To the Graduate Council:

I am submitting herewith a dissertation written by Steven A. Pax entitled "Sewings of Closed Four-Cell Complements." I recommend that it be accepted in partial fulfillment of the requirements for the degree of Doctor of Philosophy, with a major in Mathematics.

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SEWINGS OF CLOSED FOUR-CELL COMPLEMENTS

A Dissertation

Presented for the

Doctor of Philosophy

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ABSTRACT

This dissertation makes four contributions to the theory of four dimensional sewings. First, two new concepts are introduced and explored extensively; one is shown to lead to a classification of closed four-cell complements which is analogous to the classification of closed n-cell complements $(n > 5)$ by types. Second, we find wide classes of closed four-cell complements for which the identity sewing yields S^4. Third, we show that suspensions of closed three-cell complements are universal, and we make other investigations into universality. Fourth and last, we adapt five pathological examples from the theory of high dimensional sewings to the case $n = 4$.

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CHAPTER I

PRELIMINARIES

Introduction

The main problem of topological embeddings may be briefly stated in this way. If $A \subset X$ and $B \subset Y$ are topological spaces with $X \approx Y$ and $A \approx B$, when is there a homeomorphism $h: X \to Y$ such that $h(A) = B$? When h exists we say that the pair (X, A) is homeomorphic to the pair (Y, B). This is about as fundamental a question as there is in the field of topology, and people have worked on it in one form or another just about ever since topology came into being. The most conspicuous case of the problem, when $X \approx Y \approx S^n$ and $A \approx B \approx S^{n-1}$, remains unsolved for each $n \geq 3$.

In only one sense has this case been successively resolved; M . Brown in [14] shows that such a homeomorphism exists whenever the embeddings of S^{n-1} in S^n are bicollared. The case when they are not bicollared has been mostly unaddressed except in one important arena. We construe the study of crumpled cubes and sewing spaces by Eaton [30], Hosay [38], Lininger [43], Daverman [26, 27], and Cannon-Daverman [18] as work directed towards this problem because it aims at classifying the embeddings of S^{n-1} in S^n ; each deals with crumpled cubes or sewings.

A <u>crumpled</u> <u>n-cube</u> C is a space homeomorphic to the union of an $(n - 1)$-sphere in S^n and one of its complementary domains. The <u>interior</u> of C, $\mathrm{Int}C$, is the set of points at which C is an n-manifold without boundary, and the <u>boundary</u> of C, $\mathrm{Bd}C$, is $C-\mathrm{Int}C$.

When crumpled cubes C_0 and C_1 and a homeomorphism $h\colon \mathrm{Bd}C_0 \to \mathrm{Bd}C_1$ (called a sewing) are given, the quotient space of the disjoint union of C_0 and C_1 by the relation generated by h ($x \in \mathrm{Bd}C_0$ is identified with $h(x) \in \mathrm{Bd}C_1$) is called a sewing space and is denoted by $C_0 \cup_h C_1$; the image of $\mathrm{Bd}C_0$ in $C_0 \cup_h C_1$ is called the seam of the sewing and we denote it by $\pi(\mathrm{Bd}C_0)$. When $C_0 \cup_h C_1 \approx S^n$, the sewing is said to yield S^n. If $C_0 \approx C_1$ and h equals the identity, the sewing is called the identity sewing.

To see how crumpled cubes and sewing spaces help classify the embeddings of S^{n-1} in S^n, consider the case where $n = 3$, which is the most thoroughly resolved. When a crumpled n-cube is embedded in S^n so that the closure of its complement is an n-cell, that image is called a closed n-cell complement (this is different from usual - in our usage the term is not a topological property). In [38, 43], it is shown that each crumpled 3-cube embeds as a closed 3-cell complement. Now if $f\colon S^2 \to S^3$ is an embedding, it determines two crumpled cubes, which by the preceding must be homeomorphic to closed 3-cell complements, C_0 and C_1, and a sewing $h\colon \mathrm{Bd}C_0 \to \mathrm{Bd}C_1$ such that the pairs $(S^3, f(S^2))$ and $(C_0 \cup_h C_1, \pi(\mathrm{Bd}C_0))$ are homeomorphic. Thus f is classified by two simpler embeddings of S^2 in S^3 (those giving C_0 and C_1) and the homeomorphism h. Once one perceives that f depends upon more than just C_0 and C_1, one naturally wonders if given C_0 and C_1 whether or not every sewing between them corresponds to an embedding in this fashion; or equivalently, whether all sewings between C_0 and C_1 yield S^3. This is the question addressed in [30]. There, the sewings that yield

S^3 are completely characterized. (Before this, Ball in [6] had produced a closed 3-cell complement C_0 and a sewing h such that $C_0 \cup_h C_0 \not\approx S^3$.) Thus the set of embeddings of S^2 in S^3 is smaller than might be expected and is classified up to embeddings of B^3 in S^3.

When $n > 4$, nearly as much is known. Daverman in [27] shows that each crumpled n-cube $(n > 4)$ embeds as a closed n-cell complement, and in [26] he conducts a lengthy investigation into the matter of which sewings yield S^n, solves it for certain kinds of closed n-cell complements and sewings, and shows in general that the problem is more complex than it is when $n = 3$.

This work is concerned with the case where $n = 4$. It is notoriously difficult to do geometric topology in E^4; moreover, the results of Siebenmann [49] and Freedman [33] which establish the relative complexity of E^4 to E^n $(n > 4$ or $n = 3)$ and the general lack of geometric tools in E^4 have led most researchers to concentrate their efforts elsewhere. It is not known if a crumpled 4-cube embeds in S^4 as a closed 4-cell complement (as a CFC - we shall use this acronymn freely) or when a sewing of CFC's yields S^4. But first steps have been taken in the second case. In [18], it is shown how a technique from [26] might be applied when $n = 4$, and some consequences are obtained. By the time that paper was complete, my advisor, R.J. Daverman, had realized that, given an appropriate setting, many of the results in [26] might be carried over to the case where $n = 4$.

The present work is devoted to describing an appropriate setting and establish as many of those results from [26] as our tools allow.

I was encouraged and sustained throughout by a great many people whom I thank most heartily. I am especially indebted to John Walsh for unlimited encouragement and confidence in me when I needed it most; to Charles Bass who saved me from error more than once; and, most of all, to my advisor, Robert Daverman - my good fortune in having such a gracious advisor can hardly be overstated.

<u>Summary of Results</u>

The dissertation makes four contributions to the theory of four dimensional sewings.

First, we define two new concepts - Geometric 1-ULC sets and flat sections. We say F is a <u>Geometric 1-ULC</u> (1-GULC) <u>set</u> for a CFC C if $F \subset \mathrm{Bd}C$ and $S^4 - (C - F)$ has a certain 2-polyhedra pushing property (see page 11). This notion is modeled after the concept in [26] and [30] of a set $F \subset \mathrm{Bd}C$ such that $F \cup \mathrm{Int}C$ is 1-ULC . It has significance only when $n = 4$. If $n \neq 4$, a set F in the boundary of a closed n-cell complement would possess this 2-polyhedra pushing property if and only if $F \cup \mathrm{Int}C$ were 1-ULC . When $n = 4$, the "only if" is trivial; the "if" is open.

A <u>section</u> for a CFC C over a set $X \subset \mathrm{Bd}C$ is a CFC determined by the graph in a collar on C of a map $f\colon \mathrm{Bd}C \to [0, 1)$ such that $f^{-1}(0) = X$; a section is <u>flat</u> if the CFC so determined is a 4-cell. This concept has not been defined before, though [18], [26], and other authors make tacit use of the idea in their arguments.

We show that if F is a 1-GULC set for C and $X \subset \mathrm{Bd}C - F$, then each section for C over X is flat, and we show that if X_i is a sequence of sets in $\mathrm{Bd}C$ such that a section over each is flat, then C has a 1-GULC set F contained in $BdC - \cup X_i$. The 1-ULC taming hypothesis of [18] may be regarded as anticipating this relationship. We also suggest a classification scheme for CFC's based upon the dimension of a 1-GULC set and its embedding properties. For example, if F may be found so that it is the union of Cantor sets which are tame in BdC , we say that C has G-type at most 1 . The scheme is completely analogous to the classification of closed n-cell complements by types in [26], and like in [26], we show where various examples and various kinds of CFC's fit within it.

Finally, the technique from [26], which [18] shows how to apply when $n = 4$, is couched in the language of Geometric 1-ULC sets to produce the following geometric mismatch theorem which is similar to theorems from [18], [26], and [30] bearing like titles.

<u>Theorem</u>. <u>If</u> $F_e \subset BdC_e$ $(e = 0, 1)$ <u>is a 1-GULC set for</u> C_e <u>and</u> $h: BdC_0 \to BdC_1$ <u>is a sewing such that</u> $h(F_0) \cap F_1 = \emptyset$, <u>then</u> $C_0 \cup_h C_1 \approx S^4$.

Our second contribution consists of showing that the identity sewing of a G-type 1 CFC which is locally flat modulo a 2-dimensional set yields S^4 . The version of the technique from [26] that we use to prove this may be of interest in its own right (see Theorem 3.3, page 39). While this result is analogous to the high dimensional result that the identity sewing of a G-type 1 closed n-cell complement to itself yields S^n , the proof is conceptually different because, at the onset, we do not find disjoint 1-GULC sets in the boundary of the CFC.

Third, we show that suspensions of closed 3-cell complements are universal in the sense that each sewing of one to another CFC yields S^4. We also show that certain CFC's which are locally flat modulo a 1-dimensional set are universal in the same sense and that all sewings of certain other CFC's to G-type 1 CFC's yield S^4.

Fourth, we establish that all of the pathological examples from [26] exist via the same constructions when $n = 4$.

Our results may be briefly stated by saying that we establish all of the theory from [26] when $n = 4$ but with less generality. In many instances we were forced to add that the CFC had a degree of local flatness. (All known pathology is represented by CFC's which are locally flat modulo Cantor sets.) This led us to routinely investigate what effect the degree of local flatness had upon our theorems, and at the end, we give a summary of our results as they apply to sewings of a CFC which is locally flat modulo a 1-dimensional set.

Definitions and Notation

E^n is used throughout to denote Euclidean n-space endowed with a fixed, complete metric; we regard $E^n = E^{n-1} \times (-\infty, \infty)$ and $E^{n-1} = E^{n-1} \times \{0\} \subset E^n$. B^n (S^{n-1}) denotes the set of points in E^n of norm ≤ 1 (norm $= 1$). An n-cell ((n - 1)-sphere) is any space homeomorphic to B^n (S^{n-1}). A map is a continuous function. $I = [0, 1]$.

Each space to be considered has a metric which is always denoted by ρ, but whenever the space is a subset of E^n, ρ is to be the restriction of E^n's fixed metric. When f and g are maps from a space X to a space Y, $\rho(f, g) = \sup\{\rho(f(x), g(x)) | x \in X\}$.

When A is understood to be a subset of a metric space X , $N_\varepsilon(A)$ denotes the set of all points in X whose distance to A is less than ε , and a map $f\colon A \to X$ is called an _ε-map_ if $\rho(f, \text{inclusion}) < \varepsilon$.

A map $f\colon X \to Y$ is called an _embedding_ if it is a homeomorphism onto its image, and called a _near-homeomorphism_ if for each $\varepsilon > 0$ there exists a homeomorphism $g\colon X \to Y$ such that $\rho(f, g) < \varepsilon$. A map $f\colon X \times I \to X$ is called an _isotopy_ of X (_pseudoisotopy_ of X) if $f(x, 0) = x$ for all $x \in X$ and $f|X \times \{t\}$ is a homeomorphism for all $t \in I$ (for all $t \in [0, 1)$); we rarely refer to a pseudoisotopy by f , but rather by f_t understanding $f_t = f|X \times \{t\}$ and t to range from 0 to 1 . An _ε-isotopy_ (_ε-pseudoisotopy_) is an isotopy (pseudoisotopy) f_t such that each f_t is an ε-map. We call an ε-map $f\colon X \to X$ which is a homeomorphism (near-homeomorphism) an _ε-self-homeomorphism_ of X (_ε-near-homeomorphism_). A map $f\colon X \to Y$ is said to be _supported_ in a set U if $f|X - U$ is the identity. A pseudoisotopy f_t is said to be _supported in_ U if each f_t is. The inverse sets of a map $f\colon X \to Y$ are the sets $f^{-1}(y)$, $y \in Y$.

Besides these terms, we have occasion to use a number of technical terms from the theory of PL and topological manifolds, and from dimension theory, more than could be adequately defined here; we refer the reader to four sources for such prerequisites: Rushing [48], Hudson [39], Edwards [32], and Hurewicz and Wallman [40]. Our notation follows that of these authors except in the usage just described and in a few other ways. First, our use of "polyhedron" is unusual. We say P is a _polyhedron in_ X whenever P is a polyhedron and $P \subset X$; thus, in our usage, the Alexander Horned Square

[1] is a polyhedron in S^3. We say P is a <u>rectilinear polyhedron in</u> X whenever X is a manifold and P is a subcomplex of some abstract PL triangulation of X - perhaps different from one already mentioned or understood. We say P is a <u>PL polyhedron in</u> X when P is a subcomplex of X with respect to a subdivision of an already specified triangulation of X; if $P \subset S^n$, and no triangulation is specified, S^n is understood to be triangulated as the suspension of S^{n-1}. Second, we do not distinguish between a complex and the underlying point set. Third, we say that P <u>is an infinite k-skeleton of a PL manifold</u> M if $P = \cup R_i$ where R_i is the k-skeleton of a triangulation T_i of M such that T_{i+1} is a subdivision of T_i and mesh $T_i \to 0$. Fourth, since a CFC, C, always has a collar $e\colon \mathrm{BdC} \times I \to S^4$ on its boundary, we usually suppress the embedding e and regard $\mathrm{BdC} \times I$ as equal to the image $e(\mathrm{BdC} \times I)$.

Finally, we shall make frequent use of a famous convergence criterion due to Bing [8] and Anderson [4]; also see Edwards [32]. The proof is straightforward.

<u>Convergence Criterion 1.1</u>. <u>Let</u> X <u>and</u> Q <u>be metric spaces with</u> Q <u>complete. Suppose one constructs a sequence of proper embeddings</u> $g_i\colon X \to Q$ <u>and a sequence of maps</u> $\varepsilon_i\colon X \to (0, \infty)$ <u>such that</u>

1) $\rho(g_{i+1}(x), g_i(x)) < \varepsilon_i(x)$ <u>for all</u> x <u>in</u> X <u>and</u>
2) $\varepsilon_i(x) < \frac{1}{4}\min\{\varepsilon_{i-1}(x), \rho(g_{i-1}(x), g_{i-1}(X - N_{1/i}(x)))\}$ <u>for all</u> x <u>in</u> X.

<u>Then</u> $g = \lim g_i$ <u>exists and is an embedding</u>. ∎

Any method of constructing the sequences $\{g_i\}$ and $\{\varepsilon_i\}$ in which one obtains successively g_{i-1} , then ε_i , and then g_i and in which the procedure for finding g_i works no matter how close to zero ε_i might be, may be presumed to satisfy condition 2 - one says that ε_i may depend upon g_{i-1} . We make this presumption again and again, but always warn the reader by referencing 1.1. Notice that if $Q = X$ and is a manifold and if each g_i is onto, then g is a homeomorphism onto. When X is compact, we may take ε_i to be a constant function and consider it a number.

CHAPTER II

SUFFICIENTLY MANY ISOTOPIES

If there is one technique that could be said to dominate geometric topology in E^n $(n > 4)$, it is engulfing. A casual examination of the book by Rushing [48] is more than sufficient to establish this fact, and since its appearance, consequential theorems have continued to surface wherein engulfing is a fundamental tool; see, for example, Chernavskii [20]. Moreover, those papers which do not apply the engulfing theorems themselves are almost always found to rely upon one that does. It is this engulfing that accounts for the widespread occurence of uniform local 1-connectedness (1-ULC) in the hypotheses of high-dimensional theorems, for it is a fortuitous characteristic of reality that in E^n $(n > 4)$ forms of 1-connectivity imply forms of engulfing. Conversely, it is the forms of 1-connectivity (like 1-ULC) that account for the pervasiveness of engulfing, because 1-connectivity in all its many forms only deals with maps which are ridiculously easy to come by compared to the isotopies engulfing makes of them.

But what is a happy marriage in high-dimensions is on the rocks in E^4 where engulfing can only rarely translate 1-connectivity into isotopy. For this reason, we abandon 1-connectivity (except in the construction of counterexamples where it is used differently) during our investigation of four-dimensional sewings. It is replaced by the concept of a 1-GULC set which more directly measures, in our specific context, the number of isotopies there are of E^4. The new tool is more sterile than the high-dimensional 1-connectivity-engulfing marriage but breeds a surprisingly large family anyway. This is mostly because the concept is designed to replace engulfing in many high-dimensional arguments, but also because even without engulfing there still exists a large collection of isotopies of E^4 constructed by more fundamental methods such as pushing along a join or product.

Geometric 1-ULC Sets and Sections

Definition 2.1. Let $C \subset S^4$ be a CFC and F be a subset of BdC. F is called a geometric 1-ULC set for C, written 1-GULC set for C, if for each $\varepsilon > 0$ there exists a neighborhood V of BdC such that for any PL 2-polyhedron P in V and any compact subset D in $IntC$, there exists an ε-pseudoisotopy $h_t\colon S^4 \to S^4$ supported in $C \cup V$, satisfying $h_1(P \cup D) \subset F \cup IntC$.

Remarks. The definition may be informally paraphrased by saying "F is a 1-GULC set for C whenever $S^4 - (F \cup IntC)$ has a sufficiently strong 2-complex pushing property". The reason that we call F a geometric 1-ULC set is that in higher dimensions, the weaker condition "$F \cup IntC$ is 1-ULC" implies that for each ε the neighborhood V exists - whether C is a closed n-cell complement or not (see [26, Theorems 3.1 and 3.4]). We have required C to be a CFC (that is, a closed four cell complement) because there is the possibility that F might vary with $S^4 - Int\, C$; indeed if $f\colon C \to S^4$ is a re-embedding, we do not know whether or not $S^4 - f(F \cup IntC)$ has any 2-complex pushing properties at all, even though $f(F \cup IntC)$ is 1-ULC.

Our first proposition shows that "1-GULC", while supplying isotopies in itself, also allows a high dimensional trick of using dual skeletons (usually associated with engulfing) to succeed in E^4, thereby producing more isotopies (compare with [26, Theorem 3.1]). Of course it was this observation (it leads to a mismatch theorem) that caused my advisor to lead me in this direction.

Proposition 2.2. *Let* C *be a* CFC *and let* F *be a* 1-GULC *set for* C. *For each* $\varepsilon > 0$ *there exists a compact subset* X *of* F *such that for each open subset* V *of* S^4 *containing* X, *there exists a* 5ε*-self-homeomorphism* H *of* S^4 *supported in the* ε*-neighborhood of* C *such that* $H(\mathrm{Bd}C) \subset V \cup \mathrm{Int}C$.

Proof. Let $\varepsilon > 0$ be given. Since F is a 1-GULC set for C, there exists a neighborhood Q of $\mathrm{Bd}C$ such that for any 2-polyhedron P in Q and any compact subset D in $\mathrm{Int}C$, there exists an ε-pseudoisotopy h_t of S^4 supported in $C \cup Q$ and satisfying $h_1(P \cup D) \subset F \cup \mathrm{Int}C$. Since $\mathrm{Bd}C$ is an ANR, we may suppose that there exists an ε-map $r\colon Q \to \mathrm{Bd}C$ which is a retraction.

Let T be a triangulation of S^4 with mesh less than ε and so much smaller that there exists a subcomplex K of T which is a PL 4-manifold, a neighborhood of $\mathrm{Bd}C$, and contained in Q. We shall be concerned with these sets, $B = \mathrm{Cl}(C - K)$, $Z = \mathrm{Cl}(S^4 - C - K)$, $P = K^{(2)}$, and R = the subcomplex of T' dual to $B \cup P$ (R is the union of all the simplexes of T', the first barycentric subdivision of T, that do not meet $B \cup P$); observe that $R - Z$ is a 1-complex in K. If, as we require, the mesh T is sufficiently fine, then $B \cup Q \cup Z = S^4$.

Apply the remarks of the first paragraph to find h_t with $h_1(P \cup B) \subset F \cup \mathrm{Int}C$. Set $X = h_1(P \cup B) \cap F$, and let V be given. Choose $s < 1$ so close to 1 that $h_s(P \cup B) \subset V \cup \mathrm{Int}C$.

Since C is a CFC there exists an open 4-cell $E \subset S^4$ with $C \subset E \subset C \cup \mathrm{Int}K$. Because $E \cap R$ is open in R and contained in $R - Z$, we may regard it as an infinite (rectilinear but not PL) 1-polyhedron in E. Let T^* be a PL triangulation of E. Now $E \cap R$ is not necessarily PL with respect to T^*, but it is locally tame - because it was a subcomplex of T; hence by the results of Homma [37] and Gluck [35, 36] (see Rushing [48, Theorem 3.6.1]) there exists an ε-self-homeomorphism g of S^4 such that $g(E \cap R)$ is PL with respect to T^* and such that g is supported in E. We also require g to be so close to the identity that $g(E \cap R) \subset Q$.

Let L be a finite subpolyhedron of $g(E \cap R)$ which contains $g(E \cap R) \cap g(C)$. Using the ε-retraction r, the fact that $g(E \cap R) \subset Q$, and the fact that $S^4 - C$ is 0-ULC (see Wilder [50]) together with simplicial approximation and general position, we may obtain a PL, with respect to T^*, ε-re-embedding $f\colon g(E \cap R) \to E$ with $fg(E \cap R) \subset E - g(C)$ and with the support of f contained in L. It follows from Bing-Kister [11] (see Rushing [48, Theorem 3.5.1]) that there exists a PL, with respect to T^*, ε-push G_t of $(E, f(L))$ with $G_1|g(E \cap R) = f$. The pushes produced by this theorem always have compact support (even though the metric on E is not complete, it is necessary to examine its proof to see this); hence we may extend G_t over S^4 via the identity.

