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I am submitting herewith a thesis written by Jennifer Katherine Aust entitled “Tree-Like Spaces with the Fixed-Point Property, Realized as Inverse Limits of Special Finite Trees.” I have examined the final electronic copy of this thesis for form and content and recommend that it be accepted in partial fulfillment of the requirements for the degree of Master of Science, with a major in Mathematics.

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Graduate School

(Original signatures are on file with official student records.)

**Tree-Like Spaces with the Fixed-Point
Property, Realized as Inverse Limits of
Special Finite Trees**

A Thesis
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Degree
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Jennifer Katherine Aust
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Abstract

In this paper, we explore some properties of inverse limit sequences on subspaces of Euclidean n -space. We address some well-known examples, in particular the example by David Bellamy of the “tree-like” continuum that does not have the fixed-point property. We highlight some spaces with the fixed-point property that are between snake-like continua and Bellamy’s example in their level of complexity. Specifically, we prove the fixed-point property for inverse limits of limit sequences on the unit interval and on the n -ad (in two configurations), and for inverse limits that can be mapped via a continuous function with small point pre-images to a generalized relative of the n -ad, which we call the (m, n) -ad. We conclude with a sufficient condition on a tree-like space for the space to have the fixed-point property.

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1 Introduction

In this paper, we explore some properties of inverse limit sequences on subspaces of $\mathbb{R}^n$. We address some well-known examples, in particular the example by David Bellamy of the “tree-like” continuum that does not have the fixed-point property. We focus on the middle ground between Bellamy’s example and snake-like continua, highlighting some spaces in this middle ground that also have the fixed-point property. The discussion of these spaces provides an estimation for the break point at which the fixed-point property no longer holds for certain spaces. We prove that inverse limit spaces of inverse limit sequences on the unit interval and inverse limit spaces of inverse limit sequences on the n -ad have the fixed-point property. We generalize this latter result to the case where each space in the inverse sequence is a different n -ad, with the number of arms increasing as the index does. We continue our exploration with inverse limits that can be mapped via a continuous function with small point pre-images to a union of two n -ads that intersect in precisely one arm; we call these relatives of the n -ad (m,n) -ads. We conclude the paper with a theorem that provides a sufficient condition on a tree-like space to ensure it has the fixed-point property.

Any discussion of the fixed-point property would be incomplete without some mention of the Brouwer Fixed-Point Theorem regarding the n -cell; one obvious consequence of this theorem is that any retract of an n -cell has the fixed-point property. Furthermore, since any finite tree can be imbedded in the plane such that each edge is a line segment, it is a retract of the disk and hence also has the fixed-point property. In the context of this fact, Bellamy’s example highlights one subtle difference between trees and spaces that are tree-like: tree-like spaces share the property of one-dimensionality with trees, but Bellamy has demonstrated a case in which the fixed-point property does not carry over from trees to tree-like spaces. In contrast to his example, we identify and discuss three examples of tree-like spaces that do have the fixed-point property.

We will use Munkres’ text as a standard reference for several basic results without citation.

2 Notation and Definitions

We will use the symbol $\mathbb{Z}_+$ to denote the positive integers. We will denote the countable product of the space X with itself, in the product topology, by X^ω . Ordered tuples will be contained in angle brackets rather than parentheses to avoid confusion with intervals, e.g. $\langle a, b \rangle$ and $\langle x_0, x_1, \dots \rangle$.

Definition 2.1. Let $X_1, X_2, \dots$ be a countable collection of spaces, and suppose that for each $n \in \mathbb{Z}_+$ such that $n \geq 2$ there is a continuous function $f_n : X_n \rightarrow X_{n-1}$. The sequence of spaces and functions $\{X_n, f_n\}$ is called an **inverse limit sequence**. We will omit the index on the space when all spaces in the sequence are the same.

Definition 2.2. The **inverse limit space** of the sequence $\{X_n, f_n\}$ is the subspace L of $\prod_{n \geq 0} X_n$ given by

$$L = \{ \langle x_1, \dots \rangle : x_n \in X_n \text{ and } x_n = f_{n+1}(x_{n+1}) \text{ for all } n \geq 1 \}.$$

We will denote this space by $\varprojlim \{X_n, f_n\}$.

Definition 2.3. A space X is said to have the **fixed point property** if every continuous function $f : X \rightarrow X$ has a fixed point.

Since we will be working mostly with metric spaces, we specify several metrics, the notation for ε -neighborhoods, and some related measures of distance below. We also specify that all ε 's, δ 's, and other small positive quantities in the paper are assumed to be less than $\frac{1}{2}$.

Metrics: We denote the standard Euclidean metric on $\mathbb{R}$ by d and its bounded version by $\bar{d}$. We denote the standard Euclidean metric on $\mathbb{R}^n$ by d_n and its bounded version by $\bar{d}_n$. We will use D for the metric on the

product $[0, 1]^\omega$ and D^* for the metric on the product B^ω , where $B = \mathbb{R}^n$. More specifically,

$$D(\mathbf{x}, \mathbf{y}) = \sup \left\{ \frac{\bar{d}(x_i, y_i)}{i} \right\} \quad \text{and} \quad D^*(\mathbf{x}, \mathbf{y}) = \sup \left\{ \frac{\bar{d}_n(x_i, y_i)}{i} \right\}.$$

In instances in which D^* is used, it should be clear from context which value of n is intended.

