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To the Graduate Council:

I am submitting herewith a thesis written by James Howard Alexander entitled "Approximately Finite Geometries and Their Coordinate Rings". I recommend that it be accepted in partial fulfillment of the requirements for the degree of Doctor of Philosophy, with a major in Mathematics.

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Major Professor

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APPROXIMATELY FINITE GEOMETRIES AND THEIR COORDINATE RINGS

A THESIS

Submitted to
The Graduate Council
of
The University of Tennessee
in
Partial Fulfillment of the Requirements
for the degree of
Doctor of Philosophy

by

James Howard Alexander

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INTRODUCTION

Von Neumann [8][1] has discovered a class of geometries in which the undefined elements are subspaces and the undefined operations are meets and joins of subspaces. His axioms require that the system consisting of the set $\mathcal{L}$ of subspaces together with the two operations form an irreducible, complete, complemented, modular lattice satisfying an additional dual pair of continuity conditions.

Since $\mathcal{L}$ is a modular lattice, all its maximal chains have the same length ν. If ν is finite then $\mathcal{L}$ is called a <u>discrete geometry</u> and is isomorphic to the lattice of all subspaces of of some $(\nu - 1)$-dimensional projective space. The atoms of $\mathcal{L}$ correspond to the points of the projective space, the joins of pairs of distinct atoms correspond to the lines of the space, etc.

If ν is infinite, that is, if $\mathcal{L}$ contains infinite chains, then $\mathcal{L}$ is called a <u>continuous geometry</u>. Because of the continuity conditions, $\mathcal{L}$ can have no atoms (and hence, nothing to correspond to points or lines) in this case.

Any discrete or continuous geometry $\mathcal{L}$ with $\nu \geqslant 4$ is isomorphic to the lattice $R_{\mathcal{V}_n}$ of principal right ideals of the ring $\mathcal{V}_n$ (called the coordinate ring of $\mathcal{L}$) of all n x n matrices over some regular ring $\mathcal{V}$, for some suitably

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[1] Numbers in square brackets refer to the bibliography at the end of the paper.

chosen positive integer n; or what amounts to the same thing, $\mathcal{L}$ is isomorphic to the lattice of all finitely generated submodules of the right module of n-tuples of elements of ϑ.[2] The classical coordinatization theorem for finite dimensional geometries of dimension ≥ 3 is a special case of this result. For if $4 \leq \nu < \infty$ and if we set $n = \nu$ (which is always permissible when ν is finite) then ϑ turns out to be a division ring, the right module of ν-tuples of elements of ϑ is a ν-dimensional vector space, and the submodules of this vector space (all of which are finitely generated) are its subspaces. Thus, the lattice of subspaces of a projective space of 3 or more dimensions, which is always a discrete geometry with $\nu - 1 \geq 3$, is isomorphic to the lattice of subspaces of a vector space, or equivalently, of a projective coordinate space.

The structure of a continuous geometry is, of course, much more complicated. However, there do exist continuous geometries which can be approximated by discrete geometries. These so-called approximately finite geometries form the principal subject of the present work.

Chapter I contains the definitions and theorems (all due to von Neumann) upon which the rest of the material is based. In Chapter II, the constructions of approximately finite geometries and their coordinate rings are described. The remaining two chapters contain applications of the results of Chapter II:

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[2] Fryer and Halperin [4,5] have given an elegant proof of this proposition.

the existence of polarities and the range of the signature of a certain type of polarity in an approximately finite geometry are discussed in Chapter III, and in Chapter IV, some light is shed on the structure of Frink's [3] projective embedding lattice for an approximately finite geometry.

CHAPTER I

BACKGROUND

The foundations of continuous geometry were presented by von Neumann in a series of lectures at the Institute for Advanced Study during the years 1935-1936 and 1936-1937, and it is the notes of these lectures [8] that are taken as the starting-point for the present discussion. Those results which are actually referred to are summarized in this Chapter, often in a somewhat simplified form.

1. <u>Axioms</u>. Let $\mathcal{L}$ be a set of elements $\mathfrak{a}, \mathfrak{b}, \ldots$ and let $\cap$ and $+$ be two binary operations on $\mathcal{L}$ having the following properties:

A.1 The system $(\mathcal{L}, \cap, +)$ is a complete, complemented modular lattice with $\cap$ and $+$ as lattice meet and join, respectively.

A.2 $\mathcal{L}$ is irreducible: that is, $0 \equiv \bigcap\{\mathfrak{a} : \mathfrak{a} \in \mathcal{L}\}$ and $1 \equiv \sum\{\mathfrak{a} : \mathfrak{a} \in \mathcal{L}\}$ are the only elements of $\mathcal{L}$ having unique complements.

A.3 If $\mathcal{C}$ is a chain (totally ordered subset) contained in $\mathcal{L}$ and $\mathfrak{a} \in \mathcal{L}$ then

$$\bigcap\{\mathfrak{a} + \mathfrak{b} : \mathfrak{b} \in \mathcal{C}\} = \mathfrak{a} + \bigcap\{\mathfrak{b} : \mathfrak{b} \in \mathcal{C}\}$$

and

$$\sum\{\mathfrak{a} \cap \mathfrak{b} : \mathfrak{b} \in \mathcal{C}\} = \mathfrak{a} \cap \sum\{\mathfrak{b} : \mathfrak{b} \in \mathcal{C}\}$$

Since $\mathcal{L}$ is modular, all its maximal chains have the same length ν. If ν is finite then $\mathcal{L}$ is a discrete geometry and is isomorphic to the lattice of all subspaces of some $(\nu - 1)$-dimensional projective space (Cf [2] and [3]), and conversely, all finite dimensional projective lattices are discrete geometries. If ν is infinite then $\mathcal{L}$ is a continuous geometry.

2. Independence. A finite subset $\{\mathfrak{a}_i : 1 \leq i \leq k\}$ of $\mathcal{L}$ is independent if $\mathfrak{a}_j \cap \sum\{\mathfrak{a}_i : 1 \leq i \leq k,\ i \neq j\} = 0$, $j = 1, 2, \ldots, k$.

3. The dimension-function. A (normalized) dimension-function on a complemented modular lattice is a real valued function d such that for all elements $\mathfrak{a}, \mathfrak{b}, \ldots$ of the lattice:

D.1 $d(\mathfrak{a}) \in [0,1]$

D.2 $d(0) = 0$ and $d(1) = 1$

D.3 $d(\mathfrak{a} + \mathfrak{b}) + d(\mathfrak{a} \cap \mathfrak{b}) = d(\mathfrak{a}) + d(\mathfrak{b})$

If a lattice $\mathcal{L}$ satisfies A.1-3. of §1 then a dimension function exists on $\mathcal{L}$, is unique, and has the following additional properties:

D.4 $d(\mathcal{L}) = \begin{cases} \{0, 1/\nu, 2/\nu, \ldots, 1\} & \text{if } \nu \text{ is finite,} \\ [0, 1] & \text{if } \nu \text{ is infinite.} \end{cases}$

D.5 If $\mathfrak{a} \subset \mathfrak{b}$ (i.e., $\mathfrak{a} = \mathfrak{a} \cap \mathfrak{b} \neq \mathfrak{b}$) then $d(\mathfrak{a}) < d(\mathfrak{b})$

D.6 $d(\mathfrak{a}) = d(\mathfrak{b})$ if and only if $\mathfrak{a}$ and $\mathfrak{b}$ have a common complement.

D.7 The function $\delta(\mathfrak{a},\mathfrak{b}) = d(\mathfrak{a}+\mathfrak{b}) - d(\mathfrak{a}\cap\mathfrak{b})$ is a metric on $\mathcal{L}$ and

(a) $\delta(\mathfrak{a}+\mathfrak{b},\mathfrak{c}+\mathfrak{d}) \leqslant \delta(\mathfrak{a},\mathfrak{c}) + \delta(\mathfrak{b},\mathfrak{d})$

(b) $\delta(\mathfrak{a}\cap\mathfrak{b},\mathfrak{c}\cap\mathfrak{d}) \leqslant \delta(\mathfrak{a},\mathfrak{c}) + \delta(\mathfrak{b},\mathfrak{d})$

(c) $|d(\mathfrak{a}) - d(\mathfrak{b})| \leqslant \delta(\mathfrak{a},\mathfrak{b})$;

hence, $+$, $\cap$, and d are uniformly continuous in the metric δ.

4. Regular rings. A ring $\mathcal{R}$ with unit 1 is regular if for every $a\in\mathcal{R}$ there exists an $x\in\mathcal{R}$ such that $axa = a$. In the rest of this chapter, $\mathcal{R}$ will be a regular ring with unit.

Since $\mathcal{R}$ has a unit, the principal right ideal of $\mathcal{R}$ generated by $a\in\mathcal{R}$ is $a\mathcal{R} = \{ax: x\in\mathcal{R}\}$. Let $R_{\mathcal{R}}$ denote the set of all principal right ideals of $\mathcal{R}$. When partly ordered by inclusion, the set $R_{\mathcal{R}}$ becomes a lattice, and
$\inf\{a\mathcal{R}, b\mathcal{R}\} = a\mathcal{R}\cap b\mathcal{R} = \{x: x\in a\mathcal{R},\ x\in b\mathcal{R}\}$,
$\sup\{a\mathcal{R}, b\mathcal{R}\} = a\mathcal{R} + b\mathcal{R} = \{x+y: x\in a\mathcal{R},\ y\in b\mathcal{R}\}$.
Every element of $R_{\mathcal{R}}$ is generated by an idempotent element of $\mathcal{R}$.

Define $(a)^r = \{y: ay = 0\}$. If $axa = a$ then $(a)^r = (1-xa)\mathcal{R}$, and hence $(a)^r \in R_{\mathcal{R}}$ for all $a\in\mathcal{R}$

A finite set $\{e_j: 1\leqslant j\leqslant n\}$ of idempotents of $\mathcal{R}$ is independent if $e_ie_j = 0 = e_je_i$ whenever $i \neq j$. If $\{\mathfrak{a}_j: 1\leqslant j\leqslant n\}$ is any independent subset of $R_{\mathcal{R}}$, that is, if $\mathfrak{a}_k \cap \sum\{\mathfrak{a}_j: 1\leqslant j\leqslant n,\ j\neq k\} = 0$ for $k = 1, 2,\dots,n$, (Cf. §2), then there exists an independent set $\{e_j: 1\leqslant j\leqslant n\}$ of idempotents of $\mathcal{R}$ such that $\mathfrak{a}_j = e_j\mathcal{R}$, $j = 1, 2, \dots, n$.

The center $\mathcal{Z}$ of $\mathcal{R}$ is a regular ring.

If $\mathcal{R}$ is a direct sum of two of its non-trivial subrings then $\mathcal{R}$ is <u>reducible</u>; otherwise, $\mathcal{R}$ is <u>irreducible</u>. $\mathcal{R}$ is reducible $\Leftrightarrow$ $\mathcal{Z}$ is a field $\Leftrightarrow$ $\mathcal{Z}$ has no idempotents other than 0 and 1.

5. <u>Rank-rings</u>. A regular ring $\mathcal{R}$ with unit 1 is a <u>rank-ring</u> if there exists a real valued function r defined on $\mathcal{R}$ and such that for all a, b, e, f $\in \mathcal{R}$,

R.1 $r(a) \in [0, 1]$,

R.2 $r(a) = 0$ if and only if $a = 0$,

R.3 $r(1) = 1$,

R.4 $r(ab) \leqslant \min \{ r(a), r(b) \}$,

R.5 if $e^2 = e$, $f^2 = f$, and $ef = 0 = fe$ then $r(e+f) = r(e) + r(f)$.

Such a function r is a <u>rank-function</u> on $\mathcal{R}$, and always has the further properties:

R.6 $r(a+b) \leqslant r(a) + r(b)$,

R.7 the function $\delta(a, b) = r(a - b)$ is a metric on $\mathcal{R}$, and

(a) $\delta(a + b, c + d) \leqslant \delta(a, c) + \delta(b, d)$

(b) $\delta(ab, cd) \leqslant \delta(a, c) + \delta(b, d)$

(c) $|r(a) - r(b)| \leqslant \delta(a, b)$.

It follows from R.7 (a) - (c) that the operations, addition and multiplication, and the function r are all uniformly continuous with respect to the metric δ.

If a rank-ring $\mathcal{R}$ is irreducible then the rank-function r is unique and has the still further property:

R.8 $r(a) = r(b)$ if and only if $b = paq$ for some p and q in $\mathcal{R}$ for which p^{-1} and q^{-1} exist.

6. Coordinatization and geometrization theorems.

(6.1) Coordinatization theorem. If $\mathcal{L}$ is a discrete or continuous geometry with $\nu \geqslant 4$ then there exists a regular ring $\mathcal{R}$, called the coordinate ring of $\mathcal{L}$, such that the lattices $\mathcal{L}$ and $R_{\mathcal{R}}$ are isomorphic. If $\nu \geqslant 3$ then the coordinate ring, when it exists, is a complete, irreducible rank-ring and is unique up to a ring-isomorphism.

The next theorem goes in the converse direction.

(6.2) Geometrization theorem. If $\mathcal{R}$ is a complete, irreducible rank-ring, then the lattice $R_{\mathcal{R}}$ is a discrete or continuous geometry. The rank-function r on $\mathcal{R}$ and the dimension-function d on $R_{\mathcal{R}}$ satisfy the equation: $d(a\mathcal{R}) = r(a)$, for all $a \in \mathcal{R}$.

CHAPTER II

APPROXIMATELY FINITE GEOMETRIES

7. Discrete geometries. According to §1, a lattice $\mathcal{L}$ is a discrete geometry if and only if it is isomorphic to the lattice of subspaces of a finite dimensional projective space. Since projective spaces of dimension ≥ 3 are isomorphic to projective coordinate spaces, it follows that a discrete geometry $\mathcal{L}$ with $\nu \geq 4$ is essentially the lattice of subspaces of some ν-dimensional vector space, and conversely. But more than the converse is true: the lattice of subspaces of any finite dimensional vector space is a discrete geometry. The purpose of the present section is, roughly, to show that the coordinate rings of vector spaces are matrix rings.

In all that follows, let ϑ be a fixed division ring; $\mathfrak{D}_n$, the right vector space of all $n \times 1$ matrices (column vectors of length n) over ϑ; $\mathcal{L}_n$, the lattice of subspaces of $\mathfrak{D}_n$; and $\mathfrak{M}_n$, the ring of all $n \times n$ matrices over ϑ.

By a standard theorem on matrices, if $A \in \mathfrak{M}_n$ then, for some non-singular $X \in \mathfrak{M}_n$, $H = AX$ is a matrix in Hermite normal form. But such a matrix H is idempotent, so $AX = H = H^2 = AXAX$ and $A = AXX^{-1} = AXAXX^{-1} = AXA$. Therefore, $\mathfrak{M}_n$ is a regular ring. As in §4, let $R\mathfrak{M}_n$ denote the lattice of principal right ideals of $\mathfrak{M}_n$.

Suppose that $\mathfrak{a}$ is a subspace of $\mathfrak{D}_n$ (that is, $\mathfrak{a} \in \mathcal{L}_n$)

and that $\{a_1, \ldots, a_n\}$ is a (not necessarily independent) set of vectors spanning $\mathfrak{a}$. Then the matrix $A = (a_1\ a_2\ \ldots\ a_n)$ is an element of $\mathfrak{M}_n$, and clearly $A\mathfrak{D}_n = \{Av : v \in \mathfrak{D}_n\} = \mathfrak{a}$. Therefore $\mathcal{L}_n = \{A\mathfrak{D}_n : A \in \mathfrak{M}_n\}$.

It is easy to see that if $A, B \in \mathfrak{M}_n$ then

(7.1) $A\mathfrak{D}_n \supseteq B\mathfrak{D}_n \iff B = AX$ for some $X \in \mathfrak{M}_n$

$\iff A\mathfrak{M}_n \supseteq B\mathfrak{M}_n$

Consequently, $A\mathfrak{D}_n = B\mathfrak{D}_n \iff A\mathfrak{M}_n = B\mathfrak{M}_n$, and so

(7.2) $\sigma_n(A\mathfrak{D}_n) = A\mathfrak{M}_n$, for all $A \in \mathfrak{M}_n$,

defines a one-to-one mapping of $\mathcal{L}_n$ onto $R_{\mathfrak{M}_n}$. In fact,(7.1) implies more: σ_n and its inverse both preserve lattice-order; hence σ_n is a lattice-isomorphism of $\mathcal{L}_n$ onto $R_{\mathfrak{M}_n}$. In view of §6, we have the

(7.3) <u>Theorem</u>. The mapping $\sigma_n : \mathcal{L}_n \to R_{\mathfrak{M}_n}$ defined by (7.2) is a lattice -isomorphism. If $n \geq 3$ then $\mathfrak{M}_n$ is a complete, irreducible rank-ring, the coordinate ring of $\mathcal{L}_n$.

Remark: It is easy to show that $\mathfrak{M}_1$ and $\mathfrak{M}_2$ are complete, irreducible rank-rings also. However the lattices $\mathcal{L}_1$ and $\mathcal{L}_2$ which correspond, respectively, to the projective point and line, are uninteresting and generally do not have well-determined coordinate rings.

If we define $D(\mathfrak{a})$, $\mathfrak{a} \in \mathcal{L}_n$, to be the ordinary dimension of $\mathfrak{a}$ (that is, the number of elements in a basis for $\mathfrak{a}$) then D satisfies condition D.3 of §3. It follows immediately that $d(\mathfrak{a}) = \frac{D(\mathfrak{a})}{n}$ is the dimension function on $\mathcal{L}_n$.

Furthermore, if $R(A)$, $A \in \mathfrak{M}_n$, denotes the ordinary rank of the matrix A, then $D(A\mathfrak{D}_n) = R(A)$, and hence, by (6.2), $r(A) = d(A\mathfrak{D}_n) = \frac{D(A\mathfrak{D}_n)}{n} = \frac{R(A)}{n}$ is the rank-function on $\mathfrak{M}_n$.

Thus we get the

(7.4) <u>Corollary</u>. (a) The dimension-function d on $\mathcal{L}_n$ satisfies the equation $d(\mathfrak{a}) = \frac{D(\mathfrak{a})}{n}$ for all $\mathfrak{a} \in \mathcal{L}_n$, where $D(\mathfrak{a})$ is the dimension of $\mathfrak{a}$.

(b) The rank-function r on $\mathfrak{M}_n$ satisfies the equation $r(A) = \frac{R(A)}{n}$ for all $A \in \mathfrak{M}_n$, where $R(A)$ is the rank of the matrix A.

8. <u>Construction of $\overline{\mathcal{L}}_\infty$</u>. Von Neumann [9] has shown how certain continuous geometries, which he calls "approximately finite" may be constructed as limits of discrete geometries. In the present section, we give what is essentially the same construction except that the terminology of vector spaces, rather than that of projective spaces, is used so that it will be easier to apply results like those of the preceding section.

