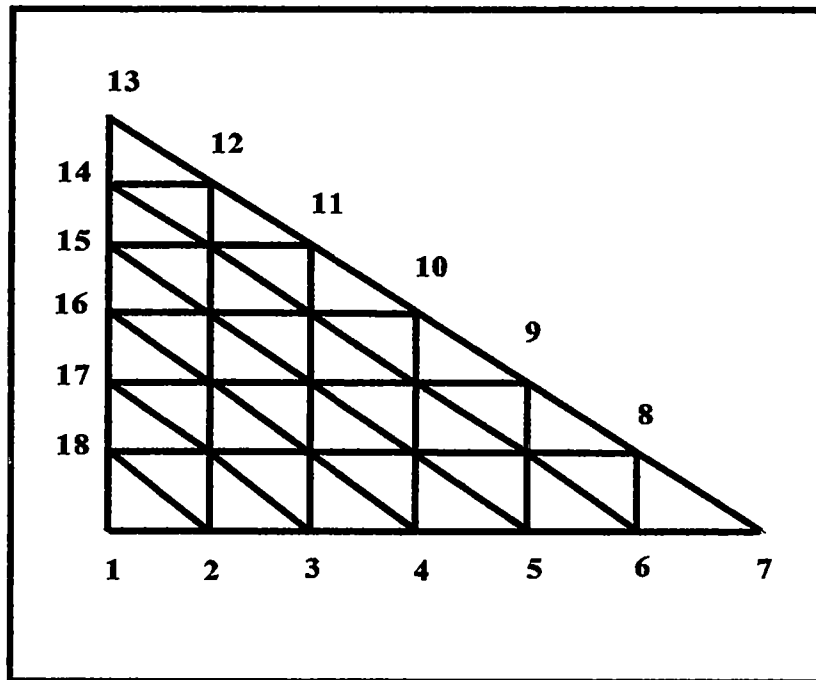


Figure 15. 18-Node Triangular Element

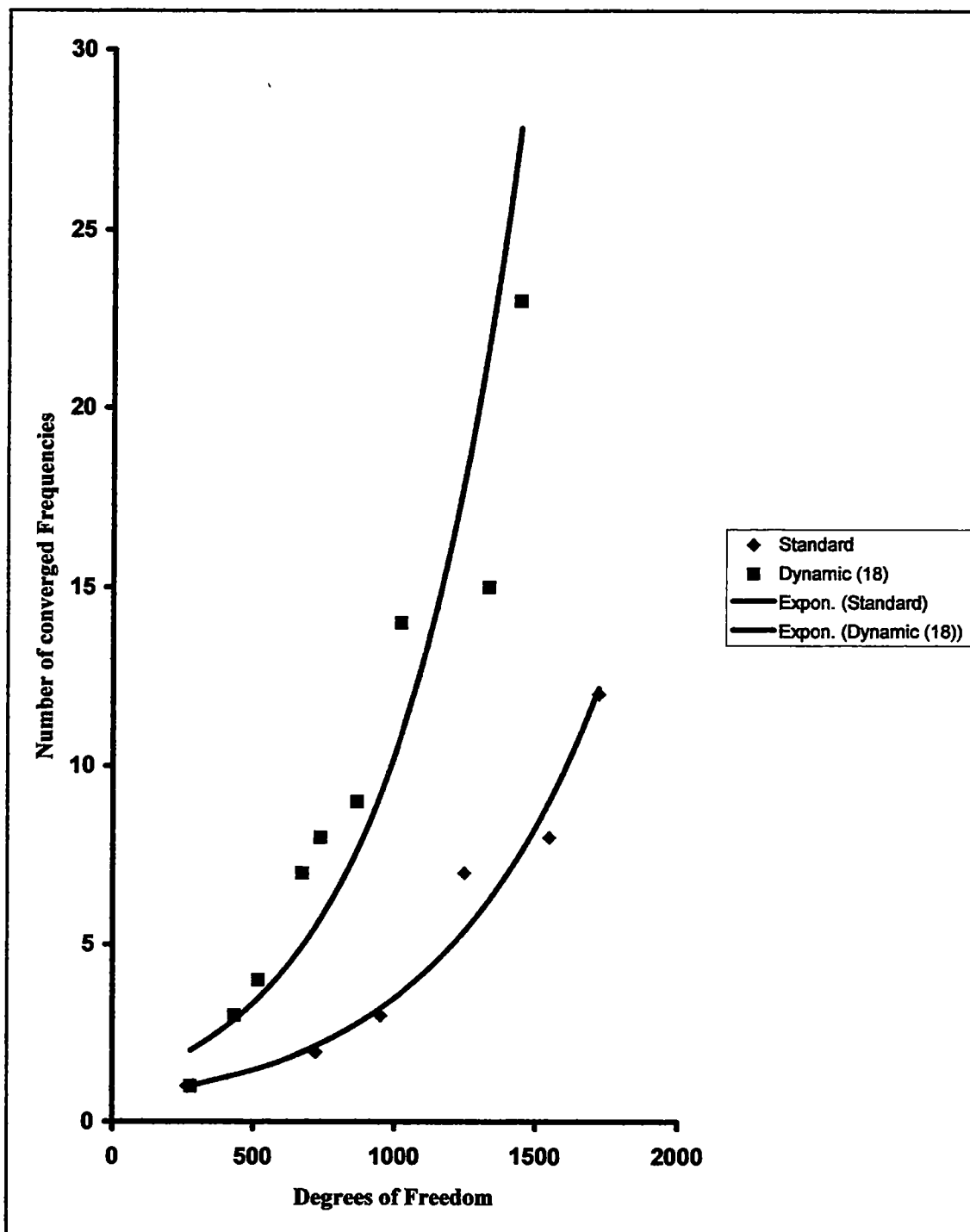


**Figure 16. Number of Converged Frequencies within 1% for
Standard vs Dynamic (18)
Triangular Element**

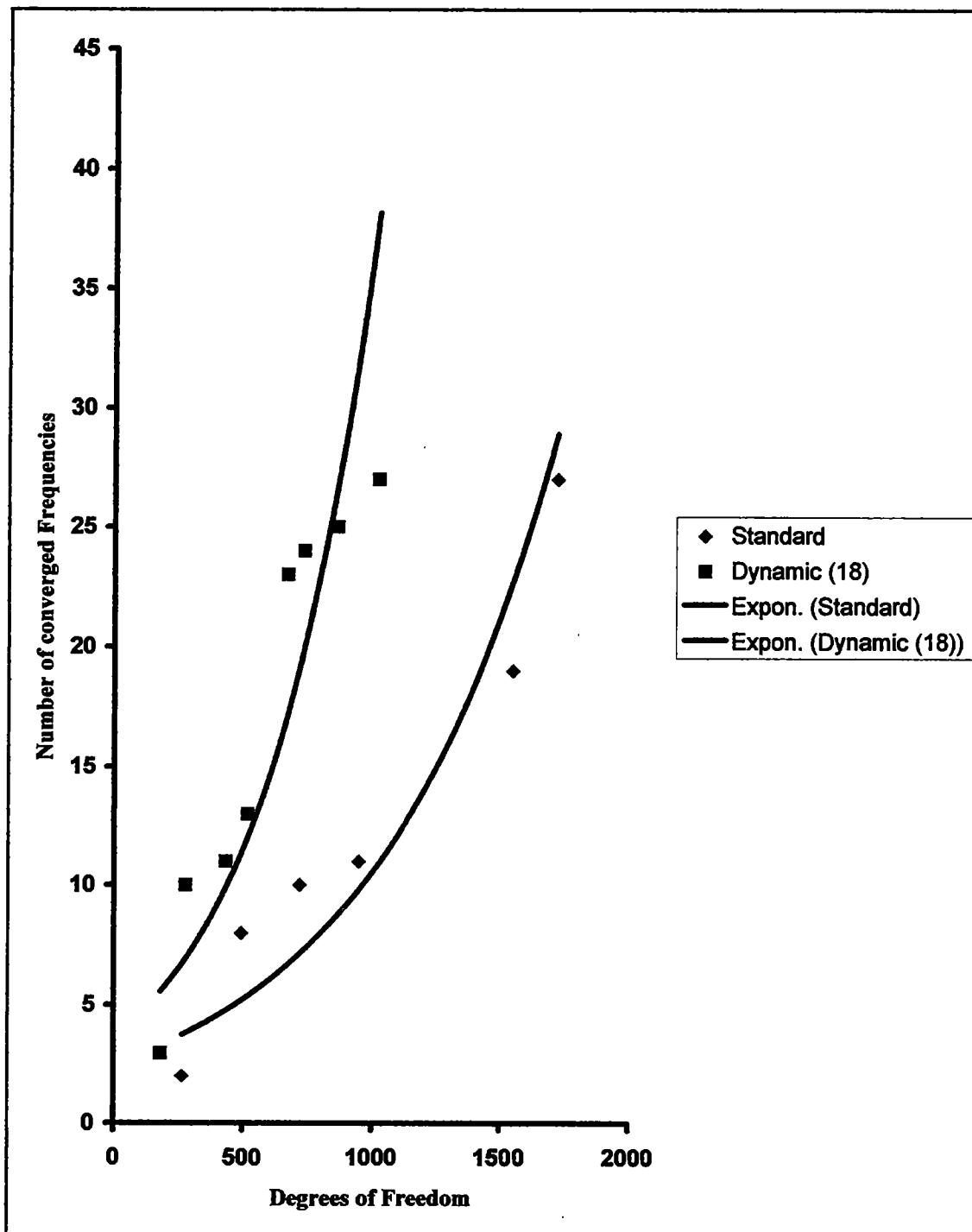


**Figure 17. Number of Converged Frequencies within 3% for
Standard vs Dynamic (18)
Triangular Element**

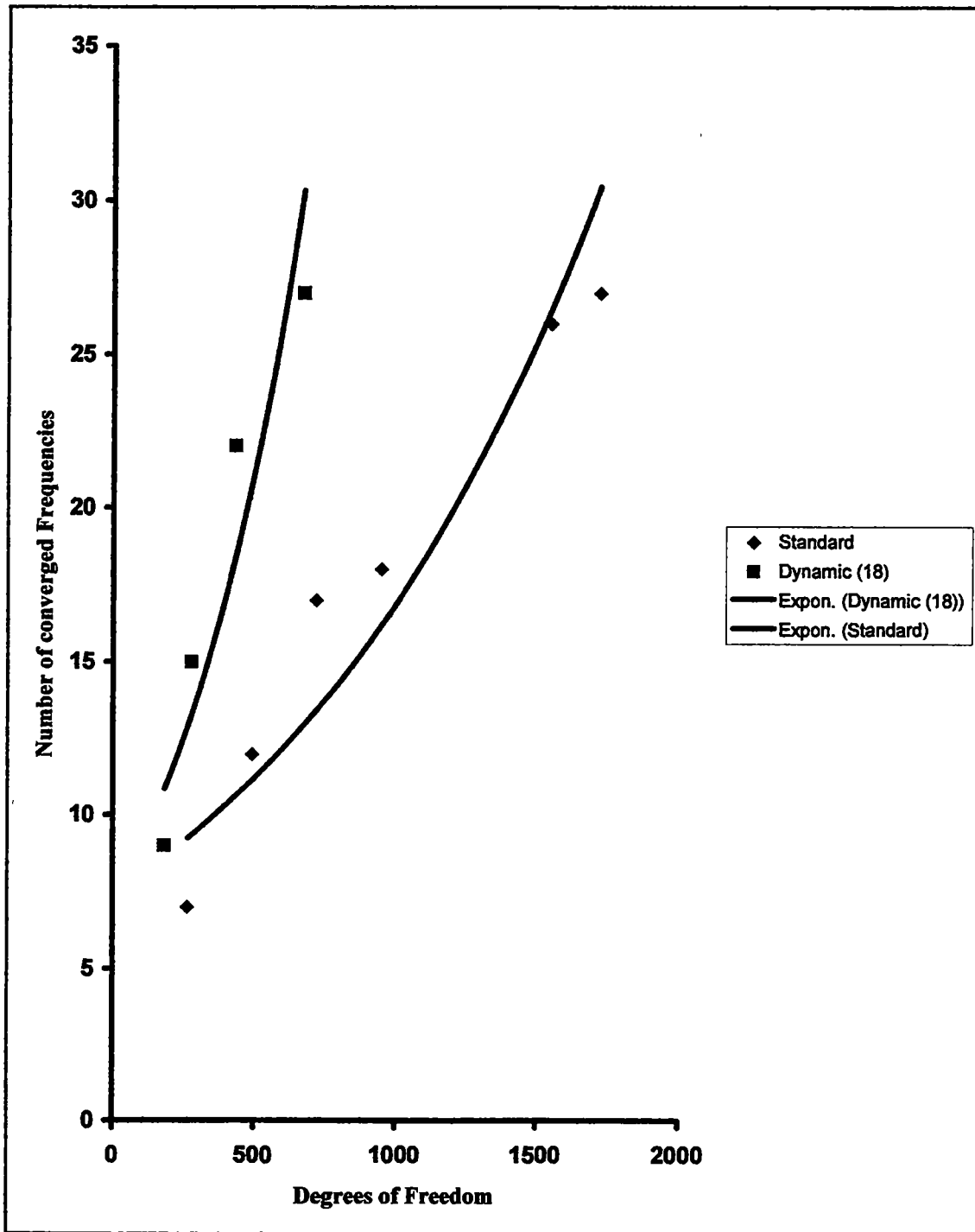


**Figure 18. Number of Converged Frequencies within 5% for
Standard vs Dynamic (18)
Triangular Element**

A comparison of the results for a percentage error of less than 1% are shown in Table 1.

The 18-Node Triangular model offers an average savings of 44% for a percentage error of less than 1 %, when compared to the standard finite element model.

Next, the comparison of the results for a percentage error of less than 3% are shown in Table 2. The 18-Node Triangular Model offers an average savings of 54% for a percentage error of less than 3% when compared to the standard finite element model.

Finally, a comparison of the results for a percentage error of less than 5% are shown in Table 3. The 18-Node Triangular Model offers an average savings of 62% for a percentage error of less than 5% when compared to the standard finite element model.

Next, the dynamic finite element model, shown in figure 19, is compared to the standard finite element model. Graphs were generated using data from the standard finite element model, and from the dynamic finite element model, for the 12-Node Triangular Element. See figures 20-22. The graphs were used to obtain the number of degrees of freedom needed for the convergence of a desired number of frequencies. A comparison of the results for a percentage error of less than 1% are shown in Table 4. The 12-Node Triangular model offers an average saving of 31% for a percentage error of less than 1%, when compared to the standard finite element model.

