October 21, 1960

To the Graduate Council:

I am submitting herewith a thesis written by Clayton D. Zerby entitled "Electromagnetic Production of Pion Pairs." I recommend that it be accepted in partial fulfillment of the requirements for the degree of Doctor of Philosophy, with a major in Physics.

R. D. Present
Major Professor

We have read this thesis and recommend its acceptance:

Edward G. Harris

John W. Neuberger

D. T. King.

T. A. Welton

Accepted for the Council:

H. E. Spivey
Acting Dean of the Graduate School

ELECTROMAGNETIC PRODUCTION OF PION PAIRS

A Dissertation

Presented to

the Graduate Council of

The University of Tennessee

In Partial Fulfillment

of the Requirements for the Degree

Doctor of Philosophy

---

by

Clayton D. Zerby

December, 1960

ACKNOWLEDGMENTS

The author is deeply indebted to Dr. T. A. Welton, who suggested this problem and under whose guidance it was carried out. The many stimulating discussions with him which were possible only by a generous allocation of his time contributed markedly to every phase of this work.

The author is also grateful to the administration of the Oak Ridge National Laboratory, who made their facilities available in the completion of the work, and is particularly appreciative of the consideration of the Director of the Neutron Physics Division, Mr. E. P. Blizard, for arranging the work schedule so that this problem could be considered and carried to completion.

479948

TABLE OF CONTENTS

CHAPTER		PAGE
I.	INTRODUCTION	1
	History of Problem	2
	Objectives of the Study	5
	Organization of the Report	6
	Notations and Units	7
II.	GENERAL CONSIDERATIONS	8
	The Charge Distribution	8
	The Heavy-Nucleus Approximation	12
III.	THE OPTICAL POTENTIAL AND ITS INTRODUCTION INTO THE FIELD EQUATIONS	14
	Justification of the Optical Model	14
	Introduction of the Complex Optical Potential into the Field Equations	17
IV.	FORM OF THE FINAL-STATE FUNCTIONS	27
V.	DEVELOPMENT OF THE PAIR PRODUCTION CROSS SECTION IN ANGULAR MOMENTUM STATES	34
VI.	METHODS OF SOLUTION	48
	General Description of Contents	48
	Racah and Clebsch-Gordon Coefficients	49
	Methods for Obtaining the Function $F(\ell_1 \ell_2 \ell_3 L)$	51
	Spherical Bessel functions	51
	The Coulomb potential	52

CHAPTER PAGE

Removing the singularity in the radial differential equation 54

Solution of the radial differential equation 56

Determination of the normalizing constant and complex phase shift 58

Performing the integration in $F(\ell_1 \ell_2 \ell_3 L)$ 62

The General Plan of the Calculation 64

VII. RESULTS 67

Calculations with the Coulomb Potential 67

The Born approximation 67

Comparison of Born approximation and calculations with exact waves 71

Calculations with a Modified Coulomb and Nuclear Potential 76

VIII. SUMMARY 83

BIBLIOGRAPHY 88

APPENDIX A 93

APPENDIX B 98

APPENDIX C 100

LIST OF TABLES

TABLE PAGE

I. The Pion Pair Production Cross Section in Units of $\alpha R^2 = 7.762 \times 10^{-33}$ cm^2 for Lead, Including the Coulomb and Optical Potential 80

LIST OF FIGURES

FIGURE PAGE

1. Values of the Real Part of a Square-Well Optical Potential as Determined from Pion-Nucleus Scattering Data 18
2. Values of the Imaginary Part of a Square-Well Optical Potential as Determined from Pion-Nucleus Scattering Data 19
3. Comparison of the Born Approximation Pion Pair Production Cross Section at a Photon Energy of 310 Mev for a Point-Charge and Distributed-Charge Lead Nucleus 70
4. Comparison of the Born Approximation and Exact Wave Pion Pair Production Cross Section for Lead at a Photon Energy of 290 Mev. The Nuclear Potential Was Neglected 72
5. Comparison of the Born Approximation and Exact Wave Pion Pair Production Cross Section for Lead at a Photon Energy of 300 Mev. The Nuclear Potential Was Neglected 73
6. Comparison of the Born Approximation and Exact Wave Pion Pair Production Cross Section for Lead at a Photon Energy of 310 Mev. The Nuclear Potential Was Neglected 74
7. Ratio of Exact Wave to Born Approximation Differential Cross Section as a Function of Z for Photon Energy of 290 Mev . . 75
8. Ratio of Exact Wave to Born Approximation Pion Pair Production Cross Section for Lead as a Function of Energy. The Nuclear Potential Was Neglected 77
9. The Differential Pion Pair Production Cross Section for Lead Including the Coulomb and Nuclear Optical Potential (P = -16 Mev, V = -5 Mev) 79

FIGURE PAGE

10. The Pion Pair Production Cross Section for Lead Including the Coulomb and Nuclear Optical Potential (P = -16 Mev, V = -5 Mev) as a Function of Photon Energy 82

CHAPTER I

INTRODUCTION

The Pauli-Weisskopf[1] theory of the electromagnetic production of pion pairs has been a subject of interest in various connections for many years. Recently new interest has been stimulated in the theory because pion pair production is a possible means of investigating the π-π interaction. This approach, which was originally suggested by Pomeranchuk,[2] is particularly valid in view of the increase in the cross section, when the strong pion-nucleus interaction is included, over that obtained by the original theory. The purpose of this work was to study the Pauli-Weisskopf theory with the objective of including the strong pion-nucleus interaction and developing an accurate method of obtaining the cross section for pair production in the experimentally attainable energy region. The π-π interaction was neglected in this work since there is insufficient information available at the present time to adequately treat this interaction.

---

[1] W. Pauli and V. Weisskopf, Helv. Phys. Acta 7, 709 (1934).

[2] Iu. Ia. Pomeranchuk, Doklady Acad. Nauk S. S. S. R. 96, 265 and 481 (1954); English translation: Proceedings of the Cern Symposium 1956 (Service d'Information, Cern, 1956), vol. II, p. 167.

I. HISTORY OF PROBLEM

Shortly after Bethe and Heitler[3] derived the cross-section formula for the electromagnetic production of electron pairs in 1934, Pauli and Weisskopf[1] evolved a similar expression, starting with a complex scalar field and using the methods of second quantization, for the production of charged pairs of Bose particles.

The Pauli-Weisskopf theory is particularly interesting since it was the first method of treating a problem where the equations of motion are given by the Klein-Gordon equation. This equation had been a subject of much dispute and little use before the Pauli-Weisskopf theory appeared, because the probability density obtained in the same manner as in non-relativistic quantum mechanics led to a quantity which was not positive-definite.

As in the Bethe-Heitler derivation, Pauli and Weisskopf had assumed that the pair production took place in the presence of an infinitely heavy charged nucleus and had considered the Coulomb interaction as part of the perturbation. This meant that the Fourier coefficients of the field satisfied the Klein-Gordon equation without the Coulomb potential and therefore could be expanded in a series of plane waves. This method of quantization was not a necessary condition, as was demonstrated by Snyder and Weinberg,[4] who showed that the Hamiltonian could be brought to diagonal form by a unitary transformation when an electrostatic field is present. From this it can be shown that

---

[3] H. Bethe and W. Heitler, Proc. Roy. Soc. (London) A146, 83 (1934).

[4] H. Snyder and J. Weinberg, Phys. Rev. 57, 307 (1940).

the second-order perturbation theory used by Pauli and Weisskopf could be relaxed to a first-order process where the perturbation was the interaction between the complex scalar field and the electromagnetic field. In this case the Fourier coefficients could be expanded in a series of exact waves.

Although the interest in the Pauli-Weisskopf theory as a tool to help investigate nuclear forces had been extensive, it was not until the discovery of the π mesons in 1947 by Lattes, Occhialini, and Powell[5] and the subsequent determination of their properties[6] that the theory was known to apply to physical particles. The spin zero of the π mesons and their appearance in positive- and negative-charged states with approximately the same rest mass clearly indicated that the theory was directly applicable and appropriate to use for the determination of the cross section for the electromagnetic production of charged pion pairs.

An essential element of the Pauli-Weisskopf theory of pair production was the presence of the Coulomb field, which provided a means of interaction between the meson field and the nucleus to transmit linear momentum to the nucleus, thereby enabling energy and momentum in the pair production process to be conserved.[7] However, the Coulomb field is not the only method of interaction since there exists a strong, nonelectromagnetic interaction between the meson field and the nucleon

---

[5] Lattes, Occhialini, and Powell, Nature 160, 453, 486 (1947).

[6] H. A. Bethe and F. de Hoffmann, Mesons and Fields (Row, Peterson and Company, Evanston, 1955), vol. II, sec. 27 and 28.

[7] It is assumed that the nucleus is infinitely heavy, so that no energy is transmitted to the nucleus.

field. Landau and Pomeranchuk[8] were the first to introduce the strong interaction into the theory of meson bremsstrahlung in a semiphenomenological way by means of an optical model for the nucleus. Using the same approach, Pomeranchuk[2] later introduced the optical model into the theory for meson pair production. In both cases the nucleus was assumed to be "black" to mesons with a cross-sectional area equal to the total inelastic cross section for incident mesons. In this manner Pomeranchuk arrived at the pair production cross section which would leave the nucleus in an unexcited state.

Vdovin,[9] using the optical model of Fernbach, Serber, and Taylor,[10] extended Pomeranchuk's work to include the strong pion-nucleon interaction in the form of a gray nucleus. Both Pomeranchuk and Vdovin neglected the Coulomb field in their derivations and used a high-energy approximation.

The inclusion of the strong interaction in the theory is an important feature since the cross section for pair production is very small, considering only the modified Coulomb potential from a finite nucleus which is necessary in this theory; the strong interaction makes the pair production much larger and therefore is of considerable interest since pair production is a possible means of investigating the π-π interaction.

---

[8] L. D. Landau and Iu. Ia. Pomeranchuk, Zhur. Eksptl. i Teort. Fiz. 24, 505 (1953); English translation: Proceedings of the Cern Symposium 1956 (Service d'Information, Cern, 1956), vol. II, p. 159.

[9] Yu. A. Vdovin, Doklady Akad. Nauk S. S. S. R. 105, 947 (1955).

[10] Fernbach, Serber, and Taylor, Phys. Rev. 75, 1352 (1949).

II. OBJECTIVES OF THE STUDY

The optical model of Fernbach, Serber, and Taylor used by Vdovin is based on the hypothesis that a complex optical potential can be introduced into the equations of motion for the pions. The potential was not explicitly introduced by Vdovin and this is a necessary step for further, more exact, calculations of the cross section. Since it is desirable ultimately to calculate the cross section as accurately as possible, one objective of this work was to investigate the introduction of such a potential into the field theoretical equations and to show that it leads, in a consistent manner, to the absorption and scattering of the pions after they are produced. The absorptive part of the potential is intended to include, in a phenomenological way, all inelastic events that leave the nucleus in an excited state. It acts to remove the pions that have been produced within the nucleus which, in effect, reduces the cross section, in addition to facilitating the momentum transfer to the nucleus.

A second objective was to find a practical method of calculating the matrix elements that arise in the formulation of the pair production cross section. Matrix elements of this type, which involve integrals where the integrand is a product of only continuum wave functions or their derivatives, arise in many cases and are particularly difficult to calculate. An accurate method of treating this problem with a minimum amount of work would find many applications not only for calculating matrix elements but also in other fields--for example, to obtain the Fourier transforms of slowly converging oscillating functions.

Since Pomeranchuk[2] and Vdovin[9] neglected the Coulomb interaction in their work and used a high-energy approximation which effectively placed their results in an energy range above that now attainable, an additional objective of this study was to investigate the effects of the distributed nuclear charge and the complex optical potential on the total and differential cross section in the experimentally attainable energy region just above threshold.

III. ORGANIZATION OF THE REPORT

Some of the general features of pion pair production, such as consideration of a nucleus with a charge distribution and justification of the heavy-nucleus approximation, are discussed in Chapter II. Chapter III presents the justification of the use of the optical model in the pair production and shows that the complex optical potential can indeed be introduced into the field equations in a consistent manner. In Chapter IV it is shown that the use of the complex optical potential leads to an ambiguity of the form of the final-state wave functions in the matrix elements and that this ambiguity can be resolved. The final set of equations for the wave functions is then expanded into angular momentum states in Chapter V, and the matrix element is shown to be reducible in a convenient manner for final numerical calculations. The methods devised for evaluating the various factors appearing in the matrix element are given in Chapter VI; the results of some calculations for lead showing the effects of the modified Coulomb potential and nuclear optical potential on the cross section are presented in Chapter VII.

IV. NOTATIONS AND UNITS

In most of this work the "natural units" $\hbar = c = 1$ are used, and when the ordinary units are used, they are so specified. In most cases the notation is in a familiar form and needs no further clarification. An exception to this perhaps is the notation used in Chapter III, where the notation and conventions used by Schweber et al.[11] in their book have been carefully adhered to. The arguments of functions are omitted altogether when there is little likelihood of misunderstanding; in some cases the arguments are included in the first step of a derivation and omitted subsequently in order to simplify the notation. The major problem of duplication of symbols could not be avoided. Where duplications appear, the symbols are redefined.

In general, m without a subscript represents the pion mass, $\lambda\!\!\!^{-} = \frac{\hbar}{mc}$ the Compton wave length, $\alpha = \frac{e^2}{\hbar c}$ the fine structure constant, $R = \frac{e^2}{mc^2} = \alpha\lambda\!\!\!^{-}$ the classical pion radius, and the electronic charge e was taken as a positive quantity. In those sections where $\hbar = c = 1$, μ is the mass of the pion, ω is an energy, and k and q are momenta.

---

[11] Schweber, Bethe, and de Hoffmann, Mesons and Fields (Row, Peterson and Company, Evanston, 1955), vol. I.

CHAPTER II

GENERAL CONSIDERATIONS

Before going on to a systematic development of the cross section for the electromagnetic production of pion pairs which includes the strong interaction in the form of an optical potential, it is necessary to review some of the assumptions and general features of the theory. It is particularly important to discuss the necessity of including a nucleus with a charge distribution and also to review the heavy-nucleus assumption.

I. THE CHARGE DISTRIBUTION

The electromagnetic production of pion pairs differs from the production of electron pairs in one important way not previously indicated: Because the mass of the pion is so much greater than that of the electrons (approximately 273 electron rest mass units), the production of the mesons takes place predominantly inside the nucleus. This can be demonstrated by investigating the second-order perturbation theory of Pauli and Weisskopf[1] and assuming for the moment that the nucleus is a point charge.[2] In the matrix element for the Coulomb potential in the second-order perturbation theory an integral of the form

---

[1] W. Pauli and V. Weisskopf, Helv. Phys. Acta 7, 709 (1934).

[2] The following analysis parallels the methods presented in the book by W. Heitler, Quantum Theory of Radiation (Oxford University Press, London, 1954), 3rd ed., p. 248.

$$\int \frac{e^{i\vec{k}_n \cdot \vec{r}}}{r} \, d^3r \qquad (2.1)$$

occurs, where $\vec{k}_n$ is the momentum transferred to the nucleus. The main contribution to this integral comes from a radius

$$r_0 \sim \frac{1}{k_n} \qquad (2.2)$$

since for distances greater than r_0 the exponential function oscillates rapidly for small changes in r^{-1}, and for distances less than r_0 the volume element d^3r is small. The radius r_0 can therefore be interpreted as the mean radius at which production takes place. The maximum value of r_0 is obtained when k_n has its minimum value which can easily be determined from the requirements of conservation of energy and linear momentum consistent with the heavy-nucleus approximation. For fixed magnitudes of $\vec{q}_\gamma$, the momentum of the incident photon, and $\vec{k}_+$ and $\vec{k}_-$, the momentum of the π^+ and π^- mesons, respectively, we must have

$$\vec{k}_n = \vec{q}_\gamma - \vec{k}_+ - \vec{k}_- \quad , \qquad (2.3)$$

where the minimum magnitude of k_n from this expression occurs when

$$k_n = q_\gamma - k_+ - k_- \quad . \qquad (2.4)$$

Allowing k_+ and k_- to be variable but consistent with the conservation of energy $\omega_\gamma = \omega_+ + \omega_-$, where $k_\pm = \sqrt{\omega_\pm^2 - \mu^2}$, we obtain the minimum value of Eq. (2.4) when $\omega_+ = \omega_- = 1/2\ \omega_\gamma = \omega$ and $k_+ = k_- = k$, which gives

$$k_n(\min) = q_\gamma - 2k = 2(\omega - k) = 2\,\frac{\mu^2}{\omega + k} \quad . \qquad (2.5)$$

Using $k_n(\min)$ for k_n in Eq. (2.2) we find that

$$r_0(\max) = \lambda\!\!\!^{-} \frac{\omega + k}{2\mu} \quad , \tag{2.6}$$

where $\lambda\!\!\!^{-}$ is the meson Compton wavelength (1.4×10^{-13} cm). For photon energies just above threshold, $\omega \cong \mu$ and $k \cong 0$; therefore the main contribution to pair production takes place less than 1 pion Compton wavelength from the center of the nucleus, which is well within the bounds of the nucleus.

Because the pair production may take place primarily inside the nucleus, it is necessary to consider the finite extent of the nuclear charge distribution. The results of the analysis of electron scattering experiments from nuclei provide a convenient function to use in the form of the distribution function

$$\rho(r) = \frac{\rho_0}{1 + \exp\left(\frac{r - \beta}{\alpha}\right)} \quad , \tag{2.7}$$

which is reasonably close to the actual charge distribution for medium-heavy and heavy nuclei.[3] The constant ρ_0 is for normalization, and the constants for the best fit to the experimental data are $\alpha = 0.546 \times 10^{-13}$ cm and $\beta = 1.07\ A^{1/3} \times 10^{-13}$ cm.

Although the use of the charge distribution will lead to some complications from a numerical standpoint, several advantages are realized which outweigh the disadvantage. It was first pointed out by Schiff *et al.*[4] that when a modified Coulomb potential is sufficiently

---

[3] R. Hofstadter, Revs. Mod. Phys. 28, 214 (1956).

[4] Schiff, Snyder, and Weinberg, Phys. Rev. 57, 315 (1940).

broad in a region where its magnitude is greater than the rest mass energy of the pion, the wave equation leads to complex eigenvalues for certain states; in fact, the Hamiltonian with the modified Coulomb potential present could not be brought to diagonal form. For the distributed charge nucleus this problem is avoided, as can be seen if a uniform charge distribution which approximates the distribution given in Eq. (2.7) is examined. In this case the maximum value of the potential well occurs at r = 0 and has the value

$$V(0) = \frac{3Ze^2}{2R_0} \quad , \tag{2.8}$$

where R_0 is the nuclear radius and will be taken as approximately $\lambda\!\!\!^{-}A^{1/3}$, with $\lambda\!\!\!^{-}$ the pion Compton wavelength. Then

$$\frac{V(0)}{\mu} = \frac{3/2(Ze^2/R_0)}{\mu} = \frac{3}{2}\frac{Z\alpha\lambda\!\!\!^{-}}{R_0} \cong \frac{3}{2}\frac{Z\alpha}{A^{1/3}} < 1 \quad , \tag{2.9}$$

where α is the fine structure constant and we see that too deep a Coulomb potential is never realized.

Another problem avoided by the charge distribution is the one that occurs in the Klein-Gordon radial equations. The radial differential equations with a point charge are of the form

$$\frac{d^2\phi}{dr^2} + \left[P(r) + \frac{(Z\alpha)^2 - \ell(\ell+1)}{r^2}\right]\phi = 0 \quad , \tag{2.10}$$

where P(r) goes to infinity as 1/r as r approaches zero; therefore, for small r, we keep only the second term in the square brackets of Eq. (2.10) and seek solutions of the form Ar^{s+1}, which leads to

$$s = -\frac{1}{2} \pm \frac{1}{2}\sqrt{(2\ell+1)^2 - 4(Z\alpha)^2} \quad . \tag{2.11}$$

The choice of solutions from Eq. (2.11) is based on the boundary condition that $\phi(0) = 0$ which is obtained for $\ell > 0$ by taking the upper sign. However, for $\ell = 0$ and $Z\alpha > 1/2$, s is complex and the choice of solutions has to be made on other grounds.[5] In any event the complex power of r leads to complications when solving the radial equation, a problem that can be avoided with the use of the distributed charge and noting that the modified Coulomb potential from a charge distribution is finite at the origin so that at small r the V^2 term corresponding to $(Z\alpha/r)^2$ in the differential equation can be dropped. The result $s = \ell$ is the desired solution.

II. THE HEAVY-NUCLEUS APPROXIMATION

In the Pauli-Weisskopf[1] theory the assumption that the nucleus is infinitely heavy means among other things that the cross section is calculated in the center-of-mass frame of reference. In a physical situation there is some interest concerning what corrections should be made, if any, to transform from the laboratory to the center-of-mass frame. Letting ω_0 be the energy of the incident photon in the laboratory frame, ω_0^c the energy of the photon in the center-of-mass frame, and M the rest mass of the nucleus, the transformation from ω_0 to ω_0^c is given by

$$\omega_0^c = \omega_0 \left(1 + \frac{2\omega_0}{M}\right)^{-1/2} . \qquad (2.12)$$

---

[5] L. I. Schiff, Quantum Mechanics (McGraw-Hill Book Company, Inc., New York, 1955), 2nd ed., p. 322.

Therefore, for moderately heavy to very heavy nuclei there is little difference between ω_0 and ω_0^c until very high photon energies are reached. In the region near threshold for lead the correction is approximately 0.2% and can be neglected. This means that the threshold energy can be taken as twice the rest mass energy of the pions. Since the pions have a rest mass equal to 273.27 electron rest mass units, the threshold is at 279.17 Mev.

The assumption that the nucleus takes up no energy also warrants a brief review here especially since the momentum transferred to the nucleus is greater in the case of pion pair production than in electron pair production. In a manner similar to that shown in Section I of this chapter the maximum linear momentum transferred to the nucleus consistent with negligible energy transfer is found to be

$$k_n(\max) = \omega_0 + 2\sqrt{4\omega_0^2 - \mu^2} \quad , \tag{2.13}$$

where μ is the mass of the pion and ω_0 is the energy of the photon. The maximum kinetic energy transfer to the nucleus is then approximately

$$\omega(\max) \cong \frac{\left(\omega_0 + 2\sqrt{4\omega_0^2 - \mu^2}\right)^2}{2M} \quad , \tag{2.14}$$

where M is the mass of the nucleus. For lead and a photon energy of 290 Mev a maximum kinetic energy of 0.35 Mev is transferred to the nucleus, which is a small percentage of the total energy available, being only 3.2% of the kinetic energy transmitted to the pions. At higher photon energies these percentages decrease and the heavy nucleus approximation seems to hold very well.

CHAPTER III

THE OPTICAL POTENTIAL AND ITS INTRODUCTION INTO THE FIELD EQUATIONS

In the latter part of this chapter it is shown that an optical potential can be introduced into the field equations in a consistent manner; however, before entering into that discussion, it is appropriate to review the optical model to establish its acceptability as a phenomenological means of introducing the scattering and absorption of pions by a complex nucleus.

I. JUSTIFICATION OF THE OPTICAL MODEL

The "optical" model of the nucleus seems to have gained its name from the model of Fernbach, Serber, and Taylor,[1] in which they treated the nucleus as a continuous medium with a complex index of refraction. The model was so designated because of the similarity of the treatment to the familiar methods in optics where an incident wave can be coherently scattered and attenuated by a medium with a complex index of refraction. Actually, the model of Fernbach, Serber, and Taylor is based on the hypothesis that a complex interaction potential can be introduced into the wave equation which depends only on the coordinates of the incident particle relative to the nucleus as a whole. Only by means of a potential of this type is one led to the concept that a complex index of refraction will apply to the nucleus. For this reason it is customary

---

[1] Fernbach, Serber, and Taylor, Phys. Rev. 75, 1352 (1949).

to refer to the complex potential for the nucleus as an optical potential and to refer to the inclusion of the potential in the wave equation as using an optical model.