Observe that $g^{-1}G_1\, g(R) \subset S^4 - C$. Turning this fact and the earlier $h_s(P \cup B) \subset V \cup \mathrm{Int}C$ around, we obtain that $R \subset g^{-1}G_1^{-1}g(S^4 - C)$ and that $P \cup B \subset h_s^{-1}(V \cup \mathrm{Int}C)$. This leaves us in a position to exploit the join structure between R and $P \cup B$ after the fashion of most engulfing arguments (each simplex σ of the first barycentric subdivision of T is the join of $\sigma \cap R$ and $\sigma \cap (P \cup B))$. That is, there exists an ε-self-homeomorphism of S^4 , F , with the property that $g^{-1} G_1 g(S^4 - C) \cup Fh_s^{-1}(V \cup \mathrm{Int}C) = S^4$ and with support in Q (because $B \cup Q \cup Z = S^4$) . Therefore $(S^4 - C) \cup g^{-1} G_1^{-1} g\ Fh_s^{-1}(V \cup \mathrm{Int}C) = S^4$. It follows that $\mathrm{Bd}C \subset g^{-1} G_1^{-1} g\ Fh_s^{-1}(V \cup \mathrm{Int}C)$, whence $h_s F^{-1} g^{-1} G_1\ g(\mathrm{Bd}C) \subset (V \cup \mathrm{Int}C)$.

Let $H = h_s F^{-1} g^{-1} G_1 g$. All the maps in this composition are ε-maps; so H is a 5ε-self-homeomorphism of S^4 . Furthermore, each map in the composition is the identity off $C \cup Q$, so H is too; therefore H is supported in the ε-neighborhood of C . ■

Remark. The proof of the proposition does not use the full strength of the hypothesis; specifically, we have proved the following:

Proposition 2.3. Let C be a CFC in S^4 . When Q is an ε-neighborhood of $\mathrm{Bd}C$, when P is the 2-complex in Q arising as in the proof of 2.2, when B is the compact set in $\mathrm{Int}C$ arising as in the proof of 2.2, when H_t is any ε-pseudoisotopy of S^4 with support in $C \cup Q$ such that $H_1(P \cup B) \subset C$, and when $X = H_1(P \cup B) \cap \mathrm{Bd}C$, then it is true that for each open subset V of S^4 containing X , there exists a 5ε-self-homeomorphism H of S^4 supported in the ε-neighborhood of C such that $H(\mathrm{Bd}C) \subset V \cup \mathrm{Int}C$. ■

Since C is collared in S^4 , $\mathrm{Bd}C$ is always a 1-GULC set for C . One can always find smaller ones. After a short space we will do so. But first we show how to find within a given 1-GULC set another which is an F_σ .

Proposition 2.4. Let C be a CFC and X be a subset of BdC such that BdC - X is a 1-GULC set for C. Then there exists an F_σ-set in BdC - X which is a 1-GULC set for C.

Proof. For each i we will find a closed set $J_i \subset C - X$ and a neighborhood V of BdC such that for any 2-polyhedron P in V and any compact set D in IntC there exists a $(6/i)$-pseudoisotopy ψ_t with support in V such that $\psi_1(P \cup D) \subset J_i \cup \mathrm{Int}C$. The proposition is proved by setting $F = \cup J_i$.

Fix i, and set $\varepsilon = 1/i$. Apply Definition 2.1 with this ε to find the neighborhood V. Since BdC is an ANR, we may assume that there exists a retraction $r: V \to \mathrm{Bd}C$ which is an ε-map. Let $M \subset V$ be a PL-manifold containing BdC in its interior and contained in the ε-neighborhood of BdC. Determine a triangulation T of S^4 in which M is a subcomplex and with mesh less than ε. Set $R = T^{(2)}$ and $B = \mathrm{Cl}(C - M)$. Then R is a 2-polyhedron in V and B a compact set in IntC; so there exists an ε-pseudoisotopy F_t of S^4 with support in $C \cup V$ such that $F_1(R \cup B) \subset C - X$. Set $J_i = F_1(R \cup B) \cap \mathrm{Bd}C$, so $J_i \subset C - X$.

Now let P be any 2-polyhedron in V and D be any compact set in IntC. Since IntC is 0-ULC (see Wilder [50]), there exists a compact set D^* in IntC with $D \subset D^*$ and $S^4 - D^*$ 0-ULC. Let Q be the subcomplex of T' dual to $R \cup B$ (Q is the union of all the simplexes in the first barycentric subdivision of T which do not meet $R \cup B$); then $Q \cap M$ is a 1-complex since it is contained in the 1-skeleton of M dual to R. Since C is a CFC there exists a 4-cell E in S^4 with $C \subset E \subset C \cup M$. By hypothesis there exists an ε-pseudoisotopy k_t of S^4 with support in $C \cup V$ such that $k_1(P \cup D^*) \subset C - X$. Choose a number $s < 1$ but so close to 1 that $k_s(P \cup D^*) \subset E$.

Now we argue like in the last proof. Let T^* be a triangulation of E. Apply the results of Homma [37] and Gluck [35, 36] to find a $\gamma(x)$-self-homeomorphism $g: E \to E$ with $\gamma: E \to (0, 1]$ chosen so that g is an ε-map and so that g may be extended over S^4 via the identity, and such that $g(Q \cap E)$ is a PL polyhedron. We also require $\gamma(x)$ to be so small that $g(Q \cap E) \subset M$ (that is, does not meet B). Let L be a finite subpolyhedron of $g(Q \cap E)$ such that $g(Q \cap E) \cap gk_s(P \cup D^*) \subset L$; see Figure 1 for a schematic representation of these sets. Now, by applying the retraction r to L and using the facts that $S^4 - gk_s(D^*)$ is 0-ULC, that P is closed and 2-dimensional, and that every point of BdC is within a distance of ε of a point in $E - gk_s(P \cup D^*)$, together with simplicial approximation and general position, we may apply a well-known, classical technique (we leave the details to the reader) to find a (T^*) PL ε-reembedding $f: g(Q \cap E) \to E$ such that $f = id$ on $g(Q \cap E) - L$ and $fg(Q \cap E) \subset E - gk_s(P \cup D^*)$. As in the last proposition, it follows from Bing-Kister [11] that there exists an ε-push G_t of $(S^4, fg(Q \cap E))$ with support in E such that G_1 extends f.

Observe that $g^{-1} G_1 g(Q) \subset S^4 - k_s(P \cup D^*)$; whence $Q \cap g^{-1} G_1^{-1} gk_s(P \cup D^*) = \emptyset$. Again we are left in a position to exploit the join structure between Q and $P \cup B$; this time we observe that there exists an ε-pseudoisotopy H_t of S^4 such that $H_1 g^{-1} G_1^{-1} gk_s(P \cup D^*) \subset R \cup B$. It follows that

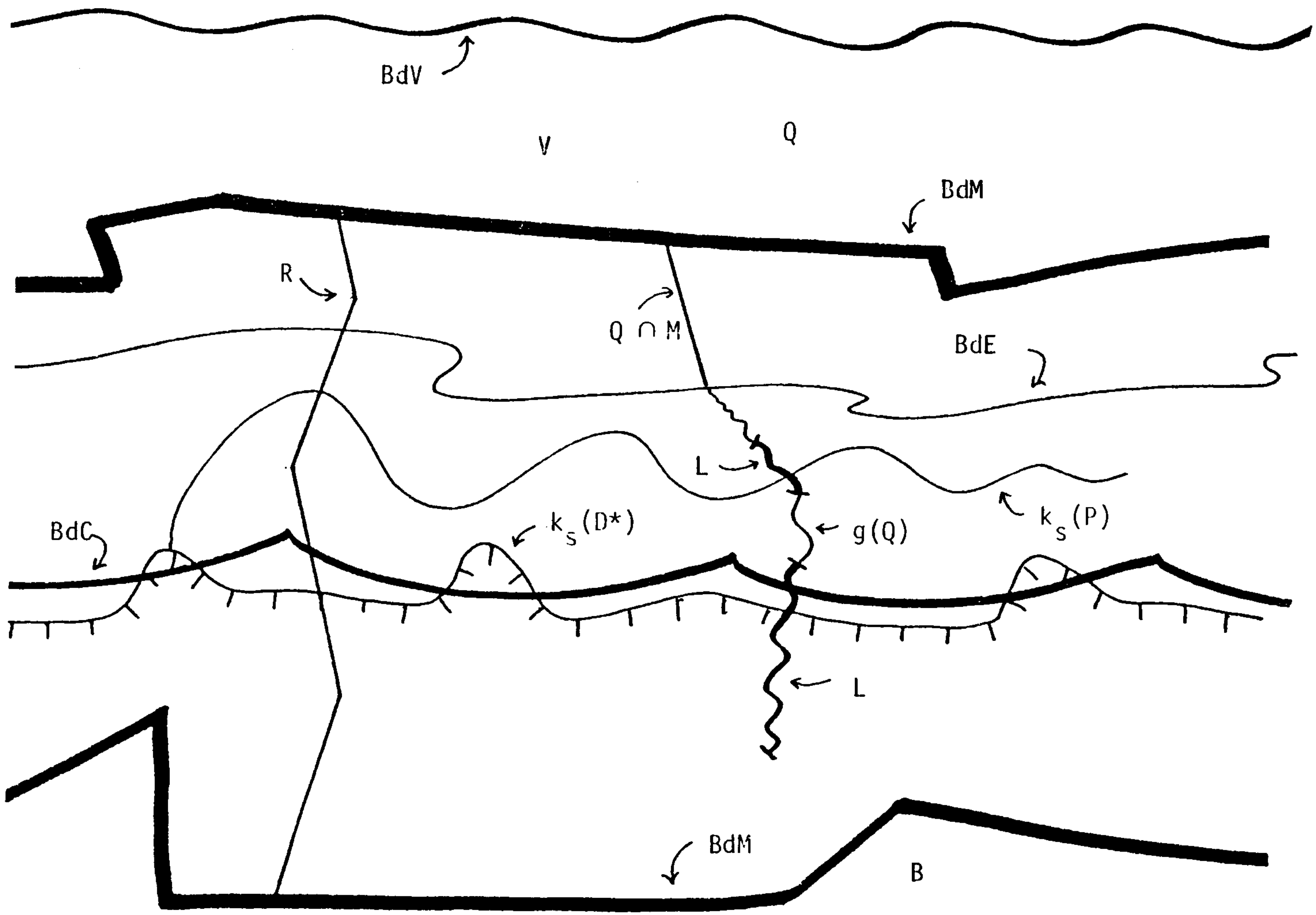


Figure 1. A Schematic for Proposition 2.4.

$F_1H_1\ g^{-1}\ G_1^{-1}\ gk_s(P \cup D^*) \subset J_i \cup \mathrm{Int}C$. Thus we set $\psi_t = F_tH_tg^{-1}\ G_t^{-1}\ gk_{st}$. The support of ψ_t is in the ε-neighborhood of BdC because this is so for each map in the composition. Likewise, because each map in the composition is an ε-homeomorphism, ψ_t is a 6ε-homeomorphism $(t \neq 1)$ as desired. ■

Remark. If the reader will carefully examine this last proof, he will see that we have revealed how to prove the following: If F is a 1-GULC set for C , then for each $\varepsilon > 0$ there exists a neighborhood V^* of BdC such that for every (not necessarily PL) 2-polyhedron P in V^* ... $h_1(P \cup D) \subset (\mathrm{Int}C) \cup F$, the omitted portion being unchanged from the definition.

Outline of proof: First, one applies the definition of 1-GULC to find V ; then one finds M as in the preceding proof, sets $V^* = \mathrm{Int}M$, and reproduces the procedure we used to pseudisotop $k_s(P \cup D^*)$ into $J_i \cup \mathrm{Int}C$ - except for replacing $k_s(P \cup D^*)$ by $P \cup D^*$. This shows that (PL) may be omitted from 2.1 without affecting the sense of the definition. Indeed, the sense is unchanged even if P is allowed to be a 2-dimensional compactum.

Now we pause in our discussion of 1-GULC sets to introduce a notion which turns out to be closely related.

Definition 2.5. Let C be a CFC in S^4 and X be a closed subset of BdC . A section of C over X is a CFC containing C determined by the graph in a collar on C of a continuous function $f: C \to [0, 1)$ with $f^{-1}(0) = X$. When the section is a 4-cell, that is, when the graph of f is bicollared, we call it flat.

Remark. This definition represents a formalization of ideas already present in Daverman [26] and Cannon-Daverman [18] (see the 1-ULC taming hypothesis of [18] for example). Clearly, neither the specific function nor the specific collar affects the topological type of the section, which is in all cases embedded as a CFC, and obviously, if one section for C over X is flat, so is every other. Equally obvious is the next proposition.

Proposition 2.6. *Let X and Y be closed subsets in the boundary of a CFC in S^4 with $Y \subset X$. If a section of C over X is flat, then each section of C over Y is flat.* ∎

The sequence of propositions to follow (principally 2.8, 2.11, and 2.16) reveals the kinship between flat sections and 1-GULC sets. But first, we establish a very useful taming theorem for S^4 first described to us by Robert Daverman in 1975.

Proposition 2.7. *Let B be a 4-cell in S^4 and D be a compact set in $\mathrm{Bd}B$. If for each $\varepsilon > 0$ there exists an ε-self-homeomorphism h of S^4 supported in the ε-neighborhood of $S^4 - B$ such that $h(D) \cap B = \emptyset$, then B is locally flat at each point of $\mathrm{Int}D$. In particular, when $D = \mathrm{Bd}B$, B is flat.*

Proof. We present only the proof of the special case where $D = \mathrm{Bd}B$, which is easier to visualize but nevertheless makes the general case obvious.

Let $e: D \times [0, 1] \to B$ be a collar on D $(= \mathrm{Bd}B)$ $(e(D \times \{0\}) = D)$, and let $s_1 > s_2 > \cdots$ be a decreasing sequence of numbers from $(0, 1)$ which converges to 0. Use the hypothesis to find a sequence of numbers, ε_i , and a sequence of ε_i-self-homeomorphisms, h_i , $(i = 1, 2, \ldots)$ of S^4 satisfying

1) $\varepsilon_i < \operatorname{dist}(e(D \times \{0\}), e(D \times \{s_i\}))$,

2) $\varepsilon_{i+1} < \operatorname{dist}(D, h_i(D))$,

3) h_i leaves $e(D \times \{s_j\})$ fixed for all $j \leq i$, and

4) $h_i(D) \cap B = \emptyset$.

Then $\varepsilon_i \to 0$, $h_i(D) \cap h_j(D) = \emptyset$ for $i \neq j$, and $h_i|D$ converges uniformly to the identity. Let $q_i \in (0, 1)$ be so close to 0 that $q_i < s_i$ and

5) $\operatorname{dist}(h_{i+1} e(d, 0), h_{i+1}e(d, q_i)) < \frac{1}{4}\operatorname{dist}(h_{i+1}(D), h_j(D))$ for all $j \neq i + 1$ and all d in D .

Observe that the spheres $h_i(D)$ and $h_i(e(D \times \{q_i\}))$ are all pairwise disjoint and "concentric" like the layers of an onion.

Now use the product structure of $h_{i+1}e(D \times [0, 1])$ to find ε_i-self-homeomorphisms, F_i , of S^4 such that

6) $F_i h_{i+1} e(d, s_i) = h_{i+1} e(d, q_i)$ for all d in D and

7) $F_i h_i e(d, q_{i-1}) = h_i e(d, q_{i-1})$ for all d in D .

Observe that $F_i h_i e$ embeds $D \times [q_{i-1}, s_i]$ as the annulus between $h_i e(D \times \{q_{i-1}\})$ and $h_{i+1} e(D \times \{q_i\})$.

Let $g_i\colon D \times [1/(i + 1), 1/i] \to D \times [q_{i-1}, s_i]$ be a homeomorphism which preserves first coordinates and takes $D \times \{1/i\}$ to $D \times \{q_{i-1}\}$. Now define $G\colon D \times [0, 1] \to S^4 - \operatorname{Int}B$ by

8) $G(d, 0) = d$ for all d in D and

9) $G(d, t) = F_i h_i e g_i(d, t)$ when $1/(i + 1) \leq t \leq i/1$.

First observe that G is continuous on $D \times (0, 1]$ because each composition $F_i h_i e g_i$ is continuous (on $D \times [1/(i+1), 1/i]$) and because 6) and 7) force these maps to agree whenever they have common domain; that is,

10) $F_{i+1}h_{i+1}e(d, q_i) = F_i h_{i+1} e(d, s_i) = F_i h_i e(d, s_i)$.

Next observe that G is continuous on $D \times [0, 1]$ because $\rho(F_i h_i e g_i(d, q), e(d, 0)) \to 0$ as $i \to \infty$. Finally, G is 1-1 because the images $F_i h_i e g_i (D \times (1/(i+1), 1/i])$ are pairwise disjoint - they lie between different pairs of the "concentric" spheres. Note: if D were as in the general proposition, then G would only be 1-1 on a neighborhood of IntD .

We have constructed a collar on B , so B is flat by [14]. ■

Proposition 2.8. *If C is a CFC in S^4 , F a 1-GULC set for C , and X a closed subset of BdC - F , then sections for C over X are flat.*

Proof. Fix a collar BdC $\times [0, 1)$ on BdC (we regard this as a subset of S^4 with BdC $\times \{0\}$ = BdC) and a function f: BdC $\to [0, 1)$ with $f^{-1}(0) = X$. Let Q denote the section determined by the graph of f in BdC $\times [0, 1)$; then $C \subset Q$ and $C \cap$ BdQ $= X$. To show Q is flat, it is sufficient by 2.7 to find for each $\varepsilon > 0$ an ε-self-homeomorphism h of S^4 supported in the ε-neighborhood of Q such that h(BdQ) $\subset$ IntQ .

Let $\varepsilon > 0$ be given. Choose a closed neighborhood N of X in BdC sufficiently close to X that there exists a function g: BdC $\to [0, 1)$ with $g^{-1}(0) = N$, $g(c) < f(c)$ for all c in BdC - X , and dist$(g(c), f(c)) < \frac{1}{2}\varepsilon$ for all c in BdC . Let Q_N denote the section obtained from g in $BdC \times [0, 1)$; then $C \subset Q_N \subset Q$. Let $\eta = \frac{1}{2}\min\{dist(BdQ_N - N, BdQ), \varepsilon\}$. According to Proposition 2.2, there exists a compact subset W of $BdC - X$ $(W \subset F)$ such that for each open subset V of S^4 containing W , there exists a η-self-homeomorphism, H , of S^4 supported in the η-neighborhood of C with $H(BdC) \subset (V \cup IntC)$. Let $\delta = dist(W, BdQ)$, V be the δ-neighborhood of W , and H be so determined. Observe that $H(BdQ_N - N) \subset IntQ$ by the definition of η and that $H(N) \subset IntQ$ by the definition of δ ; thus $H(BdQ_N) \subset IntQ$. From the construction of g , it follows that given any neighborhood U of BdQ_N there exists a $\frac{1}{2}\varepsilon$-self-homeomorphism, G , of S^4 supported in the $\frac{1}{2}\varepsilon$-neighborhood of Q_N such that $G(BdQ) \subset U$. Let $U = H^{-1}(IntQ)$ and let G be so determined. Then $h = HG$ has the properties required by 2.7. ∎

The next two propositions are easily proved using the fact that the boundary of a flat section is always bicollared, and the third shows part of the relation between sections and 1-GULC sets.

Proposition 2.9. *Let C be a CFC in S^4 . If X is a closed subset of BdC , Q a flat section over X , and D' a compact subset of $IntC$, then for each $\delta > 0$ there exists a δ-isotopy f_t supported in the δ-neighborhood of X and leaving D' fixed such that $f_1(BdC) \subset IntQ$.* ∎

Proposition 2.10. *If C is a CFC in S^4 , X a closed subset of BdC , and a section over X is flat, then there exists a 1-GULC set for C in $BdC - X$.* ∎

Proposition 2.11. Suppose C *is a* CFC *in* S^4 *and* $\{X_i\}$ *is a sequence of closed subsets in* BdC . *If for each* i *there exists a flat section* Q_i *over* X_i , *then there exists an* F_σ*-set in* BdC $- \cup X_i$ *which is a* 1-GULC *set for* C .

Proof. Let BdC $\times [0, 1) \subset S^4$ be a collar on BdC with BdC $\times \{0\}$ identified with BdC , and let p: BdC $\times [0, 1) \to$ BdC be projection; extend p over C by the identity. By Proposition 2.6 we may assume that all the Q_i are sections with respect to this collar. In view of Proposition 2.4, it is sufficient to find for each $\varepsilon > 0$ a neighborhood V of BdC such that for any 2-polyhedron $P \subset V$ and any compact subset $D \subset$ IntC , there exists an ε-pseudoisotopy h_t with support in $C \cup V$ such that $h_1(P \cup D) \subset C - \cup X_i$.

Let $\varepsilon > 0$ be given, and let $\{a_n\}$ be a decreasing sequence of numbers with $\sum a_n < \frac{1}{2}\varepsilon$. Set $V = C \cup (\text{BdC} \times [0, t))$ where t is chosen so that diam$(\{c\} \times [0, t)) < \frac{1}{2}\varepsilon$ for all c in BdC , and let P be any 2-polyhedron in V . We will find a sequence of isotopies $\{h_t(n)\}$ and a sequence of closed subsets $\{Z_n\}$ of BdC satisfying six properties.

1) $\rho(h_t(n)(c), h_t(n - 1)(c)) < 2a_n$ for all c and all t ,

2) $h_t(n)$ is supported in $C \cup V$ for all t ,

3) $h_t(n) = h_t(n - 1)$ for all t in $[0, 1 - (1/n)]$,

4) $Z_n \subset \text{Int}Z_{n-1} - \bigcup_{i=1}^{n} X_i$,

5) $ph_1(n)(P \cup D) \subset (\text{IntC}) \cup (\text{Int}Z_n)$, and

6) $h_1(n)(P \cup D) \subset C \cup \text{BdC} \times [0, t_n)$,

where t_n is chosen so that diam$(\{c\} \times [0, t_n)) < a_{n+1}$ for all c in BdC . This will prove the proposition because we may set

$h_t = \lim h_t(n)$ for each t in I. Then 1), 2), and 3) insure that h_t is well-defined and that it is an ε-pseudoisotopy with support in $C \cup V$. Because the set D' (in the induction to follow) is left fixed, conditions 4) and 5) insure that $ph_1(P \cup D) \subset (\mathrm{Int}C) \cup (\cap \mathrm{Int}Z_n)$, while condition 6) insures that $h_1(P \cup D) \subset C$; taken together we find that $h_1(P \cup D) \subset (\mathrm{Int}C) \cup (\cap \mathrm{Int}Z_n)$. Thus, by 4), $h_1(P \cup D) \subset C - \cup X_n$.

It remains to find the sequences $\{h_t(n)\}$ and $\{Z_n\}$ satisfying 1)-6). For the purpose of starting the induction, let $a_0 = \frac{1}{2}\varepsilon$, $X_0 = \emptyset$, $Z_0 = \mathrm{Bd}C$, and $h_t(0) = \mathrm{id}$; then 1)-6) hold for $n \leq 0$.

