Analysis of Weierstrass' Function and its effect on Loewner Hulls

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David Nielsen Horton

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Dedicated to my wife, without whom I could not have come this far; and to all persons without whom I would not have come this far.

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Abstract

The Loewner equation gives a correspondence between subsets of $\mathbb{H}$ [the upper half-plane], called hulls, and continuous functions, called drivers. Schramm-Loewner evolution (SLE) refers to the Loewner hulls when the driver is scaled Brownian motion. SLE, used to study problems in physics and probability, is well understood. We know that a.s. the trace exists (i.e. the hulls are described by curves), and we know that the trace exhibits three distinct phases [13]. However, the hulls produced by drivers which are not Brownian motion are not as well understood, and the goal of this dissertation is to further this understanding.

We begin by looking at the question of which drivers generate a trace. We generalize a result of Lawler to give criteria for Lip(δ) [Lipschitz with exponent delta] drivers to admit a trace. We give both deterministic and stochastic results.

The main focus of the work is on the Weierstrass function, $W(t)$. Because it is continuous, nowhere differentiable, and Lip(1/2), similar to Brownian motion, it has been used to explore the phases in the Loewner equation driven by deterministic functions [11],[3],[16]. Many of these results rely on suboptimal bounds on the Lip(1/2) behavior of $W(t)$. In chapter 3, we sharpen the bounds on the Lip(1/2) behavior of $W(t)$. We improve on the results in [11] and [3] and obtain that when $c < .55338486$, the hull generated by $cW(t)$ is a simple curve and when $c > \frac{8\sqrt{2\pi}}{3\sqrt{3}}$, then the hull is not a simple curve.

Friz and Shekhar have used rough path theory to explore Loewner hulls [2]. In chapter 4, we apply their results to the Loewner equation driven by $W(t)$. We find values for the quadratic variation in the sense of Föllmer of the Weierstrass function along specific partitions. Then we explore the possibility of using a rough path approach to showing that $cW(t)$ admits a trace.

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Chapter 1

Background and Research Questions

1.1 The Loewner Equation

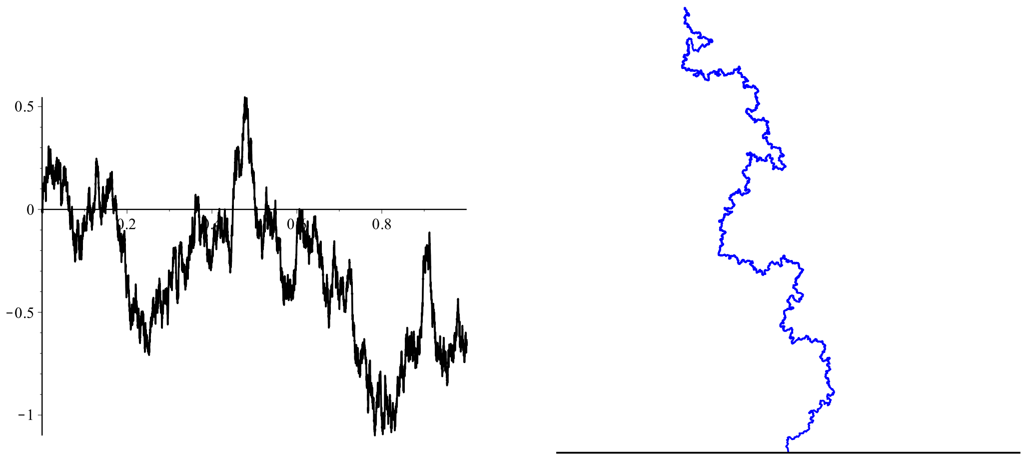
The Loewner equation uses continuous functions to define a collection of conformal maps from subsets of $\mathbb{H} := \{z = x + iy : y > 0\}$ onto $\mathbb{H}$. For a continuous real valued function $\lambda_t : [0, T] \to \mathbb{R}$, the chordal Loewner equation is given by

$$\partial_t g_t(z) = \frac{2}{g_t(z) - \lambda_t}, \; g_0(z) = z, \text{ for } z \in \mathbb{H}. \tag{1.1}$$

We say that the collection of maps $\{g_t(z)\}$ are driven by λ_t. We define the capture time $T_z = \sup\{s \in [0, T] : g_t(z) \text{ exists on } [0, s)\}$. We define the hull to be the set $K_t = \{z : T_z \leq t\}$.

As an example, take $\lambda_t \equiv 0$. It can be shown that the hulls are $K_t = \{iy : 0 \leq y \leq 2\sqrt{t}\}$. Here the hulls are described by a curve in $\bar{\mathbb{H}}$ since $K_t = \gamma[0, t]$ for $\gamma(t) = 2i\sqrt{t}$, but this is not always the case. The driving function is said to generate a curve (also called a trace) if there exists a curve $\gamma(t) \subset \bar{\mathbb{H}}$ such that $\mathbb{H} \setminus K_t$ is the unbounded component of $\mathbb{H} \setminus \gamma[0, t]$. Specifically, this happens if and only if

$$\gamma(t) := \lim_{y \to 0^+} g_t^{-1}(\lambda_t + iy) \tag{1.2}$$



(a) driver

(b) hull

Figure 1.1: A sample of SLE_2. Image courtesy of Lind and Robbins

exists and is continuous in t. In [12], Marshall and Rohde provide an example of a driving function which does not generate a curve. This example has a hull which is a logarithmic spiral circling a disc infinitely many times, adding the disc to the hull at time t_0. Such a hull cannot be described by a curve at time t_0.

When showing results about the trace, we will often work with $g_t^{-1}(z)$ instead of $g_t(z)$ directly. We will call the function $f_t(z) := g_t^{-1}(z)$ an inverse Loewner chain. An inverse Loewner chain is the solution to the differential equation:

$$\partial_t f_t(z) = -f_t'(z)\frac{2}{z-\lambda_t},\ \ f_0(z) = z,\ \text{ for } z \in \mathbb{H}. \tag{1.3}$$

Often one considers the Loewner equation driven by a random process. Schramm-Loewner evolution with parameter $\kappa \geq 0$ (SLE_κ) is the collection of maps $\{g_t(z)\}$ driven by $\sqrt{\kappa}B_t$ for a standard one-dimensional Brownian motion B_t. Various values of κ correspond to different models including: loop-erased random walks ($\kappa = 2$), the scaling limit of self-avoiding random walks ($\kappa = 8/3$ conjectured), the limit of interfaces for the Ising model ($\kappa = 3$), and the scaling limit of critical percolation ($\kappa = 6$) to name a few. One of the properties that allows SLE_κ to be so applicable is that a.s. it admits a trace ([13] $\kappa \neq 8$, [8] $\kappa = 8$). The nature of this trace depends on the value of κ. In [13] it is shown that the trace for SLE_κ exhibits the following phase changes:

- For $\kappa \in [0,4]$, $\gamma(t)$ is a.s. a simple path contained in $\mathbb{H} \cup \{0\}$.
- For $\kappa \in (4,8)$, $\gamma(t)$ is a.s. a non-simple path.
- For $\kappa \in [8,\infty)$, $\gamma(t)$ is a.s. a space-filling curve.

It is natural to wonder whether deterministic drivers exhibit a similar phase change. We typically consider drivers in the class of Lipschitz functions with exponent 1/2, defined Lip(1/2)= $\{\lambda : |\lambda_t - \lambda_s| \le c|t-s|^{1/2} \text{ for } c < \infty\}$. The smallest such c determines the (semi)norm, denoted $||\lambda||_{1/2}$. Lip(1/2) functions are used due to the fact that if K_t is generated by λ_t, then rK_{t/r^2} is generated by $r\lambda_{t/r^2}$. So like SLE$_\kappa$, Lip(1/2) functions are invariant under this scaling. The next theorem, due to Marshall, Rohde, and Lind, provides a partial analogue for the "first" phase.

Theorem 1.1 ([12],[10]). *If the driving function λ has Lip(1/2) norm less than 4, then the chordal Loewner equation generates a simple curve.*

It should be noted that there are drivers with arbitrarily large norm which generate simple curves. For example, $\lambda_t = c\sqrt{t}$ generates a simple curve for all values of c. So, we should take the 4 in Theorem 1.1 to be the start of a second phase rather than the end of a first phase. In [9] the existence of a third phase is found:

Theorem 1.2 ([9]). *If λ is a Lip(1/2) that generates a curve with non-empty interior, then $||\lambda||_{1/2} \ge 4.0001$.*

This bound is admittedly not optimal, but it does tell us that if a Lip(1/2) function, λ, generates a space-filling curve, then $||\lambda||_{1/2}$ is bounded away from 4. This shows that if we focus on the Lip(1/2) norms, we can expect distinct starting points for each phase.

In order to further study the phases, we are also interested in conditions that guarantee that the hull in not a simple curve. The next result indicates that the lower Lipschitz bound (c such that $|\lambda_t - \lambda_s| \ge c|t-s|^{1/2}$ for t and s close) is as important as the norm (upper Lipschitz bound). The Lower Lipschitz bound is also called the modulus of 1/2-Hölder roughness.

Theorem 1.3 ([11]). *Let λ_t be a continuous function. If $|\lambda_T - \lambda_t| \geq 4\sqrt{T-t}$ on $[T-\epsilon, T]$ for some $\epsilon > 0$, then the hull generated by λ at time $t = T$ is non-simple.*

Zhang and Zinsmeister use both the upper and lower Lip(1/2) bounds to extend the conclusion in Theorem 1.1 regarding the trace being a simple curve. Instead of requiring the Lip(1/2) norm to be bounded by a constant, we only need the upper Lipschitz bound from the left, arbitrarily close to the point in question, to be bounded by a function of the corresponding lower Lip(1/2) bound. This theorem does not replace Theorem 1.1 since it does not guarantee the existence of the trace.

Theorem 1.4 ([16]). *Let $\lambda : [0, T] \to \mathbb{R}$ be continuous such that the corresponding process is generated by a curve γ. For $t > 0$, define*

$$a(t) = \liminf_{s \to t^-} \frac{|\lambda_t - \lambda_s|}{\sqrt{t-s}}, \quad b(t) = \limsup_{s \to t^-} \frac{|\lambda_t - \lambda_s|}{\sqrt{t-s}}.$$

If $b(t) < f(a(t))$ for all $t \in (0, T]$ with

$$f(a) = \begin{cases} 4, & 2 \leq a < 4 \\ a + \frac{4}{a}, & 0 < a < 2 \\ +\infty, & a = 0 \end{cases}$$

then the curve γ is simple. Also, $f(a(t))$ is the optimal bound.

In that same paper, the Zhang and Zinsmeister use the quantities $a(t)$ and $b(t)$ to give conditions for a hull to be a non-simple curve–something that cannot be determined with the norm alone.

Theorem 1.5 ([16]). *Let $\lambda : [0, T] \to \mathbb{R}$ be continuous such that the corresponding process is generated by a curve on $[0, T)$. If $a(T) > \max\{4, (b(T) + \sqrt{b(T)^2 - 16})/2\}$, then the Loewner process is generated by a curve γ in $[0, T]$ and $\gamma(T) \in \mathbb{R}$ or $\gamma(T) \in \gamma([0, T))$.*

In the previous results, the authors assume the existence of the trace (or a partial existence). It remains an open question to understand what functions generate a trace. In [2] Friz and Shekhar use rough path theory to give novel conditions which will guarantee that a particular class of random driving functions generates a trace. We go over the details more carefully in chapter 4.

Definition 1.6. *We say that a function $\lambda : [0,T] \to \mathbb{R}$ has finite quadratic-variation in the sense of Föllmer if along some fixed sequence of partitions $P = (P_n)$ of $[0,T]$ with mesh size going to zero*

$$[\lambda]_t^P = \lim_{n\to\infty} \sum_{[r,s]\in P_n} (\lambda_{s\wedge t} - \lambda_{r\wedge t})^2 \tag{1.4}$$

exists and is continuous in t.

Theorem 1.7 ([2])**.** *Let $T > 0$ and consider random $\lambda_t = \lambda_t(\omega)$ where λ_t satisfies that:*

1. *λ has finite quadratic-variation in the sense of Föllmer,*
2. *The bracket, $[\lambda]_t^P$ is Lipschitz with Lipschitz constant $\kappa < 2$ a.s.,*
3. *λ_t is a.s. weak Hölder-1/2.*

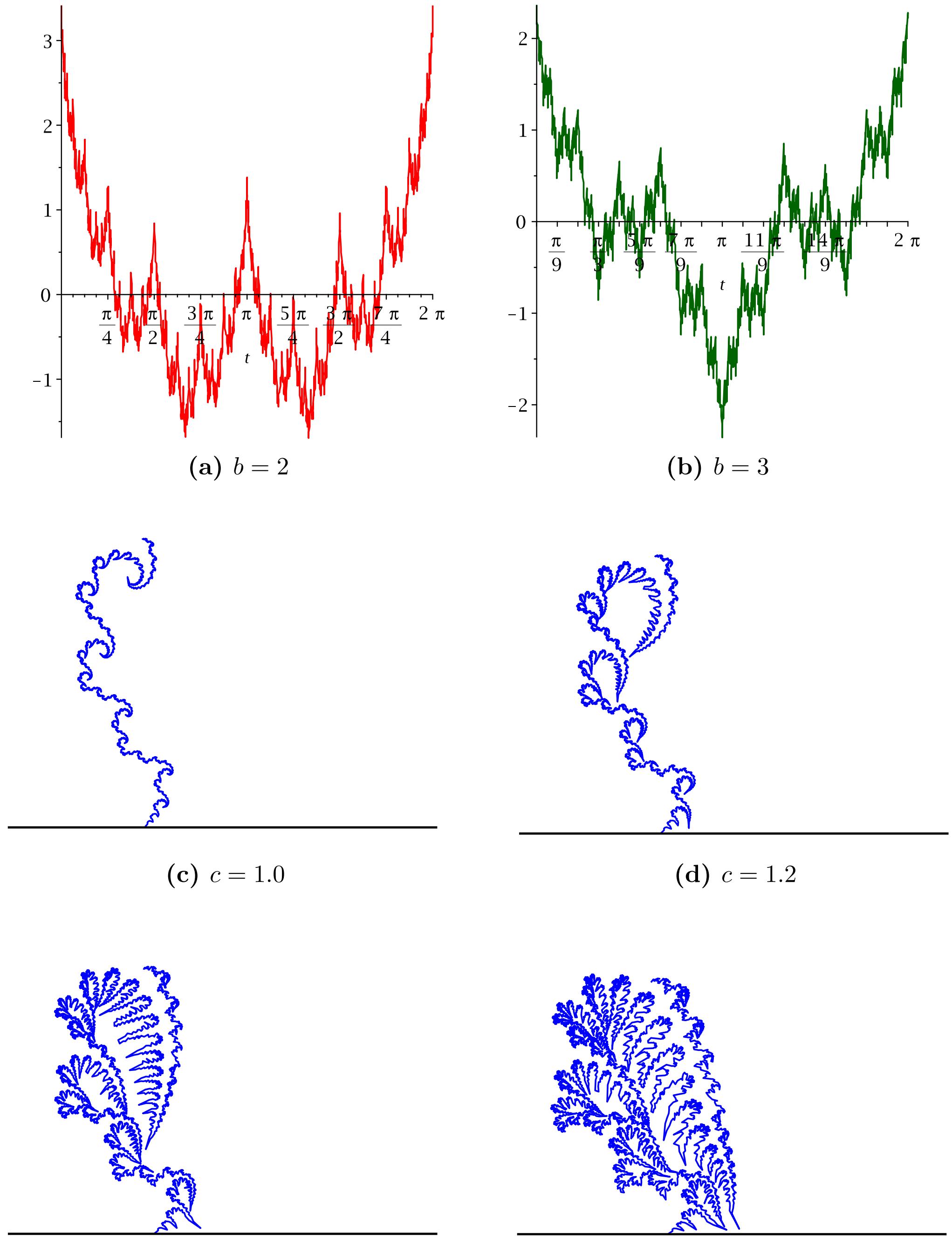
For fixed $t \in [0,T]$, define $\beta_s := \lambda_t - \lambda_{t-s}$ and assume β is a continuous semimartingale with respect to some filtration, with canonical decomposition $\beta = N + A$ into local martingale N and bounded variation part A, so that $||A||_t^2 := \int_0^t |\dot{A}_s|^2 ds$ has sufficiently high (depending only on κ) exponential moments finite uniform in t. Then the trace exists.

1.2 The Weierstrass Function

Define the Weierstrass function by

$$W_b(t) = \sum_{j=0}^{\infty} b^{-j/2} \cos\left(b^j t\right). \tag{1.5}$$

The Weierstrass function is the first published example of a continuous function which is not differentiable on any interval [15]. In [4], Hardy shows that this function is Lip(1/2). Hence the Weierstrass function shares some characteristics with Brownian motion: it is Lip(1/2) while Brownian motion is weak Lip(1/2), and they are also both continuous and nowhere



(a) $b = 2$ **(b)** $b = 3$

(c) $c = 1.0$ **(d)** $c = 1.2$

(e) $c = 1.4$ **(f)** $c = 1.6$

Figure 1.2: (top) The Weierstrass function with different values of b. (below) Hulls driven by $cW_2(t)$ with different values of c. As c increases, the hulls look less like simple curves. Hull images courtesy of Lind and Robbins.

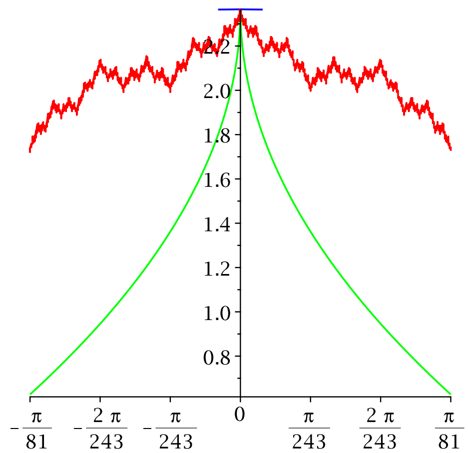


Figure 1.3: $W_3(t)$ (red) with the upper Lip(1/2) bound from Theorem 1.9 (green) and the lower Lip(1/2) bound from Theorem 1.10 (blue) centered at 0.

differentiable. As such, several papers have used $cW_b(t)$ as a driving function to study the effects on hulls of using different values of c.

In [11], Robins and Lind were motivated by the phase changes in SLE$_\kappa$ to explore possible phase changes in the hulls generated by $cW_2(t)$. Ultimately, they proved Theorem 1.8 for $b = 2$. Later in [3], Glenn generalized the proofs in [11] to extend Theorem 1.8 to all integer values of b greater than 1.

Theorem 1.8 ([11],[3])**.** *Let $b > 1$ be an integer. When c is small enough, the hull generated by $cW_b(t)$ is a simple curve in $\mathbb{H} \cup \{W_b(0)\}$, and this is not the case when c is large enough.*

One reason Theorem 1.8 is worded so vaguely is because relies on the following two suboptimal bounds:

Theorem 1.9 ([3])**.** *Let $b > 1$. Then the norm of $W_b(t)$ satisfies*

$$||W_b||_{1/2} \leq \frac{b + 2\sqrt{b}}{\sqrt{b} - 1}$$

Theorem 1.10 ([3])**.** *Let $b > 1$ be an integer. Fix $m \in \mathbb{Z}$ and $k \in \mathbb{N}$. Then there is an $\eta = \eta(b) \in \mathbb{N}$ such that if $0 < |h| \leq b^{-(k+\eta)}$ then $W_b\left(\frac{2m\pi}{b^k}\right) - W_b\left(\frac{2m\pi}{b^k} + h\right) \geq \kappa(b, \eta)\sqrt{|h|}$.*

The points $\frac{2m\pi}{b^k}$ are local maximums of W_b. This can be intuited since $b^j \frac{2m\pi}{b^k}$ is an even multiple of π for $j \geq k$. Hence, the tail of the series for $W_b(\frac{2m\pi}{b^k})$ is as large as possible.

When b is even, all points of the form $\frac{m\pi}{b^k}$ are local maximums, and when b is odd, the points $\frac{(2m+1)\pi}{b^k}$ are local minimums. We can get a similar inequality for the points $\frac{(2m+1)\pi}{b^k}$.

One way to read Theorem 1.10, is that the modulus of 1/2-Hölder roughness of W_b at $\frac{2m\pi}{b^k}$ on scale smaller than $b^{-(k+\eta)}$ is bounded from below by κ. The inequality can be rewritten as $W_b\left(\frac{2m\pi}{b^k}+h\right) \leq -\kappa\sqrt{|h|} + W_b\left(\frac{2m\pi}{b^k}\right)$, telling us that κ is the coefficient of a square root curve beginning at $W_b\left(\frac{2m\pi}{b^k}\right)$ and extending out to a length of $b^{-(k+\eta)}$ that remains above $W_b(t)$. This is illustrated by the blue curve in Figure 1.3. When $b = 3$, $\eta = 5$ and $\kappa \approx 0.0218$, hence the blue curve is short and nearly flat.

In [16], Zhang and Zinsmeister computed an upper bound on the local lower Lip(1/2) behavior so that they could apply Theorem 1.4 to the driving function $cW_b(t)$.

Theorem 1.11 ([16]). *Define* $W_b^N(t) = \sum_{j=0}^{N} b^{-j/2}\cos(b^j t)$. *Then for all* N *and* T *we have*

$$\liminf_{t\to T^-} \frac{|W_b^N(T) - W_b^N(t)|}{\sqrt{T-t}} < \left(\sqrt{\pi} + \frac{1}{\sqrt{\pi}}\right)\frac{\sqrt{2}}{\sqrt{b}-1} \sim \frac{c}{\sqrt{b}}$$

We can contrast this with Theorem 1.10 which gives a lower bound on the lower Lip(1/2) behavior of $cW_b(t)$ at local maximums.

1.3 Research Questions

Despite all that we know about the Loewner equation with deterministic drivers, several questions remain unanswered.

Question 1.12. *Given a driving function, when can we say the trace exists?*

When the conditions from Theorem 1.1 are not met, we must find different criteria. In Lemma 4.33 from [6], Lawler gives sufficient conditions to guarantee that a trace exists. Notably, one of his conditions is that the driving function be Lip(1/2). In chapter 2, we extend this result in Theorem 2.1 to a larger class of functions, namely Lip(δ) functions where $\delta \in (0, 1/2]$.

Question 1.13. *What are the hulls driven by the Weierstrass function?*

At present, Theorem 1.8 is the best answer we have. However, the range of values of c for which the theorem applies depends on the bounds from Theorems 1.9 and 1.10, and those bounds are far from optimal (see Theorem 3.1). In chapter 3, we will sharpen the bounds from Theorems 1.9 and 1.10 in the $b = 3$ case (see Theorems 3.2, 3.5, and Lemma 3.11). This will allow the upper bound on c-values which guarantee a simple hull to be increased from around 0.453 to around 0.553, and the lower bound on c-values which guarantee a non-simple hull to be decreased from around 182.709 to around 3.859 (see Theorem 3.3).

Combining both questions, we could ask: does the trace exist for the Weierstrass function? In chapter 4 we will show that $W(t)$ has finite quadratic variation in the sense of Föllmer (see Theorem 4.2) and explore the possibility of using rough path theory to ascertain the existence of a trace.

Chapter 2

Lip(δ) Drivers

In chapter 4 of [6], Lawler gives sufficient criteria for a weak Lip(1/2) driving function to generate a trace. The scope was limited to weak Lip(1/2) functions because he would ultimately use his result to show that Brownian motion generates a trace w.p.1. We are interested in more general results that imply the existence of the trace. For instance, in [5] Kobayashi, Lind, and Starnes study the Loewner hulls driven by time-changed Brownian motion. It is an open question to determine if time-changed Brownian motion admits a trace. We contribute to this effort by proving an adaptation of the previous results for Lip(δ) functions, since time-changed Brownian motion is a.s. Lip(δ) for $\delta < 1/2$.

In [6], Lawler proves the analytic conditions on the driver for the existence of the trace in a single lemma (Lemma 4.33) and leaves many of the details to the reader. To better understand where δ plays a role, each of the rudiments of the main proof will be given its own lemma, following the precedent in Johansson Viklund and Lawler's *Optimal Hölder Exponent for the SLE Path* [14].

Let us nail down some notation: $f(z)$ with no dependence on t is a conformal map; $f_t(z)$ is an inverse Loewner chain (i.e. a solution to equation 1.3) with driving function $\lambda_t \in \text{Lip}(\delta)$; and $\hat{f}_t(z) = f_t(\lambda_t + z)$.

Theorem 2.1. *Suppose $f_t(z)$ is an inverse Loewner chain with driving function $\lambda_t \in \text{Lip}(\delta)$. Assume there exists a sequence of positive numbers r_n such that*

$$|\hat{f}'_{k2^{-n/\delta}}(i2^{-n})| \leq 2^n r_n, \quad k = 0, 1, ..., \lfloor 2^{n/\delta} \rfloor$$

and

$$r_n \leq c2^{-\epsilon n} \qquad (or\ \{r_n\} \in l^1(\mathbb{N}))$$

for some $c, \epsilon > 0$. Then the trace exists.

Corollary 2.2. *Suppose $f_t(z)$ is an inverse Loewner chain with driving function $\lambda_t \in \mathrm{Lip}(\delta)$. If there exists $\{a_n\}_{n=1}^{\infty}, \{b_n\}_{n=1}^{\infty} \in l^1(\mathbb{N})$ such that*

$$\mathbb{P}(\{|\hat{f}'_{k2^{-n/\delta}}(i2^{-n})| \geq 2^n a_n\}) \leq 2^{-n/\delta} b_n$$

for $n = 1, 2, 3, \ldots$ and $k = 0, 1, .., \lfloor 2^{n/\delta} \rfloor$, then the trace exists a.s.

Note that Theorem 2.1 is entirely deterministic. Most of this chapter is dedicated to building the results we need to prove Theorem 2.1 concisely. For stochastic drivers we have Corollary 2.2. The proofs of Theorem 2.1 and Corollary 2.2 are at the end of this chapter. Everything else will be used to build these proofs. We start by articulating sufficient conditions for the trace to exist.