Bethe[2] was the first to show the significance of a complex optical potential some years before the model of Fernbach, Serber, and Taylor appeared. He showed that the complex potential would introduce the coherent scattering of an incident particle and that the imaginary part of the potential was responsible for its absorption. The function of the imaginary part of the potential as a means of introducing absorption is very effectively demonstrated when the continuity equation is derived from the Schrödinger wave equation as was done by Mott and Massey.[3]

The optical model is not without some justification from a microscopic point of view. In one series of papers[4] treating the multiple scattering of particles by complex nuclei, the multiple-scattering formalism which leaves the nucleus in an unexcited state was shown to ultimately lead to an optical potential. Although the optical potential as derived by this method is not necessarily proportional to the density of nucleons, Frank, Gammel, and Watson[5] showed

---

[2] H. A. Bethe, Phys. Rev. 57, 1125 (1940).

[3] N. F. Mott and H. S. W. Massey, The Theory of Atomic Collisions (Oxford University Press, London, 1940), 2nd ed., p. 12.

[4] K. M. Watson, Phys. Rev. 89, 575 (1953); N. C. Francis and K. M. Watson, Phys. Rev. 92, 291 (1953); G. Takeda and K. M. Watson, Phys. Rev. 97, 1336 (1955); K. M. Watson, Phys. Rev. 105, 1388 (1957).

[5] Frank, Gammel, and Watson, Phys. Rev. 101, 891 (1956).

explicitly that in the limit of large nuclei the proportionality is obtained.

The optical potential and the optical model of Fernbach, Serber, and Taylor have been useful tools for analyzing scattering data. Many workers have presented their analysis of meson-nucleus scattering in terms of these models and obtained reasonably good fits to the experimental data. In most cases the potential was assumed to be proportional to the nucleon density;[6] however, in one case analyses were made with the more complicated Kisslinger potential[7] based on Watson's multiple-scattering theory.[8]

One of the most significant results of the analysis of pion-nucleus scattering relating to this work is that the real part of the optical potential is attractive and of the same magnitude for both π^- and π^+ mesons. Also, inclusion of a complicated well shape and the more complicated Kisslinger potential to obtain the correct back scattering from light nuclei is not necessary for heavy nuclei such as lead, which was used in this work. The assumption that the potential is proportional to the nucleon density is very good for lead; in fact, only a square-well potential need be used since the nucleon density is almost uniform over much of the nuclear volume. An additional

---

[6] References to the original papers on the scattering of mesons from nuclei containing analyses in terms of a nuclear optical potential proportional to the nucleon density are included in a paper by G. Saphir, Phys. Rev. 104, 535 (1956).

[7] L. S. Kisslinger, Phys. Rev. 98, 761 (1955).

[8] Baker, Rainwater, and Williams, Phys. Rev. 112, 1763 (1958).

justification for use of the square-well approximation for the nuclear optical potential in this work is that the experimental data have an uncertainty which does not warrant the use of a more detailed potential.

Values for the real and imaginary part of a square-well optical potential which leads to a best fit of the low-energy experimental pion-nucleus scattering data are given in Figures 1 and 2. The dashed curve reflects the analysis of Stork,[9] who compared the low-energy data and the high-energy data to determine which points were the most reliable. Although there are no data below an incident energy of 33 Mev to include in Figures 1 and 2, it is reasonable to assume that the curves are flat in this region. This assumption is based on the calculations of Frank, Gammel, and Watson,[5] who obtained the potentials from the formulas relating the optical potential to the multiple-scattering theory and included the effects of the nucleon motion in the nucleus.

II. INTRODUCTION OF THE COMPLEX OPTICAL POTENTIAL INTO THE FIELD EQUATIONS

Once the use of an optical potential was justified as a tool for describing the coherent scattering and absorption of the pion pairs after production occurs, the problem remaining was how to introduce dissipation into the Pauli-Weisskopf theory which is normally nondissipative. Dissipation as used here is meant to refer to the absorption of particles by a means such as the complex potential. In the analysis of pion-nucleus scattering this problem did not arise because of the nature of the process and because most of the analyses were done with a quasi-Schrödinger

[9] D. H. Stork, Phys. Rev. 93, 868 (1954).

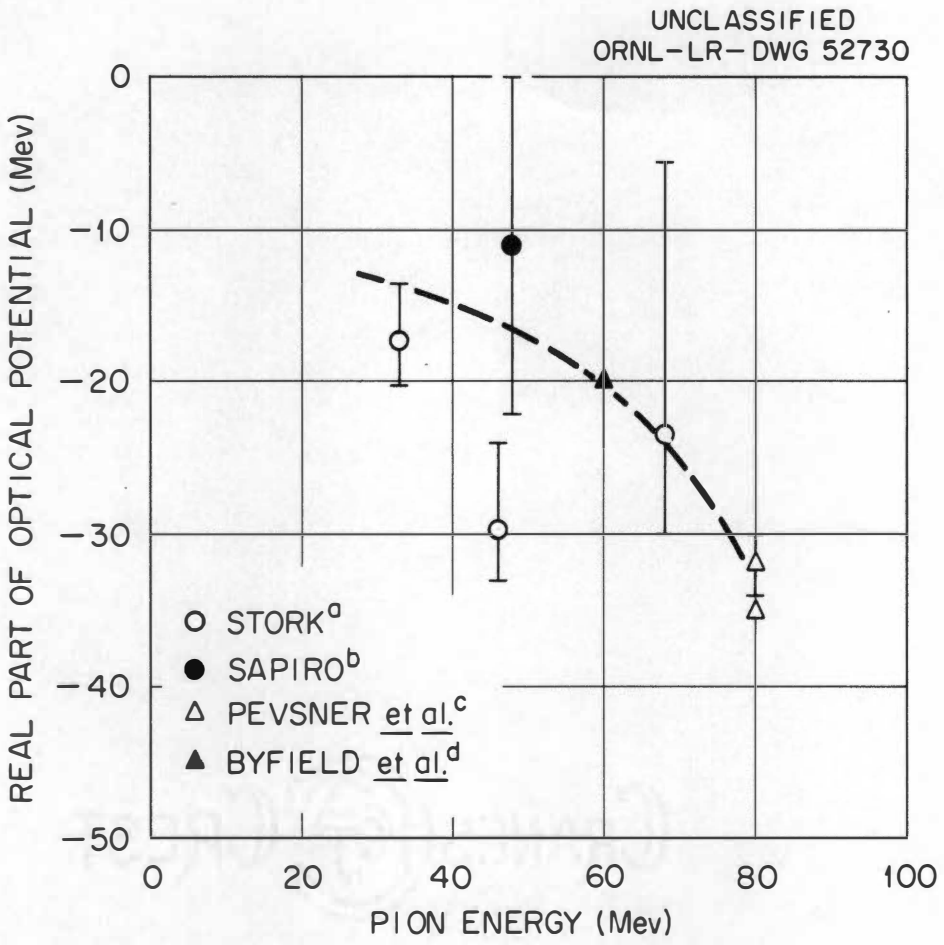

Figure 1. Values of the Real Part of a Square-Well Optical Potential as Determined from Pion-Nucleus Scattering Data.

[a]D. H. Stork, Phys. Rev. 93, 868 (1954).

[b]A. M. Sapiro, Phys. Rev. 84, 1063 (1951) and H. A. Bethe and R. R. Wilson, Phys. Rev. 83, 690 (1951).

[c]Pevsner et al., Phys. Rev. 100, 1419 (1955).

[d]Byfield, Kessler, and Lederman, Phys. Rev. 86, 17 (1952).

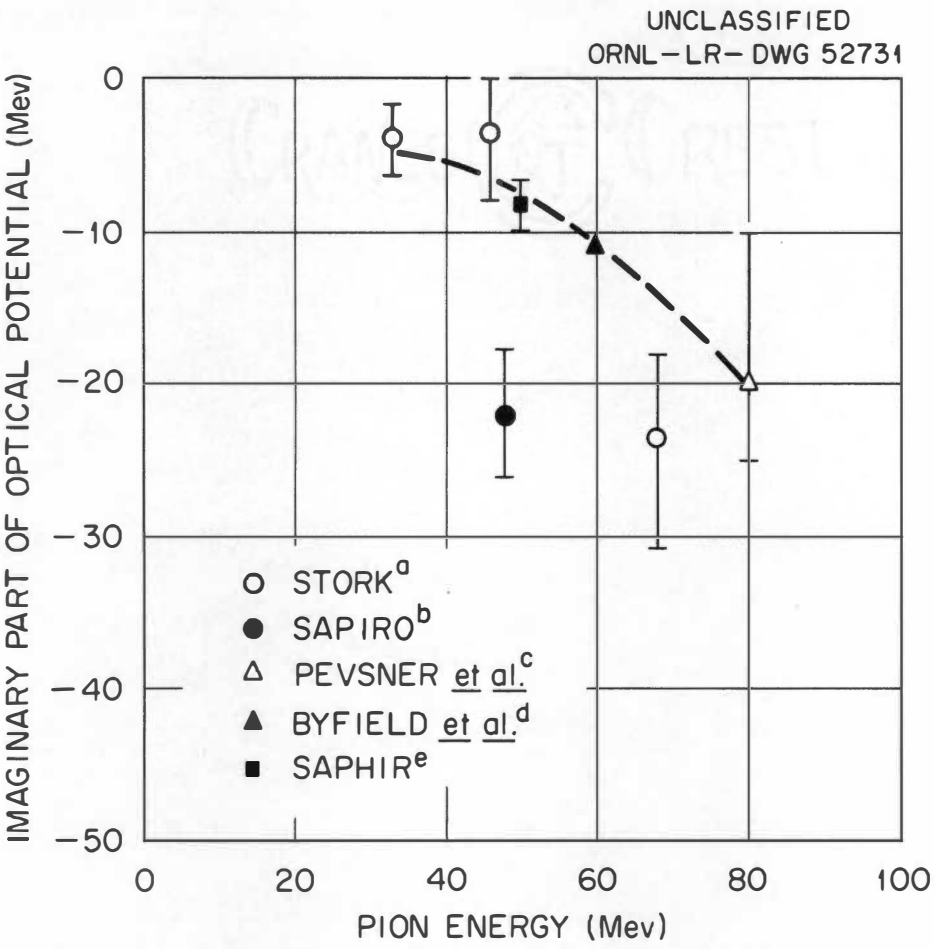

Figure 2. Values of the Imaginary Part of a Square-Well Optical Potential as Determined from Pion-Nucleus Scattering Data.

[a] D. H. Stork, Phys. Rev. 93, 868 (1954).

[b] A. M. Sapiro, Phys. Rev. 84, 1063 (1951) and H. A. Bethe and R. R. Wilson, Phys. Rev. 83, 690 (1951).

[c] Pevsner et al., Phys. Rev. 100, 1419 (1955).

[d] Byfield, Kessler, and Lederman, Phys. Rev. 86, 17 (1952).

[e] G. Saphir, Phys. Rev. 104, 535 (1956).

wave equation for π^+ or π^- mesons, where the potential was nominally the time component of a four-vector. In pair production, however, great care must be taken to assure that the optical potentials enter the field equations in the proper way to obtain dissipation (absorption) of both π^+ and π^- mesons since they enter the theory in a symmetric way. In addition, scattering experiments are essentially nonrelativistic, whereas pair production is a relativistic phenomenon; therefore it is essential that the relativistic aspects of the problem be maintained throughout this study.

For purposes of discussing the proper form of the complex optical potential in the field equations, it is necessary to introduce two real world scalar potentials, P and P_1, and a complex time component of a four-vector V_0. At first, to display the co-variance of the equations, the potential V will be represented as a four-vector, and the space components will be set to zero at the end. With the potentials present in the equations, the variation of the action for the complex scalar field and electromagnetic field can be written as

$$\delta I = \delta \int \mathcal{L} d^4x + \int (Q\delta\phi + Q^*\delta\phi^*)\, d^4x = 0 \quad , \tag{3.1}$$

where the Lagrangian density $\mathcal{L}$ must be a real function and is given along with the field components Q and Q* by

$$\mathcal{L} = -\frac{1}{2}\frac{\partial A_\mu}{\partial x_\nu}\frac{\partial A^\mu}{\partial x^\nu} - \left[(\mu + P)^2 - P_1^2\right]\phi^*\phi + \left(\frac{\partial}{\partial x^\mu} - ieA_\mu\right)\phi^*\left(\frac{\partial}{\partial x_\mu} + ieA^\mu\right)\phi \quad , \tag{3.2}$$

$$Q = \left[\left(\frac{\partial}{\partial x^\mu} - ieA_\mu\right)V^{*\mu} + V^*_\mu\left(\frac{\partial}{\partial x_\mu} - ieA^\mu\right) - V^*_\mu V^{*\mu} + 2i(\mu + P)\,P_1\right]\phi^* \quad , \tag{3.3}$$

$$Q^* = \left[\left(\frac{\partial}{\partial x^\mu} + ieA_\mu\right) V^\mu + V_\mu \left(\frac{\partial}{\partial x_\mu} + ieA^\mu\right) - V_\mu V^\mu - 2i(\mu + P)\, P_1\right]\phi \quad . \tag{3.4}$$

This form of the variation of the action is patterned after the equation that appears in classical mechanics when nonconservative forces are present and will lead to a nonconservation of particles and charge.[10] Certain terms in Eq. (3.3) and (3.4) can logically be included in the Lagrangian density and will lead to the same results given below. However, for purposes of simplicity, because of the convenient notation, these terms are retained in the expressions for Q and Q*.

Performing the variation indicated in Eq. (3.1) leads to the following equations of motion:

$$\frac{\partial \mathcal{L}}{\partial \phi} - \frac{\partial}{\partial x^\mu} \frac{\partial \mathcal{L}}{\partial \phi_\mu} + Q = 0 \quad , \tag{3.5}$$

$$\frac{\partial \mathcal{L}}{\partial \phi^*} - \frac{\partial}{\partial x^\mu} \frac{\partial \mathcal{L}}{\partial \phi^*_\mu} + Q^* = 0 \quad , \tag{3.6}$$

$$-\frac{\partial}{\partial x^\nu} \frac{\partial \mathcal{L}}{\partial A^\mu_\nu} = \Box A_\mu = -\frac{\partial \mathcal{L}}{\partial A^\mu} \quad . \tag{3.7}$$

It should be noted that Eq. (3.7) gives the correct definition for the current four-vector as

$$-\frac{\partial \mathcal{L}}{\partial A^\mu} = j_\mu \quad , \tag{3.8}$$

---

[10] H. Goldstein, Classical Mechanics (Addison-Wesley Publishing Company, Inc., Cambridge, 1953), p. 38.

which, in the present case, is

$$j_\mu = -ie\left[\left(\frac{\partial\phi^*}{\partial x^\mu} - ieA_\mu\phi^*\right)\phi - \phi^*\left(\frac{\partial\phi}{\partial x^\mu} + ieA_\mu\phi\right)\right] \quad . \tag{3.9}$$

This expression will be useful later.

Equation (3.5) should be examined at this point to determine the way that the potentials appear in the equations of motion. To this end, Eqs. (3.2), (3.3), and (3.4) are used in Eq. (3.5) to obtain

$$\left(\frac{\partial}{\partial x_\mu} + ieA^\mu - V^\mu\right)\left(\frac{\partial}{\partial x^\mu} + ieA_\mu - V_\mu\right)\phi + (\mu + P + iP_1)^2\phi = 0 \quad . \tag{3.10}$$

The complex conjugate of Eq. (3.10) is obtained from Eq. (3.6).

To simplify the discussion that follows, the space components of the electromagnetic potential and of the potential V will be set to zero. In addition, the Fourier transform of $\phi(\vec{r}, t)$ with respect to time will be taken as

$$\phi(\vec{r}, t) = \int_{-\infty}^{+\infty} e^{-i\omega' t}\,\Psi(\vec{r}, \omega')\,d\omega' \quad , \tag{3.11}$$

so that Eq. (3.10) leads to the following equation to be satisfied by the Fourier coefficients

$$\left[(\omega' - eA_0 - iV_0)^2 + \nabla^2 - (\mu + P + iP_1)^2\right]\Psi(\vec{r}, \omega') = 0 \quad . \tag{3.12}$$

In order to explicitly show the equations of motion which apply to the π^- and π^+ mesons, the frequency ω' will be redefined so that $\omega = \omega'$ for $\omega' > 0$ and $\omega = -\omega'$ for $\omega' < 0$, which results in an always

positive frequency ω and the two equations

$$\left[(\omega - eA_0 - iV_0)^2 + \nabla^2 - (\mu + P + iP_1)^2\right] \psi(\vec{r}, \omega) = 0 \quad , \tag{3.13}$$

$$\left[(\omega + eA_0 + iV_0)^2 + \nabla^2 - (\mu + P + iP_1)^2\right] \psi(\vec{r}, -\omega) = 0 \quad . \tag{3.14}$$

Equations (3.13) and (3.14) apply to the motion of the π^- and π^+ mesons, respectively.

Examination of Eqs. (3.13) and (3.14) shows that the imaginary part of V_0 does not introduce a complex factor into the equations and therefore cannot represent absorption. It can be treated as the real part of the optical potential, however, although it enters the equations with opposite sign and will be an attractive potential to one of the pair of pions and will be repulsive to the other. In contrast to this, the potential P can also represent the real part of the optical potential and appear in both equations with the same sign. Since the experiments indicate that the real part of the optical potential should have the same sign and approximately the same magnitude for both π^+ and π^-, it can be assumed that the imaginary part of V_0 is very much smaller than P. For simplicity, the imaginary part of V_0 will be set to zero, and V_0 will be considered a real quantity in the subsequent development. This leaves P to represent the real part of the optical potential and also leaves the task of determining whether V_0 or P_1 should represent the imaginary part.

To proceed, it will be noticed that the Lagrangian density given in Eq. (3.2) is invariant to the transformation

$$\phi \rightarrow \phi e^{i\lambda} \cong (1 + i\lambda)\, \phi \quad , \tag{3.15}$$

where λ is a real infinitesimal constant. Therefore, under this transformation, we must have that

$$\delta \mathcal{L} = 0 \quad . \tag{3.16}$$

By using the expression for the Lagrangian density given in Eq. (3.2), the following expression is obtained from Eq. (3.16):

$$\left(\frac{\partial \mathcal{L}}{\partial \phi} - \frac{\partial}{\partial x^\mu}\frac{\partial \mathcal{L}}{\partial \phi_\mu}\right)\delta\phi + \frac{\partial}{\partial x^\mu}\left(\frac{\partial \mathcal{L}}{\partial \phi_\mu}\delta\phi\right) + \left(\frac{\partial \mathcal{L}}{\partial \phi^*} - \frac{\partial}{\partial x^\mu}\frac{\partial \mathcal{L}}{\partial \phi^*_\mu}\right)\delta\phi^* + \frac{\partial}{\partial x^\mu}\left(\frac{\partial \mathcal{L}}{\partial \phi^*_\mu}\delta\phi^*\right) +$$

$$+\left(\frac{\partial \mathcal{L}}{\partial A^\mu} - \frac{\partial}{\partial x^\nu}\frac{\partial \mathcal{L}}{\partial A^\mu_\nu}\right)\delta A^\mu + \frac{\partial}{\partial x^\nu}\left(\frac{\partial \mathcal{L}}{\partial A^\mu_\nu}\delta A^\mu\right) = 0 \quad , \tag{3.17}$$

which can, with the use of equations of motion given in Eqs. (3.5), (3.6), and (3.7), be written as

$$-Q\delta\phi + \frac{\partial}{\partial x^\mu}\left(\frac{\partial \mathcal{L}}{\partial \phi_\mu}\delta\phi\right) - Q^*\delta\phi^* + \frac{\partial}{\partial x^\mu}\left(\frac{\partial \mathcal{L}}{\partial \phi^*_\mu}\delta\phi^*\right) + \frac{\partial}{\partial x^\nu}\left(\frac{\partial \mathcal{L}}{\partial A^\mu_\nu}\delta A^\mu\right) = 0 \quad . \tag{3.18}$$

The variation of ϕ necessary in Eq. (3.18) can be obtained from Eq. (3.15) as

$$\delta\phi = i\lambda\phi \quad , \tag{3.19}$$

$$\delta\phi^* = -i\lambda\phi^* \quad . \tag{3.20}$$

The variation of A^μ is, of course, zero under the transformation. Using these expressions in Eq. (3.18),

$$-ie(Q\phi - Q^*\phi^*) = -ie\frac{\partial}{\partial x^\mu}\left(\frac{\partial \mathcal{L}}{\partial \phi_\mu}\phi - \frac{\partial \mathcal{L}}{\partial \phi^*_\mu}\phi^*\right) \quad , \tag{3.21}$$

after multiplying through by e/λ. Substituting Eq. (3.2) for $\mathcal{L}$ in Eq. (3.21) yields

$$-ie(Q\phi - Q^*\phi^*) = \frac{\partial}{\partial x^\mu}\left\{-ie\left[\left(\frac{\partial \phi^*}{\partial x_\mu} - ieA^\mu\phi^*\right)\phi - \phi^*\left(\frac{\partial \phi}{\partial x_\mu} + ieA^\mu\phi\right)\right]\right\} \quad , \tag{3.22}$$

where the quantity in the braces will be recognized, on comparison with Eq. (3.9), as the current four-vector. Hence, for the continuity equation,

$$-ie(Q\phi - Q^*\phi^*) = \frac{\partial j^\mu}{\partial x^\mu} \quad , \tag{3.23}$$

which will clearly indicate the roll of the potentials when the expressions for Q and Q^* are introduced. Before doing this, however, the space components of the potential V will again be set to zero, so that when Eqs. (3.3) and (3.4) are substituted into Eq. (3.23),

$$2V_0\left\{-ie\left[\left(\frac{\partial \phi^*}{\partial x_0} - ieA_0\phi^*\right)\phi - \phi^*\left(\frac{\partial \phi}{\partial x_0} + ieA_0\phi\right)\right]\right\} + 4e(\mu + P)\,P_1\phi\phi^*$$

$$= \frac{\partial j^\mu}{\partial x^\mu} = \vec{\nabla}\cdot\vec{j} + \frac{\partial \rho}{\partial t} \quad , \tag{3.24}$$

where the time and space components of the current four-vector have been separated to show the familiar form of the continuity equation. Here, the time component of the currect four-vector, the charge density, is represented by ρ.

Comparison of the expression in the braces in Eq. (3.24) with Eq. (3.9) will show that it is in fact the charge density ρ. Hence, Eq. (3.24) can be written as

$$\vec{\nabla}\cdot\vec{J} - 2V_0\rho - 4e(\mu + P)\, P_1\phi\phi^* + \frac{\partial\rho}{\partial t} = 0 \quad . \tag{3.25}$$

This form of the continuity equation clearly indicates that P_1 must be set equal to zero since it introduces an absorptive term which is not proportional to the charge density, leaving

$$\vec{\nabla}\cdot\vec{J} - 2V_0\rho + \frac{\partial\rho}{\partial t} = 0 \quad , \tag{3.26}$$

which, it is interesting to note, is exactly the same form of continuity equation derived from the Schrödinger equation when a complex optical potential is present.[11]

The result of this analysis is that when a real potential V_0 and potential P are retained, it is possible to get the proper absorption and scattering of the pions; the potential P determines the scattering and the potential V_0 determines the amount of absorption when it is a negative quantity.

---

[11] N. F. Mott and H. S. W. Massey, The Theory of Atomic Collisions (Oxford University Press, London, 1949), 2nd ed., p. 12.