Suppose that the $h_t(n)$ and the Z_n have been found through $n = k$ and satisfying 1)-6) for $n \leq k$. Choose t_{k+1} satisfying the condition in 6). Let $\delta = \min\{a_{k+1}, \mathrm{dist}(p^{-1}\, ph_1(k)(P \cup D)\,,\, p^{-1}(\mathrm{Bd}C - Z_k))\}$, $X = X_{k+1}$, and $D' = h_1(k)(P \cup D) - N_{\frac{1}{2}\delta}(h_1(k)(P \cup D) \cap \mathrm{Bd}C \times (0, 1))$ in Proposition 2.9 to obtain a δ-isotopy f_t of S^4 supported in the δ-neighborhood of X_{k+1}, leaving D' fixed, and such that $f_1(\mathrm{Bd}C) \cap \mathrm{Bd}Q_{k+1} = \emptyset$. (Assume without loss of generality that $f_t = \mathrm{id}$ for t in $[0, 1 - (1/k)]$ and that the support of f is in $C \cup (\mathrm{Bd}C \times [0, t_{k+1}))$. It follows that there exists a neighborhood W of $\mathrm{Bd}C$ in S^4 such that $f_1(W) \cap \mathrm{Bd}Q_{k+1} = \emptyset$ (i.e., $pf_1(W) \subset C - X_{k+1}$) and $f_1(W) \subset C \cup \mathrm{Bd}C \times [0, t_{k+1})$.

By 6), we may find an a_{k+1}-isotopy g_t of S^4 (obtained by pushing along the fibers of $\mathrm{Bd}C \times [0, 1)$) with support in $C \cup V$ such that $g_t = \mathrm{id}$ for t in $[0, 1 - (1/k)]$ and $g_1h_1(k)(P \cup D) \subset W$.

Set $h_t(k + 1) = f_tg_th_t(k)$. Then 1), 2), and 3) hold for $n \leq k + 1$. Observe that $h_1(k + 1)(P \cup D) \subset f_1(W)$; so 6) holds too. Furthermore

7) $(ph_1(k + 1)(P \cup D)) \cap BdC$

$$\subset BdC \cap pf_1(W) \cap pp^{-1}(N_\delta(h_1(k)(P \cup D) \cap BdC \times [0, 1))$$

$$\subset BdC \cap pf_1(W) \cap pp^{-1}(IntZ_k)$$

$$\subset (BdC - X_{k+1}) \cap IntZ_k \ .$$

So, since $ph_1(k + 1)(P \cup D)$ is compact and $(BdC - X_{k+1}) \cap IntZ_k$ is open, there exists a closed set Z_{k+1} in BdC with

8) $(ph_1(k + 1)(P \cup D)) \cap BdC \subset IntZ_{k+1} \subset Z_{k+1} \subset (BdC - X_{k+1}) \cap IntZ_k$.

Now 4) and 5) hold for $n \leq k + 1$ too; the proof is complete. ■

Corollary 2.12. *If* C *is a* CFC *in* S^4 *with* X_1 *and* X_2 *closed subsets of* BdC *over which there exist flat sections, then there exists a flat section over* $X_1 \cup X_2$.

Proof. Apply 2.11 and 2.8. ■

Corollary 2.13. *Under the hypotheses of* 2.11, *if* Y *is any closed subset of* $\cup X_i$, *then sections over* Y *are flat*. ■

Corollary 2.14. *If* C *is a* CFC *in* S^4 *such that* C *has disjoint* 1-GULC *sets* F_1 *and* F_2 *which are* G_δ's , *then* C *is a* 4-*cell*.

Proof. Write $BdC - F_1$ as $\cup E_i$ and $BdC - F_2$ as $\cup K_i$ with E_i and K_i closed for all i . By 2.8, sections on E_i and K_i are flat. Hence there exists a 1-GULC set for C in $BdC - (\cup E_i \cup F_i)$. which is empty. Proposition 2.8 shows that such CFC's are 4-cells, or apply 2.7 ■

Remark. Both F_1 and F_2 need to be G_δ's for the preceding to be true. To see this, consider the suspension of the one-point Fox-Artin sphere [34] in S^4, and let C denote the CFC it determines and L denote the suspension of the wild point. It can be shown that any dense subset of L is a 1-GULC set for C (examine the proof of Theorem 5.6 if you are skeptical). In particular, we might take F_2 to be countable and dense in L, and F_1 to be $L - F_2$. Then F_1 is a G_δ, and $F_1 \cap F_2 = \emptyset$; yet C is not a 4-cell.

Corollary 2.15. Each CFC *has a 2-dimensional* 1-GULC *set which is an* F_σ.

Proof. Let C be a CFC. Since sections over points are flat by Cantrell [19], it follows from Proposition 2.11 that for each countable set D in $\mathrm{Bd}C$, there exists a 1-GULC F_σ-set F for C in $\mathrm{Bd}C - D$. Choose D to be dense; then F is 2-dimensional by Hurewicz and Wallman [40]. ∎

Proposition 2.16. Let C *be a* CFC, F *be a* 1-GULC *set for* C, *and* $X \subset \mathrm{Bd}C$ *be a closed set. Suppose* X *can be written as a countable union of sets,* $X = \cup X_i$, *such that for each* i *there exists a flat section* Q_i *over* X_i. *Then there exists an* F_σ*-set* $F' \subset F - X$ *which is a* 1-GULC *set for* C.

Proof. Let $3\varepsilon > 0$, C, F, X, and the X_i be given. Apply Proposition 2.11 to find a 1-GULC set F^* for C in $\mathrm{Bd}C - X$. We shall apply Proposition 2.4. Let V be the neighborhood of $\mathrm{Bd}C$

such that for every 2-polyhedron P and every compact set $D \subset \text{Int}C$ there exists an ε-pseudoisotopy f_t of S^4 with $f_1(P \cup D) \subset \text{Int}C \cup F^*$ and support in $C \cup V$. An examination of the proof of 2.11 will show that f_t may be taken to be the identity on D for all t. Now, let P and D be arbitrary such sets, and let f_t be such a pseudoisotopy. Then $Z = f_1(P \cup D) \subset C - X$; hence $\text{dist}(Z, X) = \eta > 0$. Set $\zeta = \frac{1}{2}\min\{\varepsilon, \eta\}$. Let U be a neighborhood of $\text{Bd}C$ such that for any 2-polyhedron P' in U (see the remark following 2.4) and any compact set D' in $\text{Int}C$ there exists an ζ-pseudoisotopy g_t of S^4 with $g_1(P' \cup D') \subset \text{Int}C \cup F$ and support in $C \cup U$. Choose $s < 1$ so close to 1 that $f_s(P \cup D) \subset \text{Int}C \cup N_{\frac{1}{2}\eta}(Z)$. Let g_t be obtained with $D' = D$ $(= f_s(D))$ and $P' = f_s(P)$. Then $g_1 f_s(P \cup D) \subset \text{Int}C \cup (F - X)$. So set $h_t = g_t f_{st}$ and apply 2.4. ■

Geometric Type

The propositions of the preceding section have many more consequences, but before proceding to them, we pause to introduce terminology analogous to the high dimensional "types" of [26].

Definition 2.17. Let C be a CFC in S^4, and let $\{X_a\}$, $X_a \subset \text{Bd}C$, be the collection of all 1-GULC F_σ-sets for C. For each row in the table below, we use the adjective in the first column to describe C whenever C satisfies the condition beside it and none of those above it.

1) G-type 0 .... Some X_a is countable
2) G-type 1TT .... Some X_a is 0-demensional in $\text{Bd}C$ and tame in S^4

3) <u>G-type</u> <u>1TB</u> .... Some X_a is 0-demensional in BdC (see [32])

4) <u>G-type</u> <u>2TS</u> .... Some X_a is 0-dimensional and tame in S^4

5) <u>G-type</u> <u>2TW</u> .... Some X_a is 0-dimensional

6) <u>G-type</u> <u>3A</u> .... Some X_a is 1-demensional in BdC (see [32])

7) <u>G-type</u> <u>3B</u> .... Some X_a is 1-dimensional

8) <u>G-type</u> <u>4</u> .... Some X_a is 2-dimensional

If one requires that $X_a \cup \text{Int}C$ be 1-ULC in place of X_a being a 1-GULC set for C, then this scheme defines types and extends the definitions found in [26]. However, we use type only once hereafter - in Proposition 3.7.

The acronymns TT, TB, TS, and TW stand for Twice Tame, Tame in the Boundary, Tame in S^4, and Twice Wild. <u>G-type</u> <u>1</u> means G-type 1TT or G-type 1TS, and similarly for <u>G-type</u> <u>2</u> and <u>G-type</u> <u>3</u>. We regard G-types as ordered by their appearance in the table; G-type 1TT is less than G-type 1TB, etc. In this terminology, Corollary 2.15 states that a CFC has G-type at most 4. Ancel and McMillan [3] shows that a crumpled 4-cube has at most type 3A; this extends a result of Daverman [22] which says that a crumpled n-cube $(n > 4)$ has type at most 3.

It is also known as a consequence of [26, section 9] that when $n > 4$ there are no type 1TB crumpled n-cubes. Finally, when $n > 4$ there is no difference between G-type and type (see our remarks following 2.1). This might also be the case when $n = 4$, and our

Propositions 2.8, 2.11, and 2.16 would show this if a 4-dimensional 1-ULC taming theorem like [7], [20], [24], or even [47] were established - of course other hard questions would then be easy too.

Geometric Type of Locally Flat CFC's

In this section, we discover what our techniques allow concerning the G-types of CFC's with varying degrees of local flatness. See Chapter V for the G-types of suspensions and inflations. Occasionally, in this chapter, we refer to some results on topological general positions which are proved in a self-contained manner at the start of Chapter IV.

Proposition 2.18. Let C be a CFC in S^4 which is locally flat modulo a set X 1-demensional in BdC. If each Cantor set of X which is tame in BdC is tame in S^4, then C has G-type at most 1TT.

Proof. Let T_i be a sequence of triangulations of BdC with mesh tending to zero and T_{i+1} a subdivision of T_i. Set $R_i = T_i^{(2)}$. Since X is at most 1-demensional, it has the arc pushing property, by McMillan [44]; therefore, we may assume that $R_i \cap X$ is 0-dimensional for all i (see Proposition 4.4 for a detailed proof of a similar assertion). It follows, from Kirby [41] and the hypotheses, that sections over $R_i \cap X$ are flat. Since X is closed, BdC - X is an F_σ-set, say $\mathrm{BdC} - X = \cup E_i$ with each E_i closed; then clearly sections over E_i are flat. By Proposition 2.11, there is an F_σ-set F in $\mathrm{BdC} - (\cup R_i \cap X) - (\cup E_i) = X - \cup R_i$ which is a 1-GULC set for C.

Because $F \cap R_i = \emptyset$ and $\text{mesh}T_i$ tends to zero, F is 0-dimensional. Now let Q be any Cantor set in F. By the preceding, Q is defined in BdC by PL 3-cells, and so tame in BdC ; therefore, by hypothesis, Q is twice tame. ■

Proposition 2.21 gives a similar condition guaranteeing at most G-type 1TT . Propositions 2.19 and 2.20 are necessary preliminaries. We are more grateful than usual to Robert Daverman at this point for suggesting how to prove 2.20, which eluded us.

Proposition 2.19. _Let_ K _be a Cantor set in_ S^4 , _and let_ $R = \cup T_i^{(2)}$ _be an infinite_ 2-_skeleton of_ S^4 . _If_ $K \cap R = \emptyset$, _then_ K _is tame_.

Proof. Since $K \cap T_i^{(2)} = \emptyset$ for each i , K has a defining sequence of PL 4-manifolds with PL 1-polyhedra for spines. That PL 1-polyhedra are unlinked in S^4 implies that K is defined by PL 4-cells, and hence tame. ■

Proposition 2.20. _Let_ X _be a_ 1-_dimensional compactum in_ S^4 . X _may be written as_ $S_0 \cup S_1$ _with each_ S_e 0-_dimensional and_ S_0 _a tame_ F_σ-_set_.

Proof. Let R be an infinite 2-skeleton of S^4 such that $R \cap X$ is 0-dimensional (see Proposition 4.4). Since X is a 1-dimensional compactum, it can be written as $Q_0 \cup Q_1$ with each Q_e 0-dimensional and Q_0 an F_σ-set. Now let $f: X \to E^5$ be a tame embedding and let R^* be an infinite 4-skeleton of E^5 . Because

$f(X)$ is tame, we may suppose that $R^* \cap f(X) \subset f(X - (Q_0 \cup (R \cap X)))$. Set $S_0 = f^{-1}(R^* \cap f(X))$ and $S_1 = X - S_0$. Then S_0 is a 0-dimensional F_σ-set because $S_0 \subset Q_1$, S_1 is 0-dimensional because $f(X) - R^* \cap f(X)$ is , and finally, each Cantor set in S_0 is tame by 2.19 $(S_0 \subset X - R)$. ∎

Proposition 2.21. *Let C be a CFC which is locally flat modulo a 1-dimensional set X. If each Cantor set in X which is tame in S^4 is tame in BdC, then C has G-type at most* 1TT.

Proof. Apply 2.20 to write X as a union of two 0-dimensional sets S_0 and S_1 with S_0 an F_σ-set tame in S^4, say $S_0 = Q_i$ with each Q_i closed and tame. By Kirby [41] and the hypotheses, sections over each Q_i are flat. Let $R = \cup T_i^{(2)}$ be an infinite 2-skeleton of S^4, and assume (see 4.4) that $X \cap R$ is 0-dimensional. Again it follows from Kirby [41] and the hypotheses, that sections over $X \cap T_i^{(2)}$ are flat. Finally write $\text{BdC} - X = \cup E_i$ with each E_i closed, and observe that sections over each E_i are flat.

Now by 2.11, there exists an F_σ-set, F, in $\text{BdC} - \cup Q_i - \cup(X \cap T_i^{(2)}) - \cup E_1$. Because $F \subset \text{BdC} - \cup E_i - \cup Q_i = X - \cup Q_i$, it is 0-dimensional. Because $F \subset S^4 - \cup T_i^{(2)}$, each Cantor set in F is tame in S^4 and so, by hypothesis, tame in BdC too. ∎

The proofs of 2.18 and 2.20 may be combined to prove the following.

Corollary 2.22. *Let C be a CFC locally flat modulo a set X. C has G-type at most* 1TT *if for each point p in BdC, one of the following is true.*

1) There exists a neighborhood U of p in BdC such that $X \cap U$ is 1-dimensional and each Cantor set in $U \cap X$ which is tame in S^4 is tame in BdC .

2) There exists a neighborhood U of p in BdC such that $X \cap U$ is 1-demensional in BdC and each Cantor set in $U \cap X$ which is tame in BdC is tame in S^4 . ∎

The same technique has a few other weak but obvious consequences such as the following.

Proposition 2.23. *If C is locally flat modulo a Cantor set, then C has G-type at most 2 .* ∎

In higher dimensions, a crumpled n-cube which is locally flat modulo an (n - 2)-manifold has type at most 2 (Daverman [29]). So it certainly seems like something stronger than 2.24 should be true, but we have not met with any success in our attempts to improve it. A similar situation prevails when C is locally flat modulo a 2-dimensional set; in the absence of other hypotheses, we can only show that C has G-type at most 3A .

Proposition 2.24. *If C is locally flat modulo a 2-dimensional set, Q , then C has G-type at most 3A .*

Proof. Let $R = \cup T_i^{(1)}$ be an infinite 1-skeleton of BdC , and let R^* be an infinite 2-skeleton of S^4 . Assume without loss of generality (see the beginning of Chapter IV for several detailed arguments of this nature) that $R^* \cap$ BdC is 1-demensional in BdC , that $R \cap R^* \cap \mathrm{Bd}C = \emptyset$, and that $R \cap Q$ is 0-dimensional. Then $Q \cap T_i^{(1)}$ is 0-dimensional and twice tame - tame in BdC because $Q \cap T_1^{(i)} \subset R$ and tame in S^4 because $Q \cap T_i^{(1)} \subset S^4 - R^*$. Hence, by Kirby [41], the section over $Q \cap T_i^{(1)}$ is flat. It now follows from Propositions 2.8 and 2.11 and the hypotheses that sections over $T_i^{(1)}$ are flat; whence, by 2.11 again, there exists a 1-GULC set, F , for C in $\mathrm{Bd}C - R$. Because R is an infinite 1-skeleton, F is 1-demensional, so C has C-type at most 3A . ■

Corollary 2.25. If C is locally flat modulo a 1-dimensional set, then C has G-type at most 3A . ■

The next proposition is an abstraction of what has gone before. Following it, Proposition 2.27 puts another twist on the technique and suggests an attack that might improve some of our earlier results. The proofs, which we leave to the reader, are just like what we have been doing except that one will need to use 2.8 in the second.

Proposition 2.26. Let C be a CFC locally flat modulo a set X . Let P be a PL 1-complex (2-complex) in BdC . If for each $\varepsilon > 0$ there exists an ε-self-homeomorphism h of BdC with $h(P) \cap X$ a twice tame Cantor set, then C has at most G-type 3 A (G-type 1) . ■

Proposition 2.27. Let C be a CFC with 1-GULC set F . Let P be a PL 1-complex (2-complex) in BdC . If for each $\varepsilon > 0$ there exists an ε-self-homeomorphism h of BdC with the section *on* $h(P) \cap F$ *flat (so* $h(P) \cap F$ *is closed) and* $h(P) - F$ *an* F_σ*-set, then* C *has at most G-type* 3 A *(at most G-type* 1) . ■

Finally, by using other techniques, we are able to establish the G-type of a CFC which is locally flat modulo a 1-demensional set.

Proposition 2.28. *Let* C *be a* CFC *locally flat modulo a set* X . *If* X *is* 1-*demensional in* BdC , *then* C *has G-type at most* 2 .

Proof. In view of the proof of Proposition 2.4, it is sufficient to show that for each $\varepsilon > 0$ there exists a neighborhood V of BdC such that for any PL 2-polyhedron P in $V \cup C$ and any compact set $D \subset IntC$ there exists an ε-pseudoisotopy h_t of S^4 supported in $C \cup V$ such that $h_1(P \cup D) \cap BdC$ is 0-dimensional. In fact, the last condition may be replaced by $h_1(P \cup D) \cap X$ is 0-dimensional - since the hypothesized local flatness shows that h_1 could be composed with a small isotopy which pushes $BdC - X$ into $IntC$.

So let $\varepsilon > 0$ be given, let $BdC \times I$ be a collar on BdC , let $t > 0$ be so close to zero that $\operatorname{diam}(\{x\} \times (0, t)) < \frac{1}{2}\varepsilon$ for all x in BdC , set $V = C \cup (BdC \times (0, t))$, and let P and D be given. We shall use p to denote the projection of $BdC \times I$ to BdC extended over C by the identity.

According to Proposition 4.4 (Propositions 4.1 - 4.6 are self-contained), we may assume that $P \cap X$ is 0-dimensional. Like in the proof of 2.18, X being 1-demensional in BdC means that there exists a 0-demensional F_σ-set Q in X such that $X - Q$ is 0-dimensional. Now according to Proposition 4.3, we may assume that $p(P \cap (BdC \times IntI) \cap Q = \emptyset$. It follows that

$$p(P \cup D) \cap X = (P \cap X) \cup (p(P \cap (BdC \times IntI) \cap (X - Q))$$

is 0-dimensional. Since p is realizable by ε-pseudoisotopy, the proposition is proved. ■

Table 1 summarizes the results of this last section. In it C is a CFC which is locally flat modulo a set X . The numbers in parentheses refer to the propositions of this section, and entries in the table which are not accompanied by such references are obvious consequences of the others.

TABLE 1

THE G-TYPE OF C WHEN
C IS LOCALLY FLAT MODULO X

	X is a Cantor set	X is 1-dem in BdC	X is 1-dimen'l	X is 2-dimen'l
Each Cantor set in X which is tame in S^4 is tame in BdC .	1TT	1TT	1TT	3A
Each Cantor set in X which is tame in BdC is tame in S^4 .	1TT	1TT (2.18)	3A	3A
C satisfies no extra hypotheses	2	2 (2.28)	3A	3A (2.24)

CHAPTER III

THE SEWINGS PROBLEM

Special Notation

During his high-dimensional investigation into sewings of crumpled n-cubes, Daverman [26] shows how to translate the problem of whether or not $C_0 \cup_h C_1$ is S^n into a decomposition problem. This transformation is so crucial that almost nothing can be accomplished without it.

Here is the method. Given two CFC's , C_e , ($e = 0, 1$ throughout the rest of the work - occasionally we remind the reader) and a sewing, $h: BdC_0 \to BdC_1$, fix collars, $f_e: BdC_e \times I \to S^4$, on C_e in S^4 such that $f_e(x, e) = x$ for all x in BdC_e . In light of the Generalized Schoenflies Theorem [14], we may assume that $C_e \cup f_e(BdC_e \times I)$ is a 4-cell. Next, define

$$f^*: f_0(BdC_0 \times \{1\}) \to f_1(BdC_1 \times \{0\})$$

by $h^* f_0(x, 1) = f_1(h(x), 0)$. Since h^* is a sewing of 4-cells, we recognize the sewing space $(C_0 \cup f_0(BdC_0 \times 1)) \cup_{h^*}(C_1 \cup f_1(BdC_1 \times I))$ to be S^4 . So, by identifying the set $f_0(BdC_0 \times I) \cup_{h^*} f_1(BdC_1 \times I)$ with $S^3 \times I$, we associate to each sewing, h , between CFC's , C_e , an embedding (suppressed) of $S^3 \times I$ in S^4 such that the closure of one complementary domain is C_0 and the other is C_1 ; whereupon, we say that $S^3 \times I$ is embedded in S^4 <u>realizing</u> the sewing h . For notational convenience, we make the convention that the complementary domains of $S^3 \times (0, 1)$ are C_0 and C_1 and that $S^3 \times \{e\} = BdC_e$. Moreover, under these identifications, $h(s, 0) = (h(s), 1)$ for all s in S^3 ; hence, if G is the decomposition of S^4 into points and the arcs $\{s\} \times I$, $s \in S^3$, then S^4/G is homeomorphic to $C_0 \cup_h C_1$. We say that this decomposition, h , the suppressed embedding of $S^3 \times I$, and $C_0 \cup_h C_1$ are <u>associated</u> to each other.

We have explained how a sewing determines an embedding of $S^3 \times I$ in S^4 ; conversely, each embedding of $S^3 \times I$ in S^4 determines a pair of CFC's and a sewing between them that would all be associated. Lastly, observe that the sewing map always extends to a first-coordinate-preserving involution of $S^3 \times I$, and (essentially) that it extends further to an involution of S^4 if and only if it is the identity sewing.