If m and n are positive integers and if $k = mn$ then there is a mapping φ_{kn} of $\mathfrak{M}_n$ into $\mathfrak{M}_k$ defined by

$$(8.1) \qquad \varphi_{kn}A = \underbrace{\begin{bmatrix} A & 0 & \dots & 0 \\ 0 & A & \dots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \dots & A \end{bmatrix}}_{m \text{ blocks}} \in \mathfrak{M}_k, \quad \text{for } A \in \mathfrak{M}_n.$$

Clearly, φ_{kn} is an isomorphism of $\mathfrak{M}_n$ onto a subring $\mathfrak{M}_{kn}$ of $\mathfrak{M}_k$. Also

$$r(\varphi_{kn}A) = \frac{R(\varphi_{kn}A)}{k} = \frac{mR(A)}{k} = \frac{R(A)}{n} = r(A) \ ;$$

that is, φ_{kn} preserves (normalized) rank.

Let $A, B \in \mathfrak{M}_n$. Since φ_{kn} is an isomorphism,

$$\begin{aligned} A\mathfrak{D}_n \supseteq B\mathfrak{D}_n &\Rightarrow B = AX \quad \text{for some } X \in \mathfrak{M}_n \\ &\Rightarrow \varphi_{kn}B = (\varphi_{kn}A)(\varphi_{kn}X) \\ &\Rightarrow (\varphi_{kn}A)\mathfrak{D}_k \supseteq (\varphi_{kn}B)\mathfrak{D}_k \ , \end{aligned}$$

and $A\mathfrak{D}_n = B\mathfrak{D}_n \Rightarrow (\varphi_{kn}A)\mathfrak{D}_k = (\varphi_{kn}B)\mathfrak{D}_k$.

Therefore, $A\mathfrak{D}_n \rightarrow (\varphi_{kn}A)\mathfrak{D}_k$ for all $A \in \mathfrak{M}_n$ is a well-defined order-preserving mapping of $\mathcal{L}_n$ into $\mathcal{L}_k$. Let

$$(8.2) \qquad \psi_{kn}(A\mathfrak{D}_n) = (\varphi_{kn}A)\mathfrak{D}_k \qquad \text{for all } A \in \mathfrak{M}_n \ .$$

The mapping ψ_{kn} preserves dimension, for

$$d\left[\psi_{kn}(A\mathfrak{D}_n)\right] = d\left[(\varphi_{kn}A)\mathfrak{D}_k\right] = r(\varphi_{kn}A) = r(A) = d(A\mathfrak{D}_n) \ .$$

Now suppose that $(\varphi_{kn}A)\mathfrak{D}_k \supseteq (\varphi_{kn}B\mathfrak{D}_n)$. Then there is an $X \in \mathfrak{M}_k$ such that $\varphi_{kn}B = (\varphi_{kn}A)X$,

and if $$X = \begin{bmatrix} X_{11} & X_{12} & \cdots & X_{1m} \\ X_{21} & X_{22} & \cdots & X_{2m} \\ \vdots & \vdots & & \vdots \\ X_{m1} & X_{m2} & \cdots & X_{mm} \end{bmatrix} \quad \text{where } X_{ij} \in \mathfrak{M}_n$$

for $i, j = 1, \ldots, m$, then $B = AX_{11}$; hence $A\mathfrak{D}_n \supseteq B\mathfrak{D}_n$.

In summary: $\psi_{kn}(A\mathfrak{D}_n) \supseteq \psi_{kn}(B\mathfrak{D}_n) \Leftrightarrow A\mathfrak{D}_n \supseteq B\mathfrak{D}_n$;

so ψ_{kn} has a well-defined order-preserving inverse on

$\psi_{kn}(\mathcal{L}_n) \subset \mathcal{L}_k$. Thus, we have proved that ψ_{kn} is a dimension-preserving order-isomorphism of $\mathcal{L}_n$ into $\mathcal{L}_k$. But we need more:

(8.3) <u>Lemma</u>. The mapping ψ_{kn} defined by (8.2) is a dimension-preserving lattice-isomorphism of $\mathcal{L}_n$ onto $\mathcal{L}_{kn} \equiv \psi_{kn}(\mathcal{L}_n) \subset \mathcal{L}_k$.

Proof: What remains to be shown is that

(a) $\psi_{kn}(\mathfrak{A} \cap \mathfrak{B}) = (\psi_{kn}\mathfrak{A}) \cap (\psi_{kn}\mathfrak{B})$ and

(b) $\psi_{kn}(\mathfrak{A} + \mathfrak{B}) = (\psi_{kn}\mathfrak{A}) + (\psi_{kn}\mathfrak{B})$ for all $\mathfrak{A}, \mathfrak{B} \in \mathcal{L}_n$.

Let $\mathfrak{A} = A\mathfrak{D}_n$, $\mathfrak{B} = B\mathfrak{D}_n$, $\mathfrak{A} \cap \mathfrak{B} = C\mathfrak{D}_n$ where $A, B, C \in \mathfrak{M}_n$. Since ψ_{kn} preserves order, $\psi_{kn}(\mathfrak{A} \cap \mathfrak{B}) \subseteq (\psi_{kn}\mathfrak{A}) \cap (\psi_{kn}\mathfrak{B})$. Suppose that $c \in (\psi_{kn}\mathfrak{A}) \cap (\psi_{kn}\mathfrak{B}) = [(\varphi_{kn}A)\mathfrak{D}_k] \cap [\varphi_{kn}B)\mathfrak{D}_k]$. Then $c = [A \cdot \times 1_m]a = [B \cdot \times 1_m]b$ for some $a, b \in \mathfrak{D}_k$.

Let $a = \begin{bmatrix} \alpha_1 \\ \vdots \\ \alpha_k \end{bmatrix}$, $b = \begin{bmatrix} \beta_1 \\ \vdots \\ \beta_k \end{bmatrix}$, $c = \begin{bmatrix} \gamma_1 \\ \vdots \\ \gamma_k \end{bmatrix}$. Clearly,

$$c^j = \begin{bmatrix} \gamma_{jn+1} \\ \vdots \\ \gamma_{(j+1)n} \end{bmatrix} = A \begin{bmatrix} \alpha_{jn+1} \\ \vdots \\ \alpha_{(j+1)n} \end{bmatrix} = B \begin{bmatrix} \beta_{jn+1} \\ \vdots \\ \beta_{(j+1)n} \end{bmatrix}$$

for $j = 0, 1, \ldots, m-1$, and $c^j \in C\mathfrak{D}_n = \mathfrak{A} \cap \mathfrak{B}$ for $j = 0, 1, \ldots, m-1$. Hence $c^j = Cd^j$ for some $d^j \in \mathfrak{D}_n$ and $c = (C \cdot \times 1_m)d \in (C \cdot \times 1_m)\mathfrak{D}_k = \psi_{kn}(\mathfrak{A} \cap \mathfrak{B})$, where $d = \begin{bmatrix} d^0 \\ \vdots \\ d^{m-1} \end{bmatrix}$. Therefore

$$\psi_{kn}(\mathfrak{A} \cap \mathfrak{B}) \supseteq (\psi_{kn}\mathfrak{A}) \cap (\psi_{kn}\mathfrak{B}) .$$

We already have the inequality in the other direction so (a) is proved.

It follows from (a) and property D.3 (§3) of the dimension function that

$$\begin{aligned} d\left[(\psi_{kn}\mathfrak{A}) + (\psi_{kn}\mathfrak{B})\right] &= d(\psi_{kn}\mathfrak{A}) + d(\psi_{kn}\mathfrak{B}) - d\left[(\psi_{kn}\mathfrak{A}) \cap (\psi_{kn}\mathfrak{B})\right] \\ &= d(\mathfrak{A}) + d(\mathfrak{B}) - d(\mathfrak{A} \cap \mathfrak{B}) \\ &= d(\mathfrak{A} + \mathfrak{B}) \\ &= d\left[\psi_{kn}(\mathfrak{A} + \mathfrak{B})\right], \end{aligned}$$

and since ψ_{kn} preserves order, $\psi_{kn}(\mathfrak{A}+\mathfrak{B}) \supseteq (\psi_{kn}\mathfrak{A}) + (\psi_{kn}\mathfrak{B})$; hence, according to D.5, $\psi_{kn}(\mathfrak{A}+\mathfrak{B}) = (\psi_{kn}\mathfrak{A}) + (\psi_{kn}\mathfrak{B})$.

Thereby, the lemma is proved.

Let $\mathbb{N}$ be a fixed infinite set of positive integers such that

(8.4) $\qquad n < k \Rightarrow n|k , \qquad$ for all $n, k \in \mathbb{N}$.

Define $\mathcal{L}_\infty = \{\mathfrak{A} : \mathfrak{A} \in \mathcal{L}_n, n \in \mathbb{N}\}$, where $\mathfrak{A} \in \mathcal{L}_n$ is to be identified with $\psi_{kn}\mathfrak{A} \in \mathcal{L}_k$ whenever $n < k$. This identification, which we denote by $\psi_{kn}\mathfrak{A} \sim \mathfrak{A}$, makes sense because $n < k < r \Rightarrow \varphi_{rk}\varphi_{kn} = \varphi_{rk} \Rightarrow \psi_{rk}\psi_{kn} = \psi_{rn}$. Furthermore, since ψ_{kn} is a dimension-preserving lattice -isomorphism, it follows that the operations $\cap$, $+$, and d are well-defined in $\mathcal{L}_\infty$.

Clearly, all relations and identities involving only a finite number of variables that hold in every $\mathcal{L}_n$; $n \in \mathbb{N}$, continue to hold in $\mathcal{L}_\infty$. Therefore, $\mathcal{L}_\infty$ is a modular lattice with dimension-function d; $\delta(\mathfrak{a}, \mathfrak{b}) \equiv d(\mathfrak{a} + \mathfrak{b}) - d(\mathfrak{a} \cap \mathfrak{b})$ is a metric on $\mathcal{L}_\infty$ (D.7 of §3); and $+$, $\cap$, and d are uniformly continuous with respect to the metric δ.

Since the function δ makes a metric space of $\mathcal{L}_\infty$, there exists a complete metric space $\overline{\mathcal{L}}_\infty$ containing $\mathcal{L}_\infty$ as a dense subspace. The uniformly continuous functions $+$, $\cap$, and d can be extended by continuity to all of $\overline{\mathcal{L}}_\infty$. Again, all finite identities that hold on $\mathcal{L}_\infty$ continue to hold on $\overline{\mathcal{L}}_\infty$; hence $\overline{\mathcal{L}}_\infty$ is also a modular lattice.

That the lattice $\overline{\mathcal{L}}_\infty$ is actually a continuous geometry will follow from §14 where it is proved that $\overline{\mathcal{L}}_\infty$ is isomorphic to the lattice of principal right ideals of a complete irreducible rank-ring $\overline{\mathfrak{M}}_\infty$.

9. <u>Construction of</u> $\overline{\mathfrak{M}}_\infty$. Let ω be the first infinite ordinal number, and let $\mathfrak{M}_\omega$ denote the ring of all $\omega \times \omega$ matrices over ϑ having only a finite number of non-zero elements in each row and column. Let $\mathfrak{M}^{(n)}$ be the set of all matrices $A = (\alpha_{ij}) \in \mathfrak{M}_\omega$ satisfying the conditions:

$$\alpha_{ij} = \alpha_{(i+n)(j+n)}, \text{ for all } i, j$$

$$(9.1) \qquad \alpha_{ij} = 0 \quad , \text{ whenever } |i - j| \geq n$$

Diagrammatically, A looks like this

$$A = \begin{bmatrix} A' & 0 & 0 & \cdots \\ 0 & A' & 0 & \cdots \\ 0 & 0 & A' & \cdots \\ \vdots & \vdots & \vdots & \end{bmatrix}$$

where 0 and A' are $n \times n$ matrices and 1_ω is the unit matrix of $\mathfrak{M}_\omega$.

The mapping $t_n : \mathfrak{M}^{(n)} \to \mathfrak{M}_n$ defined by

$$(9.2) \qquad t_n(\alpha_{ij}) = (\alpha_{ij} : i, j \leq n) , \quad (\alpha_{ij}) \in \mathfrak{M}^{(n)} ,$$

is obviously a ring-isomorphism of $\mathfrak{M}^{(n)}$ onto $\mathfrak{M}_n$ (referring to the diagram, $t_n(A) = A'$) . Hence $\mathfrak{M}^{(n)}$ is a regular subring of $\mathfrak{M}_\omega$ and $r_n = rt_n$ (r on $\mathfrak{M}_n$ is given by (7.4)(b)) is the rank function on $\mathfrak{M}^{(n)}$.

Next, define $\mathfrak{M}_\infty = \bigcup \{\mathfrak{M}^{(n)} : n \in \mathbb{N}\}$ where $\mathbb{N}$ is the set of integers fixed in §8 . By (8.4), if $n, k \in \mathbb{N}$ and $n < k$ then $n | k$, and by (9.1), if $n | k$ then $\mathfrak{M}^{(n)} \subseteq \mathfrak{M}^{(k)}$; hence the set $\{\mathfrak{M}^{(n)} : n \in \mathbb{N}\}$ is totally ordered by inclusion. It follows that if $A , B \in \mathfrak{M}_\infty$ then $A , B \in \mathfrak{M}^{(n)}$ for some $n \in \mathbb{N}$, and hence $\mathfrak{M}_\infty$ is closed under addition, subtraction, and multiplication. Therefore $\mathfrak{M}_\infty$ is a subring of $\mathfrak{M}_\omega$. By a similar argument $\mathfrak{M}_\infty$ is regular.

By (9.1), the mappings t_n of (9.2) and φ_{kn} of (8.1) satisfy the equation $t_k = \varphi_{kn} t_n$ on $\mathfrak{M}^{(n)}$ when $n | k$ (i.e., when $\mathfrak{M}^{(n)} \subset \mathfrak{M}^{(k)}$) , and, since φ_{nk} preserves rank, the rank-functions r_n and r_k satisfy $r_k = rt_k = r\varphi_{kn} t_n = rt_n = r_n$ on

$\mathfrak{M}^{(n)}$ $\left(\subset \mathfrak{M}^{(k)}\right)$. Thus, the functions r_n ; $n \in \mathbb{N}$, are extensions (or contractions) of one another. Therefore,

$$r^{*}(A) = rt_n(A) , \qquad \text{if } A \in \mathfrak{M}^{(n)} \subset \mathfrak{M}_\infty ,$$

is a well-defined function on $\mathfrak{M}_\infty$. The fact that $A, B \in \mathfrak{M}_\infty$ $\Rightarrow$ $A, B \in \mathfrak{M}^{(n)}$ for some $n \in \mathbb{N}$, can be used again to show that r^* is a rank-function on $\mathfrak{M}_\infty$. Suppose that r' is an arbitrary rank-function on $\mathfrak{M}_\infty$, and let A be any element of $\mathfrak{M}_\infty$ -- say $A \in \mathfrak{M}^{(n)}$. Clearly, the restrictions of r' and r^* to $\mathfrak{M}^{(n)}$ are rank-functions on $\mathfrak{M}^{(n)}$, but $\mathfrak{M}^{(n)}$, being isomorphic to $\mathfrak{M}_n$, has a unique rank-function (by (7.3) and § 6). Therefore, $r'(A) = r^*(A)$. But A was an arbitrary element of $\mathfrak{M}_\infty$, so $r' = r^*$, and $\mathfrak{M}_\infty$ has a unique rank-function. This justifies our setting $r = r^*$ on $\mathfrak{M}_\infty$.

According to § 5, any rank-function on a regular ring satisfies R.7. Therefore $\delta(A, B) \equiv r(A - B)$ is a metric on $\mathfrak{M}_\infty$, and addition and multiplication are uniformly continuous in the metric space ($\mathfrak{M}_\infty$, δ) . Since $-A = (-1) \cdot A$, subtraction is also uniformly continuous.

Proceeding as we did in the construction of $\overline{\mathcal{L}}_\infty$, let $\overline{\mathfrak{M}}_\infty$ be a complete metric space of which $\mathfrak{M}_\infty$ is a dense subspace. Since the operations addition, subtraction, and multiplication, and the rank-function r are all uniformly continuous in $\mathfrak{M}_\infty$, they can be extended by continuity to all of $\overline{\mathfrak{M}}_\infty$. If r' is the extension of r to $\overline{\mathfrak{M}}_\infty$ and if the notation for the algebraic operations is preserved, then by the continuity of the extension it is easily seen that

(9.3) for all $A, B, C, D \in \overline{\mathfrak{M}}_\infty$,

(a) $A + B = B + A$

(b) $A + (B + C) = (A + B) + C$, $A(BC) = (AB)C$

(c) $A + 0 = 0$

(d) $A + (-A) = A - A = 0$

(e) $A(B + C) = AB + AC$, $(B + C)A = BA + CA$

(f) $r'(A) \in [0,1]$

(g) $r'(A) = 0 \Leftrightarrow A = 0$

(h) $r'(AB) \leqslant \min \{r'(A), r'(B)\}$

(i) $r'(A + B) \leqslant r'(A) + r'(B)$

(j) $\delta(A,B) = r'(A - B)$

(k) $\delta(A + B, C + D) \leqslant \delta(A,C) + \delta(B, D)$

(ℓ) $\delta(AB, CD) \leqslant \delta(A, C) + \delta(B, D)$

(m) $|r'(A) - r'(B)| \leqslant \delta(A, B)$.

The identities (a - e) imply that $\overline{\mathfrak{M}}_\infty$ is a ring.

Now, our task is to show that $\overline{\mathfrak{M}}_\infty$ is a rank-ring. This problem is treated in the next few sections.

10. <u>Regularity of</u> $\overline{\mathfrak{M}}_\infty$. If $A \in \mathfrak{M}_n$ then $v \to Av$, $v \in \mathfrak{D}_n$, is a linear transformation in $\mathfrak{D}_n$. Furthermore, the mapping $A \overset{\mu}{\to} (v \to Av, v \in \mathfrak{D}_n)$ is an isomorphism of the ring $\mathfrak{M}_n$ onto the ring $\mathcal{J}_n$ of all linear transofrmations in $\mathfrak{D}_n$. In the discussion below, it will be convenient to ignore the distinction between a matrix $A \in \mathfrak{M}_n$ and the transformation it determines under the mapping μ just described.