CHAPTER IV

FORM OF THE FINAL-STATE FUNCTIONS

The matrix element for the electromagnetic production of charged pion pairs given by the equation

$$M = -ie \int \hat{A}_{\vec{q}} \left[\psi^*_{\vec{k}_+} \overleftrightarrow{\nabla} \psi^*_{\vec{k}_-} - \psi^*_{\vec{k}_-} \overleftrightarrow{\nabla} \psi^*_{\vec{k}_+} \right] d^3x \qquad (4.1)$$

arises from a first-order perturbation-theory treatment where the perturbation is the interaction between the meson and electromagnetic fields.[1] It is assumed here, as it was by Vdovin,[2] that this matrix element prevails when the imaginary part of the optical potential is introduced into the theory. The functions $\psi_{\vec{k}_-}$ and $\psi_{\vec{k}_+}$ that appear in Eq. (4.1) satisfy the equations

$$\left[(\omega_{k_-} - eA_0 - iV_0)^2 + \nabla^2 - (\mu + P)^2 \right] \psi_{\vec{k}_-} = 0 \quad , \qquad (4.2)$$

$$\left[(\omega_{k_+} + eA_0 + iV_0)^2 + \nabla^2 - (\mu + P)^2 \right] \psi_{\vec{k}_+} = 0 \quad , \qquad (4.3)$$

where $\omega_k^2 = k^2 + \mu^2$ and ω_k is always positive.[3] The optical potential is explicitly shown in these two equations.

---

[1] W. Pauli and V. Weisskopf, Helv. Phys. Acta. 7, 709 (1934). In Appendix A it is shown that the matrix element of Eq. (4.1) with the correct forms of the final state wave functions reduces to the one given by the second-order perturbation-theory treatment of Pauli and Weisskopf.

[2] Yu. A. Vdovin, Doklady Akad. Nauk S. S. S. R. 105, 947 (1955).

[3] The subscript on the wave numbers in Eqs. (4.1), (4.2), and (4.3) indicate their association with π^+ or π^- mesons.

The proper form of the solution required of Eqs. (4.2) and (4.3) depends on the asymptotic form of the final-state functions required in the matrix element and is the subject of this chapter.

It has been well established for some years that the final-state continuum functions appearing in a matrix element should have the asymptotic form of a plane wave plus a spherically converging wave. The derivation of this fact has been given by Mott and Massey[4] and discussed in some detail by Breit and Bethe.[5] Landau and Lifshitz[6] have also presented an interesting derivation bearing on this subject. In these derivations, waves having the asymptotic form of a plane wave plus a spherically converging wave,

$$\psi_{\vec{k}}^{-} \sim e^{i\vec{k}\cdot\vec{r}} + f(\theta)\,\frac{e^{-ikr}}{r} \quad , \tag{4.4}$$

and waves with the asymptotic form of a plane wave plus a spherically diverging wave,

$$\psi_{\vec{k}}^{+} \sim e^{i\vec{k}\cdot\vec{r}} + g(\theta)\,\frac{e^{ikr}}{r} \quad , \tag{4.5}$$

are related by the equation[7]

$$\psi_{\vec{k}}^{-} = \psi_{-\vec{k}}^{+*} \quad . \tag{4.6}$$

---

[4] N. F. Mott and H. S. Massey, The Theory of Atomic Collisions (Oxford Univeristy Press, London, 1949), 2nd ed., pp. 111 and 353.

[5] G. Breit and H. A. Bethe, Phys. Rev. 93, 888 (1954).

[6] L. D. Landau and E. M. Lifshitz, Quantum Mechanics (Addison-Wesley Publishing Company, Inc., Reading, 1958), p. 423.

[7] Ibid., p. 422.

This occurs because of the assumption that the radial part of $\psi_{\vec{k}}$ and the phase shift were real. Therefore, either $\psi^{-}_{\vec{k}}$ or $\psi^{+*}_{-\vec{k}}$ could be used as the final-state functional form since they are indistinguishable. In the case here, however, the complex coefficients occurring in the wave equation lead to complex radial functions and phase shifts so that Eq. (4.6) does not hold, and a closer examination of the form of the final state wave function is therefore required.

In the following the wave equation for the π^{-} meson, given by Eq. (4.2), will be examined explicitly, and the first objective will be to obtain the form of $\psi^{-}_{\vec{k}}$ and $\psi^{+}_{\vec{k}}$ for that equation. No generality will be lost if we assume for the moment that A_0 is zero.

After making the substitution

$$\nabla^2 \psi = \frac{1}{r} \frac{\partial^2 (r \psi)}{\partial r^2} - \frac{\hat{\mathcal{L}}^2}{r^2} \psi \tag{4.7}$$

in Eq. (4.2), where $\hat{\mathcal{L}}$ is the angular momentum operator, one obtains

$$\left[\frac{\partial^2}{\partial r^2} + (\omega_k - iV_0)^2 - (\mu + P)^2 - \frac{\hat{\mathcal{L}}^2}{r^2} \right] r \psi_{\vec{k}} = 0 \quad . \tag{4.8}$$

Equation (4.8) can be separated in the spherical coordinate system if V_0 and P are only functions of r. This is the case in the problem here and therefore enables the expansion

$$\psi_{\vec{k}} = \sum_{\ell m} A_{\ell m} Y_{\ell,m}(\hat{r}) \frac{\chi_\ell (r)}{r \sqrt{2\omega_k}} \quad , \tag{4.9}$$

where $A_{\ell m}$ are constants, the functions $Y_{\ell,m}(\hat{r})$ are the spherical harmonics, and $\vec{k}$ is assumed to lie along the z axis. The factor $1/\sqrt{2\omega_k}$ was introduced into Eq. (4.9) to obtain the proper normalization. Substitution of

Eq. (4.9) into Eq. (4.8) leads to the following equation, which must be satisfied by the function $\chi_\ell(r)$:

$$\left\{\frac{d^2}{dr^2} + \left[k^2 - 2\mu P - V_0^2 - P^2 - \frac{\ell(\ell+1)}{r^2}\right] - 2iV_0\omega_k\right\} \chi_\ell(r) = 0 \quad . \tag{4.10}$$

If V_0 and P go to zero faster than 1/r as r approaches infinity and, in addition, if $\chi_\ell(r)/r$ is required to be finite everywhere, then the asymptotic form of $\chi_\ell(r)$ is

$$\chi_\ell(r) \sim \sin(kr - 1/2\ell\pi + \eta_\ell) \quad , \tag{4.11}$$

where η_ℓ is the complex phase shift. Assuming that $\psi_{\vec{k}}$ is independent of the azimuthal angle, we have, from Eq. (4.9),

$$\psi_{\vec{k}} \sim \sum_\ell A_\ell P_\ell(\mu) \frac{\sin(kr - 1/2\ell\pi + \eta_\ell)}{kr\sqrt{2\omega_k}} \quad , \tag{4.12}$$

where μ is the cosine of the polar angle.

The constants that appear in Eq. (4.12) can be determined so as to obtain $\psi^-_{\vec{k}}$ or $\psi^+_{\vec{k}}$ if we first subtract the asymptotic form of the Rayleigh expansion for a plane wave given by

$$\frac{e^{ikz}}{\sqrt{2\omega_k}} \sim \sum_\ell i^\ell (2\ell+1)\, P_\ell(\mu) \frac{\sin(kr - 1/2\ell\pi)}{kr\sqrt{2\omega_k}} \tag{4.13}$$

from the expression given in Eq. (4.12) to obtain

$$\psi_{\vec{k}} - \frac{e^{ikz}}{\sqrt{2\omega_k}} \sim \sum_\ell \frac{P_\ell(\mu)}{2ikr\sqrt{2\omega_k}} \left\{ e^{i(kr-1/2\ell\pi)} \left[A_\ell e^{i\eta_\ell} - i^\ell(2\ell+1)\right] - \right.$$
$$\left. - e^{-i(kr-1/2\ell\pi)} \left[A_\ell e^{-i\eta_\ell} - i^\ell(2\ell+1)\right] \right\}. \tag{4.14}$$

By choosing $A_\ell = (2\ell + 1)i^\ell e^{-i\eta_\ell}$ in Eq. (4.14), the coefficient of the diverging spherical wave is set to zero and the converging spherical wave is retained so that $\psi_{\vec{k}}$ will have the form $\psi^-_{\vec{k}}$ as defined in Eq. (4.4). If A_ℓ had been taken as $(2\ell + 1)i^\ell e^{+i\eta_\ell}$, then $\psi_{\vec{k}}$ would have had the form $\psi^+_{\vec{k}}$. From Eq. (4.14) we see that

$$\psi^-_{\vec{k}} \sim \sum_\ell \frac{(2\ell + 1)\, i^\ell P_\ell(\mu)}{2ikr\sqrt{2\omega_k}} \left[e^{i(kr - 1/2\ell\pi)} - e^{-2i\eta_\ell}\, e^{-i(kr - 1/2\ell\pi)} \right], \tag{4.15}$$

and

$$\psi^+_{\vec{k}} \sim \sum_\ell \frac{(2\ell + 1)\, i^\ell P_\ell(\mu)}{2ikr\sqrt{2\omega_k}} \left[e^{2i\eta_\ell}\, e^{i(kr - 1/2\ell\pi)} - e^{-i(kr - 1/2\ell\pi)} \right]. \tag{4.16}$$

To obtain $\psi^+_{-\vec{k}}$ from Eq. (4.16) it is only necessary to make the transformation $\mu \to -\mu$ and to note that $P_\ell(-\mu) = (-1)^\ell P_\ell(\mu)$ so that

$$\psi^+_{-\vec{k}} \sim \sum_\ell \frac{(2\ell + 1)(-i)^\ell P_\ell(\mu)}{2ikr\sqrt{2\omega_k}} \left[e^{2i\eta_\ell}\, e^{i(kr - 1/2\ell\pi)} - e^{-i(kr - 1/2\ell\pi)} \right]. \tag{4.17}$$

The decision as to which of the two alternative forms of the final-state function is the correct one can be made only by examining their complex conjugates, since this is the way they would appear in the matrix element. Therefore, we should not examine $\psi^-_{\vec{k}}$ and $\psi^{+*}_{-\vec{k}}$, but rather $\psi^{-*}_{\vec{k}}$ and $\psi^+_{-\vec{k}}$. Equation (4.17) already gives the asymptotic form of $\psi^+_{-\vec{k}}$, and the asymptotic form of $\psi^{-*}_{\vec{k}}$ can easily be obtained from Eq. (4.15):

$$\psi^{-*}_{\vec{k}} \sim \sum_\ell \frac{(2\ell + 1)(-i)^\ell P_\ell(\mu)}{2ikr\sqrt{2\omega_k}} \left[e^{2i\eta^*_\ell}\, e^{i(kr - 1/2\ell\pi)} - e^{-i(kr - 1/2\ell\pi)} \right]. \tag{4.18}$$

Equations (4.17) and (4.18) differ only in the way the phase shift enters the expressions. The phase shift appears in the asymptotic form of $\psi^{+}_{-\vec{k}}$, while its complex conjugate appears in the asymptotic form of $\psi^{-*}_{\vec{k}}$. In both expressions there is a linear combination of ingoing and outgoing spherical waves; the one required will be the expression in which the square modulus of the amplitude of the outgoing wave is less than that of the ingoing wave. This condition causes the matrix element to be reduced in value and is consistent with the physical condition that the pions be absorbed by the nucleus. The decision will therefore depend on the sign of the imaginary part of the phase shift.

In Eq. (4.10) the complex part of the operator acting on $\chi_{\ell}(r)$ is positive since V_0 is a negative quantity. This leads to a phase shift which has a positive imaginary part and indicates that

$$\left| e^{2i\eta_{\ell}} \right|^2 < 1 \quad , \tag{4.19}$$

whereas

$$\left| e^{2i\eta^{*}_{\ell}} \right|^2 > 1 \quad . \tag{4.20}$$

Therefore, it is clear that $\psi^{+*}_{-\vec{k}}$ is the desired form of the final-state function for π^- mesons. A similar analysis shows that $\psi^{-}_{\vec{k}}$ is the proper form for the π^+ mesons.

In the development of the two equations of motion from the Pauli-Weisskopf theory the sign of the electronic charge, e, determines which equation of motion is associated with the π^+ meson and which is associated with the π^- meson. There is no a priori method of choosing the sign since the equations are completely symmetric to the simultaneous transformation $e \rightarrow -e$, $\pi^+ \rightarrow \pi^-$, and $\pi^- \rightarrow \pi^+$. Since the subsequent

derivation of the matrix element for pair production maintains this symmetry, we have an additional check on our final wave form by testing the simultaneous transformation, and indeed the choice of final wave forms meets the requirements.

CHAPTER V

DEVELOPMENT OF THE PAIR PRODUCTION CROSS SECTION IN ANGULAR MOMENTUM STATES

In the following the wave functions in the matrix element will be expanded into angular momentum states and the cross section for pair production reduced in this formalism. This is done as a convenient means of obtaining an accurate value for the cross section in the energy region just above threshold where few angular momentum states are expected to contribute.

In Chapter IV it was shown that the matrix element for the electromagnetic production of charged pion pairs was given by the equation

$$M = -ie \int \vec{A}_{\vec{q}} \left[\psi^{-*}_{\vec{k}_+} \vec{\nabla} \psi^{+}_{-\vec{k}_-} - \psi^{+}_{-\vec{k}_-} \vec{\nabla} \psi^{-*}_{\vec{k}_+} \right] d^3x \quad . \tag{5.1}$$

This equation was obtained with the use of the solenoidal gauge, which is maintained throughout this development.[1]

Equation (5.1) can be simplified by means of the relation

$$\vec{\nabla}\left(\vec{A}_{\vec{q}} \psi^{-*}_{\vec{k}_+} \psi^{+}_{-\vec{k}_-}\right) = \vec{\nabla} \cdot \vec{A}_{\vec{q}} \left(\psi^{-*}_{\vec{k}_+} \psi^{+}_{-\vec{k}_-}\right) + \vec{A}_{\vec{q}} \left(\psi^{-*}_{\vec{k}_+} \vec{\nabla} \psi^{+}_{-\vec{k}_-}\right) + \vec{A}_{\vec{q}} \left(\psi^{+}_{-\vec{k}_-} \vec{\nabla} \psi^{-*}_{\vec{k}_+}\right) \quad , \tag{5.2}$$

---

[1] No particular singularities at the origin arise from the use of the solenoidal gauge in the case of pion pair production as they do in the internal conversion, because (1) the radial meson wave functions are not as singular as the electron wave functions even in the case of a point charge, and (2) the standing wave radial functions of the electromagnetic field arising from the plane wave expansion do not have singularities in contrast to the outgoing (Hankel) radial waves necessary for a description of internal conversion. The problems connected with the solenoidal gauge in electric multipole conversion are thoroughly discussed by M. E. Rose, Multipole Fields (John Wiley and Sons, Inc., New York, 1955), p. 59.

where the first term on the right is zero because of the use of the solenoidal gauge. Substituting Eq. (5.2) into Eq. (5.1) gives

$$M = -2ie\int \vec{A}_{\vec{q}}\left(\psi^{-*}_{\vec{k}_+}\vec{\nabla}\psi^{+}_{-\vec{k}_-}\right) d^3x + ie\int \vec{\nabla}\left(\vec{A}_{\vec{q}}\,\psi^{-*}_{\vec{k}_+}\psi^{+}_{-\vec{k}_-}\right) d^3x \quad . \tag{5.3}$$

The second integral on the right of Eq. (5.3) can be transformed to a surface integral over an infinite sphere:

$$\int \vec{\nabla}\left(\vec{A}_{\vec{q}}\,\psi^{-*}_{\vec{k}_+}\psi^{+}_{-\vec{k}_-}\right) d^3x = \lim_{r\to\infty}\oint \left(\vec{A}_{\vec{q}}\,\psi^{-*}_{\vec{k}_+}\psi^{+}_{-\vec{k}_-}\right)\hat{r}\, r^2 dr \quad , \tag{5.4}$$

which has the value zero since the product $\vec{A}_{\vec{q}}\,\psi^{-*}_{\vec{k}_+}\psi^{*}_{-\vec{k}_-}$ goes as $1/r^3$ as r approaches infinity. Therefore, the expression for the matrix element is

$$M = -2ie\int \vec{A}_{\vec{q}}\left(\psi^{-*}_{\vec{k}_+}\vec{\nabla}\psi^{+}_{-\vec{k}_-}\right) d^3x \quad , \tag{5.5}$$

where

$$\psi^{+}_{-\vec{k}_-} = \sum_{\ell_1} \frac{(2\ell_1+1)(-i)^{\ell_1} e^{i\eta_{\ell_1}}}{k_- r\sqrt{2\omega_{k_-}}}\,\chi_{\ell_1} P_{\ell}\left(\frac{\vec{k}_-\cdot\vec{r}}{k_- r}\right)$$

$$= \sum_{\ell_1 m_1} \frac{4\pi(-i)^{\ell_1} e^{i\eta_{\ell_1}}}{k_- r\sqrt{2\omega_{k_-}}}\,\chi_{\ell_1} Y^{*}_{\ell_1,m_1}(\hat{k}_-)\, Y_{\ell_1,m_1}(\hat{r}) \tag{5.6}$$

and

$$\psi^{-*}_{\vec{k}_+} = \sum_{\ell_3} \frac{(2\ell_3+1)(-i)^{\ell_3} e^{i\eta^{*}_{\ell_3}}}{k_+ r\sqrt{2\omega_{k_+}}}\,\chi^{*}_{\ell_3} P_{\ell_3}\left(\frac{\vec{k}_+\cdot\vec{r}}{k_+ r}\right)$$

$$= \sum_{\ell_3} \frac{4\pi(-i)^{\ell_3} e^{i\eta^{*}_{\ell_3}}}{k_+ r\sqrt{2\omega_{k_+}}}\,\chi^{*}_{\ell_3} Y_{\ell_3,m_3}(\hat{k}_+)\, Y^{*}_{\ell_3,m_3}(\hat{r}) \quad . \tag{5.7}$$

Equations (5.6) and (5.7) express the wave functions for the π^- and π^+ wave functions with an arbitrary direction of $\vec{k}$ with respect to the axis of quantization. The last term on the right of these two equations was obtained by using the addition theorem for the Legendre polynomials.[2]

The radial functions χ_{ℓ_1} and $\chi^*_{\ell_3}$ in Eqs. (5.6) and (5.7) are obtained from the differential equations

$$\left\{\frac{d^2}{dr^2} + \left[k_-^2 - 2\omega_{k_-} eA_0 - 2\mu P - V_0^2 - P^2 + e^2A_0^2 - \frac{\ell_1(\ell_1+1)}{r^2}\right] - 2iV_0(\omega_{k_-} + eA_0)\right\}\chi_{\ell_1} = 0 \quad , \tag{5.8}$$

and

$$\left\{\frac{d^2}{dr^2} + \left[k_+^2 + 2\omega_{k_+} eA_0 - 2\mu P - V_0^2 - P^2 + e^2A_0^2 - \frac{\ell_3(\ell_3+1)}{r^2}\right] - 2iV_0(\omega_{k_+} - eA_0)\right\}\chi^*_{\ell_3} = 0 \quad , \tag{5.9}$$

with the boundary condition that at r = 0 they have the value zero. In Eqs. (5.8) and (5.9) the function A_0 represents the Coulomb field of the nucleus and so χ_{ℓ_1} and $\chi^*_{\ell_3}$ must have the asymptotic forms

$$\chi_{\ell_1} \sim \sin\left(k_-r - \frac{1}{2}\ell_1\pi - \frac{\omega_{k_-} Z\alpha}{k_-} \log 2k_-r + \eta_{\ell_1}\right) \tag{5.10}$$

and

$$\chi^*_{\ell_3} \sim \sin\left(k_+r - \frac{1}{2}\ell_3\pi + \frac{\omega_{k_+} Z\alpha}{k_+} \log 2k_+r + \eta^*_{\ell_3}\right) \quad , \tag{5.11}$$

where α is the fine structure constant.

---

[2] M. E. Rose, Elementary Theory of Angular Momentum (John Wiley and Sons, Inc., New York, 1957), p. 60.

In Eq. (5.5) there is no loss of generality--in fact, there is considerable simplification if circularly polarized incident photons directed along the axis of quantization are used in the derivation. With these conditions and the Rayleigh expansion for a plane wave, the following equation[3] is obtained:

$$\vec{A}_{\vec{q}} = -p\,\hat{\xi}_p \frac{4\pi}{\sqrt{2\omega_q}} \sum_{\ell_2} i^{\ell_2}(2\ell_2 + 1)^{1/2}\, j_{\ell_2}(qr)\, Y_{\ell_2,0}(\hat{r}) \quad , \tag{5.12}$$

where the factor $1/\sqrt{2\omega_q}$ is required for proper normalization. In Eq. (5.12) $\hat{\xi}_p$ is a spherical basis vector, and p = +1 corresponds to left circular polarization and p = -1 corresponds to right circular polarization.

For unpolarized incident photons the differential cross section for meson pair production is given by

$$d\sigma = 2\pi \left(\frac{1}{2} \sum_{p=\pm 1} |M|^2 \right) \rho \quad , \tag{5.13}$$

where the density of states, ρ, is given in ordinary units by

$$\rho = \frac{P_+ \mathcal{E}_+ P_- \mathcal{E}_- d\Omega_+ d\Omega_- dE_-}{(2\pi\hbar c)^6} \quad . \tag{5.14}$$

In natural units Eq. (5.13) becomes

$$d\sigma = \left(\frac{1}{2} \sum_{p=\pm 1} |M|^2 \right) \frac{k_+ \omega_+ k_- \omega_- k_-\, d\Omega_+ d\Omega_- d\omega_-}{(2\pi)^5} \quad . \tag{5.15}$$

---

[3] Ibid., p. 136.

In order to obtain a workable expression for Eq. (5.15), it is necessary to reduce the expression for the matrix element given in Eq. (5.5). When Eqs. (5.6), (5.7), and (5.12) are substituted into Eq. (5.5), a term of the form

$$\vec{\nabla} \frac{\chi_{\ell_1}}{r} Y_{\ell_1, m_1}(\hat{r}) \tag{5.16}$$

appears. This can be evaluated by using the gradient formula:[4]

$$\vec{\nabla} R(r)\, Y_{\ell, m}(\hat{r}) = \sum_L (-1)\, C(\ell 1 L; 00) \times$$

$$\times \left\{ \frac{dR(r)}{dr} + \frac{1}{2}\left[\ell(\ell+1) + 2 - L(L+1)\right] \frac{R(r)}{r} \right\} T_{\ell L m} \quad , \tag{5.17}$$

where $T_{\ell L m}$ is an irreducible tensor of rank L, and $C(\ell 1 L; 00)$ is a Clebsch-Gordon coefficient.[5] The term given in Eq. (5.16) then becomes

$$\vec{\nabla} \frac{\chi_{\ell_1}}{r} Y_{\ell_1, m_1}(\hat{r}) = \sum_L (-1) \frac{C(\ell 1 L; 00)}{r} \times$$

$$\times \left\{ \frac{d\chi_{\ell_1}}{dr} + \frac{1}{2}\left[\ell_1(\ell_1+1) - L(L+1)\right] \frac{\chi_{\ell_1}}{r} \right\} T_{\ell_1 L m_1} \quad , \tag{5.18}$$

---

[4] Ibid., p. 124.