<u>Concerning Mismatched Sewings</u>

The value gained by looking at the decomposition G is that geometric tools exist to discover when $S^4 \approx S^4/G$ (or $C_0 \cup_h C_1 \approx S^4$), but for our purposes, the literature is succinctly summarized in this powerful theorem from Cannon-Daverman [18, Theorem 2.1], a so-called mismatch theorem.

<u>Theorem 3.1</u>. (<u>Cannon-Daverman</u>) <u>Let</u> $S^3 \times I$ <u>be embedded in</u> S^4 , <u>and let</u> G, C_0, C_1 <u>and</u> h <u>be the associated objects</u>. <u>If for each</u> $\varepsilon > 0$ <u>there exists an</u> ε-<u>self-homeomorphism</u>, H , <u>of</u> S^4 <u>such that, for each</u> $x \in S^3$, $H(\{x\} \times I)$ <u>is disjoint from either</u> C_0 <u>or</u> C_1 , <u>then</u> $S^4 \approx S^4/G$. ■

So it is, but simplest is not weakest, for the following corollary, a restatement in terms of sections of [18, Theorem 4.1], is their strongest four-dimensional tool; seeing that this is merely a restatement involves an argument virtually identical to the proof of 2.8, which we do not repeat.

Corollary 3.2. *Let* $S^3 \times I$ *be embedded in* S^4 *, and let* C_0, C_1, *and* h *be the associated objects. Suppose for each* $\varepsilon > 0$ *there exists a closed set,* F *, in* S^3 *such that*

1) *sections for* C_0 *over* $F \times \{0\}$ *are flat and*
2) *for each neighborhood* V *of* F *in* S^3 *, there exists an* ε*-self-homeomorphism* g *of* S^4 *fixed outside the* ε*-neighborhood of* BdC_1 *such that* $g((S^3 - V) \times I) \cap C_1 = \emptyset$.

Then $S^4 \approx S^4/G$.

Proof. We intend to apply 3.1, so let $\varepsilon > 0$ be given and obtain F from the hypotheses. Now use the first hypothesis to find an ε-self-homeomorphism f of S^4 fixed outside the ε-neighborhood of BdC_0 such that $f(F \times I) \cap BdC_0 = \emptyset$. Let V be a neighborhood of F in S^3 such that $f(V \times I) \cap BdC_0 = \emptyset$, and apply the second hypothesis to obtain g . When ε is sufficiently small, f and g have disjoint support, and the composition fg will do for H in 3.1. ∎

The following theorem sets forth more symmetric and seemingly weaker conditions under which Theorem 3.1 might be applied; we shall make use of it in the next chapter.

Theorem 3.3. (Geometric Mismatch Theorem) Let $S^3 \times I$ be embedded in S^4, and let G, C_0, C_1, and h be the associated objects. Suppose for each $\varepsilon > 0$ there exist disjoint neighborhoods V_e of BdC_e such that for any pair of PL 2-polyhedra $P_e \subset V_e$ and any pair of compact sets D_e contained in $IntC_e$ there exists an ε-pseudoisotopy H_t of S^4 such that each H_t is supported in $C_0 \cup V_0 \cup C_1 \cup V_1$, $H_1(P_e \cup D_e) \subset C_e$, and

$$h(H_1(P_0 \cup D_0) \cap BdC_0) \cap H_1(P_1 \cup D_1) = \emptyset \ .$$

Then $S^4 \approx S^4/G$ and $C_0 \cup_h C_1 \approx S^4$.

Proof. Again we shall use 3.1, so let $\varepsilon > 0$ be given and determine the neighborhoods V_e. We want to apply Proposition 2.3 twice, once for each V_e. According to that proposition, when the present hypotheses are applied to the proper pair of 2-polyhedra P_e' and the proper pair of compact sets D_e' and when $X_e = H_1(P_e' \cup D_e') \cap BdC_e$, it is true that for any open subsets U_e of S^4 with $X_e \subset U_e$ there exist 5ε-self-homeomorphisms h_e of S^4 supported in the ε-neighborhood of C_e with $h_e(BdC_e) \subset U_e \cup IntC_e$. In view of the special property H_1 has by hypothesis, the X_e satisfy $h(X_0) \cap X_1 = \emptyset$. Thus we may choose the U_e so that

1) $(\{s\} \times I) \cap U_0 \neq \emptyset$ implies that $(\{s\} \times I) \cap U_1 = \emptyset$, for all s in S^3.

Let the h_e be obtained with respect to such U_e. If ε is sufficiently small, then the h_e have disjoint support and the composition

$g = h_0 h_1$ is a 5ε-self-homeomorphism of S^4 satisfying $g(BdC_e) \subset U_e \cup IntC_e$. It follows from 1) that each arc $\{s\} \times I$ is disjoint from one of $g(C_0)$ or $g(C_1)$. Let $H = g^{-1}$. Then $H(\{s\} \times I)$ is disjoint from one of C_0 or C_1 for all s in S^3; so 3.1 applies to complete the proof. ∎

The following corollary, a geometric mismatch theorem more completely analogous to those found in Daverman [26] and Cannon-Daverman [18] than the preceding, is now obvious.

Corollary 3.4. _Let_ $S^3 \times I$ _be embedded in_ S^4, _and let_ C_0, C_1, _and_ h _denote the associated objects_. _If_ $F_e \subset BdC_e$ _is a_ 1-GULC _set for_ C_e _such that_ $(\{s\} \times I) \cap F_0 \neq \emptyset$ _implies that_

$$(\{s\} \times I) \cap F_1 = \emptyset$$

for all s _in_ S^3, _then_ $S^4 \approx S^4/G$. ∎

Remark. The hypotheses of 3.3 seem weaker than those of 3.4, but, in fact they are equivalent; however, we omit the proof, saying only that the proof of Proposition 2.4 is relevant (or see the proof of Proposition 4.9 which includes the argument).

Existence of Sewings Which Yield S^4

Happily, this is one question which is better answered in S^4 than in S^n, where given two crumpled n-cubes, C and D, one must be of type 1 before it is known that there exists a sewing which yields S^n (Daverman [26] shows if S is the set of homeomorphisms between BdC and BdD with the sup-norm metric, that then there is a dense G_δ G in S such that $C \cup_g D \approx S^n$ for all g in G). The type 1 hypothesis is needed because in S^n $(n \geq 4)$ it is not known if one wild Cantor set can be pushed off another by applying an ε-self-homeomorphism to S^n. In S^3, they can, since each wild Cantor set in S^3 is defined by cubes with handles; i.e., has embedding dimension 1 - this is the relevant fact. So, it happens that the techniques of Daverman [26, Theorem 7.1] suffice (given 3.4) to establish the following.

<u>Theorem 3.5</u>. <u>For</u> CFC's C_e, <u>let</u> S <u>denote the space of all homeomorphisms</u> $h: BdC_0 \to BdC_1$ <u>under the sup-norm metric</u>. <u>If each</u> C_e <u>has at most G-type</u> 3A, <u>then</u> S <u>contains a dense</u> G_δ<u>-set</u> G <u>such that</u> $C_0 \cup_g C_1 \approx S^4$ <u>for each</u> g <u>in</u> G.

<u>Proof</u>. Let $F_e \subset BdC_e$ be an at most 1-demensional 1-GULC F_σ set for C_e. Let $F_e = \cup F_{ei}$ with each F_{ei} compact. It follows that the subset $G_i \subset S$ of homeomorphisms $h: BdC_0 \to BdC_1$ with $h(F_{0i}) \cap F_{1i} = \emptyset$ is open and dense in S. Set $G = \cap G_i$. Then G is a G_δ-set; while S is not complete, G is still dense by the criterion 1.1. Finally, if $g \in G$, then $g(F_0) \cap F_1$ is empty, so $C_0 \cup_g C_1 \approx S^4$ by 3.4. ∎

There is another circumstance when one knows that a sewing yields S^4.

<u>Theorem 3.6</u>. <u>Suppose</u> h <u>is a sewing of</u> CFC's, C_e. <u>If</u> C_0 <u>has G-type</u> 1 <u>and</u> C_1 <u>has any G-type, then for each</u> $\varepsilon > 0$ <u>there</u>

exists a homeomorphism $g\colon BdC_0 \to BdC_1$ with $\rho(g, h) < \varepsilon$ and $C_0 \cup_g C_1 \approx S^4$.

Proof. Let $\varepsilon > 0$ be given, and let F_e be a 1-GULC set for C_e . Assume that F_0 can be written as $\cup F_{0i}$ with each F_{0i} a Cantor set tame in BdC_0 , and assume that F_1 can be written as $\cup F_{1i}$ with each F_{1i} an at most 2-dimensional compactum in BdC_1 (see 2.4 and 2.15). Now using the fact that $h(F_{0i})$ is tame in BdC_1 , find a sequence of numbers ε_i and a sequence of ε_i-self-homeomorphisms g_i of BdC_1 such that

1) $\rho(g_{i+1}h(x), g_i h(x)) < \varepsilon/2^i$ for all x in BdC_0 ,

2) $g_i h(\bigcup_{j=1}^{i} F_{0j}) \cap (\bigcup_{j=1}^{i} F_{1j}) = \emptyset$, and

3) $\varepsilon_i < \frac{1}{2}\text{dist}(g_i h(\bigcup_{j=1}^{i} F_{0j}) , \bigcup_{j=1}^{i} F_{1j})$.

Then the convergence criterion 1.1 shows that we may assume that $g = \lim g_i h$ exists and is a homeomorphism. Condition 1) insures that $\rho(g, h) < \varepsilon$, while 2) and 3) insure that $g(F_0) \cap F_1 = \emptyset$; whereupon $C_0 \cup_g C_1 \approx S^4$ by 3.4. ■

Examples

We conclude this chapter with two examples. One is of a sewing of a G-type 1TT CFC to itself that does not yield S^4 , and the other is of a sewing between CFC's which does yield S^4 but which does not satisfy the hypotheses of Proposition 3.3. Both examples are adapted from Daverman's work in [26].

In the first example, the construction and argument from [26] may be applied in a virtually unaltered state once the following proposition is established. The proposition itself is a four-dimensional version of a Theorem (9.1) from [26] which we noticed could be proved when $n = 4$ by adding some local flatness - thus our proof reproduces much of the proof in [26]. Observe that the proposition deals with a 3-sphere in S^4 rather than a CFC and with "1-ULC"-ness rather than "1-GULC"-ness.

Proposition 3.7. *If a 3-sphere S in S^4 bounds two crumpled cubes, C_e, of type 1 and is locally flat modulo a 2-dimensional set Q, then there exist F_σ-sets, F_e, in $\mathrm{Bd}C_e$ (= S) such that $F_e \cup \mathrm{Int}C_e$ is 1-ULC and $F_0 \cap F_1 = \emptyset$.*

Proof. Let D be a 2-cell and let $f_e\colon D_e \to C_e$ be maps such that $f_e(\mathrm{Bd}D) \subset \mathrm{Int}C_e$. If for each $\varepsilon > 0$ there exist ε-approximations $g_e\colon D_e \to C_e$ to the f_e such that $g_0(D) \cap g_1(D) = \emptyset$, then the theorem is proved by [26, 5.2]. So let $\varepsilon > 0$ be given.

To find the approximations, g_e, we improve the f_e in four stages.

Before making the first improvement, assume, by [26, 2.1], that $f_e^{-1}(S)$ is 0-dimensional. Since C_e has type 1, there exist sets $Z_e \subset Q$, unions of Cantor sets tame in S, such that $Z_e \cup \mathrm{Int}C_e$ is 1-ULC - of course these need not be disjoint. Let $\delta > 0$ be so small that any δ-subset of S is contained in a 3-cell in S of diameter less than $\frac{1}{4}\varepsilon$, and cover $f_0^{-1}(S) \cup f_1^{-1}(S)$ by the interiors of pairwise disjoint 2-cells $B_1,\ldots, B_k$ in $\mathrm{Int}D$ so small that $\mathrm{diam} f_e(B_j) < \delta$ for all j. To make the first improvement, find approximations $s_e\colon D \to S^4$ (notice the range is S^4) such that

1) $s_e | D - \cup B_j = f_e | D - \cup B_j$,

2) $\rho(s_e, f_e) < \frac{1}{4}\varepsilon$,

3) $\operatorname{diam} s_e(B_j) < \delta$ for all j , and

4) $s_0(D) \cap s_1(D) \cap S = \emptyset$.

One way to accomplish this is to take PL approximations, make $s_0(D) \cap s_1(D)$ finite by general position, and use the fact that S is nowhere dense in S^4 .

Before making the second improvement, set $D_e = s_e^{-1}(s_e(D) \cap S)$ and $U_e = s_e^{-1}(s_e(D) - C_e)$. Then $\operatorname{Fr}U_e \subset D_e$, and because each component of U_e is a subset of some B_k , each component of $s_e(\operatorname{Fr}U_e)$ is a δ-subset of S . Hence, by applying the Tietze Extension Theorem and the choice of δ , one can find approximations, $t_e\colon D \to C_e$, to the s_e such that

5) $t_e | D - U_e = s_e | D - U_e$,

6) $t_e(D) \cap S = t_e(U_e \cup D_e)$, and

7) $\rho(t_e, s_e) < \frac{1}{4}\varepsilon$.

Furthermore, by taking PL approximations and using general position and demension theory, we may suppose that the intersections between $t_e(U_e)$, Q, and Z_e are improved to the point that

8) $t_0(U_0) \cap t_1(U_1)$ is a PL 1-polyhedron in S ,

9) $t_0(U_0) \cap t_1(U_1) \cap Q$ is 0-dimensional, and

10) $t_0(U_0) \cup t_1(U_1) \subset S - (Z_0 \cup Z_1)$ (however $t_e(D_e)$ could still meet Z_e) .

Notice that $t_0(D - U_0) \cap t_1(D - U_1) = s_0(D - U_0) \cap s_1(D - U_1) = \emptyset$. Hence there exist open sets $V_e \subset \operatorname{Cl}V_e \subset U_e$ such that

11) $t_0(D - V_0) \cap t_1(D - V_1) = \emptyset$,

and, by using 9), such that

12) $t_0(\mathrm{BdV}_0) \cap t_1(\mathrm{ClV}_1) \cap Q = \emptyset$ and $t_1(\mathrm{BdV}_1) \cap t_0(\mathrm{ClV}_0) \cap Q = \emptyset$.

In the third improvement, we use the Z_e to "cut off" $w_e(D - \mathrm{ClV}_e)$; this is the common high-dimensional technique and we refer the reader to [26] for particulars. Accordingly, we take V_e so large - or BdV_e so close to BdU_e - that like in [26], there exist approximations $w_e\colon D \to C_e$ such that

13) $w_e|\mathrm{ClV}_e = t_e|\mathrm{ClV}_e$,

14) $\rho(w_e, t_e) < \frac{1}{4}\varepsilon$,

15) $w_e(D - \mathrm{ClV}_e) \cap \mathrm{BdC}_e \subset Z_e$, and

16) $w_0(D - V_0) \cap w_1(D - V_1) = \emptyset$.

Because $w_e(D - V_e)$ will meet S in Z_e and only near $t_e(D_e)$, 16) follows from taking BdV_e close enough to D_e and from the consequence of 4) that $t_0(D - V_0) \cap t_1(D - V_1) = \emptyset$.

The first three improvements are complete. Figure 2 is a schematic representation in the 3-sphere S of the sets we have constructed so far. In that figure, Q , and $w_e(\mathrm{ClV}_e)$ are represented by disks and $w_0(\mathrm{ClV}_0) \cap w_1(\mathrm{ClV}_1)$ by an arc. We now make two observations. The first is that

17) $w_0(D) \cap w_1(D) = w_0(\mathrm{ClV}_0) \cap w_1(\mathrm{ClV}_1)$.

This is true because

$$\begin{aligned} w_0(D) \cap w_1(D) = {} & [w_0(D - \mathrm{ClV}_0) \cap w_1(D - \mathrm{ClV}_1)] \\ & \cup [w_0(D - \mathrm{ClV}_0) \cap w_1(\mathrm{ClV}_1)] \cup [w_1(D - \mathrm{ClV}_1) \cap w_0(\mathrm{ClV}_0)] \\ & \cup [w_0(\mathrm{ClV}_0) \cap w_1(\mathrm{ClV}_1)] , \end{aligned}$$

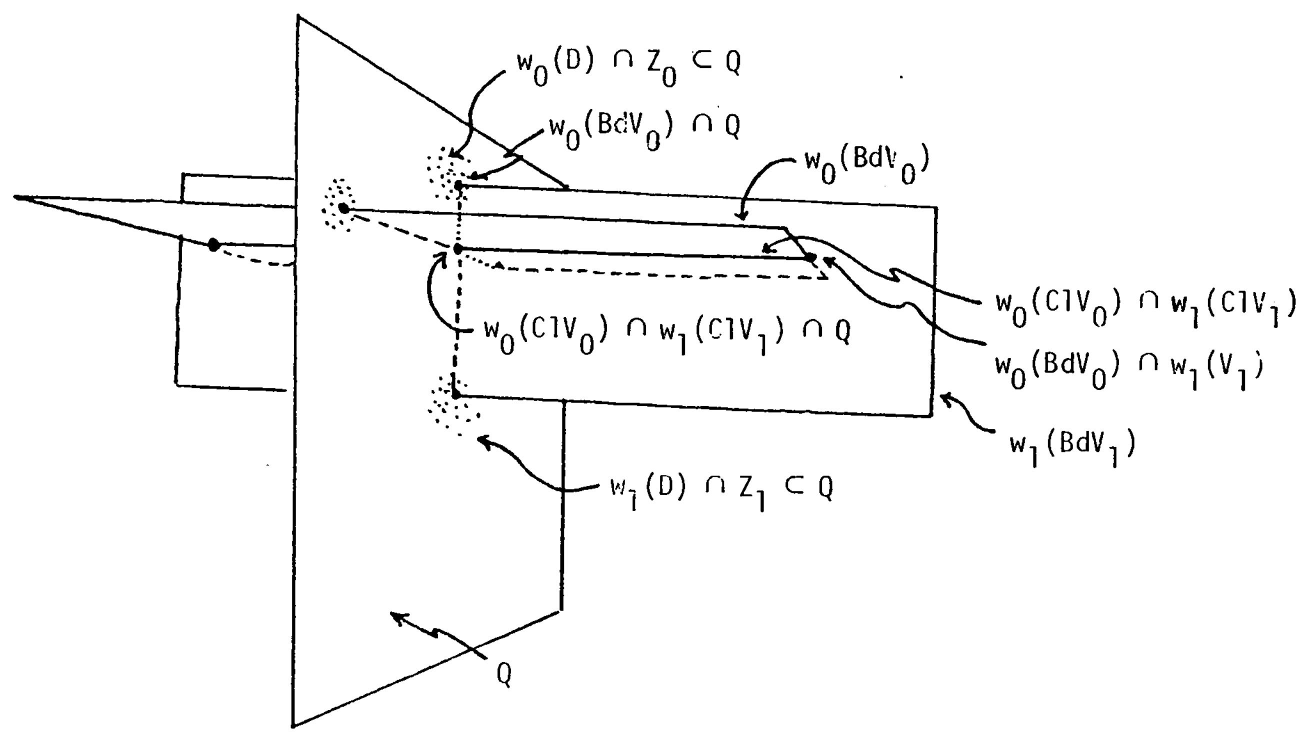


Figure 2. A Schematic Diagram Showing the Sets After the Third Improvement.

and the first three of these unionands are empty by 16), 15), and 10). The second observation makes essential use of the hypothesized local flatness and is the center of the proof. It is that the distance

18) $\text{dist}(w_0(\text{Cl}V_0) \cap w_1(\text{Cl}V_1)\ ,\ w_0(D) \cap Z_0)$

is greater than zero. This is a consequence of condition 12), specifically, 18) is true because $w_0(\text{Cl}V_0) \cap w_1(\text{Cl}V_1)$ is compact and disjoint from $[w_0(D) \cap Z_0] \cup [Q \cap w_0(\text{Bd}V_0)]$ which is also compact.

To make the last improvement, let γ denote the distance in 18). We do not change w_0 ; that is, let $g_0\colon D \to C_0$ be w_0 . To find g_1 , let R be a neighborhood of $w_1^{-1}(w_0(\text{Cl}V_0) \cap w_1(\text{Cl}V_1))$ in D so close to $w_1^{-1}(w_0(\text{Cl}V_0) \cap w_1(\text{Cl}V_1))$ that $\text{dist}(w_1(R), w_0(D) \cap Z_0) > \frac{1}{2}\gamma$ and that there exists an approximation $g_1\colon D \to C_1$ such that

19) $\rho(g_1, w_1) < \frac{1}{2}\min\{\gamma, \frac{1}{4}\varepsilon\}$,

20) $g_1|D - R = w_1|D - R$, and

21) $g_1(R) \cap S \subset Z_1$.

That is, we have obtained g_1 by the high-dimensional "cutting off" technique mentioned earlier.

All that remains is to observe that $g_0(D) \cap g_1(D) = \emptyset$. Specifically,

$$g_0(D) \cap g_1(D) = [g_0(D) \cap g_1(D - R)] \cup [g_0(\text{Cl}V_0) \cap g_1(R)]$$
$$\cup\ [g_0(D - \text{Cl}V_0) \cap g_1(R)]\ ,$$

and each of these unionands is empty. The first is empty by 20), 17), and the definition of R. The second is empty by 10) - that is, $g_0(\mathrm{Cl}V_0) \subset S - Z_1$ - and 21). Finally, the last is empty by 18), 19), the value of γ, and the fact that

$$g_0(D - \mathrm{Cl}V_0)) \cap Z_1 \subset w_0(D) \cap Z_0 . \quad \blacksquare$$

Replacing [26, Theorem 9.2] by a reference to the proposition just proved in the proof of [26, Example 13.1] will establish the following, though examples of this sort were already described in Eaton [31].

Example 3.8. There exists a CFC , C , which is locally flat modulo a Cantor set tame in $\mathrm{Bd}C$ (so C has G-type 1TT) and a sewing $h\colon \mathrm{Bd}C \to \mathrm{Bd}C$ such that $C \cup_h C \neq S^4$. $\blacksquare$

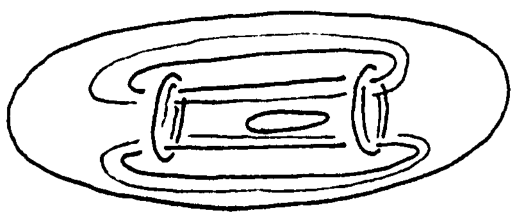
The proof given in [26] relies upon the technique of Mixing Cantor Sets which Daverman first presented in [23]. The particulars of this technique are fascinating, though it is not necessary for us to go into them. But we do need to know something of its terminology and results in order to adapt other examples from [26] to the case $n = 4$.