Neighborhoods: We will denote the ε -neighborhood (or ε -ball) about the point x under the metric ρ by $B_\rho(x, \varepsilon)$; in cases where the metric is clear, we will omit it. Similarly, we will denote the ε -neighborhood of the set Y under the metric ρ by $B_\rho(Y, \varepsilon) = \bigcup_{y \in Y} B_\rho(y, \varepsilon)$.

Definition 2.4. Given a metric space (X, ρ) , the **diameter** of a subset A of X is defined to be: $\text{diam}(A) = \sup\{\rho(a_1, a_2) : a_1, a_2 \in A\}$.

Definition 2.5. Given a metric space (X, ρ) , the **distance $\hat{D}$ between two subsets** A and B of X is defined to be: $\hat{D}(A, B) = \inf\{\rho(a, b) : a \in A, b \in B\}$.

Definition 2.6. The **mesh** of an open cover $\mathcal{U}$ of the space X is the number $\mu = \sup\{\text{diam}(U_\alpha) : U_\alpha \in \mathcal{U}\}$.

We now provide formal definitions for some of the spaces and characteristics that are the focus of this paper:

Definition 2.7. The **n-ad** is defined formally as the wedge of n unit intervals. We will view it as the following subspace of $\mathbb{R}^2$, using polar coordinates:

$$T_n = \left\{ \langle r, \theta \rangle \in \mathbb{R}^2 : r \in [0, 1] \text{ and } \theta = \frac{2\pi i}{n}, \text{ for some } i = 1, 2, \dots, n \right\}.$$

In several results below, we will use θ_i to denote the angle $\frac{2\pi i}{n}$ in an n-ad. The value of n should be clear from context because it is simply the number

of arms of the n -ad in question. If it is necessary to specify this value, we will do so by noting the bounds on the values of i , the upper bound of which will be the value of n .

Remark 2.8. We will use the following notation for interval-like subsets of the arms of the n -ad: $\langle(a, b), \theta\rangle := \{\langle x, \theta\rangle \in T_n : x \in (a, b)\}$. The sets $\langle[a, b], \theta\rangle$, $\langle[a, b), \theta\rangle$, and $\langle(a, b], \theta\rangle$ are defined similarly in the obvious way.

Note. In reference to T_n , we will use A_i to denote the i^{th} arm of the n -ad, i.e., $A_i = \langle[0, 1], \theta_i\rangle$. The notation A_i will also denote the i^{th} arm of a similar space in another context; see the note after Definition 2.9.

Definition 2.9. For $m, n \in \mathbb{Z}_+$, the **($\mathbf{m}, \mathbf{n}$)-ad** $S_{m,n}$ is the union of the $(m+1)$ -ad and the $(n+1)$ -ad such that their intersection is precisely one shared arm. In order to view this space in the plane, we express it as a bicentered tree with center vertices b and c such that b and c share an edge and m other edges emanate from b and n other edges emanate from c . We imbed this tree in the plane as follows: b is the origin $\langle 0, 0 \rangle$, c is the point $\langle 1, 0 \rangle$, and the non-shared edges emanating from them are confined to the half disks $\{\langle x, y \rangle \in \mathbb{R}^2 : x^2 + y^2 \leq 1 \text{ and } x \leq 0\}$ and $\{\langle x, y \rangle \in \mathbb{R}^2 : (x-1)^2 + y^2 \leq 1 \text{ and } x \geq 1\}$, respectively, and spaced evenly within them. See Figure 1.

Note. In reference to the (m, n) -ad, we will use A_i to denote the i^{th} arm of $S_{m,n}$, following the convention that A_0 is the shared arm, $A_1, \dots, A_m$ are the arms emanating from $\langle 0, 0 \rangle$, and $A_{m+1}, \dots, A_{m+n}$ are the arms emanating from $\langle 1, 0 \rangle$. It is important to note that these arms are closed sets homeomorphic to $[0, 1]$.

Definition 2.10. A space X is called **tree-like** if and only if, given $\varepsilon > 0$, there exists a map $f : X \rightarrow T_\varepsilon$ for some tree T_ε such that $\text{diam}(f^{-1}(t)) < \varepsilon$ for all $t \in T_\varepsilon$.