[5] The notation used here for the Clebsch-Gordon coefficients is that of M. E. Rose, Elementary Theory of Angular Momentum (John Wiley and Sons, Inc., New York, 1957). The notation $C(j_1 j_2 j; m_1 m_2 m)$ is abbreviated to $C(j_1 j_2 j; m_1 m_2)$ when no explicit sum over m is required and with the understanding that $m = m_1 + m_2$ for non-zero results. The connection between this notation and that of E. U. Condon and G. H. Shortley, Theory of Atomic Spectra (Cambridge University Press, Cambridge, 1935) is $C(j_1 j_2 j; m_1 m_2 m) \equiv (j_1 j_2 m_1 m_2 | j_1 j_2 j m)$.

and the matrix element becomes

$$M = \frac{-2iep(4\pi)^3}{k_- k_+ \sqrt{8\omega_q \omega_{k_-} \omega_{k_+}}} \times$$

$$\times \sum Y^*_{\ell_1, m_1}(\hat{k}_-)\, Y_{\ell_3, m_3}(\hat{k}_+)\, I(\ell_1 \ell_2 \ell_3 L)\, J(\ell_1 \ell_2 \ell_3 L p m_1 m_3) \quad , \tag{5.19}$$

where the sum is over all possible values of ℓ_1, ℓ_2, ℓ_3, L, m_1, and m_3. The functions I and J introduced into Eq. (5.19) are defined as follows:

$$I(\ell_1 \ell_2 \ell_3 L) = i^{\ell_2}(-i)^{\ell_1+\ell_3}(2\ell_2+1)^{1/2}\, C(\ell_1 1 L; 00)\, e^{i(\eta_{\ell_1}+\eta^*_{\ell_3})} \times$$

$$\times \int_0^\infty j_{\ell_2}(qr)\, \chi^*_{\ell_3} \left\{ \frac{d\chi_{\ell_1}}{dr} + \frac{1}{2}\left[\ell_1(\ell_1+1) - L(L+1)\right] \frac{\chi_{\ell_1}}{r} \right\} dr \tag{5.20}$$

$$J(\ell_1 \ell_2 \ell_3 L p m_1 m_3) = \int Y^*_{\ell_3, m_3}(\hat{r})\, Y_{\ell_2, 0}(\hat{r})\, \hat{\xi}_p\, T_{\ell_1 L m_1}\, d\Omega \quad . \tag{5.21}$$

The integrand in the expression for J can be put in a better form for evaluation of the integral by substituting the following expression[6]

$$T_{\ell_1 L m_1} = \sum_{m_2} C(L 1 \ell_1; m_1 - m_2, m_2)\, Y_{L, m_1 - m_2}(\hat{r})\, \hat{\xi}_{m_2} \tag{5.22}$$

and noting that[7]

$$\hat{\xi}_p \cdot \hat{\xi}_{m_2} = (-1)^p\, \delta(-m_2, p) \quad , \tag{5.23}$$

---

[6] M. E. Rose, Elementary Theory of Angular Momentum (John Wiley and Sons, Inc., New York, 1957), p. 106.

[7] Ibid., p. 104.

where $\delta(-m_2, p)$ is the Kronecker δ symbol. Hence, Eq. (5.21) becomes

$$J = (-1)^p\, C(L1\ell_1; m_1 + p, -p) \int Y^*_{\ell_3, m_3}(\hat{r})\, Y_{\ell_2, 0}(\hat{r})\, Y_{L, m_1+p}(\hat{r})\, d\Omega \quad . \tag{5.24}$$

The integral in this equation can now be evaluated to give[8]

$$\int Y^*_{\ell_3, m_3}(\hat{r})\, Y_{\ell_2, 0}(\hat{r})\, Y_{L, m_1+p}(\hat{r})\, d\Omega = \left[\frac{(2L+1)(2\ell_2+1)}{4\pi(2\ell_3+1)}\right]^{1/2} \times$$

$$\times\ C(L\ell_2\ell_3; m_1 + p, 0)\, C(L\ell_2\ell_3; 00)\, \delta(m_3, m_1 + p) \quad , \tag{5.25}$$

where $C(L\ell_2\ell_3; m_1 + p, 0, m_3)$ was replaced by $C(L\ell_2\ell_3; m_1 + p, 0)\, \delta(m_3, m_1 + p)$ since m_3 must be equal to $m_1 + p$ for non-zero results. The expression for J now has the convenient form

$$J = (-1)^p \left[\frac{(2L+1)(2\ell_2+1)}{4\pi(2\ell_3+1)}\right]^{1/2} C(L1\ell_1; m_1 + p, -p)\, C(L\ell_2\ell_3; m_1 + p, 0)$$

$$C(L\ell_2\ell_3; 00)\, \delta(m_3, m_1 + p) \quad . \tag{5.26}$$

Inserting Eq. (5.26) into the expression for the matrix element given by Eq. (5.19) and performing the sum over m_3 yields

$$M = \frac{-2iep(-1)^p\, (4\pi)^{5/2}}{k_- k_+ \sqrt{8\omega_q \omega_{k_-} \omega_{k_+}}} \sum \left\{ Y^*_{\ell_1, m_1}(\hat{k}_-)\, Y_{\ell_3, m_1+p}(\hat{k}_+)\, I_1(\ell_1\ell_2\ell_3 L) \times \right.$$

$$\left. \times\ C(L1\ell_1; m_1 + p, -p)\, C(L\ell_2\ell_3; m_1 + p, 0) \right\} \quad , \tag{5.27}$$

---

[8] Ibid., p. 62.

where the new function I_1 is related to I given in Eq. (5.20) by

$$I_1 = \left[\frac{(2L+1)(2\ell_2+1)}{(2\ell_3+1)}\right]^{1/2} C(L\ell_2\ell_3;00)\, I \quad . \tag{5.28}$$

The term in the parentheses in Eq. (5.15) can now be evaluated to give

$$\frac{1}{2}\sum_{p=\pm 1}|M|^2 = \frac{e^2(4\pi)^5}{4k_-^2k_+^2\omega_q\omega_{k_-}\omega_{k_+}} \sum\Big\{Y^*_{\ell_1,m_1}(\hat{k}_-)\, Y_{\ell_1',m_1'}(\hat{k}_-)\, Y_{\ell_3,m_1+p}(\hat{k}_+) \times$$

$$\times\; Y^*_{\ell_3,m_1'+p}(\hat{k}_+)\, I_1(\ell_1\ell_2\ell_3L)\, I_1^*(\ell_1'\ell_2'\ell_3'L')\, C(L1\ell_1;m_1+p,-p) \times$$

$$\times\; C(L\ell_2\ell_3;m_1+p,0)\, C(L'1\ell_1';m_1'+p,-p)\, C(L'\ell_2'\ell_3';m_1+p,0)\Big\}\,, \tag{5.29}$$

where the sum is over the values +1 and -1 for p, all possible values of ℓ_1, ℓ_2, ℓ_3, L, m_1, and the corresponding primed quantities.

Before Eq. (5.29) is substituted into Eq. (5.15), we shall first change to a dimensionless variable

$$v = \frac{\omega_{k_-} - \mu}{\omega_q - 2\mu} \tag{5.30}$$

in Eq. (5.15), which results in

$$d\sigma = R^2\left[\frac{8\left(1-\frac{2\mu}{\omega_q}\right)}{k_+k_-\alpha\lambda\!\!\!^{-2}}\, S\, dv\, d\Omega_+ d\Omega_-\right] \quad , \tag{5.31}$$

where the quantity in the brackets is dimensionless and, in ordinary units, R is the classical radius of the meson, $R = e^2/mc^2$; α is the fine structure constant, $\alpha = e^2/\hbar c$; and $\lambda\!\!\!^{-}$ is the meson Compton wavelength,

$\lambda\!\!\!^{-} = \hbar c/mc^2$. The function S in Eq. (5.31) is given by

$$S = \sum C(Ll\ell_1;m_1 + p,-p)\, C(L\ell_2\ell_3;m_1 + p,0) \times$$

$$\times\, C(L'1\ell_1;m_1' + p,-p)\, C(L'\ell_2'\ell_3';m_1' + p,0)\, I_1(\ell_1\ell_2\ell_3L)\, I_1^*(\ell_1'\ell_2'\ell_3'L) \times$$

$$\times\; Y^*_{\ell_1,m_1}(\hat{k}_-)\, Y_{\ell_1',m_1'}(\hat{k}_-)\, Y_{\ell_3,m_1+p}(\hat{k}_+)\, Y^*_{\ell_3',m_1'+p}(\hat{k}_+) \quad , \qquad (5.32)$$

where the sum is over the same values as indicated for Eq. (5.29).

For the purposes of the present problem we want to integrate S over all possible directions of $\vec{k}_+$ and $\vec{k}_-$; hence, we want to obtain the solution of

$$K = \iint S\, d\Omega_+\, d\Omega_- \quad . \qquad (5.33)$$

The integrals over products of spherical harmonics appearing in this equation are easily evaluated because of their orthogonal property, given by

$$\int Y^*_{\ell,m}(\hat{k})\, Y_{\ell',m'}(\hat{k})\, d\Omega_k = \delta(m,m')\, \delta(\ell,\ell') \quad . \qquad (5.34)$$

Therefore after integration the product of the four Kronecker delta symbols, $\delta(\ell_1,\ell_1')\, \delta(m_1,m_1')\, \delta(\ell_3,\ell_3')\, \delta(m_1' + p, m_1 + p)$, remains, which permits the sums over ℓ_1', ℓ_3' and m_1' to be performed in Eq. (5.33), yielding

$$K = \sum V(LL'\ell_2\ell_2'\ell_1\ell_3)\, I_1(\ell_1\ell_2\ell_3L)\, I_1^*(\ell_1\ell_2'\ell_3'L') \quad . \qquad (5.35)$$

The sum over m_1 and p is included in V, which is given by

$$V(LL'\ell_2\ell_2'\ell_1\ell_3) = \sum_{\substack{m_1 \\ p=\pm 1}} C(L1\ell_1;m_1+p,-p)\, C(L\ell_2\ell_3;m_1+p,0) \times$$

$$\times\, C(L'1\ell_1';m_1+p,-p)\, C(L'\ell_2'\ell_3;m_1+p,0) \quad , \qquad (5.36)$$

and the summation in Eq. (5.35) retains the sum over all possible values of L, L', ℓ_1, ℓ_2, ℓ_2', and ℓ_3. Equation (5.36) can be simplified to a considerable extent by the recoupling of the angular momentum. Noting certain properties of the quantum numbers will facilitate this reduction. The coefficients $C(L\,\ell_2\ell_3;00)$ and $C(\ell_1 1L;00)$ appearing in the function $I_1(\ell_1\ell_2\ell_3L)$ require $L+\ell_2+\ell_3$ and ℓ_1+1+L to be even for non-zero results, and, similarly, from $I_1^*(\ell_1\ell_2'\ell_3L')$, $L'+\ell_2'+\ell_3$ and ℓ_1+1+L' must be even. These conditions lead to the fact that $(\ell_1+1+L)+(\ell_1+1+L')$ and therefore $L+L'$ must be even and that $(L+\ell_2+\ell_3)+(L'+\ell_2'+\ell_3)$ and therefore $\ell_2+\ell_2'$ must be even.

The details of the recoupling procedure shown in Appendix C lead to the expression

$$V(LL'\ell_2\ell_2'\ell_1\ell_3) = \frac{2}{3}\,\frac{(2\ell_1+1)(2\ell_3+1)}{(2\ell_2+1)(2L+1)}\,\delta(L,L')\,\delta(\ell_2,\ell_2') +$$

$$+ (2\ell_1+1)(2\ell_3+1)(-1)^{\ell_2}\sqrt{\tfrac{2}{3}}\,C(\ell_2'\ell_2 2;00)\,W(LlL'1;\ell_1 2) \times \qquad (5.37)$$

$$\times\, W(2L\ell_2'\ell_3;L'\ell_2) \quad ,$$

where W is a Racah coefficient.

An additional simplification can be obtained by examining the product $I_1 I_1^*$ that appears in Eq. (5.35). In that factor it is found that

$$I_1 I_1^* \propto (-i)^{\ell_1+\ell_3} (i)^{\ell_1+\ell_3} e^{i(\eta_{\ell_1}+\eta^*_{\ell_3})} e^{-i(\eta^*_{\ell_1}+\eta_{\ell_3})} \quad . \tag{5.38}$$

So, if the real and imaginary parts of the phase shift are separated by letting

$$\eta_\ell = \alpha_\ell + i\,\delta_\ell \quad , \tag{5.39}$$

then, from Eq. (5.38),

$$I_1 I_1^* \propto e^{-2\delta_{\ell_1}} e^{+2\delta_{\ell_3}} \quad , \tag{5.40}$$

and there is no need to obtain the real part of the phase shift.[9] With this development in mind it is best to redefine K as

$$K = \sum V(LL'\ell_2\ell_2'\ell_1\ell_3)\, G(\ell_1\ell_2\ell_3 L)\, G^*(\ell_1\ell_2'\ell_3 L') \quad , \tag{5.41}$$

where G is obtained from I_1 given in Eq. (5.28) after the appropriate factors are dropped:

$$G(\ell_1\ell_2\ell_3 L) = i^{\ell_2} \left[\frac{(2L+1)}{(2\ell_3+1)}\right]^{1/2} (2\ell_2+1)\, C(\ell_1 1 L;00)\, C(L\ell_2\ell_3;00) \times$$

$$\times\ F(\ell_1\ell_2\ell_3 L) \quad , \tag{5.42}$$

---

[9] From Eqs. (5.8) - (5.11) we see that $\delta_{\ell_1} > 0$ and $\delta_{\ell_3} < 0$; so both factors appearing on the right of Eq. (5.40) are less than unity.

in which

$$F(\ell_1\ell_2\ell_3 L) = e^{-i(\delta_{\ell_1}-\delta_{\ell_3})} \int_0^\infty j_{\ell_2}(qr)\, \chi^*_{\ell_3} \left\{ \frac{d\chi_{\ell_1}}{dr} + \right.$$

$$\left. + \frac{1}{2}\left[\ell_1(\ell_1+1) - L(L+1)\right] \frac{\chi_{\ell_1}}{r} \right\} dr \quad . \quad (5.43)$$

The expression for K can be reduced still further if it is observed that V is real and the equation is written with the sum over ℓ_1 and ℓ_3 suppressed. For clarity it is best to represent the pair (ℓ_2, L) with the single index i and the pair (ℓ_2', L') with the single index j. Examination of V shows that it is symmetric to the interchange of i and j; therefore, after first interchanging summation indices,

$$\begin{aligned} K &= \sum_i \sum_j V(i,j)\, G(i)\, G^*(j) \\ &= \sum_j \sum_i V(j,i)\, G(j)\, G^*(i) \quad . \end{aligned} \quad (5.44)$$

The last term in Eq. (5.44) leads to

$$K = \sum_i \sum_j V(i,j)\, G^*(i)\, G(j) \quad (5.45)$$

by using the symmetry property of V; therefore, K = K*, as it must be. Therefore,

$$K = \sum_i \sum_j V(i,j)\, \mathcal{R}\left[G(i)\, G^*(j)\right] \quad , \quad (5.46)$$

where $\mathcal{R}$ indicates the real part is to be taken. The entire function $V(i,j)\mathcal{R}\left[G(i)\, G^*(j)\right]$ is now observed to be symmetric to the interchange

of i and j. This observation leads to a reduction in the number of terms in the sum.

Still suppressing the sum over ℓ_1 and ℓ_3,

$$K = \sum_{\ell_2} \sum_{\ell_2'} \sum_{L} \sum_{L'} Z(LL'\ell_2\ell_2') \quad , \tag{5.47}$$

where

$$Z(LL'\ell_2\ell_2') = V(LL'\ell_2\ell_2') \, \mathcal{R}\left[G(\ell_2 L) \, G^*(\ell_2' L')\right] \quad , \tag{5.48}$$

and using the symmetry property of Z,

$$K = \sum_{\ell_2} \sum_{L} Z(LL\ell_2\ell_2) + 2 \sum_{\ell_2} \sum_{L > L'} \sum Z(LL'\ell_2\ell_2) +$$

$$+ 2 \sum_{\ell_2 < \ell_2'} \sum_{L} \sum_{L'} Z(LL'\ell_2\ell_2') \quad . \tag{5.49}$$

Equation (5.49) can be compressed by introducing the function $Q(\ell_2\ell_2'LL')$ defined by

$$\begin{aligned} Q(\ell_2\ell_2'LL') &= 1 \quad \text{if } \ell_2' = \ell_2 \text{ and } L' = L \quad , \\ &= 2 \quad \text{if } \ell_2' = \ell_2 \text{ and } L' < L \quad , \\ &= 2 \quad \text{if } \ell_2' > \ell_2 \quad , \\ &= 0 \quad \text{otherwise.} \end{aligned} \tag{5.50}$$

Equation (5.41) now becomes

$$K = \sum Q(\ell_2\ell_2'LL') \, V(LL'\ell_2\ell_2'\ell_1\ell_3) \, \mathcal{R}\left[G(\ell_1\ell_2\ell_3 L) \, G^*(\ell_1\ell_2'\ell_3 L')\right] \quad , \tag{5.51}$$

where the sum is over all possible values of all the indicated quantum numbers.

Although the sum in Eq. (5.51) is still formidable, there is a great reduction in the allowed values of the quantum numbers because of the Clebsch-Gordon and Racah coefficients that appear in the terms. If ℓ_1 and ℓ_3 are considered to be unrestricted, than L and L' can only have the values $\ell_1 \pm 1$; and ℓ_2 and ℓ_2' must satisfy the triangle conditions for the addition of angular momentum represented by $T(L, \ell_2, \ell_3)$ and $T(L', \ell_2', \ell_3)$, along with the condition that $L + \ell_2 + \ell_3$, $L' + \ell_2' + \ell_3$, and $\ell_2 + \ell_2'$ are even. With the restriction on the quantum numbers in mind, the only term in the sum of terms remaining which is zero is for $\ell_1 = \ell_3 = 0$. In this case the only allowed value of L, L', ℓ_2 and ℓ_2' is 1, and examination of that term in the sum shows it to be zero. This simply reflects the conservation of angular momentum where the incident photon which has unit angular momentum cannot lead to two pions with vanishing angular momentum.

CHAPTER VI

METHODS OF SOLUTION

In order to obtain values from the differential cross-section formula for the electromagnetic production of pion pairs given by

$$d\sigma = R^2 \left[\frac{8\left(1 - \frac{2u}{\omega_q}\right) K}{k_+ k_- \alpha^2} \right] \quad dv \quad , \tag{6.1}$$

methods of calculation for the various terms had to be devised. Of particular difficulty and interest was the method of evaluating the integral that appears in the function $F(\ell_1 \ell_2 \ell_3 L)$, which is described below in Section III-F. This method and the other methods of obtaining the factors appearing in the function K are described in this chapter.

I. GENERAL DESCRIPTION OF CONTENTS

The function K given by Eq. (5.51) is a sum of products of three other functions of which the first, $Q(\ell_2 \ell_2' LL')$ given in Eq. (5.50), is trivial; the second, $V(LL'\ell_2\ell_2')$ given in Eq. (5.37), is not difficult once the Racah and Clebsch-Gordon coefficients are obtained; and the third, $G(\ell_1 \ell_2 \ell_3 L)$ given in Eq. (5.42), presents the most difficulty.

The method of obtaining the Clebsch-Gordon and Racah coefficients will be presented first. This essentially describes the method of obtaining $V(LL'\ell_2\ell_2')$ and two coefficients appearing in $G(\ell_1 \ell_2 \ell_3 L)$, leaving the more detailed methods of obtaining $F(\ell_1 \ell_2 \ell_3 L)$ which also appears in $G(\ell_1 \ell_2 \ell_3 L)$ to the remainder of the chapter.

II. RACAH AND CLEBSCH-GORDON COEFFICIENTS

Although tables of the Clebsch-Gordon[1] and Racah[2,3] coefficients are available, they do not contain values of the coefficients for all values of the indices anticipated in this problem. This left the task of either hand-calculating the coefficients for the remaining values of the coefficients not included in the tables and tabulating all of them or calculating them from first principles on the computing machine as they were needed. The latter course of action was taken.

Following the methods suggested by Simon,[1] the equation for the Clebsch-Gordon coefficients given by Racah,[4]

$$C(j_1 j_2 j; m_1 m_2 m) = \delta(m, m_1+m_2)\ \Delta(j_1 j_2 j)\ X$$

$$X \sum_K \frac{(-1)^K \sqrt{(j_1+m_1)!\ (j_1-m_1)!\ (j_2+m_2);\ (j_2-m_2)!\ (j+m)!\ (j-m)!\ (2j+1)}}{K!\ (j_1+j_2-j-K)!\ (j_1-m_1-K)!\ (j_2+m_2-K)!\ (j-j_2+m_1+K)!\ (j-j_1-m_2+K)!}\ , \tag{6.2}$$

where

$$\Delta(j_1 j_2 j) = \left[\frac{(j_1+j_2-j)!\ (j_1-j_2+j)!\ (-j_1+j_2+j)!}{(j_1+j_2+j+1)!}\right]^{1/2}\ , \tag{6.3}$$

---

[1] A. Simon, Numerical Tables of the Clebsch-Gordon Coefficients, USAEC Report ORNL-1718, Oak Ridge National Laboratory (1954).

[2] L. C. Biedenharn, Tables of the Racah Coefficients, USAEC Report ORNL-1098, Oak Ridge National Laboratory (1952).

[3] Simon, Vander Sluis, and Biedenharn, Tables of the Racah Coefficients, USAEC Report ORNL-1679, Oak Ridge National Laboratory (1954).

[4] G. Racah, Phys. Rev. 62, 438 (1942); Phys. Rev. 63, 367 (1943).

was solved for each coefficient. The summation index K can take on all integral values such that no factorial term becomes negative in the denominator. This is consistent with the factorial of negative integers having the value of infinity. The values of the arguments in Eq. (6.2) can be integral or half-odd integral values subject to the conditions of the vector addition of angular momentum which leads to

$$\left.\begin{aligned} &j_1 + j_2 + j \\ &j_1 + j_2 - j \\ &j_1 - j_2 + j \\ &-j_1 + j_2 + j \end{aligned}\right\} = \text{a positive integer or zero,} \qquad (6.4)$$

and

$$\begin{aligned} |m_1| &\leq j_1 \ , \\ |m_2| &\leq j_2 \ , \\ |m_3| &\leq j_3 \ , \end{aligned} \qquad (6.5)$$

which in turn leads to

$$\left.\begin{aligned} &j_1 + m_1 \\ &j_2 + m_2 \\ &j_3 + m_3 \end{aligned}\right\} = \text{a positive integer or zero.} \qquad (6.6)$$

The equation solved for the Racah coefficients was the one used by Biedenharn[2] and deduced by Racah:[4]

$$W(abcd;ef) = \Delta(abc)\,\Delta(cde)\,\Delta(acf)\,\Delta(bdf) \times$$

$$\sum_K \frac{(-1)^K\,(a+b+c+d+1-K)!}{(a+b-e-K)!\,(c+d-e-K)!\,(a+c-f-K)!\,(b+d-f-K)!\,K!\,(e+f-a-d+K)!\,(e+f-b-c+K)!} \qquad (6.7)$$

where the Δ factors are defined in Eq. (6.3). Each set of triads appearing in each Δ factor must satisfy the conditions given in Eq. (6.4), and the summation index K in Eq. (6.7) is restricted the same as in Eq. (6.2).

Equations (6.2) and (6.7) are not difficult nor time-consuming to solve on a computing machine if the values of all the factorials anticipated in these equations are kept as constants to be used as needed.

III. METHODS FOR OBTAINING THE FUNCTION $F(\ell_1\ell_2\ell_3 L)$

A. Spherical Bessel Functions

One of the functions appearing in the integrand of the integral in $F(\ell_1\ell_2\ell_3 L)$ is the spherical Bessel Function. It can be obtained from the recursion relation[5]

$$j_{\ell+1}(\rho) = \frac{(2\ell+1)}{\rho} j_\ell(\rho) - j_{\ell-1}(\rho) \quad , \tag{6.8}$$

with the starting conditions that

$$j_0(\rho) = \frac{\sin\rho}{\rho} \tag{6.9}$$

and

$$j_1(\rho) = \frac{\sin\rho}{\rho^2} - \frac{\cos\rho}{\rho} \quad . \tag{6.10}$$

However, when ℓ becomes large, $j_\ell(\rho)$ becomes a monotonically decreasing function of ℓ, and each succeeding value of $j_\ell(\rho)$ from the recursion

---

[5] L. I. Schiff, Quantum Mechanics (McGraw-Hill Book Company, Inc., New York, 1955), 2nd ed., p. 78.

relation is the difference between two larger values. This means that the number of significant figures deteriorates rapidly. The value of ℓ for which this condition arises decreases as ρ decreases, and therefore if $j_\ell(\rho)$ is to be obtained for all ℓ up to a fixed maximum value and for all ρ, a different method from that given in Eq. (6.8) must be used for the smaller values of ρ.

The method appearing to be the most useful for small ρ was that of inverting the recursion relation so that

$$j_{\ell-1}(\rho) = \frac{(2\ell + 1)}{\rho} j_\ell(\rho) - j_{\ell+1}(\rho) \tag{6.11}$$

and starting the process with $\ell = m$, where m is a large value. The starting conditions are that $j_{m+1}(\rho) = 0$ and that $j_m(\rho)$ equals some small value; the recursion relation is used to obtain $j_0(\rho)$. The value of $j_0(\rho)$ obtained in this way is compared with Eq. (6.9) to obtain a normalizing constant to be applied to all $j_\ell(\rho)$ in the sequence.