We say that the trace, γ, exists if for all t,

$$\gamma(t) = \lim_{y \to 0^+} \hat{f}_t(iy)$$

exists and is continuous in t. Since $\hat{f}_t(iy) = \hat{f}(iy, t)$ is clearly continuous on $(0, \infty) \times [0, \infty)$, to establish the continuity of $\hat{f}_t(iy)$ on $[0, \infty) \times [0, t_0]$ it suffices to find $\eta(\epsilon)$ such that $\eta(0+) = 0$ and such that

$$|\hat{f}_t(iy) - \hat{f}_s(iy_1)| \leq \eta(y + y_1 + |t - s|), \quad 0 \leq t, s \leq t_0, \quad y, y_1 \geq 0. \tag{2.1}$$

Lemma 2.3 (Lemma 3.2, [14])**.** *Let S be the rectangle $S = \{x + iy : -1 < x < 1, 0 < y < 1\}$. There exist $c, \alpha < \infty$ such that if f is a conformal map defined on $2S$, $z, w \in S$ and $\mathrm{Im}\{z\}, \mathrm{Im}\{w\} \geq 1/r$, then*

$$|f'(z)| \leq cr^{\alpha}|f'(w)|.$$

Proof. Assume that $\text{Im } z = 1/r$. Take a Whitney decomposition of S, $\{S_{j,k}\}$ where

$$S_{j,k} = \{x + iy : j2^{-k} \leq x \leq (j+1)2^{-k}, 2^{-(k+1)} \leq y \leq 2^{-k}\},$$

for $j = -2^k, \ldots, 2^k - 1$, and $k \in \mathbb{N}$. Let u, v be in $S_{j,k}$, that is, the same Whitney square. Using the Koebe Distortion Theorem with $\rho = 1/2$, we see that

$$|f'(u)| \leq \left(\frac{1+\rho}{(1-\rho)^3}\right)^2 |f'(v)| = 12^2 |f'(v)|$$

when $\text{Re } u = \text{Re } v$ by daisy-chaining two discs, and when $\text{Im } u = \text{Im } v$ we have

$$|f'(u)| \leq 12^4 |f'(v)|$$

by daisy-chaining four discs. To get sharper results, we will use only vertical and horizontal paths instead of the hyperbolic path used in the proof of Lemma 3.2 in [14]. Specifically, we will take a path straight up from z until we land in the square $S_{j,k}$ with both z and w lying directly below. Specifically, $j2^{-k} \leq \text{Re } z$, $\text{Re } w \leq (j+1)2^{-k}$. Then our path will move horizontally until we are directly above w, then we move vertically toward w. In the worst-case scenario, we move up vertically until we intersect the set $\{z : \text{Im } z = 1\}$, and we have that $\text{Im } z = \text{Im } w$ so that we vertically traverse the most squares. In this case, we move vertically through $2(k+1)$ squares and horizontally through one square so that

$$|f'(z)| \leq 12^4 (12^2)^{2(k+1)} |f'(w)| = 12^8 (12^{4\lfloor \log_2(r) \rfloor}) |f'(w)| \leq 12^8 r^{4 \log_2(12)} |f'(w)|.$$

$\square$

The proof of Lemma 2.3 was modified from the version appearing in [14] to better keep track of constants.

Lemma 2.4 (Lemma 3.5, [14])**.** *Suppose f_t is the inverse of a Loewner chain, i.e.,*

$$\partial_t f_t = -f_t' \frac{a}{z - \lambda_t}, \quad f_0(z) = z,$$

with $z = x + iy \in \mathbb{H}$. Then for $s \geq 0$

$$e^{-5as/y^2}|f_t'(z)| \leq |f_{t+s}'(z)| \leq e^{5as/y^2}|f_t'(z)|.$$

In particular, if $s \leq y^{1/\delta} < 1$,

$$e^{-5a}|f_t'(z)| \leq |f_{t+s}'(z)| \leq e^{5a}|f_t'(z)|.$$

Also, if $0 \leq s \leq y^{1/\delta} < 1$,

$$|f_{t+s}(z) - f_t(z)| \leq \frac{y}{5}[e^{5a} - 1]|f_t'(z)|.$$

Lemma 2.5 (Lemma 3.6, [14])**.** *There exist constants $c, \alpha < \infty$ such that the following holds. Let (g_t) be a Loewner chain corresponding to the continuous function λ_t. Let $s \in [0, y^{1/\delta}]$ for $y > 0$. Then*

$$|f_{t+s}'(\lambda_{t+s} + iy)| \leq cM^\alpha |f_{t+s}'(\lambda_t + iy)|,$$

where $M = \max\{|\lambda_{t+s} - \lambda_t|/y, 1\}$.

The following result is a generalization of [14, Lemma 3.8]. Here we ask for a finer mesh of points that we can control ($k2^{-n/\delta}$ instead of $k2^{-2n}$), but we better preserve the control we start with.

Lemma 2.6. *Let (g_t) be the Loewner chain corresponding to $\lambda_t \in \mathrm{Lip}(\delta)$ for some $0 \leq \delta \leq \frac{1}{2}$. Suppose that there exists a function $\phi(n)$ such that for all $n \geq 1$*

$$|f_{t_k}'(\lambda_{t_k} + i2^{-n})| \leq \phi(n)$$

where $t_k = k2^{-n/\delta}$, $k = 0, 1, \ldots, \lfloor 2^{n/\delta} \rfloor$. Then there exists a constant $c_4 < \infty$ such that

$$|f_t'(\lambda_t + i2^{-n})| \leq c_4\phi(n)$$

for $t \in [0, 1]$.

Proof. Let $t \in [0,1]$ and $y = 2^{-n}$. Then there exists $k \in \{0, 1, \ldots, \lfloor 2^{n/\delta} \rfloor\}$ and $s \in [0, 2^{-n/\delta}]$ so that $t = t_k + s$. By Lemma 2.4 and our assumption,

$$|f_t'(\lambda_{t_k} + i2^{-n})| \le e^{10} |f_{t_k}'(\lambda_{t_k} + i2^{-n})| \le e^{10}\phi(n).$$

Then by Lemma 2.5 and the previous line,

$$|f_t'(\lambda_t + i2^{-n})| \le c_1 M^\alpha |f_t'(\lambda_{t_k} + i2^{-n})| \le c_2 M^\alpha \phi(n).$$

Now, since $\lambda_t \in \mathrm{Lip}(\delta)$, $|\lambda_{t+s} - \lambda_t| \le c_3 s^\delta$, so we have

$$M = \max\{|\lambda_{t_k+s} - \lambda_{t_k}|/y, 1\} \le \max\{c_3 s^\delta / y, 1\} \le \max\{c_3 2^{n(1-\delta/\delta)}, 1\}.$$

Hence,

$$|f_t'(\lambda_t + i2^{-n})| \le c_4 \phi(n).$$

$\square$

Lemma 2.7. *If $|\hat{f}_t'(i2^{-n})| \le \phi(n)$, then $|\hat{f}_t'(iy)| \le c\phi(n)$ for $2^{-n} \le y \le 2^{-n+1}$.*

Proof. Using the idea from the proof of Lemma 2.3, $\lambda_t + i2^{-n}$ and $\lambda_t + iy$ are in the same Whitney square, with one right above the other, so we can use the distortion theorem in the same way. In particular, $c = 12^2$. $\square$

The above proposition is what we need to fully exploit the bounds on which ever mesh of points we have. Proposition 2.7 along with Lemma 2.6 extend the inequalities from Theorem 2.1 to the set $\{x + iy : |x| \le 1, 0 < y \le 1\}$.

The proof of Theorem 2.1 seems involved enough to necessitate the next Lemma. In the proof of [6, Lemma 4.33], Lawler assumes that $s \le 2^{-2n}$ which is the same as taking $\delta = 1/2$. Hence it seems natural to change our assumptions to fit this situation. In addition, this proof will not work if we do not make that assumption.

Lemma 2.8. *Let $\lambda_t \in \mathrm{Lip}(\delta)$. With the notation that $\hat{f}_t(z) = f_t(\lambda_t + z)$ we have*

$$|\hat{f}_t(i2^{-n}) - \hat{f}_{t+s}(i2^{-n})| \le c2^{-n}|\hat{f}_t'(i2^{-n})| + c2^{-n}|\hat{f}_{t+s}'(i2^{-n})|$$

for some $c > 0$ *and* $s \le 2^{-n/\delta}$.

Proof. The triangle inequality gives

$$|\hat{f}_t(i2^{-n}) - \hat{f}_{t+s}(i2^{-n})| \le |f_t(\lambda_t + i2^{-n}) - f_{t+s}(\lambda_t + i2^{-n})| + |f_{t+s}(\lambda_t + i2^{-n}) - f_{t+s}(\lambda_{t+s} i2^{-n})|.$$

The first term is bounded using Lemma 2.4,

$$|f_t(\lambda_t + i2^{-n}) - f_{t+s}(\lambda_t + i2^{-n})| \le c2^{-n}|\hat{f}_t'(i2^{-n})|.$$

For the second term, notice that

$$|f_{t+s}(\lambda_t + i2^{-n}) - f_{t+s}(\lambda_{t+s} i2^{-n})| = \left| \int_{[\lambda_t + i2^{-n}, \lambda_{t+s} + i2^{-n}]} f_{t+s}'(z) dz \right| \le s^\delta \sup_{t \le r \le t+s} |f_{t+s}'(\lambda_r + i2^{-n})|$$

Then by Lemma 2.5,

$$s^\delta \sup_{t \le r \le t+s} |f_{t+s}'(\lambda_r + i2^{-n})| \le s^\delta c \left(\frac{s^\delta}{2^{-n}} \right)^\alpha |\hat{f}_{t+s}'(i2^{-n})| \le 2^{-n} c |\hat{f}_{t+s}'(i2^{-n})|$$

Where the last inequality follows from our assumption on the range of s. □

Proof of Theorem 2.1. By 2.1, it suffices to bound $|\hat{f}_t(iy) - \hat{f}_{t+s}(iy_1)|$ for $s \le 2^{-n/\delta}$ and $y, y_1 \le 2^{-n}$. Using the triangle inequality we have

$$|\hat{f}_t(iy) - \hat{f}_{t+s}(iy_1)| \le |\hat{f}_t(iy) - \hat{f}_t(i2^{-n})| + |\hat{f}_t(i2^{-n}) - \hat{f}_{t+s}(i2^{-n})| + |\hat{f}_{t+s}(iy_1) - \hat{f}_{t+s}(i2^{-n})|.$$

Focusing on the first term, we use Lemma 2.6 ($\phi(n) = 2^n r_n$) to apply our bound from the hypothesis to all times t and Proposition 2.7 to expand it to all y-values satisfying $2^{-j-1} \le y \le 2^{-j}$. This way

$$|\hat{f}_t(iy) - \hat{f}_t(i2^{-n})| = \left| \int_{[iy, i2^{-n}]} \hat{f}_t'(z) dz \right| \le \sum_{l=n}^{\infty} \int_{[i2^{-l-1}, i2^{-l}]} |\hat{f}_t'(z)| dz \le \sum_{l=n}^{\infty} 2^{-l-1} c 2^l r_l,$$

which goes to zero in n. The third term is bounded similarly, and the middle term is handled by Lemma 2.8. Finally, 2.1 gives us the result: $y, y_1 \to 0$ and $|t - (t+s)| \to 0$ as $n \to \infty$, so we can take

$$\eta(1/n) = |\hat{f}_t(iy) - \hat{f}_t(i2^{-n})| + |\hat{f}_t(i2^{-n}) - \hat{f}_{t+s}(i2^{-n})| + |\hat{f}_{t+s}(iy_1) - \hat{f}_{t+s}(i2^{-n})|.$$

□

Proof of Corollary 2.2. By Theorem 2.1, it suffices to show that

$$|\hat{f}'_{k2^{-n/\delta}}(i2^{-n})| \leq 2^n a_n, \quad k = 0, 1, ..., \lfloor 2^{n/\delta} \rfloor, \quad a.s.$$

for a_n summable. This is equivalent to showing that $\bigcap_{k=0}^{\lfloor 2^{n/\delta}\rfloor} \{|\hat{f}'_{k2^{-n/\delta}}(i2^{-n})| \leq 2^n a_n\}$ occurs all but finitely often. So let us do that: the hypothesis gives us

$$\mathbb{P}\left(\bigcup_{k=0}^{\lfloor 2^{n/\delta}\rfloor} \{|\hat{f}'_t(i2^{-n})| \geq 2^n a_n\}\right) \leq \sum_{k=0}^{\lfloor 2^{n/\delta}\rfloor} \mathbb{P}\left(\{|\hat{f}'_t(i2^{-n})| \geq 2^n a_n\}\right) \leq (\lfloor 2^{n/\delta} \rfloor + 1) 2^{-n/\delta} b_n.$$

Then by the Borel-Cantelli Lemma, $\mathbb{P}\left(\bigcup_{k=0}^{\lfloor 2^{n/\delta}\rfloor} \{|\hat{f}'_t(i2^{-n})| \geq 2^n a_n\} i.o.\right) = 0$ which implies $\bigcap_{k=0}^{\lfloor 2^{n/\delta}\rfloor} \{|\hat{f}'_t(i2^{-n})| \leq 2^n a_n\}$ occurs all but finitely often with probability one. □

Chapter 3

Lipschitz Bounds for the Weierstrass Function

In this chapter we focus on our goal of better understanding the Loewner hulls generated by $cW_3(t) = c\sum_{j=0}^{\infty} 3^{-j/2}\cos(3^j t)$ (the Weierstrass function with $b = 3$). We will accomplish this by improving on the bounds obtained by Glenn in Theorems 1.9 and 1.10. In particular, Glenn's work yields the following theorem about the $b = 3$ case:

Theorem 3.1 ([11],[3]). *When $c < \frac{4\sqrt{3}-4}{3+2\sqrt{3}} \approx .453$, the hull generated by $cW_3(t)$ is a simple curve in $\mathbb{H} \cup \{W_3(0)\}$, and this is not the case when $c \geq 4 \Big/ \left(2\sin^2\left(\frac{1}{2}\right)\frac{1-3^{-15/2}}{3^{3/2}-1} - \frac{3^{-5/2}}{\sqrt{3}-1}\right) \approx 182.709$.*

The bound of 0.453 is obtained from Theorem 1.1 by dividing 4 by Glenn's upper bound on $||W_3||_{1/2}$ (which is approximately 8.83). Hence we will improve on the bound of 0.453 by improving the bound on the Lip(1/2) norm. We accomplish this in the following theorem which will be proved in Section 3.2.

Theorem 3.2 (An upper bound on $||W_3||_{1/2}$). *The Lip(1/2) norm of $W_3(t)$ is less than $\sqrt{3}\sigma \approx 7.228$, for $\sigma = \frac{3+\sqrt{3}}{\sqrt{\pi}} + \frac{2}{\sqrt{\pi}}\sum_{j=0}^{\infty}\sqrt{3}^j \sin\left(\frac{\pi}{(2)3^j}\right)\sin\left(\frac{\pi}{2} - (-1)^j\frac{\pi}{4} + \frac{\pi}{(4)3^j}\right)$.*

As a result, we are able to conclude that $cW_3(t)$ generates a simple curve for all $c < \frac{4}{\sqrt{3}\sigma} \approx .553$.

The bound of 182.709 in Theorem 3.1 comes from Theorem 1.3 and only depends on the existence of a large lower Lip(1/2) bound at some point. In Section 3.1, we give lower

Lip(1/2) bounds at all local extrema of $W_3(t)$. The best such bound comes from Lemma 3.11 and is $\frac{3\sqrt{3}}{2\sqrt{2\pi}} \approx 1.036$. This allows us to conclude that cW_3 does not generate a simple curve for $c > \frac{8\sqrt{2\pi}}{3\sqrt{3}} \approx 3.859$. We note that a bound of 3.859 is a large improvement over Glenn's bound of 182.709.

In summary, the improved bounds on the Weierstrass function, which we obtain in Sections 3.1 and 3.2, yield:

Theorem 3.3. *When $c < \frac{4}{\sqrt{3}\sigma}$, the hull generated by $cW_3(t)$ is a simple curve in $\mathbb{H} \cup \{W_3(0)\}$, and this is not the case when $c \geq \frac{8\sqrt{2\pi}}{3\sqrt{3}}$.*

3.1 Lower Lipschitz Bounds

This section will prove a "lower Lipschitz" bound for the Weierstrass function. For convenience, we will use the following definition.

Definition 3.4. *Define $L_b(t,h) = \frac{W_b(t) - W_b(t+h)}{\sqrt{|h|}}$.*

It is already known ([4],[11] and [3]) that the Weierstrass function exhibits Lip(1/2) behavior. For our purposes their results are too weak in the sense that their lower bound is smaller than optimal and the interval that they consider is too small. We will sharpen their results by proving the following:

Theorem 3.5. *Let $n \in \mathbb{N}$. Then points of the form $\frac{2m\pi}{3^n}$ are local maxes and*

$$L_3\left(\frac{2m\pi}{3^n}, h\right) \geq \begin{cases} \begin{cases} 0.42 & , h \in \left(0, \frac{\pi}{3^n}\right] \\ 0.077 & , h \in \left[-\frac{\pi}{3^{n+1}}, 0\right) \end{cases} & , m \equiv 1 \pmod 3 \\ \begin{cases} 0.42 & , h \in \left[-\frac{\pi}{3^n}, 0\right) \\ 0.077 & , h \in \left(0, \frac{\pi}{3^{n+1}}\right] \end{cases} & , m \equiv -1 \pmod 3. \end{cases}$$

Also, points of the form $\frac{(2m+1)\pi}{3^n}$ *are local minimums and*

$$-L_3\left(\frac{(2m+1)\pi}{3^n}, h\right) \geq \begin{cases} \begin{cases} 0.42 & , h \in \left(0, \frac{\pi}{3^n}\right] \\ 0.077 & , h \in \left[-\frac{\pi}{3^{n+1}}, 0\right) \end{cases} & , m \equiv -1 \pmod 3 \\ \begin{cases} 0.42 & , h \in \left[-\frac{\pi}{3^n}, 0\right) \\ 0.077 & , h \in \left(0, \frac{\pi}{3^{n+1}}\right] \end{cases} & , m \equiv 0 \pmod 3. \end{cases}$$

In order to prove Theorem 3.5, we will focus on showing that $W_3(t)$ has Lip(1/2) behavior at its local maximums and the analogous results for the local minimums will follow easily.

Our first result is used to show that we can extend bounds from a given interval to one which is three times as large (or b times as large).

Lemma 3.6 (Inward Expansion)**.** *Let* $b \in \mathbb{Z}$ *with* $b > 1$*. Then*

$$L_b(0, t) \leq L_b(0, t/b).$$

In particular, if $L \leq L_b(0, t)$ *then,* $L \leq L_b(0, t/b)$*.*

Proof. Noting that $W_b(0) = \frac{\sqrt{b}}{\sqrt{b}-1}$ we have,

$$\begin{aligned} [W_b(0) - W_b(t)] \frac{\sqrt{|t/b|}}{\sqrt{|t|}} &= \left[\frac{\sqrt{b}}{\sqrt{b}-1} - \sum_{n\geq 0}\left(\tfrac{1}{\sqrt{b}}\right)^n \cos(b^n t)\right]\frac{1}{\sqrt{b}} \\ &= \frac{1}{\sqrt{b}-1} - \sum_{n\geq 0}\left(\tfrac{1}{\sqrt{b}}\right)^{n+1} \cos\left(b^{n+1}t/b\right) \\ &= \frac{\sqrt{b}}{\sqrt{b}-1} - \sum_{n\geq 1}\left(\tfrac{1}{\sqrt{b}}\right)^{n} \cos\left(b^{n}t/b\right) - \frac{\sqrt{b}-1}{\sqrt{b}-1} \\ &= \frac{\sqrt{b}}{\sqrt{b}-1} - \sum_{n\geq 0}\left(\tfrac{1}{\sqrt{b}}\right)^{n} \cos\left(b^{n}t/b\right) + (\cos(t/b) - 1) \\ &\leq W_b(0) - W_b(t/b) \end{aligned}$$

Since $\cos(t/b) \leq 1$. ∎

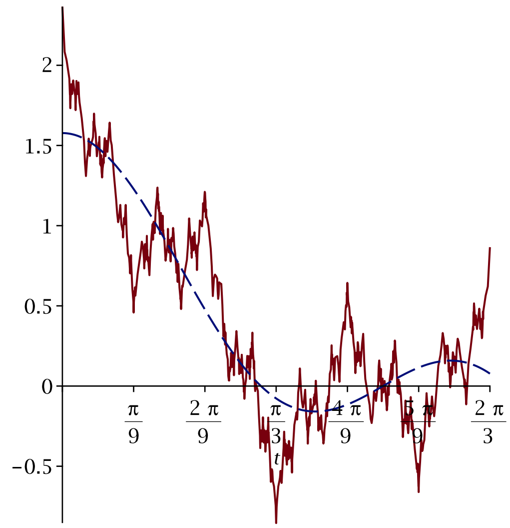


Figure 3.1: solid=$W(t)$, dashed= $\cos(t) + \frac{1}{\sqrt{3}}\cos(3t)$

We will focus on the $b = 3$ case, and so we will drop the subscript and write $W(t) = W_3(t)$ and $L(t, h) = L_3(t, h)$. The idea behind the next two results is to leverage the similarity of the Weierstrass function on different scales. It scales as

$$W(t) = \cos(t) + \frac{1}{\sqrt{3}}W(3t), \tag{3.1}$$

so we can think of $W(t)$ as a scaled down version of itself following the path of the function $\cos(t)$ or $\cos(t)+\frac{1}{\sqrt{3}}\cos(3t)$ as opposed to following the constant 0 function. Notice how each third of the interval of Figure 3.1 looks like a period of $W(t)$ that is following the dashed line.

Lemma 3.7. *(i)* $\cos(t) + \frac{1}{\sqrt{3}}\cos(3t)$ *is decreasing on* $[\pi/6, \pi/3]$.

(ii) $\cos(t) + \frac{1}{\sqrt{3}}\cos(3t) \leq 0$ *on* $\left[\frac{\pi}{3}, \frac{\pi}{2}\right]$

(iii) $\frac{d}{dh}\frac{\cos\left(\frac{2\pi}{3}\right)-\cos\left(\frac{2\pi}{3}+h\right)}{\sqrt{|h|}} = \frac{\sqrt{-h}\sin\left(\frac{2\pi}{3}+h\right)+\left(\cos\left(\frac{2\pi}{3}\right)-\cos\left(\frac{2\pi}{3}+h\right)\right)\frac{1}{2\sqrt{-h}}}{-h}$ *for* $h \in (-\infty, 0)$ *and* $\frac{\cos\left(\frac{2\pi}{3}\right)-\cos\left(\frac{2\pi}{3}+h\right)}{\sqrt{|h|}}$ *is increasing for* $h \in [-\pi/6, 0)$.

Proof. To prove the first part, note that

$$\frac{d}{dt}\left[\cos(t) + \frac{1}{\sqrt{3}}\cos(3t)\right] = -\sin(t) - \sqrt{3}\sin(3t) = -\sin(t) - \sqrt{3}\left[3\sin(t) - 4\sin^3(t)\right]$$

by the triple angle formula. Setting this expression equal to zero and solving for $\sin(t)$, we find that the only critical points between 0 and π are $t = 0, \arcsin\left(\frac{\sqrt{9+\sqrt{3}}}{2\sqrt{3}}\right), \pi + \arcsin\left(-\frac{\sqrt{9+\sqrt{3}}}{2\sqrt{3}}\right)$. We can see that $t = 0 < \pi/6$, but we also find that $\arcsin\left(\frac{\sqrt{9+\sqrt{3}}}{2\sqrt{3}}\right) - \pi/3 \approx 0.192526160$ and $\pi + \arcsin\left(-\frac{\sqrt{9+\sqrt{3}}}{2\sqrt{3}}\right) - \pi/3 \approx 0.854671392$. Hence, none of these points are contained in $[\pi/6, \pi/3]$. Because $-\sin(\pi/3) - \sqrt{3}\sin(\pi) = -\sqrt{3}/2$, we know that $\cos(t) + \frac{1}{\sqrt{3}}\cos(3t)$ is decreasing on $[\pi/6, \pi/3]$.

To prove the second part, note that $\cos(t) + \frac{1}{\sqrt{3}}\cos(3t) = \cos(t) + \frac{1}{\sqrt{3}}\left[4\cos^3(t) - 3\cos(t)\right]$ by the triple angle formula. Setting this expression equal to zero and solving for $\cos(t)$ we find that this expression changes sign when $t = \pi/2, \arccos\left(\pm\frac{\sqrt{3-\sqrt{3}}}{2}\right)$. There is no problem with $\pi/2$ and we find that $\pi/3 - \arccos\left(\frac{\sqrt{3-\sqrt{3}}}{2}\right) \approx 0.0744321709$ and $\arccos\left(-\frac{\sqrt{3-\sqrt{3}}}{2}\right) - \pi/2 \approx 0.5980309469$. Hence none of these points are contained in $[\pi/3, \pi/2]$. Since $\cos(\pi/3) + \frac{1}{\sqrt{3}}\cos(\pi) = \frac{\sqrt{3}-2}{2\sqrt{3}}$, we have $\cos(t) + \frac{1}{\sqrt{3}}\cos(3t) \leq 0$ on $[\pi/3, \pi/2]$.

To prove the final part, note that the formula for the derivative follows from the quotient rule. In order to show that $\frac{\cos\left(\frac{2\pi}{3}\right) - \cos\left(\frac{2\pi}{3}+h\right)}{\sqrt{|h|}}$ is increasing, we will first show that it has no critical points in the interval $[-\pi/6, 0)$.