Suppose X is a Cantor set in a k-manifold Q . A defining sequence for X is a sequence $\{M_n\}$ of k-manifolds in Q such that $M_{n+1} \subset \mathrm{Int}M_n$, X meets each component of each M_n , and $X = \cap M_n$. It is said to have 2-cell factors if each M_n may be written as the product of a 2-cell and a (k - 2)-manifold. A subset, C , of X is called admissible with respect to M_n if $C \cap M \neq \emptyset$ implies $C \cap M' \neq \emptyset$ whenever M is a component of M_n , M' is a component of M_{n+1} , $M' \subset M$, and n is odd. The "with respect to M_n" is omitted whenever there is no ambiguity. When Y is another Cantor set for which the context supplies a defining sequence, a homeomorphism $h: X \to Y$ is said to mix their admissible sets if $h(C) \cap C' \neq \emptyset$ whenever C is an admissible set of X and C' is an admissible set of Y . The last term, ramification, refers to a conceptually simple - but notationally difficult - procedure for modifying Cantor sets possessing defining sequences with 2-cell factors.

Let $\{M_n\}$, defining X , have 2-cell factors, and let M_{ji} be a component of M_j . The basic element of the modification is the k-fold replication of M_{ji} by which one obtains a new defining sequence defining a new Cantor set. The k-fold replication is found by writing $M_{ji} = D^2 \times P$, finding k pairwise disjoint disks, $B_1 \ldots B_k$, in D^2 , replacing M_{ji} by $\cup B_t \times P$, and reproducing $\{M_n \cap M_{ji}\}$ in each $B_t \times P$ - for precision, name a homeomorphism $h_t: D \to B_t$ and change M_n to $(M_n - (M_n \cap M_{ji})) \cup (\underset{t}{\cup}(h_t \times \mathrm{id})(M_n \cap M_{ji}))$ for $n \geq j$ and leave M_n alone when $n < j$.

The ramified Cantor set is now found by an inductive procedure which one begins by replicating each component of M_1 (with k varying from component to component however one pleases). After this stage is complete, one has a new defining sequence $\{M_n^1\}$ (of a new Cantor set) with probably vastly more components to M_2^1 than there were to M_2 . Next replicate each component of M_2^1 (with k varying from component to component however one pleases). This produces a third defining sequence $\{M_n^2\}$ (defining a third Cantor set) which agrees with $\{M_n^1\}$ at the first stage $(M_1^1 = M_1^2)$ but probably differs elsewhere. One continues indefinitely (replicating the components of M_3^2 next) and afterwards sets $R_n = M_n^n$. Then, provided the components of R_n get small as $n \to \infty$, $\{R_n\}$ is a defining sequence for a Cantor set Y which is a ramification of X .

We shall be particularly interesting in ramifying Antoine's Necklace [5]. If one were always to choose $k = 2$, then ramifying Antoine's Necklace would produce a Cantor set with a defining sequence $\{R_n\}$ in which each component of R_n contained precisely eight components of R_{n+1} linked exactly as shown.



The results from [23] which we shall need may now be summarized in two propositions.

Proposition 3.9. *If X and Y are Cantor sets in manifolds Q^k and Q^n with defining sequences $\{M_n\}$ and $\{N_n\}$ such that both have 2-cell factors, then there exist ramifications X' of X and Y' of Y and a homeomorphism $h: X' \to Y'$ mixing their admissible sets.* ■

Proposition 3.10. *Suppose X is a Cantor set in Q and $\{M_n\}$ is a defining sequence for X with 2-cell factors such that $j: \pi_1(\mathrm{Bd}M) \to \pi_1(Q - M_{i+1})$ is 1-1 for each component M of M_i . If J is a simple closed curve in $Q - M_1$ which is contractible in*

Q <u>but not contractible in</u> $Q - M_1$, <u>then each contraction of</u> J <u>in</u> Q <u>contains an admissible subset of each ramification of</u> X . ■

In the course of our further work, we shall adapt four more examples from [26] to the case $n = 4$. Three depend upon the existence of a suitably equipped crumpled 3-cube to subject to constructions based upon mixing. We next construct such a crumpled 3-cube. Afterwards, we present the second example that was promised earlier, but save the other examples for later chapters.

The suitably equipped crumpled 3-cube will be obtained from a ramified Antoine's Necklace by a method originally described in Antoine [5] and Alexander [2]; we give a brief description.

Let X be a Cantor set and $\{M_n\}$ be a defining sequence for X . Express M_n as $\bigcup_i T_{ni}$ where each T_{ni} is a component of M_n . Let Q_n be the sequence of flat 3-manifolds in S^3 defined inductively as follows. First, $Q_1 = M_1 \cup [\bigcup_i R_{1i}] \cup B'$ where B' is a cell and the R_{1i} are pairwise disjoint 3-cells (which we shall call "tubes") in $M_1 - \mathrm{Int}M_2$ such that $R_{1i} \cap B'$ is a 2-cell in $\mathrm{Bd}B'$ and $R_{1i} \cap T_{1j}$ is a disk in $\mathrm{Bd}T_{1j}$ when $i = j$ and empty when $i \neq j$. Once Q_n has been obtained, set $Q_{n+1} = M_{n+1} \cup [\bigcup_i R_{(n+1)i}] \cup Q_n$ where the $R_{(n+1)i}$ are pairwise disjoint 3-cells ("tubes") contained in $M_n - \mathrm{Int}\, M_{(n+1)}$ with $R_{(n+1)i} \cap Q_n$ a 2-cell in $\mathrm{Bd}Q_n$ and $R_{(n+1)i} \cap T_{(n+1)j}$ a disk in $\mathrm{Bd}T_{(n+1)j}$ when $i = j$ and empty when $i \neq j$. When the Q_n are defined in this manner, each is connected, $\cap Q_n = X \cup B' \cup [\bigcup_{ni} R_{ni}]$ is a 3-cell B , and $X \subset \mathrm{Bd}B$. We will say that the crumpled 3-cube $C = S^3 - \mathrm{Int}B$ is <u>obtained from</u> $\{M_n\}$ (<u>or obtained from</u> X) <u>by the method of Alexander</u>.

When X is a ramified Antoine's Necklace, each T_{ni} is a solid torus and it is possible to find those $R_{(n+1)j}$ which would lie in T_{ni} so that there exist two "plugs" F_{ni} and E_{ni} in $T_{ni} - \bigcup_j R_{(n+1)j}$ placed as is shown in Figure 3 - the essential property of the plugs is that no torus $T_{(n+1)j}$ meets both. Obviously, these plugs may always be found whatever the ramification (of Antoine's Necklace), and we always presume that they are there - especially in the next proposition.

Proposition 3.11. *If C is a crumpled 3-cube obtained by the method of Alexander from a ramified Antoine's Necklace, X, then there exist disjoint sets F_0 and F_1 in X such that $F_e \cup \text{Int}C$ is 1-ULC.*

Proof. Let D be a 2-cell and retain the notation above. It is sufficient (Cannon [17]) to show for each $\varepsilon > 0$ that when $f_e\colon D \to C$ are maps such that $F_e(\text{Bd}D) \subset \text{Int}C$ there exist 3ε-approximations $g_e\colon D \to C$ such that $g_e|\text{Bd}D = f_e|\text{Bd}D$ with $g_0(D) \cap g_1(D) \cap \text{Bd}C = \emptyset$. So let ε and the f_e be given. Choose N so large that each torus T_{Ni} and each tube R_{Ni} has diameter less than ε. Use general position and the fact that each T_{Ni} has a 1-dimensional spine to find ε-approximations $h_e\colon D \to S^3$ such that

1) $h_e|\text{Bd}D = f_e|\text{Bd}D$,
2) $h_e|\text{Int}D$ is PL,
3) $h_e(D) \subset S^3 - (B' \cup (\bigcup_{n=0}^{N} \bigcup_i R_{ni}))$, and
4) $h_e(D) \cup T_{Ni}$ is a finite union of disks with meridians for boundaries (see the diagram) for each i.

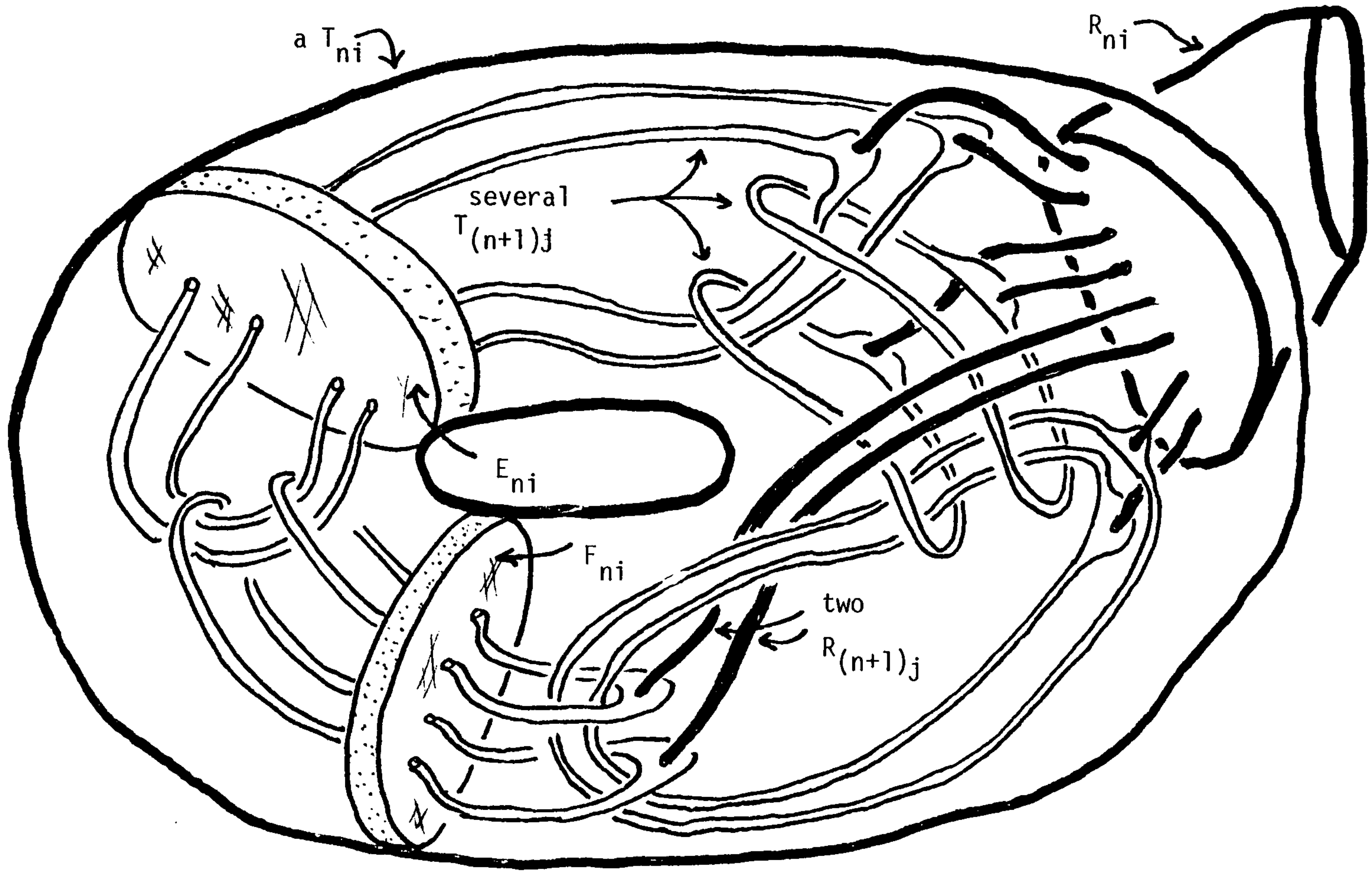
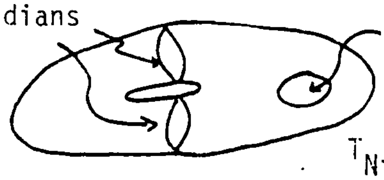


Figure 3. The Location of the Plugs E_{ni} and F_{ni} .



Now using 4) and the fact that a meridian of T_{Ni} is contractible in either of $(BdT_{Ni}) \cup E_{Ni} - R_{Ni}$ or $(BdT_{Ni}) \cup F_{Ni} - R_{Ni}$ find ε-approximations $s_e\colon D \to S^3$ to the h_e such that

6) $s_e|D - h_e^{-1}(h_e(D) \cap M_N) = h_e|D - h_e^{-1}(h_e(D) \cap M_N)$,

7) $s_0(h_0^{-1}(h_0(D) \cap T_{Ni})) \subset (E_{Ni} \cup BdT_{Ni}) - R_{Ni}$ for all i , and

8) $s_1(h_1^{-1}(h_1(D) \cap T_{Ni})) \subset (F_{Ni} \cup BdT_{Ni}) - R_{Ni}$ for all i .

Then $s_0(D) \cap B \subset \bigcup_i E_{Ni}$ and $s_1(D) \cap B \subset \bigcup_i F_{Ni}$; hence $s_e(D) \cap B \subset \bigcup_{n>N+1} \bigcup_i R_{ni}$.

Let $r\colon \bigcup_{n>N+1} R_{ni} \to \bigcup_{n>N+1} R_{ni} \cap BdC$ be a retraction (this range is a finite collection of disks in BdC which contains X) , and set $g_e = rs_e$. Then $g_e(D) \subset C$. Furthermore, $g_e(D) \subset (W_e \cap BdC) \cup IntC$ where W_0 is the union of all the $T_{(N+1)j}$ that meet E_{Ni} and W_1 is the union of all the $T_{(N+1)j}$ that meet F_{Ni} . The defining property of the E_{Ni} and F_{Ni} was that $W_0 \cap W_1 = \emptyset$; hence $g_0(D) \cap g_1(D) = \emptyset$. Moreover, g_e is an ε-approximation to s_e , and so g_e is a 3ε-approximation to h_e . ■

We need one last term. Let C be a closed n-cell complement, and let $f\colon C \to [0, 1)$ be a continuous function with $f^{-1}(0) = BdC$. The set $InfC = \{(x, y) \in S^n \times E^1 | x \in C$ and $-f(x) \leq y \leq f(x)\}$ is called the <u>inflation</u> of C ; the particular function f does not affect InfC topologically nor determine how C is embedded in $S^n \times E^1$ - therefore, we speak of <u>the</u> inflation of C. If one takes the two point compactification of $S^n \times E^1$, then, provided that $C \cup_{id} C \approx S^n$, InfC is a closed n-cell complement - this and other facts about inflations are established in [26] for $n > 4$ and in [18] for $n = 4$.

The next proposition exhibits the suitably equipped crumpled 3-cube that we have been aiming at.

<u>Proposition 3.12. There exists a closed 3-cell complement, C, in S^3 which is locally flat modulo a Cantor set, Y, and such that $C \cup_{id} C \approx S^3$, a Cantor set Z in S^4, a homeomorphism $h: Y \to Z$ mixing their admissible sets (each has a fixed defining sequence), a loop L in IntC such that every contraction of L in S^3 contains an admissible subset of Y, and a loop L' in S^4 such that every contraction of L' in S^4 contains an admissible subset of Z.</u>

<u>Proof</u>. Let X be Antoine's Necklace, W be the Blankinship [12] Cantor set in S^4, L be a loop in $S^3 - X$ such that every contraction of L in S^3 meets X, and L' be a loop in $S^4 - W$ such that every contraction of L' in S^4 meets W (see [12]). X, as we know, has a defining sequence with 2-cell factors. Blankinship's construction of W shows that it does too. Thus 3.9 applies, and there exist ramifications Y of X and Z of W and a homeomorphism $h: Y \to Z$ mixing their admissible sets. The defining sequences of Y and Z also satisfy the hypotheses of 3.10, so each contraction of L (L') in S^3 (S^4) contains an admissible subset of Y (Z). Finally, find C with $Y \subset \mathrm{Bd}C$ by the method of Alexander. By general

position, all the tubes, R_{ni}, can be found in $S^3 - L$, so we may assume that $L \subset \mathrm{Int}C$. Proposition 3.11, together with Eaton [30], shows that $C \cup_{id} C \approx S^3$. ■

This proposition enables us to adapt another example from [26] to the case $n = 4$, this being the one which shows the converse of 3.3 to be false.

Example 3.13. There exists a G-type 1TT CFC C_2, a G-type 2 CFC D_2, a sewing $h\colon \mathrm{Bd}C_2 \to \mathrm{Bd}D_2$ which yields S^4, and loops $L \subset \mathrm{Int}C_2$ and $L' \subset \mathrm{Int}D_2$ such that, whenever $f\colon D \to C_2$ is a contraction of L and $g\colon D \to D_2$ is a contraction of L', $h(f(D) \cap \mathrm{Bd}C_2) \cap g(D) \neq \emptyset$.

Proof. One lets C_2 be the inflation of C in 3.12. Then our notation in 3.12 agrees with the first two paragraphs of the proof of Example 13.2 in [26], and aside from the determination of the G-types of C_2 and D_2, the rest of that proof applies unaltered to establish the "such that" of the statement we are proving. C_2, being an inflation, satisfies the hypotheses of Proposition 2.18, so it has G-type 1TT. D_2, being locally flat modulo a Cantor set, has at nost G-type 2. If it had G-type 1 or 0, then it would have type at most 1 and Proposition 3.7 would imply the existence of mismatched 1-ULC sets in the sewing space - contradicting the "such that" in the statement we are proving.

Remark. Daverman states his example for $n \geq 6$. All that was needed to extend it to the case $n = 4$ was a way to obtain C_2. This is also true for the case $n = 5$. However, this may also be accomplished by Daverman's original proof now that Cannon-Daverman [18, Theorem 7.8] is available to show that an inflation of a Blankinship crumpled 4-cube is a crumpled 5-cube.

CHAPTER IV

THE IDENTITY SEWING

The sewing space $C \cup_{id} C$, where $id: BdC \to BdC$ is the identity, is particularly interesting. The diagrams which follow display what we know about it. The first one summarizes the relevant work of Daverman in [26] and is valid when $n \geq 5$ - the crucial implications are the darker. The second shows to what degree we have been able to extend this work to CFC's - our principal results being the darker implications, and the numbers refering to theorems in this work. Most of these are situated later in this chapter, where they are preceded by preliminaries concerning general position in E^4, which we include for the sake of completeness.

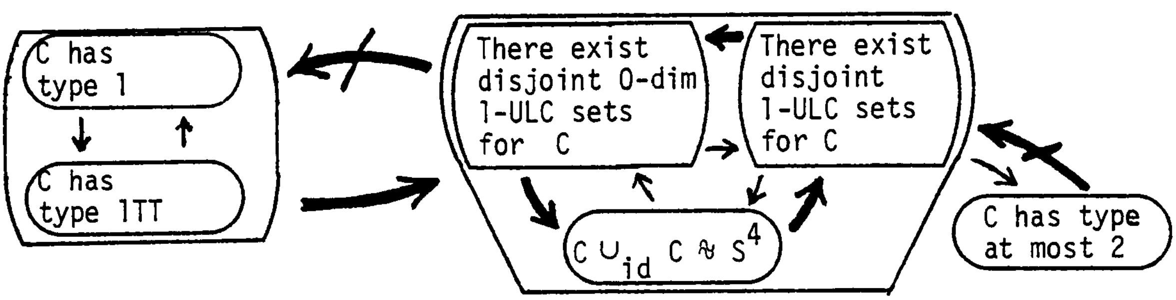


Diagram 1. The Identity Sewing for $n \geq 5$.

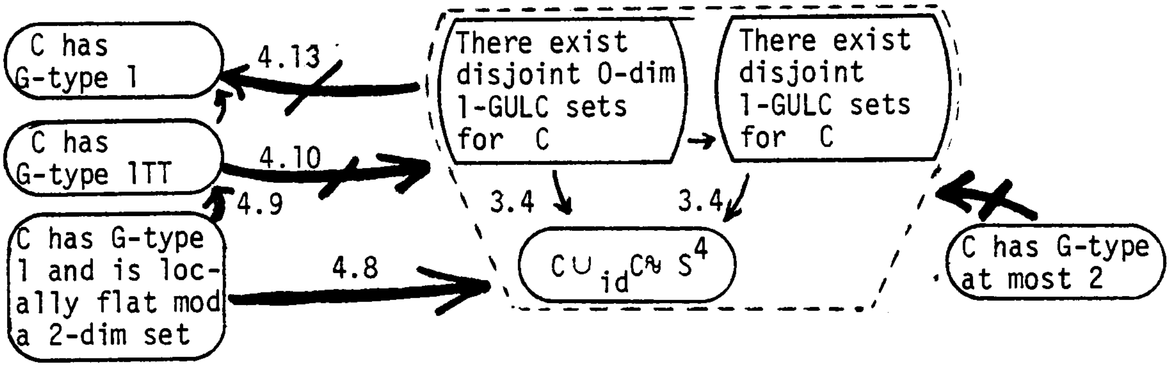


Diagram 2. The Identity Sewing for $n = 4$.

General Positions in E^4

This section is self-contained, except for references; its chief function is to use demension theory to control the size of the projection to S^3 of certain sets in $S^3 \times I$ - see 4.6, which is a preparation for the next section. Before that, we give proofs of some facts sometimes taken for granted - indeed, we did just that in Chapter II, but referenced this section - and which we include only for the sake of completeness. The basic tool used here is the following proposition from Bing-Kister [11] (paraphrased and weakened); see also Rushing [48, Theorem 3.5.1] and its proof.

Proposition 4.1. (Bing-Kister) *Suppose that* P *is a possibly infinite* PL 1-*polyhedron in* E^4 *embedded as a closed subset, that* $\varepsilon > 0$, *and that* $f: P \to E^4$ *is a* PL ε-*re-embedding of* P . *Then there is an* ε-*push* h_t *of* (E^4, P) *such that* $h_1|P = f$. ■

There are four statements to be proved, 4.3 through 4.6, each of which is self-explanatory - 4.2 is a lemma for 4.3.

Lemma 4.2. Let C be a tame Cantor set in E^3, J a compact interval in $(0, 1)$, and D a closed 2-dimensional subset of $E^3 \times (0, 1)$. For each $\varepsilon > 0$, there exists an ε-push, h_t, of $(E^3 \times I, C \times J)$ fixed on $E^3 \times \{0, 1\}$ such that $h_1(D) \cap (C \times J) = \emptyset$.

