

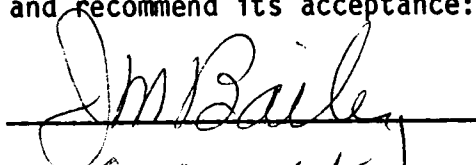
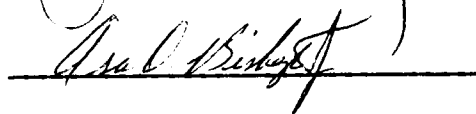
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Date August, 1985

ANALYSIS OF NONLINEAR RC OP-AMP OSCILLATORS
UTILIZING PHASE SPACE TECHNIQUES

A Thesis
Presented for the
Master of Science
Degree
The University of Tennessee, Knoxville

David L. Macintire
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ABSTRACT

This thesis presents the results of a study of applying phase space and describing function analysis to a class of nonlinear oscillators. The oscillators are nonlinear because they use thyristor elements in the feedback path of the operational amplifier. The op-amp is the active device used in the oscillator model. Phase space analysis is applied directly to second and third-order RC models in conjunction with computer solution of the state equations and computer plots of the output state variable. A fourth-order phase shift model is also considered. A design criterion is developed based upon the peak value of the output state variable and the input resistor to the op-amp. Describing function analysis demonstrates that the linearized differential equation closely follows a linear version of the nonlinear oscillator considered.

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CHAPTER 1

INTRODUCTION

General

A mathematical description of a physical process from which some output is achieved for a given input describes a physical system. In order to quantitatively study such systems, mathematical models must be formulated describing the dynamics of the system. Certain numerical quantities must be defined which are inherent in these models. For example, in an electronic circuit, these quantities are current and voltage [5, p. 1]. In actual mathematical models of circuits, capacitor voltages and inductor currents are suitable to describe the system under consideration. Since these quantities change with time, time also is a quantity describing the dynamics of the system.

An electronic feedback oscillator circuit is a physical system. A mathematical model describing the dynamics of the system is the differential equations relating the three quantities previously mentioned: capacitor voltages, inductor currents and time. A differential equation relates the change of each capacitor and inductor voltage and current with respect to time to all other capacitor voltages and inductor currents as well as time explicitly. There is one first-order equation for each of these currents and voltages; and a set of these equations describing the dynamics of the system is known as the system state equations since these quantities describe the energy state of the system [16, p. 567].

Depending on whether these first-order differential equations are of the first power of the dependent variable (the state variables) determines linearity. A set of linear state equations describes a linear system. If powers other than the first appear in any combination of state variables, the equation is nonlinear [5, p. 2]. For example, a Taylor series or other type of series of higher powers of the state variable can expand a transcendental function. Any equation having a transcendental function of the state variable must be a nonlinear equation.

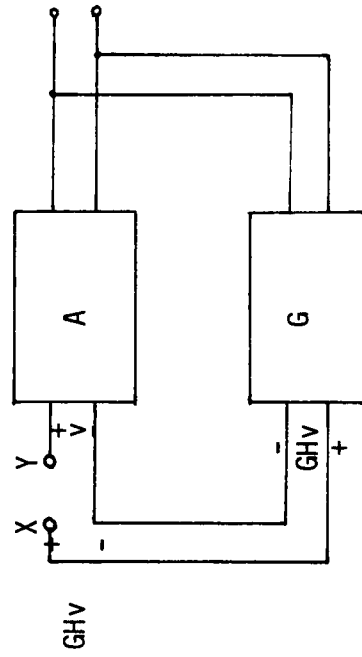
The model for an electronic feedback oscillator can be linear or nonlinear depending on the set of differential equations describing the system. Generally, the oscillator consists of an active device (an amplifier) and additional circuit elements. If the active device operates linearly as do most useful amplifiers, and if the circuit elements are linear devices such as capacitors, inductors and resistors, then the circuit is a linear oscillator. A phase-shift oscillator consisting of an amplifier and resistors and capacitors is an example. Including nonlinear devices such as diodes and nonlinear resistors is a common method of designing nonlinear oscillators. An example of an oscillator with nonlinear elements is the Wein-bridge oscillator which is discussed in detail in the next chapter. It is a second-order nonlinear oscillator described by two state equations with at least one nonlinear equation. Nonlinear devices frequently used are the Zener diode and an incandescent lamp both which exhibit nonlinear volt-amp characteristic curves [13, p. 385].

Linear Oscillator Models

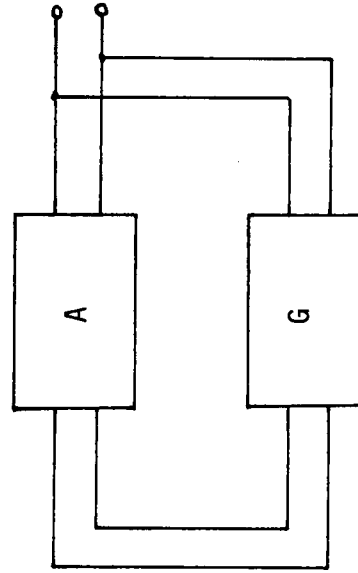
Linear system theory can be used to describe the action of linear feedback oscillators. If the method of state variable analysis is applied, the method naturally extends at least partially to nonlinear oscillator analysis. However, other methods of analysis extending from the frequency domain techniques of Bode, Nyquist and others are also partially applicable [9].

Figure 1.1 illustrates basic oscillator operation [10, p. 603; 11, p. 564]. Since the system is connected in series-parallel feedback configuration, A_v , the amplified signal at the output is fed to the feedback element producing the feedback voltage AG_v . Naturally, phase shifting occurs in many amplifiers and feedback circuits, but if the phase shift at point x is 360 degrees (or 0 degrees), the feedback voltage is in phase with the error voltage v . Positive feedback results. In practical oscillator operation, the initial loop gain AG must be greater than unity. The starting voltages are inherent in the circuit elements, e.g., resistor thermal noise. As the oscillations increase, the loop gain must reduce to unity therefore either A or G must decrease. Unity loop gain then enables sustained sinusoidal oscillations in most practical cases.

General feedback system theory can also explain oscillator action in a slightly different manner. Figure 1.2 represents a general negative feedback control system. The blocks represented are amplifier gains, transfer functions, etc. More specifically, each block represents a ratio of the Laplace transform of the output variable to



a. Feedback Before Oscillations



b. System Oscillates

Figure 1.1 Block Diagram of Basic Oscillator Model

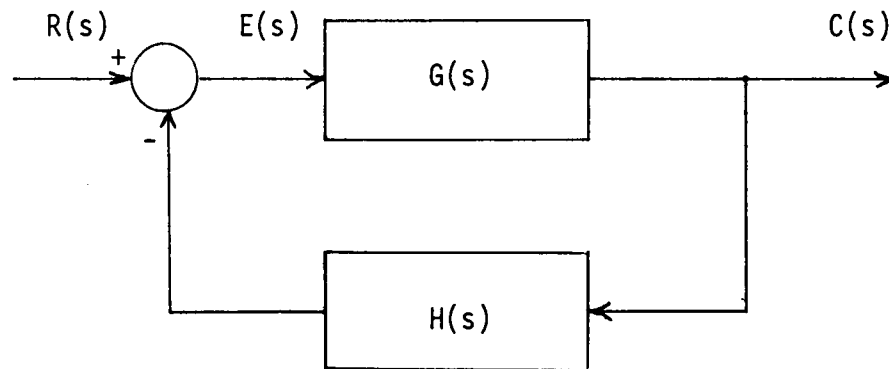


Figure 1.2 Block Diagram of Generalized Negative Feedback Control System

the input variable. Each is a function of the complex variable s , and so s -plane or frequency domain analysis is utilized.

The overall input-output transfer function of the system shown in figure 1.2 is well known and is

$$\frac{C(s)}{R(s)} = \frac{G(s)}{1 + G(s)H(s)} \quad (1.1)$$

If $G(s)$ represents an amplifier of the inverting type, such as an inverting operational amplifier or a multistage transistor amp with an odd number of common-emitter stages, then

$$G(s) = -KF(s) \quad (1.2)$$

where K is the amplifier gain and $F(s)$ is the poles and zeroes of the amplifier [17, p. 548]. The sign of the denominator of equation 1.1 is

is then minus and figure 1.2 is described as

$$G = \frac{G}{1 - GH} \quad (1.3)$$

with the minus sign incorporated and each variable as a function of s implied.

Oscillation implies that $R(s) = 0$ in equation 1.1 so

$$1 - GH = 0 \quad (1.4)$$

or

$$GH = 1 \quad (1.5)$$

Equation (1.5) is the characteristic equation of the system of figure 1.2. A root-locus plot of this equation in the $s = \sigma \pm j\omega$ plane with G defined by equation 1.2 is a solution as K varies from zero to infinity. Oscillator operation lies on this locus, and the roots must be on the $j\omega$ -axes for sustained sinusoidal oscillations to occur. If these roots were located in the right half or in the left half of the s -plane, exponentially growing or decaying output would occur respectively. The system will no longer oscillate in a useful manner.

To illustrate the ideas mentioned thus far, consider a simple RC op-amp oscillator as shown in figure 1.3 [17, p. 550]. Idealized op-amp models will be discussed in the next chapter with applications applied to state equation models of oscillator circuits. Basic op-amp operation enables one to write the output of each stage as a function of each input. The equations are [17, p. 549]:

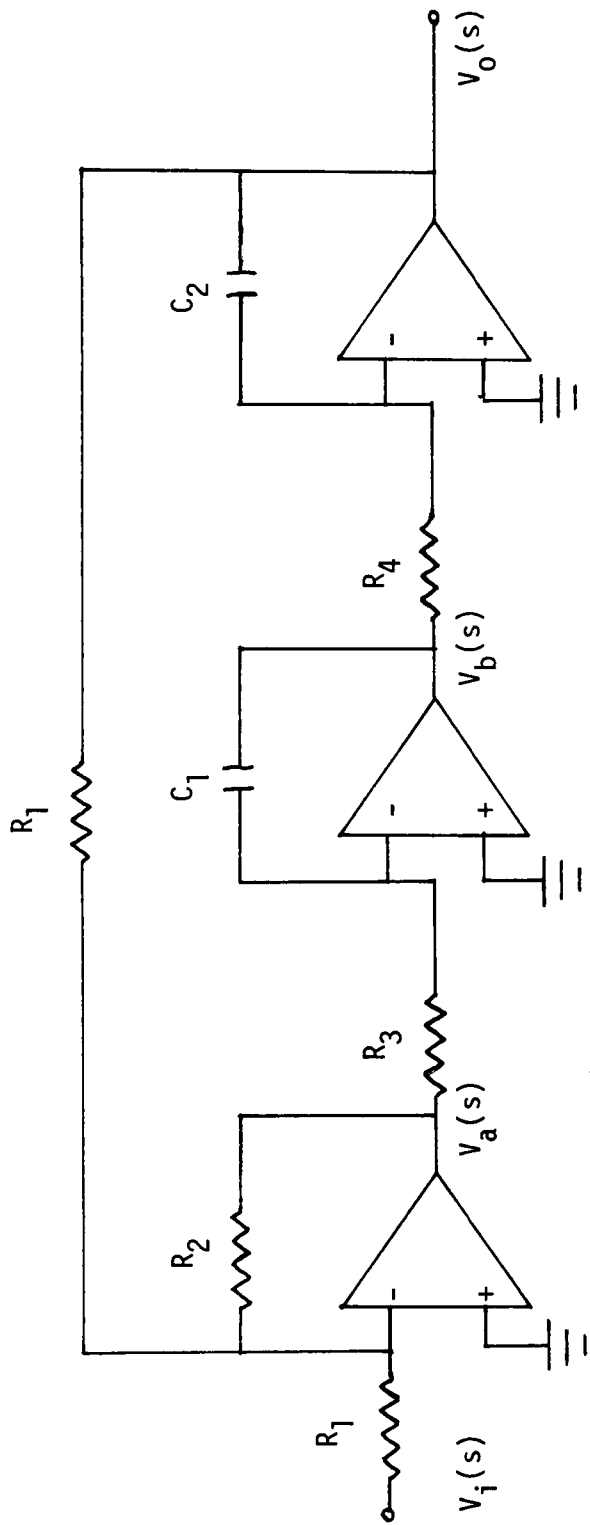


Figure 1.3 A Simple 3-Stage Op-Amp Oscillator

$$V_a(s) = - \frac{R_2}{R_1} (V_1(s) + V_0(s)) \quad (1.6)$$

$$V_b(s) = - \frac{C_1 s}{R_3} V_a(s) \quad (1.7)$$

$$V_0(s) = - \frac{C_2 s}{R_4} V_b(s) \quad (1.8)$$

The transfer function $A(s)$ is $V_0(s)/V_i(s)$ and is written by eliminating V_a and V_b in the set of equations above giving

$$\frac{V_0}{V_i}(s) = - \frac{R_2/R_1 R_3 R_4 C_1 C_2}{s^2 + R_2/R_1 R_3 R_4 C_1 C_2} \quad (1.9)$$

which is of the standard form of second-order system

$$s^2 + 2\zeta\omega_N s + \omega_N^2 \quad (1.10)$$

with $\zeta = 0$, i.e., no damping. Thus, oscillation occurs and the frequency is

$$\omega_N = \sqrt{\frac{R_2}{R_1 R_3 R_4 C_1 C_2}} \quad (1.11)$$

which is an equivalent statement of the inverse Laplace transform of equation 1.9 being

$$\begin{aligned} \frac{V_0}{V_i}(t) &= \mathcal{L}^{-1} \left\{ \frac{-\omega}{s^2 + \omega^2} \right\} \\ &= -\sin \omega t \end{aligned} \quad (1.12)$$

with ω defined as in equation 1.9. It is shown, then, that the circuit in figure 1.3 oscillates with a frequency ω_N . Equivalently, the roots of equation 1.9 are at

$$j\omega = \pm j\omega_N \quad (1.13)$$

The circuit in figure 1.3 applies to the system of figure 1.2. The transfer function GH would be

$$GH = \left(\frac{-R_2}{R_1} \right) \left(\frac{-1}{R_3 C_1 s} \right) \left(\frac{-1}{R_4 C_2 s} \right) \quad (1.14)$$

Applying the procedure described by Van Valkenburg and others, assume the characteristic equation 1.4 can be written as [17, p. 553; 9, p. 32]

$$1 - GH = 1 - \frac{Ks}{s^2 + 2\zeta\omega_N s + \omega_N^2} = 0 \quad (1.15)$$

or

$$1 - GH = \frac{s^2 + (2\zeta\omega_N - K)s + \omega_N^2}{s^2 + 2\zeta\omega_N s + \omega_N^2} = 0 \quad (1.16)$$

Roots on the $j\omega$ -axis gives the following condition for oscillation:

$$2\zeta\omega_N - K = 0 \quad (1.17)$$

or

$$K = 2\zeta\omega_N \quad (1.18)$$

and the circuit oscillates at ω_N , the frequency as shown before.

These techniques apply to a linear system. In Chapter 3, a non-linear second-order oscillator is analyzed first by attempting linear systems analysis. It will be shown that using these methods, one will not achieve proper oscillator analysis. Unfortunately, the methodical techniques of linear systems analysis generally do not apply to nonlinear systems. This fact will be further reinforced in subsequent chapters.

CHAPTER 2

NONLINEAR ANALYSIS

General

Linear systems analysis is a useful and practical engineering tool applicable to linear systems and a lesser extent to many nonlinear systems. Its usefulness extends from the fact that time domain analysis, transform calculus and computer simulation readily obtain solutions to linear differential equations which model these systems. Test input signals and small perturbations reveal global behavior in linear systems.

The engineer is not as fortunate when analyzing nonlinear systems. Local behavior does not necessarily reveal global characteristics. Nonlinear differential equations do not lend themselves to systematic procedures for determining solutions. Linear equations which frequently can represent nonlinear systems and hence reveal characteristics often are not applicable to an inherently nonlinear system.

Analysis Techniques

Techniques for analyzing nonlinear systems generally reduce to procedures for finding solutions to the differential equations describing the system. At least some qualitative or hopefully quantitative information results. These procedures fall into three broad areas: topological methods involving analysis of phase plane, or more generally, phase space portraits; approximate methods which include numerical and some graphical techniques; and stability analysis usually

attributed to Liapunov [1, p. 108]. The approximate methods usually include the so-called analytic methods but exact solutions to a very limited class of nonlinear differential equations do exist [5, p. 55].

Phase space methods are an outgrowth of studies pursued by Poincare and Bendicson during the late nineteenth century [1,2,4]. Techniques producing phase plane portraits utilize graphical and numerical procedures such as the isocline method [4,5] and generalized Lienard methods [4] including the delta method [2,4,25]. Since these utilize the phase plane, these techniques are limited to first and second-order systems. Also, phase plane methods do not produce explicit solutions in second-order systems. This method derives meaningful qualitative and some quantitative information. The next section discusses phase plane analysis in more detail since later chapters analyze nonlinear oscillators by applying this technique.