First note that any such critical point must satisfy

$$\sqrt{-h}\sin\left(\tfrac{2\pi}{3} + h\right) + \left(\cos\left(\tfrac{2\pi}{3}\right) - \cos\left(\tfrac{2\pi}{3} + h\right)\right)\frac{1}{2\sqrt{-h}} = 0$$

or equivalently,

$$-2h\sin\left(\tfrac{2\pi}{3} + h\right) + \cos\left(\tfrac{2\pi}{3}\right) - \cos\left(\tfrac{2\pi}{3} + h\right) = 0.$$

Since $h = 0$ yields 0, it will be enough to show that $-2h\sin\left(\frac{2\pi}{3} + h\right) + \cos\left(\frac{2\pi}{3}\right) - \cos\left(\frac{2\pi}{3} + h\right)$ is strictly decreasing on $[-\pi/6, 0]$. Indeed,

$$\frac{d}{dh}\left[-2h\sin\left(\tfrac{2\pi}{3} + h\right) + \cos\left(\tfrac{2\pi}{3}\right) - \cos\left(\tfrac{2\pi}{3} + h\right)\right] = -\sin\left(\tfrac{2\pi}{3} + h\right) - 2h\cos\left(\tfrac{2\pi}{3} + h\right) < 0$$

for $h \in [-\pi/6, 0]$. Hence, $\frac{\cos\left(\frac{2\pi}{3}\right) - \cos\left(\frac{2\pi}{3}+h\right)}{\sqrt{|h|}}$ has no critical points in the interval $[-\pi/6, 0)$.

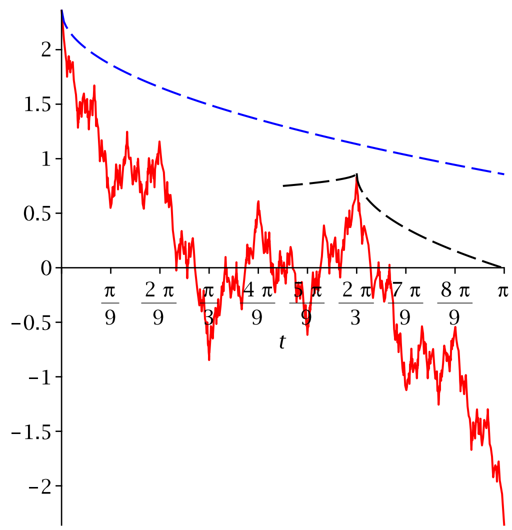


Figure 3.2: solid=$W(t)$, dashed=weak bounds at $t = 0$ and $t = \frac{2\pi}{3}$

Finally,

$$\frac{d}{dh} \frac{\cos\left(\frac{2\pi}{3}\right) - \cos\left(\frac{2\pi}{3} + h\right)}{\sqrt{|h|}}\Bigg|_{h=-\frac{\pi}{6}} \approx 0.7221313776$$

which gives the conclusion.

$\square$

The next few results will show our weak bound: $W(t) \leq -\frac{6-\sqrt{3}}{2\sqrt{2\pi}}\sqrt{t} + W(0)$ for $t \in [0, \pi]$. Just as in Figure 3.2, we will show that the bound holds out to $\pi/2$, then we will scale the bound around $\frac{2\pi}{3}$ to extend the range of the bound around 0 out to $\frac{2\pi}{3}$. Then we can let our bounds at 0 and $\frac{2\pi}{3}$ push each other out to π. We start with the linchpin of showing our bound out to $\pi/2$.

Lemma 3.8. *$W(t) \leq -\frac{6-\sqrt{3}}{2\sqrt{2\pi}}\sqrt{t} + W(0)$ for $t \in [0, \pi/2]$.*

Proof. The case for $t = 0$ is clear and the case for $t \in (0, \pi/6]$ will follow from the case for $t \in [\pi/6, \pi/2]$ by using Lemma 3.6 to extend our result to $[\frac{\pi}{2\cdot 3^n}, \frac{\pi}{2\cdot 3^{n-1}}]$ for all n.

So first let $t \in [\pi/6, 2\pi/9]$. Since $\cos(t) + \frac{1}{\sqrt{3}}\cos(3t)$ is decreasing here by Lemma 3.7(i), we know that

$$W(t) = \cos(t) + \tfrac{1}{\sqrt{3}}\cos(3t) + \tfrac{1}{3}W(9t)$$

$$
\begin{aligned}
&\leq \cos(\pi/6) + \tfrac{1}{\sqrt{3}}\cos(\pi/2) + \tfrac{1}{3}W(0) \\
&= \tfrac{1}{2}\left(\sqrt{3} + 1 + \tfrac{1}{\sqrt{3}}\right) \\
&= -\tfrac{1}{2}\left(3 + \sqrt{3} - \left(\sqrt{3} + 1 + \tfrac{1}{\sqrt{3}}\right)\right) + W(0) \\
&= -\frac{6-\sqrt{3}}{2\sqrt{2\pi}}\sqrt{\frac{2\pi}{9}} + W(0) \\
&\leq -\frac{6-\sqrt{3}}{2\sqrt{2\pi}}\sqrt{t} + W(0).
\end{aligned}
$$

Next let $t \in [2\pi/9, \pi/3]$. Again, since $\cos(t) + \frac{1}{\sqrt{3}}\cos(3t)$ is decreasing here by Lemma 3.7(i), we know that

$$
\begin{aligned}
W(t) &\leq \cos(2\pi/9) + \tfrac{1}{\sqrt{3}}\cos(2\pi/3) + \tfrac{1}{3}W(0) \\
&= \cos(2\pi/9) + \tfrac{1}{2} \\
&\leq \cos(\pi/6) + \tfrac{1}{2} \\
&\leq -\frac{6-\sqrt{3}}{2\sqrt{2\pi}}\sqrt{\frac{\pi}{3}} + W(0) \\
&\leq -\frac{6-\sqrt{3}}{2\sqrt{2\pi}}\sqrt{t} + W(0).
\end{aligned}
$$

Finally, let $t \in [\pi/3, \pi/2]$. Since $\cos(t) + \frac{1}{\sqrt{3}}\cos(3t) \leq 0$ by Lemma 3.7(ii), we know that

$$
\begin{aligned}
W(t) &\leq 0 + \tfrac{1}{3}W(0) \\
&\leq -\frac{6-\sqrt{3}}{2\sqrt{2\pi}}\sqrt{\frac{\pi}{2}} + W(0) \\
&\leq -\frac{6-\sqrt{3}}{2\sqrt{2\pi}}\sqrt{t} + W(0).
\end{aligned}
$$

□

In order to extend this bound to $[\frac{\pi}{2}, \frac{2\pi}{3}]$ we will scale our result from 3.8 at the point $t = \frac{2\pi}{3}$ in the direction of $t = \frac{\pi}{2}$. Lemma 3.7(iii) shows that our original weak bound may not be small enough when considering points to the left of $t = \frac{2\pi}{3}$, since $\cos(t)$ is decreasing near $\frac{2\pi}{3}$. As such, we will need to compensate for this.

Lemma 3.9. *$W(\frac{2\pi}{3}+h) \leq -\left(\frac{6-\sqrt{3}}{2\sqrt{2\pi}} + \frac{\cos\left(\frac{2\pi}{3}\right)-\cos\left(\frac{\pi}{2}\right)}{\sqrt{\frac{\pi}{6}}}\right)\sqrt{|h|} + W\left(\frac{2\pi}{3}\right)$, $h \in [-\pi/6, 0]$. Furthermore, $\frac{6-\sqrt{3}}{2\sqrt{2\pi}} + \frac{\cos\left(\frac{2\pi}{3}\right)-\cos\left(\frac{\pi}{2}\right)}{\sqrt{\frac{\pi}{6}}} > 0$, and $W(t) \leq W\left(\frac{2\pi}{3}\right)$ for $t \in [\frac{\pi}{2}, \frac{2\pi}{3}]$.*

Proof. Let $h \in [-\pi/6, 0]$. Then by scaling at $\frac{2\pi}{3}$, Lemma 3.8, and Lemma 3.7(iii) we have

$$\begin{aligned}
L\left(\frac{2\pi}{3}, h\right) &= \frac{\frac{1}{\sqrt{3}}W(2\pi) - \frac{1}{\sqrt{3}}W(2\pi+3h)}{\sqrt{|h|}} + \frac{\cos\left(\frac{2\pi}{3}\right) - \cos\left(\frac{2\pi}{3}+h\right)}{\sqrt{|h|}} \\
&\geq \frac{6-\sqrt{3}}{2\sqrt{2\pi}} + \frac{\cos\left(\frac{2\pi}{3}\right) - \cos\left(\frac{2\pi}{3}+h\right)}{\sqrt{|h|}} \\
&\geq \frac{6-\sqrt{3}}{2\sqrt{2\pi}} + \frac{\cos\left(\frac{2\pi}{3}\right) - \cos\left(\frac{\pi}{2}\right)}{\sqrt{\frac{\pi}{6}}} \\
&\approx 0.1603443922.
\end{aligned}$$

Therefore $-\left(\frac{6-\sqrt{3}}{2\sqrt{2\pi}} + \frac{\cos\left(\frac{2\pi}{3}\right)-\cos\left(\frac{\pi}{2}\right)}{\sqrt{\frac{\pi}{6}}}\right)\sqrt{|h|} + W\left(\frac{2\pi}{3}\right)$ is increasing in h, and in particular, $W(t) \leq W\left(\frac{2\pi}{3}\right)$ for $t \in [\frac{\pi}{2}, \frac{2\pi}{3}]$. □

Now we can show our weak bound up to $t = \frac{2\pi}{3}$. We'll get our bound up to π by scaling the current scope of our weak bound around $t = \frac{2\pi}{3}$ and showing that our scaled bound is less than our goal bound. By induction we'll get the entire interval.

Lemma 3.10 (Weak Bound)**.** *$W(t) \leq -\frac{6-\sqrt{3}}{2\sqrt{2\pi}}\sqrt{t} + W(0)$ for $t \in [0, \pi]$.*

Proof. Lemma 3.8 gives the result for $t \in [0, \pi/2]$, and the case for $t \in [\pi/2, 2\pi/3]$ follows from Lemma 3.9 since $-\frac{6-\sqrt{3}}{2\sqrt{2\pi}}\sqrt{\frac{2\pi}{3}} + W(0) - W(\frac{2\pi}{3})(\approx 0.267949193) > 0$.

We will finish by showing that

$$W(t) \leq -\frac{6-\sqrt{3}}{2\sqrt{2\pi}}\sqrt{t} + W(0) \text{ for } t \in \left[0, \sum_{j=1}^{k} \frac{2\pi}{3^j}\right] \tag{3.2}$$

, for all $k \geq 1$. Assume, for proof by induction, that there exists a $k \in \mathbb{N}$ such that equation 3.2 holds. Then by scaling our induction hypothesis about the point $t = \frac{2\pi}{3}$ we get that for $h \in \left[0, \frac{1}{3}\sum_{j=1}^{k} \frac{2\pi}{3^j}\right]$,

$$L\left(\frac{2\pi}{3}, h\right) = \frac{\frac{1}{\sqrt{3}}W(2\pi) - \frac{1}{\sqrt{3}}W(2\pi+3h)}{\sqrt{|h|}} + \frac{\cos\left(\frac{2\pi}{3}\right) - \cos\left(\frac{2\pi}{3}+h\right)}{\sqrt{|h|}} \geq \frac{6-\sqrt{3}}{2\sqrt{2\pi}},$$

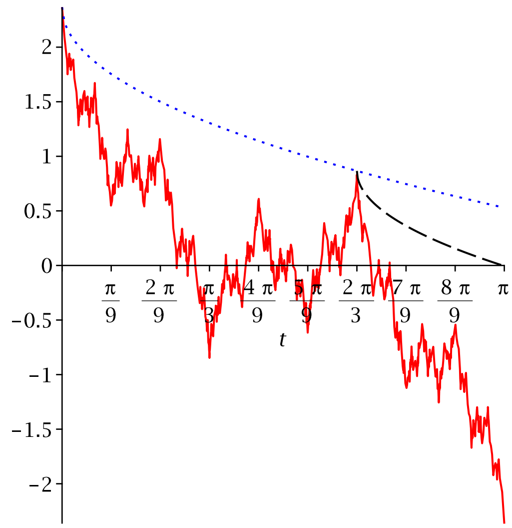


Figure 3.3: solid=$W(t)$, dashed=weak bound, dot=sharp bound

since $\frac{\cos\left(\frac{2\pi}{3}\right)-\cos\left(\frac{2\pi}{3}+h\right)}{\sqrt{|h|}} \geq 0$. This means that $W\left(\frac{2\pi}{3}+h\right) \leq -\frac{6-\sqrt{3}}{2\sqrt{2\pi}}\sqrt{h} + W\left(\frac{2\pi}{3}\right)$ for $h \in \left[0, \frac{1}{3}\sum_{j=1}^{k}\frac{2\pi}{3^j}\right]$. In addition, for all $h > 0$

$$\begin{aligned}
&\frac{d}{dh}\left[-\frac{6-\sqrt{3}}{2\sqrt{2\pi}}\sqrt{\frac{2\pi}{3}+h} + W(0) - \left(-\frac{6-\sqrt{3}}{2\sqrt{2\pi}}\sqrt{h} + W\left(\frac{2\pi}{3}\right)\right)\right] \\
&= \frac{6-\sqrt{3}}{4\sqrt{2\pi}}\left(-\frac{1}{\sqrt{\frac{2\pi}{3}+h}} + \frac{1}{\sqrt{h}}\right) > 0
\end{aligned}$$

which implies that $-\frac{6-\sqrt{3}}{2\sqrt{2\pi}}\sqrt{\frac{2\pi}{3}+h} + W(0) > -\frac{6-\sqrt{3}}{2\sqrt{2\pi}}\sqrt{h} + W\left(\frac{2\pi}{3}\right)$ (since we know that this inequality holds for $h = 0$). Hence,

$$W\left(\frac{2\pi}{3}+h\right) \leq -\frac{6-\sqrt{3}}{2\sqrt{2\pi}}\sqrt{h} + W\left(\frac{2\pi}{3}\right) \leq -\frac{6-\sqrt{3}}{2\sqrt{2\pi}}\sqrt{\frac{2\pi}{3}+h} + W(0),$$

or rather, $W(t) \leq -\frac{6-\sqrt{3}}{2\sqrt{2\pi}}\sqrt{t} + W(0)$ for $t \in \left[0, \sum_{j=1}^{k+1}\frac{2\pi}{3^j}\right]$. Therefore, the result holds for all $k \in \mathbb{N}$ and hence for all $t \in [0, \pi]$. □

The work we did with the weak bound will do most of the heavy lifting for us. It allows us to get the bound on $[\frac{2\pi}{3}, \pi]$ very easily. Notice how in Figure 3.3, the dotted line remains above the dashed line out to π. This will let us use Lemma 3.6 on the interval $[\pi/3, \pi]$ instead of the more difficult interval $\left[\frac{2\pi}{9}, \frac{2\pi}{3}\right]$.

Lemma 3.11 (Sharp Bound at 0)**.** $W(t) \leq -\frac{3\sqrt{3}}{2\sqrt{2\pi}}\sqrt{t} + W(0)$ *for* $t \in [0, \pi]$*. This result is sharp in the sense that* $\frac{3\sqrt{3}}{2\sqrt{2\pi}}$ *is the largest coefficient that will work on this interval.*

Proof. By Lemma 3.6, it will suffice to show that the inequality holds on the interval $\left[\frac{\pi}{3}, \pi\right]$. For $t \in \left[\frac{\pi}{3}, \frac{\pi}{2}\right]$, we know that $\cos(t) + \frac{1}{\sqrt{3}}\cos(3t) \leq 0$, so $W(t) \leq \frac{1}{3}W(0) \leq W\left(\frac{2\pi}{3}\right)$. For $t \in \left[\frac{\pi}{2}, \frac{2\pi}{3}\right]$ Lemma 3.9 gives $W(t) \leq W\left(\frac{2\pi}{3}\right)$. Since $-\frac{3\sqrt{3}}{2\sqrt{2\pi}}\sqrt{t} + W(0)$ is decreasing and equal to $W(t)$ at $t = \frac{2\pi}{3}$ (see Figure 3.3), we know that $W(t) \leq -\frac{3\sqrt{3}}{2\sqrt{2\pi}}\sqrt{t} + W(0)$ for $t \in \left[\frac{\pi}{3}, \frac{2\pi}{3}\right]$.

Now we'll handle the case for $t \in \left[\frac{2\pi}{3}, \pi\right]$ by comparing our sharp bound to our scaled weak bound (See Figure 3.3). Note that for $h = 0$

$$-\frac{3\sqrt{3}}{2\sqrt{2\pi}}\sqrt{\frac{2\pi}{3} + h} + W(0) = -\frac{6-\sqrt{3}}{2\sqrt{2\pi}}\sqrt{h} + W\left(\frac{2\pi}{3}\right),$$

and

$$\begin{aligned}&\frac{d}{dh}\left[-\frac{3\sqrt{3}}{2\sqrt{2\pi}}\sqrt{\frac{2\pi}{3} + h} + W(0) - \left(-\frac{6-\sqrt{3}}{2\sqrt{2\pi}}\sqrt{h} + W\left(\frac{2\pi}{3}\right)\right)\right]\\ &= -\frac{3\sqrt{3}}{4\sqrt{2\pi}}\frac{1}{\sqrt{\frac{2\pi}{3} + h}} + \frac{6-\sqrt{3}}{4\sqrt{2\pi}}\frac{1}{\sqrt{h}}.\end{aligned}$$

This is positive if and only if

$$0 < h < \frac{\pi(13 - 4\sqrt{3})}{6(\sqrt{3}-1)} \approx 4.342847954.$$

This means that our weak bound scaled down at $\frac{2\pi}{3}$ decreases faster than our unscaled sharp bound from $\frac{2\pi}{3}$ to (beyond) π. Combining our weak bound with this comparison to our sharp bound we find that $W(t) \leq -\frac{6-\sqrt{3}}{2\sqrt{2\pi}}\sqrt{t - \frac{2\pi}{3}} + W\left(\frac{2\pi}{3}\right) \leq -\frac{3\sqrt{3}}{2\sqrt{2\pi}}\sqrt{t} + W(0)$ for $t \in \left[\frac{2\pi}{3}, \pi\right]$. $\square$

This sharp bound at 0 will function like a base case to help us find a lower Lipschitz bound for any local max. In general, these maxes behave like a scaled down version of $W(t)$ plus a sum of cosine terms. Sharpness in the base case will give us some wiggle room when trying to handle the additional cosine terms, however these additional cosine terms will limit the general results that we can show. Specifically, we need to weaken the bound itself and

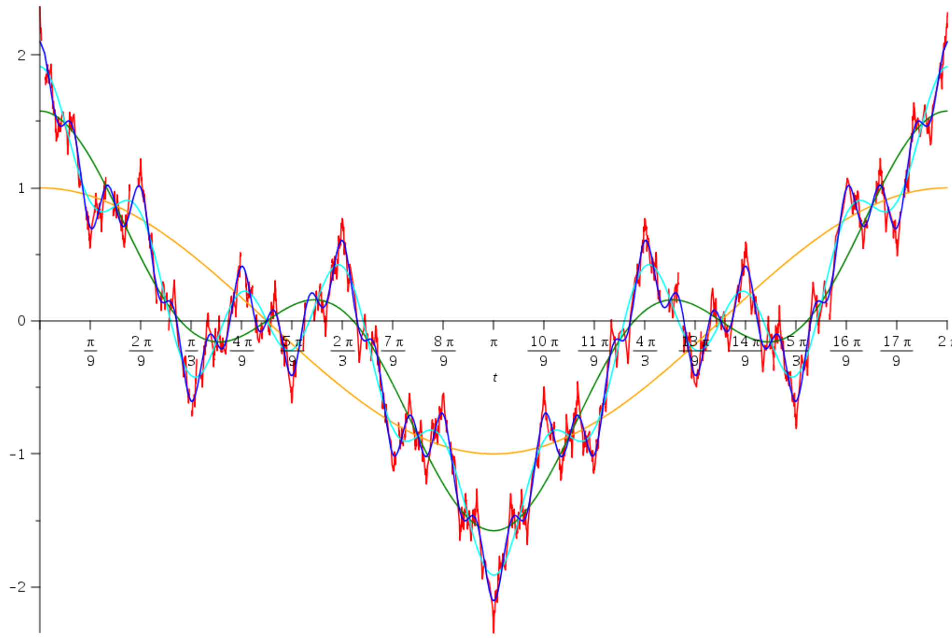


Figure 3.4: $W(t)$, $W^0(t)$, $W^1(t)$, $W^2(t)$, and $W^3(t)$

we need to determine the direction the bound will hold. Indeed, Figure 3.4 indicates that maxes have a "preferred direction." This can be accounted for by the increasing or decreasing nature of the additional cosine terms.

Definition 3.12. *Let $W^n(t)$ be the partial sum $W^n(t) = \sum_{j=0}^{n} 3^{-n/2} \cos(3^j t)$.*

From the definition of $W(t)$, we can see that $W(t) = W^0(t) + \frac{1}{\sqrt{3}} W(3t) = W^1(t) + \frac{1}{3} W(9t)$. At points of the form $\frac{2m\pi}{3^n}$ we can see that $W\left(\frac{2m\pi}{3^n}\right) = W^{n-1}\left(\frac{2m\pi}{3^n}\right) + \frac{1}{\sqrt{3}^n} W(0)$, and at points of the form $\frac{(2m+1)\pi}{3^n}$ we have that $W\left(\frac{(2m+1)\pi}{3^n}\right) = W^{n-1}\left(\frac{(2m+1)\pi}{3^n}\right) - \frac{1}{\sqrt{3}^n} W(0)$. Of particular interest is that around points like $\frac{2\pi}{3}$, $W(t)$ acts like a scaled down version of itself added to $\cos(t)$, giving a "full period" from $\pi/3$ to π (see Figure 3.4). With this in mind, we will show a one-sided (in the direction in which the first $n-1$ cosine terms are decreasing) analogue of our base case in the ball centered at $\frac{2m\pi}{3^n}$ with radius $\frac{\pi}{3^n}$. This means thinking of $W(t)$ as $W^{n-1}(t) + \frac{1}{3^{-n/2}} W(3^n t)$ and using our base case to handle the scaled part.

Lemma 3.13. *Let $m \equiv 1 \pmod 3$ and $n \in \mathbb{N}$. Then in general $W\left(\frac{2m\pi}{3^n} + h\right) \leq -0.42\sqrt{h} + W\left(\frac{2m\pi}{3^n}\right)$ for $h \in \left[0, \frac{\pi}{3^n}\right]$. Further, when $W^{n-1}\left(\frac{2m\pi}{3^n}\right) > W^{n-1}\left(\frac{2m\pi}{3^n} + \frac{2\pi}{3^{n+1}}\right)$, we can use the same Lip(1/2) bound from Lemma 3.11.*

Figure 3.5: The interval $\left[\frac{86\pi}{243}, \frac{260\pi}{729}\right]$ with $W(t)$ (red) compared to $\frac{1}{\sqrt{243}}W(243t) + W^4(\frac{86\pi}{243})$ (cyan), and $W^4(t)$ (green) compared to $W^4(\frac{86\pi}{243})$ (blue). In general, when the green curve stays below the blue curve, then our base case bound can be used in place of the number -0.42 from Lemma 3.13. This image illustrates a value of m where our base case bound will not suffice for a local maximum which satisfies the hypotheses of Lemma 3.13.

Proof. We will start by showing that $W^{n-1}(t)$ has its absolute maximum over the interval $\left[\frac{2m\pi}{3^n}, \frac{2m\pi}{3^n} + \frac{2\pi}{3^{n+1}}\right]$ at one of the endpoints. Since $W^{n-1}(t)$ is smooth, then the max can only occur at one of the endpoints or at a point where $\frac{d}{dt}W^{n-1}(t) = 0$. We will show that $W^{n-1}(t)$ is concave up on $\left[\frac{2m\pi}{3^n}, \frac{2m\pi}{3^n} + \frac{2\pi}{3^{n+1}}\right]$, so that by the second derivative test, any point where $\frac{d}{dt}W^{n-1}(t) = 0$ will correspond to a local minimum (and hence not a global maximum).

Let $m \equiv 1 \pmod 3$ and $t \in \left[\frac{2m\pi}{3^n}, \frac{2m\pi}{3^n} + \frac{2\pi}{3^{n+1}}\right]$. Then

$$\begin{aligned}
\frac{d^2}{dt^2}W^{n-1}(t) &= -\sum_{j=0}^{n-1}(3\sqrt{3})^j \cos(3^j t) \\
&\geq -\sum_{j=0}^{n-2}(3\sqrt{3})^j - (3\sqrt{3})^{n-1}\cos\left(\frac{2m\pi}{3}\right) \\
&= -\frac{(3\sqrt{3})^{n-1} - 1}{3\sqrt{3} - 1} - (3\sqrt{3})^{n-1}\cos\left(\frac{2\pi}{3}\right) \\
&\geq (3\sqrt{3})^{n-1}\left(\frac{-1}{3\sqrt{3} - 1} + \frac{1}{2}\right) > 0.
\end{aligned}$$

Hence $W^{n-1}(t)$ is concave up on $\left[\frac{2m\pi}{3^n}, \frac{2m\pi}{3^n} + \frac{2\pi}{3^{n+1}}\right]$, and its maximum occurs at one of the endpoints.

If the max occurs at $\frac{2m\pi}{3^n}$, then $W(t) = W^{n-1}(t) + [W(t) - W^{n-1}(t)] \leq W^{n-1}\left(\frac{2m\pi}{3^n}\right) + [W(t) - W^{n-1}(t)]$. That is, the Weierstrass function is less than or equal to a scaled version of itself and the lower Lipschitz bound is at least $\frac{3\sqrt{3}}{2\sqrt{2\pi}}$ as in the sharp bound at 0. But since this doesn't always happen, we must find a universal bound.