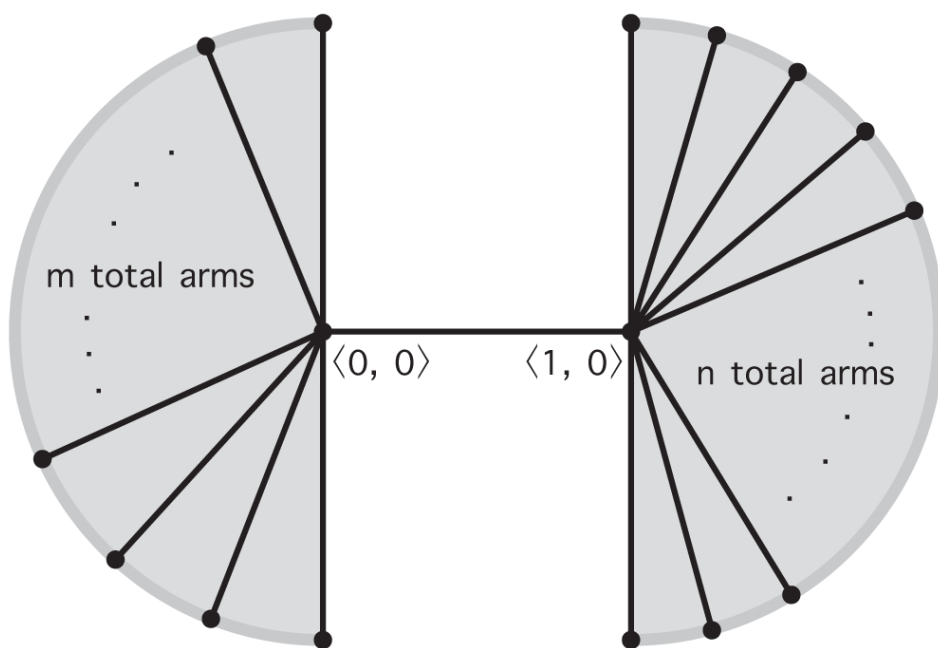


Figure 1: Imbedding of the (m, n) -ad in the plane.

3 Preliminary Lemmas

Lemma 3.1. *Suppose that each space X_n in the inverse limit sequence $\{X_n, f_n\}$ is a compact Hausdorff space. Then the limit space $\varprojlim\{X_n, f_n\}$ is not empty.*

We refer the reader to [3], p. 92, for a proof of this lemma.

Example 3.2. If the compactness condition does not hold, it is possible that the limit space is empty. We construct such an example: For each $n \in \mathbb{Z}_+$, let $X_n = (0, 1)$; for each $n \geq 2$, let $f_n : X_{n+1} \rightarrow X_n$ be given by $f_n(x) = \frac{x}{n}$. We note that f_n is a homeomorphism of $(0, 1)$ with $(0, \frac{1}{n})$, so that points in $L = \varprojlim\{X_n, f_n\}$ are completely determined by any single coordinate. Now suppose that $x_0 > \frac{a}{b} > 0$ is the first coordinate of an element of L , for some $a, b \in \mathbb{Z}_+$ with $a < b$. Then we have that $x_n > \frac{a \cdot n!}{b}$ for all $n \in \mathbb{Z}_+$. But we also have that $0 < x_n < 1$ for all $n \in \mathbb{Z}_+$, which implies that $n! < b$ for all $n \in \mathbb{Z}_+$, contradiction. Therefore $L = \emptyset$. $\square$

Lemma 3.3. *Let $B = \varprojlim\{X_n, f_n\}$, where (X_n, ρ_n) is a compact metric space. For each $n \in \mathbb{Z}_+$, let $\pi_n : B \rightarrow X$ denote projection onto the n^{th} coordinate. For each $n \in \mathbb{Z}_+$, $\text{diam}(\pi_n^{-1}(x_n)) \leq \frac{1}{n+1} < \frac{1}{n}$, for all $x_n \in X$.*

Proof. Fix $n \in \mathbb{Z}_+$. By definition of inverse limit, the values $x_1, \dots, x_{n-1}, x_n$ are fixed for all $\mathbf{x} = \langle x_1, \dots, x_n, \dots \rangle \in \pi_n^{-1}(x_n)$. Thus, if $\mathbf{x}, \mathbf{y} \in \pi_n^{-1}(x_n)$, then, if $\overline{\rho_n}$ denotes the bounded version of the metric ρ_n ,

$$\overline{D}(\mathbf{x}, \mathbf{y}) = \sup \left\{ \frac{\overline{\rho_n}(x_i, y_i)}{i} \right\} \leq \sup \left\{ \frac{1}{i} : i \geq n+1 \right\} = \frac{1}{n+1}.$$

Hence, for all $n \in \mathbb{Z}_+$, $\text{diam}(\pi_n^{-1}(x_n)) \leq \frac{1}{n+1} < \frac{1}{n}$ for all $x_n \in X$. $\square$

Lemma 3.4. *Let $B = \varprojlim\{X_n, f_n\}$, where X_n is a compact subspace of $\mathbb{R}^k$, for some $k \in \mathbb{Z}_+$. Then B is metrizable and compact.*

Proof. We consider B as a subspace of $\prod X_n$; B is therefore metrizable (in the product topology). Let $\mathbf{x}$ be a limit point of B . Since $\prod X_n$ is metrizable, there is a sequence of points $\mathbf{b}_k$ in B converging to $\mathbf{x}$. Since sequences in product spaces are coordinate-wise convergent in the product topology, the sequence $\pi_n(\mathbf{b}_1), \pi_n(\mathbf{b}_2), \dots$ converges to $\pi_n(\mathbf{x})$ for all $n \in \mathbb{Z}_+$. But then, for each $n \geq 2$, we must have that $f_{n-1}(\pi_n(\mathbf{b}_k)) = \pi_{n-1}(\mathbf{b}_k)$ converges to $\pi_{n-1}(\mathbf{x})$; hence, by uniqueness of limits in Hausdorff spaces, $f_{n-1}(\pi_n(\mathbf{x})) = \pi_{n-1}(\mathbf{x})$ for each $n \geq 2$. But then $\mathbf{x}$ is an element of the inverse limit space B by definition. Thus B is closed in $\prod X_n$, which is compact by the Tychonoff Theorem; B is therefore compact. $\square$

Lemma 3.5. *For all $n \in \mathbb{Z}_+$, let X_n be a compact, connected metric space. The inverse limit space $L = \varprojlim \{X_n, f_n\}$ is connected.*

Proof. Suppose that there exists a separation $B \cup C$ of L ; note that both B and C are both open and closed in L . Let $\pi_n : L \rightarrow X_n$ be projection onto the n^{th} coordinate. Since projection maps are open, the sets $\pi_n(B)$ and $\pi_n(C)$ are open in $\pi_n(L)$, for all $n \in \mathbb{Z}_+$. Furthermore, since B and C are closed in the compact space L , they are compact; thus their images under the continuous function π_n are compact; since X_n is Hausdorff, the sets $\pi_n(B)$ and $\pi_n(C)$ are closed in X_n , for all $n \in \mathbb{Z}_+$.