Next, a comparison of the results for a percentage error of less than 3% are shown in Table 5. The 12-Node Triangular model offers an average saving of 26% for a percentage error of less than 3%, when compared to the standard finite element model.

Finally, a comparison of the results for a percentage error of less than 5% are shown in Table 6. The 12-Node Triangular model offers an average saving of 23% for a

Table 1 **Converged Frequencies for Free-Free Plate within 1%error**
Standard verse Dynamic Method
(18-Node Triangular Element)

Converged Frequencies	Degrees of Freedom for Standard Finite Element Method	Degrees of Freedom for Dynamic Finite Element Method	% Saving
2	688	313	55%
4	1112	594	47%
8	1452	875	40%
10	1572	958	39%
12	1688	1000	41%

Table 2 **Converged Frequencies for Free-Free Plate within 3%error**
Standard verse Dynamic Method
(18-Node Triangular Element)

Converged Frequencies	Degrees of Freedom for Standard Finite Element Method	Degrees of Freedom for Dynamic Finite Element Method	% Savings
6	573	219	62%
10	980	438	55%
15	1260	625	50%
20	1446	708	51%
25	1568	750	52%

**Table 3 Converged Frequencies for Free-Free Plate within 5%error
Standard verse Dynamic Method
(18-Node Triangular Element)**

Converged Frequencies	Degrees of Freedom for Standard Finite Element Method	Degrees of Freedom for Dynamic Finite Element Method	% Saving
15	854	313	63%
20	1178	469	60%
25	1438	531	63%
30	1688	620	63%