If $A \in \mathfrak{M}_n$ then the <u>null space</u> of A is the subspace

$\mathfrak{N}(A) = \{v : Av = 0, v \in \mathfrak{V}_n\} \in \mathcal{L}_n$. If $A \in \mathfrak{M}_n$ and $\mathcal{X} \in \mathcal{L}_n$ then the <u>restriction</u> of A to $\mathcal{X}$ is the linear transformation $A|\mathcal{X} : x \to Ax$, all $x \in \mathcal{X}$. It follows from well-known facts about vector spaces and their linear transformations that

(10.1) for all $A \in \mathfrak{M}_n$ and all $\mathcal{X}, \mathcal{Y} \in \mathcal{L}_n$

(a) $d(A\mathfrak{V}_n) + d(\mathfrak{N}(A)) = 1$;

(b) $A|\mathcal{X}$ is non-singular $\Leftrightarrow \mathfrak{N}(A) \cap \mathcal{X} = 0 \Leftrightarrow d(A\mathcal{X}) = d(\mathcal{X})$ $\Leftrightarrow \mathcal{Y} \to A\mathcal{Y}$ is an isomorphism of the lattice of subspaces of $\mathcal{X}$ onto the lattice of subspaces of $A\mathcal{X}$;

(c) if $\mathcal{X} \cap \mathcal{Y} = 0$ and if X and Y are linear transformations of $\mathcal{X}$ and $\mathcal{Y}$, respectively, into $\mathfrak{V}_n$, then there exists a $C \in \mathfrak{M}_n$ such that $C|\mathcal{X} = X$ and $C|\mathcal{Y} = Y$.

(10.2) <u>Lemma</u>. If $A, X \in \mathfrak{M}_n$ then the following statements are equivalent:

(a) $AXA = A$,

(b) $A|(XA\mathfrak{V}_n) = [X|(A\mathfrak{V}_n)]^{-1}$,

(c) $(AX)|(A\mathfrak{V}_n)$ is the identity map on $A\mathfrak{V}_n$.

Proof: Obviously (b) $\Rightarrow$ (c) $\Rightarrow$ (a). Now suppose that $AXA = A$. If x is any vector in $A\mathfrak{V}_n$, say $x = Av$, then $[A|(XA\mathfrak{V}_n)][X|(A\mathfrak{V}_n)]x = AXx = AXAv = Av = x$, so $[A|(XA\mathfrak{V}_n)][X|(A\mathfrak{V}_n)]$ is the identity map on $A\mathfrak{V}_n$. Also, if $y = XAw$ then $[X|(A\mathfrak{V}_n)][A|(XA\mathfrak{V}_n)]y = XAy =$ $= XAXAw = XAw = y$, so $[X|(A\mathfrak{V}_n)][A|(XA\mathfrak{V}_n)]$ is the identity map on $(XA\mathfrak{V}_n)$. Thus it follows that $A|(XA\mathfrak{V}_n)$ and $X|(A\mathfrak{V}_n)$ are mutually inverse. Therefore (a) $\Rightarrow$ (b), which completes the proof of the lemma.

The critical step in the proof of the regularity is the

(10.3) Lemma. Let $A, B, X \in \mathfrak{M}_n$, and let $AXA = A$. Then there exists a $Y \in \mathfrak{M}_n$ such that $BYB = B$ and $r(X - Y) \leqslant 2r(A - B)$.

Proof: Clearly $X|\mathfrak{N}(BX - 1) = \left[B \mid X\mathfrak{N}(BX - 1)\right]^{-1}$; hence $\mathfrak{N}(B) \cap X\mathfrak{N}(BX - 1) = 0$. Since $\mathcal{L}_n$ is a complemented modular lattice, there exists a complement $\mathfrak{X}$ of $\mathfrak{N}(B)$ which contains $X\mathfrak{N}(BX - 1)$: that is,

$$\begin{aligned} &\mathfrak{X} \cap \mathfrak{N}(B) = 0 \\ \text{(a)} \qquad &\mathfrak{X} + \mathfrak{N}(B) = \mathfrak{V}_n \\ &\mathfrak{X} \supseteq X\mathfrak{N}(BX - 1). \end{aligned}$$

Now let $\mathfrak{Y}$ be a relative complement of $X\mathfrak{N}(BX - 1)$ in $\mathfrak{X}$, so that

$$\begin{aligned} &\mathfrak{Y} \cap X\mathfrak{N}(BX - 1) = 0 \\ \text{(b)} \qquad &\mathfrak{Y} + X\mathfrak{N}(BX - 1) = \mathfrak{X} \end{aligned}$$

According to (10.1)(b), $B|\mathfrak{X}$ is a lattice isomorphism so

$$\text{(c)} \qquad \begin{aligned} &(B\mathfrak{Y}) \cap \mathfrak{N}(BX - 1) = (B\mathfrak{Y}) \cap BX\mathfrak{N}(BX - 1) = B\left[\mathfrak{Y} \cap X\mathfrak{N}(BX - 1)\right] = 0, \\ &B\mathfrak{Y} + \mathfrak{N}(BX - 1) = B\left[\mathfrak{Y} + X\mathfrak{N}(BX - 1)\right] = B\mathfrak{X} = B\mathfrak{V}_n. \end{aligned}$$

Finally, let $\mathfrak{Z}$ be a complement of $B\mathfrak{V}_n$, so that

$$\text{(d)} \qquad \begin{aligned} &\mathfrak{Z} \cap B\mathfrak{V}_n = 0 \\ &\mathfrak{Z} + B\mathfrak{V}_n = \mathfrak{V}_n. \end{aligned}$$

Since $\mathfrak{Y} \cap \mathfrak{N}(B) = \mathfrak{Y} \cap \mathfrak{X} \cap \mathfrak{N}(B) = \mathfrak{Y} \cap 0 = 0$, $(B|\mathfrak{Y})^{-1}$ exists and is a linear transformation of $B\mathfrak{Y}$ onto $\mathfrak{Y}$.

If a, b, and c are elements of a modular lattice and if $(a + b) \cap c = 0$ then $a \cap (b + c) = a \cap b$. Hence it follows from (c) and (d) that

$$B\mathcal{Y} \cap \left[\mathcal{N}(BX - 1) + \mathcal{Z}\right] = B\mathcal{Y} \cap \mathcal{N}(BX - 1) = 0 .$$

and so, by (10.1)(c), there exists a $Y \in \mathcal{M}_n$ such that

$$Y|B\mathcal{Y} = (B|\mathcal{Y})^{-1}$$

(e) $$Y|\left[\mathcal{N}(BX - 1) + \mathcal{Z}\right] = X|\left[\mathcal{N}(BX - 1) + \mathcal{Z}\right]$$

But $X|\left[\mathcal{N}(BX - 1)\right] = \left[B|X\mathcal{N}(BX - 1)\right]^{-1}$ so

$$Y|B\mathcal{D}_n = Y|\left[B\mathcal{Y} + \mathcal{N}(BX - 1)\right] = \left\{B|\left[\mathcal{Y} + X\mathcal{N}(BX - 1)\right]\right\}^{-1} .$$

Therefore $BYB = B$, by (10.2).

That $\mathcal{N}(X - Y) \supseteq \mathcal{N}(BX - 1) + \mathcal{Z}$ is an immediate consequence of the definition of Y. This yields

$$\begin{aligned} r(X - Y) &= d\left[(X - Y)\mathcal{D}_n\right] \\ &= 1 - d\left[\mathcal{N}(X - Y)\right] && \text{by (10.1)(a)} \\ \text{(f)} \qquad &\leqslant 1 - d\left[\mathcal{N}(BX - 1) + \mathcal{Z}\right] \\ &= 1 - d\left[\mathcal{N}(BX - 1)\right] - d\mathcal{Z} && \text{by (c) and (d)} \\ &= d(B\mathcal{D}_n) - d\left[\mathcal{N}(BX - 1)\right] && \text{by (d)} \\ &\leqslant d(B\mathcal{D}_n) - d\left[\mathcal{N}(BX - 1) \cap A\mathcal{D}_n\right] \end{aligned}$$

Since $\mathcal{N}(BX - 1) \cap A\mathcal{D}_n \subseteq (B\mathcal{H}) \cap (A\mathcal{D}_n) =$
$= (B\mathcal{D}_n) \cap (A\mathcal{D}_n)$, we may let $\mathcal{U}$ be a relative complement of $\mathcal{N}(BX - 1) \cap A\mathcal{D}_n$ in $B\mathcal{D}_n \cap A\mathcal{D}_n$. Let u be an arbitrary element of $\mathcal{U}$. Since $\mathcal{U} \subseteq A\mathcal{D}_n$ and $AXA = A$, $u = AXu$ by (10.2). Also $\mathcal{U} \cap \mathcal{N}(BX - 1) =$
$= \mathcal{U} \cap A\mathcal{D}_n \cap \mathcal{N}(BX - 1) = 0$, so if $u \neq 0$ then

$(BX - 1)u \neq 0$, $BXu \neq u = AXu$, and $(A - B)Xu \neq 0$. Therefore $(X\mathcal{U}) \cap \mathcal{N}(A - B) = 0$. By (10.1)(a) and (10.2), $(X|\mathcal{U}) = (A|X\mathcal{U})^{-1}$ and hence $d(\mathcal{U}) = d(X\mathcal{U})$. Thus it follows from (10.1) and the definition of $\mathcal{U}$, that

(g) $$d(B\mathcal{D}_n \cap A\mathcal{D}_n) - d[\mathcal{N}(BX - 1) \cap A\mathcal{D}_n] = d(\mathcal{U}) = d(X\mathcal{U}) \leqslant$$
$$\leqslant 1 - d\mathcal{N}(A - B) = r(A - B) .$$

On the other hand, let $\mathcal{M}$ be a relative complement of $(B\mathcal{D}_n) \cap (A\mathcal{D}_n)$ in $B\mathcal{D}_n$. Since $\mathcal{M} \subseteq B\mathcal{D}_n$ and $BYB = B$, (10.2) tells us that $Y|\mathcal{M} = (B|Y\mathcal{M})^{-1}$, and $d(\mathcal{M}) = d(Y\mathcal{M})$ by (10.1)(b). Hence $v \in Y\mathcal{M} \cap \mathcal{N}(A - B)$ implies that $Av = Bv \in A\mathcal{D}_n \cap B\mathcal{D}_n \cap BY\mathcal{M} = A\mathcal{D}_n \cap B\mathcal{D}_n \cap \mathcal{M} = 0$; so $v = YBv = Y0 = 0$. Therefore $Y\mathcal{M} \cap \mathcal{N}(A - B) = 0$, and

(h) $$d(B\mathcal{D}_n) - d[(B\mathcal{D}_n) \cap (A\mathcal{D}_n)] = d(\mathcal{M}) = d(Y\mathcal{M}) \leqslant$$
$$\leqslant 1 - d[\mathcal{N}(A - B)] = r(A - B) .$$

From (f), (g), and (h), we get $r(X - Y) \leqslant 2r(A - B)$, and the proof of the lemma is complete.

Remark: One can exhibit matrices A , B , X such that $AXA = A$, and $BYB = B \Rightarrow r(X - Y) \geqslant 2r(A - B)$. For example, if

$$A = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} , \quad B = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} , \quad X = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \quad \text{then}$$

$$BYB = B \Rightarrow Y = B \Rightarrow r(X - Y) = r \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} = 2r \begin{bmatrix} 0 & 0 \\ 0 & -1 \end{bmatrix} = 2r(A - B) .$$

The corollary that follows is an immediate consequence of the lemma.

(10.4) <u>Corollary</u>. If $A , B , X \in \mathcal{M}_\infty$ and $AXA = A$ then there exists a $Y \in \mathcal{M}_\infty$ such that $BYB = B$ and $\delta(X,Y) \leqslant 2\delta(A,B)$.

(10.5) <u>Theorem</u>. $\overline{\mathfrak{M}}_\infty$ is regular.

Proof: Let A be any element of $\overline{\mathfrak{M}}_\infty$. Since $\mathfrak{M}_\infty$ is dense in $\overline{\mathfrak{M}}_\infty$, there exists a sequence $(A_n)_n$ in $\mathfrak{M}_\infty$ such that $A_n \to A$. Assume, as we may, that $\delta(A; A_n) < 2^{-n}$. Then $\delta(A_n, A_{n+1}) \leqslant \delta(A; A_n) + \delta(A; A_{n+1}) < 2^{-n} + 2^{-n-1} < 2^{-n+1}$. Let X_1 be an element of $\mathfrak{M}_\infty$ such that $A_1X_1A_1 = A_1$. By (10.4), $X_2, X_3, \ldots, X_n, \ldots$ may be chosen successively so that $A_nX_nA_n = A_n$ and $\delta(X_n, X_{n+1}) \leqslant 2\delta(A_n, A_{n+1}) < 2^{-n+2}$. The sequence $(X_n)_n$ so constructed satisfies the inequality

$$\delta(X_n, X_{n+k}) \leqslant \sum_{i=0}^{k-1} \delta(X_{n+i}, X_{n+i+1}) < \sum_{i=0}^{k-1} 2^{-n-i+2} < 2^{-n+3},$$

and is, therefore, fundamental. Since $\overline{\mathfrak{M}}_\infty$ is complete, $X = \lim_n X_n$ exists. By continuity of multiplication, $AXA = \lim_n A_nX_nA_n = \lim_n A_n = A$. The existence of such an $X \in \overline{\mathfrak{M}}_\infty$ for every $A \in \overline{\mathfrak{M}}_\infty$ implies that $\overline{\mathfrak{M}}_\infty$ is regular, and the proof is complete.

11. $\overline{\mathfrak{M}}_\infty$ <u>is a rank-ring</u>. The statements (9.3)(f-h) assert that the function r' on $\overline{\mathfrak{M}}_\infty$ satisfies R.1, 2, and 4 of §5. Of course, $r'(1) = r(1) = 1$, since 1 is already an element of $\mathfrak{M}_\infty$. Therefore to show that r' is a rank-function, it suffices to show that R.5 is also satisfied by r'.

(11.1) <u>Lemma</u>. If $A \in \mathfrak{M}_n$ then there exists an idempotent $E \in \mathfrak{M}_n$ such that $r(A - E) \leqslant r(A - A^2)$.

Proof: Let $\mathfrak{A}$ be some complement of $\mathfrak{N}(A^2 - A)$ and let E be the linear transformation defined by

$$E = \begin{cases} A & \text{on } \mathfrak{N}(A^2 - A) \\ 0 & \text{on } \mathfrak{A}. \end{cases}$$

That is, $E \mid \mathfrak{N}(A^2 - A) = A \mid \mathfrak{N}(A^2 - A)$ and $E \mid \mathfrak{A} = 0$.

Now $A\,\mathfrak{N}(A^2 - A) = A\,\mathfrak{N}[(A - 1)A] \subseteq \mathfrak{N}(A - 1) \subseteq \mathfrak{N}[A(A - 1)]$

$$= \mathfrak{N}(A^2 - A),$$

so $E \mid \mathfrak{N}(A^2 - A) = A \mid \mathfrak{N}(A^2 - A)$ is an idempotent linear transformation in $\mathfrak{N}(A^2 - A)$; also, $E \mid \mathfrak{A} = 0$ is idempotent in $\mathfrak{A}$. Therefore E is idempotent in $\mathbb{D}_n$. Furthermore $\mathfrak{N}(A - E) \supseteq$ $\supseteq \mathfrak{N}(A^2 - A)$, so

$$r(A - E) = 1 - d[\mathfrak{N}(A - E)] \leqslant 1 - d[\mathfrak{N}(A^2 - A)] = r(A^2 - A).$$

Therefore E has all the desired properties.

(11.2) <u>Corollary</u>. If $A \in \mathfrak{M}_\infty$ then there exists an idempotent E in $\mathfrak{M}_\infty$ such that $r(A - E) \leqslant r(A^2 - A)$.

(11.3) <u>Definition</u>. If $\mathfrak{A}$ is any subset of $\overline{\mathfrak{M}}_\infty$ then $\overline{\mathfrak{A}}$ is the topological closure of $\mathfrak{A}$.

(11.4) <u>Definition</u>. If $\mathfrak{A}$ is any subset of $\overline{\mathfrak{M}}_\infty$ or $\mathfrak{M}_n$ then $\mathcal{J}(\mathfrak{A})$ is the set of all idempotent elements of $\mathfrak{A}$.

(11.5) <u>Lemma</u>. $\mathcal{J}(\overline{\mathfrak{M}}_\infty) = \overline{\mathcal{J}(\mathfrak{M}_\infty)}$.

Proof: By the continuity of multiplication, any limit of idempotents is an idempotent; hence, $\mathcal{J}(\overline{\mathfrak{M}}_\infty) \supseteq \overline{\mathcal{J}(\mathfrak{M}_\infty)}$. On the other hand, suppose that $E \in \mathcal{J}(\overline{\mathfrak{M}}_\infty)$; and let $(A_i)_i$ be a sequence in $\mathfrak{M}_\infty$ such that $E = \lim_i A_i$. By (11.2), for each A_i, we may choose an $E_i \in \mathcal{J}(\mathfrak{M}_\infty)$ such that $\delta(A_i, E_i) \leqslant \delta(A_i, A_i^2)$. The sequence $(E_i)_i$ so selected is contained in $\mathcal{J}(\mathfrak{M}_\infty)$ and converges to E, because

$$\begin{aligned}\delta(E,E_1) &\leqslant \delta(E,A_1) + \delta(A_1,E_1)\\ &\leqslant \delta(E,A_1) + \delta(A_1,A_1^2)\\ &\leqslant 2\,\delta(E,A_1) + \delta(E,A_1^2)\\ &= 2\,\delta(E,A_1) + \delta(E^2,A_1^2)\\ &\leqslant 4\,\delta(E,A_1)\,, \qquad \text{(by R.7 (b)).}\end{aligned}$$

Therefore, $E = \lim_i E_i \in \overline{\mathcal{J}(\mathcal{M}_\infty)}$, and since E was an arbitrary element of $\mathcal{J}(\overline{\mathcal{M}_\infty})$, it follows that $\mathcal{J}(\overline{\mathcal{M}_\infty}) \subseteq \overline{\mathcal{J}(\mathcal{M}_\infty)}$. This completes the proof of the lemma.

Next we list for ready reference a few well-known results about idempotents in $\mathcal{M}_n$.

(11.6) If $A \in \mathcal{M}_n$ then $A \in \mathcal{J}(\mathcal{M}_n) \Leftrightarrow (1 - A) \in \mathcal{J}(\mathcal{M}_n)$.

(11.7) If $E \in \mathcal{J}(\mathcal{M}_n)$ then

(a) $E\mathcal{D}_n = \{v : Ev = v\}$,

(b) $E\mathcal{D}_n + \mathcal{N}(E) = \mathcal{D}_n$ and $E\mathcal{D}_n \cap \mathcal{N}(E) = 0$,

(c) $\mathcal{N}(E) = (1 - E)\mathcal{D}_n$.

(11.8) <u>Lemma</u>. If $A \in \mathcal{M}_n$ and $E \in \mathcal{J}(\mathcal{M}_n)$ then

(a) $A\mathcal{D}_n \subseteq E\mathcal{D}_n \Leftrightarrow EA = A$,

(b) $\mathcal{N}(A) \supseteq \mathcal{N}(E) \Leftrightarrow AE = A$,

(c) $\mathcal{N}(AE) = \mathcal{N}(E) + [\mathcal{N}(A) \cap E\mathcal{D}_n]$.