The value of ρ at which the change is made from Eq. (6.11) to Eq. (6.8) depends on the number of significant figures carried and the number of significant figures required in the answer and is not fixed.

B. The Coulomb Potential

Before the differential equations can be solved for the radial functions, the Coulomb potential from the distributed nuclear charge has to be obtained. The charge density of the nucleus was discussed in Chapter II and is represented by the equation

$$\rho(r) = \frac{\rho_0}{1 + \exp\left(\frac{r - \beta}{\alpha}\right)} \quad , \tag{6.12}$$

where the values of the constants α and β for best fit to the experimental data are $\alpha = 0.546 \times 10^{-13}$ cm and $\beta = 1.07\ A^{1/3} \times 10^{-13}$ cm. The constant ρ_0 is chosen so that

$$\rho_0 = \left[\int \frac{d^3r}{1 + \exp\left(\frac{r - \beta}{\alpha}\right)} \right]^{-1} \tag{6.13}$$

and the nuclear charge Z is expressed separately.

The potential is obtained from the spherically symmetric charge density by the equation

$$\frac{A_0(r)}{-Ze} = \frac{4\pi}{r} \int_0^r \rho(r')\, r'^2\, dr' + 4\pi \int_r^\infty \frac{\rho(r')\, r'^2\, dr'}{r'} \ , \tag{6.14}$$

which can also be expressed by

$$\frac{A_0(r)}{-Ze} = \frac{4\pi\rho_0}{r} \int_0^r \frac{r'^2\, dr'}{1 + \exp\left(\frac{r' - \beta}{\alpha}\right)} + 4\pi\rho_0 \int_r^\infty \frac{r'\, dr'}{1 + \exp\left(\frac{r' - \beta}{\alpha}\right)} \ . \tag{6.15}$$

This function is best defined in terms of the new functions

$$F_n(x) = \int_0^x \frac{u^n\, du}{1 + e^u} \ . \tag{6.16}$$

The dependence of the potential on the functions F_n can be obtained by changing variables in Eq. (6.15) to $u = (r' - \beta)/\alpha$, which gives

$$\begin{aligned} \frac{A_0(r)}{-Ze} = 4\pi\rho_0\alpha \Bigg(\frac{1}{r} \Bigg\{ & \alpha^2 \left[F_2\left(\frac{r - \beta}{\alpha}\right) - F_2\left(-\frac{\beta}{\alpha}\right) \right] + 2\alpha\beta \left[F_1\left(\frac{r - \beta}{\alpha}\right) - F_1\left(-\frac{\beta}{\alpha}\right) \right] + \\ & + \beta^2 \left[F_0\left(\frac{r - \beta}{\alpha}\right) - F_0\left(-\frac{\beta}{\alpha}\right) \right] \Bigg\} + \\ & + \left\{ \alpha \left[F_1(\infty) - F_1\left(\frac{r - \beta}{\alpha}\right) \right] + \beta \left[F_0(\infty) - F_0\left(\frac{r - \beta}{\alpha}\right) \right] \right\} \Bigg) \ , \end{aligned} \tag{6.17}$$

where

$$F_0(\infty) = \log_e 2 \quad ,$$
$$F_1(\infty) = \pi^2/12 \quad , \tag{6.18}$$
$$F_2(\infty) = 3/2\ (1.202056903) \quad .$$

In the same way the constant ρ_0 is found to be

$$\rho_0^{-1} = 4\pi\alpha\left\{\alpha^2\left[F_2(\infty) - F_2\left(-\frac{\beta}{\alpha}\right)\right] + 2\alpha\beta\left[F_1(\infty) - F_1\left(-\frac{\beta}{\alpha}\right)\right] + \right.$$
$$\left. + \beta^2\left[F_0(\infty) - F_0\left(-\frac{\beta}{\alpha}\right)\right]\right\} . \tag{6.19}$$

Although the functions $F_0(x)$, $F_1(x)$, and $F_2(x)$ have to be tabulated for use, the advantage of this formalism is that $A_0(r)$ can be obtained for any value of A which appears in the constant β from the same set of tables.

C. Removing the Singularity in the Radial Differential Equation

The solutions of the two radial equations (5.8) and (5.9) are essentially the same; therefore the discussion will be restricted to considering only Eq. (5.8), which is repeated here for convenience without subscripts:

$$\left\{\frac{d^2}{dr^2} + \left[k^2 - 2\omega eA - 2\mu P - V^2 - P^2 + e^2A^2 - \frac{\ell(\ell+1)}{r^2}\right] - 2iV(\omega + eA)\right\}\chi = 0. \tag{6.20}$$

Before the solution of Eq. (6.20) can be effected, it is necessary to examine carefully the various terms that appear. The quantities k, ω, e, μ, and ℓ are constants and V and P, representing the square-well nuclear optical potential, are constant for r less than the nuclear

radius and zero for r greater than the nuclear radius.[6] The Coulomb potential A resulting from distributed nuclear charge is a slowly varying function in the region of small r, having a finite value and a slope or zero at r = 0.[7] For large r, of course, the Coulomb potential goes as 1/r.

To obtain starting conditions for the solution of Eq. (6.20), the equation will be examined at small values of r. Therefore, we consider the solution of the equation

$$\frac{d^2\chi}{dr} - \frac{(\ell + 1)}{r^2}\chi = 0 \quad , \tag{6.21}$$

where the boundary condition requires χ to be zero at r = 0. By letting $\chi = Br^{s+1}$ and substituting it into Eq. (6.21), it is found that this solution can only hold for $s = \ell$. Hence, for small r

$$\chi = Br^{\ell+1} \quad , \tag{6.22}$$

and

$$\frac{d\chi}{dr} = B(\ell + 1)r^{\ell} \quad . \tag{6.23}$$

Equations (6.22) and (6.23) present a problem in that for $\ell > 0$ they are both zero at r = 0 and are not adequate to start a numerical solution from the origin. Another numerical difficulty is the singularity $\ell(\ell + 1)/r^2$ for $\ell > 0$. These difficulties can be removed, however,

---

[6] The nuclear radius for purposes of defining the square-well optical potential was taken at $1.4\ A^{1/3} \times 10^{-13}$ cm.

[7] These observations can be substantiated by examining the Coulomb potential resulting from a uniform charge density.

by making the transformation

$$\theta = r^{-(\ell+1)} \chi \quad , \tag{6.24}$$

$$\frac{d\theta}{dr} = r^{-(\ell+1)} \left[\frac{d\chi}{dr} - \frac{(\ell+1)}{r} \chi \right] , \tag{6.25}$$

so that the differential equation becomes

$$\left\{ \frac{d^2}{dr^2} + \frac{2(\ell+1)}{r} \frac{d}{dr} + \left[k^2 - 2\omega eA - 2\mu P - V^2 - P^2 + e^2 A^2 \right] - 2iV(\omega + eA) \right\} \theta = 0, \tag{6.26}$$

where, for small r,

$$\theta = B \tag{6.27}$$

$$\frac{d\theta}{dr} = 0 \quad . \tag{6.28}$$

In practice, the differential equation for θ is solved numerically until some convenient value of r is reached; then the transformation back to the function χ is made, and one continues with the solution of its differential equation.

D. Solution of the Radial Differential Equation

The Runge-Kutta method of integrating the radial differential equations, and in particular the version given by Gill,[8] has many advantages for the present problem. With this method it is possible to obtain values of the derivatives of the radial functions necessary in the integrand in the function $F(\ell_1 \ell_2 \ell_3 L)$ and the starting conditions are particularly simple since only the initial values of the radial function

[8] S. Gill, Proc. Cambridge Phil. Soc. 47, 96 (1951).

and its derivative are needed.

In general, the Runge-Kutta method is a means of solving any number of coupled first-order differential equations, and it is a fourth-order process which has a truncation error in one step of the order of h^5, where h is the interval width.

To show how Eq. (6.26) can be put in the form of coupled first-order equations we consider the simplified notation of the second-order equation

$$\frac{d^2\theta}{dr^2} + R(r)\frac{d\theta}{dr} + \left[P(r) + iQ(r)\right]\theta = 0 \tag{6.29}$$

and note that θ is a complex function. By making the substitution

$$\theta = \theta_1 + i\theta_2 \tag{6.30}$$

the real and imaginary parts of Eq. (6.29) can be separated to give

$$\frac{d^2\theta_1}{dr^2} + R\frac{d\theta_1}{dr} + P\theta_1 - Q\theta_2 = 0 \tag{6.31a}$$

and

$$\frac{d^2\theta_2}{dr^2} + R\frac{d\theta_2}{dr} + P\theta_2 + Q\theta_1 = 0 \quad . \tag{6.31b}$$

Each of these equations can now be reduced to first-order equations by the substitution

$$y_1 = \theta_1 \quad , \tag{6.32a}$$

$$y_2 = \frac{d\theta_1}{dr} \quad , \tag{6.32b}$$

$$y_3 = \theta_2 \quad , \tag{6.32c}$$

$$y_4 = \frac{d\theta_2}{dr} \quad , \tag{6.32d}$$

so that Eqs. (6.31a) and (6.31b) become the set of four coupled equations:

$$\frac{dy_1}{dr} - y_2 = 0 \quad , \tag{6.33a}$$

$$\frac{dy_2}{dr} + Ry_2 + Py_1 - Qy_3 = 0 \quad , \tag{6.33b}$$

$$\frac{dy_3}{dr} - y_4 = 0 \quad , \tag{6.33c}$$

$$\frac{dy_4}{dr} + Ry_4 + Py_3 + Qy_1 = 0 \quad . \tag{6.33d}$$

Hence, each solution of Eq. (6.26) involves the solution of four coupled first-order equations, which is also true of Eq. (6.20).

To start the integration from r = 0, the conditions given in Eq. (6.27) and (6.28) are used:

$$y_1(0) = D \quad , \tag{6.34a}$$

$$y_2(0) = 0 \quad , \tag{6.34b}$$

$$y_3(0) = D \quad , \tag{6.34c}$$

$$y_4(0) = 0 \quad , \tag{6.34d}$$

where D is an arbitrary real constant.

E. Determination of the Normalizing Constant and Complex Phase Shift

The function resulting from the integration of the radial differential equation is not normalized properly as a result of the arbitrary

constant used in the starting conditions. This means that a numerically determined radial function will differ from the required asymptotic forms given in Eqs. (5.10) or (5.11) by a complex constant which must be determined. To obtain the constant without integrating to very large values of r, the radial differential equation is written as

$$\chi'' + p^2(r)\chi = 0 \quad ; \tag{6.35}$$

for r greater than the nuclear radius, p^2 is real, and for r sufficiently large, p^2 is positive as well as real. In the latter case the W.K.B. approximation ϕ for the solution to Eq. (6.35) is

$$\chi \cong \phi = \left[\frac{k}{K(r)}\right]^{1/2} \sin\left[\int K(r)\, dr + \eta\right] , \tag{6.36}$$

where K(r) is a function approximated by p(r), η is the complex phase shift, and the asymptotic form of χ and ϕ is given by

$$\chi = \phi \sim \sin\left[G(r) + \eta\right] \quad . \tag{6.37}$$

In Eq. (6.37), the function G(r) has the form appearing in Eq. (5.10) or (5.11).

The approximation $\chi \cong \phi$ in Eq. (6.36) can be achieved to a higher degree of accuracy for intermediate values of r if a better approximation than p(r) for the function K(r) is used. The better approximation for K(r) can be obtained by noting that ϕ satisfies the differential equation

$$\phi'' + \left[K^2 + \frac{K''}{2K} - \frac{3}{4}\left(\frac{K'}{K}\right)^2\right]\phi = 0 \tag{6.38}$$

and that for ϕ to be also a solution to Eq. (6.35),

$$p^2 = K^2 + \frac{K''}{2K} - \frac{3}{4}\left(\frac{K'}{K}\right)^2 \quad . \tag{6.39}$$

Now, if p is slowly varying with respect to r, the iteration

$$K_{i+1} = \sqrt{p^2 - \frac{K_i''}{2K_i} + \frac{3}{4}\left(\frac{K_i'}{K_i}\right)^2} \quad , \tag{6.40}$$

with the starting condition that $K_0 = p$, will converge very rapidly. In fact, it is satisfactory to carry the iteration only one step, which results in

$$K = \sqrt{p^2 - \frac{p''}{2p} + \frac{3}{4}\left(\frac{p'}{p}\right)^2} \quad , \tag{6.41}$$

and therefore making ϕ a very good approximation to χ.

For intermediate values of r the functions θ and θ' resulting from the integration of Eq. (6.35) differ from χ and χ' respectively by a complex constant, A, which can be determined from Eq. (6.36) by first setting

$$\theta = A\left(\frac{k}{K}\right)^{1/2} \sin\left(\int K dr + \eta\right) \tag{6.42a}$$

and then obtaining

$$\theta' = A(kK)^{1/2} \cos\left(\int K dr + \eta\right) - \frac{1}{2}\,\frac{K'}{K}\,\theta \quad . \tag{6.42b}$$

From these two equations we can write

$$B \equiv J_1 + iJ_2 = \left(\frac{K}{k}\right)^{1/2} \theta = A \sin\left(\int K dr + \alpha + i\delta\right) \quad , \tag{6.43a}$$

$$C \equiv I_1 + iI_2 = \frac{\theta'}{(kK)^{1/2}} + \frac{K'\theta}{2K(kK)^{1/2}} = A \cos\left(\int K dr + \alpha + i\delta\right) \quad , \tag{6.43b}$$

where the complex phase shift is written as $\eta = \alpha + i\delta$. By summing the squares of Eqs. (6.43a) and (6.43b),

$$A^2 = \left(J_1^2 - J_2^2 + I_1^2 - I_2^2\right) + i(2J_1J_2 + 2I_1I_2) \equiv N_1 + iN_2 \quad , \tag{6.44}$$

which leads to

$$|A|^2 = \sqrt{N_1^2 + N_2^2} \tag{6.45}$$

and finally to

$$A = \sqrt{\frac{|A|^2 + N_1}{2}} + i\,\frac{N_2}{|N_2|}\sqrt{\frac{|A|^2 - N_1}{2}} \quad , \tag{6.46}$$

which gives the normalizing constant.

The imaginary part of the complex phase shift can now be easily obtained in the form of $e^{-i\delta}$, which is the form required in $F(\ell_1\ell_2\ell_3 L)$, by using Eqs. (6.43a) and (6.43b) to write

$$B - iC = A\left[\sin\left(\int K dr + \alpha + i\delta\right) - i\cos\left(\int K dr + \alpha + i\delta\right)\right] =$$

$$= (J_1 + I_2) + i(J_2 - I_1) \quad , \tag{6.47a}$$

$$B^* + iC^* = A^*\left[\sin\left(\int K dr + \alpha - i\delta\right) + i\cos\left(\int K dr + \alpha - i\delta\right)\right] =$$

$$= (J_1 + I_2) - i(J_2 - I_1) \quad . \tag{6.47b}$$

By multiplying these two equations together,

$$(J_1 + I_2)^2 + (J_2 - I_1)^2 = |A|^2\left[\cos(2i\delta) + i\sin(2i\delta)\right] = |A|^2 e^{-2\delta} \quad , \tag{6.48}$$

which gives the desired term as

$$e^{-\delta} = \sqrt{\frac{(J_1 + I_2)^2 + (J_2 - I_1)^2}{|A|^2}} \quad , \tag{6.49}$$

where $|A|^2$ is given in Eq. (6.45).

In practice, the solutions to the radial differential equations were terminated when the value of A obtained from Eq. (6.46) converged to a constant value, indicating that the approximation for χ given in

Eq. (6.36) was sufficiently accurate.

F. Performing the Integration in $F(\ell_1\ell_2\ell_3 L)$

The integral in the function $F(\ell_1\ell_2\ell_3 L)$ is of considerable interest from the standpoint of obtaining the matrix elements for processes involving only continuum wave functions. From a numerical point of view little has been done to develop methods of obtaining the infinite integrals of slowly converging oscillating functions, and little has been published on this subject.

The problem is to find an integral of the form

$$T = \lim_{r\to\infty} \int_0^r f(x)\,dx = \lim_{r\to\infty} I(r) \quad , \tag{6.50}$$

where $f(x)$ is a slowly converging function which oscillates about zero, by solving for $I(r)$ and deducing its limiting value. Since $f(x)$ oscillates about zero and converges to zero, the function $I(r)$ for large r will oscillate about some mean value and converge to that value. The first approach then is to integrate to some value r for which the magnitude of the oscillations of $I(r)$ is small compared with the mean value. However, in the present problem this value was prohibitively large. This means that for economy, the asymptotic value of the integral must be determined in the range where the oscillations of $I(r)$ are a significant fraction of the mean value. Coupled with the fact that $I(r)$ may not be very regular in its oscillations about the asymptotic value, this value is very difficult to derive with any degree of accuracy. Additional methods of smoothing out the oscillations by taking the average value of

$I(r)$ as shown by Titchmarsh[9] still result in an oscillating function $\overline{I(r)}$ which converges to a limit in a not too regular way. Therefore, still another method must be devised.

The method devised for solving the integrals was to put a convergence factor in the integrand of the form $e^{-\alpha x}$. Then, for selected values of α, the integral

$$T(\alpha, r) = \int_0^r e^{-\alpha x} f(x)\, dx \tag{6.51}$$

will converge quite rapidly to the value of the integral

$$T(\alpha) = \int_0^\infty e^{-\alpha x} f(x)\, dx \quad . \tag{6.52a}$$

To find the value of $T(0)$, which is the required value, the value of the integral $T(\alpha_i)$ is found at $n + 1$ values of α_i, $i = 1, 2, \ldots, n + 1$, and extrapolated to $\alpha = 0$, the method of extrapolation depending on the function $T(\alpha)$. A convenient method of extrapolation is to fit an nth-order polynomial to the values $T(\alpha_i)$ and solve for $\alpha = 0$. This can be done with the aid of the Lagrange[10] formula

$$T(0) = \sum_{i=1}^{n+1} A_i\, T(\alpha_i) \quad , \tag{6.52b}$$

where the constants A_i are given by

$$A_i = \prod_{j=1}^{i-1} \frac{\alpha_j}{(\alpha_i - \alpha_j)} \prod_{k=i+1}^{n+1} \frac{\alpha_k}{(\alpha_i - \alpha_k)} \quad . \tag{6.52c}$$

---

[9] F. C. Titchmarsh, Introduction to the Theory of Fourier Integrals (Oxford University Press, London, 1948), 2nd ed., p. 26.

[10] W. E. Milne, Numerical Calculus (Princeton University Press, Princeton, 1949), p. 83.

The convenience of Eqs. (6.52) is made very clear when the α_i are equally spaced and we let $y = e^{-\alpha_1 x}$, where α_1 is the smallest value of α_i. In this case, it is found that

$$T(0) = \int_0^\infty \sum_{i=1}^{n+1} A_i y^i f(x)\, dx \quad . \tag{6.53}$$

The sacrifice of calculating the exponential and polynomial in the weighting factor of Eq. (6.53) at each step of the integration is more than compensated for by the rapid convergence of the integral and the resulting time saved in computing f(x). An even greater advantage is that $T(\alpha)$ is usually a slowly varying function of α and may be monotonic, which means that it can easily be extrapolated to zero, and such was not the case in the original problem. The slowly varying nature of $T(\alpha)$ can be observed from the following two examples:

$$\int_0^\infty e^{-ax} \frac{\sin x}{x}\, dx = \tan^{-1} \frac{1}{a} \quad , \tag{6.54}$$

and

$$\int_0^\infty e^{-ax} \sin x\, dx = \frac{1}{a^2 + 1} \quad . \tag{6.55}$$

IV. THE GENERAL PLAN OF THE CALCULATION

Because of the duplication of functions that appear in the terms of the sum in the function K given in Eq. (5.51), the most economical method of solution was to carry forward the integration of the differential equations for all the required radial functions one step at a time. At the end of each step the values for the functions and the values of the spherical Bessel functions at that step were distributed

into the integrations appearing in all the functions $F(\ell_1\ell_2\ell_3 L)$, and these integrations were carried forward one step. Returning to the radial equations, the step-by-step process was continued until the functions $F(\ell_1\ell_2\ell_3 L)$ converged and all the normalizing constants and phase shifts for the radial equations were obtained. The functions $F(\ell_1\ell_2\ell_3 L)$ were then renormalized and multiplied by the exponential of the phase shifts, and the final sum in the function K was performed. The value of K obtained in this way was then used in the cross-section formula Eq. (6.1).

The actual calculation was complicated by the complex nature of the functions. In order to obtain each radial equation four coupled first-order differential equations had to be integrated, and in order to find each integral in the functions $F(\ell_1\ell_2\ell_3 L)$ a real and imaginary integral had to be obtained. These complications were intensified by the necessity for economy and space limitations in the computer which made it mandatory that no extra integrals be found even though the allowed values of ℓ_1, ℓ_2, ℓ_3, and L did not index in a regular way. The solution to the indexing problem was found in a special routine which controlled storage and the distribution of the wave functions among the integrals in the process of integration. In effect, it acted as a speedometer which would produce only allowed combinations of the quantum numbers in a given sequence that covered all possibilities.

In the sum in the function K the two quantum numbers ℓ_1 and ℓ_3 were allowed to be unrestricted and all other quantum numbers were then restricted by the Clebsch-Gordon coefficients and the Racah coefficients. In order to cut off the sums at maximum values of ℓ_1 and ℓ_3 for the

cases where the optical potential was present, it was assumed that production of pions takes place mainly within the bounds of the potential. Under these circumstances, the maximum value of these numbers should be such that the first maximum of the corresponding radial equations lies just outside the bounds of the potential. This means that

$$\ell_1(\text{Max}) \cong \left(k_-^2 + \frac{2\omega_{k_-} Z\alpha}{r_0} - 2\mu P - V^2 - P^2 + \frac{Z^2\alpha^2}{r_0^2} \right)^{1/2} \qquad (6.54)$$

and

$$\ell_3(\text{Max}) \cong \left(k_+^2 - \frac{2\omega_{k_+} Z\alpha}{r_0} - 2\mu P - V^2 - P^2 + \frac{Z^2\alpha^2}{r_0^2} \right)^{1/2} , \qquad (6.55)$$

where r_0 is the radius of the nuclear potential ($1.4\ A^{1/3} \times 10^{-13}$ cm). This turned out to be a very effective means for choosing the maximum values. The same method also worked for those cases where the nuclear potential was neglected but the modified Coulomb potential was retained; however, the radius, r_0, in this case was approximately two and one half times that indicated above.

As an example, consider a case where the incident photon energy was 310 Mev, the excess kinetic energy was equally distributed between the pair of pions, the nucleus was lead, and the nuclear potential was included. The maximum values of ℓ_1 and ℓ_3 were then chosen to be 5 and 3, respectively, based on Eqs. (6.54) and (6.55), and the calculation involved the integration of forty first-order differential equations and the evaluation of 192 integrals. The running time on the IBM-704 was 20 minutes per case.

CHAPTER VII

RESULTS

Calculations of the differential cross section for the electromagnetic production of pion pairs were performed, with exact wave functions used to determine the effects of the distributed nuclear charge and the nuclear optical potential. The equations were first solved with the optical potential neglected and the results compared with the Born approximation, and, finally, both the modified Coulomb and nuclear potential were included in the computation. The results of these calculations and the comparisons are presented in this chapter.

I. CALCULATIONS WITH THE COULOMB POTENTIAL

A. The Born Approximation[1]

In Chapter I attention was called to the fact that if the Hamiltonian included a Coulomb potential which is sufficiently broad and with magnitude greater than the rest mass of the pion it could not be brought to diagonal form and led to complex eigenvalues. This is not the case when the Coulomb potential is included in the perturbation--the resulting Hamiltonian can be diagonalized, and leads to the Born approximation. Then the resulting cross-section formula can be investigated without concern about the magnitude of the Coulomb potential. This point is made because there is some interest in investigating the effect on the value of the cross section as one goes from a point charge nucleus to a

---

[1] In this section the equations will be given in ordinary units, where P is the momentum and E is the energy.

distributed charge. That the point charge is not a good approximation for the nuclear charge in the case of pair production of pions, since the production occurs predominantly inside the nucleus, is clearly indicated by this comparison.