In the case that the max of W^{n-1} occurs at $t = \frac{2m\pi}{3^n} + \frac{2\pi}{3^{n+1}}$, it will be sufficient to show that our lower Lipschitz bound, 0.42, holds at this endpoint. This is because $W\left(\frac{2m\pi}{3^n} + h\right) \leq -\frac{3\sqrt{3}}{2\sqrt{2\pi}}\sqrt{h} + W\left(\frac{2m\pi}{3^n}\right) + \left[W^{n-1}\left(\frac{2m\pi}{3^n} + h\right) - W^{n-1}\left(\frac{2m\pi}{3^n}\right)\right]$ and $-\frac{3\sqrt{3}}{2\sqrt{2\pi}}\sqrt{h} + W\left(\frac{2m\pi}{3^n}\right) + \left[W^{n-1}\left(\frac{2m\pi}{3^n} + h\right) - W^{n-1}\left(\frac{2m\pi}{3^n}\right)\right] \leq L\left(\frac{2m\pi}{3^n}, \frac{2\pi}{3^{n+1}}\right)\sqrt{h} + W\left(\frac{2m\pi}{3^n}\right)$, where it will be shown that $L\left(\frac{2m\pi}{3^n}, \frac{2\pi}{3^{n+1}}\right) \geq 0.42$. Indeed when we carry out the calculation, we get:

$$\begin{aligned}
&\sqrt{\tfrac{3^{n+1}}{2\pi}}\left[W\left(\tfrac{2m\pi}{3^n}\right) - W\left(\tfrac{2m\pi}{3^n} + \tfrac{2\pi}{3^{n+1}}\right)\right] \\
&= \sqrt{\tfrac{3^{n+1}}{2\pi}} \sum_{j=0}^{n-1} \left(\tfrac{1}{\sqrt{3}}\right)^j \left[\cos\left(3^j \tfrac{2m\pi}{3^n}\right) - \cos\left(3^j \tfrac{2m\pi}{3^n} + 3^j \tfrac{2\pi}{3^{n+1}}\right)\right] + \tfrac{3\sqrt{3}}{2\sqrt{2\pi}},
\end{aligned}$$

since the sharp bound at 0 shows up in the tail of the series. But then the $(n-1)^{st}$ term can be determined exactly, and the other terms can use the bound $\cos\left(3^j \frac{2m\pi}{3^n}\right) - \cos\left(3^j \frac{2m\pi}{3^n} + 3^j \frac{2\pi}{3^{n+1}}\right) \geq -1 \cdot |3^j \frac{2m\pi}{3^n} - 3^j\left(\frac{2m\pi}{3^n} + \frac{2\pi}{3^{n+1}}\right)|$ so that we can get

$$\begin{aligned}
L\left(\frac{2m\pi}{3^n}, \frac{2\pi}{3^{n+1}}\right) &\geq \sqrt{\tfrac{3^{n+1}}{2\pi}} \sum_{j=0}^{n-2} \left(\tfrac{1}{\sqrt{3}}\right)^j \left[-3^j \tfrac{2\pi}{3^{n+1}}\right] + \sqrt{\tfrac{9}{2\pi}}\left[\cos\left(\tfrac{2m\pi}{3}\right) - \cos\left(\tfrac{2m\pi}{3} + \tfrac{2\pi}{9}\right)\right] + \tfrac{3\sqrt{3}}{2\sqrt{2\pi}} \\
&= -\sqrt{\tfrac{2\pi}{3^{n+1}}} \sum_{j=0}^{n-2} \sqrt{3}^j + \tfrac{3}{\sqrt{2\pi}}\left[\cos\left(\tfrac{2\pi}{3}\right) - \cos\left(\tfrac{8\pi}{9}\right)\right] + \tfrac{3\sqrt{3}}{2\sqrt{2\pi}} \\
&= -\sqrt{\tfrac{2\pi}{3^{n+1}}}\, \tfrac{\sqrt{3}^{n-1} - 1}{\sqrt{3} - 1} + \tfrac{3}{\sqrt{2\pi}}\left[-\tfrac{1}{2} - \cos\left(\tfrac{8\pi}{9}\right)\right] + \tfrac{3\sqrt{3}}{2\sqrt{2\pi}} \\
&\geq -\tfrac{\sqrt{2\pi}}{3}\tfrac{1}{\sqrt{3}-1} + \tfrac{3}{\sqrt{2\pi}}\left[-\tfrac{1}{2} - \cos\left(\tfrac{8\pi}{9}\right)\right] + \tfrac{3\sqrt{3}}{2\sqrt{2\pi}} \approx 0.4213457452 > 0.
\end{aligned}$$

Now we will expand the interval on which this bound holds the same way we did in the proof of Lemma 3.10. Let $m = 3q + 1$ for some $q \in \mathbb{Z}$. Then $\frac{2(3q+1)\pi}{3^l} + \frac{2\pi}{3^{l+1}} = \frac{2(9q+3+1)\pi}{3^{l+1}}$, and $9q + 3 + 1 \equiv 1 \pmod 3$ which implies that we can use the domain on the bound that we have so far to show that we can extend the domain of our bound to $\left[\frac{2m\pi}{3^n}, \frac{2m\pi}{3^n} + \frac{2\pi}{3^n}\sum_{j=1}^{N}\frac{1}{3^j}\right]$ for all $N > 0$ by using an induction argument. $\square$

With the hard part done, we easily get the same bound for $k \equiv -1 \pmod 3$ but in the opposite direction (to the left of the max). We stop to prove it now so that we can use it in the proof of Lemma 3.15.

Lemma 3.14. *Let $m \equiv -1 \pmod 3$. Then for all $n \in \mathbb{N}$, $W\left(\frac{2m\pi}{3^n} + h\right) \geq 0.42\sqrt{|h|} + W\left(\frac{2m\pi}{3^n}\right)$ for $h \in \left[-\frac{\pi}{3^n}, 0\right]$.*

Proof. Since $\cos(2\pi - t) = \cos(2\pi)\cos(t) + \sin(2\pi)\sin(t) = \cos(t)$, which implies $W(t) = W(2\pi - t)$. If we let $m = 3q - 1$, then notice that $2\pi - \frac{2(3q-1)\pi}{3^n} = \frac{2(3^n-3q+1)\pi}{3^n}$ which puts us in the case where $n \equiv 1 \pmod 3$. Hence, $L\left(\frac{2m\pi}{3^n}, -h\right) = L\left(\frac{2(3^n-3q+1)\pi}{3^n}, h\right) \geq 0.42$ for $h \in \left[-\frac{\pi}{3^n}, 0\right)$ by Corollary 3.13. $\square$

The next result gives us control to the left of the local maximums of Corollary 3.13. However, the domain of control will be smaller, so we will compare it to $W(t) - W^n(t)$ instead of $W(t) - W^{n-1}(t)$ like we did before.

Lemma 3.15. *Let $m \equiv 1 \pmod 3$ and $n \in \mathbb{N}$. Then $W\left(\frac{2m\pi}{3^n} + h\right) \leq -0.077\sqrt{h} + W\left(\frac{2m\pi}{3^n}\right)$ for $h \in \left[-\frac{\pi}{3^{n+1}}, 0\right]$.*

Proof. Just like the last bound, we will show this result on two thirds of the interval and then we will use induction to get the rest. So let $t \in \left[\frac{2m\pi}{3^n} - \frac{2\pi}{3^{n+2}}, \frac{2m\pi}{3^n}\right]$. First we will bound $W^n(t) - W^n\left(\frac{2m\pi}{3^n}\right)$ so that we can get an upper bound on the amount $W^n(t)$ "pushes up" on $[W(t) - W^n(t)]$. Let's bound the first and second derivatives:

$$\begin{aligned}\frac{d}{dt}W^n\left(\frac{2m\pi}{3^n}\right) &= -\sum_{j=0}^{n}\sqrt{3}^j\sin\left(3^j\frac{2m\pi}{3^n}\right) = -\sum_{j=0}^{n-1}\sqrt{3}^j\sin\left(3^j\frac{2m\pi}{3^n}\right)\\ &\geq -\sum_{j=0}^{n-1}\sqrt{3}^j = -\frac{\sqrt{3}^n - 1}{\sqrt{3}-1} \geq -\frac{\sqrt{3}^n}{\sqrt{3}-1}\end{aligned}$$

and

$$\frac{d^2}{dt^2}W^n(t) = -\sum_{j=0}^{n}(3\sqrt{3})^j\cos\left(3^j t\right)$$

$$\begin{aligned}
&\leq -\sum_{j=0}^{n-2}(3\sqrt{3})^j(-1) - (3\sqrt{3})^{n-1}\cos\left(\frac{2\pi}{3}\right) - (3\sqrt{3})^n\cos\left(-\frac{2\pi}{9}\right) \\
&\leq \frac{(3\sqrt{3})^{n-1}}{3\sqrt{3}-1} + \frac{(3\sqrt{3})^{n-1}}{2} - (3\sqrt{3})^n\cos\left(-\frac{2\pi}{9}\right) \leq (-0.6239)\,(3\sqrt{3})^n.
\end{aligned}$$

Then we can bound the Taylor series expansion by a second degree polynomial, we see that for $t \in \left[\frac{2m\pi}{3^n} - \frac{2\pi}{3^{n+2}}, \frac{2m\pi}{3^n}\right]$,

$$W^n(t) - W^n\left(\frac{2m\pi}{3^n}\right) \leq -\frac{\sqrt{3}^n}{\sqrt{3}-1}\left(t - \frac{2m\pi}{3^n}\right) + \frac{1}{2}\,(-0.6239)\,(3\sqrt{3})^n\left(t - \frac{2m\pi}{3^n}\right)^2. \quad (3.3)$$

Now we will show that $-0.077\sqrt{\frac{2m\pi}{3^n} - t} + W\left(\frac{2m\pi}{3^n}\right) - W(t) \geq 0$. First note that $W(t) = \frac{1}{\sqrt{3}^{n+1}}W\left(3^{n+1}t\right) - \frac{1}{\sqrt{3}^{n+1}}W\left(6k\pi\right) + W\left(\frac{2k\pi}{3^n}\right) + W^n(t) - W^n\left(\frac{2k\pi}{3^n}\right)$. Then using equation 3.3 and Lemma 3.11,

$$W(t) \leq \frac{1}{\sqrt{3}^{n+1}}\frac{3\sqrt{3}}{2\sqrt{2\pi}}\sqrt{3^{n+1}\left(\frac{2k\pi}{3^n} - t\right)} + W\left(\frac{2k\pi}{3^n}\right) + \frac{\sqrt{3}^n}{\sqrt{3}-1}\left(\frac{2k\pi}{3^n} - t\right) + \frac{1}{2}\left(\frac{2-3\sqrt{3}}{3\sqrt{3}-1}\right)(3\sqrt{3})^n\left(\frac{2k\pi}{3^n} - t\right)^2$$

for $\left(\frac{2m\pi}{3^n} - t\right) \in \left[0, \frac{2\pi}{3^{n+2}}\right]$. Now we can show that $-0.077\sqrt{-t + \frac{2m\pi}{3^n}} + W\left(\frac{2m\pi}{3^n}\right) - W(t) \geq 0$ by replacing the $W(t)$ term by the larger expression we just found. Then this is equivalent to showing that

$$\frac{1}{\sqrt{3}}^n\left[\left(\frac{3\sqrt{3}}{2\sqrt{2\pi}} - 0.077\right)\sqrt{3^n\left(\frac{2m\pi}{3^n} - t\right)} - \frac{1}{\sqrt{3}-1}3^n\left(\frac{2m\pi}{3^n} - t\right) - \frac{1}{2}\,(-0.6239)\left(3^n\left(\frac{2m\pi}{3^n} - t\right)\right)^2\right] \geq 0$$

for $y = 3^n\left(\frac{2m\pi}{3^n} - t\right) \in \left[0, \frac{2\pi}{9}\right]$ which is the form we will actually work with. Then

$$\frac{d}{dy}\left[\left(\frac{3\sqrt{3}}{2\sqrt{2\pi}} - 0.077\right)\sqrt{y} - \frac{y}{\sqrt{3}-1} - \frac{1}{2}\,(-0.6239)\,(y)^2\right] = \frac{\frac{3\sqrt{3}}{4\sqrt{2\pi}} - \frac{0.077}{2}}{\sqrt{y}} - \frac{1}{\sqrt{3}-1} - (-0.6239)\,y.$$

This has a critical point when $y = 0$ and otherwise when

$$\left(\frac{3\sqrt{3}}{4\sqrt{2\pi}} - \frac{0.077}{2}\right) - \frac{\sqrt{y}}{\sqrt{3}-1} - (-0.6239)\,\sqrt{y}^3 = 0.$$

We will show that if some value of $\sqrt{y} \in \left[0, \sqrt{\frac{2\pi}{9}}\right]$ corresponds to a critical point, then it corresponds to a local maximum.

If we let $x = \sqrt{y}$ then it is apparent that we are trying to find the zeros of a cubic with a positive leading coefficient. Once we show that this cubic has two roots which go from negative to positive outside of the interval we care about, then by the first derivative test the other root (it exists) will correspond to a local maximum. Let $f(x) = \left(\frac{3\sqrt{3}}{4\sqrt{2\pi}} - \frac{0.077}{2}\right) - \frac{x}{\sqrt{3}-1} - (-0.6239)\, x^3$. Then $f(-1.5) \approx -0.04191242480 < 0$, $f(-1) \approx 1.084080182 > 0$, $f(1) \approx -0.1245977334 < 0$ and $f(1.5) \approx 1.001394873 > 0$ so the two roots which correspond to local minimums are outside of our interval.

So the minimum of $\left(\frac{3\sqrt{3}}{2\sqrt{2\pi}} - 0.077\right)\sqrt{y} - \frac{y}{\sqrt{3}-1} - \frac{1}{2}(-0.6239)(y)^2$ occurs at one of its endpoints. $y = 0$ gives 0, and $y = \frac{2\pi}{9}$ gives approximately 0.0000636193, which is positive. Hence, $W(t)$ stays below $-0.077\sqrt{\frac{2m\pi}{3^n} - t} + W\left(\frac{2m\pi}{3^n}\right)$ for $t \in \left[\frac{2m\pi}{3^n} - \frac{2\pi}{3^{n+2}}, \frac{2m\pi}{3^n}\right]$.

We extend the range of the bound using Lemma 3.14 at our end point since $\frac{2m\pi}{3^n} - \frac{2\pi}{3^{n+2}} = \frac{2(9k-1)\pi}{3^{n+2}}$ and $9k - 1 \equiv -1 \pmod 3$. An argument like the one that appears in the proof of Lemma 3.10 result shows that the extended domain shares the same bound. □

Now we will show the equivalent bound for the other maximums, the proof of which is the same as Lemma 3.14.

Lemma 3.16. *Let $m \equiv -1 \pmod 3$. Then for all $n \in \mathbb{N}$, $W\left(\frac{2m\pi}{3^n} + h\right) \leq -0.077\sqrt{|h|} + W\left(\frac{2m\pi}{3^n}\right)$ for $h \in \left[0, \frac{\pi}{3^{n+1}}\right]$.*

Now that we have our bounds for the local maximums, we can easily get the equivalent bounds for the local minimums by exploiting the fact that $W(t) = -W(\pi - t)$.

Proof of Theorem 3.5. Lemmas 3.13, 3.14, 3.15, and 3.16 take care of the points $\frac{2m\pi}{3^n}$. For the points $\frac{(2m+1)\pi}{3^n}$, note that $\cos(\pi - t) = -\cos(t)$ which implies $W(t) = -W(\pi - t)$. Then

$$\frac{-W\left(\frac{(2m+1)\pi}{3^n}\right) + W\left(\frac{(2m+1)\pi}{3^n} + h\right)}{\sqrt{|h|}} = \frac{W\left(\frac{(3^n-2m-1)\pi}{3^n}\right) - W\left(\frac{(3^n-2m-1)\pi}{3^n} - h\right)}{\sqrt{|-h|}}.$$

Since $3^n - 2m - 1$ is even, we are now at a local maximum, and we can use one of the above Lemmas. □

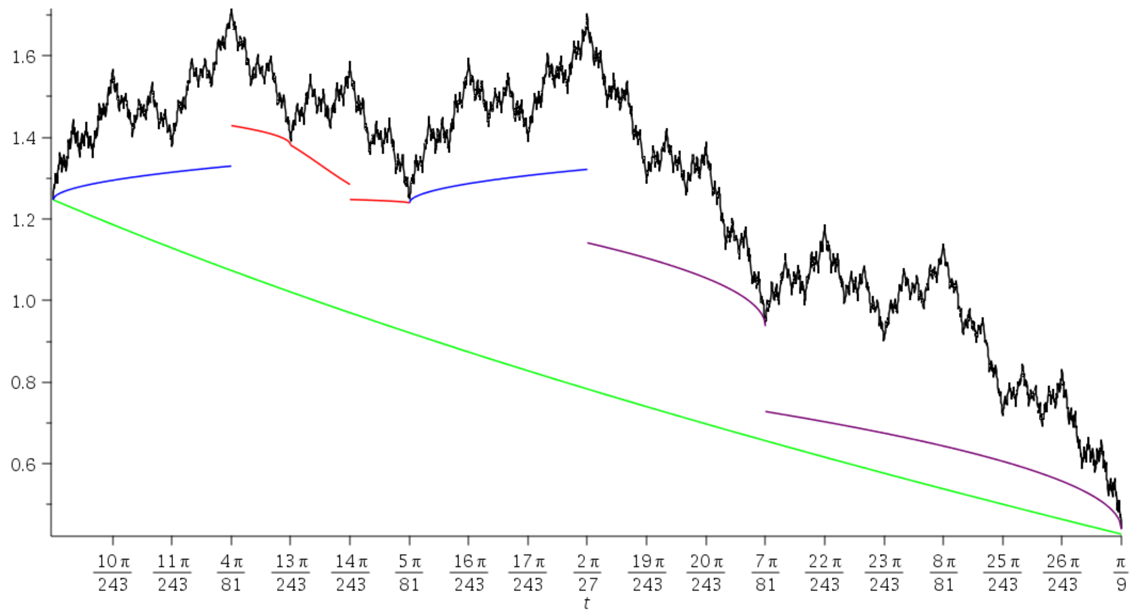


Figure 3.6: The interval $\left[\frac{\pi}{27}, \frac{\pi}{9}\right]$ with $W(t)$ shown in black, $-\sigma_2\sqrt{t}+W(0)$ shown in green, the bounds from Lemma 3.19 shown in red, the bounds from Lemma 3.20 shown in purple, and simple bounds shown in blue.

3.2 Upper Lipschitz Bounds

First we will find an upper Lip(1/2) bound at 0. The values $L\left(0, \frac{\pi}{3^n}\right)$ approach this bound.

Definition 3.17. *For ease of notation, let $\sigma_n = L\left(0, \frac{\pi}{3^{n+1}}\right)$ for $n \geq 1$. By Lemma 3.6, we know that σ_n is increasing. Define $\sigma_0 = \lim_{n\to\infty} \sigma_n$.*

We will show that $\sigma_n \geq L(0,t)$ for $t \geq \frac{\pi}{3^{n+1}}$, i.e., σ_n describes the upper Lip(1/2) between 0 and $t \geq \frac{\pi}{3^{n+1}}$. Just like the proof of the base case for the lower Lip-1/2 norm, the base case for the upper Lip-1/2 norm will be done in several cases. One key difference is that we cannot leverage the Lemma 3.6 to get a bound on $[0,\pi]$ from a bound on $[\pi/3,\pi]$, for instance, so we will have to cover all of $[0,\pi]$ at once using a limit argument.

The proof approach is illustrated in Figure 3.6. That is, we will show that the green curve lies below the red curves, the purple curves and the blue curves. That the curves in question lie below $W(t)$ follows from Theorem 3.5. The next result will be used multiple times.

Lemma 3.18. *Let* $\frac{a\pi}{3^{n+3}}, \frac{b\pi}{3^{n+3}} \in \left[\frac{\pi}{3^{n+1}}, \frac{\pi}{3^n}\right]$ *and* $t_0 \geq 0$*. Then for* $n \geq 1$ *and* $R \in \mathbb{R}$*,*

$$\sqrt{3}^{n+3} \left\{ R\sqrt{t_0} + W\left(\frac{b\pi}{3^{n+3}}\right) - \left[-\sigma_n \sqrt{\frac{a\pi}{3^{n+3}}} + W(0) \right] \right\}$$
$$\geq \left(\sqrt{a} + \tfrac{(-1)^b - 1}{2}\right)(3+\sqrt{3}) + R\sqrt{3^{n+3}t_0} + 2\sqrt{a}\sum_{j=1}^{2} \sqrt{3}^j \sin^2\left(\tfrac{\pi}{2(3)^j}\right) - 2\sum_{j=1}^{\infty} \sqrt{3}^j \sin^2\left(\tfrac{b\pi}{2(3)^j}\right).$$

Proof. First we group together $W(0)$ and $W\left(\frac{b\pi}{3^{n+3}}\right)$ and cancel the denominator in σ_n with $\sqrt{\frac{a\pi}{3^{n+3}}}$:

$$\begin{aligned} &R\sqrt{t_0} + W\left(\frac{b\pi}{3^{n+3}}\right) - \left[-\sigma_n \sqrt{\frac{a\pi}{3^{n+3}}} + W(0) \right] \\ &= R\sqrt{t_0} + \sqrt{\frac{a}{3^2}}\left[W(0) - W\left(\frac{\pi}{3^{n+1}}\right)\right] - \left[W(0) - W\left(\frac{b\pi}{3^{n+3}}\right)\right]. \end{aligned}$$

Then we write out W in its series expansion and combine the grouped differences into a single series:

$$= R\sqrt{t_0} + \sqrt{\frac{a}{3^2}} \sum_{j=0}^{\infty} \sqrt{3}^{-j}\left[\cos(0) - \cos\left(\frac{\pi}{3^{n+1-j}}\right)\right] - \sum_{j=0}^{\infty} \sqrt{3}^{-j}\left[\cos(0) - \cos\left(\frac{b\pi}{3^{n+3-j}}\right)\right].$$

Now we factor out $\frac{1}{\sqrt{3}^{n+3}}$, compute the tail of each series, and use the half angle formula on the remaining difference of cosine terms:

$$\begin{aligned} &= \tfrac{1}{\sqrt{3}^{n+3}} \Bigg\{ R\sqrt{3^{n+3}t_0} + \sqrt{a}\left[\sum_{j=0}^{n} \sqrt{3}^{(n+1-j)} \sin^2\left(\frac{\pi}{2(3)^{n+1-j}}\right) + 3 + \sqrt{3}\right] \\ &- \sum_{j=0}^{n+2} \sqrt{3}^{n+3-j} \sin^2\left(\frac{b\pi}{2(3)^{n+3-j}}\right) - \tfrac{(-1)^b - 1}{2}(3+\sqrt{3}) \Bigg\}. \end{aligned}$$

Finally, we reverse the order of each sum and bound from above by using the number of terms in each sum:

$$\begin{aligned} &= \tfrac{1}{\sqrt{3}^{n+3}} \Bigg\{ \left(\sqrt{a} + \tfrac{(-1)^b - 1}{2}\right)(3+\sqrt{3}) + R\sqrt{3^{n+3}t_0} \\ &+ 2\sqrt{a}\sum_{j=1}^{n+1} \sqrt{3}^j \sin^2\left(\tfrac{\pi}{2(3)^j}\right) - 2\sum_{j=0}^{n+3} \sqrt{3}^j \sin\left(\tfrac{b\pi}{2(3)^j}\right) \Bigg\} \end{aligned}$$

$$\geq \frac{1}{\sqrt{3}^{n+3}} \left\{ \left(\sqrt{a} + \frac{(-1)^b - 1}{2}\right)(3+\sqrt{3}) + R\sqrt{3^{n+3}t_0} \right.$$
$$\left. +2\sqrt{a}\sum_{j=1}^{2} \sqrt{3}^j \sin^2\left(\frac{\pi}{2(3)^j}\right) - 2\sum_{j=0}^{\infty} \sqrt{3}^j \sin\left(\frac{b\pi}{2(3)^j}\right) \right\}.$$

□

With Lemma 3.18, we can easily show the bounds in Figure 3.6. The next lemma will show that the red curves in Figure 3.6 are above the green curve.

Lemma 3.19. *Let $n \geq 1$. Then the following inequalities hold:*

(i) $W(t) \geq 0.42\sqrt{\frac{13\pi}{3^{n+3}} - t} + W\left(\frac{13\pi}{3^{n+3}}\right) \geq -\sigma_n\sqrt{t} + W(0)$ *for* $t \in \left[\frac{4\pi}{3^{n+2}}, \frac{13\pi}{3^{n+3}}\right]$,

(ii) $W(t) \geq W^{n+2}(t) - \frac{1}{\sqrt{3}^{n+3}}W(0) \geq -\sigma_n\sqrt{t} + W(0)$ *for* $t \in \left[\frac{13\pi}{3^{n+3}}, \frac{14\pi}{3^{n+3}}\right]$,

(iii) $W(t) \geq 0.077\sqrt{\frac{5\pi}{3^{n+2}} - t} + W\left(\frac{5\pi}{3^{n+2}}\right) \geq -\sigma_n\sqrt{t} + W(0)$ *for* $t \in \left[\frac{14\pi}{3^{n+3}}, \frac{5\pi}{3^{n+2}}\right]$.

Proof. Parts (i) and (iii) are similar. The first inequality in each part follows from Theorem 3.5. The second inequality in each part will follow from an application of Lemma 3.18 showing that the minimum of the left expression is greater than the maximum of the right expression.