Stability analysis of nonlinear systems leads to qualitative information about the system under study. However, this analysis alone is not sufficient to provide design criteria. This thesis follows a procedure of analysis and attempts to discover design conditions. If one assumes initially that a design is stable and if one can verify that it is stable, then the stability analysis of Liapunov becomes unnecessary. Utilizing graphical procedures as discussed above verifies stability. In other words, if stability is assumed and graphical analysis confirms this assumption, then analysis and design proceed without concern for specific stability criteria.

Approximate analysis encompasses several techniques. Numerical and graphical methods are included in this category. Analytical

methods are best utilized in conjunction with computer algorithms and graphical procedures. With the modern availability and use of digital computers, numerical methods once used laboriously now prove simple and quick. Computer simulation in conjunction with approximate methods enable the engineer to arrive at a suitable solution of sufficient accuracy using an iterative procedure.

The approximate method of analysis as applied to the analytic approach used in this instance is known as the describing function method [1, p. 121; 2, p. 343]. Used in conjunction with the parameter plane analysis of Siljak [1], higher-order nonlinear differential equations (greater than order two) are analyzed. This method is directly applicable to harmonic free symmetrical oscillations which describe the nonlinear feedback oscillators of this work [1, p. 152].

Phase Plane Analysis

Phase plane techniques are widely discussed in the literature. [1,2,4,5,12,14,25]. They can be used to analyze linear or nonlinear systems, but their useful analysis is limited mainly to second-order differential equations or a second-order system of equations. Also, the analysis does not directly lead to explicit solutions of the independent variable of the differential equation or the state variables if a system is considered.

Manual graphical methods of obtaining the phase plane diagram have been traditionally relied upon and these methods will be briefly mentioned. These graphical methods can utilize software and plotting routines to obtain the diagram, but an iterative plotting method using a numerical approximation is more efficient and gives good results.

Cunningham details a most easily understood method of isocline construction [5, p. 32], and the method is summarized here. A second-order differential equation of the form

$$\frac{d^2x}{dt^2} + \frac{f(dx,x)}{dt} = 0 \quad (2.1)$$

is used as a general example. The equation is autonomous since it is independent of the time variable t . This example is easily applied to the isocline method of analysis. First, reduce to a first-order equation by change of variable. If $v = dx/dt$, then

$$\frac{d^2x}{dt^2} = \frac{dv}{dt} = \frac{dv}{dx} \frac{dx}{dt} = v \frac{dv}{dx} \quad (2.2)$$

by the chain rule [21, p. 86]. Equation 2.1 is then reduced to a first-order equation

$$\frac{dv}{dx} = - \frac{f(v,x)}{v} \quad (2.3)$$

which can be solved for v in terms of x by the isocline method.

For the example above, let

$$m = - \frac{f(v,x)}{v} = \frac{dv}{dx} \quad (2.4)$$

and so for different slopes, a set of isocline curves are developed.

Due to the fact that the entire plane must be filled with line segments indicating slopes at particular points makes the isocline method

tedious. Other graphical methods are more direct but they still require considerable time and effort. One must also realize that for a second-order system, an actual solution is not obtained. Only an implicit solution with respect to time is available because transforming the second-order equation to a first-order one eliminates the variable time. An additional graphical integration in the phase plane is required for an explicit solution with respect to time.

Obviously, only a limited amount of quantitative information is derived from the phase plane portrait. An actual "solution" is not obtained for the dependent variable. This does not diminish the value of the procedure. Much qualitative information can be derived. For example, a sustained oscillation in a second-order nonlinear system produces a limit cycle in the phase plane. This is a closed curve to which the trajectory approaches a limit cycle spiraling either outward for initial conditions inside the limit cycle or inward for initial conditions external to the limit cycle. The existence of a limit cycle in the phase plane is a very important fact regardless if other information is obtained from the phase plane plot. Certainly, the maximum amplitude of the variables can be read directly from the graph. Also, the shape of the limit cycle indicates the relative nonlinearity of the dependent variable in the steady state. In other words, the more elliptical (the general case) the limit cycle is in the phase plane portrait, the more sinusoidal the sustained oscillations.

The importance of phase plane analysis cannot be overemphasized. The engineer can derive much information by examining phase plane portraits of second-order systems. Chapter 4 utilizes

an extension to phase plane analysis by applying this technique to third-order oscillators and examining the limit cycle in the phase space.

CHAPTER 3

A SECOND-ORDER NONLINEAR OSCILLATOR

General

As mentioned in the first chapter, oscillators require an active element such as a transistor or operational amplifier in order to work with resistor, capacitor and inductor networks. Many useful oscillators generate sinusoidal outputs. Oscillators are deemed marginally stable since the poles of the characteristic equation of the system model lie on the imaginary axis.

The active element in an oscillator can be a vacuum tube, a transistor or other type electronic amplifier [10, p. 596]. This work will concentrate on the operational amplifier (op-amp) oscillator configuration.

When utilizing operational amplifiers in analysis and design, certain idealizations expedite the analysis and do not significantly perturb results. The op-amp is a transistor differential amplifier [11, p. 505]. The amplifier configuration is the inverting amplifier and it is shown in figure 3.1. Since this is an inverting amp, v_0 in terms of node y is

$$v_0 = -Av_y \quad (3.1)$$

and the voltage at point y is related to the input voltage as

$$v_i = i_i R_i + v_y \quad (3.2)$$

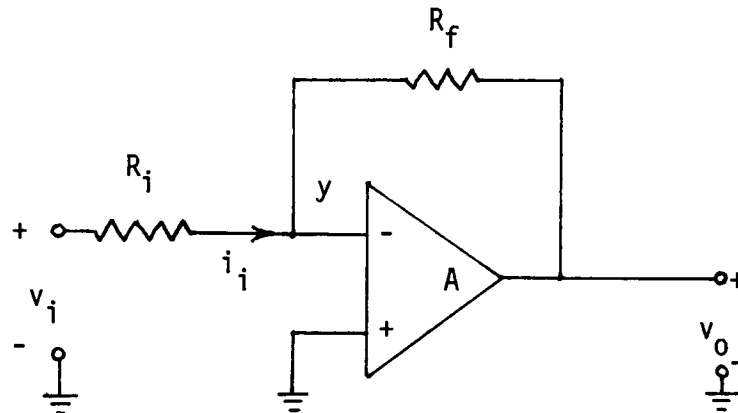


Figure 3.1 The Inverting Op-Amp

The current i_i does not flow into the inverting input because infinite input impedance is assumed. It must flow through R_f also and

$$v_i - i_i R_i - i_i R_f - v_o = 0 \quad (3.3)$$

or

$$i_i = \frac{v_i - v_o}{R_i + R_f} \quad (3.4)$$

which permits expressing v_y in terms of the input and output voltages

$$\begin{aligned} v_y &= v_i - i_i R_i \\ &= v_i - \left(\frac{v_i - v_o}{R_i + R_f} \right) R_i \\ v_y &= \frac{R_f}{R_i + R_f} v_i + \frac{R_i}{R_i + R_f} v_o \end{aligned} \quad (3.5)$$

Equation 3.2 permits reintroducing the gain into the mathematical expression and eliminating v_y to give

$$-\frac{v_o}{A} = \frac{R_f}{R_i + R_f} v_i + \frac{R_i}{R_i + R_f} v_o \quad (3.6)$$

Consolidating input and output voltages and impressing them as a ratio gives

$$\frac{v_o}{v_i} = \frac{A \cdot R_f}{-A \cdot R_f - R_i - R_f} \quad (3.7)$$

As A increases without bound, an ideal approximation of the op-amp, equation 3.7 reduces to

$$\frac{v_o}{v_i} = -\frac{R_f}{R_i} \quad (3.8)$$

The important idealizations for the model of the inverting op-amp amplifier now are summarized:

$$\begin{aligned} A &\rightarrow \infty \\ Z_{in} &\rightarrow \infty \\ Z_{out} &\rightarrow 0 \\ i_{in} &\rightarrow 0 \\ v_- &\rightarrow v_+ \\ Z_{cm} &\rightarrow \infty \end{aligned}$$

The idealizations are shown in figure 3.2.

These important idealizations apply if sufficient negative feedback is employed. In subsequent analysis, these idealizations will be utilized.

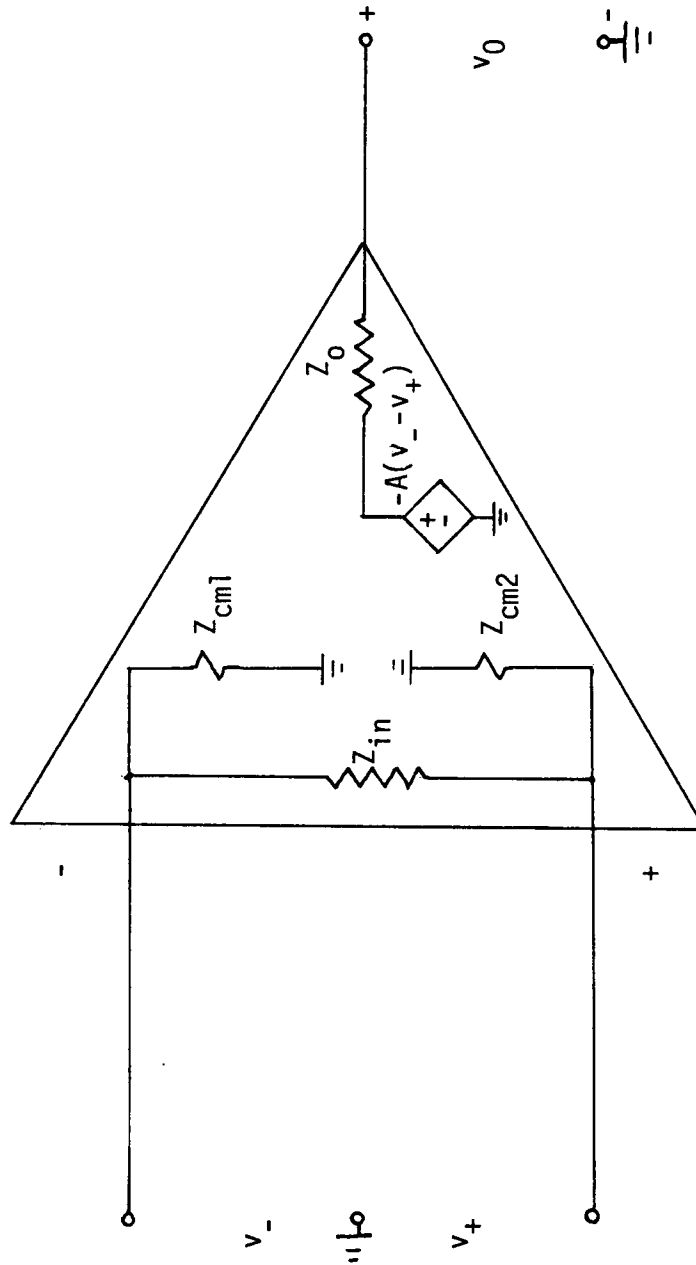


Figure 3.2 Schematic Representation of the Generalized Op-Amp Model

The Wein-Bridge Oscillator

The Wein-bridge oscillator is a standard second-order oscillator circuit used extensively in commercial audio generation circuits for low frequencies in the range of five Hertz (Hz) to one megahertz (1 MHz) [11, p. 568]. Figure 3.3 is the Wein bridge [22]. If $v_2 = 0$, the bridge is balanced. An op-amp is inserted at these terminals and an oscillator circuit results with sinusoidal oscillations at terminals v_1 . See figure 3.4. If all resistors are linear and the amplifier operates in the usual useful linear region, then the oscillator is linear and can be analyzed using different linear systems analysis techniques. Positive feedback through the lead-lag network and negative feedback through the voltage divider network are employed.

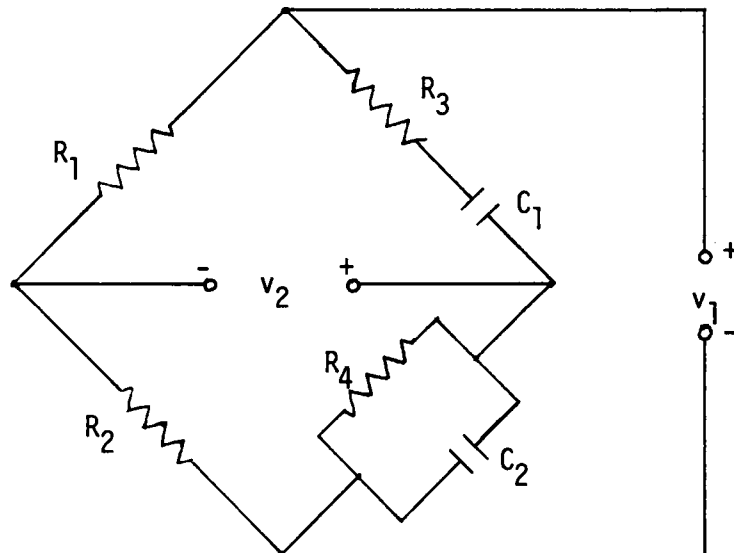


Figure 3.3 The Wein Bridge

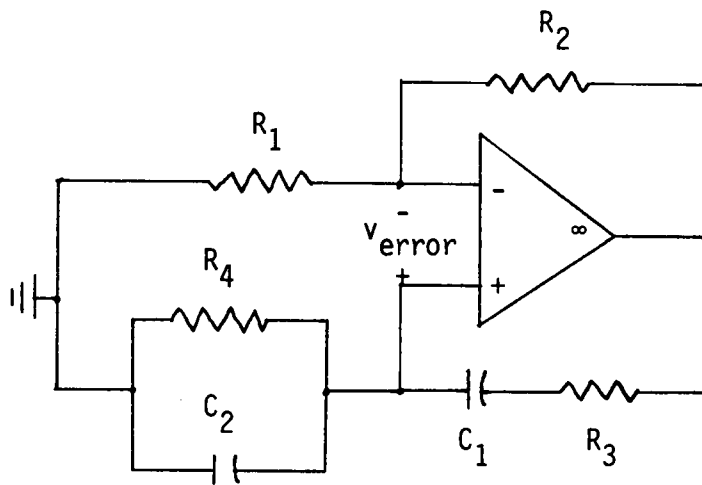


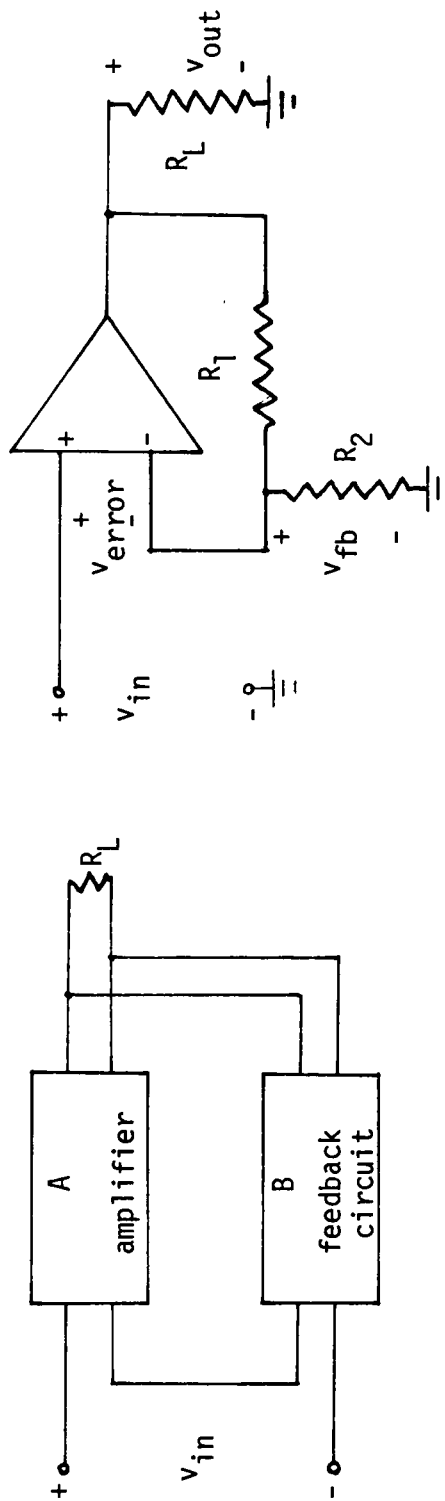
Figure 3.4 An Op-Amp Wein-Bridge Oscillator

Since v_{error} is ideally zero, R_1 and R_2 constitute the noninverting series-parallel negative feedback amplifier of figure 3.5 with gain of

$$K = \frac{R_2}{R_1} + 1 \quad (3.10)$$

as long as the ideal conditions of equations 3.9 apply.