To investigate the change in the cross section as one goes from a point charge nucleus to a distributed charge, we shall consider the Born approximation differential cross section given by Pauli and Weisskopf:[2]

$$d\sigma = \frac{\pi e^4 |\tilde{\Phi}(p)|^2 \, P_+ P_- \sin\theta_- \sin\theta_+ \, d\theta_- \, d\theta_+ \, d\phi \, dE_-}{\hbar^5 c E_0^3 (2\pi)^4} (W) \quad , \tag{7.1}$$

where E_0 is the energy of the incident photon, θ and ϕ are the polar and azimuthal angles with respect to the incident direction of the photon, and

$$W = \frac{E_-^2 P_+^2 \sin^2\theta_+}{(E_+ - cP_+\cos\theta_+)^2} + \frac{E_+^2 P_-^2 \sin^2\theta_-}{(E_- - cP_-\cos\theta_-)^2} + \frac{2E_+ E_- P_+ P_- \sin\theta_+ \sin\theta_- \cos\phi}{(E_+ - cP_+\cos\theta_+)(E_- - cP_-\cos\theta_-)} \quad . \tag{7.2}$$

$\tilde{\Phi}(p)$ is the Fourier transform of the Coulomb potential given by

$$\tilde{\Phi}(p) = \int A(x) \, e^{i\vec{p}\cdot\vec{x}} \, d^3x \quad . \tag{7.3}$$

To maintain conservation of energy and linear momentum consistent with the heavy nucleus assumption,

$$E_0 = E_+ + E_- \tag{7.4a}$$

and

$$\hbar\vec{p} = \vec{P} - \vec{P}_- - \vec{P}_+ \quad , \tag{7.4b}$$

---

[2] Equation (7.1) has been corrected by a factor of 1/2 missing in the equation given by W. Pauli and V. Weisskopf, Helv. Phys. Acta 7, 709 (1934).

where $\hbar\vec{p}$ is the momentum transmitted to the nucleus and $\vec{P}$ is the momentum of the photon.

For a point charge Eq. (7.3) gives

$$\bar{\Phi}(p) = -\frac{4\pi eZ}{p^2} \quad , \tag{7.5}$$

in which case Eq. (7.1) can be integrated over all angles. After a tedious calculation, it is found that

$$d\sigma = \alpha Z^2 \frac{4}{3} R^2 \left(\frac{E_+ E_-}{P_+ c P_- c} \; L - 1 \right) \; dE_- \quad , \tag{7.6a}$$

where α is the fine structure constant, R is the classical meson radius, and

$$L = \log_e \left| \frac{P_+ c P_- c + E_+ E_- + m^2 c^4}{-P_+ c P_- c + E_+ E_- + m^2 c^4} \right| = 2 \log_e \left| \frac{P_+ c P_- c + E_+ E_- + m^2 c^4}{E_0 m c^2} \right| \quad . \tag{7.6b}$$

For the case of a distributed charge with density $Z\rho(x)$, Eq. (7.3) can be shown to lead to

$$\bar{\Phi}(p) = -\frac{4\pi Ze}{p^2} \int \rho(x) \; e^{i\vec{p}\cdot\vec{x}} \; d^3x \quad , \tag{7.7}$$

and Eq. (7.1) has to be integrated numerically.

Equation (7.1) was solved for lead (Z = 82), with the distributed charge density given in Eq. (2.7) used, and compared with a point charge lead nucleus. This comparison is given in Figure 3 at a photon energy of 310 Mev. The two curves have similar shapes, but the point-charge cross section is a factor of approximately 5×10^3 to 10^4 larger than that for the distrubuted charge. This large factor persists in the range of photon energies investigated in this calculation (280 to 310 Mev).

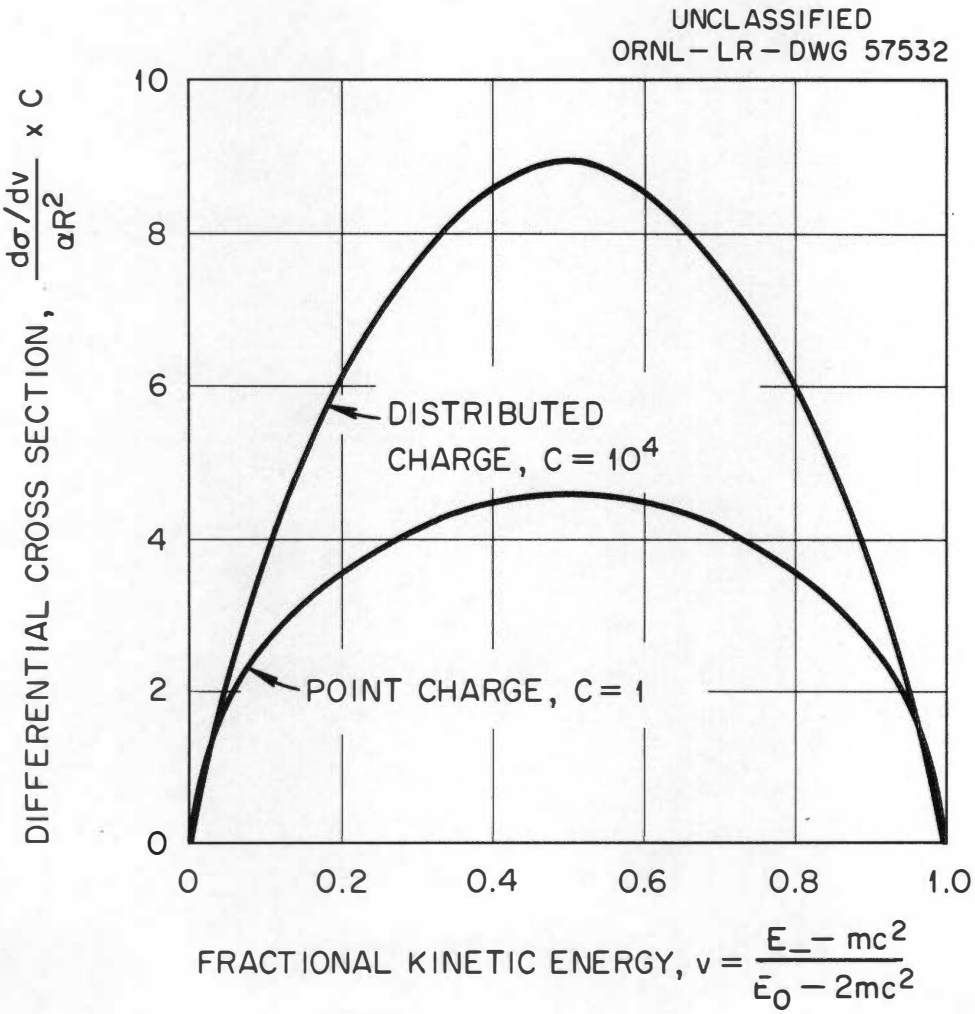

Figure 3. Comparison of the Born Approximation Pion Pair Production Cross Section at a Photon Energy of 310 Mev for a Point-Charge and Distributed-Charge Lead Nucleus.

B. Comparison of Born Approximation and Calculations with Exact Waves

A comparison of the calculation of the pion pair production cross section using exact waves with the Born approximation solution for a distributed charge nucleus is presented in Figures 4, 5, and 6. At the photon energies of 290, 300, and 310 Mev a striking difference in the spectral shapes between the Born approximation and exact wave calculation will be noticed. The enhancement of the cross section for higher energy π^+ mesons is well known and occurs because of the repulsive Coulomb potential.[3]

The difference between the Born approximation and exact wave calculations can be expected in the cases just cited because the conditions for the Born approximation to hold are certainly not obtained:

$$\frac{Z\alpha}{P_+c/E_+} \quad , \quad \frac{Z\alpha}{P_-c/E_-} \ll 1 \quad . \tag{7.8}$$

In fact, for lead, where $Z = 82$, $E_0 = 310$ Mev, and $(E_- - Mc^2)/(E_0 - 2mc^2) = 0.5$, then $Z\alpha E/Pc = 1.4$. It is of some interest then to examine the spectral shape of the exact wave cross section to see how it approaches that of the Born approximation as $Z \to 0$. Figure 7 shows the ratio of the differential cross section for the exact wave solution to that of the Born approximation as a function of Z for $v = 0.5$ and 0.1 at photon energy 290 Mev. In this comparison the value of A, which determines the

[3] A discussion of this point is given for electron pair production by H. Heitler, The Quantum Theory of Radiation (Oxford University Press, London, 1954), 3rd ed., p. 259.

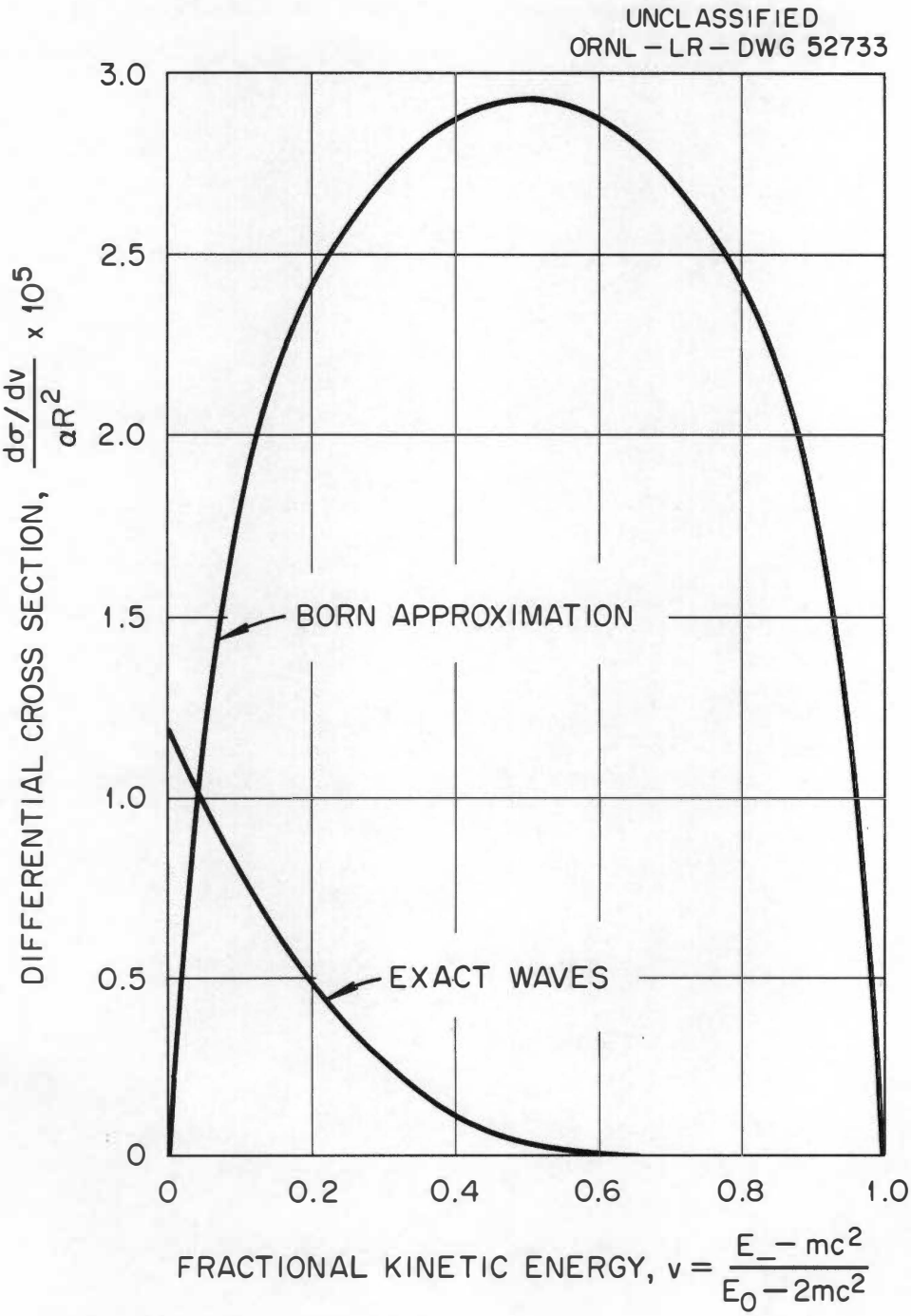

Figure 4. Comparison of the Born Approximation and Exact Wave Pion Pair Production Cross Section for Lead at a Photon Energy of 290 Mev. The Nuclear Potential Was Neglected.

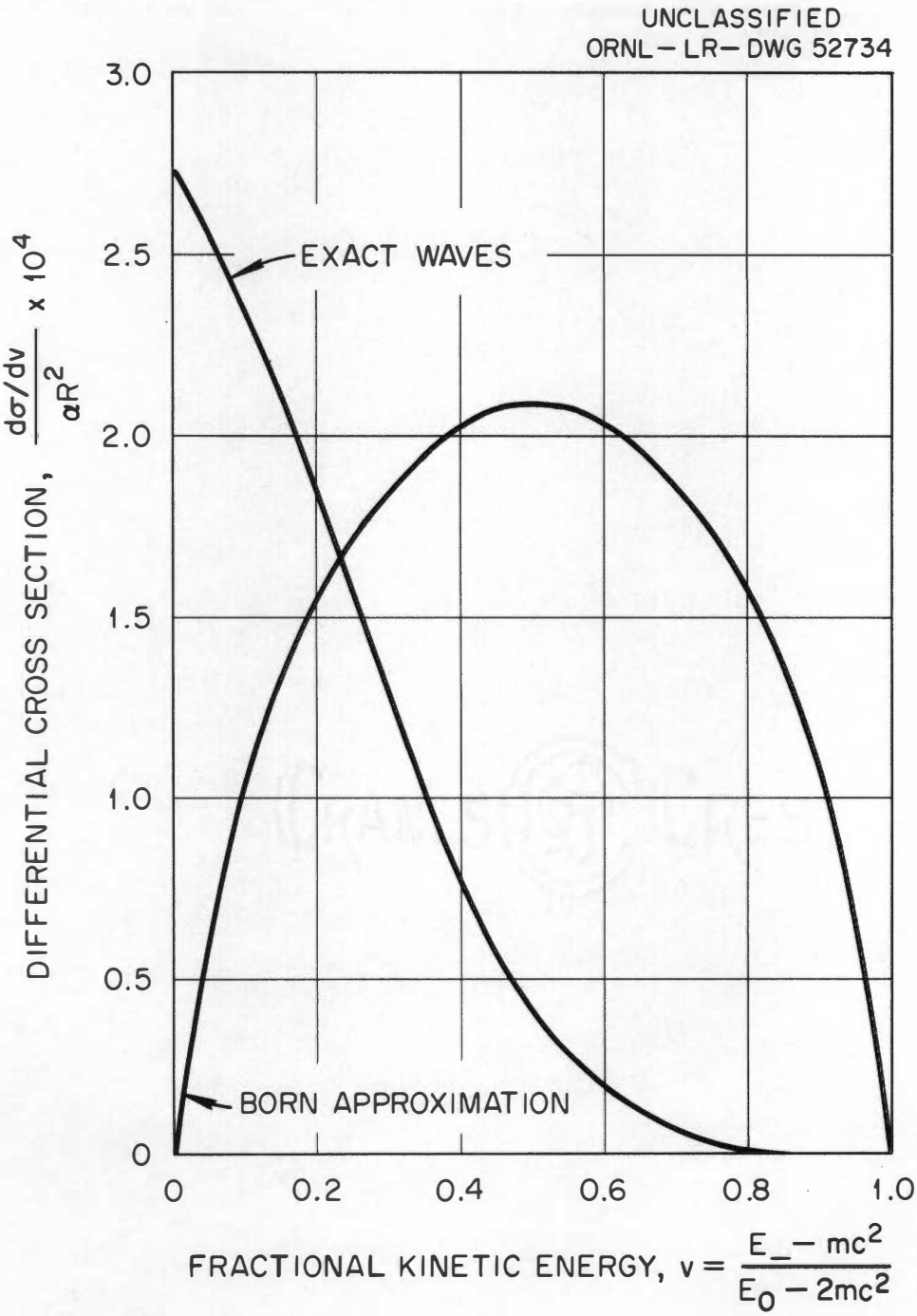

Figure 5. Comparison of the Born Approximation and Exact Wave Pion Pair Production Cross Section for Lead at a Photon Energy of 300 Mev. The Nuclear Potential Was Neglected.

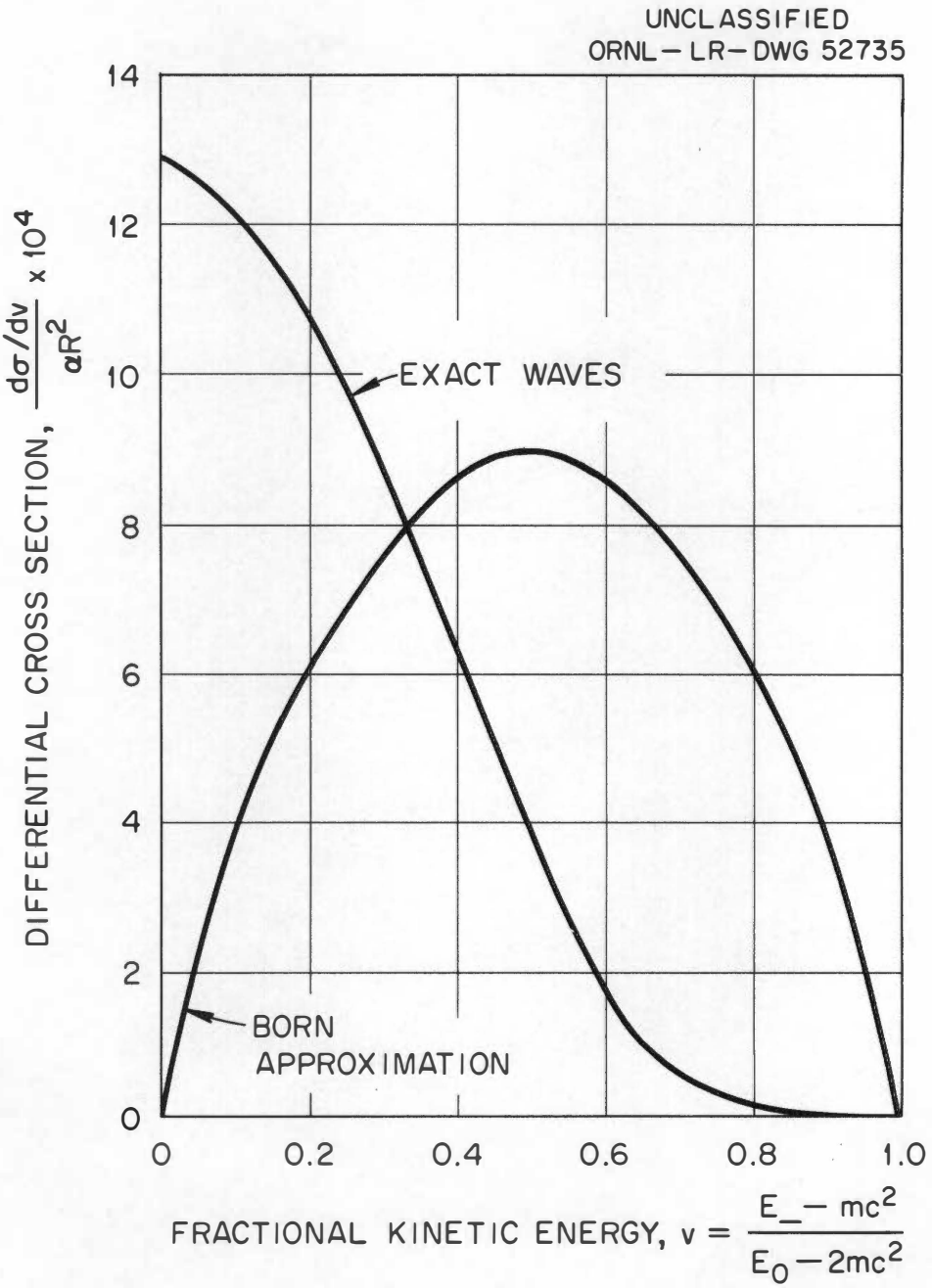

Figure 6. Comparison of the Born Approximation and Exact Wave Pion Pair Production Cross Section for Lead at a Photon Energy of 310 Mev. The Nuclear Potential Was Neglected.

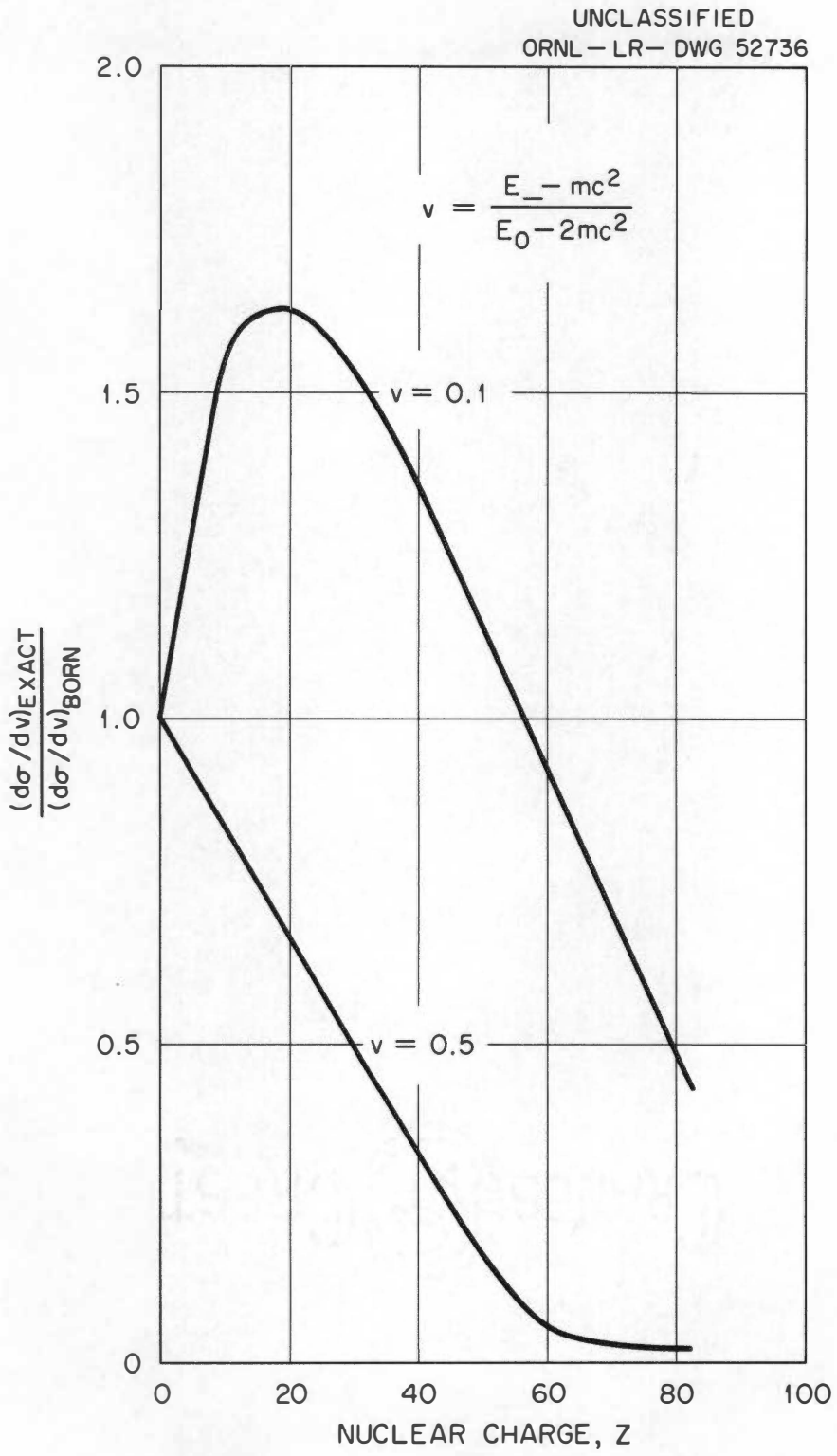

Figure 7. Ratio of Exact Wave to Born Approximation Differential Cross Section as a Function of Z For Photon Energy of 290 Mev.

extent of the charge distribution, was kept constant; so the numbers refer to fictitious nuclei for $Z < 82$. Referring back to Figure 4 and assuming that a Z^2 factor has been removed from the ordinate to make the Born approximation independent of charge, as Z decreases the exact wave curve increases steadily relative to the Born approximation, maintaining its shape until Z = 20; for small v it starts to decrease, while for large v it continues to increase until Z = 0 where the curve has assumed the Born approximation shape.