For part (i) we have $a = 12$, $b = 13$, $R = 0.42$ and $t_0 = 0$ which yields that

$$\min \mathrm{LHS} - \max \mathrm{RHS} = 0.42\sqrt{0} + W\left(\frac{13\pi}{3^{n+3}}\right) - \left(-\sigma_n\sqrt{\frac{4\pi}{3^{n+2}}} + W(0)\right) \geq \frac{1}{\sqrt{3^{n+3}}}(4.6139143) > 0.$$

For part (iii), $a = 14$, $b = 15$, $R = .077$ and $t_0 = 0$ which yields that

$$\min \mathrm{LHS} - \max \mathrm{RHS} = 0.077\sqrt{0} + W\left(\frac{15\pi}{3^{n+3}}\right) - \left(-\sigma_n\sqrt{\frac{14\pi}{3^{n+3}}} + W(0)\right) \geq \frac{1}{\sqrt{3^{n+3}}}(3.9857801) > 0.$$

Part (ii) is slightly different. The first inequality is clear since it follows from bounding $\cos(3^j t) \geq -1$ for $j \geq n+3$. The second inequality will follow as in the preceding parts by showing that the minimum of the left side is greater than the maximum of the right side. In order to find the minimum of the left side, we must first show that $W_{n+2}(t)$ is decreasing:

$$\frac{d}{dt}W^{n+2}(t) = -\sum_{j=0}^{n+2} \sqrt{3}^j \sin(3^j t) \leq -\sum_{j=0}^{n} \sqrt{3}^j \sin(3^j t) - \sqrt{3}^{n+1}\left[-1 + \frac{3}{2}\right]$$

since $3^{n+2}t \in \left[\frac{13\pi}{3}, \frac{14\pi}{3}\right]$ and the minimum of sin on this interval is $\frac{\sqrt{3}}{2}$. But we also know that $3^j \left[\frac{13\pi}{3^{n+3}}, \frac{14\pi}{3^{n+3}}\right] \subset [0, \pi]$ for $j \leq n$. Hence

$$= -\sum_{j=0}^{n} \sqrt{3}^j \sin(3^j t) - \sqrt{3}^{n+1} \left[-1 + \frac{3}{2}\right] \leq -\sqrt{3}^{n+1} \left[-1 + \frac{3}{2}\right] \leq 0.$$

Since we have $W^{n+2}\left(\frac{14\pi}{3^{n+3}}\right) = W\left(\frac{14\pi}{3^{n+3}}\right) - \frac{1}{\sqrt{3}^{n+3}} W(0)$, then we can use Lemma 3.18 with $a = 13$, $b = 14$, $R = -\frac{2}{\sqrt{3}^{n+3}} W(0)$, and $t_0 = 1$ to get that

$$\begin{aligned}
\min \mathrm{LHS} - \max \mathrm{RHS} &= W^{n+2}\left(\tfrac{14\pi}{3^{n+3}}\right) - \tfrac{1}{\sqrt{3}^{n+3}} W(0) - \left[\sigma_n \sqrt{\tfrac{13\pi}{3^{n+3}}} + W(0)\right] \\
&= W\left(\tfrac{14\pi}{3^{n+3}}\right) - \tfrac{2}{\sqrt{3}^{n+3}} W(0) - \left[\sigma_n \sqrt{\tfrac{13\pi}{3^{n+3}}} + W(0)\right] \\
&\geq \tfrac{1}{\sqrt{3}^{n+3}} (3.9190349) > 0.
\end{aligned}$$

□

Now we will show that the purple curves in Figure 3.6 are above the green curve.

Lemma 3.20. *Let $n \geq 1$. Then the following inequalities hold:*

(i) $W(t) \geq \frac{3\sqrt{3}}{2\sqrt{2\pi}} \sqrt{\frac{\pi}{3^n} - t} + W\left(\frac{\pi}{3^n}\right) \geq -\sigma_n \sqrt{t} + W(0)$ *for* $t \in \left[\frac{7\pi}{3^{n+2}}, \frac{\pi}{3^n}\right]$,

(ii) $W(t) \geq \frac{3\sqrt{3}}{2\sqrt{2\pi}} \sqrt{\frac{7\pi}{3^{n+2}} - t} + W\left(\frac{7\pi}{3^{n+2}}\right) \geq -\sigma_n \sqrt{t} + W(0)$ *for* $t \in \left[\frac{2\pi}{3^{n+1}}, \frac{7\pi}{3^{n+2}}\right]$.

Proof. First we must justify the fact that we can use the sharper bound, $\frac{3\sqrt{3}}{2\sqrt{2\pi}}$, from Lemma 3.13 (and by extension, Theorem 3.5). For the first part, we must show that the minimum of $W_{n-1}(t)$ over $\left[\frac{7\pi}{3^{n+2}}, \frac{\pi}{3^n}\right]$ occurs at $t = \frac{\pi}{3^n}$. Indeed, $W'_{n-1}(t) = -\sum_{j=0}^{n-1} \sqrt{3}^j \sin(3^j t) \leq 0$ since $\left[3^j \frac{7\pi}{3^{n+2}}, 3^j \frac{\pi}{3^n}\right] \subset [0, \pi]$ for all $j \leq n-1$.

For the second part, we will show that the minimum of $W_{n+1}(t)$ over $\left[\frac{2\pi}{3^{n+1}}, \frac{7\pi}{3^{n+2}}\right]$ occurs at $t = \frac{7\pi}{3^{n+2}}$. Indeed, $W'_{n+1}(t) = -\sum_{j=0}^{n+1} \sqrt{3}^j \sin(3^j t) \leq 0$ since $\left[3^j \frac{2\pi}{3^{n+1}}, 3^j \frac{7\pi}{3^{n+2}}\right] \subset [0, \pi]$ for all $j \leq n$ and since $\sin(t) \geq 0$ for $t \in \left[2\pi, \frac{7\pi}{3}\right]$. Hence the first inequality in both parts follows from Theorem 3.5 using sharper the Lip(1/2) bound noted in Lemma 3.13.

In both parts, the second inequality will follow from the fact that the difference of the left-hand side and the right-hand side is concave down. So, if we can show that the difference at

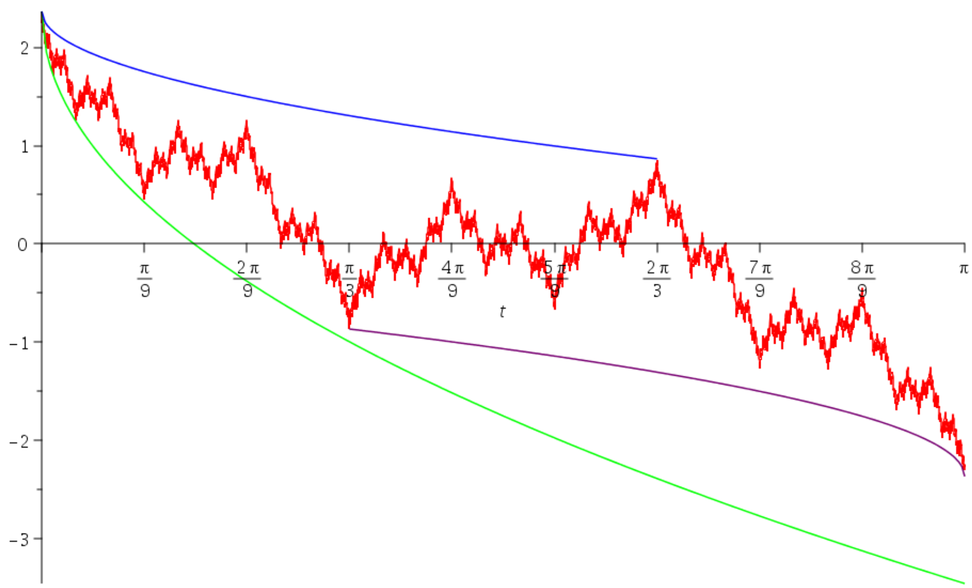


Figure 3.7: The lower Lip(1/2) bound of Theorem 3.11 is shown in blue, and by symmetry it gives the purple bounding curve on $\left[\frac{\pi}{3}, \pi\right]$, which is greater than $-\sigma_0\sqrt{t} + W(0)$ shown in green.

both endpoints (for both parts) is positive, then the difference at every point will be positive as well.

For (i) at $t = \frac{\pi}{3^n}$, the inequality holds by Lemma 3.6. At $t = \frac{7\pi}{3^{n+2}}$ we can use Lemma 3.18 with $a = b = 21$, $R = \frac{3\sqrt{3}}{2\sqrt{2\pi}}$ and $3^{n+3}t_0 = 6\pi$ to get that

$$\text{RHS} - \text{LHS} \geq \frac{1}{\sqrt{3}^{n+3}}(8.5782552) > 0.$$

For (ii) at $t = \frac{7\pi}{3^{n+2}}$, the inequality holds because it holds for (i), which is stronger. At $t = \frac{2\pi}{3^{n+1}}$ we use Lemma 3.18 with $a = b = 18$, $R = \frac{3\sqrt{3}}{2\sqrt{2\pi}}$ and $3^{n+3}t_0 = 3\pi$ to get that

$$\text{RHS} - \text{LHS} \geq \frac{1}{\sqrt{3}^{n+3}}(17.2796282) > 0.$$

$\square$

Now we can combine Lemmas 3.19 and 3.20 to show that σ_0 acts as a local Lip(1/2) norm at 0.

Lemma 3.21 (Lip(1/2) norm at 0)**.** *$W(t) \geq -\sigma_0\sqrt{|t|} + W(0)$ for all $t \in \mathbb{R}$.*

Proof. It is sufficient to show the result for $t \geq 0$ since $W(-t) = W(t)$. Also, it is sufficient to show the result on $[0, \pi]$ since $-\sigma_0\sqrt{|t|} + W(0)$ is decreasing and $\min_{t\in\mathbb{R}} W(t) = W(\pi)$. The strategy is to use our previous results to show the inequality on $\left[\frac{\pi}{3^{n+1}}, \frac{\pi}{3^n}\right]$ for all $n \geq 1$.

For $n = 0$ we will use Theorem 3.11 at π as shown by the purple curve in Figure 3.7 to see that $W(t) \leq -\frac{3\sqrt{3}}{2\sqrt{2\pi}}\sqrt{-t}+W(0)$ for $t \in \left[-\frac{2\pi}{3}, 0\right]$. This implies that $-W(t) = W(\pi - t) \leq -\frac{3\sqrt{3}}{2\sqrt{2\pi}}\sqrt{\pi - t} + W(0)$ for $t \in \left[\frac{\pi}{3}, \pi\right]$ which in turn means that $W(t) \geq \frac{3\sqrt{3}}{2\sqrt{2\pi}}\sqrt{\pi - t} + W(\pi)$ for $t \in \left[\frac{\pi}{3}, \pi\right]$. We also know that $-\sigma_0\sqrt{t} + W(0) \leq -\sigma_1\sqrt{t} + W(0) \leq \frac{3\sqrt{3}}{2\sqrt{2\pi}}\sqrt{\pi - t} + W(\pi)$. The second inequality follows since $\frac{3\sqrt{3}}{2\sqrt{2\pi}}\sqrt{\pi - t} + W(\pi)$ is concave down, $-\sigma_1\sqrt{t} + W(0)$ is concave up, they are equal at $\frac{\pi}{3}$, and $-\sigma_1\sqrt{\pi} + W(0) \leq W(\pi)$ (see Figure 3.7).

For $n \geq 1$, we can use Lemmas 3.19 and 3.20 to get the result on $\left[\frac{\pi}{3^{n+1}}, \frac{\pi}{3^n}\right] \setminus \left(\left[\frac{\pi}{3^{n+1}}, \frac{4\pi}{3^{n+2}}\right) \cup \left(\frac{5\pi}{3^{n+2}}, \frac{2\pi}{3^{n+1}}\right)\right)$, or everything *not* covered by the blue lines in Figure 3.6. For $t \in \left[\frac{\pi}{3^{n+1}}, \frac{4\pi}{3^{n+2}}\right)$, we know that $W(t) \geq 0.117\sqrt{t - \frac{\pi}{3^{n+1}}} + W\left(\frac{\pi}{3^{n+1}}\right)$ and for $t \in \left(\frac{5\pi}{3^{n+2}}, \frac{2\pi}{3^{n+1}}\right)$, we know $W(t) \geq 0.117\sqrt{t - \frac{5\pi}{3^{n+2}}} + W\left(\frac{5\pi}{3^{n+2}}\right)$ by using their lower Lipschitz bounds. Because $-\sigma_0\sqrt{|t|} + W(0)$ is decreasing, our lower Lipschitz bounds are increasing, and the inequality holds for $t = \frac{\pi}{3^{n+1}}$ and $\frac{5\pi}{3^{n+2}}$, then $W(t) \geq -\sigma_0\sqrt{|t|}+W(0)$ holds wherever the lower Lipschitz bound holds. Hence $W(t) \geq -\sigma_0\sqrt{|t|} + W(0)$ for all $t \in [0, \pi]$. $\square$

Now we will show a local Lip(1/2) norm for all local maximums. First let us identify a particular sequence of local maximums:

$$c_n := \frac{\pi}{2} - (-1)^n\frac{\pi}{4} - \frac{\pi}{4 \cdot 3^n}. \tag{3.4}$$

This sequence has the property that $3c_n \equiv c_{n-1} \pmod{2\pi}$ (even for negative values of n) and $c_{n+1} = \frac{c_n}{3}$ when $n > 0$ is odd and $c_{n+1} = \frac{c_n}{3} + \frac{2\pi}{3}$ when $n \geq 0$ is even. We will show that $L(\frac{2m\pi}{3^n}, h) \leq \sigma := \frac{3+\sqrt{3}}{\sqrt{\pi}} + \frac{2}{\sqrt{\pi}}\sum_{j=1}^{\infty}\sqrt{3}^j \sin\left(\frac{\pi}{2\cdot 3^j}\right)\sin\left(c_j + \frac{\pi}{2\cdot 3^j}\right)$ for $|h| \leq \frac{3\pi}{3^n}$. The value of σ is approximately 4.173, while $\sigma_0 \approx 3.285$. In addition, σ is computed in a similar manner to σ_0.

Proposition 3.22. $\sigma = \lim\limits_{n\to\infty} L(c_n, \frac{\pi}{3^n})$

Proof. First we write $W(t)$ as a series and notice that for $j \geq n$, $\cos(c_{n-j}) - \cos\left(c_{n-j} + \frac{\pi}{3^{n-j}}\right) = 2$. So we get

$$\begin{aligned}
&\lim_{n\to\infty} \frac{W(c_n) - W(c_n + \frac{\pi}{3^n})}{\sqrt{\frac{\pi}{3^n}}} \\
&= \lim_{n\to\infty} \sqrt{\frac{3^n}{\pi}} \frac{1}{\sqrt{3}^n}(3+\sqrt{3}) + \sqrt{\frac{3^n}{\pi}} \sum_{j=0}^{n-1} \sqrt{3}^{-j} \left[\cos(c_{n-j}) - \cos\left(c_{n-j} + \frac{\pi}{3^{n-j}}\right)\right] \\
&= \frac{3+\sqrt{3}}{\sqrt{\pi}} + \lim_{n\to\infty} \frac{2}{\sqrt{\pi}} \sum_{j=0}^{n-1} \sqrt{3}^{n-j} \sin\left(\frac{\pi}{(2)3^{n-j}}\right) \sin\left(c_{n-j} + \frac{\pi}{(2)3^{n-j}}\right) \\
&= \frac{3+\sqrt{3}}{\sqrt{\pi}} + \lim_{n\to\infty} \frac{2}{\sqrt{\pi}} \sum_{j=1}^{n} \sqrt{3}^{j} \sin\left(\frac{\pi}{(2)3^{j}}\right) \sin\left(c_{j} + \frac{\pi}{(2)3^{j}}\right)
\end{aligned}$$

where we used the fact that $\cos(y) - \cos(x) = -2\sin(\frac{x+y}{2})\sin(\frac{x-y}{2})$ and we made the substitution $n - j_1 = j_2$ so that we can more easily express the limit. $\square$

Interestingly, we get the same value if we take the limit as $n \to \infty$ of $L(c_n, \frac{\pi}{3^{n+1}})$ because we consider only the sine of the angles rather than the angles themselves. The next result can be used to show that $L(c_n, \frac{\pi}{3^{n+k}}) \leq ||W||_{1/2}$, but it also shows that the maximums c_n have different behavior (toward the right) from the behavior 0 exhibits in Lemma 3.6.

Lemma 3.23 (Inward Contraction)**.** *For $n \geq 1$ and $t \in \left[0, \frac{\pi}{3^{n+1}}\right]$, we have that*

$$\frac{W(c_n) - W(c_n + t)}{\sqrt{t}} \geq \frac{W(c_n) - W\left(c_n + \frac{t}{3}\right)}{\sqrt{t/3}}$$

Proof. We proceed by induction on n. Using the scaling property of the Weierstrass function

$$\begin{aligned}
&\frac{W\left(\frac{2\pi}{3}\right) - W\left(\frac{2\pi}{3} + t\right)}{\sqrt{t}} - \frac{W\left(\frac{2\pi}{3}\right) - W\left(\frac{2\pi}{3} + t/3\right)}{\sqrt{t/3}} \\
&= \frac{W(0) - W(3t)}{\sqrt{3t}} + \frac{\cos\left(\frac{2\pi}{3}\right) - \cos\left(\frac{2\pi}{3} + t\right)}{\sqrt{t}} - \frac{W(0) - W(t)}{\sqrt{t}} - \frac{\cos\left(\frac{2\pi}{3}\right) - \cos\left(\frac{2\pi}{3} + t/3\right)}{\sqrt{t/3}} \\
&= t^{-1/2}\left[\cos(t) - 1 + \sqrt{3}\left(\frac{1}{2} + \cos\left(\frac{2\pi}{3} + t/3\right)\right) - \frac{1}{2} - \cos\left(\frac{2\pi}{3} + t\right)\right].
\end{aligned}$$

This is nonnegative since $\cos(t) - 1 + \sqrt{3}\left(\frac{1}{2} + \cos\left(\frac{2\pi}{3} + t/3\right)\right) - \frac{1}{2} - \cos\left(\frac{2\pi}{3} + t\right) \geq 0$ for $t = 0, \frac{\pi}{9}$ and its second derivative

$$-\frac{\sqrt{3}}{9}\cos\left(\frac{2\pi}{3} + t/3\right) + \cos\left(\frac{2\pi}{3} + t\right) - \cos(t) < 0$$

for $t \in \left[0, \frac{\pi}{9}\right]$.

Now assume the result is true for $n = k-1$. Then by using the scaling property and the induction hypothesis,

$$\begin{aligned}
&\frac{W(c_k) - W(c_k + t)}{\sqrt{t}} - \frac{W(c_k) - W(c_k + t/3)}{\sqrt{t/3}} \\
&= \frac{W(c_{k-1}) - W(c_{k-1} + 3t)}{\sqrt{3t}} - \frac{W(c_{k-1}) - W(c_{k-1} + t)}{\sqrt{t}} \\
&+ \frac{\cos(c_k) - \cos(c_k + t)}{\sqrt{t}} - \frac{\cos(c_k) - \cos(c_k + t/3)}{\sqrt{t/3}} \\
&\geq t^{-1/2}\left[\sqrt{3}\left(-\cos(c_k) + \cos(c_k + t/3)\right) + \cos(c_k) - \cos(c_k + t)\right]
\end{aligned}$$

To show that this is positive, we will optimize $f(x,t) = \sqrt{3}\left(-\cos(x) + \cos(x + t/3)\right) + \cos(x) - \cos(x + t)$ over the set $\left[\frac{2\pi}{9}, \frac{3\pi}{4}\right] \times \left[0, \frac{\pi}{9}\right]$. We have that

$$f_t(x,t) = \sin(x+t) - \frac{1}{\sqrt{3}}\sin(x + t/3).$$

Since $f_t \neq 0$ on this set, we only have to check when $t = 0$. Indeed, $f(x,0) = 0$ for all x. Hence $f(x,t) \geq 0$, and the result holds for all $n \geq 1$. $\square$

The next few bounds along with Lemma 3.23 will show that $\frac{W(c_n) - W(c_n + t)}{\sqrt{t}}$ is largest for $t \geq \frac{\pi}{3^{n+1}}$.

Lemma 3.24. *Let $\frac{a\pi}{3^{n+4}}, \frac{b\pi}{3^{n+4}} \in \left[c_n + \frac{\pi}{3^{n+2}}, c_n + \frac{\pi}{3^{n+1}}\right]$ and $t_0 \geq 0$. Then for $n \geq 1$, $R \in \mathbb{R}$, and $s_n = L(c_n, \frac{\pi}{3^{n+1}})$ we have*

$$\sqrt{3}^{n+4}\left\{R\sqrt{t_0} + W\left(\frac{b\pi}{3^{n+4}}\right) - \left[s_n\sqrt{\frac{a\pi}{3^{n+4}}} + W(0)\right]\right\}$$

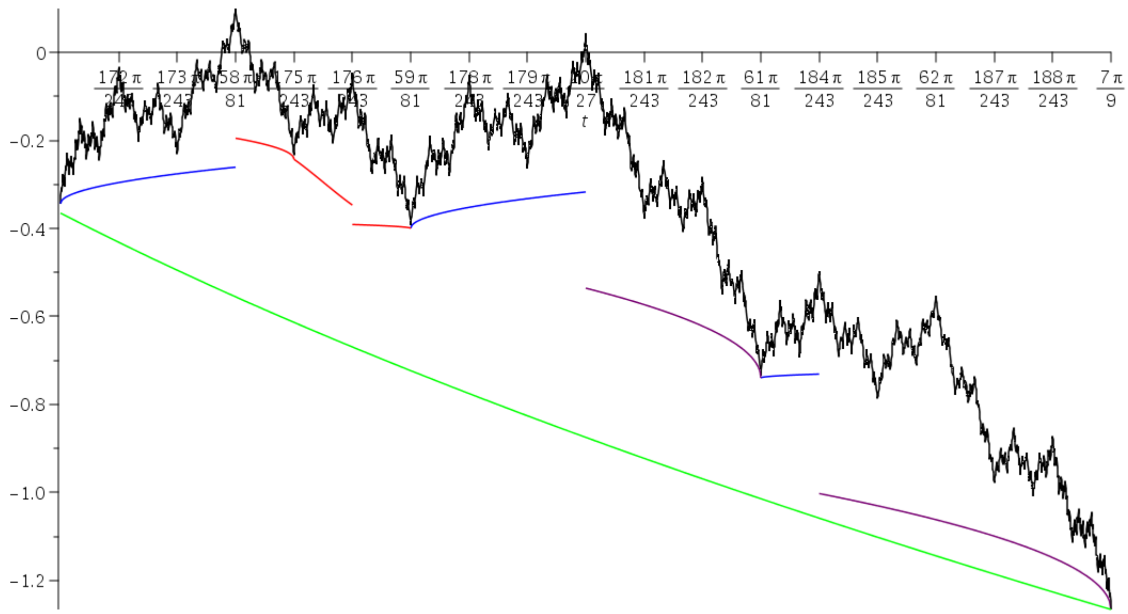


Figure 3.8: The interval $\left[\frac{19\pi}{27}, \frac{7\pi}{9}\right]$ (so $n = 1$) with $W(t)$ shown in black, $-\sigma_2\sqrt{t} + W(0)$ shown in green, the bounds from Lemma 3.25 shown in red, the bounds from Lemma 3.20 shown in purple, and simple bounds shown in blue.

$$\geq \left(\sqrt{a} + \tfrac{(-1)^b - 1}{2}\right)(3+\sqrt{3}) + R\sqrt{3^{n+4}t_0} + 2\sqrt{a}\sum_{j=1}^{2}\sqrt{3}^j \sin\left(\tfrac{\pi}{2(3)^j}\right)\sin\left(c_{j-1} + \tfrac{\pi}{2(3)^j}\right)$$
$$- 2\sum_{j=1}^{\infty}\sqrt{3}^j \sin\left(\tfrac{b\pi}{2(3)^j}\right)\sin\left(c_{j-4} + \tfrac{b\pi}{2(3)^j}\right).$$

The proof has the same steps as the proof of Lemma 3.18. The only difference is we keep track of different numbers. Lemma 3.25 is similar to Lemma 3.19. It shows that the red curves in Figure 3.8 are above the green curve.

Lemma 3.25. *Let $n \geq 1$. Then the following inequalities hold:*

(i) $W(t) \geq 0.42\sqrt{c_n + \frac{13\pi}{3^{n+4}} - t} + W(c_n + \frac{13\pi}{3^{n+4}}) \geq -s_n\sqrt{t - c_n} + W(c_n)$, *for* $t \in \left[c_n + \frac{12\pi}{3^{n+4}}, c_n + \frac{13\pi}{3^{n+4}}\right]$

(ii) $W(t) \geq W^{n+3}(t) - \frac{1}{\sqrt{3}^{n+4}}W(0) \geq -s_n\sqrt{t - c_n} + W(c_n)$ *for* $t \in \left[c_n + \frac{13\pi}{3^{n+4}}, c_n + \frac{14\pi}{3^{n+4}}\right]$

(iii) $W(t) \geq 0.077\sqrt{\frac{15\pi}{3^{n+4}} + c_n - t} + W\left(c_n + \frac{15\pi}{3^{n+4}}\right) \geq -s_n\sqrt{t - c_n} + W(c_n)$ *for* $t \in \left[c_n + \frac{14\pi}{3^{n+4}}, c_n + \frac{15\pi}{3^{n+4}}\right]$

Proof. This proof will proceed in a similar manner to the proof of Lemma 3.19. Parts (i) and (iii) are similar. The first inequality in each part follows from Theorem 3.5. The second inequality in each part will follow from an application of Lemma 3.24 showing that the minimum of the left expression is greater than the maximum of the right expression.

For part (i) we have $a = 12$, $b = 13$, $R = 0.42$ and $t_0 = 0$ which yields that

$$\min \text{LHS} - \max \text{RHS} \geq \tfrac{1}{\sqrt{3^{n+4}}}(2.4002619) > 0.$$

For part (iii), $a = 14$, $b = 15$, $R = .077$ and $t_0 = 0$ which yields that

$$\min \text{LHS} - \max \text{RHS} =\geq \tfrac{1}{\sqrt{3^{n+4}}}(1.340740) > 0.$$

Part (ii) is slightly different. The first inequality is clear since it follows from bounding $\cos(3^j t) \geq -1$ for $j \geq n+4$. The second inequality will follow as in the preceding parts by showing that the minimum of the left side is greater than the maximum of the right side. In order to find the minimum of the left side, we must first show that $W_{n+3}(t)$ is decreasing:

$$\frac{d}{dt}W^{n+3}(t) = -\sum_{j=0}^{n+3} \sqrt{3}^j \sin(3^j t) \leq -\sum_{j=0}^{n+1} \sqrt{3}^j \sin(3^j t) - \sqrt{3}^{n+2}\left[-1 + \frac{3}{2}\right]$$

since $3^{n+3}t \in \left[\frac{13\pi}{3}, \frac{14\pi}{3}\right]$ and the minimum of sine on this interval is $\frac{\sqrt{3}}{2}$. But we also know that $3^j\left[c_n + \frac{13\pi}{3^{n+3}}, c_n + \frac{14\pi}{3^{n+3}}\right] \subset [0, \pi]$ for $j \leq n+1$. Hence $\frac{d}{dt}W^{n+3}(t) \leq 0$.