Proof. Let $\varepsilon > 0$ be given, and assume without loss of generality that $\varepsilon < \operatorname{dist}(C \times J, E^3 \times \{0, 1\})$. Because C is tame, it is defined by cells, so let $B_1 \ldots B_n$ be pairwise disjoint, flat cells in E^3 such that $C \subset \cup \operatorname{Int}B_i$ and $\operatorname{diam}(B_i \times \{t\}) < \frac{1}{2}\varepsilon$ for all t in J. Let $q_i \in \operatorname{Int}B_i$, and observe that $\cup\{q_i\} \times J$ is a PL 1-polyhedron in $E^3 \times (0, 1)$ (given the product triangulation). Since D is 2-dimensional, it does not locally separate $E^3 \times (0, 1)$ [40], so, since D is closed too, $E^3 \times (0, 1) - D$ is locally arcwise connected; hence there exists a PL $\frac{1}{2}\varepsilon$-re-embedding

$$f: \cup \{q_i\} \times J \to ((E^3 \times (0, 1)) - D .$$

By Proposition 4.1, there exists a $\frac{1}{2}\varepsilon$-push, g_t, of $(E^3 \times I, \cup\{q_i\} \times J)$ such that $g_1 | \cup\{q_i\} \times J = f$.

Because the B_i are flat, there exists a second $\frac{1}{2}\varepsilon$-push, k_t, of $(E^3 \times I, \cup B_i \times J)$ such that $k_1 g_1(B_i \times J) \subset (E^3 \times I) - D$; that is, k_1 squeezes $g_1(B_i \times J)$ close to $g_1(\{q_i\} \times J)$. Now $k_t g_t$ is an ε-push of $(E^3 \times I, C \times J)$ such that $k_1 g_1(C \times J) \cap D = \emptyset$, so let $h_t = g_t^{-1} k_t^{-1}$; then $h_1(D) \cap (C \times J) = \emptyset$, and h_t is fixed on $E^3 \times \{0, 1\}$ by the size of ε. ■

Proposition 4.3. Let P be a closed 2-dimensional subset of $S^3 \times (0, 1)$ and $F = \cup F_n$ be a countable union of tame Cantor sets in S^3. For each $\varepsilon > 0$, there exists an ε-isotopy, h_t, of $S^3 \times I$ fixed on $S^3 \times \{0, 1\}$ such that $h_1(P) \cap (F \times I) = \emptyset$.

Proof. Let $\varepsilon > 0$ be given, and let J_k be a sequence of compact intervals in $(0, 1)$ such that $\cup J_k = (0, 1)$. We shall find, by induction, a sequence of numbers, ε_i, and a sequence of ε_i-self-homeomorphisms, g_i, of $S^3 \times I$ such that

1) $g_i | S^3 \times \{0, 1\} = \mathrm{id}$,

2) $g_i(P) \cap (\overset{i}{\cup} F_k \times \overset{i}{\cup} J_k) = \emptyset$, and

3) $\varepsilon_i < \frac{1}{2} \min\{\varepsilon_{i-1}, \mathrm{dist}(g_{i-1}(P), \overset{i-1}{\cup} F_k \times \overset{i-1}{\cup} J_k)\}$.

Moreover, the procedure allows ε_i to depend upon g_{i-1}, so the Convergence Criterion 1.1 shows that $h_1 = \lim g_i$ exists and is an ε-self-homeomorphism of $S^3 \times I$. Condition 1) shows that h_1 is fixed on $S^3 \times \{0, 1\}$, while the last two conditions show that $h_1(P) \cap (F \times I) = \emptyset$. Finally, since g_{i+1} will be obtained from g_i by composition with ε_{i+1}-push, h_1 is realizable by ε-isotopy.

For the purposes of starting the induction, let $J_0 = F_0 = \emptyset$ and $g_0 = \mathrm{id}$. Assume that $g_0, \ldots, g_n$ have been obtained satisfying 1) - 3). Let $\varepsilon_{n+1} < \frac{1}{2} \mathrm{dist}(g_n(P), (S^3 \times \{0, 1\}) \cup \overset{n}{\cup} F_k \times \overset{n}{\cup} J_k)$, and apply Proposition 4.2 with $\varepsilon = \varepsilon_{n+1}$, $C = F_{n+1}$, $J = J_{n+1}$, and $D = g_n(P)$ to find an ε_{n+1}-push, H_t, of $(S^3 \times I, F_{n+1} \times J_{n+1})$ such that $(F_{n+1} \times J_{n+1}) \cap H_1 g_n(P) = \emptyset$; then $g_{n+1} = H_1 g_n$ satisfies 1) - 3). ∎

Proposition 4.4. Let M be a PL 4-manifold, let P be a closed, possibly infinite, rectilinear 2-polyhedron in $\mathrm{Int}M$, and let T be a closed 2-dimensional subset of $\mathrm{Int}M$. For each $\varepsilon > 0$, there exists an ε-self-homeomorphism, h, of M fixed on $\mathrm{Bd}M$ such that $h(P) \cap T$ is 0-dimensional.

Proof. Let $\varepsilon > 0$ be given. Since P is rectilinear (with respect to a triangulation of $\mathrm{Int}M$), there exists an infinite 1-skeleton, $R = \cup T_i^{(1)}$, of M such that $P - R$ is 0-dimensional. Write $T_i^{(1)} = \bigcup_k T_{ik}$ where each T_{ik} is a compact 1-polyhedron, and set $P_n = \bigcup_{i+k\leq n} T_{ik}$. Like in the proof of 4.2, one can use 4.1 to find for each $\gamma > 0$ and each integer $n \geq 0$, a γ-push, h_t, of (M, P_n) such that $h_1(P_n) \cap T = \emptyset$, and, like in the proof of 4.3, one can use this to find a sequence of numbers, ε_i, and a sequence of ε_i-self-homeomorphisms, h_i, of M such that

1) $h_i|\mathrm{Bd}M = \mathrm{id}$,

2) $h_i(T) \cap P_n = \emptyset$,

3) $\varepsilon_i < \frac{1}{2}\min\{\varepsilon_{i-1}, \mathrm{dist}(h_{i-1}(T), P_n)\}$, and

4) $g = \lim h_i$ exists and is an ε-self-homeomorphism of M.

Because $R \cap g(T) = \emptyset$, $P \cap g(T)$ is 0-dimensional, and one sees that $h = g^{-1}$ has the required properties. ■

Proposition 4.5. Let P_e ($e = 0, 1$ as usual) be closed, possibly infinite rectilinear 2-polyhedra in $S^3 \times (0, 1)$, $p: S^3 \times I \to S^3$ be projection, $Z \subset S^3$ be a 1-demensional F_σ-set, and F be a countable union of tame Cantor sets in S^3. For each $\varepsilon > 0$, there exist

ε-isotopies, h_t and g_t, of $S^3 \times I$ fixed on $S^3 \times \{0, 1\}$ such that $ph_1(P_0) \cap pg_1(P_1) \cap Z = \emptyset$ and $F \cap ph_1(P_0) = F \cap pg_1(P_1) = \emptyset$.

Proof. Let $\varepsilon > 0$ be given. Because Z is a 1-demensional F_σ-set, there exists a tame 0-dimensional F_σ-set, G, in Z such that $Z - G$ is 0-dimensional and such that every compactum in $Z - G$ is tame - one way to find G is to let $R = \cup T_i^{(2)}$ be an infinite 2-skeleton of S^3 such that the infinite 1-skeleton $\cup T_i^{(1)}$ is contained in $S^3 - Z$ and to set $G = R \cap Z$. By Proposition 4.3, there exists an ε-isotopy, h_t, of $S^3 \times I$ fixed on $S^3 \times \{0, 1\}$ such that $h_1(P_0) \cap ((F \cup G) \times I) = \emptyset$. Set $W = h_1(P_0) \cap (Z \times I)$; then $p(W)$ is a 0-dimensional F_σ-set in $Z - G$. Hence another application of 4.3 shows that there exists an ε-isotopy, g_t, of $S^3 \times I$ fixed on $S^3 \times \{0, 1\}$ such that $g_1(P_1) \cap ((F \cup p(W)) \times I) = \emptyset$. The conclusion follows. ∎

Proposition 4.6. Let $P_e \subset S^3 \times (0, 1)$ be a pair of possibly infinite, rectilinear 2-polyhedra, $p: S^3 \times I \to S^3$ be projection, F be a countable union of tame Cantor sets in S^3 and X be a compact 2-dimensional set in S^3. For each $\varepsilon > 0$, there exist ε-isotopies, h_t and g_t, of $S^3 \times I$ fixed on $S^3 \times \{0, 1\}$ such that $Q = ph_1(P_0) \cap pg_1(P_1)$ is 1-demensional, $Q \cap X$ is 0-demensional, and $F \cap ph_1(P_0) = F \cap pg_1(P_1) = \emptyset$.

Proof. Let $\varepsilon > 0$ be given, R be an infinite 1-skeleton of S^3, and Z be a 1-demensional F_σ-set in X such that $X - Z$ is 0-demensional - one way to find Z is to let R^* be an infinite 2-skeleton of S^3 with $R^* \cap X$ 1-dimensional; since $R^* \cap X \subset R^*$ it is 1-demensional, so one lets $Z = R^* \cap X$. Now $R \cup Z$ is a 1-demensional F_σ-set, so 4.5 produces ε-isotopies, h_t and g_t , of $S^3 \times I$ fixed on $S^3 \times \{0, 1\}$ such that $Q \cap (Z \cup R) = \emptyset$ and $F \cap ph_1(P_0) = F \cap pg_1(P_1) = \emptyset$. Because $Q \cap R = \emptyset$, Q is 1-demensional; because $Q \cap Z = \emptyset$, $Q \cap X$ is 0-demensional. ■

<u>When</u> C <u>has</u> G-<u>type</u> 1

In high dimensions, the identity sewing of a type 1 closed n-cell complement to itself always yields S^n . After the following lemma, we present a similar result for CFC's possessing enough local flatness.

<u>Lemma</u> <u>4.7</u>. <u>Let</u> C <u>be a</u> CFC , $\varepsilon > 0$, $\mathrm{Bd}C \times I$ <u>a collar on</u> C <u>with</u> $\mathrm{diam}(\{x\} \times I) < \varepsilon$ <u>for all</u> x <u>in</u> $\mathrm{Bd}C$, F <u>a</u> 1-GULC <u>set for</u> C , P <u>a compact rectilinear</u> 2-<u>polyhedron in</u> $C \cup (\mathrm{Bd}C \times [0, 1))$, U <u>and</u> Z <u>open sets in</u> $\mathrm{Bd}C$ <u>with</u> $P \cap \mathrm{Bd}C \subset U \subset \mathrm{Cl}U \subset Z$, <u>and</u> $p\colon C \cup (\mathrm{Bd}C \times I) \to C$ <u>projection from</u> $\mathrm{Bd}C \times I$ <u>to</u> $\mathrm{Bd}C$ <u>extended by the identity</u>. <u>For each</u> $\gamma > 0$, <u>there exists a</u> 2ε-<u>pseudoisotopy</u>, k_t , <u>of</u> S^4 <u>fixed off</u> $C \cup (\mathrm{Bd}C \times [0, 1))$ <u>such that</u> $k_1(x) \in F \cup \mathrm{Int}C$ <u>whenever</u> $x \in P \cap (p^{-1}(U) \cup C)$, $\rho(p(x), pk_1(x)) < \gamma$ <u>whenever</u> $x \in C \cup (\mathrm{Bd}C \times I)$, <u>and</u> $k_1(x) = x$ <u>whenever</u> $x \in P \cap p^{-1}(\mathrm{Bd}C - Z)$.

<u>Proof</u>. Assume the hypotheses, let P^* be a subpolyhedron of P such that $P \cap (p^{-1}(U) \cup C) \subset P^* \subset P \cap (p^{-1}(Z) \cup C)$, and let V and W be open sets in $\mathrm{Bd}C$ such that $\mathrm{Cl}U \subset W \subset \mathrm{Cl}W \subset V \subset \mathrm{Cl}V \subset Z$, $P^* \subset p^{-1}(V) \cup C$, and $\mathrm{Cl}(P - P^*) \subset p^{-1}(\mathrm{Bd}C - \mathrm{Cl}W)$. Next let ζ be a positive number less than $\mathrm{dist}(\mathrm{Bd}C , P \cap p^{-1}(\mathrm{Bd}C - Z))$ and

less than $\operatorname{dist}(p^{-1}(U), p^{-1}(\operatorname{Bd}C - W))$, and let $\eta > 0$ be a number less than ζ such that $x, y \in \operatorname{Bd}C \times I$ with $\rho(x, y) < \eta$ implies that $\rho(p(x), p(y)) < \gamma$.

Now let g_t be an isotopy of S^4 supported in $p^{-1}(V)$ which pushes P^* down the fibers of $\operatorname{Bd}C \times I$ so close to $\operatorname{Bd}C$ that there exists (from the definition of 1-GULC) an η-pseudoisotopy, h_t , supported in the η-neighborhood of C such that $h_1 g_1(P^*) \subset F \cup \operatorname{Int}C$. The reader should have no trouble verifying that $k_t = h_t g_t$ meets the requirements of the lemma; for example, $k_1(x) = x$ whenever $x \in P \cap p^{-1}(\operatorname{Bd}C - Z)$ because g_t is supported in $p^{-1}(U)$, which is disjoint from $p^{-1}(\operatorname{Bd}C - Z)$, and h_t is supported in the η-neighborhood of C which, because $\eta < \zeta$, is disjoint from $P \cap p^{-1}(\operatorname{Bd}C - Z)$. ■

<u>Theorem 4.8</u>. <u>If</u> C <u>is a</u> G-<u>type</u> 1 CFC <u>which is locally flat modulo a</u> 2-<u>dimensional set</u>, X , <u>then</u> $C \cup_{id} C \approx S^4$.

<u>Proof</u>. Let $S^3 \times I$ be embedded in S^4 so that it realizes the identity sewing of C to itself, and let C_0, C_1 , and $h\colon \operatorname{Bd}C_0 \to \operatorname{Bd}C_1$ be the associated objects. Since h is the identity sewing, it extends to an involution, also denoted h , of S^4 ; thus $h(C_0) = C_1$ and $h(C_1) = C_0$ - we also require that $h(s, t) = (s, 1 - t)$ for all s in S^3 and t in I .

We shall prove the theorem by applying Theorem 3.3, so let $\varepsilon > 0$ be given. Let $x \in (0, \frac{1}{2})$ be so close to zero that each arc of the form $\{s\} \times [0, x]$ or $\{s\} \times [1 - x, 1]$, $s \in S^3$, has diameter less than ε , and set $V_0 = S^3 \times (0, x)$ and $V_1 = S^3 \times (1 - x, 1)$:

then $V_0 \cap V_1 = \emptyset$, $h(V_0) = V_1$, and $h(V_1) = V_0$. Let $p: C_e \cup V_e \to C_e$ be projection to BdC_e on V_e extended by the identity. Now let $D_e \subset IntC_e$ be compact sets and $P_e \subset V_e \cup C_e$ be PL 2-polyhedra. According to Theorem 3.3, it is sufficient to find a 4ε-pseudoisotopy, f_t , of S^4 with support in $C_0 \cup V_0 \cup C_1 \cup V_1$ such that $f_1(P_e \cup D_e) \subset C_e$ and

1) $(f_1(P_1 \cup D_1) \cap BdC_1) \cap h(f_1(P_0 \cup D_0) \cap BdC_0) = \emptyset$.

Before starting to find f_t , let $X_e \subset BdC_e$ be 2-dimensional sets such that $h(X_0) = X_1$, $h(X_1) = X_0$, and C_e is locally flat modulo X_e ; likewise, let $F_e \subset BdC_e$ be a countable union of Cantor sets tame in BdC_e such that $h(F_0) = F_1$, $h(F_1) = F_0$, and F_e is a 1-GULC set for C_e (C is G-type 1). Notice that it is sufficient to find f_t such that

2) $(f_1(P_1 \cup D_1) \cap X_1) \cap h(f_1(P_0 \cup D_0) \cap X_0) = \emptyset$,

for 1) may be obtained from 2) by pushing $f_1(P_e \cup D_e)$ through the local flatness of C_e at points of $BdC_e - X_e$ (the composition of a pseudoisotopy and is an isotopy is a pseudoisotopy). Notice further that it is sufficient to find f_1 such that

3) $(pf_1(P_1 \cup D_1) \cap X_1) \cap h(pf_1(P_0 \cup D_0) \cap X_0) = \emptyset$,

because replacing f_1 by pf_1 produces 2). Finally, by taking 4ε to be smaller than $\mathrm{dist}(D_0 \cup D_1 , S^3 \times I)$, we remove the D_e from consideration and make it sufficient to find f_1 such that

4) $(pf_1(P_1) \cap X_1) \cap h(pf_1(P_0) \cap X_0) = \emptyset$.

We shall find f_t as the composition of two pseudoisotopies, $f_t = k_t g_t$, and satisfying 4).

By general position in S^4 , we may assume, without loss of generality, that $P_1 \cap h(P_0) \cap BdC_1 = \emptyset$. Let $\zeta < \varepsilon$, and let ζ

be small enough that $h\psi_t h$ is an ε-isotopy of V_0 whenever ψ_t is a ζ-isotopy of V_1. Now apply Proposition 4.6 in V_1, with respect to $P_1 \cap V_1$, $h(P_0) \cap V_1$ (this set equals $h(P_0 \cap V_0)$), p, F_1, and X_1, to find ζ-isotopies, ϕ_t and ψ_t, of S^4 fixed off V_1 such that $p\,\phi_1(P_1 \cap V_1) \cap F_1 = p\,\psi_1(h(P_0) \cap V_1) \cap F_1 = \phi$ and such that $p\,\phi_1(P_1 \cap V_1) \cap p\,\psi_1(h(P_0) \cap V_1) \cap X_1$ is 0-demensional in BdC_1. Define g_t to be ϕ_t on V_1, $h\psi_t h$ on V_0, and id otherwise. Then the following are true:

5) g_t is an ε-isotopy of S^4 supported in $V_0 \cup V_1$,

6) $g_1(P_1) \cap hg_1(P_0) \cap BdC_1 = \phi$,

7) $Q \cap X_1$ is 0-demensional in BdC_1 where
$Q = p(g_1(P_1) \cap V_1) \cap hp(g_1(P_0) \cap V_0)$, and

8) $pg_1(P_1 \cap V_1) \cap F_1 = pg_1(h(P_0) \cap V_1) \cap F_1 = \phi$.

In view of 6), we observe that the compact set

$$R = pg_1(P_1 \cap ClV_1) \cap phg_1(P_0 \cap ClV_0),$$

which is a subset of BdC_1, may be written as the union of three sets - Q, $Q_1 = g_1(P_1) \cap phg_1(P_0 \cap V_0)$ and $Q_0 = hg_1(P_0) \cap pg_1(P_1 \cap V_1)$; that $Q_0 \cap Q_1 = \phi$; and that Q_0 and Q_1 are compact. Thus, by 7) (only 0-dimensionality is used), we may write $R = h(W_0) \cup W_1$ with W_e compact sets, $Q_0 \subset h(W_0)$, $Q_1 \subset W_1$, and $h(W_0)$ disjoint from W_1. Set $L_e = W_e \cup (g_1(P_e \cap BdC_e))$ (the set $g_1(P_e \cap BdC_e)$ equals $P_e \cap BdC_e$); since $h(L_0) \cap L_1 = \phi$, let U_e and Z_e be open sets in BdC_e such that $L_e \subset U_e \subset ClU_e \subset Z_e$ and

9) $h(ClZ_0) \cap ClZ_1 = \phi$;

see Figure 4 which illustrates the location of the Z_e.

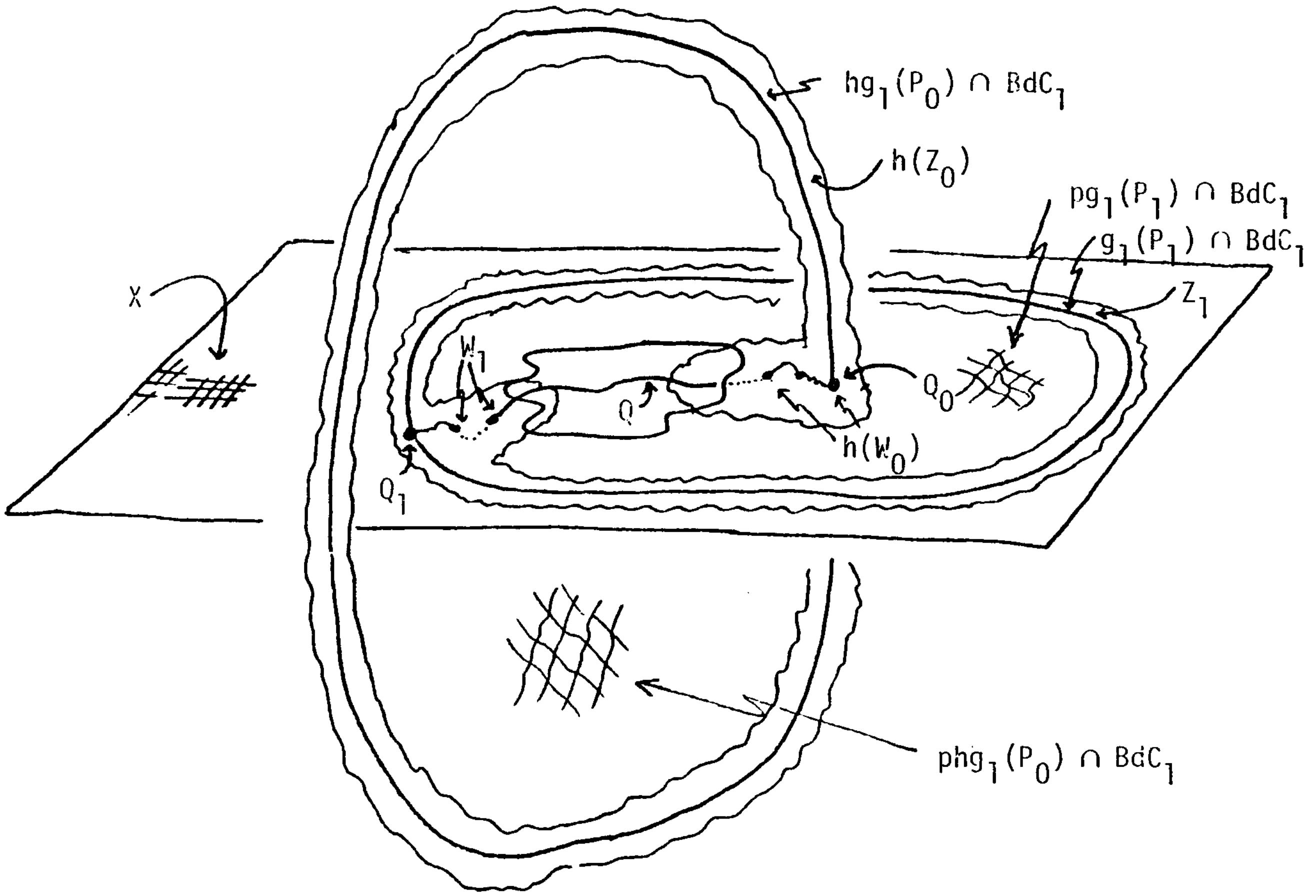


Figure 4. A Schematic Representation of the Z_e .