The ratio of the total pair production cross section obtained by exact waves and distributed charge to that obtained by Born approximation can be determined after integrating the curves shown in Figures 4, 5, and 6. This ratio is shown in Figure 8. The most interesting fact to be observed from these data is that the ratio is less than 1 as the photon energy increases above threshold, reaching and crossing the value 1 at approximately 317 Mev. This is the opposite behavior from that obtained in electron pair production with a point charge where the ratio is greater than 1 just above threshold and decreases and finally crosses the value 1 as the photon energy increases.[4]

II. CALCULATIONS WITH A MODIFIED COULOMB AND NUCLEAR POTENTIAL

In order to select the nuclear optical potential parameters for the present calculation from the data shown in Figures 1 and 2, it is necessary to briefly review the way that the data were obtained.

---

[4] A discussion of this point for electron pair production is given by I. E. Dayton, Phys. Rev. 89, 544 (1953).

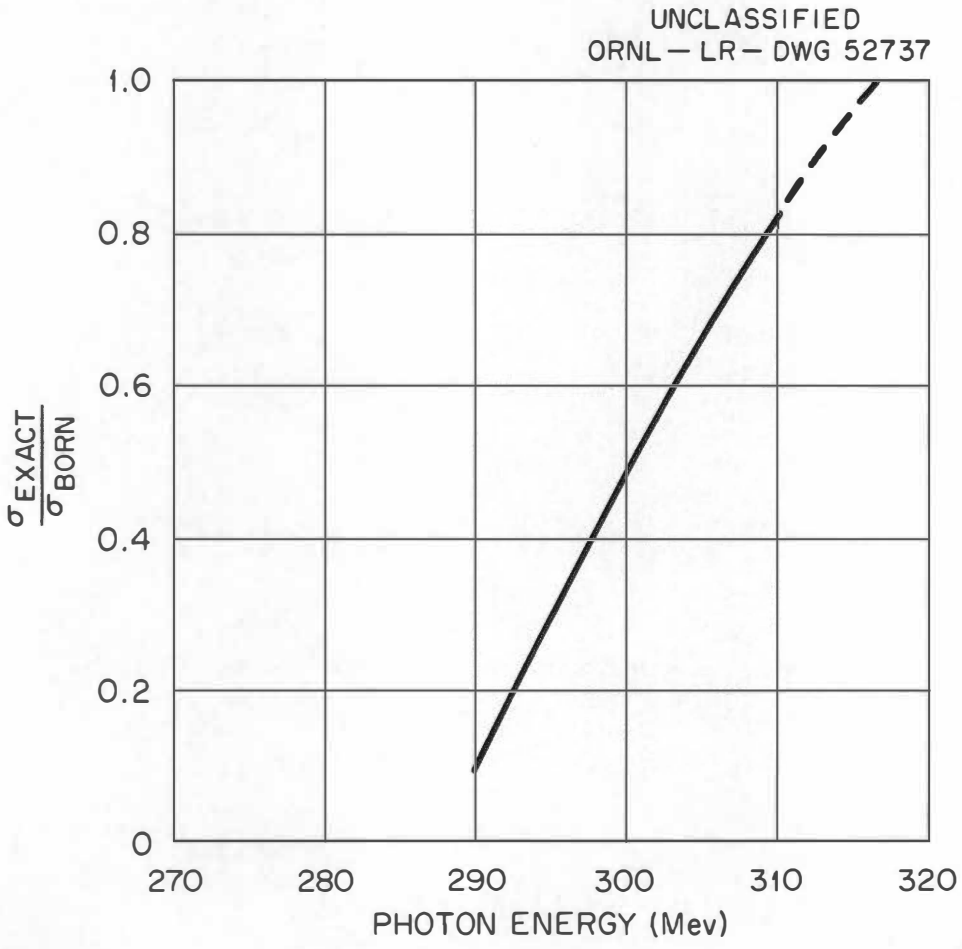

Figure 8. Ratio of Exact Wave to Born Approximation Pion Pair Production Cross Section for Lead as a Function of Energy. The Nuclear Potential Was Neglected.

In the analysis of the pion scattering data from complex nuclei it has been customary to introduce both the real and imaginary part of the optical potential as a time component of a four-vector. In contrast to this, in Chapter III, it was shown that only the imaginary part of the optical potential should be a time component of a four-vector, the real part being a world scalar. In effect this means that the data presented in Figures 1 and 2 were obtained by using a radial differential equation where the factor $-2\omega_k(P + iV)$ rather than $-2\mu P - 2i\omega_k V$ [which was in Eqs. (5.8) and (5.9)] appeared in the operator. This means that the real part of the potential in Figure 1 should be corrected by a factor $\omega k/\mu$ in order to make it applicable to this calculation. The imaginary part of the potential need not be corrected.

In the calculations to be described, incident photon energies up to 310 Mev were considered, and therefore since threshold is at approximately 279 Mev, the optical potential in the range 0 - 30 Mev pion kinetic energy was required. From the data given in Figures 1 and 2 and from the assumption that the potential curves are flat in the lower energy region as discussed in Chapter II the values of -16 Mev for the real part of the potential and -5 Mev for the imaginary part of the potential were selected.

The differential cross sections obtained from the present calculation including both the Coulomb potential from the distributed charge in the nucleus and the nuclear optical potential are presented in Figure 9 and in Table I. These data still show the effects of the strong Coulomb potential, although the cross section has been considerably

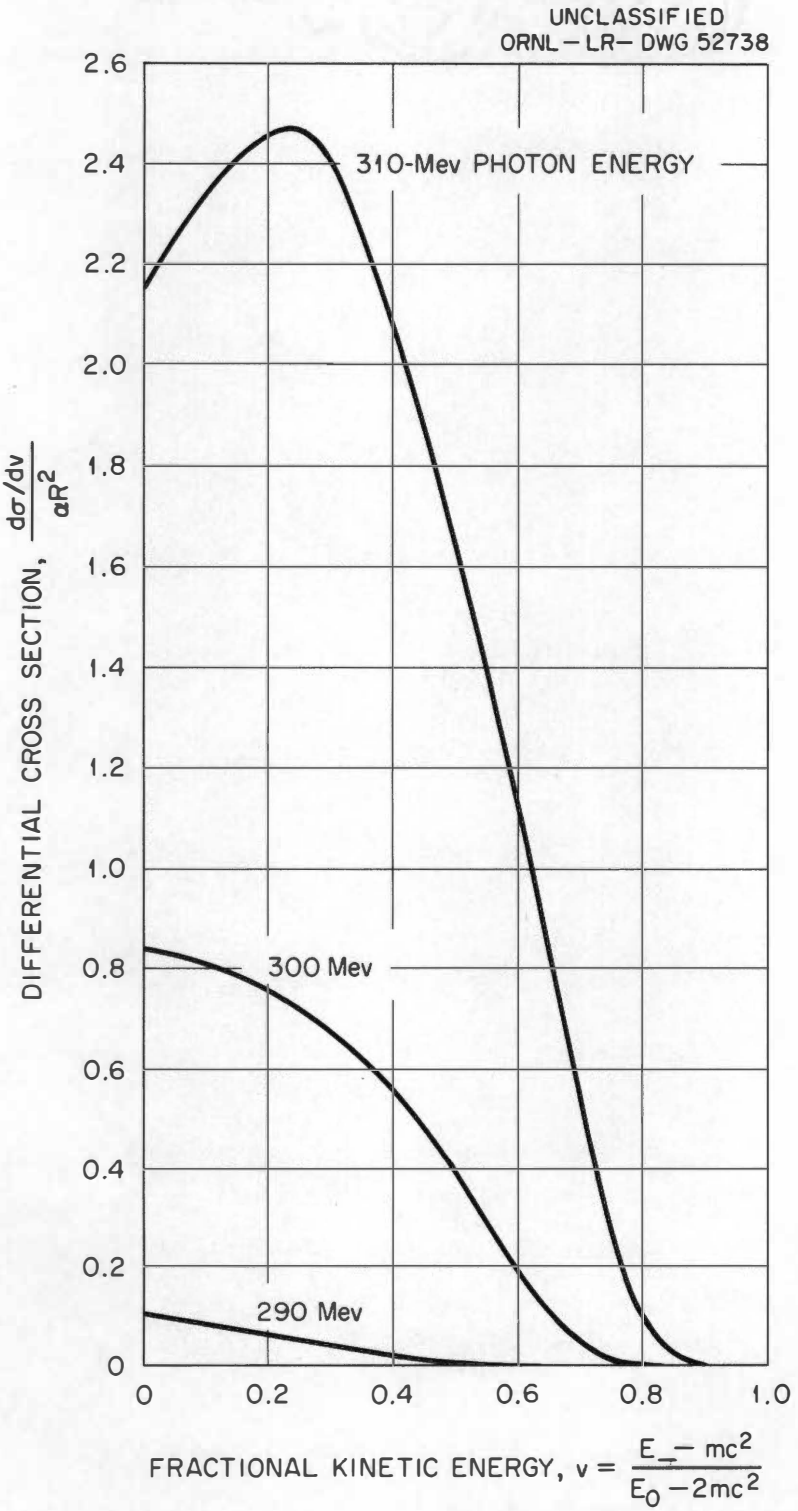

Figure 9. The Differential Pion Pair Production Cross Section for Lead Including the Coulomb and Nuclear Optical Potential (P = −16 Mev, V = −5 Mev).

TABLE I

THE PION PAIR PRODUCTION CROSS SECTION IN UNITS OF $\alpha R^2 = 7.762 \times 10^{-33}$ cm^2 FOR LEAD, INCLUDING THE COULOMB AND OPTICAL POTENTIAL (P = -16 Mev, V = -5 Mev)

$v = \frac{E_- - mc^2}{E_0 - 2mc^2}$	Cross section at photon energy of		
	290 Mev	300 Mev	310 Mev
	Differential Cross Section, $\frac{d\sigma/dv}{\alpha R^2}$		
0.001	1.06×10^{-1}	8.31×10^{-1}	2.16
0.125	7.66×10^{-2}	7.91×10^{-1}	2.38
0.250	4.59×10^{-2}	7.21×10^{-1}	2.47
0.375	1.99×10^{-2}	5.79×10^{-1}	2.17
0.500	5.57×10^{-3}	4.00×10^{-1}	1.63
0.625	7.92×10^{-4}	1.41×10^{-1}	9.89×10^{-1}
0.750	2.73×10^{-5}	1.27×10^{-2}	2.53×10^{-1}
0.875		4.91×10^{-5}	2.68×10^{-3}
0.95			2.66×10^{-7}
	Total Cross Section, $\frac{\sigma}{\alpha R^2}$		
	2.49×10^{-2}	3.81×10^{-1}	1.38

increased from the corresponding cases without the optical potential. The data at photon energy 310 Mev show a hump which can be accounted for on the basis that within the nuclear well the effective momentum of the pions is increased whereas the effects of the Coulomb potential are reduced and therefore the spectrum would be expected to tend toward the symmetric shape given by the Born approximation. Figure 10 and Table I present the integration of the differential cross section as a function of the incident photon energy indicating a sharp increase in the cross section about 15 Mev above threshold.

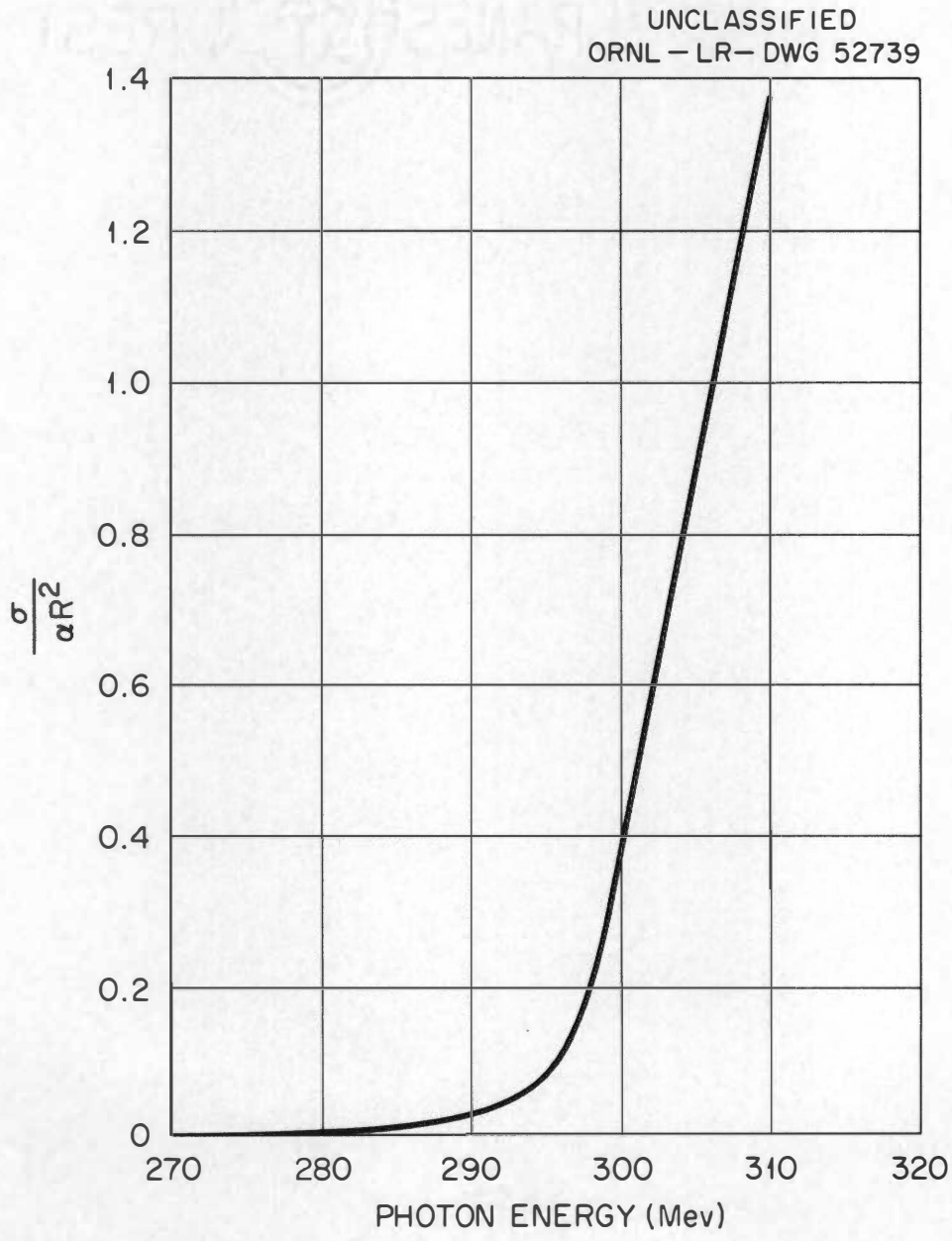

Figure 10. The Pion Pair Production Cross Section for Lead Including the Coulomb and Nuclear Optical Potential ($P = -16$ Mev, $V = -5$ Mev) as a Function of Photon Energy.

CHAPTER VIII

SUMMARY

The Pauli-Weisskopf[1] theory of the electromagnetic production of pion pairs has been investigated with the objective of including in it the strong pion-nucleus interaction which ultimately leads to an enhancement of the cross section. This was accomplished by the semiphenomenological method of including the interaction into the rigorous theory in the form of an optical model, a method first suggested by Landau and Pomeranchuk.[2]

In Chapter III it was shown that the optical model in the form of a complex optical potential could be explicitly introduced into the field equations in a consistent manner which leads to the absorption and scattering of the pions from the nucleus. The consistency of the theory is maintained only in the case where the real part of the potential is a world scalar and the imaginary part a time component of a four-vector.

Absorption of mesons in the nucleus as introduced into the theory by the imaginary part of the optical potential includes all inelastic events as well as true absorption which leads to an excited state of the nucleus. The real part of the potential, on the contrary, introduces the coherent scattering of the mesons from the nucleus as a whole. The resulting cross section with these potentials therefore represents the

---

[1] W. Pauli and V. Weisskopf, Helv. Phys. Acta 7, 709 (1934).

[2] L. D. Landau and I. I. Pomeranchuk, Zhur. Eksptl. i Teort. Fiz. 24, 505 (1953); English translation: Proceedings of the Cern Symposium 1956 (Service d'Information; Cern, 1956), vol. II, p. 159.

pion pair production cross section where the nucleus is left in its ground state.

When the optical potential is present, the final-state wave forms for the pion wave functions appearing in the matrix element are not fixed by existing theory. Therefore a choice between two possible forms had to be made. It was possible to make the selection unambiguously, however, as was shown in Chapter IV. The resulting matrix element was then expressed in angular momentum states for ease of solution, and methods of solution were devised for the various terms. These last two steps are shown in Chapters V and VI. Of particular interest in Chapter VI is the method devised for finding certain infinite integrals that appear in the matrix element, where the integrand is a product of continuum wave functions or their derivatives. This problem was resolved by the introduction of a weighting factor in the integrand which made the integral converge quite rapidly.

Since the production of pion pairs occurs predominantly inside the nucleus, it was necessary to consider the modified Coulomb potential resulting from a finite nucleus with a distributed charge density. The form of the charge density used was that obtained from electron scattering experiments. The complication of the distributed charge has its compensation in that now the ambiguity occurring in the equations of motion for $\ell = 0$ is avoided, as are the complex eigenvalues that could arise if the Coulomb potential is sufficiently broad and has too large a magnitude.

The results of the calculations for photon energies just above threshold shown in Chapter VII indicate a considerable change in the

spectral shape of the differential cross section when exact waves are used as compared with the Born approximation results. The enhancement of the cross section for π^+ mesons with energy greater than the π^- meson is very evident and can be attributed to the repulsive Coulomb potential for the π^+ mesons. Although the cross section is markedly reduced as one goes from a point charge to a distributed charge for the nucleus, the loss is regained when the strong pion-nucleus interaction is included in the form of the optical potential. This increase in the cross section is significant, putting the values in an experimentally determinable region especially for photon energies above 295 Mev.

It was pointed out in Chapter I that the π-π interaction was neglected in this calculation. This was done because at the present time there is insufficient information available for an adequate treatment of the interaction.

In the present state of affairs, it is difficult to speculate as to the effect of the π-π interaction on the spectral distribution of the pion pairs, or, for that matter, on the cross section for the production process. Only a few qualitative statements can be made.

First, it should be pointed out that at gamma-ray energies just above threshold such as those considered in this paper, the maximum center-of-mass energy of the pion pair is relatively low (approximately 2.2 μ). This is well below the start of the proposed resonance in the isotopic spin t = 1 state of the π-π system which is located at an energy of approximately 4 μ.[3,4] It would also be below the t = 0 resonance

---

[3] W. R. Frazer and J. R. Fulco, Phys. Rev. Letters 2, 365 (1959).

[4] P. Curruthers and H. A. Bethe, Phys. Rev. Letters 4, 536 (1960).

suggested by Curruthers and Bethe[4] which, if the resonance actually exists, would be located at an approximate energy of 2.5 μ. For these reasons, serious fluctuations from the cross section as calculated in this paper should not be expected.

It was pointed out by Curruthers and Bethe[4] that the t = 0 interaction might be expected to dominate at the lower center-of-mass energies of the π-π system. Rodberg[5] has reported some impulse approximation calculations of the $\pi^- + p \rightarrow \pi^+ + \pi^- + n$ reaction which seem to substantiate this. In the lower energy range he assumed the dominance of the t = 0 interaction and obtained reasonable agreement with the experimental results of Perkins et al.[6] Assuming that the t = 0 interaction is dominant at low center-of-mass energies of the π-π system and noting that the π^+-π^- pair is a mixture to t = 0, 1, and 2, the interaction might well affect the results presented in this paper and be observable by comparison with experimental data.

Although the present calculations maintained at least 1% numerical accuracy, a much larger error no doubt occurred because of the uncertainty in the optical potential parameters selected from the spread of experimental data. A systematic study of the effect of changing the values of the potential parameters and shape would give an indication of how sensitive the present results are to these factors. However, an improvement in the analysis of the experimental data and more values from the

---

[5] L. S. Rodberg, Phys. Rev. Letters 3, 58 (1959).

[6] Perkins et al., Phys. Rev. Letters 3, 56 (1959).

low-energy pion-nucleus scattering experiments would ultimately be necessary to improve the values of the potential parameters and lead to a more accurately calculated pion pair production cross section. An improved analysis of the scattering data would result if the radial equations of the form given in Eqs. (5.8) and (5.9), which includes the effects of the modified Coulomb potential, were used.

BIBLIOGRAPHY

BIBLIOGRAPHY

Baker, Rainwater, and Williams, Phys. Rev. 112, 1763 (1958).

Bethe, H. A., Phys. Rev. 57, 1125 (1940).

Bethe, H. A., and F. de Hoffmann, Mesons and Fields (Row, Peterson and Company, Evanston, 1955), vol. II.

Bethe, H. A., and W. Heitler, Proc. Roy. Soc. (London) A146, 83 (1934).

Bethe, H. A., and R. R. Wilson, Phys. Rev. 83, 690 (1951).

Biedenharn, L. C., Tables of the Racah Coefficients, USAEC Report ORNL-1098, Oak Ridge National Laboratory (1952).

Breit, G., and H. A. Bethe, Phys. Rev. 93, 888 (1954).

Byfield, Kessler, and Lederman, Phys. Rev. 86, 17 (1952).

Condon, E. U., and G. H. Shortley, Theory of Atomic Spectra (Cambridge University Press, Cambridge, 1935).

Curruthers, P., and H. A. Bethe, Phys. Rev. Letters 4, 536 (1960).

Dayton, I. E., Phys. Rev. 89, 544 (1953).

Fernbach, Serber, and Taylor, Phys. Rev. 75, 1352 (1949).

Frank, Gammel, and Watson, Phys. Rev. 101, 891 (1956).

Francis, N. C., and K. M. Watson, Phys. Rev. 92, 291 (1953).

Frazer, W. R., and J. R. Fulco, Phys. Rev. Letters 2, 365 (1959).

Gill, S., Proc. Cambridge Phil. Soc. 47, 96 (1951).

Goldstein, H., Classical Mechanics (Addison-Wesley Publishing Company, Inc., Cambridge, 1953).

Heitler, W., Quantum Theory of Radiation (Oxford University Press, London, 1954).

Hofstadter, R., Revs. Mod. Phys. 28, 214 (1956).

Kisslinger, L. S., Phys. Rev. 98, 761 (1955).

Landau, L. D., and E. M. Lifshitz, Quantum Mechanics (Addison-Wesley Publishing Company, Inc., Reading, 1958).

Landau, L. D., and Iu. Ia. Pomeranchuk, Zhur. Eksptl. i Teort. Fiz. 24, 505 (1953); English translation: Proceedings of the Cern Symposium 1956 (Service d'Information, Dern, 1956), vol. II, p. 159.

Lattes, Occhialini, and Powell, Nature 160, 453 (1947).

Milne, W. E., Numerical Calculus (Princeton University Press, Princeton, 1949).

Mott, N. F., and H. S. W. Massey, The Theory of Atomic Collisions (Oxford University Press, London, 1940).

Pauli, W., and V. Weisskopf, Helv. Phys. Acta 7, 709 (1934).