Proof: (a) By (11.7) (a),

$$A\mathcal{D}_n \subseteq E\mathcal{D}_n \Leftrightarrow E(Av) = Av \text{ for all } v \in \mathcal{D}_n \Leftrightarrow EA = A.$$

(b) By (11.7)(c)

$$\mathcal{N}(A) \supseteq \mathcal{N}(E) = (1 - E)\mathcal{D}_n \Leftrightarrow A(1 - E)\mathcal{D}_n = 0 \Leftrightarrow AE = A.$$

(c) $AE\{\mathcal{N}(E) + [\mathcal{N}(A) \cap E\mathcal{D}_n]\} = 0$; hence

$$\mathcal{N}(AE) \supseteq \mathcal{N}(E) + [\mathcal{N}(A) \cap E\mathcal{D}_n].$$

On the other hand, if $AEv = 0$ then $Ev \in \mathfrak{N}(A) \cap E\mathfrak{D}_n$ and hence, by (11.7)(c) $v = (1 - E)v + Ev \in \mathfrak{N}(E) + [\mathfrak{N}(A) \cap E\mathfrak{D}_n]$. Therefore $\mathfrak{N}(AE) \subseteq \mathfrak{N}(E) + [\mathfrak{N}(A) \cap E\mathfrak{D}_n]$, and (c) is proved.

(11.9) <u>Lemma</u>. If $E, F \in \mathcal{I}(\mathfrak{M}_\infty)$ then there exists an $E' \in \mathcal{I}(\mathfrak{M}_\infty)$ such that $E'E = E' = EE'$, $E'F = 0$, and $r(E - E') = r(EF)$.

Proof: Suppose first that $E, F \in \mathcal{I}(\mathfrak{M}_n)$ (instead of $\mathcal{I}(\mathfrak{M}_\infty)$). By (11.7)(b), $\mathfrak{N}(E) + E\mathfrak{D}_n = \mathfrak{D}_n$; hence there exists a complement $\mathfrak{A}$ of $\mathfrak{N}(E) + F\mathfrak{D}_n$ such that $\mathfrak{A} \subseteq E\mathfrak{D}_n$. Let E' be the linear transformation defined by

$$E' = \begin{cases} 1 & \text{on } \mathfrak{A} \\ 0 & \text{on } \mathfrak{N}(E) + F\mathfrak{D}_n. \end{cases}$$

Clearly, $E' \in \mathcal{I}(\mathfrak{M}_n)$. Also, $E'\mathfrak{D}_n = \mathfrak{A} \subseteq E\mathfrak{D}_n$ and $\mathfrak{N}(E') = \mathfrak{N}(E) + F\mathfrak{D}_n \supseteq \mathfrak{N}(E)$; hence, by (11.8) $E'E = E' = EE'$. Since $\mathfrak{N}(E') \supseteq F\mathfrak{D}_n$, we get $E'F\mathfrak{D}_n = 0$ and $E'F = 0$.

As for $r(E - E')$, $E'(E - E') = E'E - E'E' = E' - E' = 0$ and $(E - E')E' = EE' - E'E' = E' - E' = 0$; hence, by R.5 of §5, $r(E) = r(E - E' + E') = r(E - E') + r(E')$. This yields

$$
\begin{aligned}
r(E - E') &= r(E) - r(E') \\
&= d(E\mathfrak{D}_n) - d(E'\mathfrak{D}_n) \\
&= d[\mathfrak{N}(E')] - d[\mathfrak{N}(E)] \\
&= d[\mathfrak{N}(E) + F\mathfrak{D}_n] - d[\mathfrak{N}(E)] \\
&= d(F\mathfrak{D}_n) - d[\mathfrak{N}(E) \cap F\mathfrak{D}_n] \\
&= 1 - d[\mathfrak{N}(F)] - d[\mathfrak{N}(E) \cap F\mathfrak{D}_n] \\
&= 1 - d\{\mathfrak{N}(F) + [\mathfrak{N}(E) \cap F\mathfrak{D}_n]\} \\
&\qquad - d[\mathfrak{N}(F) \cap \mathfrak{N}(E) \cap F\mathfrak{D}_n] \\
&= 1 - d\{\mathfrak{N}(F) + [\mathfrak{N}(E) \cap F\mathfrak{D}_n]\},
\end{aligned}
$$

because $\mathfrak{N}(F) \cap \mathfrak{N}(E) \cap F\mathfrak{D}_n \subseteq \mathfrak{N}(F) \cap F\mathfrak{D}_n = 0$, by (11.7)(b). By (11.8)(c), $\mathfrak{N}(F) + [\mathfrak{N}(E) \cap F\mathfrak{D}_n] = \mathfrak{N}(EF)$, and

$$
\begin{aligned}
r(E - E') &= 1 - d\,\{\mathfrak{N}(F) + [\mathfrak{N}(E) \cap F\mathfrak{D}_n]\} \\
&= 1 - d\,[\mathfrak{N}(EF)] \\
&= d(EF\mathfrak{D}_n) \\
&= r(EF).
\end{aligned}
$$

Thereby, the lemma is proved with "$\mathfrak{M}_n$" in place of "$\mathfrak{M}_\infty$". But this implies that the lemma is true as stated.

(11.10) <u>Lemma</u>. If $E, F \in \mathcal{J}(\overline{\mathfrak{M}}_\infty)$ and if $EF = 0 = FE$ then $r'(E + F) = r'(E) + r'(F)$.

Proof: If $E, F \in \mathcal{J}(\overline{\mathfrak{M}}_\infty)$ and if $\varepsilon > 0$ then by (11.5) there exist matrices $E^{\#}, F^{\#} \in \mathcal{J}(\mathfrak{M}_\infty)$ such that

(a) $$\delta(E, E^{\#}) + \delta(F, F^{\#}) < \varepsilon.$$

By (11.9), there exist matrices $E', F' \in \mathcal{J}(\mathfrak{M}_\infty)$ such that

(b) $$E'E^{\#} = E' = E^{\#}E', \qquad F'F^{\#} = F' = F^{\#}F';$$

(c) $$E'F^{\#} = 0, \qquad F'E^{\#} = 0;$$

(d) $\quad r(E^* - E') = r(E^*F^*)\ , \quad r(F^* - F') = r(F^*E^*)\ .$

It follows from (b) and (c) that $E'F' = E'F^*F' = 0$ and similarly that $F'E' = 0$; hence, by R.5 which holds on $\mathfrak{M}_\infty$, we get

(e) $$r(E' + F') = r(E') + r(F')\ .$$

From this, we get

$$\begin{aligned}
\Delta \equiv\ & |r'(E+F) - r'(E) - r'(F)| \\
\leqslant\ & |r'(E+F) - r'(E'+F')| + |r'(E) - r'(E')| + \\
& + |r'(F) - r'(F')|\ , \text{ by (e) }, \\
\leqslant\ & \delta(E+F\ , E'+F') + \delta(E\ , E') + \delta(F\ , F')\ , \text{ by (9.3)(m) }, \\
\leqslant\ & 2\delta(E\ , E') + 2\delta(F\ , F')\ , \text{ by (9.3)(k) }, \\
\leqslant\ & 2\delta(E\ , E^*) + 2\delta(E^*, E') + 2\delta(F\ , F^*) + 2\delta(F^*, F') \\
<\ & 2\varepsilon + 2r(E^*F^*) + 2r(F^*E^*)\ , \text{ by (a) and (d)} \\
=\ & 2\varepsilon + 2\delta(E^*F^*, 0) + 2\delta(F^*E^*, 0) \\
=\ & 2\varepsilon + 2\delta(E^*F^*, EF) + 2\delta(F^*E^*, FE)\ , \text{ since } EF = 0 = FE\ , \\
=\ & 2\varepsilon + 4\delta(E^*, E) + 4\delta(F^*, F)\ , \text{ by (9.3)}(\ell) \\
<\ & 6\varepsilon\ , \text{ by (a) }.
\end{aligned}$$

But ε is an arbitrary positive number, so $\Delta = 0$: that is, $r'(E+F) = r'(E) + r'(F)$, which is the desired result.

The lemma asserts that the function r' satisfies condition R.5, which is all that was needed to show that r' is a rank-function on $\overline{\mathfrak{M}}_\infty$. We are in a position to prove the

(11.11) <u>Theorem</u>. $\overline{\mathfrak{M}}_\infty$ is a rank-ring.

Proof: By (10.5), $\overline{\mathfrak{M}}_\infty$ is a regular ring and by the result above,

r' is a rank-function on $\overline{\mathfrak{M}}_\infty$; therefore, $\overline{\mathfrak{M}}_\infty$ has all the properties of a rank-ring.

In view of this result, we shall set $r = r'$ on $\overline{\mathfrak{M}}_\infty$. Though it still remains to be shown that there are no other rank-functions on $\overline{\mathfrak{M}}_\infty$ no confusion is possible because the equation $r(A) = \delta(A,0)$ ((9.3)(j)) determines r uniquely.

12. <u>The center of</u> $\overline{\mathfrak{M}}_\infty$. In this section, it is proved that the center $\mathfrak{Z}(\overline{\mathfrak{M}}_\infty)$ of $\overline{\mathfrak{M}}_\infty$ is isomorphic to the center $\mathfrak{Z}(\vartheta)$ of ϑ.

(12.1) <u>Lemma</u>. If Z is an element of $\overline{\mathfrak{M}}_\infty$ that commutes with the diagonal idempotents of $\mathfrak{M}^{(n)}$, for some $n \in \mathbb{N}$, then, given any $\varepsilon > 0$, there exists an $A = (a_{ij}) \in \mathfrak{M}_\infty$ such that $\delta(A, Z) < \varepsilon$ and $a_{ij} = 0$ unless $i \equiv j \pmod n$.

Proof: Define the matrix $E^{kn} \in \mathfrak{M}_\omega$ by

$$(E^{kn})_{ij} = \delta_{ij}\, \delta^n_{ik}$$

where

$$\delta_{ij} = \begin{cases} 0 & \text{if } i \neq j \\ 1 & \text{if } i = j \end{cases}$$

and

$$\delta^n_{ik} = \begin{cases} 0 & \text{if } i \not\equiv k \pmod n \\ 1 & \text{if } i \equiv k \pmod n . \end{cases}$$

Clearly, E^{kn} is a diagonal idempotent of $\mathfrak{M}^{(n)}$. Next, let p be the polynomial on $\overline{\mathfrak{M}}_\infty$ defined by

$$p(Y) = Y - \sum_{k=1}^{n} E^{kn}Y(1 - E^{kn}) .$$

If $Y = (y_{ij}) \in \mathfrak{M}_\infty$ then $(E^{kn}Y)_{ij} = \sum_s \delta_{is}\, \delta^n_{ik} y_{sj} = \delta^n_{ik} y_{ij}$,

$$\left[E^{kn}Y(1 - E^{kn})\right]_{ij} = \delta^n_{ik}y_{ij} - \sum_s \delta^n_{ik}y_{is}\delta_{sj}\delta^n_{sk}$$
$$= \delta^n_{ik}y_{ij} - \delta^n_{ik}y_{ij}\delta^n_{jk}$$
$$= \delta^n_{ik}y_{ij}(1 - \delta^n_{jk}) ,$$

$$\left[\sum_{k=1}^{n} E^{kn}Y(1 - E^{kn})\right]_{ij} = \sum_{k=1}^{n}\delta^n_{ik}y_{ij}(1 - \delta^n_{jk}) = y_{ij}(1 - \delta^n_{ij})$$

and therefore $\left[p(Y)\right]_{ij} = y_{ij}\delta^n_{ij}$. Thus

(a) if $Y = (y_{ij}) \in \mathfrak{M}_\infty$ and if $A = (a_{ij}) = p(Y)$ then $a_{ij} = 0$ unless $i \equiv j \pmod n$.

The polynomial p also has the following properties:

(b) if $X \in \overline{\mathfrak{M}}_\infty$ then $r\left[p(X)\right] \leqslant (n+1)r(X)$,

(c) if $Z \in \overline{\mathfrak{M}}_\infty$ and if Z commutes with the diagonal idempotents of $\mathfrak{M}^{(n)}$ then $p(Z) = Z$, and

(d) if $W, X \in \overline{\mathfrak{M}}_\infty$ then $p(W \pm X) = p(W) \pm p(X)$.

Finally, given Z and ε as in the hypotheses of the lemma, let Y be any element of $\mathfrak{M}_\infty$ such that $\delta(Y,Z) < \dfrac{\varepsilon}{n+1}$, and define $A = p(Y)$. Then

$$\text{(e)} \qquad \begin{aligned} \delta(Z, A) &= r(Z - A) \\ &= r\left[p(Z) - p(Y)\right], \quad \text{by (c)} \\ &= r\left[p(Z - Y)\right] , \quad \text{by (d)} \\ &\leqslant (n+1)r(Z - Y), \quad \text{by (b)} \\ &= (n+1)\,\delta(Z, Y) < \varepsilon \end{aligned}$$

The desired result is contained in statements (a) and (e) .

(12.2) <u>Lemma</u>. Let $A \in \mathfrak{M}_n$ and let $\mathfrak{D}_n$ be the set of all diagonal matrices in $\mathfrak{M}_n$. If, for some positive integer k ,

$\min\{R(A + D) : \text{ all } D \in \mathcal{D}_n\} \geqslant 3k$ then A has a non-singular $k \times k$ minor consisting entirely of off-diagonal elements.

Proof: The lemma is obvious when $k = 1$. Assume the lemma holds for $k < t$ and that $\min\{R(A + D) : D \in \mathcal{D}_n\} \geqslant 3t$. It is always possible to rearrange the rows and columns of A so as to leave diagonal elements on the diagonal and to carry any given off-diagonal minor to the lower left hand corner. It follows from this and the inductive hypothesis that we may suppose that $A = (a_{ij})$ has the non-singular $(t - 1) \times (t - 1)$ minor

$$\begin{bmatrix} a_{(n-t+2)1} & \cdots & a_{(n-t+2)(t-1)} \\ \vdots & & \vdots \\ a_{n1} & \cdots & a_{n(t-1)} \end{bmatrix}$$

Suppose further that A has no non-singular, off-diagonal $t \times t$ minor and hence that

$$R\begin{bmatrix} a_{t1} & \cdots & a_{t(t-1)} & a_{tj} \\ \vdots & & \vdots & \vdots \\ a_{(j-1)1} & \cdots & a_{(j-1)(t-1)} & a_{(j-1)j} \\ a_{(j+1)1} & \cdots & a_{(j+1)(t-1)} & a_{(j+1)j} \\ \vdots & & \vdots & \vdots \\ a_{n1} & \cdots & a_{n(t-1)} & a_{nj} \end{bmatrix} = t - 1,$$

$j = t, t + 1, \ldots, n - t + 1$. Then we may choose a^*_{jj} so that

$$R\begin{bmatrix} a_{t1} & \cdots & a_{t(t-1)} & a_{tj} \\ \vdots & & \vdots & \vdots \\ a_{(j-1)1} & \cdots & a_{(j-1)(t-1)} & a_{(j-1)j} \\ a_{j1} & \cdots & a_{j(t-1)} & a^{*}_{jj} \\ a_{(j+1)1} & \cdots & a_{(j+1)(t-1)} & a_{(j+1)j} \\ \vdots & & \vdots & \vdots \\ a_{n1} & \cdots & a_{n(t-1)} & a_{nj} \end{bmatrix} = t - 1 ,$$

$j = t, t + 1 , \ldots, n - t + 1$, and we get

$$R\begin{bmatrix} a_{t1} & \cdots & a^{*}_{tt} & \cdots & a_{t(n-t+1)} \\ \vdots & & \vdots & & \vdots \\ a_{(n-t+1)1} & \cdots & a_{(n-t+1)t} & \cdots & a^{*}_{(n-t+1)(n-t+1)} \\ \vdots & & \vdots & & \vdots \\ a_{n1} & \cdots & a_{nt} & \cdots & a_{n(n-t+1)} \end{bmatrix} = t - 1 .$$

Adding $t - 1$ rows to a matrix raises its rank by no more than $t - 1$, so

$$R\begin{bmatrix} a^{*}_{11} & a_{12} & \cdots & a_{1(n-t+1)} \\ a_{21} & a^{*}_{22} & \cdots & a_{2(n-t+1)} \\ \vdots & \vdots & & \vdots \\ a_{(n-t+1)1} & a_{(n-t+1)2} & \cdots & a^{*}_{(n-t+1)(n-t+1)} \\ \vdots & \vdots & & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{n(n-t+1)} \end{bmatrix} \leqslant 2(t - 1),$$

regardless of the choice of $a^{*}_{11} , a^{*}_{22} , \ldots, a^{*}_{(t-1)(t-1)}$.

Adding $t - 1$ columns gives

$$R \begin{bmatrix} a_{11}^{*} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22}^{*} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn}^{*} \end{bmatrix} \leqslant 3(t - 1) ,$$

for arbitrary $a^{*}_{(n-t+2)(n-t+2)}, \ldots, a^{*}_{nn}$. Therefore, if we set $d_j = a^{*}_{jj} - a_{jj}$ and $D = \operatorname{diag}(d_1, d_2, \ldots, d_n)$ then $R(A + D) \leqslant 3(t - 1) < 3t$. But this is contrary to the inductive hypothesis, and therefore the lemma is proved.

Remark: The last result can be slightly improved: in fact, the argument given above may be used to show that if $R(A + D) \geqslant 3k - 2$ for all diagonal matrices D then A still has a non-singular $k \times k$ off-diagonal minor. Little more strengthening is possible, however -- for let $K_0 = 1 \in \mathfrak{M}_1$, and having defined K_{j-1}, define

$$K_j = \begin{bmatrix} 0 & 1 \\ K_{j-1} & 0 \end{bmatrix} \in \mathfrak{M}_{2^j} ,$$

where $0, 1, K_{j-1} \in \mathfrak{M}_{2^{j-1}}$, and finally, define

$$A_j = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ K_j & 0 & 0 \end{bmatrix} \in \mathfrak{M}_{3 \cdot 2^j} ,$$

$j = 0, 1, \ldots$, where $0, 1, K_j \in \mathfrak{M}_{2^j}$. It is easy to see that the largest off-diagonal non-singular minor of A_j has rank 2^j whereas the first $3 \cdot 2^j - 1$ rows of A_j are linearly independent regardless of how the diagonal elements be altered.

Before stating the next theroem, we make two definitions.

(12.3) <u>Definition.</u> $\mathcal{D}$ is the set of all diagonal matrices of $\mathfrak{M}_\infty$.

(12.4) <u>Definition.</u> If $\mathcal{R}$ is any ring and if $\mathcal{T} \subseteq \mathcal{R}$ then $\mathcal{L}(\mathcal{T}) = \{x : x \in \mathcal{R}, xs = sx \text{ for all } s \in \mathcal{T}\}$ is the centralizer of $\mathcal{R}$. $\mathfrak{Z}(\mathcal{R}) = \mathcal{L}(\mathcal{R})$ is the <u>center</u> of $\mathcal{R}$.