Since we have $W^{n+3}\left(\frac{14\pi}{3^{n+4}}\right) = W\left(\frac{14\pi}{3^{n+4}}\right) - \frac{1}{\sqrt{3}^{n+4}}W(0)$, we can use Lemma 3.24 with $a = 13$, $b = 14$, $R = -\frac{2}{\sqrt{3}^{n+4}}W(0)$, and $t_0 = 1$ to get that

$$\min \text{LHS} - \max \text{RHS} \geq \tfrac{1}{\sqrt{3}^{n+3}}(1.4890564) > 0.$$

$\square$

Now we will show that the purple curves in Figure 3.8 are above the green curve.

Lemma 3.26. *Let $n \geq 1$. Then the following inequalities hold:*

(i) $W(t) \geq \frac{3\sqrt{3}}{2\sqrt{2\pi}}\sqrt{c_n + \frac{21\pi}{3^{n+4}} - t} + W\left(c_n + \frac{21\pi}{3^{n+4}}\right) \geq -s_n\sqrt{t-c_n} + W(c_n)$ *for* $t \in \left[c_n + \frac{18\pi}{3^{n+4}}, c_n + \frac{21\pi}{3^{n+4}}\right]$,

(ii) $W(t) \geq \frac{3\sqrt{3}}{2\sqrt{2\pi}}\sqrt{c_n + \frac{\pi}{3^{n+1}} - t} + W\left(c_n + \frac{\pi}{3^{n+1}}\right) \geq -s_n\sqrt{t-c_n} + W(c_n)$ *for* $t \in \left[c_n + \frac{22\pi}{3^{n+4}}, c_n + \frac{\pi}{3^{n+1}}\right]$.

Proof. The first inequality in both parts follows from the fact that the larger bound from Lemma 3.13 works in each case. For the first part: $\frac{d}{dt}W^{n+2}(t) = -\sum_{j=0}^{n+2}\sqrt{3}^j\sin(3^jt) < 0$ since $3^jt \in [0,\pi]$ for $j \leq n+2$ (when $j = n+2$, $3^jt \in 2k\pi + [2\pi, 7\pi/3]$). Hence the max of $W^{n+2}(t)$ occurs at the left most endpoint. And for the second part: $\frac{d}{dt}W^n(t) = -\sum_{j=0}^{n}\sqrt{3}^j\sin(3^jt) < 0$ since $3^jt \in [0,\pi]$ for $j \leq n$. Hence the max of $W^n(t)$ occurs at the left most endpoint.

Because the difference in the LHS and the RHS is concave down for both parts, the second inequality in each part will follow from showing that the difference in the LHS and the RHS is positive at the left and right endpoints. For part (ii), we can use Lemma 3.23 to show the inequality holds at $c_n + \frac{\pi}{3^{n+1}}$. To show that part (ii) holds at $c_n + \frac{22\pi}{3^{n+4}}$ we can use Lemma 3.24 with $a = b = 22$, $R = \frac{3\sqrt{3}}{2\sqrt{2\pi}}$, and $3^{n+4}t_0 = 5\pi$ to get

$$\text{LHS} - \text{RHS} \geq \frac{1}{\sqrt{3}^{n+4}}(8.7169375) > 0.$$

To show that part (i) holds, first we use Lemma 3.24 with $a = b = 21$, $R = \frac{3\sqrt{3}}{2\sqrt{2\pi}}$, and $t_0 = 0$ to get

$$\text{LHS} - \text{RHS} \geq \frac{1}{\sqrt{3}^{n+4}}(.2547703) > 0.$$

Then we set $a = b = 18$, $R = \frac{3\sqrt{3}}{2\sqrt{2\pi}}$, and $3^{n+4}t_0 = 3\pi$ to get

$$\text{LHS} - \text{RHS} \geq \frac{1}{\sqrt{3}^{n+4}}(14.0741814) > 0.$$

□

Lemmas 3.25 and 3.26 along with Lemma 3.23 show that $L(c_n, h) \leq \sigma$ for all $h < \frac{\pi}{3^{n+1}}$:

Lemma 3.27. *Let* $n \geq 0$*. Then* $L(c_n, h) < \sigma$ *for all* $h \in \left(0, \frac{\pi}{3^{n+1}}\right]$.

Proof. If we can show the statement to be true for $t \in \left[c_n + \frac{\pi}{3^{n+2}}, c_n + \frac{\pi}{3^{n+1}}\right]$, then 3.23 will give us the result on the rest of the interval. Lemmas 3.25 and 3.26 give us the result everywhere except $\left[c_n + \frac{\pi}{3^{n+2}}, c_n + \frac{12\pi}{3^{n+4}}\right] \cup \left[c_n + \frac{15\pi}{3^{n+4}}, c_n + \frac{18\pi}{3^{n+4}}\right] \cup \left[c_n + \frac{21\pi}{3^{n+4}}, c_n + \frac{22\pi}{3^{n+4}}\right]$. Since $W(t)$ can be bounded on each of these intervals by an increasing function given in Theorem 3.5, it will be enough to know that the Weierstrass function at the left most endpoint is bounded above the curve $-s_n\sqrt{t - c_n} + W(c_n)$ (since $s_n < \sigma$). Indeed the points $t = c_n + \frac{15\pi}{3^{n+4}}, c_n + \frac{21\pi}{3^{n+4}}$ are shown in Lemmas 3.25 and 3.26, and $t = c_n + \frac{\pi}{3^{n+2}}$ follows from 3.23. □

Due to the limited scope of 3.23, we are unable to capture the points between $c_n + \frac{\pi}{3^{n+1}}$ and $c_n + \frac{\pi}{3^n}$ the same way we did the other points closer than $\frac{\pi}{3^{n+1}}$. We must take care of those points separately.

Lemma 3.28. *Let* $\frac{a\pi}{3^{n+3}}, \frac{b\pi}{3^{n+3}} \in \left[c_n + \frac{\pi}{3^{n+1}}, c_n + \frac{\pi}{3^n}\right]$ *and* $t_0 \geq 0$*. Then for* $n \geq 1$*,* $R \in \mathbb{R}$*, and* $t_0 \geq 0$ *we have*

$$\sqrt{3}^{n+3}\left\{R\sqrt{t_0} + W\left(c_n + \tfrac{b\pi}{3^{n+3}}\right) - \left(-\sigma\sqrt{\tfrac{a\pi}{3^{n+3}}} - W(c_n)\right)\right\}$$
$$\geq R\sqrt{3^{n+3}t_0} + \sigma\sqrt{a\pi} + \frac{(-1)^b - 1}{2}(3 + \sqrt{3}) - 2\sum_{j=1}^{\infty}\sqrt{3}^j \sin\left(\tfrac{b\pi}{(2)3^j}\right)\sin\left(c_{j-3} + \tfrac{b\pi}{(2)3^j}\right).$$

Again, the proof of Lemma 3.28 is nearly identical to the proof of Lemma 3.18 so we will omit it. Lemma 3.28 uses σ instead of an approximation of σ (as we do in Lemmas 3.18 and 3.24) because the "correct" approximation (either $L(c_n, \frac{\pi}{3^n})$ or $L(c_n, \frac{\pi}{3^{n+1}})$) depends on n. With Lemma 3.28, we can prove bounds analogous to those in Lemmas 3.25 and 3.26.

Lemma 3.29. *Let* $n \geq 1$*. Then the following inequalities hold:*

(i) $W(t) \geq 0.42\sqrt{c_n + \frac{13\pi}{3^{n+3}} - t} + W(c_n + \frac{13\pi}{3^{n+3}}) \geq -\sigma\sqrt{t - c_n} + W(c_n)$, *for* $t \in \left[c_n + \frac{12\pi}{3^{n+3}}, c_n + \frac{13\pi}{3^{n+3}}\right]$

(ii) $W(t) \geq W^{n+2}(t) - \frac{1}{\sqrt{3}^{n+3}}W(0) \geq -\sigma\sqrt{t - c_n} + W(c_n)$ *for* $t \in \left[c_n + \frac{13\pi}{3^{n+3}}, c_n + \frac{14\pi}{3^{n+3}}\right]$

(iii) $W(t) \geq 0.077\sqrt{\frac{15\pi}{3^{n+3}} + c_n - t} + W\left(c_n + \frac{15\pi}{3^{n+3}}\right) \geq -\sigma\sqrt{t - c_n} + W(c_n)$ *for* $t \in \left[c_n + \frac{14\pi}{3^{n+3}}, c_n + \frac{15\pi}{3^{n+3}}\right]$

Proof. This proof will proceed in a similar manner to the proof of Lemma 3.19, so we will omit some details. For part (i) we have $a = 12$, $b = 13$, $R = 0.42$ and $t_0 = 0$ which yields that

$$\min \mathrm{LHS} - \max \mathrm{RHS} \geq \tfrac{1}{\sqrt{3^{n+3}}}(4.0276297) > 0.$$

For part (iii), $a = 14$, $b = 15$, $R = .077$ and $t_0 = 0$ which yields that

$$\min \mathrm{LHS} - \max \mathrm{RHS} =\geq \tfrac{1}{\sqrt{3^{n+3}}}(3.2664) > 0.$$

Part (ii) is works the same as it did in Lemma 3.24. The proof that $W_{n+3}(t)$ is decreasing is the same: just make the substitution $n_2 = n_1 - 1$ where n_1 was used in the previous proof and n_2 is the value of n that we use in this proof. We then use Lemma 3.28 with $a = 13$, $b = 14$, $R = -\frac{2}{\sqrt{3}^{n+3}} W(0)$, and $t_0 = 1$ to get that

$$\min \mathrm{LHS} - \max \mathrm{RHS} \geq \tfrac{1}{\sqrt{3}^{n+3}}(3.2579079) > 0.$$

□

Lemma 3.30. *Let $n \geq 1$. Then the following inequalities hold:*

(i) $W(t) \geq \frac{3\sqrt{3}}{2\sqrt{2\pi}}\sqrt{c_n + \frac{21\pi}{3^{n+3}} - t} + W\left(c_n + \frac{21\pi}{3^{n+3}}\right) \geq -\sigma\sqrt{t - c_n} + W(c_n)$ *for* $t \in \left[c_n + \frac{18\pi}{3^{n+3}}, c_n + \frac{21\pi}{3^{n+3}}\right]$,

(ii) $W(t) \geq \frac{3\sqrt{3}}{2\sqrt{2\pi}}\sqrt{c_n + \frac{\pi}{3^n} - t} + W\left(c_n + \frac{\pi}{3^n}\right) \geq -\sigma\sqrt{t - c_n} + W(c_n)$ *for* $t \in \left[c_n + \frac{22\pi}{3^{n+3}}, c_n + \frac{\pi}{3^n}\right]$.

Proof. The proof of Lemma 3.30 is similar to the proof of Lemma 3.26, so we will omit details where the proof is nearly identical. The first inequality in both parts follows from the fact that the larger bound from Lemma 3.13 works in each case.

Because the difference in the LHS and the RHS is concave down for both parts, the second inequality in each part will follow from showing that the difference in the LHS and the RHS is positive at the left and right endpoints. For part (ii), we can use the fact that the limit from Proposition 3.22 is increasing to show the inequality holds at $c_n + \frac{\pi}{3^n}$. To show that part (ii) holds at $c_n + \frac{22\pi}{3^{n+3}}$ we can use Lemma 3.28 with $a = b = 22$, $R = \frac{3\sqrt{3}}{2\sqrt{2\pi}}$, and $3^{n+3}t_0 = 5\pi$ to get

$$\text{LHS} - \text{RHS} \geq \frac{1}{\sqrt{3}^{n+3}}(12.3091762) > 0.$$

To show that part (i) holds, first we use Lemma 3.24 with $a = b = 21$, $R = \frac{3\sqrt{3}}{2\sqrt{2\pi}}$, and $t_0 = 0$ to get

$$\text{LHS} - \text{RHS} \geq \frac{1}{\sqrt{3}^{n+3}}(7.0404745) > 0.$$

Then we set $a = b = 18$, $R = \frac{3\sqrt{3}}{2\sqrt{2\pi}}$, and $3^{n+3}t_0 = 3\pi$ to get

$$\text{LHS} - \text{RHS} \geq \frac{1}{\sqrt{3}^{n+4}}(16.6868460) > 0.$$

□

Lemma 3.31. *$L(c_n, h) \leq \sigma$ for all $h \in \left(0, \frac{\pi}{3^n}\right]$ and all n.*

Proof. Lemma 3.27 and Lemmas 3.29 through 3.30 give us the result everywhere except $\left[c_n + \frac{\pi}{3^{n+1}}, c_n + \frac{12\pi}{3^{n+3}}\right] \cup \left[c_n + \frac{15\pi}{3^{n+3}}, c_n + \frac{18\pi}{3^{n+3}}\right] \cup \left[c_n + \frac{21\pi}{3^{n+3}}, c_n + \frac{22\pi}{3^{n+3}}\right]$. Since $W(t)$ can be bounded on each of these intervals by an increasing function given by Theorem 3.5, it will be enough to know that the Weierstrass function at the left most endpoint is bounded above the curve $-\sigma\sqrt{t - c_n} + W(c_n)$. Indeed, the points $t = c_n + \frac{15\pi}{3^{n+3}}, c_n + \frac{21\pi}{3^{n+3}}, c_n + \frac{\pi}{3^{n+1}}$ are shown in Lemmas 3.29 and 3.30. □

In order to show that this bound holds for other local maximums, we need to consider maximums that scale up to the same value. That is maximums m_i so that $3^{n_i}m_i$ are equivalent modulo 2π. For example, c_n and $c_n + (-1)^n\frac{2\pi}{3}$ yield the same angle when multiplied by 3. Call such numbers *descendants* of the value that they scale up to. Call the value that they scale up to their *ancestor*. Each ancestor a has 3 first-generation descendants: $a/3 + \theta(j)$, where $\theta(j) \in \{0, 2\pi/3, 4\pi/3\}$. The sequence c_n was chosen as the sequence of descendants of 0 where each descendant is as close to $\pi/2$ as possible given that their ancestors were also as close as possible to $\pi/2$. First, we will show that the descendants that result from dividing by 3 and adding $4\pi/3$ have a smaller upper Lip(1/2) bound than the maximums c_n.

Lemma 3.32. *Let $\frac{2m\pi}{3^n}$ be a descendant of $\frac{c_k}{3} + \frac{4\pi}{3}$. Then $L\left(\frac{2m\pi}{3^n}, h\right) \leq \sigma$ for $h \in (0, \pi/3^n]$.*

Proof. The proofs for $k = 1$ and $k \geq 2$ are different. For the $k = 1$ case, let $h \in [\pi/9, \pi/3]$. Then we can use the scaling property of the Weierstrass function and the bound $\frac{\cos(x)-\cos(x+y)}{\sqrt{y}} \leq \sqrt{y}$ to get

$$L\left(\frac{4\pi}{3^n} + \sum_{j=0}^{n-1} \frac{\theta(j)}{3^j}, h\right) \leq \frac{W\left(\frac{4\pi}{3}\right) - W\left(\frac{13\pi}{9}\right)}{\sqrt{\pi/9}} + \sum_{j=2}^{\infty} \sqrt{\frac{\pi}{3^j}}$$

where $\theta(j) \in \{0, 2\pi/3, 4\pi/3\}$ identifies the particular choice of descendant $n - j$ generations past $\frac{4\pi}{3}$. However, this is approximately 4.0038 which is less than σ. For $h \in [\pi/27, \pi/9]$, we use a similar argument to show that the local upper Lip(1/2) behavior is less than about 3.762. For $h \in [0, \pi/27]$, we note that since $W^0\left(\frac{4\pi}{3} + h\right)$ is increasing here, the local Lip(1/2) norm at $\frac{4\pi}{3}$ is the same as the Lip(1/2) norm at 0 given in Lemma 3.21. Hence for $h \in [0, \pi/27]$ we get

$$L\left(\frac{4\pi}{3^n} + \sum_{j=0}^{n-1} \frac{\theta(j)}{3^j}, h\right) \leq \sigma_0 + \sum_{j=3}^{\infty} \sqrt{\frac{\pi}{3^j}} \leq 4.092074357 < \sigma.$$

Now let $k \geq 2$ and $h \in (0, \pi/3^n]$. Because $c_k + \frac{\pi}{3^k} \in \left[\frac{2\pi}{9}, \frac{7\pi}{9}\right]$, and $\int_l^r \sin(t)dt \geq \min_{t\in[l,r]} \sin(t)|r - l|$ we have that

$$\frac{\cos(c_k) - \cos(c_k + h)}{\sqrt{h}} \geq \sin\left(\frac{2\pi}{9}\right) \frac{h}{\sqrt{h}}.$$

By contrast, we can omit the "wrong" descendant of c_{k-1} from our calculation of the upper Lip(1/2) bound because

$$\frac{\cos\left(\frac{c_{k-1}}{3} + \frac{4\pi}{3}\right) - \cos\left(\frac{c_{k-1}}{3} + \frac{4\pi}{3} + h\right)}{\sqrt{h}} \leq 0$$

Hence,

$$\sigma - \frac{W\left(\frac{2m\pi}{3^n}\right) - W\left(\frac{2m\pi}{3^n} + t\right)}{\sqrt{t}}$$

$$
\geq \left[\frac{W(c_k) - W(c_k + 3^{n-k}t)}{\sqrt{3^{n-k}t}} + \sin\left(\frac{2\pi}{9}\right) \sum_{j=n+1}^{\infty} \sqrt{\frac{t}{3^{j-k}}}\right]
$$

$$
- \left[\frac{W(c_k) - W(c_k + 3^{n-k}t)}{\sqrt{3^{n-k}t}} - \sum_{j=n+2}^{\infty} \sqrt{\frac{t}{3^{j-k}}}\right]
$$

$$
= \sum_{j=n+1}^{\infty} \sqrt{\frac{t}{3^{j-k}}} \left(\sin\left(\frac{2\pi}{9}\right) - \frac{1}{\sqrt{3}}\right) > 0.
$$

□

We still must show that the other local maximums which are not c_n do not have upper Lip(1/2) bounds greater than σ. We will utilize the fact that these maxes have ancestors in $\{c_n\}$. However at the k^{th} generation their ancestor is $c_k + (-1)^k \frac{2\pi}{3}$ instead of c_k.

Lemma 3.33. *Let $n \geq 2$ and let $\frac{2m\pi}{3^k}$ be a descendant of $c_n + (-1)^n \frac{2\pi}{3}$. Then $L\left(\frac{2m\pi}{3^k}, h\right) \leq \sigma$ for $h \in (0, \frac{\pi}{3^k}]$.*

Proof. First we will note the restriction of the range of the first-generation descendants of $c_n + (-1)^n \frac{2\pi}{3}$. Since $c_n + (-1)^n \frac{2\pi}{3} \in \left[0, \frac{\pi}{12}\right] \cup \left[\frac{8\pi}{9}, \frac{11\pi}{12}\right]$ then its first-generation descendants are in the set $\left[0, \frac{\pi}{36}\right] \cup \left[\frac{8\pi}{27}, \frac{11\pi}{36}\right] \cup \left[\frac{2\pi}{3}, \frac{25\pi}{36}\right] \cup \left[\frac{26\pi}{27}, \frac{35\pi}{36}\right] \cup \left[\frac{4\pi}{3}, \frac{49\pi}{36}\right] \cup \left[\frac{44\pi}{27}, \frac{59\pi}{36}\right]$. Even after we account for adding $\frac{\pi}{3^{n+1}}$, the closest these values come to $\frac{\pi}{2}$ is either $\frac{\pi}{3}$ or $\frac{2\pi}{3}$. Hence if we let $\hat{c}_{n+1}$ be a first-generation descendant of $c_n + (-1)^n \frac{2\pi}{3}$, then for $h \in \left(0, \frac{\pi}{3^{n+1}}\right]$ we can say that

$$
\cos(\hat{c}_{n+1}) - \cos(\hat{c}_{n+1} + h) = 2\sin(h/2)\sin(\hat{c}_{n+1} + h/2) \leq \frac{\pi}{3^{n+1}} \cos\left(\frac{\pi}{2(3)^{n+1}}\right) \sin\left(\frac{2\pi}{3}\right). \tag{3.5}
$$

Now we will show that $L\left(\frac{2m\pi}{3^k}, h\right) \leq \sigma$ in a manner similar to how we proved Lemma 3.32. We will show that

$$
\sigma - \frac{W\left(c_n + (-1)^n \frac{2\pi}{3}\right) - W\left(c_n + (-1)^n \frac{2\pi}{3} + \frac{t}{3^n}\right)}{\sqrt{t/3^n}} - \frac{\sqrt{3}}{2} \cos\left(\frac{t}{2(3)^{n+1}}\right) \sqrt{\frac{t}{3^{n+1}}} - \sum_{j=n+2}^{\infty} \sqrt{\frac{t}{3^j}} \geq 0
$$

for $t \in [0, \pi]$. Indeed,

$$
\sigma - \frac{W\left(c_n + (-1)^n \frac{2\pi}{3}\right) - W\left(c_n + (-1)^n \frac{2\pi}{3} + \frac{t}{3^n}\right)}{\sqrt{t/3^n}} - \frac{\sqrt{3}}{2} \cos\left(\frac{t}{2(3)^{n+1}}\right) \sqrt{\frac{t}{3^{n+1}}} - \sum_{j=n+2}^{\infty} \sqrt{\frac{t}{3^j}}
$$

$$
\begin{aligned}
&\geq \frac{2\sqrt{3}^n}{\sqrt{\pi}} \sin\left(\frac{\pi}{2(3)^n}\right) \sin\left(c_n + \frac{\pi}{2(3)^n}\right) - \frac{2\sqrt{3}^n}{\sqrt{t}} \sin\left(\frac{t}{2(3)^n}\right) \sin\left(c_n + (-1)^n\frac{2\pi}{3} + \frac{t}{2(3)^n}\right) \\
&+ \frac{2\sqrt{3}^{n+1}}{\sqrt{\pi}} \sin\left(\frac{\pi}{2(3)^{n+1}}\right) \sin\left(c_{n+1} + \frac{\pi}{2(3)^{n+1}}\right) - \frac{\sqrt{3}}{2} \cos\left(\frac{t}{2(3)^{n+1}}\right)\sqrt{\frac{t}{3^{n+1}}} \\
&+ \frac{2}{\sqrt{\pi}} \sum_{j=n+2}^{\infty} \sqrt{3}^j \sin\left(\frac{\pi}{2(3)^j}\right) \sin\left(c_j + \frac{\pi}{2(3)^j}\right) - \sqrt{\frac{t}{3^{n+2}}}\frac{3+\sqrt{3}}{2}
\end{aligned}
$$

by writing out the series expansion for σ and cancelling the terms obtained from the common ancestry (c_{n-1}). Then we factor out $\frac{1}{\sqrt{3}^{n+2}}$, use 3.5 and set $t = \pi$ in the series to get that

$$
\begin{aligned}
&\sigma - L\left(\frac{2m\pi}{3^k}, h\right) \\
&\geq \frac{1}{\sqrt{3}^{n+2}} \left\{ \frac{2(3)^{n+1}}{\sqrt{\pi}} \sin\left(\frac{\pi}{2(3)^n}\right) \left[\sin\left(c_n + \frac{\pi}{2(3)^n}\right) - \sin\left(c_n + (-1)^n\frac{2\pi}{3} + \frac{t}{2(3)^n}\right)\right] \right. \\
&+ \frac{2\sqrt{3}(3)^{n+1}}{\sqrt{\pi}} \sin\left(\frac{\pi}{2(3)^{n+1}}\right) \left[\sin\left(c_{n+1} + \frac{\pi}{2(3)^{n+1}}\right) - \frac{\sqrt{3}}{2}\right] \\
&\left. + \frac{2\sqrt{3}^{n+2}}{\sqrt{\pi}} \sum_{j=n+2}^{\infty} \sqrt{3}^j \sin\left(\frac{\pi}{2(3)^j}\right) \sin\left(c_j + \frac{\pi}{2(3)^j}\right) - \frac{\sqrt{\pi}}{2}(3+\sqrt{3}) \right\}.
\end{aligned}
$$

Then we can find bounds for the alternating angles depending on whether n is odd or even to get that

$$
\begin{aligned}
&\sigma - L\left(\frac{2m\pi}{3^k}, h\right) \\
&\geq \frac{1}{\sqrt{3}^{n+2}} \left\{ \frac{2(3)^{n+1}}{\sqrt{\pi}} \sin\left(\frac{\pi}{2(3)^n}\right) \left[\sin\left(c_n + \frac{\pi}{2(3)^n}\right) - \begin{cases} \sin\left(c_n - \frac{2\pi}{3} + \frac{\pi}{2(3)^n}\right), n \text{ odd} \\ \sin\left(c_n + \frac{2\pi}{3}\right), n \text{ even} \end{cases} \right] \right. \\
&+ \frac{2\sqrt{3}(3)^{n+1}}{\sqrt{\pi}} \sin\left(\frac{\pi}{2(3)^{n+1}}\right) \left[\left\{ \begin{array}{l} \sin\left(\frac{\pi}{4}\right), n \text{ odd} \\ \sin\left(\frac{3\pi}{4} + \frac{\pi}{4(3)^{n+1}}\right), n \text{ even} \end{array} \right\} - \frac{\sqrt{3}}{2}\right] \\
&\left. + \frac{2\sqrt{3}^{n+2}}{\sqrt{\pi}} \sum_{j=n+2}^{\infty} \sqrt{3}^j \sin\left(\frac{\pi}{2(3)^j}\right) \sin\left(c_j + \frac{\pi}{2(3)^j}\right) - \frac{\sqrt{\pi}}{2}(3+\sqrt{3}) \right\}.
\end{aligned}
$$

Now we use the bound $\cos(x) \geq \frac{\sin(x)}{x}$:

$$\begin{aligned}
&\sigma - L\left(\frac{2m\pi}{3^k}, h\right) \\
&\geq \tfrac{1}{\sqrt{3}^{n+2}} \left\{ 6\sqrt{\pi}\cos\left(\frac{\pi}{2(3)^n}\right) \begin{cases} 2\cos\left(c_n - \frac{\pi}{3} + \frac{\pi}{2(3)^n}\right)\sin\left(\frac{\pi}{3}\right), n \text{ odd} \\ 2\cos\left(c_n + \frac{\pi}{3} + \frac{\pi}{4(3)^n}\right)\sin\left(-\frac{\pi}{3} + \frac{\pi}{4(3)^n}\right), n \text{ even} \end{cases} \right. \\
&+ \sqrt{3\pi}\cos\left(\frac{\pi}{2(3)^{n+1}}\right)\left[\begin{Bmatrix} \sin\left(\frac{\pi}{4}\right), n \text{ odd} \\ \sin\left(\frac{3\pi}{4} + \frac{\pi}{4(3)^{n+1}}\right), n \text{ even} \end{Bmatrix} - \frac{\sqrt{3}}{2}\right] \\
&\left. + \sqrt{3}^{n+2} \sum_{j=n+2}^{\infty} \frac{\sqrt{\pi}}{\sqrt{3}^j}\cos\left(\frac{\pi}{2(3)^j}\right)\sin\left(c_j + \frac{\pi}{2(3)^j}\right) - \frac{\sqrt{\pi}}{2}(3+\sqrt{3}) \right\}.
\end{aligned}$$

Finally, we can bound the trig function in the series to get

$$\begin{aligned}
&6\sqrt{\pi}\cos\left(\frac{\pi}{2(3)^n}\right) \begin{cases} 2\cos\left(c_n - \frac{\pi}{3} + \frac{\pi}{2(3)^n}\right)\sin\left(\frac{\pi}{3}\right), n \text{ odd} \\ 2\cos\left(c_n + \frac{\pi}{3} + \frac{\pi}{4(3)^n}\right)\sin\left(-\frac{\pi}{3} + \frac{\pi}{4(3)^n}\right), n \text{ even} \end{cases} \\
&+ \sqrt{3\pi}\cos\left(\frac{\pi}{2(3)^{n+1}}\right)\left[\begin{Bmatrix} \sin\left(\frac{\pi}{4}\right), n \text{ odd} \\ \sin\left(\frac{3\pi}{4} + \frac{\pi}{4(3)^{n+1}}\right), n \text{ even} \end{Bmatrix} - \frac{\sqrt{3}}{2}\right] \\
&+ \frac{\sqrt{\pi}}{2}(3+\sqrt{3})\left[\cos\left(\frac{\pi}{2(3)^{n+2}}\right)\begin{Bmatrix} \cos\left(\frac{\pi}{4} + \frac{\pi}{4(3)^{n+2}}\right), n \text{ odd} \\ \cos\left(\frac{\pi}{4} + \frac{\pi}{4(3)^{n+3}}\right), n \text{ even} \end{Bmatrix} - 1\right] \\
&\geq \begin{cases} 0.88, n \text{ odd} \\ 0.98, n \text{ even} \end{cases} > 0
\end{aligned}$$

because the pieces still depending on n are increasing in n. $\square$

Note that we do not need to show the $n = 1$ case above since the only maxes at that level are $\frac{2\pi}{3}$ and $\frac{4\pi}{3}$. Now we will show that we can extend this upper Lip(1/2) bound out farther than $\frac{\pi}{3^n}$ past an n^{th} generation maximum.