More than one method can be used to analyze the linear Wein-bridge oscillator. Perhaps the simplest approach considers that the magnitude of the transfer function $v_{\text{out}}/v_{\text{in}}$ for the lead-lag network is one-third with zero phase shift at that frequency. Therefore, figure 3.5(b) shows that the negative feedback amplifier of figure 3.6 must have a gain of three to sustain oscillations. This means that



a. Block Diagram

b. Schematic

Figure 3.5 Block Diagram and Schematic of a Series-Parallel Noninverting Amplifier

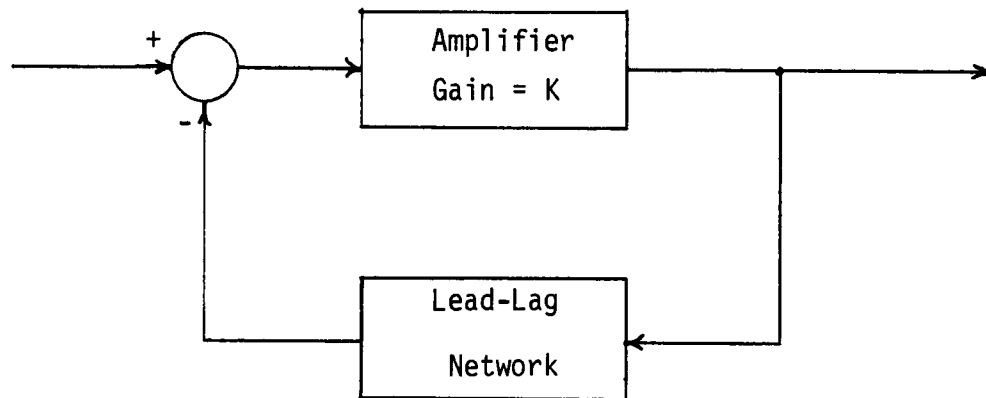


Figure 3.6 Block Diagram of Inverting Negative Feedback Amplifier

$$K = \frac{R_1}{R_2} + 1 = 3 \quad (3.11)$$

so

$$\frac{R_1}{R_2} = 2$$

which is

$$R_1 = 2R_2 \quad (3.12)$$

to sustain oscillations.

Using feedback analysis with the root-locus method provides another method of analysis and also gives insight into the operation of the Wein bridge oscillator.

Referring to figure 3.7, the feedback gain through the lead-lag network as a function of s is

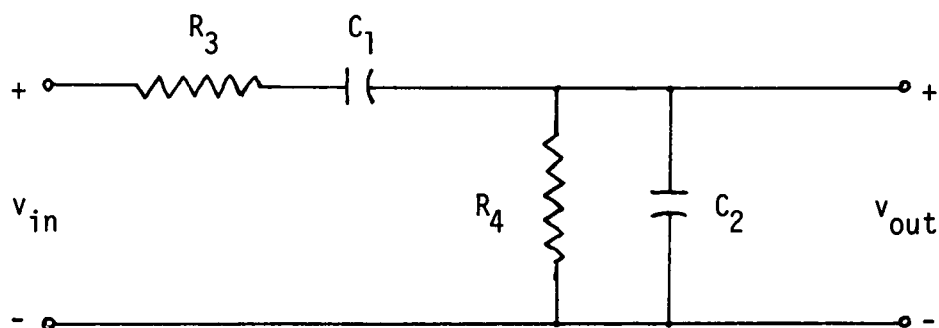


Figure 3.7 A RC Lead-Lag Network

$$\frac{v_{out}}{v_{in}} = \frac{\frac{R_4 \cdot 1/sC_2}{R_4 + 1/sC_2}}{\frac{R_4 \cdot 1/sC_2}{R_4 + 1/sC_2} + R_3 + \frac{1}{sC_2}} \quad (3.13)$$

This can be expressed in terms of the standard second-order system function as

$$\frac{v_{out}}{v_{in}} = A \frac{s}{s^2 + Bs + C} \quad (3.14)$$

with

$$A = \frac{1}{R_3 C_2}$$

$$B = \frac{R_3 C_1 + R_4 C_2 + R_4 C_1}{R_3 R_4 C_1 C_2} \quad (3.15)$$

$$C = \frac{1}{R_3 R_4 C_1 C_2}$$

As is usually done in this case, let $C_1 = C_2 = C$, $R_4 = R_3 = R$, so that (3.14) becomes

$$\frac{v_{out}}{v_{in}} = \frac{1}{RC} \frac{s}{s^2 + \frac{3}{RC} s + \frac{1}{R^2 C^2}} \quad (3.16)$$

The gain of the noninverting amplifier is K from equation 3.10. Since the overall feedback system as shown in figure 3.6 uses positive feedback, the characteristic equation from Chapter 1 is repeated here and is

$$1 - GH = 0 \quad (3.17)$$

or in terms of this circuit

$$0 = 1 - K \frac{1}{RC} \frac{s}{s^2 + \frac{3}{RC} s + \left(\frac{1}{RC}\right)^2} \quad (3.18)$$

$$0 = \frac{s^2 + \frac{(3-K)}{RC} s + \left(\frac{1}{RC}\right)^2}{s^2 + \frac{3}{RC} s + \left(\frac{1}{RC}\right)^2} \quad (3.19)$$

The roots of the numerator of equation 3.19 are the poles of the overall transfer function of the feedback oscillator circuit. The

locus of these roots is shown in figure 3.8. When $K=3$, the locus crosses the $j\omega$ -axis indicating that the circuit will oscillate at this gain. Also, $\omega = 1/RC$ at this point which verifies previous analysis.

A point which now must be considered is this: what happens to the oscillator when the circuit initially powers up and how does the circuit achieve oscillation and maintain this condition?

As mentioned in Chapter 1, initially the overall gain of the oscillator AB must be greater than one. As oscillations build, the gain must be reduced. To do this, one of the two configurations is possible. In figure 3.9, a nonlinear device (a nonlinear resistor or other device with a nonlinear volt-ampere relationship) either can be

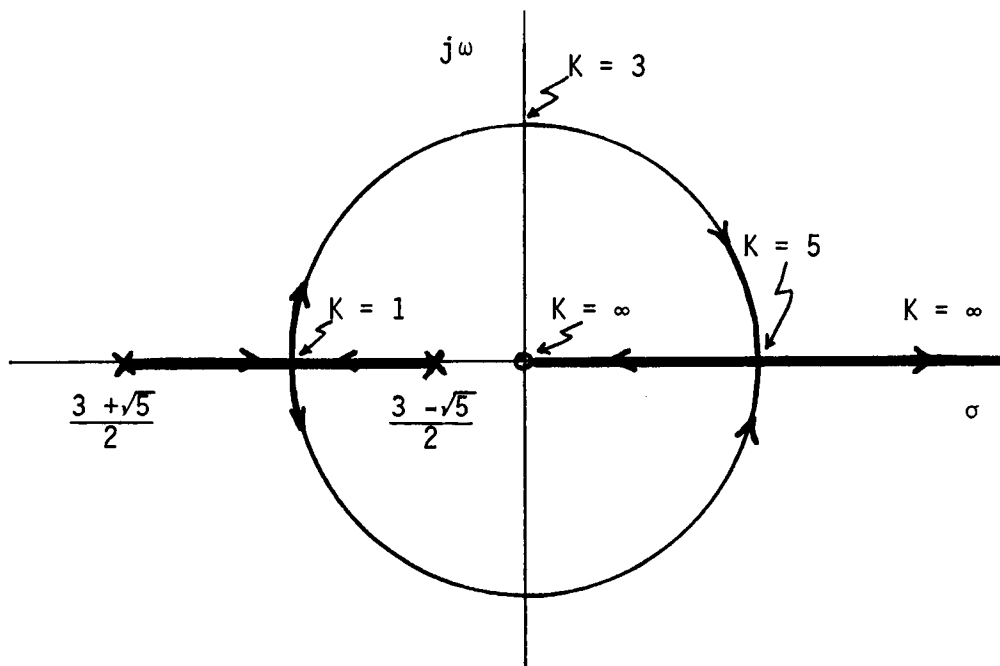


Figure 3.8 Root-Locus of Second-Order Oscillator

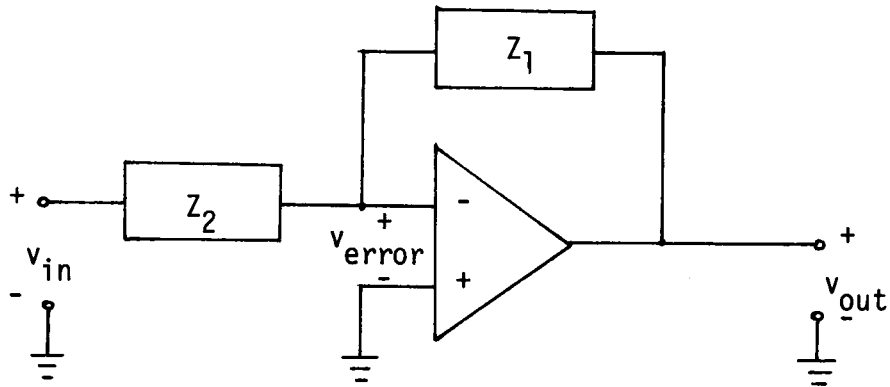
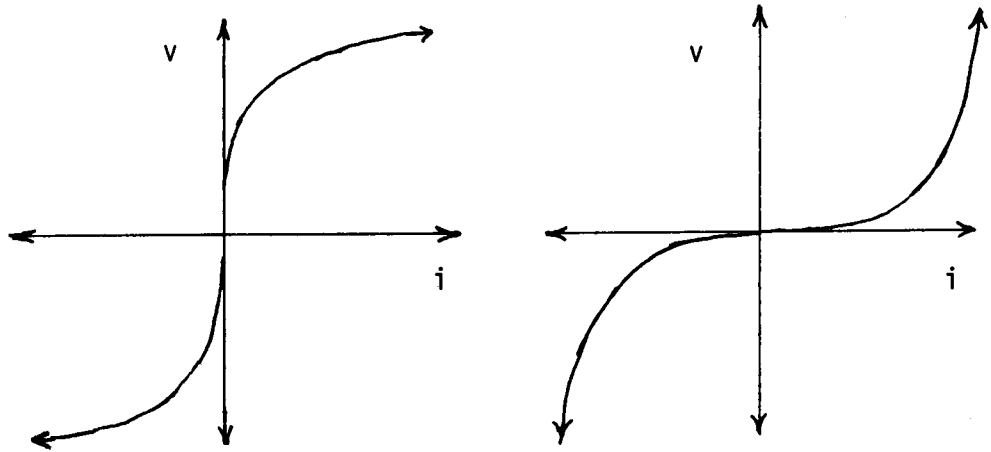


Figure 3.9 Standard Diagram Of Inverting Op-Amp

applied at z_1 or at z_2 . If the device is applied at z_1 , the characteristic curve must be similar to that of figure 3.10(a). The instantaneous resistance r is the slope of the curve at a given point. Initially, the resistance is high permitting high gain for the amp of figure 3.9. As the current increases, the value of r decreases and the gain decreases. The converse is true if a device characterized by 3.10(b) is used. This second device would be placed at z_2 of figure 3.9 giving high gain initially but reducing as current increases through the device. Figure 3.10(b) characterizes a device such as a lamp filament which has increasing resistance with increasing current. Examples of figure 3.10(a) are varistors of different types.

Linear analysis cannot be used for a complete analysis of a nonlinear oscillator. Linear techniques simply will not predict



a. Thyristor Type Device

b. Filament Type Device

Figure 3.10 Characteristic Volt-Amp Curves of Nonlinear Devices

accurate results for such parameters as gain and amplitude of oscillation [2, p. 344].

To proceed with analysis, a state variable method will be utilized [20,22]. (Note: a similar analysis was conducted for the Wein-bridge oscillator and is outlined in reference [22]. Examining figure 3.11, one sees that this circuit is the Wein-bridge oscillator redrawn slightly indicating that v_{error} goes to zero. The resistor R_2 has the nonlinear characteristics of a lamp. Since no current ideally flows into the op-amp on the left side, i_L must flow through R_2 . Summing currents at the bottom node v_2 gives

$$C \dot{v}_2 + \frac{(v_2 - 0)}{R} + \frac{v_R}{R} = 0 \quad (3.20)$$

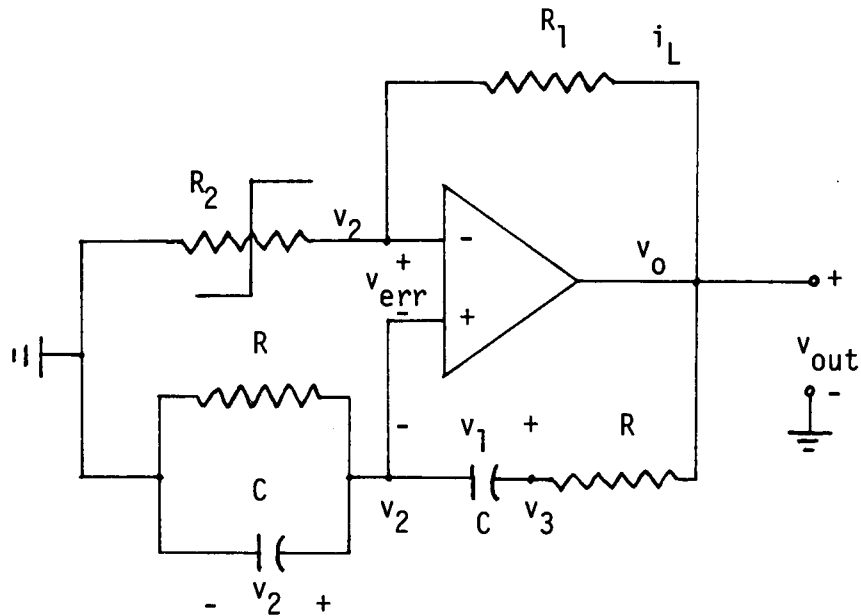


Figure 3.11 A Complete Wein-Bridge Oscillator Model

Capacitor voltages and inductor currents are the desired state variables. The voltage v_2 is the capacitor voltage across the capacitor in the parallel RC branch. The voltage across the other capacitor is $v_3 - v_2$ and it will be called v_1 . Noting also that $v_R/R = i_L$ which is the lamp current, and

$$-v_1 + v_R + v_{R1} = 0$$

implying

$$v_R = v_1 - v_{R1} \quad (3.21)$$

equation 3.20 becomes

$$\dot{v}_2 = -\frac{v_1}{RC} - \frac{v_2}{RC} + \frac{v_{R1}}{RC} \quad (3.22)$$

The other state equation is in terms of v_1 and is found by summing currents at v_3 giving

$$C\dot{v}_1 + \frac{v_R}{R} = 0$$

or

$$\dot{v}_1 = -\frac{v_1}{RC} + \frac{v_{R1}}{RC} \quad (3.23)$$

To express the term involving v_{R1} as a function of state variables and the resistor R_1 , write v_{R1} as

$$v_{R1} = i_L R_1 \quad (3.24)$$

and

$$i_L = f(v_L) = f(v_2) \quad (3.25)$$

Using these two expressions, the state equations can be expressed after normalizing the time scale as

$$\dot{v}_1 = -v_1 + R_1 \cdot f(v_2) \quad (3.26)$$

$$\dot{v}_2 = -v_1 - v_2 + R_1 \cdot f(v_2) \quad (3.27)$$

A second-order differential equation results if v_1 is eliminated from the above system. Differentiating equation 3.27 yields

$$\ddot{v}_2 = -\dot{v}_1 - \dot{v}_2 + R_1 \left(\frac{\partial f(v_2)}{\partial v_2} \right) \dot{v}_2 \quad (3.28)$$

Substituting equation 3.26 into 3.28 gives

$$\ddot{v}_2 = -[-v_1 + R_1 \cdot f(v_2)] - \dot{v}_2 + R_1 \left(\frac{\partial f(v_2)}{\partial v_2} \right) \dot{v}_2 \quad (3.29)$$

Solving for v_1 in equation 3.27 and substituting this result into 3.29 finally gives the second-order equation

$$\ddot{v}_2 = [-\dot{v}_2 - v_2 + R_1 \cdot f(v_2)] - R_1 \cdot f(v_2) - \dot{v}_2 + R_1 \left(\frac{\partial f(v_2)}{\partial v_2} \right) \dot{v}_2 \quad (3.30)$$

which simplifies to

$$\ddot{v}_2 + \left[2 - R_1 \left(\frac{\partial f(v_2)}{\partial v_2} \right) \right] \dot{v}_2 + v_2 = 0 \quad (3.31)$$

At this time, it is appropriate to consider what happens if the nonlinear device were a thyristor and what would occur if it were placed in the circuit at R_1 , the feedback resistor. This configuration was analyzed in reference [22]. The resulting state equations are

$$\dot{v}_1 = -v_1 + v_T \quad (3.32)$$

$$\dot{v}_2 = -v_1 - v_2 + v_T \quad (3.33)$$

Since the nonlinear resistor replaces R_1 , v_T is used instead of v_{R1} and is the voltage across the thyristor. The current i_T flows through the

thyristor and R_2 , so the equation

$$v_T = f(i_T) = f(v_2/R_2) \quad (3.34)$$

which relates v_T and a state variable is valid. Equations 3.32 and 3.33 can now be written as

$$\dot{v}_1 = -v_1 + f(v_2/R_2) \quad (3.35)$$

$$\dot{v}_2 = -v_1 - v_2 + f(v_2/R_2) \quad (3.36)$$

These two equations can be combined to give the single second-order equation

$$\ddot{v}_2 + \left[2 - \frac{\partial f(v_2/R_2)}{\partial v_2} \right] \dot{v}_2 + v_2 = 0 \quad (3.37)$$

Equations 3.37 and 3.31 are alike except for the damping term.