(12.5) <u>Theorem.</u> $\mathcal{L}(\mathcal{J}(\mathcal{D})) \subseteq \overline{\mathcal{D}}$; that is, if $Z \in \overline{\mathfrak{M}}_\infty$ and if Z commutes with every diagonal idempotent of $\mathfrak{M}_\infty$, then Z is an accumulation point of the set of diagonal matrices of $\mathfrak{M}_\infty$.

Proof: Suppose that $Z \in \mathcal{L}(\mathcal{J}(\mathcal{D})) \setminus \overline{\mathcal{D}}$. $Z \notin \overline{\mathcal{D}}$ implies that $\delta(Z, \mathcal{D}) = \inf\{\delta(Z, D) : D \in \mathcal{D}\} > 0$, and hence $\delta(Z, \mathcal{D}) > \frac{10}{n}$ for some sufficiently large positive integer n. Let $A = (a_{ij})$ be any matrix in $\mathfrak{M}_\infty$ such that $\delta(A, Z) < \frac{1}{n}$. Then

$$\text{(a)} \qquad \delta(A, \mathcal{D}) > \delta(Z, \mathcal{D}) - \delta(A, Z) > \frac{9}{n} .$$

Since $A \in \mathfrak{M}_\infty$, $A \in \mathfrak{M}^{(m)}$ for some $m \in \mathbb{N}$ such that $m \geqslant 2n$. Let k be the integer satisfying

$$\text{(b)} \qquad kn \leqslant m < (k+1)n .$$

Let t_m be the mapping of $\mathfrak{M}^{(m)}$ onto $\mathfrak{M}_m$ defined by (9.2) and let $A' = (a_{ij}) = t_m(A) \in \mathfrak{M}_m$. Then $a_{ij} = a'_{ij}$ for all $i, j \leqslant m$. From (a) and (b), it follows that

$R(A' + D) = mr(A' + D) > m\left(\frac{9}{n}\right) \geqslant 9k$ for all $D \in \mathcal{D}_m$; hence, by lemma (12.2), there exists a non-singular $3k \times 3k$ minor A'' of A' consisting entirely of off-diagonal elements. Clearly, the minor A'' appears in every set of m consecutive rows of A.

By lemma (12.1), on the other hand, $Z \in \mathcal{L}(\mathcal{J}(\mathcal{D}))$ implies that there exists a $B = (b_{ij}) \in \mathfrak{M}_\infty$ such that $b_{ij} = 0$ whenever $i \not\equiv j \pmod{m}$ and $\delta(Z,B) < \frac{1}{n}$. Obviously

$$\text{(c)} \qquad \delta(A, B) < \frac{2}{n}.$$

Since $b_{ij} = 0$ whenever $i \not\equiv j \pmod{m}$, it follows that $b_{ij} = 0$ whenever $i \neq j$ and $|i - j| < m$; Hence $A \in \mathfrak{M}^{(m)}$ implies that the minor A'' still appears in every set of m consecutive rows of $A - B$. Therefore

$$\text{(d)} \qquad \delta(A, B) = r(A - B) \geqslant \frac{3k}{m} > \frac{3k}{n(k+1)} \geqslant \frac{2}{n},$$

the last two inequalities following from (b) and the fact that $m \geqslant 2n$ (so that $k \geqslant 2$.)

Statements (c) and (d) are contradictory; hence the set $\mathcal{L}(\mathcal{J}(\mathcal{D})) \setminus \overline{\mathcal{D}}$ must be empty. Thereby, the theorem is proved.

Obviously $\mathcal{L}(\mathcal{J}(\mathcal{D})) \supseteq \mathcal{D}$, and since multiplication is continuous, the centralizer of any subset of $\overline{\mathfrak{M}}_\infty$ is closed; hence, $\mathcal{L}(\mathcal{J}(\mathcal{D})) = \overline{\mathcal{L}(\mathcal{J}(\mathcal{D}))} \supseteq \overline{\mathcal{D}}$, which is the reverse of the inclusion asserted in the theorem. Thus we have the

(12.6) <u>Corollary.</u> $\mathcal{L}(\mathcal{J}(\mathcal{D})) = \overline{\mathcal{D}}$.

(12.7) Lemma. Let Z be in the center $\mathfrak{Z}(\overline{\mathfrak{M}}_\infty)$ of $\overline{\mathfrak{M}}_\infty$, let $n \in \mathbb{N}$, and let $\varepsilon > 0$. Then there exists a matrix $D = \mathrm{diag}(d_1, \ldots, d_i, \ldots) \in \mathcal{D}$ such that $\delta(Z, D) < \varepsilon$ and $d_i = d_j$ whenever $\left[\frac{i}{n}\right] = \left[\frac{j}{n}\right]$, where $[x]$ denotes the greatest integer less than x.

Proof: Define $U = (u_{ij})$ and $V = (v_{ij})$ as follows:

$$u_{ij} = v_{ji} = \begin{cases} 1 & \text{if } j = i + 1 \not\equiv 1 \pmod{n} \\ 0 & \text{otherwise} \end{cases}$$

Clearly $U, V \in \mathfrak{M}^{(n)}$. Next, define $q(X) = X + V(XU - UX)$ for all $X \in \overline{\mathfrak{M}}_\infty$. It is easy to see that for all $X, X_1, X_2 \in \overline{\mathfrak{M}}_\infty$ and all $Z \in \mathfrak{Z}(\overline{\mathfrak{M}}_\infty)$

(a) $$r(q(X)) \leqslant 3r(X)$$

(b) $$q(X_1 \pm X_2) = q(X_1) \pm q(X_2), \text{ and}$$

(c) $$q(Z) = Z.$$

It follows immediately from (a) and (b) that

(d) $$\delta(q^n(X_1), q^n(X_2)) \leqslant 3^n \delta(X_1, X_2)$$

If $D = \mathrm{diag}(d_1, \ldots, d_i, \ldots)$ then $(DU)_{ij} = d_i u_{ij}$, $(UD)_{ij} = u_{ij} d_j$, and $\left[V(DU - UD)\right]_{ij} = \sum_s v_{is} u_{sj}(d_s - d_j)$

$$= v_{i(i-1)} u_{(i-1)j}(d_{i-1} - d_j)$$

$$= \begin{cases} d_{i-1} - d_i & \text{if } i = j \not\equiv 1 \pmod{n} \\ 0 & \text{otherwise} \end{cases}$$

Hence

$$\left[q(D)\right]_{ij} = \left[D + V(DU - UD)\right]_{ij}$$

$$= \begin{cases} d_i & \text{if } i = j \equiv 1 \pmod{n} \\ d_{i-1} & \text{if } i = j \not\equiv 1 \pmod{n} \\ 0 & \text{if } i \neq j, \end{cases}$$

and by finite induction

$$\left[q^n(D)\right]_{ij} = \begin{cases} d_i & \text{if } i = j \equiv 1 \pmod{n} \\ d_{i-1} & \text{if } i = j \equiv 2 \pmod{n} \\ \vdots & \qquad\qquad \vdots \\ d_{i-n+1} & \text{if } i = j \equiv n \pmod{n} \\ 0 & \text{if } i \neq j. \end{cases}$$

that is, $q^n(D) = \operatorname{diag}(d_1^*, \ldots, d_i^*, \ldots) \in \mathcal{D}$ and $d_i^* = d_k$ where $k = n\left[\frac{i}{n}\right] + 1$. Then clearly $\left[\frac{i}{n}\right] = \left[\frac{j}{n}\right] \Rightarrow$ $\Rightarrow d_i^* = d_j^*$. Furthermore, by (c) and (d),

$$\delta(q^n(D), Z) = \delta(q^n(D), q^n(Z)) \leqslant 3^n\, \delta(D, Z).$$

Therefore, if we choose $D \in \mathcal{D}$ such that $\delta(D, Z) < 3^{-n}\varepsilon$ (as we may, by (12.5)) then $q^n(D)$ has the required properties, and the lemma is proved.

If Z is in the center of $\overline{\mathfrak{M}}_\infty$ and if $\varepsilon > 0$, then by (12.5) there is a $D = \operatorname{diag}(d_1, \ldots, d_i, \ldots) \in \mathcal{D}$ such that $\delta(Z, D) < \varepsilon$. Since $\mathcal{D} \subset \mathfrak{M}_\infty$, $D \in \mathfrak{M}^{(n)}$ for some $n \in \mathbb{N}$. By (12.7), there is a $D' = \operatorname{diag}(d_1', \ldots, d_i', \ldots) \in \mathcal{D}$ such that $\delta(Z, D') < \varepsilon$ and $\left[\frac{i}{n}\right] = \left[\frac{j}{n}\right] \Rightarrow d_i' = d_j'$.

Since $\delta(D, D') \leqslant \delta(D, Z) + \delta(Z, D') < 2\varepsilon$, there must be a d'_j such that $d_i = d'_j$ for $i = 1, 2, \ldots, n$ with fewer than $2n\varepsilon$ exceptions. Setting $d'' = d'_j$ and $D'' = d'' \cdot 1_\omega = \operatorname{diag}(d'', d'', \ldots, d'', \ldots)$, the scalar matrix whose diagonal elements are all $= d''$, yields $\delta(D,D'') < 2\varepsilon$ and $\delta(Z, D'') \leqslant \delta(Z, D) + \delta(D, D'') < 3\varepsilon$; hence Z is a limit of scalar matrices. But a sphere of radius $< 1/2$ can contain at most one scalar matrix, so Z itself is scalar. Clearly, a scalar matrix $S = s \cdot 1_\omega$ is in the center $\mathfrak{Z}(\overline{\mathfrak{M}}_\infty)$ of $\overline{\mathfrak{M}}_\infty$ only if s is in the center $\mathfrak{Z}(\vartheta)$ of ϑ. Thus, we have proved that

$$\mathfrak{Z}(\overline{\mathfrak{M}}_\infty) = \{s \cdot 1_\omega : s \in \mathfrak{Z}(\vartheta)\}$$

(12.8) <u>Theorem</u>. $\mathfrak{Z}(\overline{\mathfrak{M}}_\infty)$ is isomorphic to $\mathfrak{Z}(\vartheta)$. Furthermore $\mathfrak{Z}(\overline{\mathfrak{M}}_\infty) = \mathfrak{Z}(\mathfrak{M}_\infty) = \mathfrak{Z}(\mathfrak{M}^{(n)})$ for all $n \in \mathbb{N}$.

Proof: It follows from what we proved that $s \cdot 1_\omega \to s$ is an isomorphism between $\mathfrak{Z}(\overline{\mathfrak{M}}_\infty)$ and $\mathfrak{Z}(\vartheta)$. Also $\mathfrak{Z}(\overline{\mathfrak{M}}_\infty) \subseteq \mathfrak{Z}(\mathfrak{M}_\infty) \subseteq \mathfrak{Z}(\mathfrak{M}^{(n)}) = \{s \cdot 1_\omega : s \in \mathfrak{Z}(\vartheta)\} = \mathfrak{Z}(\overline{\mathfrak{M}}_\infty)$, which completes the proof.

13. $R_{\overline{\mathfrak{M}}_\infty}$ <u>is a continuous geometry</u>.

According to (11.11), $\overline{\mathfrak{M}}_\infty$ is a rank-ring. By §4 and (12.8), $\overline{\mathfrak{M}}_\infty$ is irreducible because its center $\mathfrak{Z}(\overline{\mathfrak{M}}_\infty)$ is isomorphic to the field $\mathfrak{Z}(\vartheta)$. By its construction, $\overline{\mathfrak{M}}_\infty$ is complete in the rank-metric $\delta(A,B) = r(A-B)$. Therefore, according to (6.3), $R_{\overline{\mathfrak{M}}_\infty}$ is a discrete or continuous geometry. That it is not discrete follows from the fact that the rank-function assumes infinitely many values. The Theorem that follows summarizes these results.

(13.1) <u>Theorem.</u> $\overline{\mathfrak{M}}_\infty$ is a complete, irreducible rank-ring. $R\,\overline{\mathfrak{M}}_\infty$ is a continuous geometry.

14. <u>The isomorphism of</u> $\overline{\mathcal{L}}_\infty$ <u>and</u> $R\,\overline{\mathfrak{M}}_\infty$.

The purpose of this section is to show that $\overline{\mathfrak{M}}_\infty$ is essentially the coordinate ring of $\overline{\mathcal{L}}_\infty$; in other words, that $\overline{\mathcal{L}}_\infty$ is isomorphic to $R\,\overline{\mathfrak{M}}_\infty$.

(14.1) <u>Lemma.</u> If $A, B \in \mathfrak{M}^{(n)}$ then the following three statements are equivalent:

(a) $$A\,\mathfrak{M}_\infty \supseteq B\,\mathfrak{M}_\infty$$

(b) $$B = AX \text{ for some } X \in \mathfrak{M}^{(n)}$$

(c) $$(t_n A)\mathbb{D}_n \supseteq (t_n B)\mathbb{D}_n$$

where $t_n : \mathfrak{M}^{(n)} \to \mathfrak{M}_n$ is the ring-isomorphism defined by (9.2).

Proof: Obviously (b) $\Rightarrow$ (a) . Since t_n is an isomorphism, (b) $\Leftrightarrow$ $t_n B = (t_n A)X'$ for some $X' \in \mathfrak{M}_n \Leftrightarrow$ (c) . Suppose that $A\,\mathfrak{M}_\infty \supseteq B\,\mathfrak{M}_\infty$; that is, $B = AY$ for some

$Y = (y_{ij}) \in \mathfrak{M}_\infty$. Let $Y' = \begin{bmatrix} y_{11} & \cdots & y_{1n} \\ \vdots & & \vdots \\ y_{n1} & \cdots & y_{nn} \end{bmatrix}$ (whether or or not $Y \in \mathfrak{M}^{(n)}$) . Then $X = t_n^{-1} Y' \in \mathfrak{M}^{(n)}$ and $B = AX$. Thus (a) $\Rightarrow$ (b) , and the proof is complete.

(14.2) <u>Lemma.</u> There is a mapping $\tau : R\,\mathfrak{M}_\infty \to \mathcal{L}_\infty$ that is one-to-one and such that $\tau(A\,\mathfrak{M}_\infty) = (t_n A)\mathbb{D}_n$ whenever $A \in \mathfrak{M}^{(n)}$ and $n \in \mathbb{N}$. (cf (7.2)).

Proof: First notice that the condition makes sense: for if $A \in \mathfrak{M}^{(n)}$ and $n \in \mathbb{N}$ then $A\mathfrak{M}_\infty \in R_{\mathfrak{M}_\infty}$, $t_nA \in \mathfrak{M}_n$, and $(t_nA)\mathcal{D}_n \in \mathcal{L}_n \subset \mathcal{L}_\infty$.

As we have already observed in §9, if $n|k$ then $t_k = \varphi_{kn}t_n$ on $\mathfrak{M}^{(n)}$; hence

$$(t_kA)\mathcal{D}_k = (\varphi_{kn}t_nA)\mathcal{D}_k = \psi_{kn}\left[(t_nA)\mathcal{D}_n\right] \sim (t_nA)\mathcal{D}_n ,$$

where φ_{kn} and ψ_{kn} are the mappings defined by (8.1) and (8.2), respectively. Furthermore, if $A\mathfrak{M}_\infty = B\mathfrak{M}_\infty$ then $A, B \in \mathfrak{M}^{(n)}$ for some $n \in \mathbb{N}$, and, by (14.1), $(t_nA)\mathcal{D}_n = (t_nB)\mathcal{D}_n$. Therefore, $A\mathfrak{M}_\infty \xrightarrow{\tau} (t_nA)\mathcal{D}_n$ (n is any element of $\mathbb{N}$ such that $A \in \mathfrak{M}^{(n)}$) is a well-defined mapping of $R_{\mathfrak{M}_\infty}$ onto $\mathcal{L}_\infty$. That the mapping τ has a well-defined inverse follows from another application of (14.1).

Lemma (14.1) implies still more: τ and τ^{-1} preserve order. Therefore we have the

(14.3) <u>Corollary</u>. The mapping τ of (14.2) is a lattice-isomorphism.

(14.4) <u>Lemma</u>. The mapping $A\mathfrak{M}_\infty \xrightarrow{\theta} A\overline{\mathfrak{M}}_\infty$, $A \in \mathfrak{M}_\infty$, is an order-isomorphism of $R_{\mathfrak{M}_\infty}$ into $R_{\overline{\mathfrak{M}}_\infty}$. This lemma is a special case of [8, vol II, Lemma 15.5, p. 145].

The next three lemmas are used to show that the mapping θ of (14.4) is not merely an order-isomorphism but also a lattice-isomorphism.

(14.5) <u>Lemma</u>. Let $E, F \in \mathcal{J}(\mathfrak{M}_n)$ and $G \in \mathfrak{M}_n$.

(a) There exists an $A \in \mathfrak{M}_n$ such that

$EA = A$, $FA = A$, and $r(A - G) \leqslant r(EG - G) + r(FG - G)$.

(b) There exists a $B \in \mathfrak{M}_n$ such that

$BE = E$, $BF = F$ and $r(B - G) \leqslant r(GE - E) + r(GF - F)$.

Proof: (a) Let $\mathfrak{N}_1 = \mathfrak{N}(EG - G)$ and $\mathfrak{N}_2 = \mathfrak{N}(FG - G)$, and let $\mathfrak{A}$ be some complement of $\mathfrak{N}_1 \cap \mathfrak{N}_2$. Then define

$$A = \begin{cases} G & \text{on } \mathfrak{N}_1 \cap \mathfrak{N}_2 \\ 0 & \text{on } \mathfrak{A}. \end{cases}$$

Any vector $v \in \mathfrak{D}_n$ may be written $v = a + n$ where $a \in \mathfrak{A}$ and $n \in \mathfrak{N}_1 \cap \mathfrak{N}_2$. Since $A = 0$ on $\mathfrak{A}$, $Aa = 0$, and since $A = G = EG$ on $\mathfrak{N}_1 \cap \mathfrak{N}_2 \subseteq \mathfrak{N}_1$, $An = EGn$; hence $Av = Aa + An = EGn$, and $EAv = E(EGn) = EGn = Av$. Therefore $EA = A$, and similarly, $FA = A$. Furthermore

$$\begin{aligned} r(A - G) &= 1 - d\left[\mathfrak{N}(A - G)\right] \\ &\leqslant 1 - d(\mathfrak{N}_1 \cap \mathfrak{N}_2) \text{ , for } \mathfrak{N}(A - G) \supseteq \mathfrak{N}_1 \cap \mathfrak{N}_2 \\ &= 1 - d(\mathfrak{N}_1) - d(\mathfrak{N}_2) + d(\mathfrak{N}_1 + \mathfrak{N}_2) \\ &= r(EG - G) + r(FG - G) - (1 - d(\mathfrak{N}_1 + \mathfrak{N}_2)) \\ &\leqslant r(EG - G) + r(FG - G) . \end{aligned}$$

Thus, A has all the desired properties, and (a) is proved.