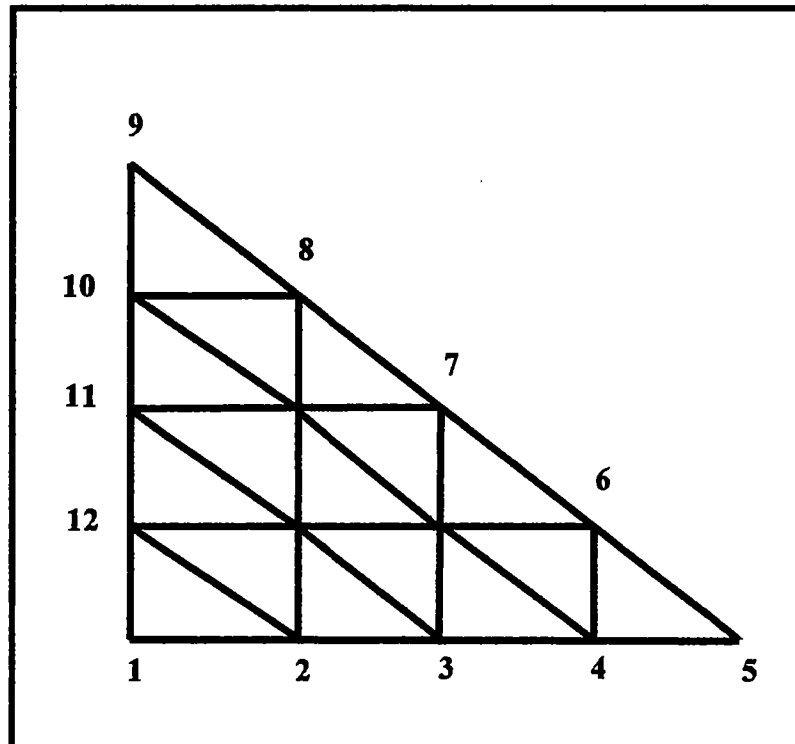


Figure 19. 12- Node Triangular Element

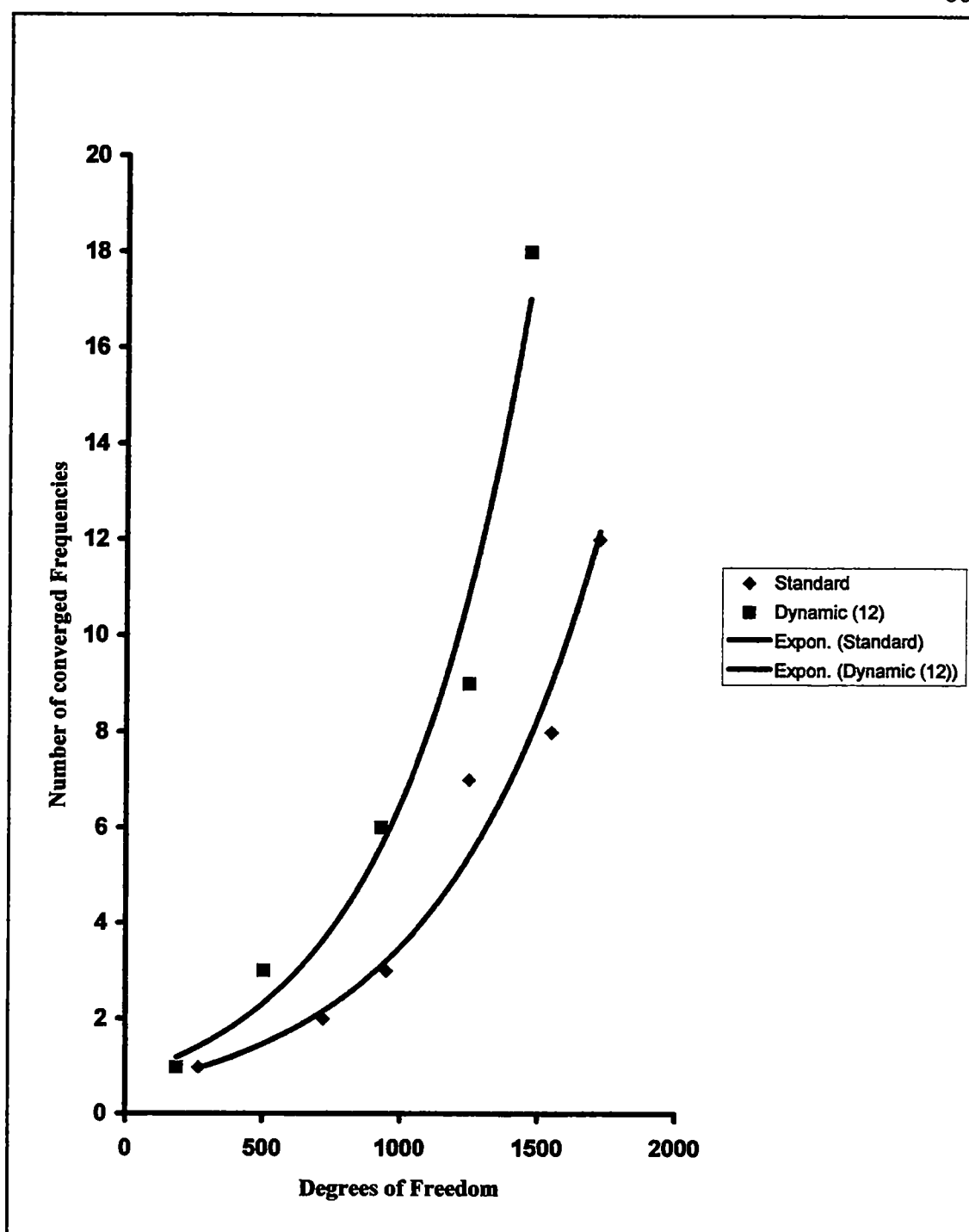


**Figure 20. Number of Converged Frequencies within 1% for
Standard vs Dynamic (12)
Triangular Element**

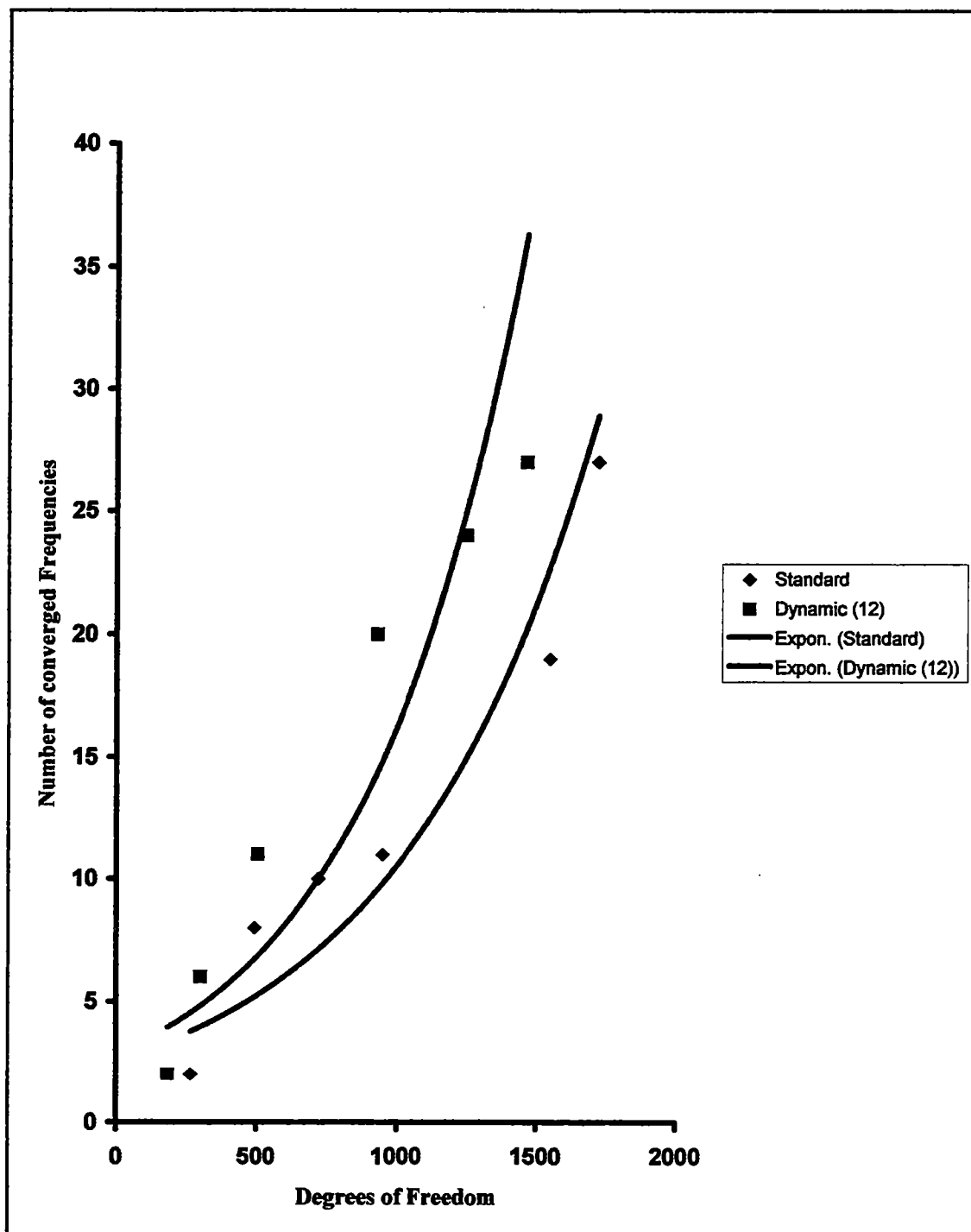


**Figure 21. Number of Converged Frequencies within 3% for
Standard vs Dynamic (12)
Triangular Element**

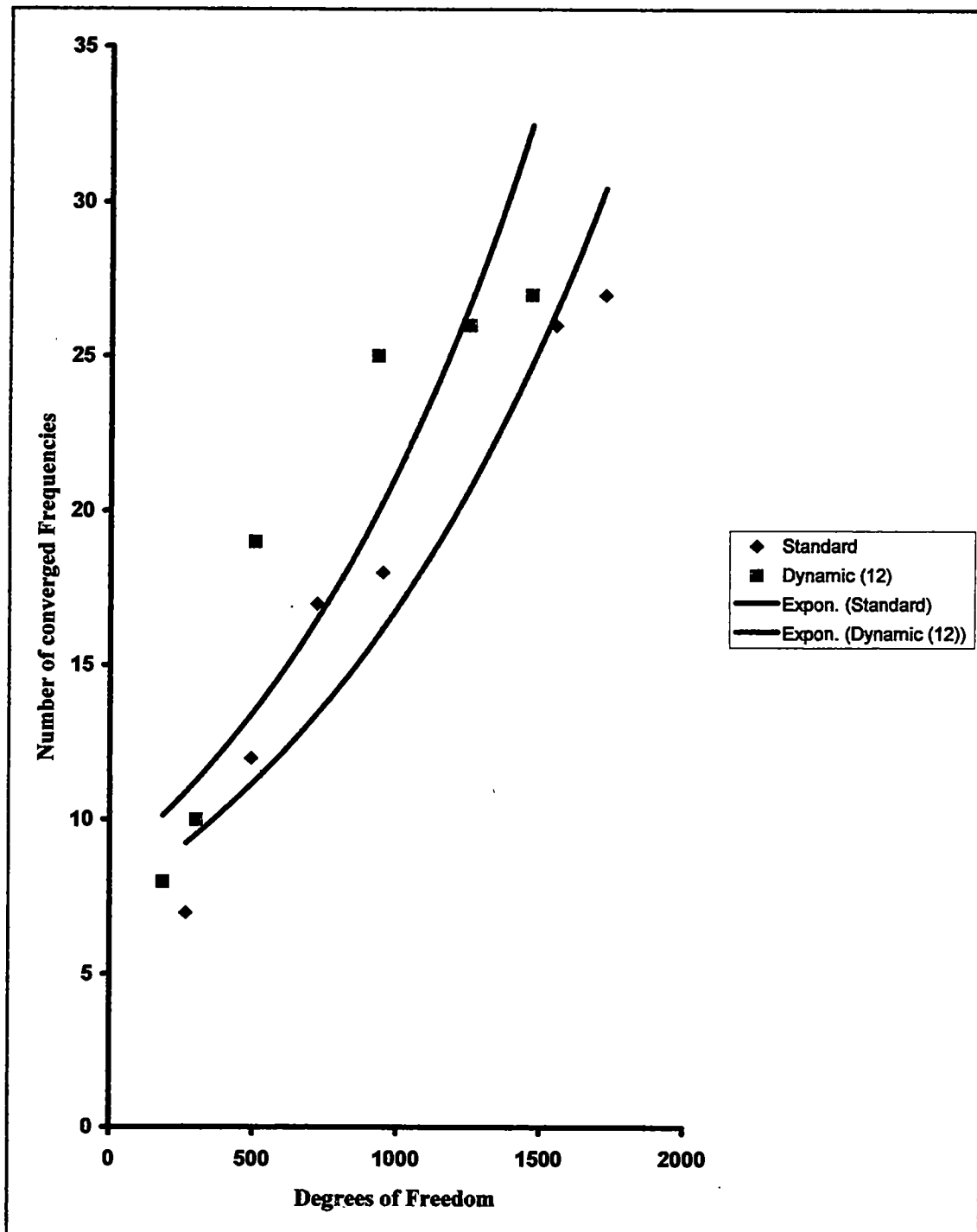


**Figure 22. Number of Converged Frequencies within 5% for
Standard vs Dynamic (12)
Triangular Element**

Table 4 **Converged Frequencies for Free-Free Plate within 1%error**
Standard verse Dynamic Method
(12-Node Triangular Element)