Claim: $\hat{D}(B, C) = \inf \{D(\mathbf{b}, \mathbf{c}) : \mathbf{b} \in B, \mathbf{c} \in C\} > 0$.

Proof of Claim: Suppose $\hat{D}(B, C) = 0$. Then zero is a limit point of the subset $\{D(\mathbf{b}, \mathbf{c}) : \mathbf{b} \in B, \mathbf{c} \in C\}$ of $\mathbb{R}$, so there exists a sequence $\{D(\mathbf{b}_n, \mathbf{c}_n)\}$ of points of this subset converging to zero (since $\mathbb{R}$ is metrizable). Now consider the sequence $\{\langle \mathbf{b}_n, \mathbf{c}_n \rangle\}$ in $B \times C$: since $B \times C$ is a closed subset of the compact, metrizable space $L \times L$, $B \times C$ is compact metric, so it is sequentially compact and limit point compact. Thus the sequence $\{\langle \mathbf{b}_n, \mathbf{c}_n \rangle\}$ has a convergent subsequence $\{\langle \mathbf{b}_{n_k}, \mathbf{c}_{n_k} \rangle\}$, which has a limit point $\langle \mathbf{b}, \mathbf{c} \rangle$ by limit point compactness. Then, since the metric D is continuous and $\langle \mathbf{b}_{n_k}, \mathbf{c}_{n_k} \rangle \rightarrow \langle \mathbf{b}, \mathbf{c} \rangle$, we have $D(\mathbf{b}_{n_k}, \mathbf{c}_{n_k}) \rightarrow D(\mathbf{b}, \mathbf{c})$ by continuity of D . Since $D(\mathbf{b}_{n_k}, \mathbf{c}_{n_k}) \rightarrow 0$, the subsequence $\{D(\mathbf{b}_{n_k}, \mathbf{c}_{n_k})\}$ also converges to zero, and therefore

$D(\mathbf{b}, \mathbf{c}) = 0$. Then $\mathbf{b} = \mathbf{c}$ since D is a metric, and thus $B \cap C \neq \emptyset$, contradiction. Hence $\hat{D}(B, C) > 0$.

Now let $\varepsilon = \hat{D}(B, C)$ and choose $m \in \mathbb{Z}_+$ such that $\frac{1}{m} < \varepsilon$.

Claim: $\pi_n(B) \cap \pi_n(C) = \emptyset$, for all $n \geq m$.

Proof of Claim: Let $x_n \in \pi_n(C)$. Recall that $\text{diam}(\pi_n^{-1}(x_n)) < \frac{1}{n}$, and let $\mathbf{c} \in C \cap \pi_n^{-1}(x_n)$; then, for all $\mathbf{a} \in \pi_n^{-1}(x_n)$, $D(\mathbf{a}, \mathbf{c}) < \frac{1}{n}$, so $\mathbf{a} \notin B$ since $\hat{D}(B, C) > \frac{1}{m} > \frac{1}{n}$. Thus $x_n \notin \pi_n(B)$, so $\pi_n(B)$ and $\pi_n(C)$ do not intersect.

As demonstrated above, $\pi_m(B)$ and $\pi_m(C)$ are both open and closed in $\pi_m(L)$; since they are disjoint, they separate $\pi_m(B \cup C)$; but

$$\pi_m(B \cup C) = \pi_m(L) = X_m \cap f_m(X_{m+1}) \cap f_{m+1}(f_m(X_{m+2})) \cap \cdots$$

by the definition of inverse limit system $\{X_n, f_n\}$. Since (by hypothesis) all the sets X_n are compact and connected for all $n \in \mathbb{Z}_+$ and all of the functions f_n are continuous, we have expressed $\pi_m(B \cup C)$ as the intersection of a nested sequence of compact connected sets; it is thus compact and connected, contradicting the existence of the separation $\pi_m(B) \cup \pi_m(C) = \pi_m(B \cup C)$. Hence the limit space L is connected. $\square$

Lemma 3.6. *Let (X, ρ) be a compact metric space, and let Y be a Hausdorff space; let $\varepsilon > 0$; let $f : X \rightarrow Y$ be a continuous function and suppose that $\text{diam}(f^{-1}(y)) < \varepsilon$ for all $y \in Y$. Then there exists an open cover $\mathcal{U}$ of Y such that $\text{diam}(f^{-1}(U)) < \varepsilon$ for all $U \in \mathcal{U}$.*