(b) Let $\mathfrak{B}$ be a complement of $E\mathfrak{D}_n + F\mathfrak{D}_n$ and define

$$B = \begin{cases} 1 & \text{on } E\mathfrak{D}_n + F\mathfrak{D}_n \\ G & \text{on } \mathfrak{B} . \end{cases}$$

If v is any vector in $\mathfrak{D}_n$ then $Ev \in E\mathfrak{D}_n \subseteq E\mathfrak{D}_n + F\mathfrak{D}_n$, hence, $BEv = Ev$ and $BE = E$. Similarly $BF = F$.

Clearly, $\mathfrak{N}(B - G) \supseteq \mathfrak{L} + \left[(E\mathfrak{D}_n + F\mathfrak{D}_n) \cap \mathfrak{N}(G - 1)\right]$, from which it follows that

$$\begin{aligned} r(B - G) &= 1 - d\left[\mathfrak{N}(B - G)\right] \\ &\leqslant 1 - d(\mathfrak{L}) - d\left[(E\mathfrak{D}_n + F\mathfrak{D}_n) \cap \mathfrak{N}(G - 1)\right] + \\ &\quad + d\left[\mathfrak{L} \cap (E\mathfrak{D}_n + F\mathfrak{D}_n) \cap \mathfrak{N}(G - 1)\right] \\ &= 1 - d(\mathfrak{L}) - d\left[(E\mathfrak{D}_n + F\mathfrak{D}_n) \cap \mathfrak{N}(G - 1)\right], \end{aligned}$$

and since $\mathfrak{L}$ and $E\mathfrak{D}_n + F\mathfrak{D}_n$ are supposed to be complementary, we get

$$r(B - G) \leqslant d(E\mathfrak{D}_n + F\mathfrak{D}_n) - d\left[(E\mathfrak{D}_n + F\mathfrak{D}_n) \cap \mathfrak{N}(G - 1)\right].$$

By (11.8)(c), $\mathfrak{N}(GE - E) = \mathfrak{N}\left[(G - 1)E\right] = \mathfrak{N}(E) + \left[\mathfrak{N}(G - 1) \cap E\mathfrak{D}_n\right]$ and similarly $\mathfrak{N}(GF - F) = \mathfrak{N}(F) + \mathfrak{N}(G - 1) \cap F\mathfrak{D}_n$. Hence

$$\begin{aligned} r(GE-E) + r(GF-F) &= \\ &= 2 - d\left[\mathfrak{N}(GE-E)\right] - d\left[\mathfrak{N}(GF-F)\right] \\ &= 2 - d\left[\mathfrak{N}(E)\right] - d\left[\mathfrak{N}(G-1) \cap E\mathfrak{D}_n\right] - d\left[\mathfrak{N}(F)\right] \\ &\qquad - d\left[\mathfrak{N}(G-1) \cap F\mathfrak{D}_n\right] \\ &= d(E\mathfrak{D}_n) + d(F\mathfrak{D}_n) - d\left[\mathfrak{N}(G-1) \cap E\mathfrak{D}_n\right] - d\left[\mathfrak{N}(G-1) \cap F\mathfrak{D}_n\right] \\ &= d(E\mathfrak{D}_n + F\mathfrak{D}_n) + d(E\mathfrak{D}_n \cap F\mathfrak{D}_n) - d\left[(\mathfrak{N}(G-1) \cap E\mathfrak{D}_n) + \right. \\ &\qquad \left. + (\mathfrak{N}(G-1) \cap F\mathfrak{D}_n)\right] - d\left[\mathfrak{N}(G-1) \cap E\mathfrak{D}_n \cap F\mathfrak{D}_n)\right] \\ &\geqslant d(E\mathfrak{D}_n + F\mathfrak{D}_n) - d\left[(\mathfrak{N}(G-1) \cap E\mathfrak{D}_n) + (\mathfrak{N}(G-1) \cap F\mathfrak{D}_n)\right] \\ &\geqslant r(B-G) + d\left[\mathfrak{N}(G-1) \cap (E\mathfrak{D}_n + F\mathfrak{D}_n)\right] - d\left[(\mathfrak{N}(G-1) \cap E\mathfrak{D}_n) + \right. \\ &\qquad \left. + (\mathfrak{N}(G-1) \cap F\mathfrak{D}_n)\right] \\ &\geqslant r(B-G), \end{aligned}$$

where the last inequality comes from the semi-distributive law of lattices: $a \cap (b + c) \geqslant (a \cap b) + (a \cap c)$. Therefore B is the matrix whose existence proves (b).

(14.6) <u>Lemma</u>. If $\mathcal{R}$ is a complete, irreducible rank-ring then $d(a\mathcal{R} + b\mathcal{R}) - d(a\mathcal{R} \cap b\mathcal{R}) \leqslant 2r(a - b)$, and hence $a \to a\mathcal{R}$ is a uniformly continuous mapping of $\mathcal{R}$ onto $R_{\mathcal{R}}$.

Proof: Let $\mathfrak{a}$ and $\mathfrak{b}$ be complements of $a\mathcal{R} \cap b\mathcal{R}$ in $a\mathcal{R}$ and $b\mathcal{R}$, respectively. Then the set $\{\mathfrak{a}, \mathfrak{b}, a\mathcal{R} \cap b\mathcal{R}\}$ is independent (cf § 2). According to §4, there exist three independent idempotents e, f, and g of $\mathcal{R}$ such that $e\mathcal{R} = \mathfrak{a}$, $f\mathcal{R} = \mathfrak{b}$, and $g\mathcal{R} = a\mathcal{R} \cap b\mathcal{R}$. Since e, f, and g are independent

$$\begin{aligned}(e + f + g)\mathcal{R} &= e\mathcal{R} + f\mathcal{R} + g\mathcal{R}\\ &= \mathfrak{a} + \mathfrak{b} + (a\mathcal{R} \cap b\mathcal{R})\\ &= a\mathcal{R} + b\mathcal{R} + (a\mathcal{R} \cap b\mathcal{R})\\ &= a\mathcal{R} + b\mathcal{R},\end{aligned}$$

and hence

$$\begin{aligned}d(a\mathcal{R} + b\mathcal{R}) &= d\left[(e + f + g)\mathcal{R}\right]\\ &= r(e + f + g) && \text{by (6.3)}\\ &= r(e) + r(f) + r(g) && \text{by R.5 of §5}\\ d(a\mathcal{R} \cap b\mathcal{R}) &= d(g\mathcal{R}) = r(g),\end{aligned}$$

and

$$d(a\mathcal{R} + b\mathcal{R}) - d(a\mathcal{R} \cap b\mathcal{R}) = r(e) + r(f).$$

Since $e\mathcal{R} = \mathfrak{a} \subseteq a\mathcal{R}$, we have $e \in a\mathcal{R}$ and $e = ax$ for some $x \in \mathcal{R}$. On the other hand, $(f + g)\mathcal{R} = f\mathcal{R} + g\mathcal{R} =$

$= b + (a\mathcal{R} \cap b\mathcal{R}) = b\mathcal{R}$; hence $(f + g)\mathcal{R} = b\mathcal{R}$ and $(f + g)b = b$. Therefore $e(a - b)x = eax - ebx = e^2 - e(f + g)bx = e$, and $r(a - b) \geqslant r(e(a - b)x) = r(e)$. Similarly $r(a - b) \geqslant r(f)$. This yields $d(a\mathcal{R} + b\mathcal{R}) - d(a\mathcal{R} \cap b\mathcal{R}) = r(e) + r(f) \leqslant 2r(a - b)$, or $\delta(a\mathcal{R}, b\mathcal{R}) \leqslant 2\,\delta(a,b)$; hence the lemma is proved.

(14.7) <u>Lemma</u>. The mapping $A\mathcal{M}_\infty \xrightarrow{\Theta} A\overline{\mathcal{M}}_\infty$, $A \in \mathcal{M}_\infty$, is a lattice-isomorphism of $R_{\mathcal{M}_\infty}$ onto a sublattice $\mathcal{L}^*$ of $R_{\overline{\mathcal{M}}_\infty}$.

Proof: (cf. Proof of (8.3)) According to (14.4), Θ is an order-isomorphism of $R_{\mathcal{M}_\infty}$ onto $\mathcal{L}^* = \Theta(R_{\mathcal{M}_\infty})$; hence it remains only to show that, for all $E, F, G \in \mathcal{M}_\infty$,

(a) $E\mathcal{M}_\infty \cap F\mathcal{M}_\infty = G\mathcal{M}_\infty \Rightarrow E\overline{\mathcal{M}}_\infty \cap F\overline{\mathcal{M}}_\infty = G\overline{\mathcal{M}}_\infty$, and

(b) $E\mathcal{M}_\infty + F\mathcal{M}_\infty = G\mathcal{M}_\infty \Rightarrow E\overline{\mathcal{M}}_\infty + F\overline{\mathcal{M}}_\infty = G\overline{\mathcal{M}}_\infty$.

To prove (a), suppose that $G\mathcal{M}_\infty = E\mathcal{M}_\infty \cap F\mathcal{M}_\infty$ and $H\overline{\mathcal{M}}_\infty = E\overline{\mathcal{M}}_\infty \cap F\overline{\mathcal{M}}_\infty$ where $E, F, G \in \mathcal{M}_\infty$ and $H \in \overline{\mathcal{M}}_\infty$. According to § 4, we may as well suppose that E, F, G and H are all idempotent; also, let $(H_i)_i$ be a sequence in $\mathcal{M}_\infty$ that converges to H . Since $H \in E\overline{\mathcal{M}}_\infty$ and since E is idempotent, $EH = H$; similarly, $FH = H$. Hence, by continuity of multiplication, $\lim_i EH_i = EH = H = \lim_i H_i$ and $\lim_i FH_i = \lim_i H_i$, so $r(EH_i - H_i) + r(FH_i - H_i) \longrightarrow 0$. But it follows easily from (14.5)(a) that for every H_i , there exists an $A_i \in \mathcal{M}_\infty$ such that $EA_i = A_i$, $FA_i = A_i$ and $r(A_i - H_i) \leqslant r(EH_i - H_i) + r(FH_i - H_i)$. Therefore,

$$\delta(A_1\overline{\mathfrak{M}}_\infty, H\overline{\mathfrak{M}}_\infty) \leq 2r(A_1 - H_1), \text{ by } (14.6)$$

$$\leq 2r(EH_1 - H_1) + 2r(FH_1 - H_1) \to 0,$$

and $\lim_1 A_1\overline{\mathfrak{M}}_\infty = H\overline{\mathfrak{M}}_\infty$. Furthermore, since $EA_1 = A_1$, it follows that $A_1\mathfrak{M}_\infty = EA_1\mathfrak{M}_\infty \subseteq E\mathfrak{M}_\infty$ and similarly $A_1\mathfrak{M}_\infty \subseteq F\mathfrak{M}_\infty$; hence $A_1\mathfrak{M}_\infty \subseteq G\mathfrak{M}_\infty$. According to (14.4), θ preserves order; hence

$$A_1\overline{\mathfrak{M}}_\infty = \theta(A_1\mathfrak{M}_\infty) \subseteq \theta(G\mathfrak{M}_\infty) = G\overline{\mathfrak{M}}_\infty,$$

and by continuity of $\bigcap$,

$$H\overline{\mathfrak{M}}_\infty \cap G\overline{\mathfrak{M}}_\infty = \lim_1(A_1\overline{\mathfrak{M}}_\infty \cap G\overline{\mathfrak{M}}_\infty) = \lim_1 A_1\overline{\mathfrak{M}}_\infty = H\overline{\mathfrak{M}}_\infty.$$

Thus we get the result: $H\overline{\mathfrak{M}}_\infty \subseteq G\overline{\mathfrak{M}}_\infty$. The fact that θ preserves order also implies that

$$H\overline{\mathfrak{M}}_\infty = E\overline{\mathfrak{M}}_\infty \cap F\overline{\mathfrak{M}}_\infty = \theta(E\mathfrak{M}_\infty) \cap \theta(F\mathfrak{M}_\infty) \supseteq \theta(E\mathfrak{M}_\infty \cap F\mathfrak{M}_\infty) =$$

$= G\overline{\mathfrak{M}}_\infty$, which is the reverse of the inclusion just proved; therefore $H\overline{\mathfrak{M}}_\infty = G\overline{\mathfrak{M}}_\infty$, and (a) is established.

A similar argument, using (14.5)(b) in place of (14.5)(a) proves (b).

(14.8) Theorem. The lattices $\mathcal{L}_\infty$ and $R_{\overline{\mathfrak{M}}_\infty}$, are isomorphic.

Proof: The lattice isomorphisms $\tau: R_{\mathfrak{M}_\infty} \to \mathcal{L}_\infty$ of (14.2) and $\theta: R_{\mathfrak{M}_\infty} \to \mathcal{L}^* = \{A\overline{\mathfrak{M}}_\infty : A \in \mathfrak{M}_\infty\}$ of (14.7) both preserve dimension: for

$d[\tau(A\mathfrak{M}_\infty)] = d[(t_nA)\mathfrak{D}_n] = r(t_nA) = r(A) = d(A\mathfrak{M}_\infty)$, and $d(A\overline{\mathfrak{M}}_\infty) = r(A) = d(A\mathfrak{M}_\infty)$. Therefore, $\mathcal{L}_\infty \xrightarrow{\tau^{-1}} R_{\mathfrak{M}_\infty} \xrightarrow{\theta} \mathcal{L}^*$ is a dimension preserving lattice-isomorphism, and hence is uniformly continuous.

Let κ be the extension-by-continuity of $\theta\tau^{-1}$ to the closure $\overline{\mathcal{L}}_\infty$ of $\mathcal{L}_\infty$. Since the operations $\cap$, $+$, d, and δ are preserved by $\theta\tau^{-1}$ and are continuous both in $\overline{\mathcal{L}}_\infty$ and $R_{\overline{\mathfrak{M}}_\infty}$, it follows that they are still preserved by κ. Hence κ is an isometry and a lattice-homomorphism. Since an isometric mapping is necessarily one-to-one, κ is a lattice-isomorphism. Since $\mathfrak{M}_\infty$ is dense in $\overline{\mathfrak{M}}_\infty$, $\mathcal{L}^*(=\theta(\mathfrak{M}_\infty))$ is dense in $R_{\overline{\mathfrak{M}}_\infty}$, by (14.6). Therefore, the image of $\overline{\mathcal{L}}_\infty$ under κ, being a complete space containing $\mathcal{L}^*$, contains all of $R_{\overline{\mathfrak{M}}_\infty}$. Thus, we have shown that κ is an isomorphism of $\overline{\mathcal{L}}_\infty$ onto $R_{\overline{\mathfrak{M}}_\infty}$, and the proof of the theorem is complete.

(14.9) Corollary. $\overline{\mathcal{L}}_\infty$ is a continuous geometry, and $\overline{\mathfrak{M}}_\infty$ is its coordinate ring.

Proof: The corollary is an immediate consequence of the theorem together with Theorem (13.1).

15. Concluding remarks. The construction of $\overline{\mathfrak{M}}_\infty$ given in §9 is essentially the same as one suggested by von Neumann in one of a series of Colloquium lectures.[1] In the same lecture, he stated further that if $\mathcal{D}$ is a division ring and if $\mathbb{N}_1$ and $\mathbb{N}_2$ are any two sets of integers satisfying (8.4) then $\overline{\mathfrak{M}}_\infty(\mathcal{D},\mathbb{N}_1)\cong\overline{\mathfrak{M}}_\infty(\mathcal{D},\mathbb{N}_2)$; and also, if $\mathcal{D}$ is finite over its center $\mathfrak{Z}(\mathcal{D})$ then $\overline{\mathfrak{M}}_\infty(\mathcal{D},\mathbb{N})\cong\overline{\mathfrak{M}}_\infty(\mathfrak{Z}(\mathcal{D}),\mathbb{N})$.

[1] These lectures on continuous geometry were delivered at Pennsylvania State College in September, 1937. Notes abstracting them were distributed at that time. A copy in Professor von Neumann's files was made available through the courtesy of Mrs. Elizabeth Gorman.

The basic problem of finding a simple criterion for determining whether two given approximately finite geometries are isomorphic is, however, still unsolved.

CHAPTER III

POLARITIES IN $\overline{\mathcal{L}}_\infty$

16. Dualities and polarities.

If $\mathcal{L}$ and $\mathcal{L}'$ are two lattices and if φ is a one-to-one mapping of $\mathcal{L}$ onto $\mathcal{L}'$ such that

$$\left.\begin{aligned} \varphi(a+b) &= \varphi(a) \cap \varphi(b) \\ \varphi(a \cap b) &= \varphi(a) + \varphi(b) \end{aligned}\right\} \text{ all } a, b \in \mathcal{L}$$

then φ is a dual-isomorphism or duality of $\mathcal{L}$ onto $\mathcal{L}'$, and $\mathcal{L}$ and $\mathcal{L}'$ are dual lattices. Every lattice possesses a dual which may be obtained from it simply by interchanging the operations of join and meet. Axioms A, 1-3 of §1 are invariant under transposition of $\cap$ and $+$, and hence the dual of a discrete or continuous geometry is again a discrete or continuous geometry. However, a projective lattice (the lattice of subspaces of a projective space) is the dual of a projective lattice if and only if it is finite dimensional (cf. [1, section IV.1, p. 95]).

A dual-automorphism φ of a discrete or continuous geometry is a polarity if it is involutoric ($\varphi^2 = 1$). This definition, which coincides with the usual one in the discrete case (that is, for finite dimensional projective lattices) has been used by Givens[1] as a basis for a theory of polarities of

[1]Wallace Givens, Seminar on Continuous Geometry, The University of Tennessee, 1955-6.

continuous geometries. It is shown in this section that polarities of approximately finite geometries do sometimes exist, and in the next section, some properties of the signature of a polarity are investigated.

The following fundamental facts about auto-dualities and polarities were proved by von Neumann [8, vol. II, Theorems 4.3 and 4.4, p. 48]:

(16.1) If $\mathcal{R}$ is a complete irreducible rank-ring and if $x \to x'$ is an anti-automorphism of $\mathcal{R}$ then

$$\varphi : x\mathcal{R} \to (x')^r = \{ y : x'y = 0 \}, \quad \text{for } x \in \mathcal{R},$$

is a dual-automorphism of $R_{\mathcal{R}}$ (according to § 4, $(a)^r \in R_{\mathcal{R}}$ for all $a \in \mathcal{R}$). Conversely, if $R_{\mathcal{R}}$ has chains of length ≥ 3 then every dual-automorphism of $R_{\mathcal{R}}$ is generated in this way by a unique anti-automorphism of $\mathcal{R}$. Furthermore, φ is a polarity if and only if $x \to x'$ is an involution ($x'' = x$ for all $x \in \mathcal{R}$).