For the latter case, the nonlinear term is a function of v_2 and R_2 whereas the former case is only a function of v_2 . Examining figure 3.12 [13, p. 260] reveals why. In (b), the resistor R_2 is included inside the radical when expressing the output voltage; but in (a), only the input voltage is raised to a power. The resistor R_1 is a linear coefficient. This explains the difference in the damping terms of equations 3.37 and 3.31.

Phase plane analysis can be utilized to demonstrate that the two circuits discussed are truly oscillators. As mentioned in Chapter 2, the designer has graphical methods available which can be used to "solve" the state equations characterizing each oscillator model configuration. Graphical methods are tedious and complicated. A clever

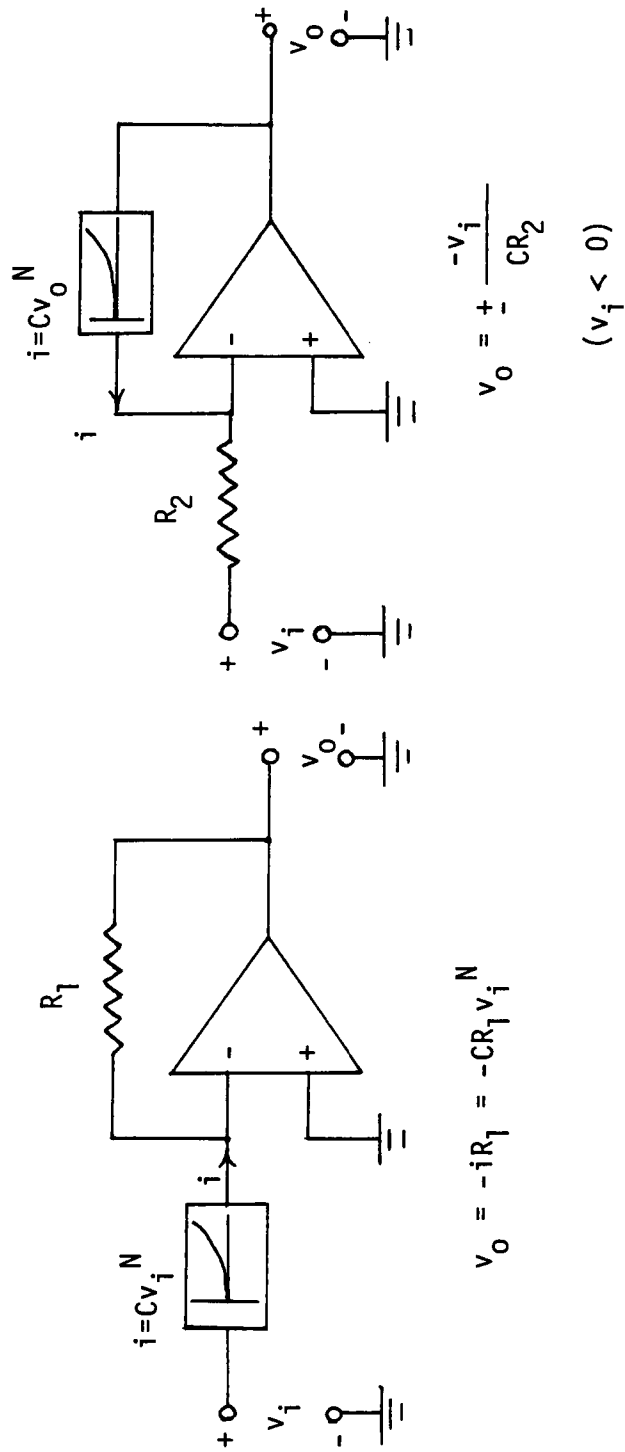


Figure 3.12 Two Possible Configurations for a Nonlinear Op-Amp Model

computer oriented version of Lienard's graphical method was developed [14] and applied to equations 3.35 and 3.36. With the wide-spread availability of small, inexpensive digital computers and associated graphics capabilities, computer time and memory are no longer critical in terms of cost. This permits the designer to utilize computers and graphics packages to quickly and easily solve systems of differential equations. Equations 3.33 and 3.34 are easily solved and a phase plane portrait is generated in this manner.

The state equations are written in normal form which is a system of equations and is written as

$$[v] = [A][v] + [f] \quad (3.37)$$

with each term being a matrix (or vector). In this case, $[f] = [0]$ and so is absent from the state equations. Applying initial conditions $[v(0)]$, the equations are solved for $[v(0)]$. The definition of the derivative gives

$$\frac{dv(t)}{dt} = \lim_{\Delta t \rightarrow 0} \frac{v(t+\Delta t) - v(t)}{\Delta t}$$

or

$$v(\Delta t) \approx v'(0) \Delta t + v(0) \quad (3.38)$$

for $t=0$ and Δt sufficiently small. Next, the value of $[v(\Delta t)]$ is computed and the process continues in increments of Δt [16, p. 594]. Of course, the smaller the increment, the more accurate the value of $[v]$ with associated penalty in computer memory requirements and run-time.

The above method was used to generate solutions to the system of equations examined. The program is included at Appendix A. A graphics package used in conjunction with the HP 9000 minicomputer and an HP 7470a plotter plotted the phase plane portrait. Figure 3.13 gives the phase plane for two different sets of initial conditions and an indicated value of R_2 . The functional relationship between v_T and i_T used was the same as used in reference [22] and is of the form

$$\begin{aligned} v_T &= C i_T^N \\ \text{with } C &= 59.4 \\ N &= 0.3 \end{aligned} \tag{3.39}$$

At this point, discussion returns to the oscillator configuration with the nonlinear element in place of R_2 . As was mentioned previously, the v - i characteristic curve for this configuration must be of the form of figure 3.10(b). Equation 3.38 states the relationship between voltage and current. A curve trace of the device derives the values of C and N . In order for this device to have the same phase portrait as the oscillator in reference [22], the value of C must be chosen so that the product of R_1 and C must be

$$R_1 \cdot C = \left(\frac{1}{R_2} \right)^{0.3} 59.4 \tag{3.40}$$

For example, using $R_2 = 100K$, the phase plane portrait illustrating the oscillator in reference [22] is in figure 3.13. This means equation 3.39 equals 1.878 to four places. If $R_1 = 100K$ for the oscillator discussed here, then with $N = 0.3$, the constant C of equation 3.38 must

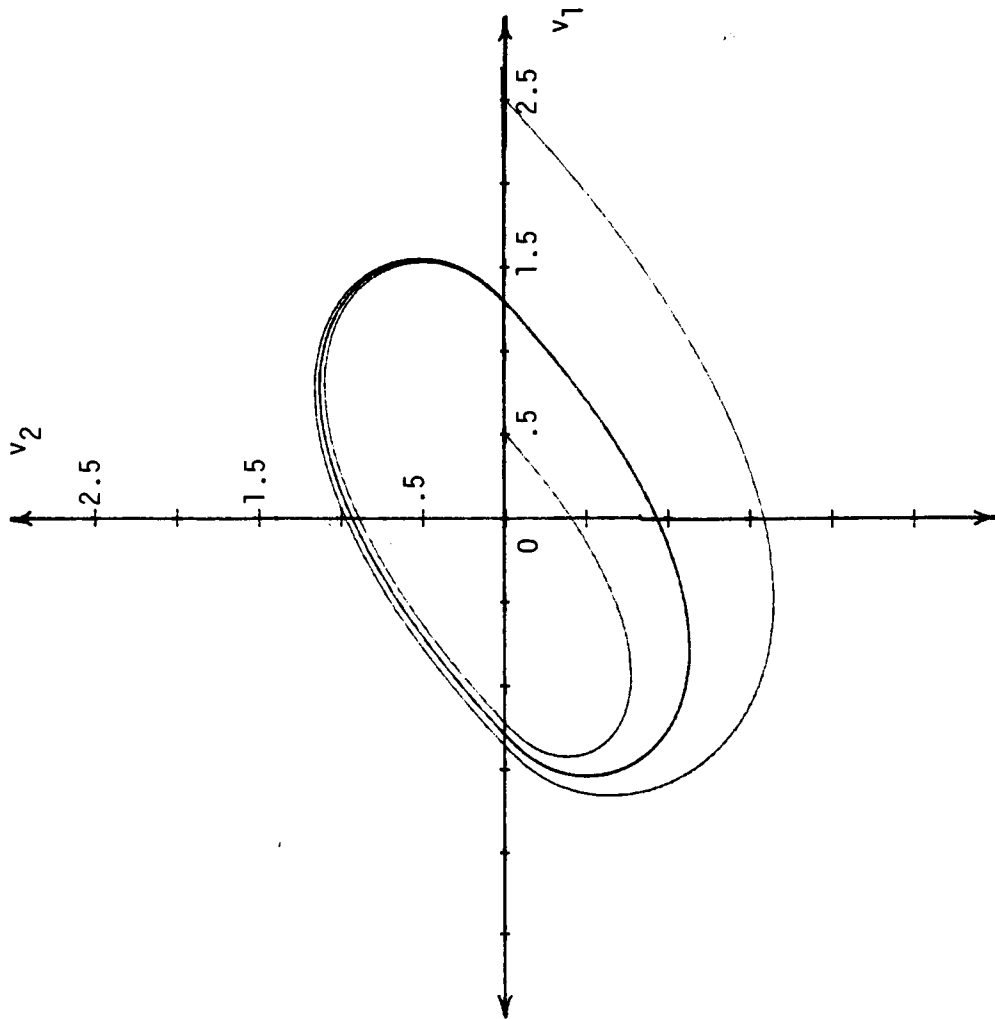


Figure 3.13 Phase Plane Portrait of Wein Bridge Showing Limit Cycle With Two Initial Conditions

be 1.878×10^{-5} for the oscillator to duplicate the phase portrait of figure 3.13.

Results and Discussions

Both configurations of the nonlinear op-amp Wein-bridge oscillator obviously can oscillate at the same frequency and amplitude. With the nonlinear element in position R_2 , the characteristic curve is not the inverse of the functional relationship characterizing the curve of the nonlinear element used in the position R_1 . Also, the oscillator with the nonlinear element in position R_1 is much less sensitive to changes in R_2 than the other oscillator is to changes in R_1 . For example, for the latter case, a ten-fold change in R_1 changes the coefficient of the nonlinear term ten-fold, but the same change in the former changes the coefficient about two times.

Interesting results occur when linear analysis and intuition are applied to these nonlinear oscillators. If a linear resistor were used instead of a thyristor in the oscillator of figure 3.18(b), the resulting linear second-order equation would be

$$\ddot{v}_2 + (2 - R/R_2)\dot{v}_2 + v_2 = 0 \quad (3.41)$$

and the condition for oscillation being

$$R_T/R_2 = 2 \quad (3.42)$$

which agrees with the earlier linear analysis. The oscillator of figure 3.10(a) also yields to this type of linear analysis if a linear resistor R_T replaces the nonlinear element. The case also gives a linear second-order equation and the condition for the damping term is

$$R_T/R_2 = 2 \quad (3.42)$$

which agrees with the earlier linear analysis. The oscillator of figure 3.10(a) also yields to this type of linear analysis if R_L , a linear resistor, replaces the nonlinear element. The case also gives a linear second-order equation and the condition for the damping term is

$$R_1/R_L = 2 \quad (3.43)$$

as expected.

Unfortunately, such straightforward analysis to a nonlinear oscillator yields erroneous results. The damping term of the second-order nonlinear equation of the oscillator of figure 3.10(b) is

$$2 - \frac{\partial f(v_2/R_2)}{\partial v_2} \quad (3.44)$$

If the incremental resistance r_T is defined as

$$r_T = \frac{\partial v_T}{\partial i_T} \quad (3.45)$$

then the damping term is

$$2 - \frac{\partial f}{\partial v_2} = 2 - \frac{r_T}{R_2} \quad (3.46)$$

and the ratio r_T/R_2 was calculated as 0.5 in reference [22] for different values of R_2 . Since this ratio is constant for the particular thyristor used, the circuit can be designed to a specific

peak value of the state variable v_2 by choosing the appropriate resistor R_2 .

The mathematical relationship between v_L and i_L for the device used in the oscillator of figure 3.10(a) was developed earlier to yield a phase plane portrait exactly as that of the oscillator of 3.10(b). It should be mentioned, though, that this is not a practical oscillator design since the mathematical relationship for the nonlinear device would need to be

$$i_L = 1.8783 \times 10^{-5} (v_L)^{0.3} \quad (3.47)$$

as mentioned. This means the curve trace equation is

$$v_L = 5.6769 \times 10^{15} \cdot i_L^{3.33} \quad (3.48)$$

which is an unrealistic incandescent lamp equation [11, p. 568].

The next chapter examines linear and nonlinear analysis as applied to a third-order oscillator circuit. Subsequent to analysis, a search is made for design considerations. Once again, linear analysis proves ineffective in predicting gain relationships as conditions for oscillation.

CHAPTER 4

RC PHASE SHIFT OSCILLATOR ANALYSIS

General

The conditions for oscillation in a third-order linear circuit differ significantly from the second-order case. Recall that the denominator of the transfer function is the characteristic equation of the system when set to zero. The equation is a function of s and for positive feedback is

$$1 - G(s)H(s) = 0 \quad (4.1)$$

where the product $G(s)H(s)$ consists of a numerator and denominator polynomial in s . Equation (4.1) can be written as

$$\frac{N(s)}{D(s)} = 0 \quad (4.2)$$

so the roots of $N(s)$ are the poles of the transfer function.

For the second-order case let $N(s)$ be

$$N(s) = a_2s^2 + a_1s + a_0 = 0 \quad (4.3)$$

in general. This can be written as

$$N(s) = a_2\left(s^2 + \frac{a_1}{a_2}s + \frac{a_0}{a_2}\right) = 0 \quad (4.4)$$

If $a_1 = 0$, this expresses the condition for oscillation. Then $N(s)$

becomes

$$N(s) = a_2(s^2 + \left[\sqrt{\frac{a_0}{a_2}}\right]^2) = 0 \quad (4.5)$$

with $\omega_0 = \sqrt{\frac{a_0}{a_2}}$.

For the third-order case, let $N(s)$ be written as a third-order polynomial in s

$$N(s) = a_3s^3 + a_2s^2 + a_1s + a_0 = 0 \quad (4.6)$$

If $s = j\omega_0$ satisfies equation 4.6 and ω_0 is the frequency of oscillation, then equation 4.6 becomes

$$N(j\omega_0) = a_3(j\omega_0)^3 + a_2(j\omega_0)^2 + a_1(j\omega_0) + a_0 = 0 \quad (4.7)$$

or

$$N(j\omega_0) = (a_0 - a_2\omega_0^2) + j(a_1\omega_0 - a_3\omega_0^3) = 0 \quad (4.8)$$

The real and imaginary parts of equation 4.8 must be zero.

This gives

$$a_0 - a_2\omega_0^2 = 0 \quad (4.9)$$

$$a_1 - a_3\omega_0^2 = 0$$

Therefore

$$\frac{a_0}{a_2} = \frac{a_1}{a_3} \quad (4.10)$$

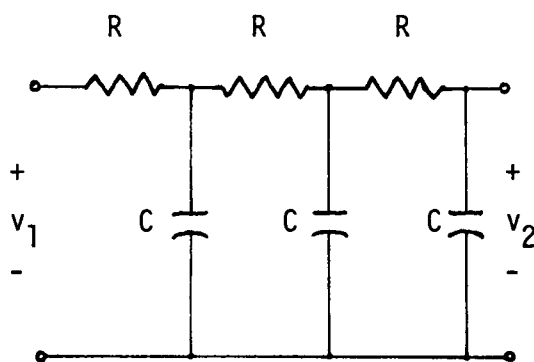
or the condition for oscillation is then

$$a_0 a_3 = a_1 a_2 \quad (4.11)$$

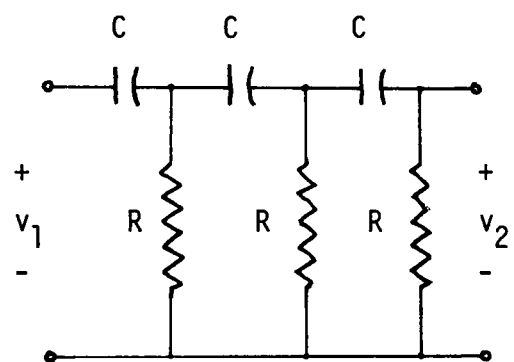
This is merely an alternative statement of Routh's criterion [17, p. 559] which is a means for determining stability of a linear system. This technique will be employed when examining the linear op-amp phase shift oscillator.