Our central observation is that the sets

$B_0 = p^{-1}(\mathrm{Bd}C_0 - h(U_1)) \cap g_1(P_0)$ and $B_1 = p^{-1}(\mathrm{Bd}C_1 - h(U_0)) \cap g_1(P_1)$ are compact and satisfy

10) $(p(B_1) \cap X_1) \cap h(p(B_0) \cap X_0) = \emptyset$.

Hence there exists a positive number γ such that

11) $\gamma < \min\{\mathrm{dist}(p(B_1) \cap X_1 , h(p(B_0) \cap X_0))$, $\mathrm{dist}(h(p(B_1) \cap X_1) , p(B_0) \cap X_0)\}$ and

12) $\gamma < \min\{\mathrm{dist}(h(\mathrm{Cl}Z_0), \mathrm{Cl}Z_1) , \mathrm{dist}(\mathrm{Cl}Z_0 , h(\mathrm{Cl}Z_1))\}$.

Now find k_t by applying Lemma 4.7 once in $V_1 \cup C_1$ with respect to γ , ε , F_1 , U_1 , and $g_1(P_1)$; and once again in $V_0 \cup C_0$ with respect to γ , ε , F_0 , U_0 , Z_0 , and $g_1(P_0)$. Specifically, the lemma produces a 2ε-pseudoisotopy, k_t , fixed off $C_0 \cup V_0 \cup C_1 \cup V_1$ such that

13) $k_1(x) \in F_0 \cup \mathrm{Int}C_0$ whenever $x \in g_1(P_0) \cap p^{-1}(U_0 \cup \mathrm{Int}C_0)$,

14) $k_1(x) \in F_1 \cup \mathrm{Int}C_1$ whenever $x \in g_1(P_1) \cap p^{-1}(U_1 \cup \mathrm{Int}C_1)$,

15) $\rho(p(x), pk_1(x)) < \gamma$ whenever $x \in C_0 \cup V_0$,

16) $\rho(p(x), pk_1(x)) < \gamma$ whenever $x \in C_1 \cup V_1$,

17) $k_1(x) = x$ whenever $x \in g_1(P_0) \cap p^{-1}(\mathrm{Bd}C_0 - Z_0)$,

and

18) $k_1(x) = x$ whenever $x \in g_1(P_1) \cap p^{-1}(\mathrm{Bd}C_1 - Z_1)$.

To see that 4) is true when $f_t = k_t g_t$, we make four observations. First, by 10), 11), 15), and 16),

$$A_1 = (pk_1(B_1) \cap X_1) \cap h(pk_1(B_0) \cap X_0) = \emptyset .$$

Second, by 9), 12), 15), and 16),

$$A_2 = [pk_1(g_1(P_1) \cap p^{-1}(Z_1 \cup \mathrm{Int}C_1)) \cap X_1]$$
$$\cap\, h[pk_1(g_1(P_0) \cap p^{-1}(Z_0 \cup \mathrm{Int}C_0)) \cap X_0] = \emptyset \, .$$

Third, A_3, the intersection between $pk_1(g_1(P_1) \cap p^{-1}(\mathrm{Bd}C_1 - Z_1)) \cap X_1$ and $h[pk_1(g_1(P_0) \cap p^{-1}(U_0)) \cap X_0]$, is empty - because the first set is contained in $\mathrm{Bd}C_1 - F_1$, by 8) and 18), while the second set is contained in F_1 by 13). The fourth and last observation, dual to the third, is that A_4, the intersection between

$$h[pk_1(g_1(P_0) \cap p^{-1}(\mathrm{Bd}C_0 - Z_0)) \cap X_0]$$

and

$$pk_1(g_1(P_1) \cap p^{-1}(U_1)) \cap X_1 \, ,$$

is empty - because the first set is contained in $\mathrm{Bd}C_1 - F_1$, by 8) and 17), while the second set is contained in F_1 by 14). Finally these observations show that 4) is true, because the set from 4), $(pf_1(P_1) \cap X_1) \cap h(pf_1(P_0) \cap X_0)$, is equal to $A_1 \cup A_2 \cup A_3 \cup A_4$, and so empty. ■

Actually, under the hypotheses of the last theorem, more is true.

Theorem 4.9. *If C is a G-type 1 CFC which is locally flat modulo a 2-dimensional set X, then C has G-type at most $1TT$.*

Proof. Let $\mathrm{Bd}C \times I$ be a collar on C. We will prove the following.

For each $\varepsilon > 0$, *there exists a number* s *such that for any* PL *polyhedron* P *in* $C \cup V$ $(V = \mathrm{Bd}C \times (0, s))$ *and any compact set* D *in* $\mathrm{Int}C$, *there exists an* ε-*pseudoisotopy* h_t *of* S^4 *supported in* $C \cup V$ *such that* $h_1(P \cup D) \subset C$ *and* $h_1(P \cup D) \cap \mathrm{Bd}C$ *is* 0-*dimensional*, *tame in* $\mathrm{Bd}C$, *and tame in space*.

The proof of Proposition 2.4 shows that this is sufficient to prove the theorem. Specifically, let $\varepsilon_i = 1/i$, and in place of using 1-GULC in the second paragraph of that proof, use instead the statement underlined above. This produces a sequence $h_t(i)$ of ε_i-pseudoisotopies such that for each i $h_t(i)$ pulls the "right" 2-complex P_i and the right compact set D_i into C . Then one sets $J_i = h_1(i)(P_i \cup D_i) \cap \mathrm{Bd}C$ and shows that $\cup J_i$ is a 1-GULC set for C , exactly like in the rest of that proof.

So it is sufficient to prove the underlined statement; therefore let $\varepsilon > 0$ be given, choose s so that each arc $\{x\} \times (0, s)$ (x in $\mathrm{Bd}C$) has diameter less than $\frac{1}{2}\varepsilon$, and let P and D be given. We concentrate on finding the ε-near-homeomorphism h_1 , which will be realizable as the end of an ε-pseudoisotopy h_t supported in $C \cup V$, but leave the verification to the reader.

Let F be the 1-GULC set for C that makes it G-type 1 , $T = \cup T_i$ to be an infinite 2-skeleton of $\mathrm{Bd}C$ with $T \cap F = \emptyset$, and $p\colon C \cup \mathrm{Bd}C \times I \to C$ be projection of $\mathrm{Bd}C \times I \to \mathrm{Bd}C$ extended over C by the identity. By induction, we will find six sequences:

1) $\{\varepsilon_i\}$, a sequence of numbers decreasing to zero,
2) $\{f_i\}$, a sequence of $3\varepsilon_i$-self-homeomorphisms of S^4 supported in $C \cup V$,

3) $\{s_i\}$, a sequence of numbers decreasing to zero with $\operatorname{diam}(\{x\} \times (0, s_i)) < \frac{1}{2}\varepsilon_i$ for all x in BdC ,

4) $\{P_i\}$, a sequence of PL polyhedra in $C \cup V_i$ $(V_i = BdC \times (0, s_i))$,

5) $\{D_i\}$, a sequence of compact sets in $IntC$, and

6) $\{g_i\}$, a sequence of ε_i-near-homeomorphisms of S^4 supported in $C \cup V_i$.

The sequences will have four properties.

7) $F^* = \cup\, g_i(P_i \cup D_i) \cap BdC$ will be a 1-GULC set for C ,

8) $f_i(P) \subset C \cup V_{i+1}$,

9) $pf_i(P) \cap BdC \subset BdC - [\bigcup_{k=1}^{i} g_k(P_k \cup D_k) \cap BdC] - \bigcup_{k=1}^{i} T_k$, and

10) $\varepsilon_{i+1} < \frac{1}{4}\min\{\frac{1}{4}\varepsilon_i$, $\operatorname{dist}(pf_i(P) \cap BdC$, $(\bigcup_{k=1}^{i} g_i(P_k \cup D_k) \cap BdC) \cup \bigcup_{k=1}^{i} T_k$, $\operatorname{dist}(pf_i(P), \cup\, g_k(P_k \cup D_k) \cap BdC)$, $\operatorname{dist}(f_i(D)$, $BdC)\}$.

Then $h_1 = \lim f_i$ exists and is an ε-near-homeomorphism of S^4 supported in $C \cup V$; moreover, because $\varepsilon_{i+1} < \operatorname{dist}(f_i(D), BdC)$, $h_1(D) \subset IntC$; and by this, together with 8), 9), 10), and the fact that $s_i \to 0$, $h_1(P \cup D) \subset C - \cup T_i - F^*$. Hence, because $h\ (P \cup D) \subset C - \cup T_i$, $h_1(P \cup D) \cap BdC$ is 0-dimensional and tame in BdC ; and because $h_1(P \cup D) \cap BdC \subset C - F^*$, Proposition 2.8 shows that it is flat in S^4 too.

For the purposes of starting the induction, let f_0 and g_0 be the identity; P_0, D_0, and T_0 be empty; and ε_0, ε_1, s_0, and s_1 satisfy 3) and 10) with $\varepsilon_0 < \frac{1}{4}\min\{\varepsilon, \operatorname{dist}(D, BdC)\}$. Suppose each sequence has been obtained for $i = 0,\ldots, n$ and that s_{n+1} and ε_{n+1} have been found satisfying 1) through 6) and 8) through 10), and suppose this has been done through use of the exact argument we are about to present for the inductive step (this is important to be sure of getting condition 7)).

Let Q be a PL 2-polyhedron in $C \cup V_{n+1}$ and D' be a compact set in $\mathrm{Int}C$. According to the proof of Theorem 4.8, with P_0 equal to Q and P_1 equal to $f_n(P)$, there exist ε_{n+1}-pseudoisotopies H_t and G_t of S^4 supported in $C \cup V_{n+1}$ such that $H_1 f_n(P \cup D) \subset C$, $G_1(Q \cup D') \subset C$, and $H_1 f_n(P \cup D) \cap \mathrm{Bd}C \cap G_1(Q \cup D') = \emptyset$. Let $g_{n+1} = G_1$. Now the proof of Proposition 2.4 shows that there exists a particular PL 2-polyhedron P_{n+1} in $C \cup V_{n+1}$ and a particular compact set D_{n+1} in $\mathrm{Int}C$ such that when g_{n+1} is obtained in this way, the set F^* from 7) will be a 1-GULC set for C. Choose a number $q < 1$ so close to 1 that $pH_q f_n(P) \cap \mathrm{Bd}C \cap g_{n+1}(P_{n+1} \cup D_{n+1}) = \emptyset$. Set $\zeta = \mathrm{dist}(pH_q f_n(P) \cap \mathrm{Bd}C\ ,\ g_{n+1}(P_{n+1} \cup D_{n+1}))$. Let π be an ε_{n+1}-self-homeomorphism of S^4 supported in the ζ-neighborhood of $p^{-1}(pH_q f_n(P) \cap \mathrm{Bd}C)$ which pushes $H_q f_n(P)$ down the fibers of $\mathrm{Bd}C \times I$ and off a section on T_{n+1} (sections on T_{n+1} are flat because $T_{n+1} \subset \mathrm{Bd}C - F$). Then if f_{n+1} were to be $\pi H_q f_n$, 9) would be true, so ε_{n+2} and s_{n+2} may be defined as if it were. Finally set $f_{n+1} = g\pi H_q f_n$ where g pushes $\pi H_q f_n(P)$ down the fibers of $\mathrm{Bd}C \times I$ so close to $\mathrm{Bd}C$ that 8) $(f_{n+1}(P) \subset C \cup V_{n+2})$ holds too. ■

The next theorem shows the relationship between 4.8 and 4.9.

<u>Theorem 4.10</u>. <u>If</u> C <u>has</u> G-<u>type</u> 1TT, <u>then it has disjoint</u> 0-<u>dimensional</u>, <u>twice tame</u> 1-GULC <u>sets</u>, <u>and</u> $C \cup_{id} C \approx S^4$.

Proof. Let F be a 1-GULC set for C which is a countable union of twice tame Cantor sets, and let R be an infinite 2-skeleton of BdC with $R \cap F = \emptyset$. Then according to Propositions 2.8 and 2.11 and to Kirby [41], there exists a 1-GULC set, F', for C in $BdC - (F \cup R)$. It follows that F' is a union of twice tame Cantor sets. Proposition 3.4 shows that $C \cup_{id} C \approx S^4$. ∎

Examples

Two of the examples in [26] pertain to the identity sewing. Both may be adapted to the case $n = 4$.

Example 4.11. There exists a G-type 2 CFC C_3 *such that* $C_3 \cup_{id} C_3 \not\approx S^4$.

Proof. As in Example 3.13, one uses the sets from Proposition 3.12 letting C_2 be the inflation of C to obtain all the sets mentioned at the onset of Daverman's proof (of his example 13.3 [26]). That proof then applies unaltered except for two instances. First, the references to Theorem 10.1 should be replaced by a reference to Corollary 10.2 (of [26]). Second, to learn that C_3 has G-type 2, one argues differently as follows. Since C_3 is locally flat modulo a Cantor set, it has at most G-type 2; by our Theorem 4.8, it cannot have G-type less than 2. ∎

The remarks following 3.13 apply here to show that such an example also exists when $n = 5$. Before presenting the second example, we prove a variation on Lemma 6.3 from Daverman [26].

Lemma 4.12. Let B^4 *denote the standard 4-cell in* E^4*, and let* Y *be a Cantor set in* BdB^4*. Then there exists a tame embedding* j *of* Y *in* BdB^4 *and there exists an embedding* θ *of* B^4 *in* E^4 *such that* θj = *inclusion and* $B^4 - Y \subset \theta(IntB^4)$*. Moreover, for each* $\varepsilon > 0$ *and each compact set* D *in* $IntB^4$ *there exists an* ε*-embedding* α *of* B^4 *in* $\theta(B^4)$ *such that* $B^4 - Y \subset \alpha(IntB^4)$, $\alpha(BdB^4) \cap B^4 = Y$, Y *is tame in* $\alpha(BdB^4)$*, and* α *is the identity on* D.

Proof. We use rB to denote the set of all points in E^4 whose distance from the origin is less than or equal to r $(B^4 = 1B)$, and denote points in rB by (x, y) where $x \in BdB^4$ and $y \in [0, r]$.

Let $Y \subset BdB^4$ be given. It is easy to find an embedding $f: Y \times I \to BdB^4$ such that $f((y, 0)) = y$ for all $y \in Y$ and such that $f|Y \times (0, 1]$ is locally flat. One way to do this is to use the method of Alexander to find a 3-cell $C \subset BdB^4$ with $Y \subset BdC$ and to take $f(Y \times I)$ to be a subset of the cone on Y in C. Let $F: Y \times I \to 2B$ be given by $F((y, t)) = (f((y, t)), 1 + t)$, $0 \leq t \leq 1$, and let G be the decomposition of 2B into points and arcs of the form $F(\{y\} \times I)$. Notice that each lifted Cantor set, $F(Y \times I) \cap Bd(rB)$, $r \in (1, 2]$, is tame.

Corollary 4 of Bryant [15] implies the existence of a homeomorphism $g: 2B \to 2B$ fixed on $Bd2B$ which carries $F(Y \times I)$ into the cone (in 2B) over $F(Y \times I) \cap Bd2B$. Let G^* be the decomposition of 2B into points and arcs (all radial) of the form $gF(\{y\} \times I)$, $y \in Y$. Obviously, there exist maps $h, k: 2B \to 2B$ which preserve

$\text{Bd}B^4$ coordinates and whose inverse sets are exactly the nondegenerate elements of G^* such that k is onto and h is the identity on $g(B^4)$ (h shrinks from the outside, and k shrinks from the inside). Let $q: B \to 2B$ be a radial homeomorphism. Set $\theta: B^4 \to 2B$ equal to the composition $g^{-1}hk^{-1}gq$, and let j be given by $j(y) = \theta^{-1}(y, 1) = \theta^{-1}(y)$, $y \in Y$. It is easy to verify that θ is an embedding, θj is inclusion, and $B^4 - Y \subset (\text{Int}B^4)$; $j(Y)$ is tame because $j|Y = F|Y \times \{1\}$.

To obtain the moreover, first let ε and D be given. Now contemplate repeating the procedure just described with exactly the same F, g, and h - but with different k and q. Specifically, replace q by a radial homeomorphism $q_r: B^4 \to rB$ such that q_r is the identity on D ; and require k to be the identity on $g(D)$. Then the composition $\alpha = g^{-1}hk^{-1}gq_r$ satisfies $\alpha(B^4) \subset \theta(B^4)$ (because h remains the same as in the construction of θ), $B^4 - Y \subset \alpha(\text{Int}B^4)$; Y is tame in $\alpha(\text{Bd}B^4)$, and α is the identity on D. Finally, by choosing r near enough to 1, α may be made an ε-embedding. ■

Here is the second example.

<u>Example 4.13</u>. <u>There exists a G-type</u> 2TS CFC , C , <u>such that</u> $C \cup_{id} C \approx S^4$.

<u>Proof</u>. Example 13.4 of [26] (originating in [23]) may be used to produce C ; however the argument that $C \cup_{id} C \approx S^4$ must be replaced by one suited to $n = 4$. This involves applying 4.13 during the construction to obtain a little more control, and for this reason, we review the entire argument.

Let B^4 denote the standard 4-cell in E^4 . By Proposition 3.12, there exist Cantor sets $Y \subset BdB^4$, $Z \subset S^4$, a homeomorphism, $h': Y \to Z$, mixing their admissible subsets, and loops $L \subset BdB^4$ and $L' \subset S^4$ such that each contraction of L in BdB^4 contains an admissible subset of Y and each contraction of L' in S^4 contains an admissible subset of Z . Apply Lemma 4.12 with respect to Y to find the embeddings j and θ , and extend h' to an embedding $H: \theta(B^4) \to S^4$ (because Y is tame in $Bd\theta(B^4)$, one can extend h' using the method of Alexander). Now, recalling that $IntB^4 \subset \theta(B^4)$, set $C = S^4 - H(IntB^4)$.

C is locally flat modulo the Cantor set Z.; hence it has at most G-type 2 . Suppose it had G-type 1 . Then there would exist a 1-GULC set $F \subset Z$ and a contraction $f: B^2 \to H(BdB^4)$ of $H(L)$ such that $f(B^2) \cap F = \emptyset$. By Proposition 2.8, the section on $f(B^2)$ is flat; in particular, $f(B^2)$ is flat. Hence, L' contracts in $S^4 - f(B^2)$; but this is impossible - because such a contraction contains an admissible subset of Z , and $f(B^2)$, by construction, must meet every such admissible set. Therefore C has G-type 2 (we will show 2TS later).

To show that $C \cup_{id} C \approx S^4$, let $S^3 \times I$ be embedded in S^4 so that it realizes this sewing, and let C_0, C_1 , and h be the associated objects with h extended to an involution of S^4 such that $h(s, t) = (s, 1 - t)$ for $s \in S^3$, $t \in I$. Let Z_e stand for the copy of Z found in BdC_e . We intend to apply Theorem 3.3, so we adopt all the notation - x, V_e, P_e, and D_e - that is found in the second paragraph of the proof of Theorem 4.8.

Let $\gamma < \frac{1}{2}\min\{\mathrm{dist}(P_e, (S^3 \times I) - V_e), \varepsilon\}$. Now, our use of Lemma 4.12 in the construction implies the existence of a γ-reembedding $\alpha: S^3 \times I \to S^4$ such that $S^3 \times I \subset \alpha(\mathrm{Int}(S^3 \times I)) \cup Z_0 \cup Z_1$, Z_e is a tame subset of $\alpha(S^3 \times \{e\})$, and $h\alpha(\{s\} \times I) = \alpha(\{s\} \times I)$ for each s in S^3 . The CFC's , K_e , determined by $\alpha(S^3 \times I)$ are homeomorphic, and $\alpha(S^3 \times I)$ realizes the identity sewing between them; moreover, since Z_e is tame in $\mathrm{Bd}K_e$, K_e has G-type 1TT . Thus, by Theorem 4.10, there exist 1-GULC sets, $F_e \subset Z_e$, for K_e such that $h(F_0) \cap F_1 = \emptyset$. Using this, it is easy to find an ε-pseudoisotopy, h_t , supported in V_e that pushes P_e down the fibers of $\alpha(S^3 \times I)$ close to $\mathrm{Bd}K_e$ and which there cuts it off in $F_e \cup \mathrm{Int}K_e$ using the definition of 1-GULC ; specifically, $h_1(P_e \cup D_e) \subset F_e \cup \mathrm{Int}K_e$. But then, since $\mathrm{Int}K_e \subset \mathrm{Int}C_e$, $h_1(P_e \cup D_e) \subset F_e \cup \mathrm{Int}C_e$; and, since $h(F_0) \cap F_1 = \emptyset$, $h(h_1(P_0 \cup D_0) \cap \mathrm{Bd}C_0) \cap (h_1(P_1 \cup D_1) \cap \mathrm{Bd}C_1) = \emptyset$. Hence Theorem 3.3 shows $C \cup_{id} C \approx S^4$.

Finally, C really has G-type 2TS because $h_1(P_e \cup D_e) \cap \mathrm{Bd}C_e \subset F_e$ and F_e is tame in space. That is, while proving Proposition 2.4, we showed how to find a 1-GULC set which was a countable union of sets of the form $h_1(P_e \cup D_e) \cap \mathrm{Bd}C_e$ - that 1-GULC set shows G-type 2TS . ∎

CHAPTER V

APPLICATIONS TO UNIVERSALITY

Universal CFC's

A CFC , C_0 , is called universal if, for each CFC , C_1 , each sewing between C_0 and C_1 yields S^4 . One class of CFC's , those we call G-type 0 , has already been shown to be universal, see Cannon-Daverman [18, 7.10]; given Example 3.8, this result is already the best possible considering G-type alone. But with other hypotheses, other classes of CFC's may also be shown universal.