Converged Frequencies	Degrees of Freedom for Standard Finite Element Method	Degrees of Freedom for Dynamic Finite Element Method	% Saving
2	698	448	36%
4	1110	760	32%
8	1438	1063	26%
10	1698	1230	28%

Table 5 **Converged Frequencies for Free-Free Plate within 3%error**
Standard verse Dynamic Method
(12-Node Triangular Element)

Converged Frequencies	Degrees of Freedom for Standard Finite Element Method	Degrees of Freedom for Dynamic Finite Element Method	% Saving
5	540	343	36%
10	938	728	22%
20	1385	1062	23%
25	1578	1250	21%

Table 6 **Converged Frequencies for Free-Free Plate within 5%error**
Standard verse Dynamic Method
(12-Node Triangular Element)

Converged Frequencies	Degrees of Freedom for Standard Finite Element Method	Degrees of Freedom for Dynamic Finite Element Method	% Saving
15	828	625	24%
20	1165	875	25%
25	1390	1113	20%
30	1682	1313	22%

percentage error of less than 5%, when compared to the standard finite element model.

A comparison between the 18-Node Triangular Model and the 12-Node Triangular Model indicate an increase in % saving, when the 18-Node Triangular Model is used. This comparison has shown the degrees of freedom to be dependent upon the size of the triangular element used for the model. This is also an indication of the effects of the stiffness, on the number of degrees of freedom needed for convergence. The optimum triangular element is the 18-Node Element.

4.1.4 Dynamic Finite Element Method (Quadrilateral Element)

Fortran programs were written using the dynamic finite element method developed in sections 3.6 and 3.7 for the quadrilateral element.

The dynamic finite element models shown in figures 23-25 , are compared to the standard finite element model. Graphs were generated using data from the standard finite element model, and from the dynamic finite element model, for the 24-Node

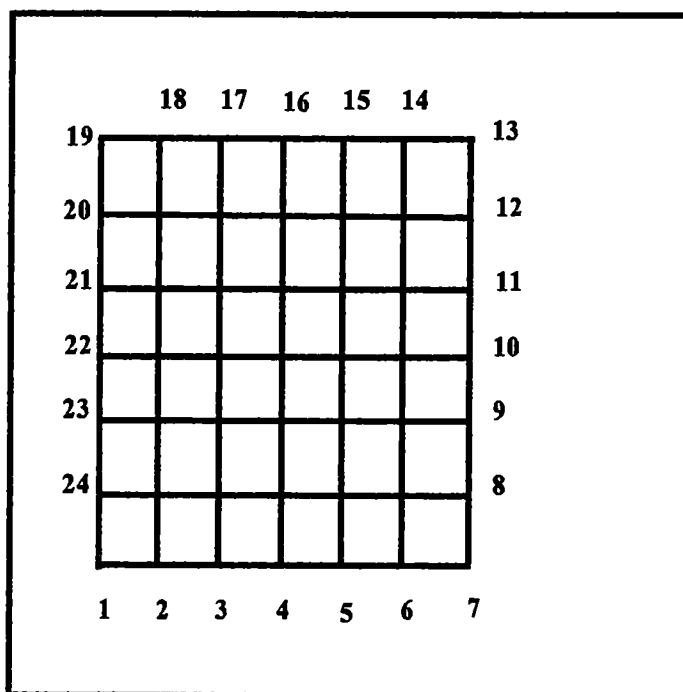


Figure 23. 24-Node Quadrilateral Element

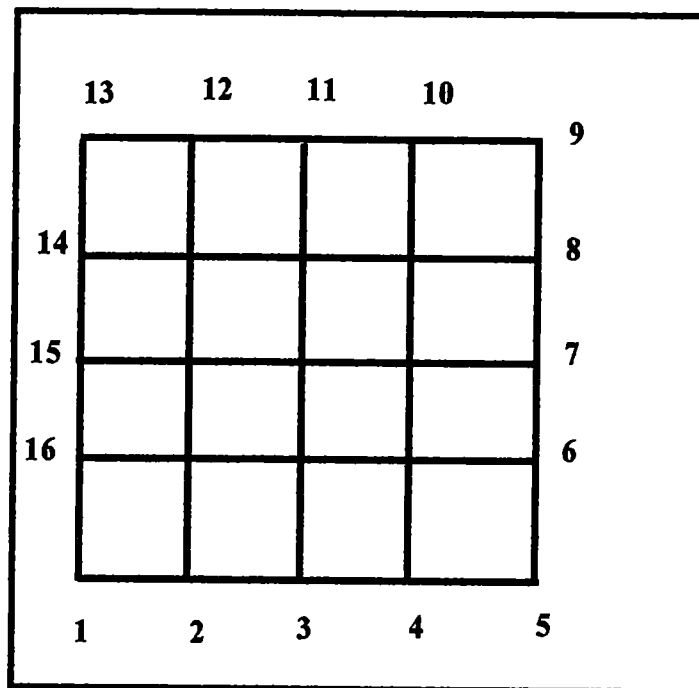


Figure 24. 16-Node Quadrilateral Element

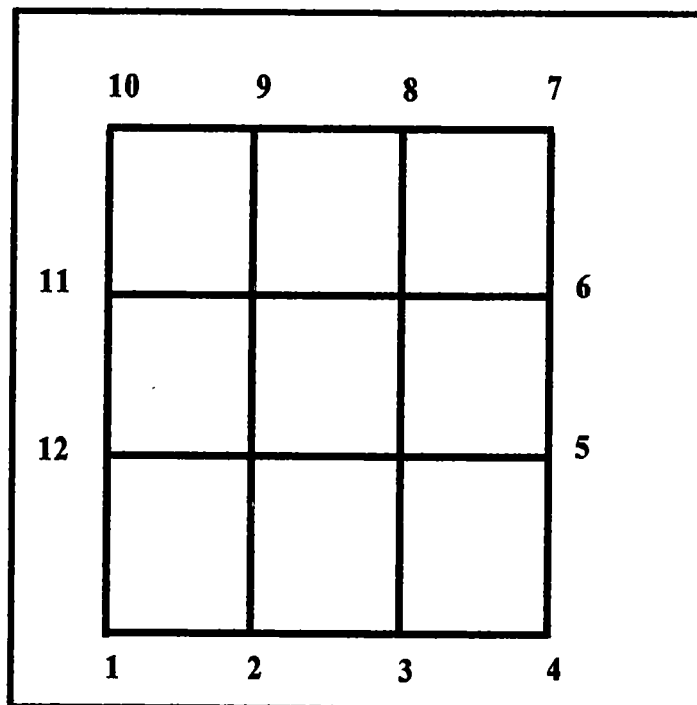


Figure 25. 12-Node Quadrilateral Element

Quadrilateral Element, the 16-Node Quadrilateral Element, and the 12-Node Quadrilateral Element. The graphs were used to obtain the number of degrees of freedom needed for the convergence of a desired number of frequencies. See figures 26-28 .