Third-Order Oscillator

Several versions of an RC ladder network can be utilized to design a phase shift oscillator [6]. Figure 4.1 shows two of the more common types where all capacitors and resistors are equal respectively. The configuration in figure 4.1(a) will be used for subsequent analysis.



a. Capacitor Shunt



b. Resistor Shunt

Figure 4.1 Third-Order RC Ladder Networks

The transfer function of the ladder network in figure 4.1(a) is derived from circuit analysis. Figure 4.2 indicates mesh currents which result in the equations relating the currents and voltages as

$$I_3 = \frac{\begin{vmatrix} R+1/sC & -1/sC & V \\ -1/sC & R+2/sC & 0 \\ 0 & -1/sC & 0 \end{vmatrix}}{\begin{vmatrix} R+1/sC & -1/sC & 0 \\ -1/sC & R+2/sC & -1/sC \\ 0 & -1/sC & R+2/sC \end{vmatrix}} \quad (4.12)$$

which reduces to

$$I_3 = \frac{V_1 sC}{(RCs)^3 + 5(RCs)^2 + 6RCs + 1} \quad (4.13)$$

Since the output voltage is related to the mesh current I_3 as

$$V_2 = I_3 \frac{1}{sC} \quad (4.14)$$

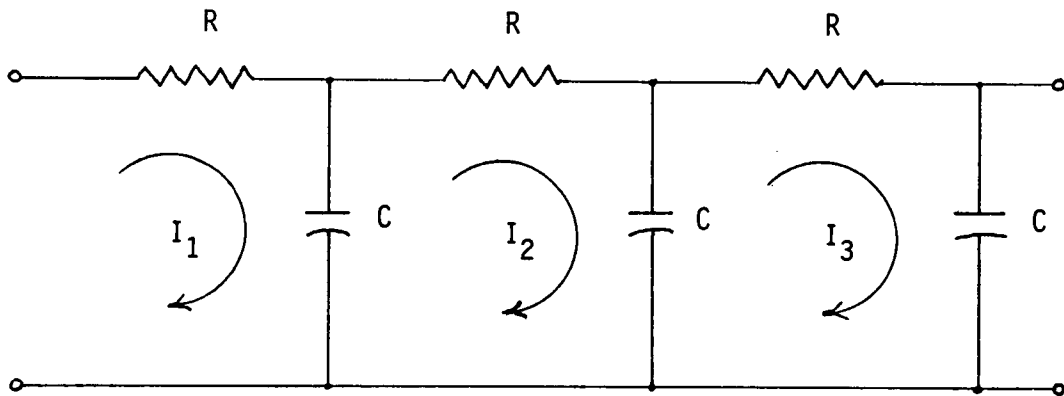


Figure 4.2 Calculating the Transfer Function of an RC Ladder Network

the transfer ratio is

$$\frac{V_2}{V_1} = \frac{1}{(RCs)^3 + 5(RCs)^2 + 6RCs + 1} \quad (4.15)$$

Connecting this circuit in cascade gives the op-amp oscillator of figure 4.3. If loading of the inverting amplifier is assumed negligible (it is not in reality), then the amp and the RC ladder network can be assumed cascaded. The resulting overall transfer function of the oscillator is the product of the gain of the amp and equation 4.15. This gives the characteristic equation as

$$0 = 1 - G(s)H(s) = 1 - (K) \frac{1}{(RCs)^3 + 5(RCs)^2 + 6RCs + 1} \quad (4.16)$$

The numerator is

$$0 = (RCs)^3 + 5(RCs)^2 + 6RCs + (1-K) \quad (4.17)$$

and the condition for oscillation is given by equation 4.11 as

$$(1-K)(RC)^3 = 5(RC)^2(6RC)$$

or

$$K = -29 \quad (4.18)$$

with the frequency of oscillation given by equation 4.10 as

$$\omega_0^2 = \frac{a_1}{a_3} = \frac{6RC}{(RC)^3}$$

or

$$\omega_0 = \frac{\sqrt{6}}{RC} \quad (4.19)$$

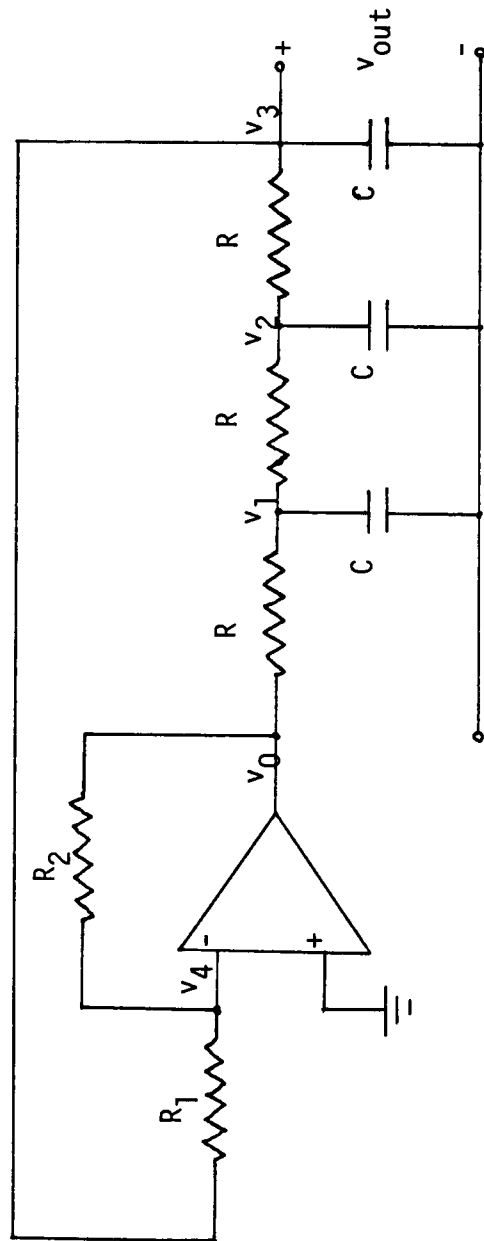


Figure 4.3 A Third-Order RC Phase Shift Oscillator Model

As was briefly mentioned, coupling is not ideal between the RC ladder network and the inverting amplifier. Some losses are expected meaning the gain of 29 is an ideally low value. The following state variable analysis shows this.

Examining figure 4.3 again, it can be seen that the state variables are the three capacitor voltages. Applying idealization to the op-amp as before, the state equations become

$$\begin{aligned} RC \dot{v}_1 &= -2v_1 + v_2 - \frac{R_2}{R_1} v_3 \\ RC \dot{v}_2 &= v_1 - 2v_2 + v_3 \\ RC \dot{v}_3 &= v_2 - v_3 \end{aligned} \quad (4.20)$$

with the system matrix

$$[a] = \begin{bmatrix} -2 & 1 & -R/R_1 \\ 1 & -2 & 1 \\ 0 & 1 & -(1+R/R_1) \end{bmatrix} \quad (4.21)$$

and the corresponding characteristic equation

$$\begin{vmatrix} -2-s & 1 & -R_2/R_1 \\ 1 & -2-s & 1 \\ 0 & 1 & -(1+R/R_1)-s \end{vmatrix} = 0 \quad (4.22)$$

This is a third-order polynomial in s , and after evaluating the indicated determinant and simplifying, the result is

$$s^3 + \left(5 + \frac{R}{R_1}\right)s^2 + \left(6 + \frac{4R}{R_1}\right)s + 1 + \frac{3R + R_2}{R_1} = 0 \quad (4.23)$$

This characteristic equation demonstrates clearly the loading effects discussed above. If R_1 is increased greatly with respect to R , then loading is minimized and equation 4.23 can be expressed as

$$s^3 + 5s^2 + 6s + (1 + K) = 0 \quad (4.24)$$

where $K = -R_2/R_1$. If K carries its own sign, then the condition for oscillation is seen as $K = 29$ which agrees with previous analysis. If $R = 10R_1$ then the characteristic equation is

$$s^3 + 5.1s^2 + 6.4s + (1.3 + K) = 0 \quad (4.25)$$

with $K = 31.34 \quad (4.26)$

which agrees with intuition since loading is now considered.

A practical phase shift oscillator must have a means to control divergence. Otherwise, the gain and frequency of oscillation will fluctuate. Using a nonlinear device can achieve stabilization so nonlinear analysis becomes important if the engineer wants to design based upon the amplitude and frequency of the output state variable.

Nonlinear Analysis

Linear analysis is often employed to predict frequency of oscillation because many nonlinearities are assumed frequency insensitive but amplitude sensitive [2, p. 344]. Other methods must be used to complete the analysis which yield amplitude and gain considerations.

Figure 4.4 is the phase shift oscillator with R_2 replaced by a nonlinear device. As with the Wein-bridge oscillator, using a nonlinear device in place of the feedback resistor of an inverting

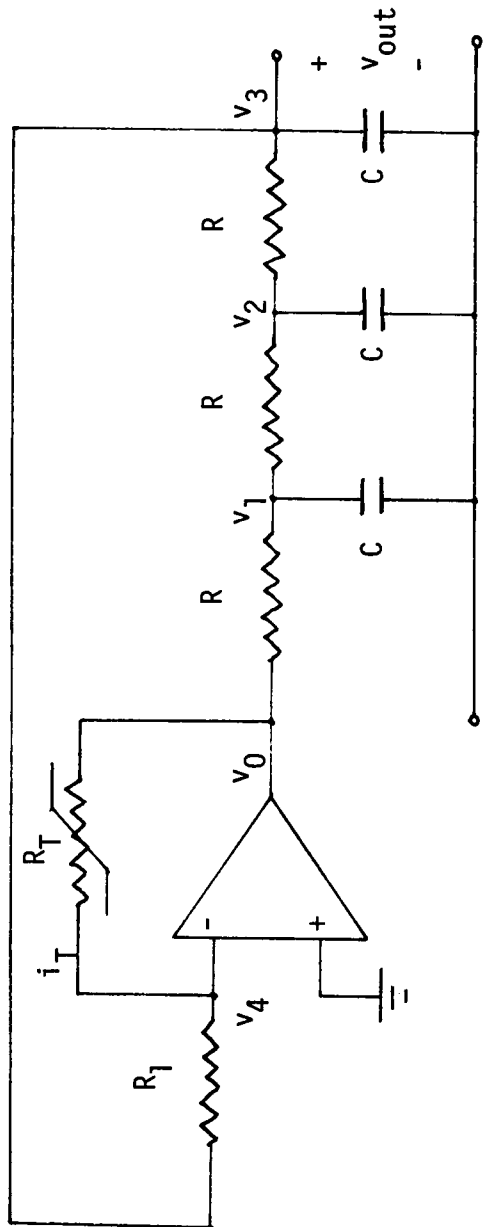


Figure 4.4 Nonlinear RC Phase Shift Oscillator

amplifier necessitates that the nonlinear device have a volt-amp curve similar to that of figure 4.5. If the incremental resistance r_T is the slope of inverse v - i curve, then the incremental resistance initially is very high when the oscillator begins operation. The amplifier gain is greater than unity. As the voltage increases, r_T becomes less and gain decreases. Stabilization is achieved.

Utilizing the state variable method of analysis, the state equations for the oscillator circuit of figure 4.4 are

$$RC \dot{v}_1 = -2v_1 + v_2 + v_0$$

$$RC \dot{v}_2 = v_1 - 2v_2 + v_3 \quad (4.27)$$

$$RC \dot{v}_3 = v_2 - \left(1 + \frac{R}{R_1}\right)v_3$$

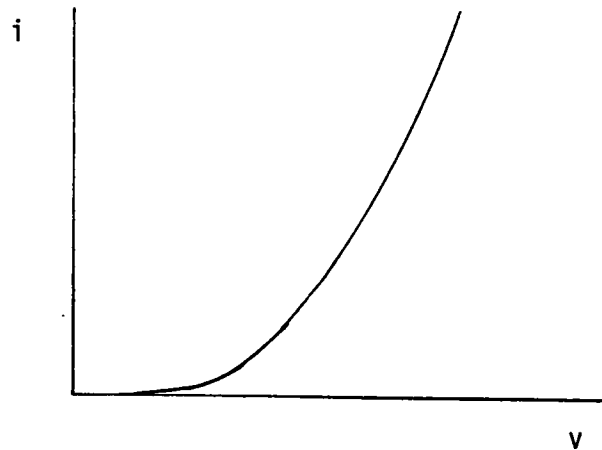


Figure 4.5 Volt-Amp Curve Of Nonlinear Element Of Oscillator In Figure 4.4

which are the same as equations (4.20) except for the term in the first equation involving v_0 . This term is a nonlinear term, and it can be expressed in terms of the state variable v_3 as

$$-\frac{v_3}{R_1} - f(v_0) = 0 \quad (4.28)$$

If i_T , the nonlinear element current, is written as

$$i_T = f(v_T) = Cv^N \quad (4.29)$$

then equation 4.28 is

$$-\frac{v_3}{R_1} - Cv_0 = 0$$

$$v_0 = -\frac{|v_3|}{CR_1} \quad (4.30)$$

since $v_T = v_0$. Let the constants N and C be equal to those in the first Wein bridge analysis. Normally, these constants will fit data derived from curve traces of the nonlinear element (a thyristor) used. Applying equation 4.30 with $1/N = 0.3$ and $(1/C)^{1/N} = 59.4$, the state equations become

$$\begin{aligned} \dot{v}_1 &= -2v_1 + v_2 - 59.4 (v_3/R_1)^{0.3} \\ \dot{v}_2 &= v_1 - 2v_2 + v_3 \\ \dot{v}_3 &= v_2 - v_3 \end{aligned} \quad (4.31)$$

if the constants R , C , and R_1 can be appropriately chosen so that $RC=1$ and $R/R_1 \ll 1$.

The state equations 4.31 were solved iteratively as explained in Chapter 3. Suitable initial conditions were applied corresponding to realistic circuit noise. The phase space portraits of figure 4.6 through 4.9 were obtained using Advanced Graphics Package [23] software routines available with the HP 9000 minicomputer. The program and discussion are at Appendix B. As shown, a limit cycle in three-dimensional phase space was obtained demonstrating that this circuit is an oscillator for the given values of the circuit components.

The state equations 4.31 can be combined into a single third-order equation in terms of state variable v_3 . The procedure is straightforward and gives

$$\ddot{v}_3 + 5\dot{v}_3 + 6v_3 + v_3 + 59.4(v_3/R_1)^{0.3} = 0 \quad (4.32)$$

This equation is identical to the characteristic equation of the linear phase shift oscillator discussed in the previous section except for the one nonlinear term involving the thyristor current. The term is analogous to K , the gain constant of the inverting amplifier. If the incremental resistance r_T of the thyristor is defined as previously, this gives

$$r_T = \frac{\partial v_T}{\partial i_T} = \frac{\partial (59.4 \cdot i_T)^{0.3}}{\partial i_T} \quad (4.33)$$

or
$$r_T = 17.8 i_T^{-0.7} = 17.8 \cdot (v_3/R_1)^{-0.7} \quad (4.34)$$

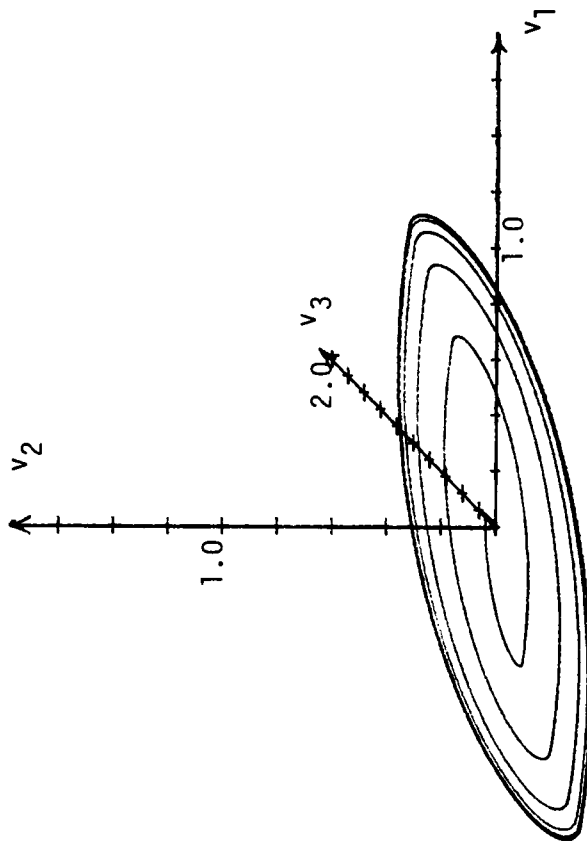


Figure 4.6 Right Perspective View of Phase Space Trajectory of Phase Shift Oscillator