Theorem 5.1. *If C_0 is a CFC which is locally flat modulo a 1-dimensional set P such that each Cantor set in P is twice tame, then C_0 is universal.*

Proof. Let $h: BdC_0 \to BdC_1$ be any sewing. By Proposition 2.20, $h(P)$ may be written as the union of two 0-dimensional sets S and Q with Q an F_σ tame in S^4 . By hypothesis, $h^{-1}(Q)$ is tame in BdC_0 ; hence Q is twice tame. It follows from Proposition 2.11 and Kirby [41] that C_1 has an F_σ 1-GULC set F_1 in $BdC_1 - Q$. Write $F_1 = \cup H_i$ with each H_i compact. Then for each i , $H_i \cap P = H_i \cap S$ is compact and 0-dimensional. So by hypothesis, $h^{-1}(H_i \cap S)$ is twice tame and, by Kirby [41], has a flat section. Besides that, $BdC_0 - P$ is an F_σ , $\cup E_i$ say, and each E_i has also a flat section. Now a second application of 2.11 shows that there exists a 1-GULC set F_0 for C_0 in $BdC_0 - \cup E_i - \cup h^{-1}(H_i \cap S) \subset BdC_0 - h^{-1}(F_1)$. That is, $h(F_0) \cap F_1 = \emptyset$, and the assertion is proved by 3.4. ∎

In high dimensions there is another class known to be universal - namely suspensions; see Daverman [26, 8.6]. We shall show likewise in S^4 . Parenthetically, we observe that the suspension of a crumpled 3-cube C is embedded in S^4 as a CFC whenever C is embedded in S^3 as a closed 3-cell complement. Throughout this section C is such and ΣC denotes its suspension. There are three lemmas.

Lemma 5.2. *For each $\delta > 0$ and each finite subset K of $(-1, 1)$ there exists a δ-self-homeomorphism h (realizable by δ-pseudoisotopy) of $S^3 \times [-1, 1]$ such that h is the identity on $S^3 \times \{-1, 1\}$, h is fixed outside an arbitrarily small neighborhood of $BdC \times K$, and $h(C \times [-1, 1]) \cap (S^3 \times K) \subset (IntC) \times K$.*

Proof. This is essentially the same as Lemma 9.2 in Cannon-Daverman [18]. ■

Lemma 5.3. *For each $\eta > 0$ there exists a neighborhood Ω of C in S^3 and a Cantor set Q in BdC such that for each compact set T in Ω there is an 2η-pseudoisotopy λ_t supported in Ω with $\lambda_1(T) \subset Q \cup IntC$.*

Proof. Let $BdC \times I$ be a collar on BdC with $diam(\{x\} \times [0, 1]) < \eta/2$ for each x in BdC . Let $p\colon BdC \times I \to BdC$ be projection, and set $\Omega = C \cup BdC \times [0, 1)$. We shall let $T = C \cup BdC \times [0, \frac{1}{2}]$ and find λ_t such that $\lambda_1(T) \subset Z \cup IntC$ with Z a Cantor set. Then clearly, if $Q = Z$ and T' is another compactum in Ω , T' may be deformed into T and thence into $Q \cup IntC$ as the lemma requires.

By induction, we shall find λ_1 as a limit $\lambda_1 = \ldots f_n \ldots f_2 f_1$ of compositions of homeomorphisms f_i. We also shall find sets Z_i in BdC such that for all i

1) f_i is supported in Ω,
2) $\mathrm{dist}(f_i, \mathrm{id}) < \eta/(2^{i+1})$,
3) f_i is realizable by $\eta/(2^{i+1})$-isotopy,
4) $Z_{i+1} \subset Z_i$,
5) Z_i is a finite union of disjoint disks of diameter less than $1/i$,
6) $p(f_i \ldots f_1(T) \cap (\mathrm{BdC} \times I)) \subset \mathrm{Int}\, Z_i$, and
7) $f_i \ldots f_1(T) \subset (\mathrm{BdC} \times [0, \eta/(2^{i+1})]) \cup C$.

The first three conditions make λ_1 realizable as an η-pseudoisotopy supported in Ω, and the next four conditions together with 10) below make $\lambda_1(T) \subset Z \cup \mathrm{IntC}$ where $Z = \cap Z_i$ and is 0-dimensional.

The inductive step may be used to start the induction; so suppose that $f_1, \ldots, f_k$ have already been found and that 1) - 7) hold for each $i \leq k$. To find f_{k+1}, first apply the theorem of Craggs [21] to find an $\varepsilon > 0$ such that whenever S_1 and S_2 are disjoint locally flat ε-approximations to BdC there exists an annulus $S \times I$ cobounded by S_1 and S_2 and whose fibers have diameter less than $\gamma > 0$ which satisfies

8) $\gamma < \eta/(2^{k+2}) < (1/i)$ (wlog $\eta < 1$),
9) $\gamma < \frac{1}{4}\,\mathrm{dist}(p(f_k \ldots f_1(T) \cap (\mathrm{BdC} \times I)), \mathrm{BdZ}_k)$, and
10) $\gamma < \mathrm{dist}(f_k \ldots f_1(T), (\mathrm{BdC} - Z_k) \times I)$.

Let $t < 0$ be a number less than $\eta/(2^{k+2})$ so close to zero that $S_1 \equiv \mathrm{BdC} \times \{t\}$ is an ε-approximation to BdC. Apply the Side

Approximation Theorem of Bing [9] to find another locally flat ε-approximation S_2 to BdC disjoint from S_1 such that

11) there exists a finite set of disjoint ε-disks $E_1 \dots E_n$ in S_2 such that $S_2 - \cup \operatorname{Int}E_i \subset \operatorname{Int}C$,

12) there exists a finite set of disjoint ε-disks D_i in BdC such that $(\mathrm{BdC} - \cup D_i) \cap S_2 = \emptyset$, and

13) $p(S_2 \cap (\mathrm{BdC} \times I)) \subset \cup D_i$.

The first two conditions are straight from Bing's theorem, and any approximation satisfying them may be modified to satisfy 13) too by disk trading with a very close locally flat approximation to the D_i . Let $S \times I$ be the annulus obtained from the theorem of Craggs with respect to these S_e , and let Z_{k+1} be the union of those disks D_i from 12) wholly contained within Z_k .

Now f_{k+1} will be the composition of two $\eta/(2^{k+2})$-homeomorphisms, $f_{k+1} = gh$, each supported in Ω and each realizable by $\eta/(2^{k+2})$-isotopy; thus 1), 2), and 3) will hold for $i = k + 1$. The first, h , is to be a push along the fibers of $\mathrm{BdC} \times I$ with the property that $hf_k \dots f_1(T)$ is contained in $C \cup (\mathrm{BdC} \times [0, t))$; by 7), h is sufficiently close to the identity. The second, g , is to be a γ-push along the fibers of $S \times I$ from S_1 to S_2 and a little beyond such that $ghf_k \dots f_1(T) \subset C \cup (\mathrm{BdC} \times [0, t)) - S \times I$; by 8), g is sufficiently close to the identity.

Observe that 7), 4), and (by 8)) 5) hold true for $i = k + 1$. Let C_S be the 3-cell bounded by S_2 which does not contain S_1 . Then, for $i = k + 1$, 6) holds because $f_{k+1} \dots f_1(T) \subset ((\operatorname{Int}C) \cup p^{-1}(Z_{k+1})) \cap C_S$, as may be set-theoretically verified through use of 9), 10), 13), and the construction. ■

Lemma 5.4. If Z is a Cantor set and C a CFC with $Z \times I \subset \mathrm{Bd}C$, then there exists a continuous function $f: Z \to I$ such that the graph of f in $Z \times I$ is twice tame.

Proof. Let T be an infinite 2-skeleton of $\mathrm{Bd}C$ and S be an infinite 2-skeleton of S^4. By McMillan [44] T may be found so that $T \cap (Z \times I)$ is 0-dimensional. Likewise, it follows from Proposition 4.4 and the Convergence Criterion 1.1 that S may be found so that $S \cap (Z \times I)$ is 0-dimensional too. Now choose $f: Z \to I$ so that $(z, f(z)) \in Z \times I - (S \cup T)$ for all z in Z. The graph of f is twice tame - see Proposition 2.19. ■

Theorem 5.5. If C is a closed 3-cell complement in S^3 and ΣC is the suspension of C, then ΣC is universal.

Proof. Let a sewing h be given and let C_0 $(= \Sigma C)$, C_1, $S^3 \times I$, and G be the associated objects. Since C_0 is ΣC there exists a map $p: S^3 \times [-1, 1] \to S^4$ with exactly two inverse sets $S^3 \times \{-1\}$ and $S^3 \times \{1\}$ satisfying $p(C \times [-1, 1]) = C_0$ and $p(\mathrm{Bd}C \times [-1, 1]) = \mathrm{Bd}C_0$.

Now, we shall prove the theorem by applying 3.3. This involves finding disjoint neighborhoods V_e of $\mathrm{Bd}C_e$ and a 5ε-pseudoisotopy H_t of S^4 possessing certain properties with respect to certain objects P_e and D_e, all as described in 3.3. Our procedure is to successively define the V_e, let P_e and D_e be given, find a pseudoisotopy G_t, find another pseudoisotopy F_t, set $H_t = F_tG_t$, and show that H_t has the required properties.

Let $\varepsilon > 0$ be given.

(Finding the V_e). Let s and t be numbers, $-1 < s < t < 1$, such that for all q in S^3, the arcs $\{q\} \times [0, s]$ and $\{q\} \times [t, 1]$ from $S^3 \times I$ have diameter less than ε, and set $V_0 = S^3 \times [0, s)$ and $V_1 = S^3 \times (t, 1]$. Let P_e be an arbitrary PL 2-polyhedron in $V_e \cup C_e$ and $D_e \subset \mathrm{Int}C_e$ be any compact sets. We assume by general position that $P_0 \cap (\{a, b\} \times I) = \emptyset$ where $p(S^3 \times \{-1, 1\}) \subset \{a, b\} \subset \mathrm{Bd}C_0$; that is, assume that P_0 does not meet two flat arcs - see Proposition 4.3.

(Finding G_t). The map p is uniformly continuous; hence there exists an $\eta > 0$ such that pfp^{-1} is ε-close to the identity whenever f is well-defined and η-close to the identity. G_t will be found as a composition, $G_t = pq_tf_tg_th_tp^{-1}r_t$ where each pseudoisotopy except r_t will be an η-pseudoisotopy of $S^3 \times I$ which leaves $S^3 \times \{-1, 1\}$ setwise invariant, and r_t will be an ε-isotopy of S^4. Then by writing $G_t = (pq_tp^{-1})(pf_tp^{-1})(pg_tp^{-1})(ph_tp^{-1})(r_t)$, one sees that G_t is a 5ε-pseudoisotopy. Moreover, we shall define these maps in such a way that G_t is supported in $C_0 \cup V_0$, but leave the actual verification to the reader. Likewise we leave the reader to verify that each map may be taken to be the identity on D_0, so that G_t is too.

(About r_t and h_t). Apply 5.3 with $\eta = \eta$ to obtain Q and Ω. We shall apply 5.2 also. Let $K = \{k_i\}_{i=1}^{n}$ and $S = \{s_i\}_{i=1}^{n+1}$ be sets of numbers such that $-1 = s_1 < k_1 < s_2 < \ldots < k_n < s_{n+1} = 1$ and $k_i - k_{i-1} < \eta$ for all i, and $k_1 + 1 < \eta$, and $1 - k_n < \eta$. Set $\delta = \min\{\eta, \mathrm{dist}(S^3 \times S, S^3 \times K), \mathrm{dist}(C \times K, \mathrm{Bd}\Omega \times [-1, 1]))\}$

and apply 5.2 with respect to δ to find a pseudoisotopy h_t of $S^3 \times [-1, 1]$ such that $h_1(C \times [-1, 1]) \cap (S^3 \times S) \subset \mathrm{Int}C \times S$. Now r_t is to push P_0 along the fibers of $S^3 \times I$ (in S^4) so close to $\mathrm{Bd}C_0$ that $h_1 p^{-1} r_1(P_0) \cap (S^3 \times S) \subset \mathrm{Int}C \times S$ and $h_1 p^{-1} r_1(P_0) \subset (\Omega \times (-1, 1))$.

(About g_t) . This pseudoisotopy is to preserve S^3 coordinates and project $h_1 p^{-1} r_1(P_0) \cap ((S^3 - \mathrm{Int}C) \times [s_i, s_{i+1}])$ to $S^3 \times \{k_i\}$ leaving $g_1 h_1 p^{-1} r_1(P_0) \subset (C \times (-1, 1)) \cup (\Omega \times K)$.

(About f_t) . Obtain the η-pseudoisotopy λ_t from 5.3 with T equal to the projection to S^3 of $g_1 h_1 p^{-1} r_1(P_0)$; T is a compactum in Ω . Set $f_t = \lambda_t \times \mathrm{id}$. Then

$$f_1 g_1 h_1 p^{-1} r_1(P_0) \subset (\mathrm{Int}C \times (-1, 1)) \cup (Q \times K) ;$$

that is, $f_1 g_1 h_1 p^{-1} r_1(P_0)$ meets $\mathrm{Bd}C \times [-1, 1]$ in an 0-dimensional set contained in $Q \times K$.

(About q_t). Finally, q_t is an isotopy on $S^3 \times I$ which preserves S^3 coordinates and ambiently re-embeds $Q \times K$ in $Q \times (-1, 1)$ in such a way that the section for C_1 over $h(pq_1(Q \times K)$ (h is the sewing map) is flat. The details of this construction are well-known; suffice it to say that one uses 5.4 to re-embed each set $Q \times \{k_i\}$ in $Q \times (s_i, s_{i+1})$ separately, and afterwards obtains q_1 by the method made famous in the Klee trick [42, or see Rushing [48]]. How to realize q_1 by η-isotopy is then obvious.

(Finding F_t). Since the section for C_1 over $h(pq_1(Q \times K))$ is flat, $\mathrm{Bd}C_1 - h(pq_1(Q \times K))$ is a 1-GULC set for C_1 . F_t is to be the composition of an isotopy which pushes P_1 along the fibers of $S^3 \times I$ so close to BdC_1 that the image, together with D_1, may be ε-pseudoisotoped into this 1-GULC set. Thus F_t is a 2ε-pseudoisotopy supported in $V_1 \cup C_1$ such that $F_1(P_1 \cup D_1) \cap BdC \subset BdC_1 - h(pq_1(Q \times K))$.

(Concerning H_t). Set $H_t = G_t F_t$. Since G_t and F_t have disjoint supports, H_t is a 5ε-pseudoisotopy. Similarly, H_t is supported in $C_0 \cup V_0 \cup C_1 \cup V_1$. Since $G_1(P_0 \cup D_0) \cap BdC_0$ (G_t is the identity on D_0) is contained in $pq_1(Q \times K)$, we have $h(H_1(P_0 \cup D_0) \cap BdC_0) \cap H_1(P_1 \cup D_1) = \emptyset$. Thus 3.3 applies to show that the sewing yields S^4; thus ΣC is universal. ■

Corollary 5.6. *If C is a closed 3-cell complement in S^3 and ΣC is its suspension, then ΣC has G-type 1TT.*

Proof. Notice that in the proof of the last theorem G_t ε-deformed the whole of $p(\Omega \times (-1, 1))$ into $Q \times K$, a twice tame 0-dimensional set, union the interior of C. It follows that by repeating that part of the argument that got G_t for each $\varepsilon = 1/i$, one obtains sets $(Q \times K)_i$ such that $F = \cup(Q \times K)_i$ is the required 1-GULC set for ΣC. ■

G-type 1 Universal CFC's

A CFC C_0 is said to be G-type 1 universal if, for each G-type 1 CFC C_1, each sewing between C_0 and C_1 yields S^4. Like in high dimensions, a lack of pathology among the Cantor sets in BdC_0 implies G-type 1 universality. See Daverman [26, section 12] for the high dimensional versions - all the proofs are the same.

Theorem 5.7. If each tame Cantor set in BdC_0 *is tame in* S^4, *then* C_0 *is G-type 1 universal.*

Proof. If C_1 is G-type 1, F, the 1-GULC set that makes it so, and $h: BdC_0 \to BdC_1$ a sewing map, then by 2.11 and Kirby [41], $BdC_0 - h^{-1}(F)$ is a 1-GULC set for C_0. Thus the sewing yields S^4 by 3.4. ∎

Remark. Similar concepts and theorems hold true for the other G-types.

Corollary 5.8. If C_0 *is locally flat modulo a set which is flat in* S^4, *then it is G-type 1 universal.* ∎

A CFC is called *self-universal* if each sewing to itself yields S^4. The next corollary is a slight improvement on Cannon-Daverman [18, 7.7].

Corollary 5.9. If C_0 *has G-type 1 and if each Cantor set tame in* BdC_0 *is tame in* S^4, *then* C_0 *is self-universal.* ∎

Applying 2.18, we have:

Corollary 5.10. If C_0 *is locally flat modulo a set* X *of embedding dimension 1 in* BdC_0 *and if each Cantor set tame in* BdC_0 *is also tame in* S^4, *then* C_0 *is self-universal.* ∎

Applying 2.21, we have:

Corollary 5.11. If C_0 is locally flat modulo a 1-dimensional set X and if each Cantor set in X which is tame in S^4 is tame in BdC, then C_0 is self-universal. ■

That these are the best possible is shown by Example 3.8.

In Chapter III we defined inflations. These are studied in Cannon-Daverman [18], where they are shown to be self-universal. In view of 5.8, this result may be obtained by showing that inflations have G-type 1 ; in fact, more is true.

Corollary 5.12. If C is an inflation, then C is G-type 1 universal. ■

Were it known that inflations are G-type 1 , then this would be the more general result. In fact, Cannon and Daverman prove their result, albeit in their own language, by establishing just that. More specifically, they show two things. First, [18, 7.3] shows, in our language, that if C is an inflation and Q a set in BdC such that $C - Q$ is 1-ULC , then sections for C over Q are flat (like in the case of 3.2, one must understand our proof of 2.8 to see that this is really so). Thus they establish a connection between 1-ULC and 1-GULC for inflations. Secondly, [18, 7.4] shows that there exists an F_σ set F in BdC which is a union of tame in BdC Cantor sets such that $F \cup IntC$ is 1-ULC . To see that this means C has G-type 1 , observe that by the second part there exists an infinite 2-skeleton T of BdC with $T \cap F = \emptyset$. Now, by the first part, there exists a 1-GULC set F^* for C in $BdC - T$. Since $F^* \subset BdC - T$, it is tame in

BdC ; since C is an inflation, F* is tame in S^4 . Thus

<u>Theorem 5.13</u>. <u>If C is an inflation, then C has G-type at most 1TT, is G-type 1 universal, and is self-universal</u>. ■

Another example from [26, 13.5] exhibits an inflation which is not universal. This construction again succeeds when $n = 4$ by 3.12. Note: a trivial alteration is needed in that proof, namely, $g_0(\Delta^2)$ and $g_1(\Delta^2)$ should be procured so that $g_0(\Delta^2) \cap g_1(\Delta^2) \cap j(Y) = \emptyset$ (rather than having $g_0(\Delta^2)$ and $g_1(\Delta^2)$ be disjoint). Since $g_0(\Delta^2) \cap g_1(\Delta^2)$ may be taken to be finitely many points and since $j(Y)$ is a Cantor set, this is always possible. Thus we have

<u>Example 5.14</u>. <u>There exists an inflation which is not universal</u>. ■

<u>When C is Locally Flat Modulo a 1-Dimensional Set</u>

We conclude the chapter with a table showing which sewings yield S^4 in the case that one CFC is locally flat modulo a 1-dimensional set P . First, we require one last result.

<u>Theorem 5.15</u>. <u>Let</u> $h: BdC_0 \to BdC_1$ <u>be a sewing and suppose that</u> C_0 <u>is locally flat modulo a set</u> Q . <u>If</u> $C_{h(Q)}$ <u>is the section for</u> C_1 <u>over</u> $h(Q)$, <u>then</u> $C_0 \cup_h C_1$ <u>and</u> $C_0 \cup_{h*} C_{h(Q)}$ <u>are homeomorphic for some sewing</u> h* .

<u>Proof</u>. Let $f: BdC_0 \times I \to C_0$ be a collar on C_0 pinched along Q . Then the complementary domains of $f(BdC_0 \times \{\frac{1}{2}\})$ in $C_0 \cup_h C_1$ are homeomorphic to C_0 and $C_{h(Q)}$. The conclusion follows. ■

We use these abbreviations in the table. When we say C is k - TA , k is a number $(k = 0, 1, 2, 3)$ denoting the dimension of the set of points Q at which C is not locally flat, and TA is one of TF, TS, TB, or TW where TF means that each Cantor set in Q is twice flat, TS means no Cantor set in Q is tame in BdC but wild in S^4 , TB means no Cantor set in Q is tame in S^4 but wild in BdC , and TW means there is no restriction on Cantor sets in Q . Neither 3-TF nor 3-TB is possible, but all the rest are meaningful and distinct adjectives.

TABLE 2

SEWINGS OF A CFC , C_0 , LOCALLY FLAT MODULO A
1-DIMENSIONAL SET

C_1 is	C_0 is 1-TF	1-TS	1-TB	1-TW
G-type 2	□	▨	▨	▨
G-type 1	□	□	▨	▨
an inflation.	□	□	□	▨
k - TW.	□	▨	▨	▨
k - TB.	□	□	▨	▨
k - TS.	□	□	□	▨
k - TF.	□	□	□	□

KEY ▨ There exists a sewing, based upon the examples, not yielding S^4

□ Every sewing yields S^4 by 5.1, 5.7, and (or) 5.15.

Note: k may take the values 0, 1, 2, or 3 throughout (Recall 3-TB and 3-TF CFC's do not exist, and a 0-TF CFC is flat).

CHAPTER VI

A SHORT LIST OF QUESTIONS

Everyone knows that in E^4 it is particularly easy to ask hard questions. Moreover, the theory of sewings is not complete even in high dimensions. Therefore we shall put an end to this work by posing some questions potentially easier than usual. At least in each case, except 6.4, stronger results have been proved in higher dimensions.

6.1. If C is a CFC which is locally flat modulo a 1-dimensional set, must C have G-type at most 2 ?

6.2. If C is a CFC which is locally flat modulo a twice flat 2-polyhedron, must C have G-type at most 1 ?

6.3. If C is a CFC , is there an arc A in BdC such that the section over A is flat?

6.4. Is there a CFC C which is not universal even though every section over every Cantor set in BdC is?

6.5. If C has disjoint 1-GULC sets, must C have at most G-type 2 ?

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