Perkins et al., Phys. Rev. Letters 3, 56 (1959).

Pevsner, et al., Phys. Rev. 100, 1419 (1955).

Pomeranchuk, Iu. Ia., Dokl. Acad. Nauk SSSR, 96, 265 and 481 (1954); English translation: Proceedings of the Cern Symposium 1956 (Service d'Information, Cern, 1956), vol. II, p. 167.

Racah, G., Phys. Rev. 62, 438 (1942).

Racah, G., Phys. Rev. 63, 367 (1943).

Rodberg, L. S., Phys. Rev. Letters 3, 58 (1959).

Rose, M. E., Elementary Theory of Angular Momentum (John Wiley and Sons, Inc., New York, 1957).

Rose, M. E., Multipole Fields (John Wiley and Sons, Inc., New York, 1955).

Saphir, G., Phys. Rev. 104, 535 (1956).

Sapiro, A. M., Phys. Rev. 84, 1063 (1951).

Schiff, Snyder, and Weinberg, Phys. Rev. 57, 315 (1940).

Schiff, L. I., Quantum Mechanics (McGraw-Hill Book Company, Inc., New York, 1955).

Schweber, Bethe, and de Hoffmann, Mesons and Fields (Row, Peterson and Company, Evanston, 1955), vol. I.

Simon, Vander Sluis, and Biedenharn, Tables of the Racah Coefficients, USAEC Report ORNL-1679, Oak Ridge National Laboratory (1954).

Simon, A., Numerical Tables of the Clebsch-Gordon Coefficients, USAEC Report ORNL-1718, Oak Ridge National Laboratory (1954).

Snyder, H., and J. Weinberg, Phys. Rev. 57, 307 (1940).

Stork, D. H., Phys. Rev. 93, 868 (1954).

Takeda, G., and K. M. Watson, Phys. Rev. 97, 1336 (1955).

Titchmarsh, F. C., Introduction to the Theory of Fourier Integrals (Oxford University Press, London, 1948), 2nd ed.

Vdovin, Yu. A., Doklady Akad. Nauk S. S. S. R. 105, 947 (1955).

Watson, K. M., Phys. Rev. 105, 1388 (1957).

Watson, K. M., Phys. Rev. 89, 575 (1953).

APPENDICES

APPENDIX A

REDUCTION OF THE FIRST-ORDER PERTURBATION THEORY MATRIX ELEMENT WITH EXACT WAVES TO THAT OBTAINED WITH SECOND-ORDER PERTURBATION THEORY WITH PLANE WAVES

It will be shown here that the matrix element

$$M = -2ie \int \vec{A}_{\vec{q}} \cdot \left(\psi^{-*}_{\vec{k}_+} \overrightarrow{\nabla} \psi^{+}_{-\vec{k}_-} \right) \, d^3r \tag{A.1}$$

[given in Eq. (5.5)] obtained from a first-order perturbation theory treatment of the interaction between the meson and electromagnetic field using exact waves reduces to the results obtained by Pauli and Weisskopf in a second-order perturbation theory treatment using plane waves if the Born approximation is made. In order to simplify the derivation and make it directly comparable with the results of Pauli and Weisskopf, the nuclear optical potential will be set to zero in the differential equations for the wave functions. Then, from Eqs. (4.2) and (4.3),

$$(-\nabla^2 - k_-^2) \, \psi^{+}_{-\vec{k}_-} = (2\omega_{k_-} eA_0 - e^2A_0^2) \, \psi^{+}_{-\vec{k}_-} \equiv V_- \psi^{+}_{-k_-} \quad , \tag{A.2}$$

$$(-\nabla^2 - k_+^2) \, \psi^{-*}_{\vec{k}_+} = (-2\omega_{k_+} eA_0 - e^2A_0^2) \, \psi^{-*}_{\vec{k}_+} \equiv V_+ \psi^{-*}_{k_+} \quad , \tag{A.3}$$

where

$$\omega_k^2 = k^2 + \mu^2 \quad . \tag{A.4}$$

The waves in Eqs. (A.2) and (A.3) have the following asymptotic forms

$$\psi^{+}_{-\vec{k}_-} \sim \phi^{*}_{\vec{k}_-} + f(\theta)\,\frac{e^{ikr}}{r\sqrt{2\omega_{k_-}}} \quad , \tag{A.5}$$

$$\psi^{-*}_{k_+} \sim \phi^{*}_{k_+} + g(\theta)\,\frac{e^{ikr}}{r\sqrt{2\omega_{k_+}}} \quad , \tag{A.6}$$

where the notations

$$\phi_{\vec{k}} = \frac{e^{i\vec{k}\cdot\vec{r}}}{\sqrt{2\omega_k}} \tag{A.7}$$

$$\phi^{*}_{\vec{k}} = \frac{e^{-i\vec{k}\cdot\vec{r}}}{\sqrt{2\omega_k}} \tag{A.8}$$

have been used. Equation (A.2) can now be written in operator form as

$$\psi^{+}_{-\vec{k}_-} = \phi^{*}_{\vec{k}_-} + \frac{V_-\,\psi^{+}_{-\vec{k}}}{H_0 - k_-^2 - i\epsilon} \quad , \tag{A.9}$$

where $H_0 = -\nabla^2$, ϵ is a real infinitesimal constant which approaches zero, and $\phi^{*}_{\vec{k}_-}$ is the solution to the homogeneous equation. To get a more explicit form for Eq. (A.9), it can be written as

$$\psi^{+}_{-\vec{k}_-}(\vec{r}) = \phi^{*}_{\vec{k}_-}(\vec{r}) + \int \frac{\delta(\vec{r} - \vec{r}')\, V_-(\vec{r}')\,\psi^{+}_{-\vec{k}_-}(\vec{r}')\, d^3r'}{H_0(r) - k_-^2 - i\epsilon} \quad , \tag{A.10}$$

and

$$\sum_k 2\omega_k\ \phi^{*}_{\vec{k}}(\vec{r}')\ \phi_{\vec{k}}(\vec{r}) = \delta(\vec{r} - \vec{r}') \quad , \tag{A.11}$$

where the waves are assumed to be quantized in a box of unit volume.

Using Eq. (A.11) in Eq. (A.10),

$$\psi^{+}_{-\vec{k}_-}(\vec{r}) = \phi^{*}_{\vec{k}_-}(\vec{r}) + \sum_{\vec{k}} \frac{2\omega_k \, \phi_{\vec{k}}(\vec{r})\left(\phi_{\vec{k}}, V_- \psi^{+}_{-\vec{k}_-}\right)}{k^2 - k_-^2 - i\epsilon} \quad , \tag{A.12}$$

where

$$\left(\phi_{\vec{k}}, V_- \psi^{+}_{-\vec{k}_-}\right) = \int \phi^{*}_{\vec{k}} V_- \psi^{+}_{-\vec{k}_-} \, d^3r \quad . \tag{A.13}$$

Similarly, the expression

$$\psi^{-*}_{\vec{k}_+}(\vec{r}) = \phi^{*}_{\vec{k}_+}(\vec{r}) + \sum_{\vec{k}'} \frac{2\omega_{k'} \phi_{\vec{k}'}(\vec{r})\left(\phi_{\vec{k}'}, V_+ \psi^{-*}_{\vec{k}_+}\right)}{k'^2 - k_-^2 - i\epsilon} \tag{A.14}$$

can be obtained.

For the potential $\vec{A}_{\vec{q}}$,

$$\vec{A}_{\vec{q}} = \sqrt{\frac{4\pi}{2\omega_q}} \; e^{i\vec{q}\cdot\vec{r}} \, \hat{e}_q \quad , \tag{A.15}$$

where unrationalized units are used and $\hat{e}_q$ is a unit vector in the direction of polarization. This wave is transverse, so that $\hat{e}_q \cdot \vec{q} = 0$. Inserting Eqs. (A.12), (A.14), and (A.16) into Eq. (A.1) yields

$$\begin{aligned} M = -2e\sqrt{4\pi} \int (\hat{e}_q \cdot \vec{k}) \, \frac{e^{i(\vec{q}-\vec{k}_+-\vec{k}_-)\cdot\vec{r}}}{\sqrt{8\omega_q \omega_{k_-} \omega_{k_+}}} \, d^3r \quad + \\ + 2e\sqrt{4\pi} \int \sum_{\vec{k}} (\hat{e}_q \cdot \vec{k}) \, \frac{2\omega_k \, e^{i(\vec{q}-\vec{k}_++\vec{k})\cdot\vec{r}} \left(\phi_{\vec{k}}, V_- \psi^{+}_{-\vec{k}_-}\right) d^3r}{\sqrt{8\omega_q \omega_{k_+} \omega_k} \left(\omega_k^2 - \omega_{k_-}^2 - i\epsilon\right)} \quad - \\ - 2e\sqrt{4\pi} \int \sum_{\vec{k}'} (\hat{e}_q \cdot \vec{k}_-) \, \frac{2\omega_{k'} \, e^{i(\vec{q}+\vec{k}'-\vec{k}_-)\cdot\vec{r}} \left(\phi_{\vec{k}'}, V_+ \psi^{-*}_{\vec{k}_+}\right) d^3r}{\sqrt{8\omega_q \omega_{k'} \omega_{k_-}} \left(\omega_{k'}^2 - \omega_{k_+}^2 - i\epsilon\right)} \quad + \text{second-order term.} \end{aligned} \tag{A.16}$$

In Eq. (A.16) the second-order term contains a factor V_+V_- and therefore is of order e^2, but it is dropped subsequently, since only terms of order e are retained. The first term on the right of Eq. (A.16) can also be dropped because it leads to the Dirac delta function $\delta(\vec{q} - \vec{k}_+ - \vec{k}_-)$, which will give no contribution because the theory requires that $\omega_q = \omega_+ + \omega_-$ and both conditions cannot be satisfied.

Before substituting the values of V_+ and V_- in the remaining terms of Eq. (A.16) the terms of order e^2 are dropped, and the Fourier transform is introduced:

$$A_0(r) = \sum_{\vec{p}} e^{-i\vec{p}\cdot\vec{r}} \Phi(\vec{p}) \quad , \tag{A.17}$$

where

$$\Phi(p) = \int e^{i\vec{p}\cdot\vec{x}} A_0(r)\, d^3r \quad , \tag{A.18}$$

so that

$$V_+ = -2\omega_{k_+} e \sum_{\vec{p}} e^{-i\vec{p}\cdot\vec{r}} \Phi(p) \tag{A.19}$$

and

$$V_- = 2\omega_{k_-} e \sum_{\vec{p}} e^{-i\vec{p}\cdot\vec{r}} \Phi(p) \quad . \tag{A.20}$$

Now, the Born approximation

$$\psi^+_{-\vec{k}_-} \rightarrow \phi^*_{\vec{k}_-} \tag{A.21}$$

$$\psi^{-*}_{\vec{k}_+} \rightarrow \phi^*_{\vec{k}_+} \tag{A.22}$$

is introduced and Eqs. (A.19) and (A.20) are used to obtain

$$\left(\phi_{\vec{k}},\ V_{-}\psi^{+}_{-\vec{k}_{-}}\right) \rightarrow \omega_{k_{-}}\, e \int \sum_{\vec{p}} \frac{e^{-i(\vec{p}+\vec{k}+\vec{k}_{-})\cdot\vec{r}}\ \bar{\Phi}(p)\ d^{3}r}{\sqrt{\omega_{k}\,\omega_{k_{-}}}}$$

$$= \omega_{k_{-}}\, e \sum_{\vec{p}} \frac{\delta(\vec{p}+\vec{k}+\vec{k}_{-})\ \bar{\Phi}(p)}{\sqrt{\omega_{k}\omega_{k_{-}}}} \qquad \text{(A.23)}$$

and

$$\left(\phi_{\vec{k}'},\ V_{+}\psi^{-*}_{k_{+}}\right) \rightarrow \omega_{k_{+}}\, e \sum_{\vec{p}} \frac{\delta(\vec{p}+\vec{k}+\vec{k}_{+})\ \bar{\Phi}(p)}{\sqrt{\omega_{k'}\omega_{k_{+}}}} \quad . \qquad \text{(A.24)}$$

Equation (A.16) now becomes in the Born approximation

$$M = 2e^{2}\sqrt{4\pi}\left[\sum_{\vec{k}}\sum_{\vec{p}} \frac{(\hat{e}_{q}\cdot\vec{k})\,\omega_{k_{-}}\,\delta(\vec{q}-\vec{k}_{+}+\vec{k})\,\delta(\vec{p}+\vec{k}+\vec{k}_{-})}{\sqrt{2\omega_{q}\omega_{k_{+}}\omega_{k_{-}}}\left(\omega_{k}^{2}-\omega_{k_{-}}^{2}\right)} - \right. \qquad \text{(A.25)}$$

$$\left. - \sum_{\vec{k}'}\sum_{\vec{p}} \frac{(\hat{e}_{q}\cdot\vec{k}_{-})\,\omega_{k_{+}}\,\delta(\vec{q}+\vec{k}'-\vec{k}_{-})\,\delta(\vec{p}+\vec{k}'+\vec{k}_{+})}{\sqrt{2\omega_{q}\omega_{k_{+}}\omega_{k_{-}}}\left(\omega_{k'}^{2}-\omega_{k_{+}}^{2}\right)}\right] ,$$

where ϵ is set equal to zero. The sums in Eq. (A.25) can now be performed, and the condition $\hat{e}\cdot\vec{q} = 0$ used to obtain

$$M = 2e^{2}\sqrt{\frac{4\pi}{2\omega_{q}}}\left\{\frac{(\hat{e}_{q}\cdot\vec{k}_{+})}{\left(\omega_{k_{-}}^{2}-\omega_{\vec{k}_{+}-\vec{q}}^{2}\right)}\sqrt{\frac{\omega_{k_{-}}}{\omega_{k_{+}}}} + \frac{(\hat{e}_{q}\cdot\vec{k}_{-})}{\left(\omega_{k_{+}}^{2}-\omega_{\vec{k}_{-}-\vec{q}}^{2}\right)}\sqrt{\frac{\omega_{k_{+}}}{\omega_{k_{-}}}}\right\}\bar{\Phi}(p) \quad , \qquad \text{(A.26)}$$

with

$$\vec{p} = \vec{q} - \vec{k}_{-} - \vec{k}_{+} \quad , \qquad \text{(A.27)}$$

which is the form of the matrix element from the Pauli-Weisskopf theory.

APPENDIX B

A LIST OF SOME USEFUL RELATIONS BETWEEN CLEBSCH-GORDON AND RACAH COEFFICIENTS

For the purposes of reference, a few equations involving the Clebsch-Gordon nd Racah coefficients are reproduced in this Appendix. These equations are contained in the book by Rose,[1] along with others that are of considerable interest but not explicitly referred to in this work.

Four useful symmetry relations of the Clebsch-Gordon coefficients are[2]

$$C(j_1 j_2 j_3; m_1 m_2 m_3) = (-1)^{j_1+j_2-j_3} C(j_1 j_2 j_3; -m_1, -m_2, -m_3) \quad , \tag{B.1}$$

$$= (-1)^{j_1+j_2-j_3} C(j_1 j_2 j_3; m_2 m_1 m_3) \quad , \tag{B.2}$$

$$= (-1)^{j_1-m_1} \left(\frac{2j_3+1}{2j_2+1}\right)^{1/2} C(j_1 j_3 j_2; m_1, -m_3, -m_2) \quad , \tag{B.3}$$

$$= (-1)^{j_2+m_2} \left(\frac{2j_3+1}{2j_1+1}\right)^{1/2} C(j_3 j_2 j_1; -m_3, m_2, -m_1) \quad , \tag{B.4}$$

---

[1] M. E. Rose, Elementary Theory of Angular Momentum (John Wiley and Sons, Inc., New York, 1957).

[2] Ibid., p. 38.

and two equations relating the Racah and Clebsch-Gordon coefficients are[3]

$$\sum_f [(2e + 1)(2f + 1)]^{1/2} W(abcd;ef)\, C(bdf;\beta\delta)\, C(afc;\alpha,\beta + \delta) = C(abe;\alpha\beta)\, C(edc;\alpha + \beta, \delta) \quad , \tag{B.5}$$

$$[(2e + 1)(2f + 1)]^{1/2} W(abcd;ef)\, C(afc;\alpha, \beta + \delta) = \sum_\beta C(abe;\alpha\beta)\, C(edc;\alpha + \beta, \delta)\, C(bdf;\beta\delta) \quad . \tag{B.6}$$

[3] Ibid., p. 110. A typographical error in the first Clebsch-Gordon coefficient on the right of Eq. (B.6) was corrected before reproducing the equation here.

APPENDIX C

REDUCTION OF THE FUNCTION $V(LL'\ell_2\ell_2'\ell_1\ell_3)$ BY ANGULAR MOMENTUM RECOUPLING

In this appendix are given the details for recoupling of the angular momentum used to perform the sum indicated in Eq. (5.36) and leading to the final expression for $V(LL'\ell_2\ell_2'\ell_1\ell_3)$.

The expression to be reduced is

$$V(LL'\ell_2\ell_2'\ell_1\ell_3) = \sum_{\substack{m_1 \\ p=\pm 1}} C(L1\ell_1;m_1+p,-p)\, C(L\ell_2\ell_3;m_1+p,0) \times$$

$$\times\; C(L'1\ell_1;m_1+p,-p)\, C(L'\ell_2'\ell_3;m_1+p,0) \quad , \qquad (C.1)$$

and the task is to obtain an expression where the quantum number p appears in only one coefficient.

The two coefficients in Eq. (C.1) which contain ℓ_1 are considered first. To put these coefficients in the proper form for recoupling of the angular momentum, we use the symmetry relations given in Eqs. (B.4) and (B.1) in that order to obtain

$$C(L'1\ell_1;m_1+p,-p) = C(\ell_1 1L';m_1 p)(-1)^{-p+\ell_1-L'}\left(\frac{2\ell_1+1}{2L'+1}\right)^{1/2} \quad , \qquad (C.2)$$

so that the product of the two coefficients becomes

$$C(L1\ell_1;m_1+p,-p)\, C(L'1\ell_1;m_1+p,-p)$$

$$= C(L1\ell_1;m_1+p,-p)\, C(\ell_1 1L';m_1,p)(-1)^{-p+\ell_1-L'}\left(\frac{2\ell_1+1}{2L'+1}\right)^{1/2} \qquad (C.3)$$

$$= \sum_r \left[\frac{(2\ell_1+1)^2(2r+1)}{(2L'+1)}\right]^{1/2} (-1)^{-p+\ell_1-L'}\, W(L1L'1;\ell_1 r)\, C(11r;-p,p)\, C(LrL';m_1+1,0) \quad .$$

Equation (B.5) has been used to obtain the last form for the product given in Eq. (C.3).

After substituting Eq. (C.3) into Eq. (C.1) and gathering the three coefficients containing m_1, we consider the sum

$$\sum_{m_1} C(LrL';m_1+p,0)\, C(L\ell_2\ell_3;m_1+p,0)\, C(L'\ell_2'\ell_3;m_1+p,0)$$

$$= \sum_{m_1} C(rLL';0,m_1+p)(-1)^{L+r-L'}\, C(L\ell_3\ell_2;m_1+p,-m_1-p)(-1)^{L-m_1-p}\left(\frac{2\ell_3+1}{2\ell_2+1}\right)^{1/2}$$

$$C(L'\ell_3\ell_2';m_1+p,-m_1-p)(-1)^{L'-m_1-p}\left(\frac{2\ell_3+1}{2\ell_2'+1}\right)^{1/2} , \qquad (C.4)$$

where the symmetry relations (B.2) have been used on the first coefficient, and (B.3) on the second and third coefficients. Because of the arrangement of the quantum numbers in the coefficients on the right of Eq. (C.4) Eq. (B.6) can now be used to obtain

$$\sum_{m_1} C(LrL';m_1+p,0)\, C(L\ell_2\ell_3;m_1+p,0)\, C(L'\ell_2\ell_3;m_1+p,0)$$

$$= \left[\frac{(2L'+1)(2\ell_3+1)^2}{(2\ell_2'+1)}\right]^{1/2} (-1)^r\, C(r\ell_2\ell_2';00)\, W(rL\ell_2'\ell_3;L'\ell_2) \quad . \qquad (C.5)$$

Gathering terms, V now becomes

$$V(LL'\ell_2\ell_2'\ell_1\ell_3) = \sum_{\substack{r \\ p=\pm 1}} \left[\frac{(2\ell_1+1)^2\,(2r+1)(2\ell_3+1)^2}{(2\ell_2'+1)}\right]^{1/2} (-1)^{-p+\ell_1-L'+r} \times$$

$$\times\ C(11r;-p,p)\, C(r\ell_2\ell_2';00)\, W(LL1\ell_1;\ell_1 r)\, W(rL\ell_2'\ell_3;L'\ell_2) \quad , \qquad (C.6)$$

and the primary objective of obtaining an expression with the quantum number p in one coefficient has been achieved.

To proceed, use is made of the fact that $\ell_2 + \ell_2'$, L + L' and L' + ℓ_1 + 1 must be even in the expression for K given in Eq. (5.35). This was established in the section following Eq. (5.36). With $\ell_2 + \ell_2'$ even and since r + $\ell_2 + \ell_2'$ must be even in the coefficient $C(r\ell_2\ell_2';00)$ appearing in Eq. (C.6) for non-zero results, the quantum number r must now be even. This condition and the condition that r must satisfy a restricted triangle condition in the coefficient C(11r;-p,p) in Eq. (C.6) leaves r with only two possible values, 0 and 2.

For r = 0,

$$\sum_{p=\pm 1} C(110;-p,p) = \frac{2}{\sqrt{3}} \quad , \tag{C.7}$$

and for r = 2

$$\sum_{p=\pm 1} C(112;-p,p) = \sqrt{\frac{2}{3}} \quad , \tag{C.8}$$

so that

$$V(LL'\ell_2\ell_2'\ell_1\ell_3) = \frac{(2\ell_1+1)(2\ell_3+1)}{(2\ell_2'+1)^{1/2}} \frac{2}{\sqrt{3}} C(0\ell_2\ell_2';00)\, W(LlL'1;\ell_1 0)\, W(0L\ell_2'\ell_3;L'\ell_2) +$$

$$+ \frac{(2\ell_1+1)(2\ell_3+1)}{(2\ell_2'+1)^{1/2}} \sqrt{\frac{10}{3}} C(2\ell_2\ell_2';00)\, W(LlL'1;\ell_1 2)\, W(2L\ell_2'\ell_3;L'\ell_2) \quad . \tag{C.9}$$

The coefficients in the first term on the right of Eq. (C.9) can be evaluated to give

$$C(0\ell_2\ell_2';00) = \delta_{\ell_2,\ell_2'} \quad ,$$

$$W(LlL'1;\ell_1 0) = \frac{\delta_{L,L'}}{[3(2L'+1)]^{1/2}} \quad ,$$

$$W(0L\ell_2'\ell_3;L'\ell_2) = \frac{\delta_{L,L'}\,\delta_{\ell_2,\ell_2'}}{\left[(2L+1)(2\ell_2+1)\right]^{1/2}} \quad ,$$

where use is made of the conditions that $\ell_2 + \ell_2'$, $L + L'$ and $L' + \ell_1 + 1$ are even.

The value of the second term on the right or Eq. (C.9) will be unaltered by using the symmetry relation (B.4) on the Clebsch-Gordon coefficient. Using this relation, the final expression for V becomes

$$V(LL'\ell_2\ell_2'\ell_1\ell_3) = \frac{2}{3}\,\frac{(2\ell_1+1)(2\ell_3+1)}{(2\ell_2+1)(2L+1)}\,\delta_{L,L'}\,\delta_{\ell_2,\ell_2'} +$$

$$+ (2\ell_1+1)(2\ell_3+1)(-1)^{\ell_2}\sqrt{\frac{2}{3}}\,C(\ell_2'\ell_2 2;00)\,W(LL'1;\ell_1 2)\;\times$$

$$\times\; W(2L\ell_2'\ell_3;L'\ell_2) \quad , \qquad (C.10)$$

which is the form used for Eq. (5.37).