Lemma 3.34. *Let $n \geq 0$. Then $L\left(\frac{2m\pi}{3^n}, h\right) \leq \sigma$ for all $h \in [0, \frac{3\pi}{3^n}]$.*

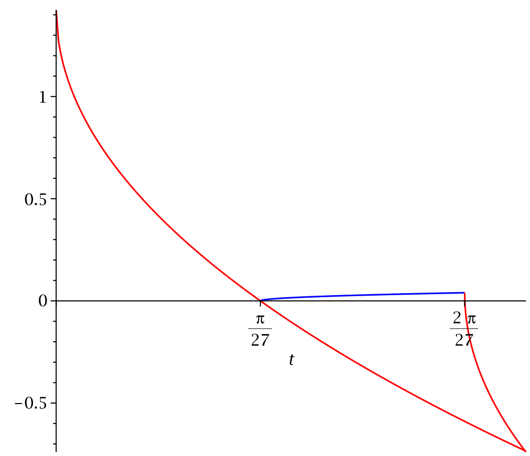


Figure 3.9: A visualization of how we are able to extend our upper Lip(1/2) bound a bit farther.

Proof. First let $m \equiv 1 \pmod 3$. Then $2m+1 \equiv 0 \pmod 3$ and by Theorem 3.5, $\frac{(2m+1)\pi}{3^n}$ is a global minimum at least $\frac{\pi}{3^n}$ toward the right. Hence, the Lip(1/2) bound holds out $\frac{2\pi}{3^n}$ to the right. We are able to reach the same conclusion when $m \equiv -1 \pmod 3$ since then we have that $2m+1 \equiv -1 \pmod 3$. However by Theorem 3.5, we can quantify how much greater $W\left(\frac{(2m+2)\pi}{3^n}\right)$ is than $W\left(\frac{(2m+1)\pi}{3^n}\right)$. In particular, $W\left(\frac{(2m+2)\pi}{3^n}\right) \geq 0.077\sqrt{\frac{\pi}{3^n}} + W\left(\frac{(2m+1)\pi}{3^n}\right) \geq 0.077\sqrt{\frac{\pi}{3^n}} - \sigma\sqrt{\frac{\pi}{3^n}} + W\left(\frac{2m\pi}{3^n}\right)$. But then we know that for $t \in \left[\frac{(2m+2)\pi}{3^n}, \frac{(2m+3)\pi}{3^n}\right]$, $W(t) \geq -\sigma\sqrt{t - \frac{(2m+2)\pi}{3^n}} + W\left(\frac{(2m+2)\pi}{3^n}\right)$. Therefore, $W(t) > -\sigma\sqrt{t - \frac{2m\pi}{3^n}} + W\left(\frac{2m\pi}{3^n}\right)$ until the time t when $-\sigma\sqrt{t - \frac{2m\pi}{3^n}} + W\left(\frac{2m\pi}{3^n}\right) = -\sigma\sqrt{t - \frac{(2m+2)\pi}{3^n}} + 0.077\sqrt{\frac{\pi}{3^n}} - \sigma\sqrt{\frac{\pi}{3^n}} + W\left(\frac{2m\pi}{3^n}\right)$. This happens for $t \geq \frac{2m\pi}{3^n} + \frac{2.26\pi}{3^n}$ see Figure 3.9.

To get the rest of the interval, first assume that $m \equiv 1 \pmod 3$. Then $\frac{2m\pi}{3^n}$ is a descendant of $\frac{2\pi}{3}$. Then if we let $t \in \left[\frac{(2m+2.26)\pi}{3^n}, \frac{3\pi}{3^n}\right]$, and we use Theorem 3.5 with the sharper bound from Lemma 3.13 we get

$$\frac{W\left(\frac{2m\pi}{3^n}\right) - W\left(\frac{2m\pi}{3^n} + t\right)}{\sqrt{t}} \leq \frac{W\left(\frac{2\pi}{3}\right) - \left(\frac{3\sqrt{3}}{2\sqrt{2\pi}}\sqrt{\frac{2\pi}{3} + \frac{2.26\pi}{3} - \pi} + W(\pi)\right)}{\sqrt{2.26\pi/3}} + \sum_{j=2}^{\infty} \frac{3\pi/3^j}{\sqrt{2.26\pi/3^j}}$$

which is about 4.116, hence less than σ.

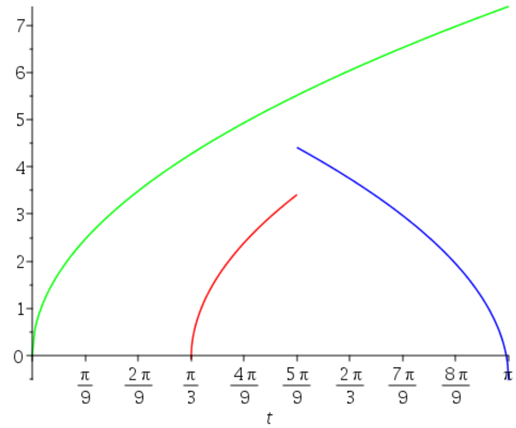


Figure 3.10: A scaled up version of the argument presented in the proof of Theorem 3.2. We want to show that $W(t)$ is below the green curve from $\frac{\pi}{3}$ to π knowing that $W(t)$ is below the red and blue curves.

Now let $m \equiv -1 \pmod 3$. Then $\frac{2m\pi}{3^n}$ is a descendant of $\frac{4\pi}{3}$, and we know that for $t \in \left[\frac{(2m+2.26)\pi}{3^n}, \frac{3\pi}{3^n}\right]$

$$\frac{W\left(\frac{2m\pi}{3^n}\right) - W\left(\frac{2m\pi}{3^n} + t\right)}{\sqrt{t}} \leq \frac{W\left(\frac{4\pi}{3}\right) - W\left(\frac{7\pi}{3}\right)}{\sqrt{2.26\pi/3}} + \sum_{j=2}^{\infty} \frac{3\pi/3^j}{\sqrt{2.26\pi/3^j}}$$

which is about 3.35, hence less than σ. □

We can easily extend the result of Lemma 3.34 to point to the left of local maximums and also to local minimums.

Theorem 3.35. *Let $n \geq 0$, $m \in \mathbb{Z}$ and let $h \in \left[-\frac{3\pi}{3^n}, \frac{3\pi}{3^n}\right]$. Then $L\left(\frac{m\pi}{3^n}, h\right) \leq \sigma$ if m is even, and $-L\left(\frac{m\pi}{3^n}, h\right) \leq \sigma$ if m is odd.*

Proof. To get the inequality for points to the left of any local maximum, we use the fact that $W(t) = W(2\pi - t)$ to get $L\left(\frac{2m\pi}{3^n}, h\right) = L\left(2\pi - \frac{2m\pi}{3^n}, -h\right)$. Then we get the analogous inequality for all local minimums by using the fact that $W(t) = -W(\pi+t)$ to get $L\left(\frac{2m\pi}{3^n}, h\right) = -L\left(\pi + \frac{2m\pi}{3^n}, h\right)$. □

Now we can improve on the Lip(1/2) norm in Theorem 1.9.

Proof of Theorem 3.2. First, let us show that $|L\left(\frac{2m\pi}{3^n}, h\right)| \leq \sigma$ for $h \in (0, \frac{\pi}{3^n}]$. We already know that $L\left(\frac{2m\pi}{3^n}, h\right) \leq \sigma$, so we just need to show that $L\left(\frac{2m\pi}{3^n}, h\right) \geq -\sigma$. If $m \equiv 1 \pmod 3$, then Lemma 3.13 gives us the stronger bound $L\left(\frac{2m\pi}{3^n}, h\right) \geq 0.42$.

Assume $m \equiv -1 \pmod 3$. Then for $h \in (0, \frac{\pi}{3^{n+1}}]$, Theorem 3.5 gives us the stronger bound $L\left(\frac{2m\pi}{3^n}, h\right) \geq 0.077$. From this, we can assert that $W(t) \leq W\left(\frac{2m\pi}{3^n} + \frac{\pi}{3^{n+1}}\right) + \sigma\sqrt{h - \frac{\pi}{3^{n+1}}} \leq W\left(\frac{2m\pi}{3^n}\right) + \sigma\sqrt{h - \frac{\pi}{3^{n+1}}}$ for $h \in \left[\frac{\pi}{3^{n+1}}, \frac{5\pi}{3^{n+2}}\right]$. But then we can show $W\left(\frac{2m\pi}{3^n}\right) + \sigma\sqrt{h} > W\left(\frac{2m\pi}{3^n}\right) + \sigma\sqrt{h - \frac{\pi}{3^{n+1}}}$ by showing that the min of the left-hand side is greater than the max of the right-hand side. Indeed, $W\left(\frac{2m\pi}{3^n}\right) + \sigma\sqrt{\frac{\pi}{3^{n+1}}} - \left(W\left(\frac{2m\pi}{3^n}\right) + \sigma\sqrt{\frac{5\pi}{3^{n+2}} - \frac{\pi}{3^{n+1}}}\right) > \frac{1}{\sqrt{3}^n}(.78362242) > 0$. This is shown by the green and red curves in Figure 3.10.

Now we will show that $W\left(\frac{2m\pi}{3^n}\right) > W\left(\frac{(2m+1)\pi}{3^n}\right)$. Indeed,

$$W\left(\frac{2m\pi}{3^n}\right) - W\left(\frac{(2m+1)\pi}{3^n}\right) = \frac{1}{\sqrt{3}^n}\left\{3 + \sqrt{3} + \sum_{j=1}^{n} \sqrt{3}^j \sin\left(\frac{\pi}{(2)3^j}\right)\sin\left(\frac{m\pi}{3^j} + \frac{\pi}{(2)3^j}\right)\right\}$$

by using arguments similar to those used in Lemma 3.19. Then we can use the bounds $-1 \leq \sin\left(\frac{m\pi}{3^j} + \frac{\pi}{(2)3^j}\right)$ for $j \geq 1$ and $\sin\left(\frac{\pi}{(2)3^j}\right) < \frac{\pi}{(2)3^j}$ for $j \geq 2$ to bound the terms in the series to get that $W\left(\frac{2m\pi}{3^n}\right) - W\left(\frac{(2m+1)\pi}{3^n}\right) > \frac{1}{\sqrt{3}^n}(0.5223) > 0$. Hence, we know that $W(t) \leq W\left(\frac{(2m+1)\pi}{3^n}\right) + \sigma\sqrt{\frac{\pi}{3^n} - h} < W\left(\frac{2m\pi}{3^n}\right) + \sigma\sqrt{\frac{\pi}{3^n} - h}$ for $h \in [\frac{5\pi}{3^{n+2}}, \frac{\pi}{3^n}]$. But we can show that $W\left(\frac{2m\pi}{3^n}\right) + \sigma\sqrt{\frac{\pi}{3^n} - h} < W\left(\frac{2m\pi}{3^n}\right) + \sigma\sqrt{h}$ for $h \in [\frac{5\pi}{3^{n+2}}, \frac{\pi}{3^n}]$ by showing that the min of the left-hand side is greater than the max of the right-hand side. Indeed, $W\left(\frac{2m\pi}{3^n}\right) + \sigma\sqrt{\frac{\pi}{3^n} - \frac{5\pi}{3^{n+2}}} - \left(W\left(\frac{2m\pi}{3^n}\right) + \sigma\sqrt{\frac{5\pi}{3^{n+2}}}\right) > \frac{1}{\sqrt{3}^n}(.58202165) > 0$. Therefore, $|L\left(\frac{2m\pi}{3^n}, h\right)| \leq \sigma$ for $h \in (0, \frac{\pi}{3^n}]$.

Now let $a < b \in \mathbb{R}$. If $|b-a| \geq \pi$ then $|L(a, |b-a|)| < \frac{3+\sqrt{3}}{\pi} < \sigma$. So assume that $|b-a| < \pi$. We may assume that $a, b \in [0, \pi]$ since $W(2k\pi + t) = W(t)$ and $W(\pi + t) = -W(t)$. If π is between a and b after moving them, then we can use the fact that $W(k\pi + h) = W(k\pi - h)$ to show that $|L(a, |k\pi + h - a|)| \leq |L(a, |k\pi - h - a|)|$. Then there exists a maximal N so that $\frac{m\pi}{3^N} \leq a < b \leq \frac{(m+1)\pi}{3^N}$. If $a < c \leq b$ for $c \in \{\frac{(3m+1)\pi}{3^{N+1}}, \frac{(3m+2)\pi}{3^{N+1}}\}$ then

$$\begin{aligned}|W(b) - W(a)| &\leq |W(b) - W(c)| + |W(c) - W(a)| \\ &\leq \sigma\sqrt{b - c} + \sigma\sqrt{c - a}\end{aligned}$$

$$
\begin{aligned}
&= \sigma\sqrt{b-a+2\sqrt{b-c}\sqrt{c-a}} \\
&\leq \sigma\sqrt{b-a+2\sqrt{\tfrac{b-a}{2}}\sqrt{\tfrac{b-a}{2}}} \\
&= \sqrt{2}\sigma\sqrt{b-a}.
\end{aligned}
$$

On the other hand, if $a < \frac{(3m+1)\pi}{3^{N+1}} = c$ and $b > \frac{(3m+2)\pi}{3^{N+1}} = d$, then

$$
\begin{aligned}
|W(b)-W(a)| &\leq |W(b)-W(d)| + |W(c)-W(d)| + |W(c)-W(a)| \\
&\leq \sigma\sqrt{b-d} + \sigma\sqrt{c-d} + \sigma\sqrt{c-a} \\
&= \sigma\sqrt{b-a+2\sqrt{b-d}\sqrt{c-a}+2\sqrt{b-d}\sqrt{d-c}+2\sqrt{d-c}\sqrt{c-a}} \\
&\leq \sigma\sqrt{b-a+2\sqrt{\tfrac{b-a}{3}}\sqrt{\tfrac{b-a}{3}}+2\sqrt{\tfrac{b-a}{3}}\sqrt{\tfrac{b-a}{3}}+2\sqrt{\tfrac{b-a}{3}}\sqrt{\tfrac{b-a}{3}}} \\
&= \sqrt{3}\sigma\sqrt{b-a}.
\end{aligned}
$$

$\square$

Chapter 4

A Rough Path Approach to Understanding the Weierstrass Function

The purpose of this chapter is to explore the possibility of understanding the trace (or lack thereof) generated by the Weierstrass function. The first section will be a brief overview of the relevant results from rough path theory. The second section will be an in-depth justification of our use of the Weierstrass function in the rough path setting. The last section will be a musing on a few findings.

4.1 Rough Path Theory Background

Let us start by carefully defining some terms. A *partition* $(P_n) = \{t_{k,n}\}$ of $[0, T]$ will be identified by points in $[0, T]$ rather than the intervals created by those points. The variable n will identify one in a sequence of partitions and the variable k will identify a particular point in a given partition. We will use P to denote that a function or quantity depends on the limit of (P_n). The *mesh size* of a partition is $\max_k |t_{k,n} - t_{k-1,n}|$.

Definition 4.1. *We say that a function $\lambda : [0, T] \to \mathbb{R}$ has finite quadratic-variation in the sense of Föllmer if along some fixed sequence of partitions $P = (P_n)$ of $[0, T]$ with mesh size* *going to zero*

$$[\lambda]_t^P = \lim_{n\to\infty} \sum_{[r,s]\in P_n} (\lambda_{s\wedge t} - \lambda_{r\wedge t})^2 \tag{4.1}$$

exists and is continuous in t. Call the function $[\lambda]_t^P$ the bracket of λ along $(P_n)_{n=0}^\infty$.

Finite quadratic-variation in the sense of Föllmer contrasts with quadratic variation in that for stochastic processes, quadratic variation does not specify a partition, and it contrasts with bounded 2-variation in that we do not consider the supremum over all partitions. We also require finite quadratic-variation in the sense of Föllmer to be continuous in t. We say that the bracket is Lip(1) if for all $t, s \in [0,T]$ there exists a $\kappa < \infty$ such that $|[\lambda]_t - [\lambda]_s| \leq \kappa|t-s|$.

As an example, when (P_n) is a nested sequence of partitions with mesh size going to zero, and B_t is a standard Brownian motion it can be shown that the bracket $[\sqrt{\kappa}B]_t^P$ is Lip(1) and $|[\sqrt{\kappa}B]_t - [\sqrt{\kappa}B]_s| \leq \kappa|t-s|$ with probability one. For a more in-depth explanation and further background, see [1].

4.2 Quadratic Variation for the Weierstrass Function

In [2], Friz and Shekhar use rough path theory to show Theorem 1.7, which guarantees the existence of a trace for the class of (random) weak Lip(1/2) functions with finite quadratic-variation in the sense of Föllmer and Lip(1) bracket with Lip(1) norm less than 2. We already know that $cW(t) = cW_3(t)$ is Lip(1/2). Here we will show that $cW(t)$ has finite quadratic-variation in the sense of Föllmer and we will show that the bracket $[cW]_t^P$ is linear (hence Lip(1)) for two choices of partition, (P_n).

Theorem 4.2. *Consider the partitions $(P_n) = \{\frac{k\pi}{3^n}\}_{k=0}^\infty$ and $(P_n') = \{\frac{2k\pi}{3^n}\}_{k=0}^\infty$. The bracket $[W]_t$ is linear with respect to both partitions and $[W]_t^P = \left(\frac{12+6\sqrt{3}}{\pi} + \sum_{j=1}^\infty \frac{(2)3^j}{\pi}\sin^2\left(\frac{\pi}{(2)3^j}\right)\right)t$ and $[W]_t^{P'} = \left(\sum_{j=1}^\infty \frac{3^j}{\pi}\sin^2\left(\frac{\pi}{3^j}\right)\right)t$ for $t>0$.*

For context, $\frac{12+6\sqrt{3}}{\pi} + \sum_{j=1}^\infty \frac{(2)3^j}{\pi}\sin^2\left(\frac{\pi}{(2)3^j}\right) \approx 7.865$ and $\sum_{j=1}^\infty \frac{3^j}{\pi}\sin^2\left(\frac{\pi}{3^j}\right) \approx 1.225$. We will focus on the partition $(P_n) = \{\frac{k\pi}{3^n}\}_{k=0}^\infty$ since the proofs for the other partition are similar. This partition (P_n) identifies points on the real line according to location of the local minimums and maximums of $W(t)$, whereas the other partition identifies only the maximums. For convenience define

$$A_n = \sum_{k=1}^{3^n} \left[W\left(\frac{k\pi}{3^n}\right) - W\left(\frac{(k-1)\pi}{3^n}\right)\right]^2. \tag{4.2}$$

Notice that we are only considering the minimums and maximums on the interval $[0, \pi]$, and as a result $\lim\limits_{n\to\infty} A_n = [W]_\pi^P$. Similarly define

$$A'_n = \sum_{k=1}^{3^n} \left[W\left(\frac{2k\pi}{3^n}\right) - W\left(\frac{2(k-1)\pi}{3^n}\right)\right]^2. \tag{4.3}$$

Here we consider the maximums on the interval $[0, 2\pi]$. Let us also define the quantities

$$Q_n = \sum_{k=1}^{3^n} \left[\cos\left(\frac{k\pi}{3^n}\right) - \cos\left(\frac{(k-1)\pi}{3^n}\right)\right]^2 \tag{4.4}$$

and

$$Q'_n = \sum_{k=1}^{3^n} \left[\cos\left(\frac{2k\pi}{3^n}\right) - \cos\left(\frac{2(k-1)\pi}{3^n}\right)\right]^2. \tag{4.5}$$

Ultimately, we will show that $[W]_\pi^P = A_0 + \sum\limits_{j>0} Q_j$ and $[W]_{2\pi}^{P'} = A'_0 + \sum\limits_{j>0} Q'_j$. Thinking of the bracket in this form will simplify the task of computing it.

The idea behind this first result is that considering minimums and maximums on the interval $[0, 3\pi]$ is the same as considering the minimums and maximums on $[0, \pi]$ three times over. Analogously, considering maximums on $[0, 6\pi]$ is the same as considering the maximums on $[0, 2\pi]$ three times over.

Lemma 4.3. *Let $n \geq 1$. Then*

$$\sum_{k=1}^{3^n} \left[W\left(\frac{k\pi}{3^{n-1}}\right) - W\left(\frac{(k-1)\pi}{3^{n-1}}\right)\right]^2 = 3A_{n-1}.$$

Similarly,

$$\sum_{k=1}^{3^n} \left[W\left(\frac{2k\pi}{3^{n-1}}\right) - W\left(\frac{2(k-1)\pi}{3^{n-1}}\right)\right]^2 = 3A'_{n-1}.$$

Proof. The proof for the (P_n) partition is a straightforward calculation. The first and last thirds are identical, and the middle third picks up a negative sign (since $\cos(\pi + x) = -\cos(x)$) which is mitigated by squaring.

$$
\begin{aligned}
&\sum_{k=1}^{3^n}\left[W\left(\frac{k\pi}{3^{n-1}}\right) - W\left(\frac{(k-1)\pi}{3^{n-1}}\right)\right]^2 \\
&= \sum_{k=1}^{3^{n-1}}\left[W\left(\frac{k\pi}{3^{n-1}}\right) - W\left(\frac{(k-1)\pi}{3^{n-1}}\right)\right]^2 + \sum_{k=3^{n-1}+1}^{2\cdot 3^{n-1}}\left[W\left(\frac{k\pi}{3^{n-1}}\right) - W\left(\frac{(k-1)\pi}{3^{n-1}}\right)\right]^2 \\
&+ \sum_{k=2\cdot 3^{n-1}+1}^{3^n}\left[W\left(\frac{k\pi}{3^{n-1}}\right) - W\left(\frac{(k-1)\pi}{3^{n-1}}\right)\right]^2 \\
&= \sum_{k=1}^{3^{n-1}}\left[W\left(\frac{k\pi}{3^{n-1}}\right) - W\left(\frac{(k-1)\pi}{3^{n-1}}\right)\right]^2 + \sum_{k=1}^{3^{n-1}}\left[W\left(\pi + \frac{k\pi}{3^{n-1}}\right) - W\left(\pi + \frac{(k-1)\pi}{3^{n-1}}\right)\right]^2 \\
&+ \sum_{k=1}^{3^{n-1}}\left[W\left(2\pi + \frac{k\pi}{3^{n-1}}\right) - W\left(2\pi + \frac{(k-1)\pi}{3^{n-1}}\right)\right]^2 \\
&= 3A_n.
\end{aligned}
$$

The proof for the (P'_n) partition is even simpler since all three thirds are identical.