A comparison of the results for the 24-Node Quadrilateral Element are shown in Table 7. The 24-Node Quadrilateral Model offers an average savings of 27% for a percentage error of less than 1%, when compared to the standard finite element method.

Next, a comparison of the results for the 16-Node Quadrilateral Element are shown in in Table 8. The 16-Node Quadrilateral offers an average savings of 11% for a percentage error of less than 1%, when compared to the standard finite element model.

Finally, a comparison of the results for the 12-Node Quadrilateral Element are shown in Table 9. The negative savings is an indication that the 12-Node dynamic finite element model does not, in this case, provide a savings when compared to the standard finite element method. A probable cause lies within the stiffness of the element. For this case, the 12-Node quadrilateral model is not as stiff as the model used for the standard finite element. For the best results, use the 24-Node or the 16-Node quadrilateral models for the analysis.

The 24-Node quadrilateral element offers an additional average savings of 16 % when compared to the 16-Node quadrilateral element. The optimum quadrilateral element is the 24-Node element.

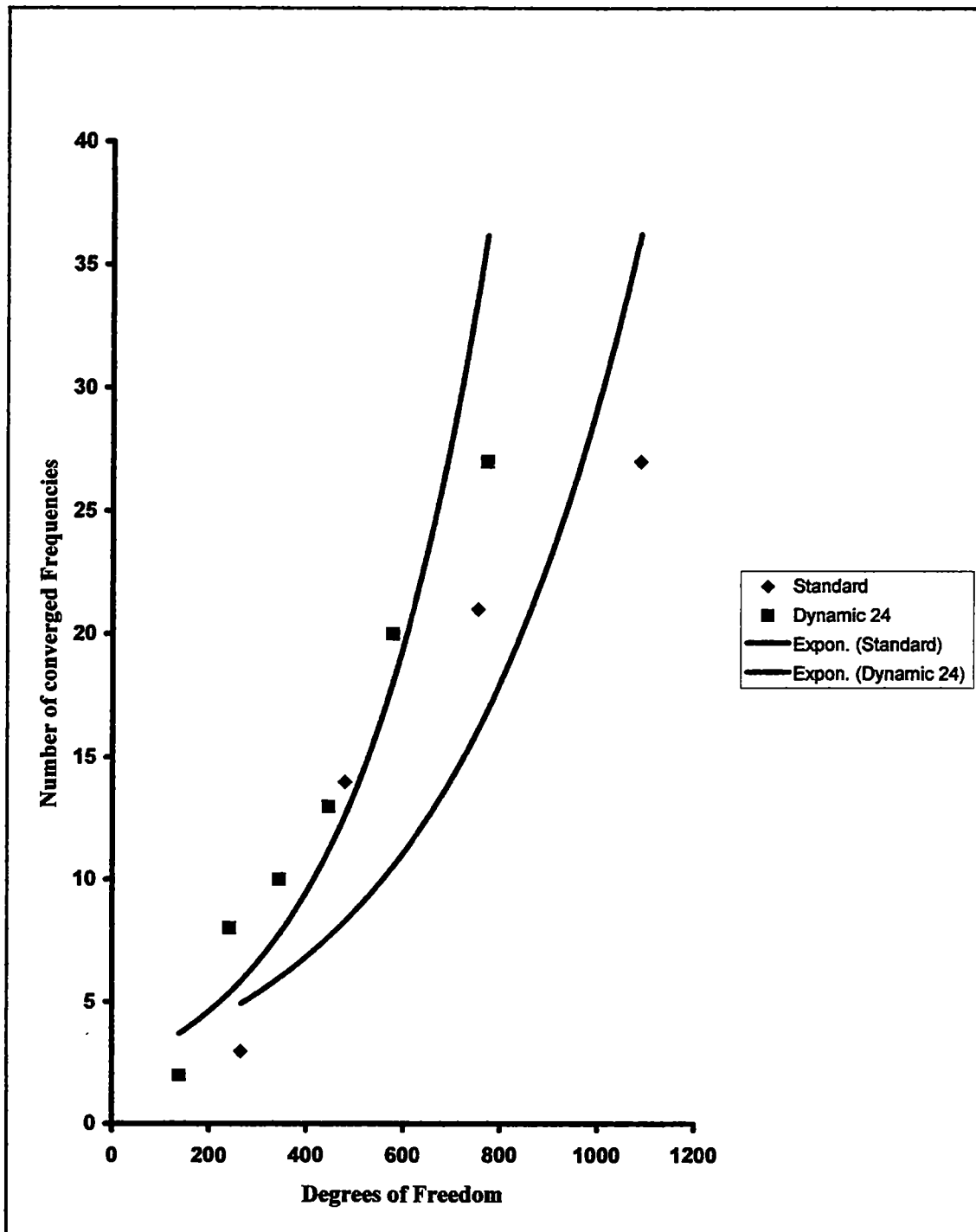


**Figure 26. Number of Converged Frequencies within 1% for
Standard vs Dynamic (24)
Quadrilateral Element**

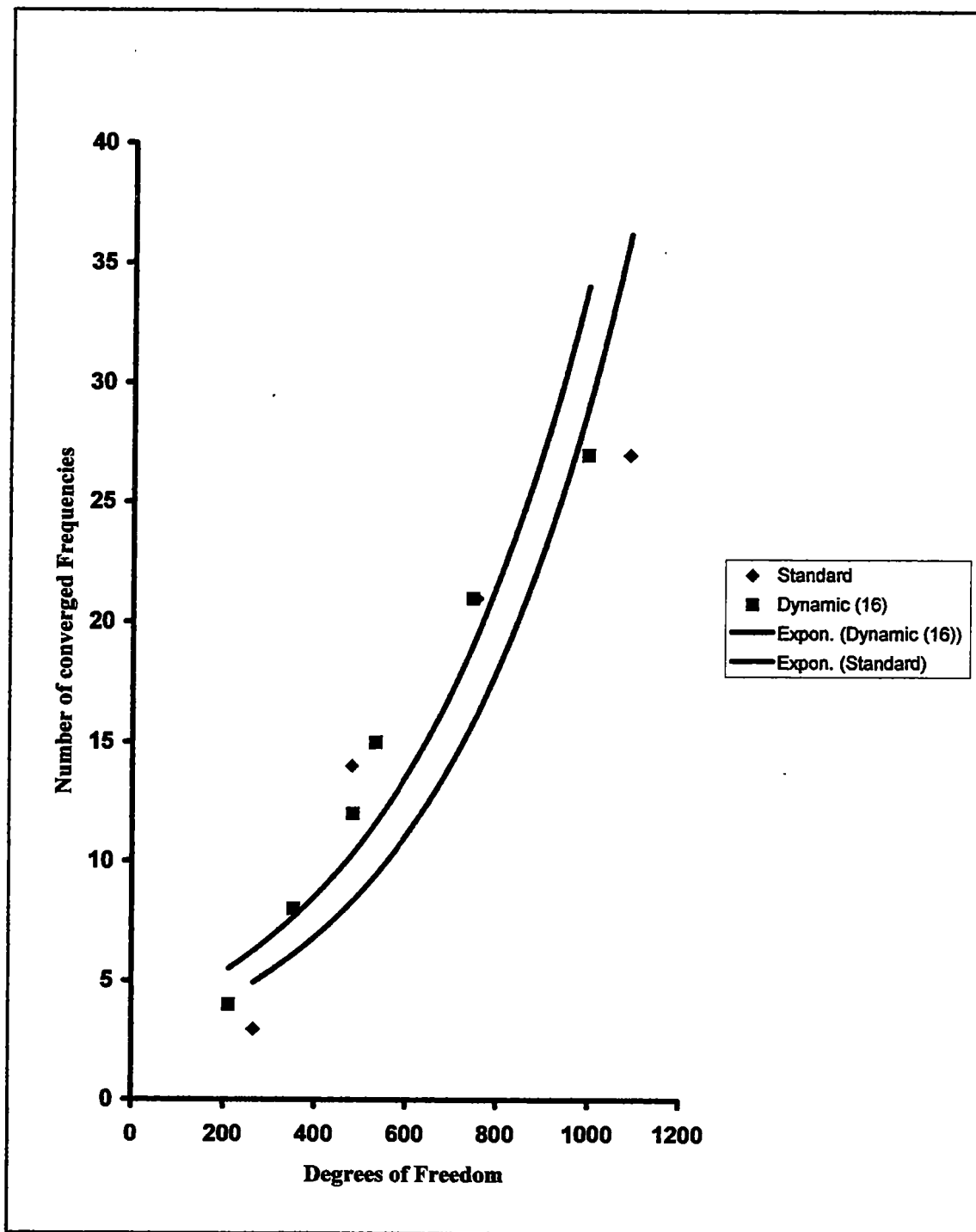


Figure 27. Number of Converged Frequencies within 1% for Standard vs Dynamic (16) Quadrilateral Element

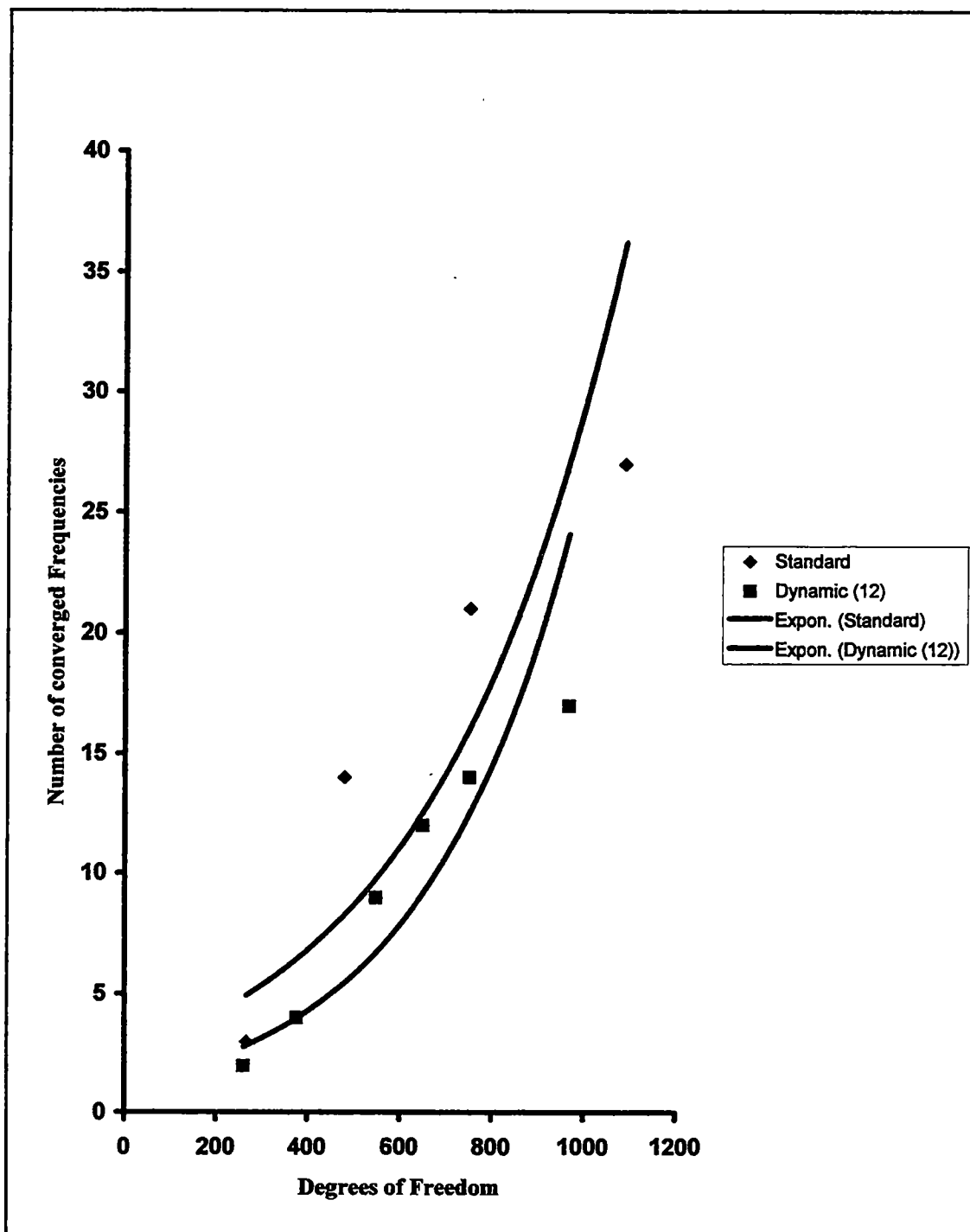


**Figure 28. Number of Converged Frequencies within 1% for
Standard vs Dynamic (12)
Quadrilateral Element**

Table 7 **Converged Frequencies for Free-Free Plate within 1%error**
Standard verse Dynamic Method
(24-Node Quadrilateral Element)

Converged Frequencies	Degrees of Freedom for Standard Finite Element Method	Degrees of Freedom for Dynamic Finite Element Method	% Savings