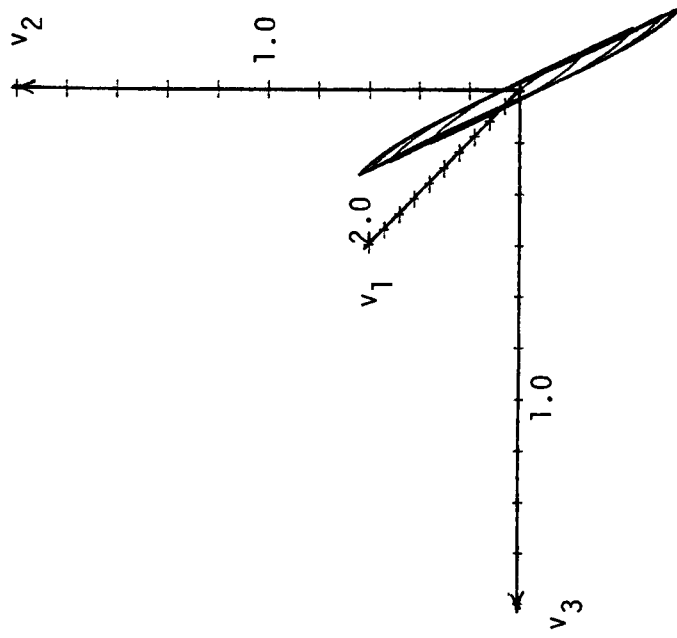


Figure 4.7 Left Perspective View of Phase Space Trajectory of Phase Shift Oscillator

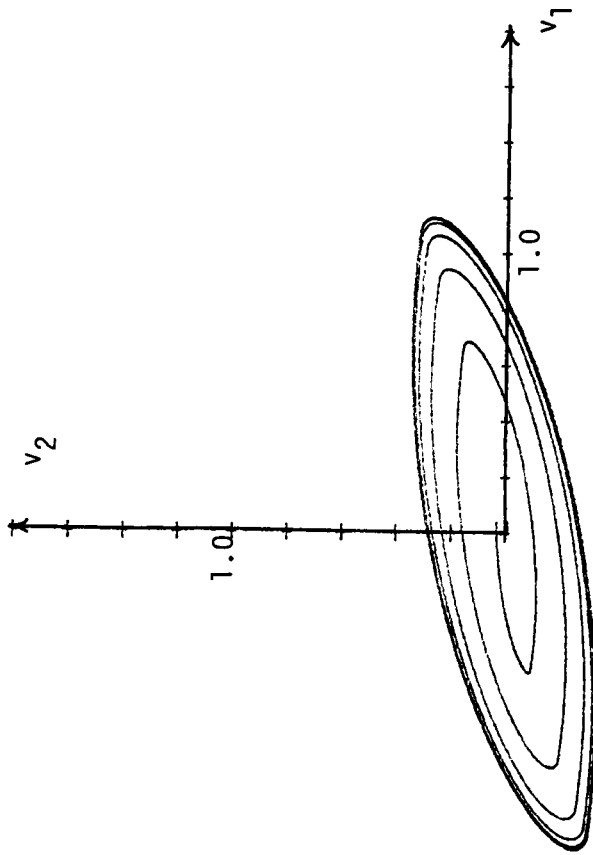


Figure 4.8 Phase Space Trajectory Projection In v_1 - v_2 Plane

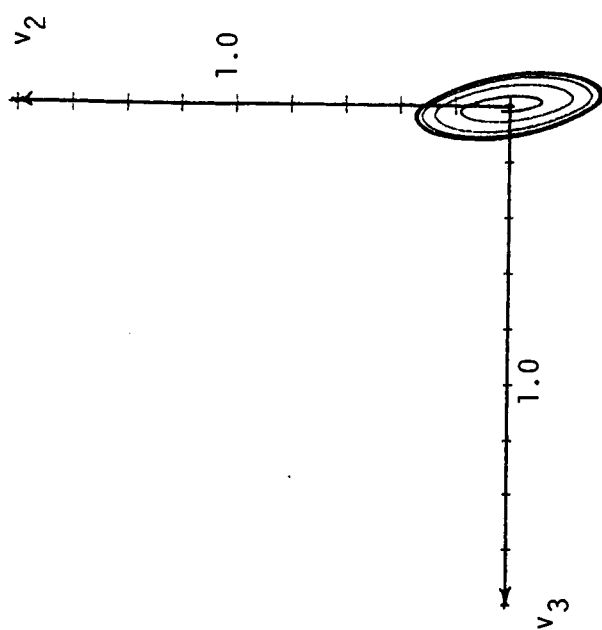


Figure 4.9 Phase Space Trajectory Projection In v_2 - v_3 Plane

The nonlinear term of equation 4.32 can be written as

$$\begin{aligned}
 59.4 \left(\frac{v_3}{R_1} \right)^{0.3} &= 3.33 \cdot 17.8 \left(\frac{1}{R_1} \right)^{0.3} \cdot (v_3)^{-0.7} (v_3)^{1.0} \\
 &= 3.33 \cdot 17.8 \frac{\left(\frac{v_3}{R_1} \right)^{-0.7}}{R_1} \cdot v_3 \\
 &= 3.33 \left(\frac{r_T}{R_1} \right) v_3
 \end{aligned} \tag{4.35}$$

Expression 4.35 means that an analogy exists between the linear characteristic equation and the equation for the nonlinear system which is

$$(1 + K) = 1 + 3.33 \left(\frac{r_T}{R_1} \right) \tag{4.36}$$

This suggests examining the term containing r_T/R_1 .

The phase space curves of the state equations can yield quantitative data enabling calculations of r_T . However, it is just as easy to plot the state variables as a function of iterative time. Now that an oscillating system is confirmed, measurements of the amplitude of the state variables can be more easily made from such plots. The program code for this routine is at Appendix B, and figure 4.10 gives the plots of the three state variables with sufficient elapsed time to yield steady state oscillations.

Results and Discussion

For various values of R_1 and R and with loading considered, i.e., using the term $(1+R/R_1)$ in the third state equation, plots were generated similar to those of figure 4.10. The vertical scale was adjusted appropriately to enable measurement. Frequency was predicted using linear analysis: equation 4.19 was applied. Necessary measurements and calculations were made to complete table 4.1.

The last column of table 4.1 shows constant values of r_T/R_1 within tolerance. This value is analogous to the gain R_2/R_1 of the linear amplifier. The value is not comparable to the value for the linear system. Even when considering equation 4.36, the value falls short of the ratio required for gain in the linear system.

Since r_T/R_1 is a constant, a design consideration exists for an oscillator of this type. The value of r_T/R_1 is characteristic of the particular thyristor used. If v_3 is chosen as the output desired, then the maximum amplitude of v_3 can be expressed as a function of R_1 , or

$$|v_3|_{\max} = f(R_1) \quad (4.37)$$

In order to verify the above analysis, an RC phase shift oscillator was constructed in the laboratory using standard 5% and 10% parts. The op-amp used was the LM 358 [24, p. 3-216]. Only one value of C was used. The resistor values were 100 and 1000 ohms. The capacitors used were 0.1 μF instead of values necessary to make $RC=1$. Capacitors with larger values of capacitance other than electrolytic

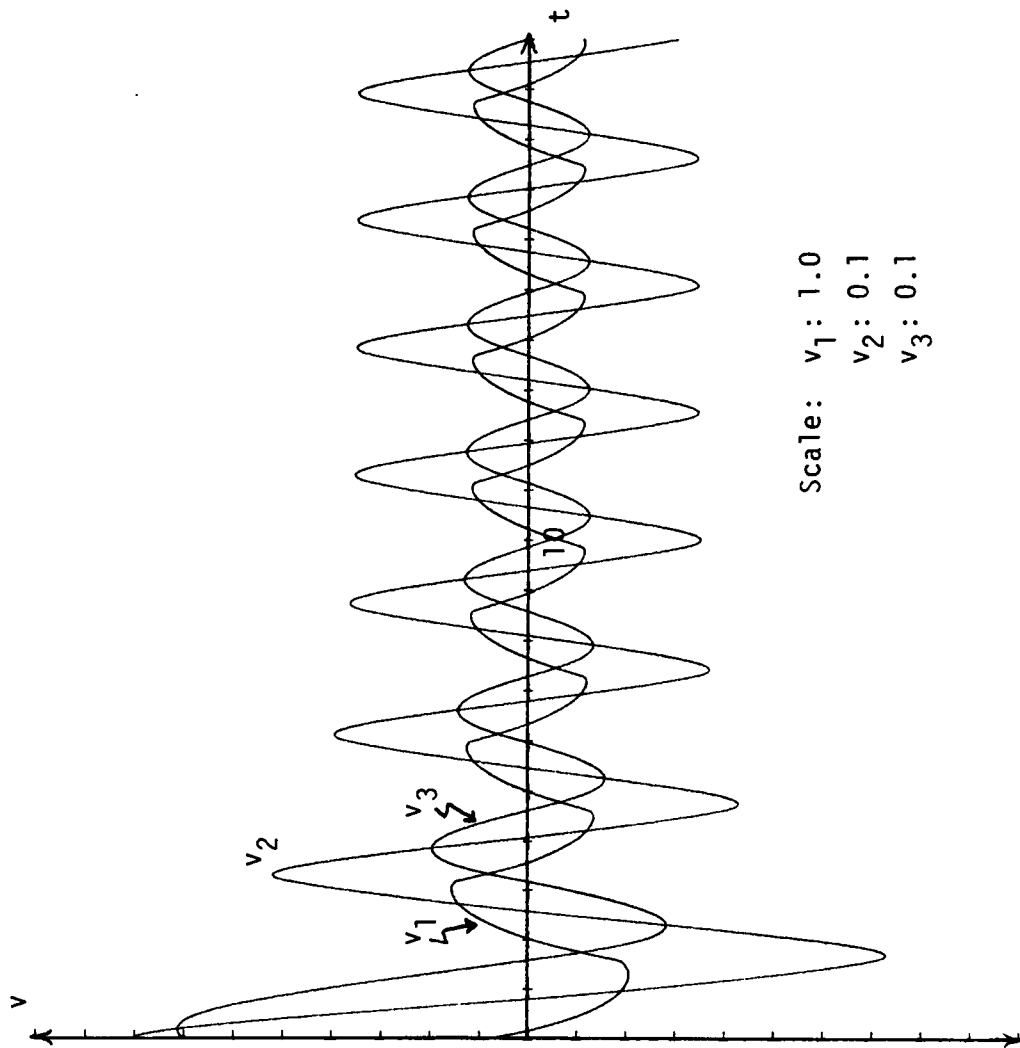
Figure 4.10 Plot of State Variables Against Time Δt

TABLE 4.1
Limit Cycle Amplitude Data

R_1	v_0^a	v_1	v_2	v_3	r_T^b	r_T/R_1
<u>$R = 100\Omega, C = 10,000 \mu F$</u>						
1K	4355	1550	460	165	.00972	7.919
5K	2241	850	245	90	.03734	7.468
10K	1688	600	185	70	.07233	7.237
50K	831	305	90	33	.3778	7.555
100K	621	225	68.5	25	.7453	7.453
500K	311	120	35	12.5	3.7355	7.471
<u>$R = 1K\Omega, C = 1000 \mu F$</u>						
10K	1612	575	175	60	.08058	8.058
50K	846	305	95	35	.3625	7.251
100K	621	230	70	25	.7453	7.453
500K	311	115	35	12.5	3.7355	7.471
1M	233	85	26	9.5	7.3536	7.354
<u>$R = 10K\Omega, C = 100 \mu F$</u>						
100K	602	220	64	22.5	.8023	8.024
500K	311	115	34	12.5	3.736	7.471
1M	229.5	85	25.5	9.1	7.608	7.608
5M	122	50	13	4.9	33.261	7.313
10M	86.5	31.5	10	3.5	74.141	7.414

^aVoltages in millivolts

^bThe instantaneous resistance r_T in megaohms

types were unavailable. Note that only the frequency of oscillation and not the amplitude is affected by varying the product RC .

Two thyristors were used to obtain data. Each has similar characteristic v - i curves. These are shown in figure 4.11. As in previous analysis, equations of the form $i=Cv^N$ describe these curves. The data are given in table 4.2

The experimental data confirms the analysis within experimental tolerances. The oscillator operated properly between the given values of R_1 in the table. Beyond these values, the circuit would not oscillate. The reason might be that the thyristor has some static resistance at a negligibly small current. Placing the thyristor in series with a one-megaohm resistor and measuring the voltage drop across the thyristor confirms this fact. The actual v - i characteristic curve does not match the modeled curve since the latter has an infinite slope at the origin and hence an infinite value of r_T there.

The important relationship is the value of r_T/R_1 . For a limited range of values of R_1 , the experimental circuit demonstrates that r_T/R_1 evaluated at the peak value of the state variable is a constant. Another RC circuit will be examined to see if this relationship also applies.

Fourth-Order Oscillator

A fourth-order RC phase shift oscillator similar to the third-order one previously discussed is now analyzed. The complete circuit is shown in figure 4.12. Except for the additional RC ladder, this circuit is the same as the third-order oscillator previously considered.

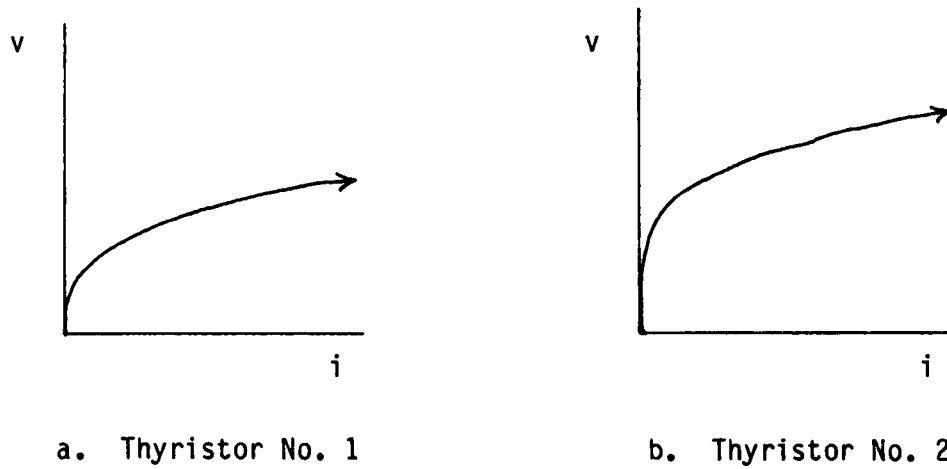


Figure 4.11 Thyristor Characteristic Curves

TABLE 4.2

Peak Values of State Variable V_3 For Each Thyristor Used

<u>Thyristor No. 1</u>			
$R = 100\Omega$		$R = 1K\Omega$	
$R_1 (K\Omega)$	$ V_3 (mV)$	$R_1 (K\Omega)$	$ V_3 (mV)$
2.25	3500	1.92	600
3.08	3000	4.10	500
6.06	2000	9.73	400
11.8	1000	17.8	300
20.3	500	30.5	200
27.2	200	43.0	100
30.4	100	50.4	50
35.3	40	51.8	35
<u>Thyristor No. 2</u>			
13.85	2000	12.5	1000
18.75	1000	27.0	600
20.7	500	38.5	400
22.8	200	47.5	200
24.9	100	51.6	100
31.5	40	53.8	50

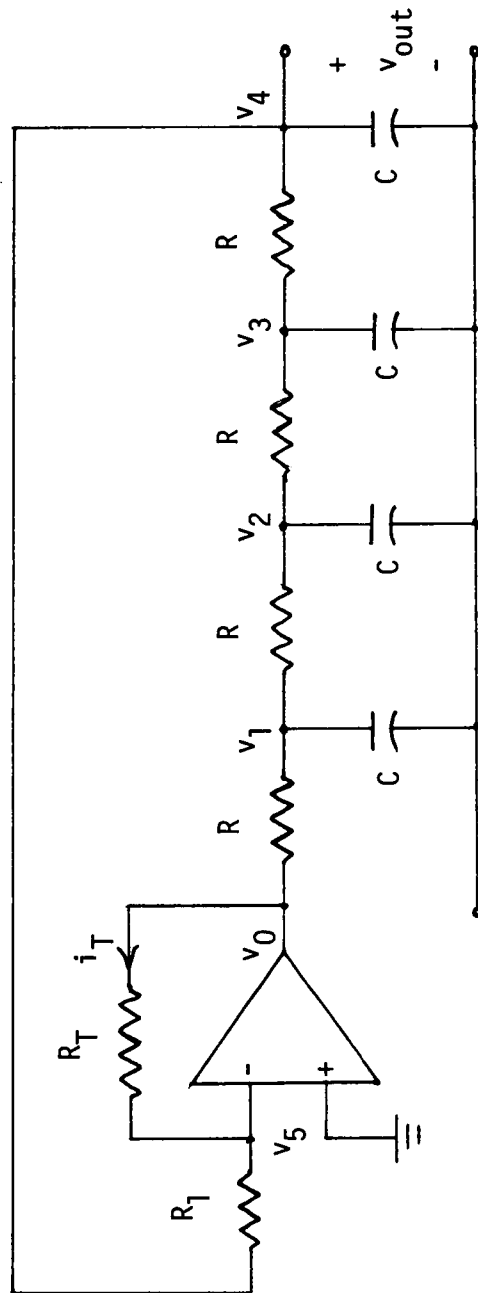


Figure 4.12 Fourth-Order Oscillator Model

Since the analysis of the fourth-order circuit proceeds similarly to the third-order case, many of the details will be omitted so that the conclusions may be emphasized.