Consider the case $\mathcal{R} = \overline{\mathfrak{M}}_\infty$, where $\overline{\mathfrak{M}}_\infty$ is determined by ϑ and $\mathbb{N}$ as in §8 and §9, and let $a \to a^*$ be an involutoric anti-automorphism (iaa, for short) of ϑ. (For commutative ϑ this is no restriction on ϑ since $a^* \equiv a$ is then permissible.) If A is a matrix in $\mathfrak{M}_\infty$, define A^* to be the matrix obtained by replacing each element a_{ij} of A by a^*_{ij}, and let A^T denote the transpose of A. Then $\alpha : A \to A' = (A^*)^T$ is an iaa of $\mathfrak{M}_\infty$. Since $\mathfrak{M}_\infty$ has a unique rank-function, any semi-automorphism of $\mathfrak{M}_\infty$ must preserve rank; hence $r(A' - B') = r\left[(A - B)'\right] = r(A - B)$, and α is uniformly continuous. Thus α may be extended by continuity to a mapping β of $\overline{\mathfrak{M}}_\infty$ into itself. By virtue of the continuity of the extended mapping β and of the algebraic operations of $\overline{\mathfrak{M}}_\infty$, β is an iaa of $\overline{\mathfrak{M}}_\infty$. In view of (16.1), this proves the

(16.2) <u>Theorem</u>. If ϑ possesses an involutoric anti-automorphism then polarities of $R_{\overline{\mathfrak{M}}_\infty}$ exist.

Remark: In the case of projective coordinate geometries of dimension ≥ 2 over ϑ, the existence of an iaa on ϑ is necessary for the existence of polarities (cf. [1, § 4.3, Theorem 2, p. 111]). The corresponding problem for approximately finite geometries is complicated by the fact that ϑ is not necessarily an invariant of $\overline{\mathfrak{M}}_\infty$, and that ϑ may fail to have an iaa even if it is finite over its center (cf. §15).

17. <u>Signature</u>. In Givens' theory of polarities in a continuous geometry, signature is defined as follows:

(17.1) <u>Definition</u>. If π is a polarity of a continuous geometry $\mathcal{L}$ then

$$s(\pi) = \sup\{d(\mathfrak{x}) : \mathfrak{x} \in \mathcal{L}, \pi(\mathfrak{x}) \supseteq \mathfrak{x}\}$$

is the <u>signature</u> of π.

Let $\mathcal{R}$ be the coordinate ring of a continuous geometry $\mathcal{L}$ and let $x \to x'$ be the iaa on $\mathcal{R}$ generating the polarity π, so that $\pi(x\mathcal{R}) = (x')^r$ (cf. (16.1)). Then

$\mathcal{T}(x\mathcal{R}) = (x')^r \supseteq x\mathcal{R}$ if and only if $x'x = 0$, and $d(x\mathcal{R}) = r(x)$; hence

$$s(\mathcal{T}) = \sup\{r(x) : x \in \mathcal{R},\ x'x = 0\}$$

If a, $a^{-1} \in \mathcal{R}$ then $x \to a^{-1}xa$ is an automorphism of $\mathcal{R}$, so $x \to a^{-1}x'a$ is an anti-automorphism. If in addition $a' = a$ -- in which case a will be called <u>hermitian</u> -- then $x \to a^{-1}x'a$ is also involutoric, for

$$a^{-1}(a^{-1}x'a)'a = a^{-1}\left[a'x''(a^{-1})'\right]a = a^{-1}axa^{-1}a = x .$$

Therefore, $x \to a^{-1}x'a$ generates a polarity $\mathcal{T}(a)$. By the <u>signature of</u> a (with respect to the iaa $x \to x'$) we shall mean $s(a) = s(\mathcal{T}(a))$. Obviously, $a^{-1}x = 0$ implies $x = aa^{-1}x = 0$, and hence

$$(17.2)\quad s(a) = \sup\{r(x) : x \in \mathcal{R},\ a^{-1}x'ax = 0\} = \sup\{r(x) : x \in \mathcal{R},\ x'ax = 0\}.$$

It is apparent that the $x'ax$ occurring in (17.2) plays a role analogous to that of a quadratic form in the classical theory of polarity. Indeed, if the geometry is n-dimensional and if x is chosen so that $R(x) = 1$, then each of the n^2 equations comprising $x'ax = 0$ is either the identity $0 = 0$ or the ordinary (scalar) equation of a quadric. In order to investigate another aspect of the analogy, let $\mathcal{V}$, $\mathcal{D}_n$, $\mathcal{L}_n$, and $\mathcal{M}_n$ be defined as in § 7; let $x \to x^*$ be an iaa of $\mathcal{V}$; and, as in § 16, let $X' = (X^*)^T$ for all $X \in \mathcal{M}_n$.

Finally, if $v = \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix} \in \mathfrak{D}_n$, define $v' = (v_1^* \; v_2^* \; \dots \; v_n^*)$.

Suppose that $A = A' \in \mathfrak{M}_n$; then $g(u, v) = u'Av$ is a hermitian bilinear form on $\mathfrak{D}_n$: for if $u, v, x \in \mathfrak{D}_n$ and $a \in \vartheta$ then

$$g(u, v + x) = u'A(v + x) = u'Av + u'Ax = g(u,v) + g(u,x),$$

$$g(v + x, u) = g(v, u) + g(x, u),$$

$$g(u, va) = u'A(va) = (u'Av)a = g(u, v)a,$$

$$g(va, u) = (va)'Au = a^*(vAu) = a^*g(v,u),$$

and

$$g(u,v) = u'Av = (v'Au)^* = \left[g(v,u)\right]^*$$

where va is, of course, the scalar product of $v \in \mathfrak{D}_n$ and $a \in \vartheta$. Define the orthogonal complement $\mathcal{H}^\perp$ of a subspace $\mathcal{H} \in \mathcal{L}_n$ in the usual way:

$$\mathcal{H}^\perp = \{y : g(x,y) = 0 \text{ for all } x \in \mathcal{H}\}.$$

Let $X \in \mathfrak{M}_n$ be a matrix whose columns span $\mathcal{H}$, so that $X\mathfrak{D}_n = \mathcal{H}$. Clearly, $y \in \mathcal{H}^\perp$ if and only if $X'Ay = 0$, and hence $Y\mathfrak{D}_n \subseteq \mathcal{H}^\perp$ if and only if $X'AY = 0$. But $X'AY = 0$ if and only if $Y \in (X'A)^r$. Thus it follows that $\sigma_n(\mathcal{H}^\perp) = (X'A)^r$ where σ_n is the mapping defined by (17.2). Since $\sigma_n(\mathcal{H}) = X\mathfrak{M}_n$, we get

$$(X'A)^r = \sigma_n(\mathcal{H}^\perp) = \sigma_n(\left[\sigma_n^{-1}(X\mathfrak{M}_n)\right]^\perp);$$ therefore

(17.3) the mapping $\mathcal{H} \to \mathcal{H}^\perp$, $\mathcal{H} \in \mathcal{L}_n$, and the mapping $(X\mathfrak{M}_n) \longrightarrow (X'A)^r$, $X\mathfrak{M}_n \in R_{\mathfrak{M}_n}$, are corresponding polarities under the lattice-isomorphism $\sigma_n : \mathcal{L}_n \to R_{\mathfrak{M}_n}$.

The relationship between the Witt signature [6, p.170] $w(A) \equiv n - 2\left[\max\left\{\dim \mathcal{H} : \mathcal{H}^{\perp} \supseteq \mathcal{H}\right\}\right]$ and Givens' signature $s(A)$ is given by

$$w(A) = n\left[1 - 2s(A)\right] ,$$

where A is any non-singular hermitian matrix.

The next theorem states that the signature of hermitian elements with respect to a fixed iaa $x \to x'$ is a continuous function. But first we prove a

(17.4) Lemma. If $a' = a \in \mathcal{R}$, then for any $x \in \mathcal{R}$ there exists a $y \in \mathcal{R}$ such that $y'ay = 0$ and $r(x - y) \leqslant r(x'ax)$.

Proof: Let $(x'ax)u(x'ax) = x'ax$. If $e = 1 - u(x'ax)$ then e is idempotent and, by § 4, $(x'ax)^r = e\mathcal{R}$.

Setting $y = xe$ gives

$$y'ay = (xe)'axe = e'(x'ax)e = e'\cdot 0 = 0$$

and

$$r(x - y) = r(x - xe) = r\left[x(1 - e)\right] = r\left[xu(x'ax)\right] \leqslant r(x'ax) ,$$

so y has the desired properties.

(17.5) Theorem. If $a' = a$ and $b' = b$ then $|s(a) - s(b)| \leqslant r(a - b)$, and hence s is uniformly continuous.

Proof: Let $\varepsilon > 0$. By definition of $s(a)$, there exists an x such that $x'ax = 0$ and $r(x) > s(a) - \varepsilon$. By (17.4) there exists a y such that $y'by = 0$ and $r(x - y) \leqslant r(x'bx)$. But $r(x'bx) = r(x'(b - a)x) \leqslant r(b - a)$. Therefore

$s(b) \geqslant r(y) \geqslant r(x) - r(x - y) > s(a) - \varepsilon - r(b - a)$, and $s(a) - s(b) < r(b - a) + \varepsilon$. Since ε was arbitrary,

$s(a) - s(b) \leqslant r(b - a)$. The remainder of the proof follows from symmetry.

Remark: The proof of the theorem is valid for singular hermitian elements provided we define the signature of such elements by

$$s(a) = \sup \{ r(x) : x'ax = 0 \}.$$

Let us now return to the case in which $\mathcal{R} = \overline{\mathfrak{M}}_\infty$, $x \to x^*$ is an iaa of $\mathcal{J}$, and $X \to X'$ is the extension-by-continuity to $\overline{\mathfrak{M}}_\infty$ of the iaa $X \to (X^*)^T$ on $\mathfrak{M}_\infty$. If $A = A' \in \mathfrak{M}^{(n)}$, let

$$s_n(A) = \sup \{ r(X) : X \in \mathfrak{M}^{(n)}, X'AX = 0 \}.$$

(17.6) <u>Theorem</u>. If $A = A' \in \mathfrak{M}_\infty$ then $s(A) = \sup \{ s_k(A) : A \in \mathfrak{M}^{(k)} \}$.

Proof: Let $\sup \{ s_k(A) : A \in \mathfrak{M}^{(k)} \} = \alpha$. Obviously, $s(A) \geqslant \alpha \geqslant s_k(A)$ whenever $A \in \mathfrak{M}^{(k)}$. Given $\varepsilon > 0$, let X be an element of $\overline{\mathfrak{M}}_\infty$ such that $X'AX = 0$ and $r(X) > s(A) - \varepsilon$, and let $Y \in \mathfrak{M}_\infty$, $r(X - Y) < \varepsilon$. Then

$r(Y'AY) = r(Y'AY - X'AX) \leqslant r(Y' - X') + r(Y - X) < 2\varepsilon$.

If $Y \in \mathfrak{M}^{(k)}$ then, by (17.4), there exists a $Z \in \mathfrak{M}^{(k)}$ such that $r(Y - Z) \leqslant r(Y'AY) < 2\varepsilon$ and $Z'AZ = 0$. Therefore $\alpha \geqslant s_k(A) \geqslant r(Z) \geqslant r(X) - r(X - Y) - r(Y - Z) \geqslant s(A) - 4\varepsilon \geqslant \alpha - 4\varepsilon$, and since ε is arbitrarily small, $s(A) = \alpha$. This completes the proof.

Let A be a symmetric permutation matrix in $\mathfrak{M}_n$, that is, a matrix having the properties: (a) $A = A^T$ (b) every element is either 0 or 1, and (c) every row (and column) of A has exactly one non-zero element. Then clearly $A = A^* = (A^*)^T$ for every iaa $x \to x^*$ of ϑ, and $r(A) = 1$. In addition, let $x \to x^*$ satisfy the condition:

(17.7) for every finite set $\{x_i : 1 \leqslant i \leqslant n\}$ of non-zero elements of ϑ, $\sum_{i=1}^{n} x_i^* x_i \neq 0$.

It follows from (17.7) that ϑ must have characteristic $\neq 2$, and that, for all $X \in \mathfrak{M}_n$, $(X^*)^T X = 0 \Leftrightarrow X = 0$, so $s(1_n) = 0$ (1_n is totally regular). If j is the number of 1's on the diagonal of A then A is cogredient to

$$\begin{bmatrix} 0 & 1_i & 0 \\ 1_i & 0 & 0 \\ 0 & 0 & 1_j \end{bmatrix};$$

hence, by [6, p. 170], $w(A) = j$ and $s(A) = \frac{1}{2}\left[1 - \frac{w(A)}{n}\right] = \frac{i}{n}$.

Furthermore, if $k = mn$ then $w(\varphi_{kn}A) = mw(A)$ so $s(\varphi_{kn}A) = \frac{im}{nm} = s(A)$. In view of the isomorphism t_k between $\mathfrak{M}^{(k)}$ and $\mathfrak{M}_k$, we have

$$s_k(t_k^{-1}A) = s(\varphi_{kn}A) = s(A)$$

for all $k \in \mathbb{N}$, $k \geqslant n$, and consequently $s(t_n^{-1}A) = s(A)$ by (17.6).

Let $\mathcal{J}_n$ denote the set of symmetric permutation matrices of $\mathfrak{M}_n$ and let $\mathcal{J} = \{t_n^{-1}A : A \in \mathcal{J}_n, n \in \mathbb{N}\}$. An easy computation shows that if $A = (a_{ij}) \in \mathcal{J}_n$, $B = (b_{ij}) \in \mathcal{J}_n$, and $a_{ii}b_{ii} = a_{ii}$ for all $i \leqslant n$ then $\delta(A,B) = s(A) - s(B)$. From this it follows that for every $s \in [0, \frac{1}{2}]$ there exists a convergent sequence $(A_i)_i$ in $\mathcal{J}$

such that $s = \lim_i s(A_i)$. Now the iaa $X \rightarrow X' = (X^*)^T$ is always continuous, so $\mathcal{J}$ consists entirely of non-singular hermitian elements. Therefore $s(\mathcal{J}) = [0, \frac{1}{2}]$. Thereby we have proved the

(17.8) Theorem. Let $x \rightarrow x^*$ be an iaa of ϑ satisfying (17.7) , let $X \rightarrow X'$ be the extension-by-continuity to $\overline{\mathfrak{M}}_\infty$ of the iaa $X \rightarrow (X^*)^T$ on $\mathfrak{M}_\infty$, and let $\mathfrak{h}$ be the set of non-singular elements of $\overline{\mathfrak{M}}_\infty$ that are hermitian relative to $X \rightarrow X'$. Then

$$s(\mathfrak{h}) = [0, \tfrac{1}{2}] .$$

CHAPTER IV

THE LATTICE $\mathbb{P}(\overline{\mathcal{L}}_\infty)$

18. <u>Frink's Lattice</u>. If $\mathcal{L}$ is any lattice and if $\mathcal{D}$ is any non-void subset of $\mathcal{L}$ such that

$$\{a \cap b : a, b \in \mathcal{D}\} \subseteq \mathcal{D}$$

and

$$\{a + v : a \in \mathcal{D}, v \in \mathcal{L}\} \subseteq \mathcal{D}$$

then $\mathcal{D}$ is a <u>dual ideal</u> of $\mathcal{L}$. If $\mathcal{L}$ has a least element 0 then set inclusion is an inductive partial ordering of the proper dual ideals of $\mathcal{L}$; hence, on the strength of Zorn's Lemma, every proper dual ideal of a lattice $\mathcal{L}$ with 0 is contained in a maximal proper dual ideal (mdi) of $\mathcal{L}$. Set $\mathbb{D}(\mathcal{L}) = \{\mathcal{D} : \mathcal{D}$ is a mdi of $\mathcal{L}\}$ and define $(\mathcal{D}_1, \mathcal{D}_2) = \{\mathcal{D} : \mathcal{D} \in \mathbb{D}(\mathcal{L}), \mathcal{D} \supseteq \mathcal{D}_1 \cap \mathcal{D}_2\}$ for all $\mathcal{D}_1, \mathcal{D}_2 \in \mathbb{D}(\mathcal{L})$. Finally, let $\mathbb{P}(\mathcal{L})$ be the class of subsets of $\mathbb{D}(\mathcal{L})$ defined by the condition: $S \in \mathbb{P}(\mathcal{L})$ if and only if $(\mathcal{D}_1, \mathcal{D}_2) \subseteq S$ whenever $\mathcal{D}_1, \mathcal{D}_2 \in S$. It is clear that $\mathbb{P}(\mathcal{L})$ contains (i) the empty set, (ii) every set $\{\mathcal{D}\}$ containing just one element $\mathcal{D}$ of $\mathbb{D}(\mathcal{L})$, (iii) the set $(\mathcal{D}_1, \mathcal{D}_2)$ for any pair $\mathcal{D}_1, \mathcal{D}_2 \in \mathbb{D}(\mathcal{L})$, and (iv), for any element a of $\mathcal{L}$, the set

(18.1) $\qquad \Phi(a) = \{\mathcal{D} : a \in \mathcal{D} \in \mathbb{D}(\mathcal{L})\}$.

Furthermore, $\mathbb{P}(\mathcal{L})$ is closed under arbitrary set-intersections and therefore becomes a complete lattice when ordered by inclusion. The lattice $\mathbb{P}(\mathcal{L})$ is atomic, and the singletons are the atoms.

Frink, who devised the construction of $\mathbb{P}(\mathcal{L})$ has proved the

(18.2) <u>Theorem</u>. [3, Theorem 14] If $\mathcal{L}$ is a complemented modular lattice then

(a) $\mathbb{P}(\mathcal{L})$ is a direct union of projective lattices, and

(b) the mapping $\Phi : \mathcal{L} \to \mathbb{P}(\mathcal{L})$ defined by (18.1) is a lattice-isomorphism of $\mathcal{L}$ onto a sublattice of $\mathbb{P}(\mathcal{L})$.

Note: Here, as always, we require that projective lattices and projective spaces be irreducible. Hence, where Frink [3] writes "projective lattice (space)" we substitute "direct union of projective lattices (spaces)".

This theorem looks like a promising tool for studying the structure of the lattice $\mathcal{L}$: for example, it shows that the class of coordinate division rings of the non-trivial projective lattices in the direct union decomposition of $\mathbb{P}(\mathcal{L})$ is an invariant of $\mathcal{L}$. Unfortunately our knowledge of $\mathbb{P}(\mathcal{L})$ is meager, but it is shown below that when $\mathcal{L}$ is an approximately finite geometry, the class of division rings just described is never vacuous.