5	295	240	19%
10	560	405	28%
15	700	525	25%
20	830	590	29%
25	895	610	32%
30	980	698	29%

Table 8 **Converged Frequencies for Free-Free Plate within 1%error**
Standard verse Dynamic Method
(16-Node Quadrilateral Element)

Converged Frequencies	Degrees of Freedom for Standard Finite Element Method	Degrees of Freedom for Dynamic Finite Element Method	% Saving
8	460	395	14%
15	710	605	15%
20	815	730	10%
25	895	840	6%

**Table 9 Converged Frequencies for Free-Free Plate within 1%error
Standard verse Dynamic Method
(12-Node Quadrilateral Element)**

Converged Frequencies	Degrees of Freedom for Standard Finite Element Method	Degrees of Freedom for Dynamic Finite Element Method	% Savings
5	260	450	-42%
10	555	700	-21%
15	710	800	-13%
20	810	910	-11%

CHAPTER 5

CONCLUSIONS AND RECOMMENDATIONS

A method to reduce the number of degrees of freedom needed for the dynamic analysis of two dimensional structures has been presented. This method is based on the inclusion of both static constraint modes (SCM) and Interface Restrained Assumed Modes (IRAM). The reduction is achieved at the element level. This is done through the convergence characteristics of the triangular and quadrilateral elements developed in Chapter 3. The assembly process for the method allows for the replacement of a large system eigenvalue problem by a series of relatively small and inexpensive eigenvalue problems. The cost of the analysis is thus improved with the reduction in the dimension of the mass and stiffness matrix.

The dynamic finite element method has consistently used fewer degrees of freedom, than the standard finite element method, to obtain a desired number of converged frequencies. It is concluded that the success of the method is based on the inclusion of Interface Restrained Assumed Modes (IRAM) and corresponding frequencies. It's success is also dependent on element stiffness, element size, and the desired percentage error.