Proof. Let $y_0 \in Y$ and consider the set $f^{-1}(y_0)$. Since $\text{diam}(f^{-1}(y_0)) < \varepsilon$ by hypothesis, there exists some $\delta > 0$ such that $\text{diam}(f^{-1}(y_0)) + \delta = \varepsilon$. Now consider the neighborhood $V_{y_0} = B_\rho(f^{-1}(y_0), \frac{\delta}{2})$ of $f^{-1}(y_0)$; then V_{y_0} has diameter less than ε . Note that V_{y_0} is open, so $X - V_{y_0}$ is closed and hence compact in X . Then $f(X - V_{y_0})$ is compact in Y and therefore closed, so $U_{y_0} = Y - f(X - V_{y_0})$ is open in Y . Note that, since $f^{-1}(y_0) \subset V_{y_0}$, $(X - V_{y_0}) \cap f^{-1}(y_0) = \emptyset$, and thus $y_0 \notin f(X - V_{y_0})$, so $y_0 \in U_{y_0}$. Note also that $W_{y_0} = f^{-1}(U_{y_0}) \subset (X - f^{-1}(f(X - V_{y_0}))) \subset (X - (X - V_{y_0})) = V_{y_0}$,

so $W_{y_0} \subset V_{y_0}$; hence W_{y_0} is a neighborhood of $f^{-1}(y_0)$ with diameter less than ε . Obtain in this manner a set U_y for each $y \in Y$ and define a cover $\mathcal{U} = \{U_y | y \in Y\}$; then $\mathcal{U}$ is the desired open cover of Y . $\square$

Corollary 3.7. *Suppose, in addition to the hypotheses of Lemma 3.6, that Y is compact. Then there exists a $\delta > 0$ such that if $\mathcal{V}$ is an open cover of Y with $\text{diam}(V_\alpha) < \delta$ for all $V_\alpha \in \mathcal{V}$, then $\text{diam}(f^{-1}(V_\alpha)) < \varepsilon$ for all $V_\alpha \in \mathcal{V}$.*

Proof. Let $\mathcal{U}$ be the open cover of Y given by Lemma 3.6. Since Y is compact, the cover $\mathcal{U}$ has a Lebesgue number $\delta > 0$. Hence any open cover $\mathcal{V}$ of Y with mesh less than δ is a refinement of $\mathcal{U}$ and thus, for all $V_\alpha \in \mathcal{V}$, $f^{-1}(V_\alpha) \subset f^{-1}(U)$ for some $U \in \mathcal{U}$ and hence $\text{diam}(f^{-1}(V_\alpha)) < \varepsilon$ for all $V_\alpha \in \mathcal{V}$. $\square$

Lemma 3.8. *Let (X, ρ) be a compact metric space; fix $k \in \mathbb{Z}_+$. If for each $\varepsilon > 0$ there exists a Hausdorff space Y_ε of dimension k and a continuous function $f : X \rightarrow Y_\varepsilon$ with $\text{diam}(f^{-1}(y)) < \varepsilon$ for all $y \in Y_\varepsilon$, then X is also finite-dimensional with dimension at most k .*

Proof. Let (X, ρ) be a compact metric space and suppose that for some $k \in \mathbb{Z}_+$ and for each $\varepsilon > 0$ there exists a Hausdorff space Y_ε of dimension k and a continuous function $f : X \rightarrow Y_\varepsilon$ with $\text{diam}(f^{-1}(y)) < \varepsilon$ for all $y \in Y_\varepsilon$. Let $\mathcal{A}$ be an open cover of X . Since X is compact, $\mathcal{A}$ has a Lebesgue number $\delta > 0$. We will now consider the space Y_δ and the function $f : X \rightarrow Y_\delta$: we have that $\text{diam}(f^{-1}(y)) < \delta$ for all $y \in Y_\delta$. We have therefore satisfied the hypotheses of Lemma 3.6, so there exists a cover $\mathcal{U}$ of Y_δ such that $\text{diam}(f^{-1}(U)) < \delta$ for all $U \in \mathcal{U}$. Now, since Y_δ is k -dimensional, we can refine $\mathcal{U}$ to an open cover $\mathcal{V} = \{V_\alpha\}_{\alpha \in J}$ of Y_δ of order $k + 1$. Now, for each $\alpha \in J$, let $W_\alpha = f^{-1}(V_\alpha)$; let $\mathcal{W} = \{W_\alpha\}_{\alpha \in J}$. Note that, since $\mathcal{V}$ refines $\mathcal{U}$, $\text{diam}(W_\alpha) < \delta$ for all α , so $\mathcal{W}$ is a refinement of $\mathcal{A}$. Now we note that, since $V_\alpha \cap V_\beta = \emptyset \implies W_\alpha \cap W_\beta = \emptyset$, $\mathcal{W}$ has order at most $k + 1$. Hence X has dimension at most k . $\square$

Corollary 3.9. *The following inverse limit spaces are one-dimensional:*

1. $A = \varprojlim \{[0, 1], f_n\}$, where for each $n \geq 2$, $f_n : [0, 1] \rightarrow [0, 1]$ is continuous.
2. $L = \varprojlim \{T_r, f_n\}$, where T_r is the r -ad, and, for each $n \geq 2$, $f_n : T_r \rightarrow T_r$ is continuous.
3. $\Lambda = \varprojlim \{T_n, f_n\}$, where for each $n \in \mathbb{Z}_+$, T_n is the n -ad and $f_{n+1} : T_{n+1} \rightarrow T_n$ is continuous.
4. Any inverse limit space X of a sequence of compact, connected metric spaces such that for each $\delta > 0$, there exists a continuous function $f : X \rightarrow S_{m,n}$ mapping X to an (m, n) -ad such that $\text{diam}(f^{-1}(s)) < \delta$ for all $s \in S_{m,n}$.