19. <u>The lattice</u> $\mathbb{P}(\overline{\mathcal{L}}_\infty)$. Before proceeding we list two facts about the dual ideals of any lattice $\mathcal{L}$ with 0.

(19.1) Let $\mathcal{J}$ be a non-void subset of $\mathcal{L}$ such that

$$\{a \cap b : a, b \in \mathcal{J}\} \subseteq \mathcal{J}.$$

Then $\mathcal{S}^* \equiv \{\mathfrak{r} : \mathfrak{r} \in \mathcal{L}, \ \mathfrak{r} \geq \mathfrak{u}$ for some $\mathfrak{u} \in \mathcal{S}\}$ is a dual ideal. If $0 \notin \mathcal{S}$ then $\mathcal{S}^*$ is proper.

(19.2) If $\mathcal{D}$ is a mdi of $\mathcal{L}$ and $\mathfrak{b} \in \mathcal{L}$ then either $\mathfrak{b} \in \mathcal{D}$ or else $\mathfrak{u} \cap \mathfrak{b} = 0$ for some $\mathfrak{u} \in \mathcal{D}$

Assertion (19.1) follows immediately from the definitions. As for (19.2), suppose that $\mathfrak{b} \notin \mathcal{D}$, and set $\mathcal{S} = \{\mathfrak{u} \cap \mathfrak{b} : \mathfrak{u} \in \mathcal{D}\}$. The set $\mathcal{S}$ satisfies the hypothesis of (19.1), so $\mathcal{S}^*$ is a dual ideal. But $\mathcal{S}^* \supseteq \mathcal{D}$ and $\mathfrak{b} \in \mathcal{S}^*$; hence $\mathcal{S}^* = \mathcal{L}$. Therefore $0 \in \mathcal{S}$, and $\mathfrak{u} \cap \mathfrak{b} = 0$ for some $\mathfrak{u} \in \mathcal{D}$. Thereby (19.2) is proved.

In all that follows, let

$$E_n = \begin{bmatrix} 1 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & \cdots & 0 \end{bmatrix} \in \mathfrak{M}_n \text{ for all } n \in \mathbb{N}$$

and let $F_n = t_n^{-1}(E_n)$ $(\in \mathfrak{M}^{(n)})$. Define

$$\mathcal{F} = \{x\overline{\mathfrak{M}}_\infty : x \in \overline{\mathfrak{M}}_\infty, \ x\overline{\mathfrak{M}}_\infty \supseteq F_n\overline{\mathfrak{M}}_\infty \text{ for some } n \in \mathbb{N}\}.$$

By (19.1), $\mathcal{F}$ is a proper dual ideal of $R_{\overline{\mathfrak{M}}_\infty}$. Let $\mathcal{E}$ be a fixed but otherwise arbitrary mdi of $R_{\overline{\mathfrak{M}}_\infty}$ that contains $\mathcal{F}$, and define $A\mathcal{E} = \{A\mathfrak{A} : \mathfrak{A} \in \mathcal{E}\}$ for all $A \in \overline{\mathfrak{M}}_\infty$. It is easy to see that $A\mathcal{E}$ is a mdi if and only if $r(A) = 1$.

(19.3) <u>Lemma.</u> Let $BF_n = (Z_1 + Z_2A)F_n$ where A and B are non-singular elements of $\overline{\mathfrak{M}}_{\infty}$ and Z_1 and Z_2 are in $\mathfrak{Z}(\mathfrak{M}_{\infty})$. Then $B\mathcal{E} \supseteq \mathcal{E} \cap A\mathcal{E}$.

Proof: By (19.2), it suffices to show that if $\mathfrak{A} \in \mathcal{E} \cap A\mathcal{E}$ and $\mathcal{L} \in \mathcal{E}$, then $\mathfrak{A} \cap B\mathcal{L} \neq 0$. Now $\mathfrak{A} \in A\mathcal{E}$, so $A^{-1}\mathfrak{A} \in \mathcal{E}$. Also $F_n\overline{\mathfrak{M}}_{\infty} \in \mathcal{F} \subset \mathcal{E}$. Therefore

$$\mathcal{H} \equiv \mathfrak{A} \cap \mathcal{L} \cap (A^{-1}\mathfrak{A}) \cap (F_n\overline{\mathfrak{M}}_{\infty}) \in \mathcal{E}.$$

Since $\mathcal{H} \subseteq F_n\overline{\mathfrak{M}}_{\infty}$; $F_n\mathcal{H} = \mathcal{H}$; hence

$$B\mathcal{H} = (Z_1 + Z_2A)\mathcal{H} \subseteq Z_1\mathcal{H} + Z_2A\mathcal{H} = \mathcal{H} + A\mathcal{H}.$$

But $\mathcal{H} + A\mathcal{H} \subseteq \mathfrak{A} + A(A^{-1}\mathfrak{A}) = \mathfrak{A}$ so $B\mathcal{H} \subseteq \mathfrak{A}$, and $B\mathcal{H} \subseteq B\mathcal{L}$; therefore $B\mathcal{H} \subseteq \mathfrak{A} \cap B\mathcal{L}$.

All that remains is to show that $B\mathcal{H} \neq 0$. Since B is non-singular and $\mathcal{E}$ is a proper dual ideal, $\mathcal{H} \in \mathcal{E} \Rightarrow \mathcal{H} \neq 0 \Rightarrow$ $\Rightarrow B\mathcal{H} \neq 0$, and the proof is complete.

(19.4) <u>Corollary</u>. If B is a non-singular element of $\overline{\mathfrak{M}}_{\infty}$ and if $BF_n = F_n$, then $B\mathcal{E} = \mathcal{E}$.

Proof: Set $Z_1 = A = 1$ and $Z_2 = 0$. Then according to the Lemma, $B\mathcal{E} \supseteq \mathcal{E} \cap 1\mathcal{E} = \mathcal{E}$. $B\mathcal{E} \supset \mathcal{E}$ is impossible because $B\mathcal{E}$ is proper and $\mathcal{E}$ is maximal.

Let $\mathfrak{D}_{\omega}$ denote the space of columns of elements of $\mathfrak{M}_{\omega}$, or in other words, the set of all $1 \times \omega$ matrices over $\mathfrak{L}$ having at most a finite number of non-zero elements. If $X \in \mathfrak{M}_{\infty}$, $r(X) = 1$, and if x is the first column of X, then set $\mathcal{E}(x) = X\mathcal{E}$. If x is also the first column of Y, then $XF_n = YF_n$ for any n such that $X, Y \in \mathfrak{M}^{(n)}$. But then

$X^{-1}YF_n = F_n$, and $X^{-1}Y\mathcal{E} = \mathcal{E}$, by the Corollary. Therefore $X\mathcal{E} = XX^{-1}Y\mathcal{E} = Y\mathcal{E}$, so $\mathcal{E}(x) \equiv X\mathcal{E}$ is determined uniquely by x .

(19.5) <u>Lemma</u>. Let $v_1, \ldots, v_t$ be non-zero vectors of $\mathcal{D}_\omega$. Then $\mathcal{E}(\sum_{i=1}^t v_i \zeta_i) \supseteq \mathcal{E}(v_1) \cap \ldots \cap \mathcal{E}(v_t)$ whenever $\zeta_i \in \mathcal{Z}(\vartheta)$, for all $i \leq t$, and $0 \neq \sum_{i=1}^t v_i \zeta_i$.

Proof: Suppose $t = 2$. Let $V_i \in \mathcal{M}_\infty$, $r(V_i) = 1$, and (first column of V_i) $= v_i$, for $i = 1, 2$. Then the first column of $V_1Z_1 + V_2Z_2$ is $v_1\zeta_1 + v_2\zeta_2$, where $Z_i = \zeta_i \cdot 1_\omega \in \mathcal{Z}(\vartheta)$. By (19.3) $(Z_1 + V_1^{-1}V_2Z_2)\mathcal{E} \supseteq \mathcal{E} \cap (V_1^{-1}V_2)\mathcal{E}$, and hence

$$
\begin{aligned}
\mathcal{E}(v_1\zeta_1 + v_2\zeta_2) &= (V_1Z_1 + V_2Z_2)\mathcal{E} \\
&= V_1(Z_1 + V_1^{-1}V_2Z_2)\mathcal{E} \\
&\supseteq V_1[\mathcal{E} \cap (V_1^{-1}V_2)\mathcal{E}] \\
&\supseteq V_1\mathcal{E} \cap V_2\mathcal{E} \\
&= \mathcal{E}(v_1) \cap \mathcal{E}(v_2) .
\end{aligned}
$$

If $t = 1$, just set $\zeta_2 = 0$.

If $t > 2$, observe that our result for $t = 2$ and

$$\mathcal{E}(v_1\zeta_1 + \ldots + v_r\zeta_r) \supseteq \mathcal{E}(v_1) \cap \ldots \cap \mathcal{E}(v_r)$$

imply that

$$
\begin{aligned}
\mathcal{E}(v_1\zeta_1 + \ldots + v_r\zeta_r + v_{r+1}\zeta_{r+1}) \supseteq \\
&\supseteq \mathcal{E}(v_1\zeta_1 + \ldots + v_r\zeta_r) \cap \mathcal{E}(v_{r+1}) \\
&\supseteq \mathcal{E}(v_1) \cap \ldots \cap \mathcal{E}(v_r) \cap \mathcal{E}(v_{r+1}) ;
\end{aligned}
$$

therefore the proof is complete by finite induction.

A partial converse of (19.5) is

(19.6) Lemma. If $v_1, \ldots, v_{t+1}$ are right linearly independent vectors of $\mathbb{D}_\omega$ then

$$\mathcal{E}(v_{t+1}) \not\supseteq \mathcal{E}(v_1) \cap \ldots \cap \mathcal{E}(v_t) .$$

Proof: As in (19.5), let $V_i \in \mathcal{M}_\infty$, $r(V_i) = 1$, and (first column of V_i) = v_i for all $i \leqslant t+1$. We may suppose that $V_i \in \mathcal{M}^{(n)}$ for all $i \leqslant t+1$. Since $F_n \overline{\mathcal{M}}_\infty \in \mathcal{J} \subset \mathcal{E}$, it follows that

$$V_i F_n \overline{\mathcal{M}}_\infty \in V_i \mathcal{E} = \mathcal{E}(v_i)$$

and hence that

$$V_1 F_n \overline{\mathcal{M}}_\infty + \ldots + V_t F_n \overline{\mathcal{M}}_\infty \in \mathcal{E}(v_1) \cap \ldots \cap \mathcal{E}(v_t) .$$

Let W be the matrix of $\mathcal{M}^{(n)}$ whose first n columns are $v_1, v_2, \ldots, v_t, 0, \ldots, 0$, (clearly $n > t$, because $v_1 \ldots v_{t+1}$ are linearly independent and are essentially of length $\leqslant n$). Then

$$(W\overline{\mathcal{M}}_\infty) \cap (V_{t+1} F_n \overline{\mathcal{M}}_\infty) = 0$$

so $W\overline{\mathcal{M}}_\infty \notin V_{t+1}\mathcal{E} = \mathcal{E}(v_{t+1})$, whereas

$$W\overline{\mathcal{M}}_\infty = V_1 F_n \overline{\mathcal{M}}_\infty + \ldots + V_t F_n \overline{\mathcal{M}}_\infty \in \mathcal{E}(v_1) \cap \ldots \cap \mathcal{E}(v_t) .$$

Thereby, the lemma is proved.

In order to apply (19.5) and (19.6), two more lemmas are required. The first of these is due to Jónsson [7, Theorem 2.3, p. 300] and is stated here without proof.

(19.7) Lemma. Let $\mathcal{L}$ be a complemented modular lattice, let A be an atom of $\mathbb{P}(\mathcal{L})$, and let F be a finite dimensional element of $\mathbb{P}(\mathcal{L})$. If $A \cap F = 0$ then there exist elements $\mathfrak{a}$ and $\mathfrak{f}$ of $\mathcal{L}$ such that $\mathfrak{a} \cap \mathfrak{f} = 0$, $\Phi(\mathfrak{a}) \supseteq A$, and $\Phi(\mathfrak{f}) \supseteq F$.

(19.8) <u>Lemma</u>. Let $\mathcal{L}$ be a complemented modular lattice and let $\mathcal{D}_1, \ldots, \mathcal{D}_k$ be mdi's of $\mathcal{L}$. Define

$$S = \left\{\mathcal{D} : \left[\bigcap_{j=1}^{k} \mathcal{D}_j\right] \subseteq \mathcal{D} \in \mathbb{D}(\mathcal{L})\right\} .$$

Then S is the join $\sum_{j=1}^{k} \{\mathcal{D}_j\}$ of $\{\mathcal{D}_1\}, \ldots, \{\mathcal{D}_k\}$ in $\mathbb{P}(\mathcal{L})$.

Proof: It is easy to see that $S \in \mathbb{P}(\mathcal{L})$, and from this, that $S \supseteq \sum_{j=1}^{k} \{\mathcal{D}_j\}$. To prove the reverse inclusion, let $\mathcal{D}_0$ be an arbitrary mdi of $\mathcal{L}$ that is not in $\sum_{j=1}^{k} \{\mathcal{D}_j\}$. Then $\{\mathcal{D}_0\} \cap \sum_{j=1}^{k} \{\mathcal{D}_j\} = 0$, so it follows from (19.7) that

$$\Phi(\mathfrak{a}) = \{\mathcal{D} : \mathfrak{a} \in \mathcal{D} \in \mathbb{D}(\mathcal{L})\} \supseteq \{\mathcal{D}_0\}, \quad \Phi(\mathfrak{f}) =$$
$$= \{\mathcal{D} : \mathfrak{f} \in \mathcal{D} \in \mathbb{D}(\mathcal{L})\} \supseteq \sum_{j=1}^{k} \{\mathcal{D}_j\}, \text{ and } \mathfrak{a} \cap \mathfrak{f} = 0,$$

for some $\mathfrak{a}, \mathfrak{f} \in \mathcal{L}$. Hence $\mathfrak{a} \in \mathcal{D}_0$ and $\mathfrak{f} \in \mathcal{D}_j$, $1 \leqslant j \leqslant k$, and by (19.2) $\mathfrak{f} \notin \mathcal{D}_0$. Thus $\bigcap_{j=1}^{k} \mathcal{D}_j \nsubseteq \mathcal{D}_0$, and $\mathcal{D}_0 \notin S$. Therefore $S \subseteq \sum_{j=1}^{k} \{\mathcal{D}_j\}$, which completes the proof.

Let $\mathfrak{U}_\omega$ be the set of all those vectors of $\mathfrak{D}_\omega$ whose elements lie in $\mathfrak{Z}(\vartheta)$. We shall look upon the elements of $\mathfrak{U}_\omega$ as the coordinate vectors of points of a countably infinite dimensional projective space Π over $\mathfrak{Z}(\vartheta)$. The mapping $u \to \{\mathcal{E}(u)\}$, $0 \neq u \in \mathfrak{U}_\omega$, carries points of Π into atoms of $\mathbb{P}(R_{\overline{\mathfrak{M}_\infty}})$, for if $\zeta \neq 0$, then by (19.5) with $t = 1$, $\mathcal{E}(u\zeta) = \mathcal{E}(u)$.

(19.9) Theorem. The vectors $u_1, \ldots, u_k$ of $\mathfrak{U}_\omega$ represent independent points of Π if and only if the atoms $\{\mathcal{E}(u_1)\}, \ldots, \{\mathcal{E}(u_k)\}$ of $\mathbb{P}(R_{\overline{\mathfrak{M}_\infty}})$ are independent.

Proof: Let $1 < j \leqslant k$. If u_j is a linear combination of $\{u_i : i < j\}$ over ϑ, then the matrices $(u_1 u_2 \ldots u_{j-1})$ and $(u_1 u_2 \ldots u_{j-1} u_j)$ have the same rank, and since the elements of these matrices all lie in $\mathfrak{Z}(\vartheta)$, the equation

$$u_j = \sum_{i<j} u_i \zeta_i$$

has a solution with $\zeta_i \in \mathfrak{Z}(\vartheta)$, all $i < j$. Therefore, it follows from (19.5) and (19.6) $u_1, \ldots, u_k$ is an independent set if and only if $\mathcal{E}(u_j) \not\supseteq \mathcal{E}(u_1) \cap \ldots \cap \mathcal{E}(u_{j-1})$, for $1 < j \leqslant k$, and this, in turn is equivalent to $\{\mathcal{E}(u_j)\} \cap \sum_{i<j} \{\mathcal{E}(u_i)\} = 0$, by (19.8). Thus the theorem is proved.

Every atom of a direct union of projective lattices is an atom of some component of the union, and two atoms lie in the same component if and only if their join contains at least three atoms. From these facts together with (19.9), it follows that the set $\{\{\mathscr{E}(u)\} : 0 = u \in \mathfrak{U}_\omega\}$ lies entirely in some one component of $\mathbb{P}(R\overline{\mathfrak{M}}_\infty)$. For let u_0 be some fixed element of $\mathfrak{U}_\omega$, let $\mathcal{L}$ be the component of $\mathbb{P}(R\overline{\mathfrak{M}}_\infty)$ that contains $\{\mathscr{E}(u_0)\}$, and let u_1 be any other element of $\mathfrak{U}_\omega$. If $u_0 = u_1\zeta$ for some $\zeta \in \mathfrak{Z}(\vartheta)$ then $\mathscr{E}(u_1) = \mathscr{E}(u_0)$. Otherwise, u_0, u_1, and $u_1 + u_0$ represent three distinct collinear points, and hence $(\mathscr{E}(u_0), \mathscr{E}(u_1))$ contains three distinct atoms $\{\mathscr{E}(u_0)\}$, $\{\mathscr{E}(u_1)\}$, and $\{\mathscr{E}(u_0 + u_1)\}$. Therefore $\{\mathscr{E}(u_1)\} \in \mathcal{L}$ and $\mathcal{L} \supseteq \{\{\mathscr{E}(u)\} : 0 \neq u \in \mathfrak{U}_\omega\}$.

Let S be a subspace of $\mathfrak{U}_\omega$ and define

$$\Psi(S) = \sum\{\{\mathscr{E}(u)\} : u \in S\}$$

It follows easily from the fact that the mapping $u \rightarrow \{\mathscr{E}(u)\}$ preserves dependence and independence, that Ψ is an isomorphism of the lattice of subspaces of $\mathfrak{U}_\omega$ onto a sublattice of $\mathcal{L}$, and hence that coordinate division ring determined by $\mathcal{L}$ contains a subring isomorphic to the coordinate division ring $\mathfrak{Z}(\vartheta)$ of $\mathfrak{U}_\omega$. Thereby, we have proved the

(19.10) <u>Theorem</u>. The direct union decomposition of $\mathbb{P}(R\overline{\mathfrak{M}}_\infty)$ contains an infinite dimensional projective lattice $\mathcal{L}$ whose coordinate division ring has a subfield isomorphic to $\mathfrak{Z}(\vartheta)$.

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