Examining the circuit yields the following state equations assuming $RC=1$

$$\dot{v}_1 = -2v_1 + v_2 + v_0 \quad (4.38)$$

$$\dot{v}_2 = v_1 - 2v_2 + v_3 \quad (4.39)$$

$$\dot{v}_3 = v_2 - 2v_3 + v_4 \quad (4.40)$$

$$\dot{v}_4 = v_3 - (1 + R/R_1)v_4 \quad (4.41)$$

For a nonlinear device placed at R_T , v_0 is related to v_4 as

$$v_0 = - \frac{|v_4|^{1/N}}{R_1 G} \quad (4.42)$$

Substituting equation 4.42 into 4.38 gives the four state equations of the fourth-order nonlinear RC phase shift oscillator.

Since a phase-space plot of a fourth-order system is meaningless, direct computer solution of one state variable was achieved. The peak value of v_4 was measured for different values of R_1 and these data are given in table 4.3. Also given in the table is the calculations for r_T/R_1 . Just as was the case for the second and third-order oscillators, r_T/R_1 or the instantaneous gain of the op-amp is a constant.

To compare this value of r_T/R_1 to the gain of the op-amp in the linear fourth-order oscillator, cascade analysis will be used to analyze the linear analogous circuit. The RC ladder network used in

the oscillator has a transfer function which is calculated exactly as the third-order ladder. Referring to figure 4.13, mesh analysis yields the equation for I_4 as

$$I_4 = \frac{\begin{vmatrix} R+1/sC & -1/sC & 0 & V \\ -1/sC & R+2/sC & -1/sC & 0 \\ 0 & -1/sC & R+2/sC & 0 \\ 0 & 0 & -1/sC & 0 \end{vmatrix}}{\begin{vmatrix} R+1/sC & -1/sC & 0 & 0 \\ -1/sC & R+2/sC & -1/sC & 0 \\ 0 & -1/sC & R+2/sC & -1/sC \\ 0 & 0 & -1/sC & R+2/sC \end{vmatrix}} \quad (4.43)$$

The algebra becomes unwieldy and is omitted. Since $V_o = I_4 \cdot 1/sC$, the expression for the transfer function is

$$\frac{V_o}{V_i} = \frac{1}{(RCs)^4 + 7(RCs)^3 + 15(RCs)^2 + 10RCs + 1} \quad (4.44)$$

The oscillator consists of an amplifier cascaded with the RC ladder network above. The characteristic equation of the cascaded system is the same as for the other RC oscillators previously discussed and is

$$1 - KG(s) = 0 \quad (4.44)$$

where K is the gain of the amp and $G(s)$ is the transfer function of the RC ladder network. Substituting equation 4.44 into 4.45 gives

$$0 = 1 - \frac{K}{(RCs)^4 + 7(RCs)^3 + 15(RCs)^2 + 10RCs + 1}$$

$$\text{or} \quad (RCs)^4 + 7(RCs)^3 + 15(RCs)^2 + 10(RCs) + (1-K) = 0 \quad (4.46)$$

Oscillations occur when the roots of equation 4.46 lie on the $j\omega$ -axis.

Substituting $s=j\omega$ in equation 4.46 and separating into real and imaginary parts yields

$$[(RC\omega_0)^4 - 15(RC\omega_0)^2 + (1-K)] + j[10RC\omega_0 - 7(RC\omega_0)^3] = 0 \quad (4.47)$$

The real and imaginary parts of equation 4.47 must both equal zero. This gives two equations in ω_0 . Solving the imaginary part for the frequency ω_0 results

$$\omega_0 = \sqrt{\frac{10}{7}} \frac{1}{RC} \quad (4.48)$$

Substituting this result into the real part of 4.47 gives the value of K as

$$K = -18.4 \quad (4.49)$$

Both of these results agree with tabulated data given but not derived in reference [6, p. 185].

Results and Discussion

Table 4.3 shows a constant instantaneous gain of approximately one-fourth the value of the gain in the linear version of this oscillator. In each oscillator considered, the linear model initially examined has an op-amp with a specified gain. In other words, the op-amp must have resistors with a specified ratio. For the Wein-bridge oscillator, this value was two. For the third and fourth-order phase shift oscillators, these values are 29 and 18.4 respectively. In each case, the instantaneous gain for the analogous nonlinear case was one-fourth the value of the linear respective model.

TABLE 4.3
State Variable Amplitude Data
 $R = 1K\Omega$, $RC = 1$

R_1	$ V_4 $ (mV)	r_T	r_T/R_1
10K	140	4.45×10^4	4.45
50K	65	2.35×10^5	4.70
100K	48	4.72×10^5	4.72
500K	27	2.17×10^6	4.35
1M	19	4.99×10^6	4.91

A few factors should be considered before attempting an explanation. First, the factor of one-fourth characterizes a particular thyristor. The thyristors used in the laboratory did not give this value. (See Table 4.3.) Second, the instantaneous resistance r_T changes during oscillation. The ratio of r_T/R_1 calculated occurs at the peak value of the state variable considered which allows the prediction of the amplitude of oscillation.

The next chapter attempts an explanation for this particular relationship discussed above. Using describing function analysis, a linearized third-order differential equation with constant coefficients replaces the nonlinear third-order equation discussed previously. The same method can be applied to higher order systems. An examination of the characteristic equation of this linearized differential equation hopefully will give insight to the idea of instantaneous gain.

CHAPTER 5

DESCRIBING FUNCTION ANALYSIS

General

The limited usefulness in applying exact methods to nonlinear engineering problems has been continually emphasized. Approximate methods of analysis utilizing assumed solutions describe the behavior of nonlinear systems. This so-called "describing function" method developed independently by Tustin, Goldfarb, Oppelt and Kochenberger [1,3] is based upon the method of equivalent linearization of Krylov and Bogoliubov [1,2,3]. The method is limited to certain classes of problems in nonlinear systems. Fortunately, systems exhibiting free oscillations of the type considered in this work are particularly suitable.

Chapter 4 considered nonlinear oscillators of order three and four. Nonlinear differential equations were derived for each system, but these equations were not solved explicitly. If these equations can be linearized, then linear systems analysis can give solutions. These solutions to the easier linear problem then can be compared to previous graphical solutions. Explanations for the discrepancy between the gain relationships of the linear and nonlinear op-amp oscillator versions are realized by examining the linear and linearized characteristic equations respectively. Utilizing describing function analysis in this manner verifies previous work.

The Method

The describing function method relies upon the assumption that the system considered can be separated into linear and nonlinear functional blocks as shown in figure 5.1. Also assumed is the input to the nonlinear block $x(t)$ must be sinusoidal, $x = A \sin \omega t$. This is equivalent to stating that the input to the nonlinear device must be oscillatory [1, p. 122; 2, p. 344]. The nonlinear block is amplitude sensitive and insensitive to frequency but the linear block is just the opposite.

Justifying the above criteria mathematically is complicated, but reference [1] gives applicability conditions in precise mathematical detail. It must be noted that some a priori knowledge of the system is necessary. Analysis in previous chapters proved the systems considered were oscillators. This essentially satisfies the applicability conditions for practical purposes.

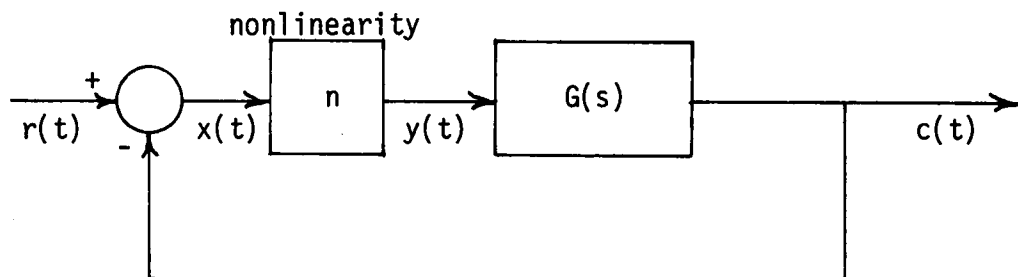


Figure 5.1 Block Diagram Of Nonlinear Control System

In general, the output y of the nonlinear block n can be expressed as a function of the input x

$$y = f(x) \quad (5.1)$$

and the functional relationship can be further expressed by dividing the function into so-called linear and distortion terms [2, p. 344]

$$y = K_{eq}x + f_d(x) \quad (5.2)$$

Since the input x is assumed sinusoidal, the terms on the right-hand side of equation 5.2 is the fundamental component of the output or the equivalent gain and a distortion component. The term $K_{eq}x$ is called the describing function of the nonlinearity $f(x)$ [2, p. 344].

The function $f(x)$ can be expressed as a Fourier series since it is periodic function of time. If the distortion term is neglected, then an approximate of $f(x)$ is derived by considering only the first few terms of the Fourier series which it represents

$$y(t) = f(x(t)) = a_0 + \sum_{n=1}^{\infty} (a_1 \sin n\omega t + a_2 \cos n\omega t) \quad (5.3)$$

If $y = A \sin \omega t$, then the normalized coefficients can be written as ($\omega t = \theta$)

$$a_0 = \frac{1}{2\pi} \int_0^{2\pi} f(A \sin \theta) d\theta \quad (5.4)$$

$$a_1 = \frac{1}{\pi A} \int_0^{2\pi} f(A \sin \theta) \sin \theta d\theta \quad (5.5)$$

$$a_2 = \frac{1}{\pi A} \int_0^{2\pi} f(A \sin \theta) \cos \theta d\theta \quad (5.6)$$

The nonlinear characteristic considered thus far is symmetric with respect to the origin; therefore, the constant term a_0 is identically zero. The describing function in a rectangular coordinate system is

$$a_1(A) + ja_2(A) = K_{eq}$$

The applicability of the above analysis to the nonlinear oscillator system considered in this work deserves several comments. Examining figure 5.1, one sees that the output of the nonlinear block $y(t)$ is periodic but nonsinusoidal. This function is composed of the fundamental and higher harmonics. If the block $G(s)$ exhibits a low-pass characteristic, then the output of $G(s)$ is sinusoidal as a good approximation. The blocks for $G(s)$ considered in this paper exhibit low-pass characteristics so the approximation that $x(t)$ is sinusoidal is realistic.

To demonstrate how the describing function is obtained, consider a simple example shown in figure 5.2.

The output is the square wave as shown in (c). Other outputs from more complicated nonlinearities can be calculated numerically. Here, the coefficient $a_1(A)$ is

$$\begin{aligned} a_1(A) &= \frac{1}{\pi A} \int_0^{2\pi} f(A \sin \theta) \sin \theta d\theta \\ &= \frac{1}{\pi A} \int_0^{2\pi} K \sin \theta d\theta + \frac{1}{\pi A} \int_0^{2\pi} -K \sin \theta d\theta \end{aligned}$$

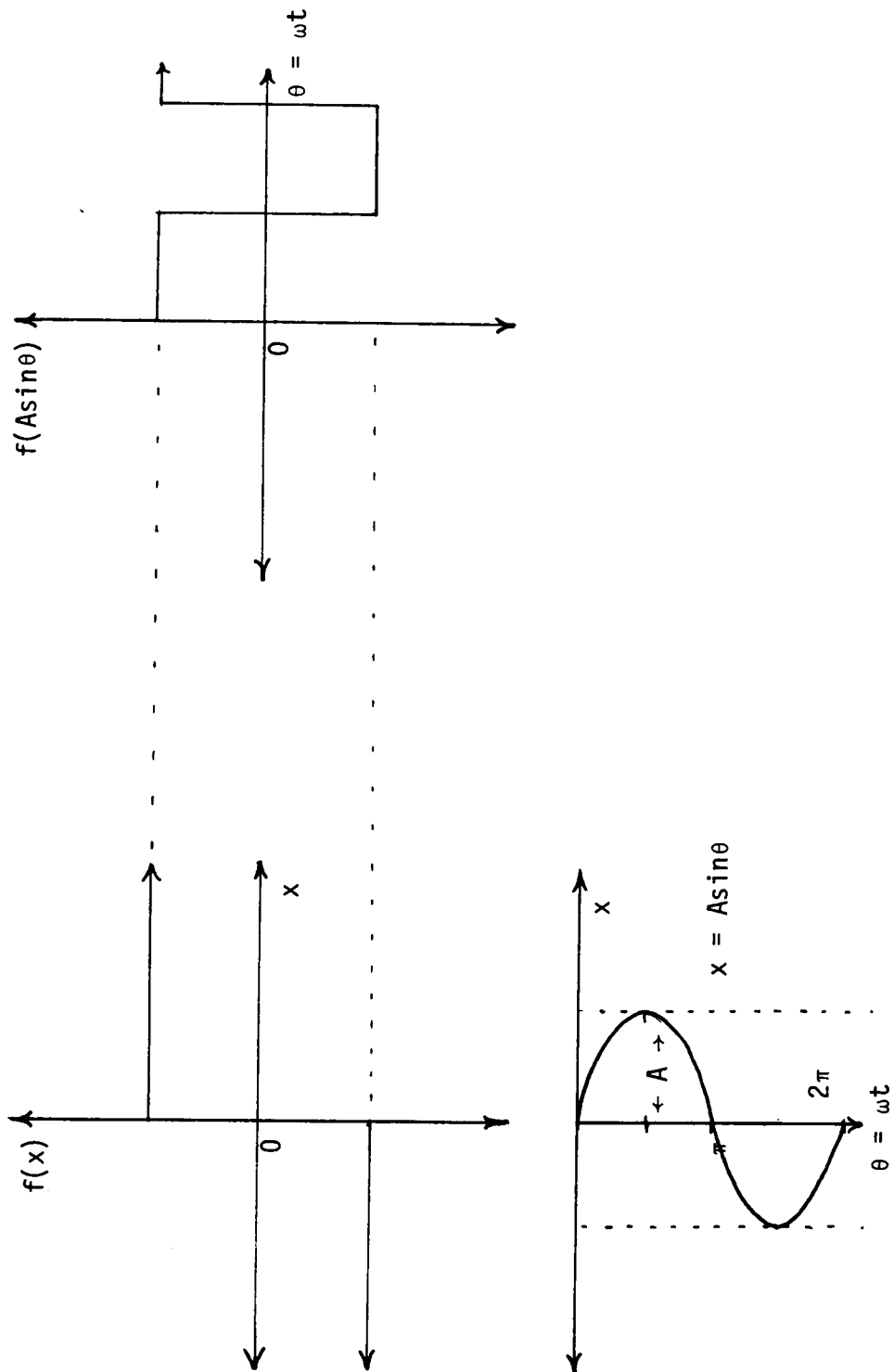


Figure 5.2 Construction of the Describing Function

$$= \frac{4K}{\pi A}$$

This result is only a function of the nonlinearity, so calculations can be done for types of nonlinearities to form a table of characteristics. These characteristics are unchanging since they are functions of the nonlinearity only.

Once the coefficients a_0 , a_1 , a_2 are determined, the nonlinear differential equation can be rewritten as a linear differential equation with constant coefficients. Consider the third-order oscillator discussed in Chapter 4. The characteristic equation was

$$s^3 + 5s^2 + 6s = 1 + KF(x) = 0$$

where $F(x)$ is the nonlinearity and K is a linear factor. The block diagram of figure 5.1 yields the transfer function

$$\frac{C(s)}{R(s)} = \frac{nG(s)}{1 + NG(s)} \quad (5.8)$$

with $1 + nG(s) = 0 \quad (5.9)$

The RC ladder has $G(s)$ as

$$G(s) = \frac{1}{(RCs)^3 + 5(RCs)^2 + 6RCs} = 1 \quad (5.10)$$

as discussed in the last chapter. If n is the nonlinearity in equation 5.9, then substituting 5.10 into 5.9 gives

$$1 + \frac{n}{s^3 + 5s^2 + 6s + 1} = 0 \quad (RC = 1)$$

$$s^3 + 5s^2 + 6s + 1 + n = 0 \quad (5.11)$$

which is the same result from Chapter 4. Hence, the block diagram of figure 5.1 accurately depicts the feedback oscillator in Chapter 4. For a given value of R_1 , equation 5.7 applies. Different values for R_1 change the value of K .

Rewriting equation 5.7 as a linearized third-order equation gives

$$s^3 + 5s^2 + 6s + 1 + K(a_1(A) + a_2s)x = 0 \quad (5.12)$$

with a_1 , a_2 defined by equations 5.5 and 5.6. The amplitude A and frequency ω are constants once the oscillations are established even though they are not given initially. Thus, periodic solutions to the system are realized by examining imaginary axis roots to the linear differential equation 5.12.

Application to Third-Order System

The utility of the describing function method now reduces to an algebraic problem in finding values of A and ω which enable equation 5.12 to describe an oscillatory system whose output is $x(t) = A \sin \omega t$. One particularly useful method is called the parameter plane method and is described by Siljak [1]. These techniques are significant in that they make the describing function method a complete approximate nonlinear analysis within itself. As discussed previously, a priori

knowledge of the system is necessary, i.e., one must assume the system under study does oscillate.