$\square$

The next result is simple, but it plays a significant role in showing Lemma 4.5.

Lemma 4.4. *For all $x \in \mathbb{R}$, the following hold:*

(i) $\cos(x) + \cos\left(\frac{2\pi}{3} + x\right) = \cos\left(\frac{\pi}{3} + x\right)$.

(ii) $[\cos(kx) - \cos((k-1)x)]^2$
$= [\cos(x) - 1]\,[2\cos^2(kx)\cos(x) + 2\sin(kx)\sin(x)\cos(kx) - \cos(x) - 1]$

Proof. The first part relies on the angle sum identity for cosine:

$$
\begin{aligned}
\cos(x) + \cos(2\pi/3 + x) &= \cos(x) + \cos(2\pi/3)\cos(x) - \sin(2\pi/3)\sin(x) \\
&= \cos(x) + -1/2\cos(x) - \sqrt{3}/2\sin(x) \\
&= 1/2\cos(x) - \sqrt{3}/2\sin(x) \\
&= \cos(\pi/3)\cos(x) - \sin(\pi/3)\sin(x)
\end{aligned}
$$

$$= \cos\left(\pi/3 + x\right).$$

The second part is more complicated. First we square the expression, then use the angle sum identity:

$$\begin{aligned}
&\left[\cos\left(kx\right) - \cos\left((k-1)x\right)\right]^2 \\
&= \cos^2(kx) + \cos^2((k-1)x) - 2\cos(kx)\cos((k-1)x) \\
&= \cos^2(kx) + \left[\cos(kx)\cos(x) + \sin(kx)\sin(x)\right]^2 - 2\cos(kx)\left[\cos(kx)\cos(x) + \sin(kx)\sin(x)\right] \\
&= \cos^2(kx) + \cos^2(kx)\cos^2(x) + \sin^2(kx)\sin^2(x) + 2\cos(kx)\cos(x)\sin(kx)\sin(x) \\
&- 2\cos^2(kx)\cos(x) - 2\cos(kx)\sin(kx)\sin(x).
\end{aligned}$$

Then we can use $\sin^2(t) = 1 - \cos^2(t)$ to obtain:

$$\begin{aligned}
&\left[\cos\left(kx\right) - \cos\left((k-1)x\right)\right]^2 \\
&= \cos^2(kx) + \cos^2(kx)\cos^2(x) + 1 - \cos^2(x) - \cos^2(kx) + \cos^2(kx)\cos^2(x) \\
&+ 2\cos(kx)\cos(x)\sin(kx)\sin(x) - 2\cos^2(kx)\cos(x) - 2\cos(kx)\sin(kx)\sin(x) \\
&= 2\cos^2(kx)\cos^2(x) + 1 - \cos^2(x) \\
&+ 2\cos(kx)\cos(x)\sin(kx)\sin(x) - 2\cos^2(kx)\cos(x) - 2\cos(kx)\sin(kx)\sin(x).
\end{aligned}$$

The rest relies on algebra in particular we reorder the terms and factor out $(\cos(x) - 1)$:

$$\begin{aligned}
&\left[\cos\left(kx\right) - \cos\left((k-1)x\right)\right]^2 \\
&= 2\cos^2(kx)\cos^2(x) + 2\cos(kx)\cos(x)\sin(kx)\sin(x) - \cos^2(x) \\
&- 2\cos^2(kx)\cos(x) - 2\cos(kx)\sin(kx)\sin(x) + 1 \\
&= 2\cos^2(kx)\cos^2(x) + 2\cos(kx)\cos(x)\sin(kx)\sin(x) - \cos^2(x) - \cos(x) \\
&- 2\cos^2(kx)\cos(x) - 2\cos(kx)\sin(kx)\sin(x) + \cos(x) + 1 \\
&= \left[\cos(x) - 1\right]\left[2\cos^2(kx)\cos(x) + 2\sin(kx)\sin(x)\cos(kx) - \cos(x) - 1\right]
\end{aligned}$$

$\square$

The next result relies on the scaling property of the Weierstrass function, i.e. $W(t) = \cos(t) + \frac{1}{\sqrt{3}}W(3t)$.

Lemma 4.5. *Let $n \geq 1$, and let A_n, Q_n, A'_n, and Q'_n be defined by 4.2-4.5. Then $A_n = A_{n-1} + Q_n$ and $A'_n = A'_{n-1} + Q'_n$.*

Proof. Like other proofs, we will go into detail with the (P_n) partition and note the differences with the (P'_n) partition. We start by taking advantage of the scaling of the Weierstrass function. This lets us separate A_n into A_{n-1}, Q_n and a remainder term, R_n:

$$\begin{aligned}
A_n &= \sum_{k=1}^{3^n} \left[W\left(\frac{k\pi}{3^n}\right) - W\left(\frac{(k-1)\pi}{3^n}\right)\right]^2 \\
&= \sum_{k=1}^{3^n} \left[\cos\left(\frac{k\pi}{3^n}\right) - \cos\left(\frac{(k-1)\pi}{3^n}\right)\right]^2 + 1/3\left[W\left(\frac{k\pi}{3^{n-1}}\right) - W\left(\frac{(k-1)\pi}{3^{n-1}}\right)\right]^2 \\
&+ 2/\sqrt{3}\left[W\left(\frac{k\pi}{3^{n-1}}\right) - W\left(\frac{(k-1)\pi}{3^{n-1}}\right)\right] \cdot \left[\cos\left(\frac{k\pi}{3^n}\right) - \cos\left(\frac{(k-1)\pi}{3^n}\right)\right] \\
&= Q_n + A_{n-1} + R_n
\end{aligned}$$

It remains to show that $R_n := \frac{2}{\sqrt{3}} \sum_{k=1}^{3^n} \left[W\left(\frac{k\pi}{3^{n-1}}\right) - W\left(\frac{(k-1)\pi}{3^{n-1}}\right)\right] \cdot \left[\cos\left(\frac{k\pi}{3^n}\right) - \cos\left(\frac{(k-1)\pi}{3^n}\right)\right]$ is zero. We do this by splitting the sum into thirds:

$$\begin{aligned}
R_n &= \frac{2}{\sqrt{3}} \sum_{k=1}^{3^{n-1}} \left[W\left(\tfrac{k\pi}{3^{n-1}}\right) - W\left(\tfrac{(k-1)\pi}{3^{n-1}}\right)\right] \cdot \left[\cos\left(\tfrac{k\pi}{3^n}\right) - \cos\left(\tfrac{(k-1)\pi}{3^n}\right)\right] \\
&+ \frac{2}{\sqrt{3}} \sum_{k=3^{n-1}+1}^{2\cdot 3^{n-1}} \left[W\left(\tfrac{k\pi}{3^{n-1}}\right) - W\left(\tfrac{(k-1)\pi}{3^{n-1}}\right)\right] \cdot \left[\cos\left(\tfrac{k\pi}{3^n}\right) - \cos\left(\tfrac{(k-1)\pi}{3^n}\right)\right] \\
&+ \frac{2}{\sqrt{3}} \sum_{k=2\cdot 3^{n-1}+1}^{3^n} \left[W\left(\tfrac{k\pi}{3^{n-1}}\right) - W\left(\tfrac{(k-1)\pi}{3^{n-1}}\right)\right] \cdot \left[\cos\left(\tfrac{k\pi}{3^n}\right) - \cos\left(\tfrac{(k-1)\pi}{3^n}\right)\right] \\
&= \frac{2}{\sqrt{3}} \sum_{k=1}^{3^{n-1}} \left[W\left(\tfrac{k\pi}{3^{n-1}}\right) - W\left(\tfrac{(k-1)\pi}{3^{n-1}}\right)\right] \left[\cos\left(\tfrac{k\pi}{3^n}\right) - \cos\left(\tfrac{\pi}{3} + \tfrac{k\pi}{3^n}\right) + \cos\left(\tfrac{2\pi}{3} + \tfrac{k\pi}{3^n}\right)\right] \\
&- \frac{2}{\sqrt{3}} \sum_{k=1}^{3^{n-1}} \left[W\left(\tfrac{k\pi}{3^{n-1}}\right) - W\left(\tfrac{(k-1)\pi}{3^{n-1}}\right)\right] \left[\cos\left(\tfrac{(k-1)\pi}{3^n}\right) - \cos\left(\tfrac{\pi}{3} + \tfrac{(k-1)\pi}{3^n}\right) + \cos\left(\tfrac{2\pi}{3} + \tfrac{(k-1)\pi}{3^n}\right)\right].
\end{aligned}$$

The middle cosine terms pick up a negative sign since we change from describing the minimums and maximums over $[\pi, 2\pi]$ to describing the minimums and maximums over $[0, \pi]$. By Lemma 4.4(i), these last two terms are zero. Hence, $A_n = A_{n-1} + Q_n$.

The proof for the (P'_n) partition is analogous except that the middle cosine term does not pick up a negative sign, and we add $\frac{2\pi}{3}$ and $\frac{4\pi}{3}$ to the angles instead of $\frac{\pi}{3}$ and $\frac{2\pi}{3}$. $\cos(x) + \cos\left(x + \frac{2\pi}{3}\right) + \cos\left(\frac{4\pi}{3}\right) = 0$ since this is the sum of the real part of three points equidistant on the unit circle. Therefore we also have $A'_n = A'_{n-1} + Q'_n$. $\square$

With Lemma 4.5 and Lemma 4.4(ii), we easily get the following:

Lemma 4.6. *The quadratic variation of $W(t)$ from 0 to π with respect to the partition $P_n = \{k\pi/3^n\}_{k=0}^{\infty}$ is finite and $[W]_\pi^P = 12 + 6\sqrt{3} + \sum_{j=1}^{\infty} (2)3^j \sin^2\left(\frac{\pi}{(2)3^j}\right)$. The quadratic variation is also finite from 0 to 2π with respect to the partition $P'_n = \{2k\pi/3^n\}_{k=0}^{\infty}$ and $[W]_{2\pi}^{P'} = \sum_{j=1}^{\infty} (2)3^j \sin^2\left(\frac{\pi}{3^j}\right)$.*

It should be noted that this does not mean that we have shown that $W(t)$ has finite quadratic-variation in the sense of Föllmer. This only means that the quadratic variation from 0 to π (respectively 0 to 2π) is finite when we restrict our attention to our partitions.

Proof. First note that by Lemma 4.5 $[W]_\pi^P = \lim_{n\to\infty} A_n = A_0 + \sum_{j>1} Q_j$. So it remains to find A_0 and $\sum Q_j$. A quick calculation shows that $A_0 = (3 + \sqrt{3})^2$, and Lemma 4.4(ii) will give us a computable form for Q_n:

$$Q_n = \sum_{j=1}^{3^n} \left[\cos\left(\tfrac{\pi}{3^n}\right) - 1\right] \left[2\cos^2\left(\tfrac{k\pi}{3^n}\right)\cos\left(\tfrac{\pi}{3^n}\right) + 2\sin\left(\tfrac{k\pi}{3^n}\right)\sin\left(\tfrac{\pi}{3^n}\right)\cos\left(\tfrac{k\pi}{3^n}\right) - \cos\left(\tfrac{\pi}{3^n}\right) - 1\right]$$
$$= 3^n\left[1 - \cos\left(\tfrac{\pi}{3^n}\right)\right] + \cos\left(\tfrac{\pi}{3^n}\right)\sum_{j=1}^{3^n}\left[2\cos^2\left(\tfrac{k\pi}{3^n}\right) - 1\right] + \sin\left(\tfrac{\pi}{3^n}\right)\sum_{j=1}^{3^n} 2\sin\left(\tfrac{k\pi}{3^n}\right)\cos\left(\tfrac{k\pi}{3^n}\right).$$

Then by the half angle formulas we get

$$= 3^n\left[1 - \cos\left(\tfrac{\pi}{3^n}\right)\right] + \cos\left(\tfrac{\pi}{3^n}\right)\sum_{j=1}^{3^n}\cos\left(\tfrac{2k\pi}{3^n}\right) + \sin\left(\tfrac{\pi}{3^n}\right)\sum_{j=1}^{3^n}\sin\left(\tfrac{2k\pi}{3^n}\right)$$
$$= 3^n\left[1 - \cos\left(\tfrac{\pi}{3^n}\right)\right] + \cos\left(\tfrac{\pi}{3^n}\right)\cdot 0 + \sin\left(\tfrac{\pi}{3^n}\right)\cdot 0.$$

The sums are zero since they are the sums of the real and imaginary parts of n^{th} roots of unity. Thus we have that

$$\begin{aligned} [W]_\pi^P &= 12 + 6\sqrt{3} + \sum_{j=1}^{\infty} 3^j \left[1 - \cos\left(\tfrac{\pi}{3^j}\right)\right] \\ &= 12 + 6\sqrt{3} + \sum_{j=1}^{\infty} (2) 3^j \sin\left(\tfrac{\pi}{(2)3^j}\right). \end{aligned}$$

The same argument shows that

$$\begin{aligned} Q'_n &= 3^n \left[1 - \cos\left(\tfrac{2\pi}{3^n}\right)\right] + \cos\left(\tfrac{2\pi}{3^n}\right) \sum_{j=1}^{3^n} \cos\left(\tfrac{4k\pi}{3^n}\right) + \sin\left(\tfrac{2\pi}{3^n}\right) \sum_{j=1}^{3^n} \sin\left(\tfrac{4k\pi}{3^n}\right) \\ &= 3^n \left[1 - \cos\left(\tfrac{2\pi}{3^n}\right)\right] + \cos\left(\tfrac{2\pi}{3^n}\right) \cdot 0 + \sin\left(\tfrac{2\pi}{3^n}\right) \cdot 0. \end{aligned}$$

This time the first half of the terms are identical to the "even numbered" roots of unity from the previous sums while the second half of the terms give the "odd numbered" roots of unity. Hence we get

$$[W]_{2\pi}^{P'} = A'_0 + \sum_{j=1}^{\infty} 3^j \left[1 - \cos\left(\tfrac{2\pi}{3^j}\right)\right] = \sum_{j=1}^{\infty} (2) 3^j \sin\left(\tfrac{\pi}{3^j}\right).$$

□

Now we will show that the quadratic variation is finite for intervals of the form $[0, t]$ for both partitions. As a first step we have the following lemma which will allow us to use the scaling of the Weierstrass function later.

Lemma 4.7. *The covariation,* $[\cos, W(3\cdot)]_{[s,t]}^P$, *of* $\cos(x)$ *and* $W(3x)$ *along either* (P_n) *or* (P'_n) *is zero.*

Proof. Let $t_{k,n}$ denote the points identified in one of the partitions (that is $\frac{k\pi}{3^n}$ or $\frac{2k\pi}{3^n}$), and let . Then we have that

$$[\cos, W(3\cdot)]_{[s,t]}^P = \lim_{n\to\infty} \sum_{t_{k,n}\in[s,t]} \left[\cos\left(t_{k,n}\right) - \cos\left(t_{k-1,n}\right)\right] \left[W\left(t_{k,n-1}\right) - W\left(t_{k-1,n-1}\right)\right]$$

$$
\begin{aligned}
&= \lim_{n\to\infty} \sum_{t_{k,n}\in[s,t]} \left[W\left(t_{k,n-1}\right) - W\left(t_{k-1,n-1}\right)\right] \int_{t_{k-1,n}}^{t_{k,n}} \sin(x)dx \\
&\leq \lim_{n\to\infty} ||W||_{1/2}\sqrt{|t_{k,n-1}-t_{k-1,n-1}|}\int_s^t \sin(x)dx \\
&= 0
\end{aligned}
$$

since the mesh size of our partitions approaches 0 in n. Similarly, $[\cos, W(3\cdot)]^P_{[s,t]} \geq \lim_{n\to\infty} -||W||_{1/2}\sqrt{|t_{k,n-1}-t_{k-1,n-1}|}\int_s^t \sin(x)dx = 0$ □

Lemma 4.8. *Let $j \geq 1$ and $l \geq 0$ be integers. Then $[W]^P_{[l\pi/3^j,(l+1)\pi/3^j]} = 3^{-j}[W]^P_\pi$ and $[W]^{P'}_{[2l\pi/3^j,2(l+1)\pi/3^j]} = 3^{-j}[W]^{P'}_{2\pi}$.*

Proof. The idea here is to leverage the scaling of $W(t)$. First note that the quadratic variation of cosine, $[\cos]_{[s,t]} = 0$ since $\cos(x)$ is continuously differentiable. This along with Lemma 4.7 implies that

$$
\begin{aligned}
[W]^P_{[l\pi/3^j,(l+1)\pi/3^j]} &= \lim_{n\to\infty}[W]^{P_n}_{\left[\frac{l\pi}{3^j},\frac{(l+1)\pi}{3^j}\right]} \\
&= [\cos]_{\left[\frac{l\pi}{3^j},\frac{(l+1)\pi}{3^j}\right]} + [\cos, W(3\cdot)]_{\left[\frac{l\pi}{3^j},\frac{(l+1)\pi}{3^j}\right]} + \frac{1}{3}\lim_{n\to\infty}[W(3\cdot)]^{P_n}_{\left[\frac{l\pi}{3^j},\frac{(l+1)\pi}{3^j}\right]} \\
&= \frac{1}{3}\lim_{n\to\infty}[W]^{P_n}_{\left[\frac{l\pi}{3^{j-1}},\frac{(l+1)\pi}{3^{j-1}}\right]}
\end{aligned}
$$

Since $\lim_{n\to\infty}[W]^{P_n}_{[l\pi,(l+1)\pi]} = \lim_{n\to\infty}[W]^{P_n}_{[0,\pi]} = [W]^P_\pi$ exists by Lemma 4.6, we get the result by induction on j. The argument for the partition (P'_n) is analogous. □

With control on small scales, we can prove Theorem 4.2.

Proof of Theorem 4.2. By Lemma 4.8,

$$
\begin{aligned}
[W]^P_t &= \lim_{j\to\infty}\lim_{n\to\infty}\sum_{k=1}^{\lfloor 3^j t/\pi\rfloor}\left[W\left(\frac{k\pi}{3^n}\right) - W\left(\frac{(k-1)\pi}{3^n}\right)\right]^2 \\
&= \lim_{j\to\infty}\lfloor 3^j t/\pi\rfloor 3^{-j}[W]^P_\pi,
\end{aligned}
$$

where $\lfloor 3^j t/\pi\rfloor$ counts how many terms we have in the sum and 3^{-j} is the appropriate scale. But $\lim_{j\to\infty}\lfloor 3^j t/\pi\rfloor 3^{-j} = t/\pi$. Hence, $[W]^P_t = \frac{t}{\pi}[W]^P_\pi$. An analogous argument shows that $[W]^{P'}_t = \frac{t}{2\pi}[W]^{P'}_{2\pi}$. The values for $[W]^P_\pi$ and $[W]^{P'}_{2\pi}$ from Lemma 4.6 complete the proof. □

4.3 A Discussion of the Friz-Shekhar Approach

For an inverse Loewner chain $f_t(z)$ defined by 1.3, the trace exists if and only if $\lim_{y\to 0^+} f_t(\lambda_t+iy)$ exists and is continuous in t. However, it is more common to use the following when trying to prove the existence of the trace:

Theorem 4.9 ([7],Proposition 2.19)**.** *Suppose that $\lambda_t : [0,T] \to \mathbb{R}$ is a continuous function and f_t is the solution to 1.3. Suppose that there exists $\nu : (0,1] \to (0,1]$ with $\nu(0+) = 0$ such that for all $0 \leq t \leq T$ and all $\epsilon < 1$,*

$$\int_0^\epsilon |f_t'(\lambda_t + iy)| dy \leq \nu(\epsilon). \tag{4.6}$$

Then the hull is generated by a curve.

In this way, one can show the existence of the trace by bounding $|f_t'(\lambda_t + iy)|$ by an integrable function of y. The next result can be viewed as a corollary to either Theorem 4.9 or Theorem 2.1.

Corollary 4.10 ([2], Lemma 2)**.** *Suppose there exists a $\theta < 1$ and $y_0 > 0$ such that for all $y \in (0, y_0]$*

$$\sup_{t\in[0,T]} |f_t'(\lambda_t + iy)| \leq y^{-\theta}$$

then the trace exists.

The following is a common way to write $|f_t'(\lambda_t + iy)|$:

Theorem 4.11 ([2], Lemma 1)**.** *Fix $t \geq 0$ and define $\beta_s = \lambda_t - \lambda_{t-s}$, $0 \leq s \leq t$. Then*

$$\log(|f_t'(x + \lambda_t + iy)|) = \int_0^t \frac{2(x_s^2 - y_s^2)}{(x_s^2 + y_s^2)^2} ds \tag{4.7}$$

where (x_s, y_s) is the solution of the ODE

$$dx_s = d\beta_s - \frac{2x_s}{x_s^2 + y_s^2} ds, \;\; x_0 = x$$

$$dy_s = \frac{2y_s}{x_s^2 + y_s^2}, \;\; y_0 = y.$$

Moreover, $G_s := \beta_s - x_s$ *defines a* C^1*-function with*

$$\dot{G}_s = \frac{2x_s}{x_s^2 + y_s^2}.$$

In [2], Friz and Shekhar then use rough path theory to get

Corollary 4.12 ([2], Proposition 1)**.** *Fix* $t \geq 0$*. Let* λ *have finite quadratic variation in the sense of Föllmer. With* G, β *as in Theorem 4.11,*

$$\log(|f_t'(x + \lambda_t + iy)|) = M_t^P + \frac{1}{2}\int_0^t \dot{G}_s' d[\beta]_s^P - \int_0^t \dot{G}_s^2 ds + \log\left(\frac{y_t}{y}\right) - \log\left(\frac{x_t^2 + y_t^2}{x^2 + y^2}\right) \quad (4.8)$$

with the Gubinelli derivative $\dot{G}_s' := \dot{y}_s/y_s - \dot{G}_s^2$ *and Föllmer-Itô integral*

$$M_t^P = \lim_n \sum \lim_{[u,v]\in P_n} \dot{G}_u(\beta_v - \beta_u) =: \int_0^t \dot{G} d^P \beta.$$

For fixed $t \in P_n$ we have $[\beta]_s^P = [W]_s^P$. Because we know that $[W]_t^P = \kappa t$ for $\kappa = [W]_1^P$, we can show that

$$\frac{1}{2}\int_0^t \dot{G}_s' d[W]_s^P = \frac{\kappa}{2}\int_0^t (\dot{y}_s/y_s - \dot{G}_s^2) ds = \frac{\kappa}{2}\log\left(\frac{y_t}{y}\right) - \frac{\kappa}{2}\int_0^t \dot{G}_s^2 ds. \quad (4.9)$$

We can combine this with Corollary 4.10 to obtain the following:

Theorem 4.13. *Let* $\theta < 1$ *and* $\beta_s = W(t) - W(t-s)$*. Then if*

$$\int_0^t \dot{G}_s d\beta_s^P - (\kappa/2 + 1)\int_0^t \dot{G}_s^2 ds \leq c + \log\left(\frac{x_t^2 + y_t^2}{y_t^{\kappa/2+1}}\right) + (\kappa/2 - 1 - \theta)\log(y)$$

for some constant c, then the trace exists. If we use Theorem 2.1, then if $y_0 = 2^{-n}$ *and*

$$\int_0^{k2^{-2n}} \dot{G}_s d\beta_s^P - (\kappa/2 + 1)\int_0^{k2^{-2n}} \dot{G}_s^2 ds \leq c + \log\left(\frac{x_{k2^{-2n}}^2 + y_{k2^{-2n}}^2}{y_{k2^{-2n}}^{\kappa/2+1}}\right) - (\kappa/2 - 1 - \theta)\log(2)n$$

for some constant c, then the trace exists.

Proof. The result follows from Corollary 4.10 and Corollary 4.12. If we wish to satisfy the hypothesis of Corollary 4.10, then we must have $|f_t'(iy + W(t))| \leq c y_0^{-\theta}$. But by Theorem

4.12, this is the same as having

$$\exp\left\{\int_0^t \dot{G}_r d\beta_r^P + \frac{1}{2}\int_0^t \dot{G}_r' d[W]_r^P + \int_0^t \dot{G}_r^2 dr\right\}\frac{y_t}{y_0}\frac{y_0^2}{x_t^2+y_t^2} \le c y_0^{-\theta},$$

since $x_0 = 0$. Then

$$\exp\left\{\int_0^t \dot{G}_r d\beta_r^P + \frac{\kappa}{2}\int_0^t \dot{y}_r/y_r - G_r^2 dr + \int_0^t \dot{G}_r^2 dr\right\}\frac{y_t}{y_0}\frac{y_0^2}{x_t^2+y_t^2} \le c y_0^{-\theta}.$$

Finally, we can use 4.9 to get

$$\exp\left\{\int_0^t \dot{G}_r d\beta_r^P - \left(\frac{\kappa}{2}+1\right)\int_0^t \dot{G}_r^2 dr\right\}\frac{y_t^{\kappa/2+1}}{y_0^{\kappa/2+1}}\frac{y_0^2}{x_t^2+y_t^2} \le c y_0^{-\theta}.$$

The proof for the combination with Theorem 2.1 follows easily from this. In Theorem 2.1, we needed the bound $|f_t'(z)| \le c2^{(1-\epsilon)n}$, but then we can set $\theta = 1-\epsilon$ and repeat the calculations.

□

Unlike 4.6, Theorem 4.13 has a clearer dependence on the properties of the driving function. However, it is still unclear if the property of having a linear bracket is sufficient to guarantee a trace.

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Vita

The Author graduated with a Bachelor of Science degree in Mathematics from the University of Tennessee at Martin in spring of 2011. In fall of 2011, the author was admitted to the Mathematics graduate program at the University of Tennessee at Knoxville. The author was awarded a Master of Science degree in Mathematics in spring of 2017 and a PhD in Mathematics in spring 2018 from the University of Tennessee at Knoxville.