The dynamic finite element method offers an average savings of 44%, for less than 1% error, an average savings of 54% for less than 3% error, and an average savings of 62% for less than 5 % error., when the 18- Node Triangular Element is used for the finite element model. The dynamic finite element method offers an average savings of 31%, for less than 1% error, an average savings of 26% for percentage error of less than 3%,

and an average savings of 23% for less than 5% error, when the 12- Node Triangular Element is used for the finite element model.

The dynamic finite element method offers an average savings of 27%, for the 24-Node quadrilateral element, and an average savings of 11% for the 16-Node quadrilateral element for less than 1% error.

The method is most successful when the 18-Node triangular element model, and the 24-Node quadrilateral element models are used for the analysis. This is due to an increase in the stiffness of the elements. The larger element is therefore recommended for the analysis of two-dimensional structures requiring a percentage error of less than 1%. The 12-Node Triangular and the 16-Node Quadrilateral models smaller models may be used for two dimensional analysis requiring percentage errors of 3 to 5 percent.

The graphs may be used as a future reference. They can provide the analyst with a starting point for a dynamic analysis. This is done by noting the number of degrees of freedom needed to obtain a specified number of converged frequencies. The analyst can then begin with a model that has, at the least, the noted number of degrees of freedom. Providing an adequate starting point, can reduce the time and cost of the dynamic analysis.

Future efforts may be directed toward the development of dynamic finite elements for three-dimensional structures.

BIBLIOGRAPHY

BIBLIOGRAPHY

1. Nagamanickam Ganesan, 1991 *A Dynamic Finite Element Modeling Technique For Frame Structures*. The University of Tennessee, Knoxville.
2. M. Geradin, 1971 *Journal of Sound and Vibrations* **19**, 111-132. Error bounds for eigenvalue analysis by elimination of variables.
3. R. C. Engels, 1986 *Final Report For Air Force Weapons Laboratory, Kirtland, New Mexico*. Reduced order structural modeling techniques for high energy laser systems.
4. R. H. MacNeal, 1971 *Computers and Structures* **1**, 581-601. A hybrid method of component mode synthesis.
5. N. Ganesan and R. C. Engels, 1990 *AIAA Space Programs and Technologies Conference, September 25-28, 1990, Huntsville, AL., Paper No. 90-3547*. A dynamic finite element model for the Timoshenko beam finite elements.
6. Daryl L. Logan, 1992 *A First Course in the Finite Element Method, Second Edition*, PWS - KENT.
7. Tirupathi R. Chandrupatla and Ashok D. Belegundu, 1992, *Introduction to Finite Elements in Engineering*.
8. Frank L. Stasa, 1985 *Applied Finite Element Analysis For Engineers*, CBS
9. Timoshenka and Woinowsky - Krieger, 1959 *Theory of Plates and Shells*, McGraw - Hill.
10. Roy R. Craig, Jr., 1981 *Structural Dynamics, An Introduction to Computer Methods*, John Wiley & Sons, Inc.
11. O. C. Zienkiewicz and J. P. de S. R. Gago, D. W. Kelly, 1983 *Computers and*

- Structures* **16**, 53-65. The hierarchical concept in finite element analysis.
12. J. K. Bennighof and L. Meirovitch, 1986 *International Journal for Numerical Methods in Engineering* **23**, 2153-2165. Eigenvalue convergence in the finite element method.
 13. A. Peano, A. Pasini, R. Riccioni and L. Sardella, 1979 *Computers and Structures* **10**, 333-342. Adaptive approximations in finite element structural analysis.
 14. W. C. Hurty, 1960 *Proceedings of the American Society of Civil Engineers, Journal of the Engineering Mechanics Divisions* **86**, 51-69. Vibrations of Structural systems by component mode synthesis.
 15. W. A. Benfield and R. F. Hruda, 1971 *American Institute of Aeronautics and Astronautics Journal* **9**, 1255-1261. Vibrations analysis of structures by component mode substitution.
 16. I. Babuska, B. A. Szabo and I. N. Katz, 1981 *Society of Industrial and Applied Mathematics Journal of Numerical Analysis*, **18**, 515-545. The p-version of the finite element method.
 17. R. R. Craig and M. C. C. Bampton, 1968 *American Institute of Aeronautics and Astronautics Journal* **6**, 1313-1319. Coupling of substructures by component mode substitution.
 18. A. H. Flax, 1975 *American Institute of Aeronautics and Astronautics Journal* **13**, 701-702. Comment on 'Reduction of structural frequency equations'.
 19. R. M. Hintz, 1975 *American Institute of Aeronautics and Astronautics Journal* **13**,

- 1007-1016. Analytical methods in component modal synthesis.
20. D. L. Thomas, 1982 *International Journal for Numerical Methods in Engineering* **18**, 1521-1527. Errors in natural frequency calculations using eigenvalue economization.
21. R. Bamford, B. K. Wada, J. A. Garba and J. Chisholm, 1971 *Synthesis of Vibrating Systems, ASME Booklet* 57-71. Dynamic analysis of large structural systems.
22. R. D. Henshell and J. H. Ong, 1975 *Earthquake Engineering and Structural Dynamics* **3**, 375-382. Automatic masters for eigenvalue economization.
23. S. Rubin, 1975 *American Institute of Aeronautics and Astronautics Journal* **13**, 995-1006. Improved component mode representation for structural dynamic analysis.
24. R. C. Engels, 1990 *Proceedings of The 31st AIAA/ ASME/ ASCE/ AHS/ ASC Structures, Structural Dynamics and Materials Conference, Long Beach, CA., Paper No 90-1038, 1922-1931*. The assumed modes method with static constraint modes.
25. R. C. Engels, 1990 *Proceedings of the 1990 ASME International Computers in Engineering Conference, Boston, MA., 1*, 493-503. Dynamic finite element modeling of some basic structural members.
26. R. L. Kidder, 1973 *American Institute of Aeronautics and Astronautics Journal* **11**, 892. Reduction of structural frequency equations.
27. R. J. Guyan, 1965 *American Institute of Aeronautics and Astronautics Journal* **3**, 380. Reduction of stiffness and mass matrices.
28. T. C. Cheu, C. P. Johnson and R. R. Craig, Jr., 1987 *International Journal for Numerical Methods in Engineering* **24**, 1841-1848. Computer algorithms for

calculating efficient initial vectors for subspace iteration method.

29. B. Nour-Omid, B. N. Parlett and R. L. Taylor, 1983 *International Journal for Numerical Methods in Engineering* **19**, 859-871. Lanczos versus subspace iteration for solution of eigenvalue problems.
30. A. Jennings, 1985 *Sparsity and its Applications*, Edited by D. J. Evans 153-183. Solution of sparse eigenvalue problems, Cambridge University Press.
31. D. L. Thomas, J. M. Wilson and R. R. Wilson, 1973 *Journal of Sound and Vibrations* **31**, 315-330. Timoshenko beam finite elements.
32. A. Jennings, 1973 *International Journal for Numerical Methods in Engineering* **6**, 543-552. Mass condensation and simultaneous iteration for vibration problems.
33. E. J. Kuhar and C. V. Stable, 1974 *American Institute of Aeronautics and Astronautics Journal* **12**, 672-8. Dynamic transformation method for modal synthesis.

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She is married to Robert Cooley, Sr., and is the mother of three children, Lakisha, Candice, and Robert , Jr.