Proof. Given $\varepsilon > 0$, choose $n \in \mathbb{Z}_+$ such that $\frac{1}{n} < \varepsilon$. Then, by Lemma 3.3, the projection maps $\pi_n : A \rightarrow [0, 1]$, $\pi_n : L \rightarrow T_r$, and $\pi_n : \Lambda \rightarrow T_n$ satisfy Lemma 3.8, with each limit space being mapped to a one-dimensional space; the space X of item 4 also clearly satisfies Lemma 3.8. We apply it to see that these limit spaces have dimension at most one. Furthermore, by Lemma 3.5, all of these limit spaces are connected, so they have dimension greater than zero. Therefore they have dimension exactly one. $\square$

Lemma 3.10. *If $g : M \rightarrow M$ is a continuous function from a compact metric space (M, ρ) to itself with no fixed points, then there exists a $\delta > 0$ such that $\rho(x, g(x)) \geq \delta$ for all $x \in M$.*

Proof. Suppose that for all $\delta > 0$, there exists an $x \in M$ such that $\rho(x, g(x)) < \delta$. Choose, for each $n \in \mathbb{Z}_+$, some $x_n \in M$ such that $\rho(x_n, g(x_n)) < \frac{1}{n}$. Then, since ρ is a continuous function and the sequence $\{\frac{1}{n}\}$ converges (to 0), the sequence of points $\{\langle x_n, g(x_n) \rangle\}$ in $M \times M$ has a convergent subsequence converging to some $\langle x, y \rangle \in M \times M$, and thus $\rho(x, y) = 0$, so $x = y$. Now, by continuity of g , and since $x_n \rightarrow x$, we have $g(x_n) \rightarrow g(x)$; but $g(x_n) \rightarrow y = x$, so $g(x) = x$, i.e. g has a fixed point, contradiction. Hence there exists a $\delta > 0$ such that $\rho(x, g(x)) \geq \delta$ for all $x \in M$. $\square$

Definition 3.11. Given a space X , a **chain cover** of X is an open cover $\mathcal{U}$ of X of order 2 that may be simply ordered in such a way that a set $U_\alpha \in \mathcal{U}$ has nonempty intersection precisely with its immediate predecessor or immediate successor, or both if they both exist.

Lemma 3.12. *Let $\varepsilon > 0$ and let $f : A \rightarrow [0, 1]$ be a continuous function defined on a compact, connected metric space A such that $\text{diam}(f^{-1}(x)) < \varepsilon$ for all $x \in [0, 1]$. Then A has a chain cover $\mathcal{U}$ such that $\text{diam}(U_\alpha) < \varepsilon$ for all $U_\alpha \in \mathcal{U}$.*

Proof. Let $\varepsilon > 0$. By Corollary 3.7, there exists a $\delta > 0$ such that any open cover of $[0, 1]$ with mesh less than δ has the property that the inverse image under f of each set in the cover has diameter less than ε . Now let $m \in \mathbb{Z}_+$ such that $\frac{1}{m} < \delta$. Then we define an open cover $\mathcal{B} = \{B_0, \dots, B_m\}$ of $[0, 1]$ with mesh less than δ :

Set $B_0 = [0, \frac{1}{2m})$, $B_{2m} = (\frac{2m}{2m+1}, 1]$, and for all i with $1 \leq i \leq 2m-1$, $B_i = (\frac{i}{2m+1}, \frac{i}{2m+1} + \frac{1}{2m})$. Note that $\mathcal{B}$ is a chain cover of $[0, 1]$ and that $\text{diam}(B_i) \leq \frac{1}{2m} < \frac{1}{m}$ for all $i = 0, 1, \dots, 2m$, so $\mathcal{B}$ has mesh $\mu \leq \frac{1}{m} < \delta$, so $\text{diam}(f^{-1}(B_i)) < \varepsilon$ for all $i = 0, 1, \dots, 2m$.

Let $J = \{i \in \{0, \dots, 2m\} : B_i \cap f(A) \neq \emptyset\}$. Note that, since $f(A)$ is connected, if $i-1, i+1 \in J$, then $i \in J$, since otherwise $(B_0 \cup \dots \cup B_{i-1}) \cap f(A)$ and $(B_{i+1} \cup \dots \cup B_m) \cap f(A)$ would form a separation of $f(A)$. Thus J is a set of k consecutive integers, for some $k \leq 2m+1$. Let $j = \min(J)$, and define, for $0 \leq i \leq k$, $U_i = f^{-1}(B_{i+j})$. Let $\mathcal{U} = \{U_0, \dots, U_k\}$.

Claim: $\mathcal{U}$ is the desired chain cover.

Proof of Claim: By definition of $\mathcal{U}$, $\text{diam}(U_i) = \text{diam}(B_{i+j}) < \varepsilon$ for each i . Also, $\mathcal{U}$ clearly covers A , since $\mathcal{U} = \{f^{-1}(B_i) : B_i \cap f(A) \neq \emptyset\}$. Furthermore, since $\mathcal{B}$ is an open cover and f is continuous, $\mathcal{U}$ is an open cover.