In order to derive the describing function for the third-order oscillator system currently considered, equation 5.4 through 5.6 must be evaluated. The nonlinear characteristic was given in chapter three and is

$$F(x) = Cx^n \quad (5.13)$$

where C and n are real. Since this nonlinear function is symmetric with respect to the origin and single valued, the coefficients a_0 and a_2 are identically zero. If n is an integer, then the nonlinearity is a polynomial, and reference [1] gives the resulting coefficient from equation 5.5 as

$$a_1 = \frac{4KA^{n-1}}{\sqrt{\pi}} \int_0^{\pi/2} \sin^{n+1} \theta d\theta$$

$$a_1 = \frac{2K}{\sqrt{\pi}} A^{n-1} \frac{\Gamma[(n+2)/2]}{\Gamma[(n+3)/2]} \quad (5.14)$$

where Γ is the gamma function. Unfortunately, if n is real, no simple analytic method is available to evaluate equation 5.5. Approximate methods must be used.

Reference [1] gives an approximate method using a graphical procedure for calculating the describing function. The procedure is utilized here. Approximating the exact formula 5.5 by utilizing the expression

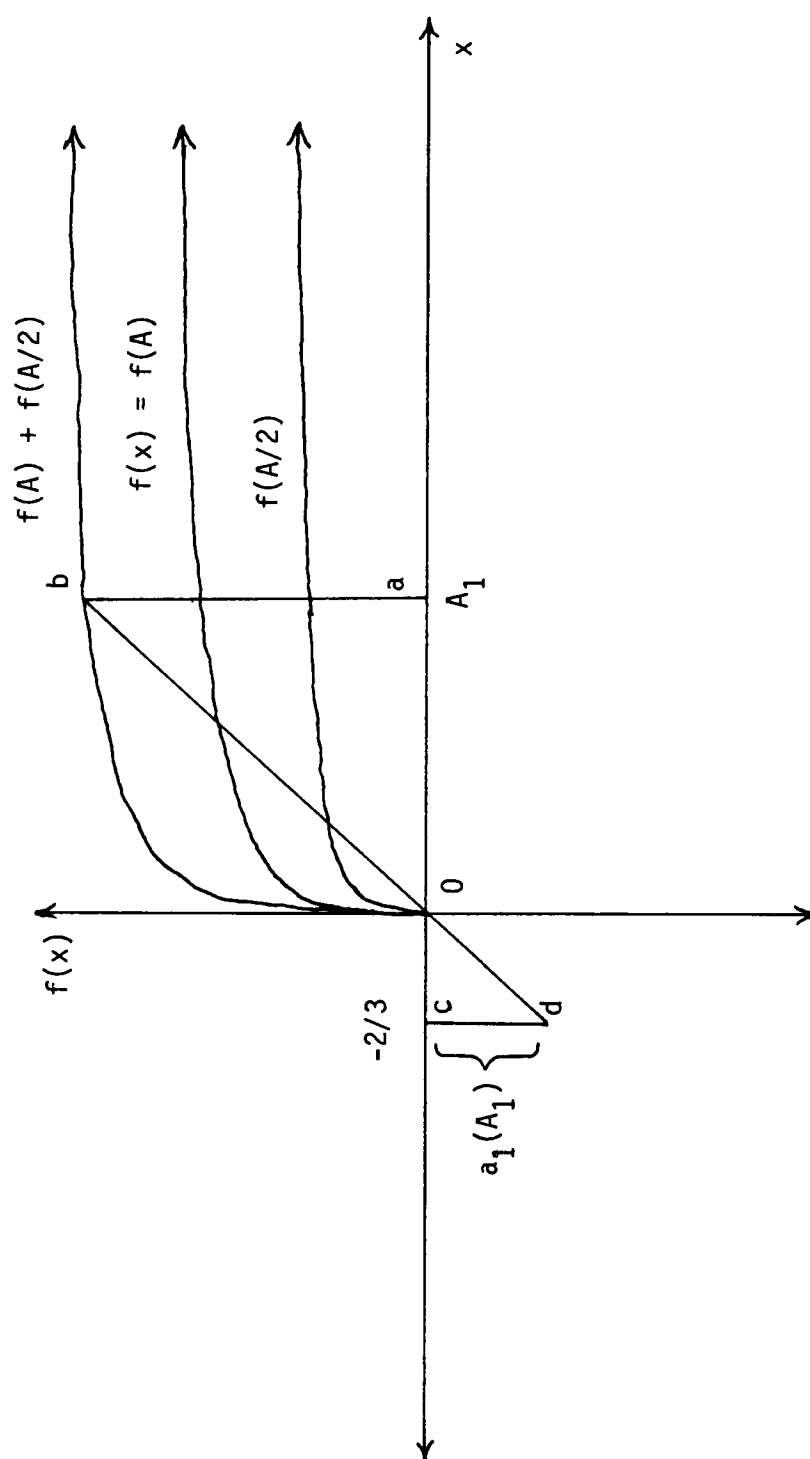


Figure 5.3 A Graphical Technique for Approximating the Describing Function

$$a_1(A) = \frac{2}{3A} \left(F(A) + F(A/2) \right) \quad (5.15)$$

enables one to evaluate $a_1(A)$ graphically from a given function $F(x)$.

Figure 5.3 shows a plot of the function representing the nonlinearity of equation 5.13 with $n = 0.3$, $C = 1.0$. If the plot of $F(x) = F(A)$ is added to $F(A/2)$, then the curve $F(A) + F(A/2)$ can be considered. Two lines are drawn parallel to the $F(x)$ axis, one through $x = -2/3$ and the other through $x = A_1$. The geometry of the diagram (similar triangles) gives

$$cd = a_1(A_1) = \frac{2}{3} \left(\frac{F(A_1) + F(A_1/2)}{A_1} \right) \quad (5.16)$$

Using this procedure, an approximate curve $a_1(A)$ can be constructed.

As mentioned previously, the describing function method yields a harmonic linearization of a nonlinear differential equation as a function of A where A is the coefficient from the assumed solution $x = A \sin \omega t$. Other algebraic methods are needed to yield a value for A which gives symmetric oscillations. These methods will not be discussed since the purpose of this chapter is to gain insight into previous analysis and not to pursue a complete autonomous describing function technique.

To verify previous work, consider values of A yielding a solution to equation 5.16. A value taken from Table 4.2 for $R = 100\Omega$ and R_1 being $10K\Omega$ for $v_3 = x = A_1$ gives

$$a_1(0.070) = 7.76 \quad (5.17)$$

Using this value in

$$s^3 + 5s^2 + 6s + 1 + K(a_1(A)) = 0 \quad (5.18)$$

and a value of K such that for $R_1 = 10K\Omega$,

$$K = \frac{59.4}{(R_1)^{0.3}} = 3.749 \quad (5.19)$$

and this finally yields

$$K(a_1(A)) = 29 \quad (5.20)$$

which corresponds with the linear analysis of the third-order oscillator in Chapter 4.

Results and Discussion

If the nonlinearity in conjunction with the associated constant K of equation 5.12 is considered analogous to the gain K of equation 4.23, then the harmonic linearization technique accurately follows the model of the linear oscillator in Chapter 4. Linear techniques could be used with the linearized characteristic equation to derive the gain, frequency of oscillation and other parameters.

The main goal of this chapter was not met. Describing function analysis did not give any additional information concerning why, for the specific nonlinear device used, there was a factor of one-fourth in each class of RC oscillator analyzed.

CHAPTER 6

CONCLUSION AND RECOMMENDATIONS

Conclusion

Using phase space analysis together with the state variable method and a digital computer, the engineer can analyze models of nonlinear oscillators. Exact analysis is impossible except for the linear versions of the oscillator models discussed in the third and fourth chapters. The phase space technique of analysis yields accurate information, but the method cannot be readily applied to systems higher than three. A phase space trajectory has little meaning for these higher systems.

Other more elegant approximate nonlinear analytical techniques are available. From a practical consideration, the engineer can apply the techniques discussed in this work easily and efficiently especially with wide availability of minicomputers and graphical output devices. Large amount of computer memory and nearly limitless available processing time enable engineers to apply approximate methods of analysis more effectively than before.

Design is the ultimate goal in engineering. The design criterion discussed for the Wein-bridge oscillator also was shown to apply to the RC phase shift oscillators. For a particular nonlinear device used, the oscillator can be designed to yield a peak value of the output state variable chosen simply by using the appropriate value of R_1 .

Recommendations

During the preparation of this work, several areas deserve consideration for further study. The first area involves the factor of one-fourth which occurred in each class of RC oscillator model. A satisfactory explanation for this factor was not found. The linearized differential equation given by utilizing the describing function method offered an alternate method for analyzing the nonlinear oscillator, and it served as verification of previous analysis; but no explanation for the factor r_T/R_1 as it relates to the classes of nonlinear oscillators was found.

Another area worth pursuing is in the describing function method utilizing the parameter plane technique. It was mentioned that this was a completely self-sufficient nonlinear analysis technique. Parameter plane analysis was not applied in this work. A recommended study might include this technique and comparing the results to those in this work would prove interesting.

Nonlinear analysis is neither straightforward nor simple. Approximate methods utilizing digital computers greatly simplifies this process. Hopefully, the reader will see the value of this technique as it is applied to nonlinear oscillator models.

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LIST OF REFERENCES

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APPENDICES

APPENDIX A

THE PROGRAM FOR GENERATING THE PHASE PLANE PLOT OF CHAPTER 3

```

10 C INITIALIZE VARIABLES AND AGP
20 C
30 $OPTION SHORT INTEGERS
40 REAL K
50 CALL JBEGN
60 CALL JDINT (1,6,6Hwsp1pt,13,13H/dev/pl7470a,0)
70 CALL JWON (1)
80 CALL JDLIM (1,50.7,257.1,38.8,169.0)
90 CALL JWIND (-3.0,3.0,-3.0,3.0)
100 C
110 C DRAW THE AXES
120 C
130 CALL J2MOV (-10.0,0.0)
140 CALL J2DRW (10.0,0.0)
150 CALL J2MOV (0.0,10.0)
160 CALL J2DRW (0.0,-10.0)
170 CALL J2MOV (0.0,0.0)
180 C
190 C AND LABEL THEM
200 C
210 G = -2.5
220 DO WHILE (G .LE. 2.5)
230 CALL J2MRK (G,0.0,2)
240 CALL J2MRK (0.0,G,2)
250 G = G + 0.5
260 END DO
270 C
280 C SET UP MAIN PART OF THE PROGRAM TO SOLVE THE SECOND ORDER DIFFERENTIAL EQUATION AND
290 C PLOT IT AS IT IS SOLVED

```

```

300 V1 = 2.5
310 V2 = 0.0
320 H = 0.05
330 E = 0.3
340
350 C
360 C THE CONSTANT K IS 59.4(V3/R1)**0.3 FOR R1 TAKEN AS 100K
370 C
380 K = 1.875
390 N = 0
400 CALL J2MOV (V1,V2)
410 DO WHILE (N .LE. 400)
420 IF (V2 .GE. 0.0) THEN
430     DV1 = -V1 + K*V2**E
440     V1 = V1 + H*DV1
450     DV2 = -V1 - V2 + K*V2**E
460     V2 = V2 + H*DV2
470 ELSE
480     DV1 = -V1 + K*(-(ABS(V2)**E))
490     V1 = V1 + H*DV1
500     DV2 = -V1 - V2 + K*V2**E
510     V2 = V2 + H*DV2
520 ENDIF
530 C
540 C NOW CALL THE DRAWING SUBROUTINE TO CONNECT THE NEXT POINT THUS MAKING A SMOOTH CURVE
550 C
560 CALL J2DRW (V1,V2)
570 N = N + 1
580 END DO
590 C
600 C NOW TERMINATE AGP AND END THE FORTRAN PROGRAM
610 C
620 CALL JWOFF (1)
630 CALL JWEND (1)
640 CALL JEND
      END

```

APPENDIX B

THE PROGRAM FOR GENERATING THE PHASE SPACE PLOTS OF CHAPTER 4

```

10 C
20 C
30 $OPTION SHORT INTEGERS
40 REAL K
50 CALL JBEGN
60 CALL JDINT (1,6,6Hwsp1pt,13,13H/dev/pl7470a,0)
70 CALL JWON (1)
80 CALL JDLIM (1,50.7,257.1,38.8,169.0)
90 C
100 C THE VIEW REFERENCE POINT IS BY DEFAULT AT THE ORIGIN (0,0,0). NOW SET THE VIEW PLANE
110 C DISTANCE FROM THE VIEW REFERENCE POINT
120 C
130 CALL JVDIS (0.1)
140 C
150 C THE VIEWPLANE NORMAL VECTOR IS BY DEFAULT THE VECTOR (0,0,1). THE VIEWPLANE IS
160 C ORIENTED WITH THE Y-AXIS UP. NOW THE TYPE OF PERJECTION MUST BE SPECIFIED. WHAT IS
170 C SELECTED IS A PARALLEL PROJECTION WITH THE CENTER OF THE PROJECTED PLANE AT THE POINT
180 C (0.3,0.3,-1.0). THE OPCODE 0 IS FOR PARALLEL PROJECTION.
190 C
200 CALL JPROJ (0,0.3,0.3,-1.0)
210 C
220 C NOW THE 3-D WINDOW MUST BE SET TO INCLUDE THE INFORMATION DESIRED.
230 C
240 CALL JWIND (-.0183,0.177,-0.183,0.177)
250 C
260 C NOW DRAW A SET OF AXES WITH THE Z-AXIS GOING INTO THE PAGE.
270 C
280 CALL J3MOV (0.0,0.0,0.2)
290 CALL J3DRW (0.0,0.0,0.0)

```

```

300 CALL J3MOV (0.0,0.2,0.0)
310 CALL J3DRW (0.0,0.0,0.0)
320 CALL J3MOV (0.2,0.0,0.0)
330 CALL J3DRW (0.0,0.0,0.0)
340
350 C
360 C
370 C
380 C
390 C
400 C
410 C
420 C
430 C
440 C
450 C
460 C
470 C
480 C
490 C
500 C
510 C
520 C
530 C
540 C
550 C
560 C
570 C
580 C
590 C
600 C
610 C
620 C
630 C
640 C

      CALL J3MOV (0.0,0.2,0.0)
      CALL J3DRW (0.0,0.0,0.0)
      CALL J3MOV (0.2,0.0,0.0)
      CALL J3DRW (0.0,0.0,0.0)

      NOW MARK THE AXES

      A = 0.0
      DO WHILE (A .LE. 0.2)
        CALL J3MRK (A,0.0,0.0,2)
        CALL J3MRK (0.0,A,0.0,2)
        CALL J3MRK (0.0,0.0,A,2)
        A = A + 0.02
      END DO

      NOW THE MAIN PART OF THE PROGRAM

      V1 = 0.004
      V2 = 0.003
      V3 = 0.002
      N = 0
      H = 0.01
      R = 100.0
      R1 = 500.0E03
      K = 59.4*(1.0/R1)**0.3
      XK = R/R1
      E = 0.3
      V = 0.0
      CALL J3MOV (V1,V2,V3)
      DO WHILE (N .LE. 1500)
        IF (V3 .LT. 0.0) THEN
          DV1 = -2.0*V1 + V2 + K*ABS(V3)**E
        ELSE
          DV1 = -2.0*V1 + V2 + (-K*V3**E)
        ENDIF
      ENDIF

```

```

650      V1 = V1 + H*DV1
660      DV2 = V1 - 2.0*V2 + V3
670      V2 = V2 + H*DV2
680      DV3 = V2 - (1.0 + XK)*V3
690      V3 = V3 + H*DV3
700      CALL J3DRW (V1,V2,V3)
710      N = N + 1
720      END DO
730      C
740      C
750      C
760      CALL JWOFF (1)
770      CALL JWEND (1)
780      CALL JEND
790      END

```

NOW END AGP AND TERMINATE THE MAIN PROGRAM

VITA

The author was born in Chester, Pennsylvania, on April 2, 1953. Being the son of a career Army officer meant moving frequently. He spent the first eighteen years of his life in several states and in Europe finally graduating from West Springfield High School in Springfield, Virginia. He attended undergraduate school at the United States Military Academy at West Point, New York. Graduating with a Bachelor of Science degree in 1975, he was commissioned a Second Lieutenant in the United States Army.

Seven and a half years of commissioned service included assignments at Fort Campbell, Kentucky, Fort Rucker, Alabama and Fort Bragg, North Carolina. He is a graduate of the Parachutist School and the Officer Rotary-Wing Aviator Course.

On June 19, 1982, he was married to the former Dianne Hutson. He left the service in the Fall of 1982, and he attended The University Of Tennessee at Knoxville graduating in August, 1985 with a Master Of Science degree with a major in electrical engineering. He is a student member of The Institute Of Electrical and Electronic Engineers and a member of the IEEE Computer Society.