Now we need to show that, for all $0 \leq i, h \leq k$ with $i \neq h$, $U_i \cap U_h \neq \emptyset$ if and only if $i = h \pm 1$. First, we show that $i \neq h \pm 1 \implies U_i \cap U_h = \emptyset$. Let $i \neq h$, $i' = i + j$ and $h' = h + j$. Note that if $f(U_i \cap U_h) = \emptyset$, then $U_i \cap U_h = \emptyset$, and that, by construction of $\mathcal{U}$,

$$f(U_i \cap U_h) \subset f(U_i) \cap f(U_h) = f(A) \cap (B_{i'} \cap B_{h'}).$$

Recall that, since $\mathcal{B}$ is a chain cover, $B_{i'} \cap B_{h'} \neq \emptyset$ if and only if $i' = h' \pm 1$. Clearly, if $B_{i'} \cap B_{h'} = \emptyset$, then $f(A) \cap (B_{i'} \cap B_{h'}) = \emptyset$ also. Conversely, if $B_{i'} \cap B_{h'} \neq \emptyset$, and $f(A) \cap B_{i'} \neq \emptyset$ and $f(A) \cap B_{h'} \neq \emptyset$, then $f(A) \cap B_{i'} \cap B_{h'} \neq \emptyset$ by connectedness of $f(A)$: otherwise we would have points $x \in f(A) \cap B_{i'}$, $y \in f(A) \cap B_{h'}$, and $z \in B_{i'} \cap B_{h'}$ such that either $x < z < y$ or $y < z < x$, with $z \notin f(A)$, so $[0, z) \cap f(A)$ and $(z, 1] \cap f(A)$ would separate $f(A)$.

Therefore, if $f(A) \cap B_{i'} \neq \emptyset$ and $f(A) \cap B_{h'} \neq \emptyset$, then $f(A) \cap (B_{i'} \cap B_{h'}) = \emptyset$ if and only if $B_{i'} \cap B_{h'} = \emptyset$. Hence, if $i \neq h \pm 1$, then $i' \neq h' \pm 1$, so $B_{i'} \cap B_{h'} = \emptyset$. By choice of i' and h' , we have that $f(A) \cap B_{i'} \neq \emptyset$ and $f(A) \cap B_{h'} \neq \emptyset$, so $f(U_i) \cap f(U_h) = f(A) \cap (B_{i'} \cap B_{h'}) = \emptyset$; since $f(U_i \cap U_h) \subset f(U_i) \cap f(U_h)$, $f(U_i \cap U_h) = \emptyset$ and therefore $U_i \cap U_h = \emptyset$. Now we show that $i = h \pm 1 \implies U_i \cap U_h \neq \emptyset$:

Suppose, for some i , $U_i \cap U_{i+1} = \emptyset$. Then, since $U_i \cap U_h = \emptyset$ for all $h > i + 1$ and $U_{i+1} \cap U_h = \emptyset$ for all $h < i$, $(U_0 \cup \dots \cup U_i)$ and $(U_{i+1} \cup \dots \cup U_k)$ form a separation of A . Hence $U_i \cap U_{i+1} \neq \emptyset$ for all $0 \leq i \leq k - 1$.

Therefore $U_i \cap U_h \neq \emptyset$ if and only if $i = h \pm 1$; hence $\mathcal{U}$ is a chain cover of A . $\square$

Lemma 3.13. *Let $\varepsilon > 0$ and let M be a compact, connected metric space; suppose M has a chain cover $\mathcal{U}$ such that $\text{diam}(U_\alpha) < \varepsilon$ for all $U_\alpha \in \mathcal{U}$. Then there exists a continuous function $f : M \rightarrow [0, 1]$ such that $\text{diam}(f^{-1}(x)) < \varepsilon$ for all $x \in [0, 1]$.*

Proof. Let $\varepsilon > 0$ and let M be a compact, connected metric space; suppose M has a chain cover $\mathcal{U}$ such that $\text{diam}(U_\alpha) < \varepsilon$ for all $U_\alpha \in \mathcal{U}$. Since M is compact, we may assume $\mathcal{U}$ is finite. There are two cases, determined by the number of elements in $\mathcal{U}$.

Case 1: If $\mathcal{U}$ has an odd number of elements, then we index them as $U_0, \dots, U_{2k}$. Then we obtain (by connectedness and compactness of M) a precise refinement $\mathcal{V} = \{V_0, \dots, V_{2k}\}$ of $\mathcal{U}$. Since $V_i \subset \overline{V_i} \subset U_i$ for all $0 \leq i \leq 2k$, if $U_i \cap U_j = \emptyset$, then $V_i \cap V_j = \emptyset$; furthermore, since $\mathcal{U}$ is a chain cover, $U_i \cap U_{i+1} \neq \emptyset$ for all $0 \leq i \leq 2k - 1$; since $\mathcal{V}$ covers M and M

is connected, $V_i \cap V_{i+1} \neq \emptyset$ for all $0 \leq i \leq 2k-1$ since otherwise $V_1 \cup \dots \cup V_i$ and $V_{i+1} \cup \dots \cup V_{2k}$ would form a separation of M . Hence a pair of sets in $\mathcal{V}$ is disjoint if and only if the corresponding pair in $\mathcal{U}$ is disjoint, so $\mathcal{V}$ is also a chain cover. Now we consider the collection $\overline{\mathcal{V}} = \{\overline{V_i} : V_i \in \mathcal{V}\}$ and construct the function $f : M \rightarrow [0, 1]$:

Let E denote the union of all elements of $\overline{\mathcal{V}}$ with even indices; we define $f_0 : E \rightarrow [0, 1]$ by $f_0(\overline{V_{2i}}) = \{\frac{2i}{2k}\}$ for all $0 \leq i \leq k$; now, we use the Urysohn Lemma to extend this function, so that $f(\overline{V_j}) = [\frac{j-1}{2k}, \frac{j+1}{2k}]$ for all odd j between 1 and $2k-1$. We do this by applying the Lemma to $\overline{V_j}$ (for j odd) and the disjoint, closed subsets $\overline{V_{j-1}} \cap \overline{V_j}$ and $\overline{V_j} \cap \overline{V_{j+1}}$ to obtain a function $f_j : \overline{V_j} \rightarrow [\frac{j-1}{2k}, \frac{j+1}{2k}]$ such that $f(x) = \frac{j-1}{2k}$ for all $x \in \overline{V_{j-1}} \cap \overline{V_j}$ and $f(x) = \frac{j+1}{2k}$ for all $x \in \overline{V_j} \cap \overline{V_{j+1}}$. Lastly, we define $f : M \rightarrow [0, 1]$ in piecewise fashion: $f(x) = f_0(x)$ for all $x \in E$ and, for each odd j , $f(x) = f_j(x)$ for all $x \in \overline{V_j}$. We note that — by construction — these functions agree on the intersections $\overline{V_i} \cap \overline{V_{i+1}}$ for all $0 \leq i \leq 2k-1$, and each is continuous where it is defined, so f is continuous by the Pasting Lemma.

Now we must show that $\text{diam}(f^{-1}(x)) < \varepsilon$ for all $x \in [0, 1]$. There are two cases. If $x = \frac{2i}{2k}$ for some $0 \leq i \leq k$, then $f^{-1}(x) = \overline{V_{2i}}$, which has diameter less than ε by hypothesis. If $x \in [0, 1]$ but $x \neq \frac{2i}{2k}$ for some $0 \leq i \leq k$, then $f^{-1}(x) \subset \overline{V_j}$ for some odd j ; since $\overline{V_j}$ also has diameter less than ε by hypothesis, we have that $\text{diam}(f^{-1}(x)) < \varepsilon$ for all $x \in [0, 1]$, as desired.

Case 2: If $\mathcal{U}$ has an even number of elements, then we index them as $U_1, \dots, U_{2k}$. If $U_1 - U_2 = \emptyset$, we eliminate U_1 and revert to Case 1, re-indexing as necessary. If $U_1 - U_2 \neq \emptyset$, we modify the process of Case 1. We again obtain a precise refinement $\mathcal{V} = \{V_1, \dots, V_{2k}\}$ of $\mathcal{U}$, which is a chain cover by the argument in Case 1. Now we consider the collection $\overline{\mathcal{V}} = \{\overline{V_i} : V_i \in \mathcal{V}\}$ and construct the function $f : M \rightarrow [0, 1]$:

First, we define $f_1(\overline{V_1} - V_2) = \{0\}$ and $f_1(\overline{V_2} - V_1) = \{\frac{2}{2k}\}$. Then we apply the Urysohn Lemma to the set $\overline{V_1} \cup \overline{V_2}$ and its closed, disjoint subsets $(\overline{V_1} - V_2)$ and $(\overline{V_2} - V_1)$ to obtain a continuous extension $f_1 : (\overline{V_1} \cup \overline{V_2}) \rightarrow [0, \frac{2}{2k}]$. Now we define, for each even index $2i$ with $2 \leq i \leq k$, $f_{2i}(\overline{V_{2i}}) = \{\frac{2i}{2k}\}$. Then we use Urysohn again to extend this function (as in Case 1) so that $f(\overline{V_j}) = [\frac{j-1}{2k}, \frac{j+1}{2k}]$ for all odd j between 3 and $2k-1$. As in Case 1, the function f is continuous.

The proof that $\text{diam}(f^{-1}(x)) < \varepsilon$ for all $x \in [0, 1]$ is identical to that in Case 1 for $x \in [\frac{2}{2k}, 1]$. We handle the case $x \in [0, \frac{2}{2k})$ here. Note that $f^{-1}(0) = (\overline{V_1} - V_2)$, which is a subset of $\overline{V_1}$ and thus has diameter less than ε . If $x \in (0, \frac{2}{2k})$, then $f^{-1}(x) \subset (V_1 \cap V_2)$, which is again a subset of $\overline{V_1}$ and thus has diameter less than ε . Hence we have that $\text{diam}(f^{-1}(x)) < \varepsilon$ for all $x \in [0, 1]$, as desired. $\square$

Definition 3.14. Given a space X , an open cover $\mathcal{U}$ of X is a **centered chain cover with n arms** if and only if the following conditions hold:

1. There is an element U_0 of $\mathcal{U}$, called the **center** of $\mathcal{U}$, that intersects precisely n other pairwise disjoint elements of $\mathcal{U}$.
2. We can index the other sets in $\mathcal{U}$ as $U_{i,\alpha}$, where $1 \leq i \leq n$ and α is an element of some index set J_i ; where $i \neq j \implies U_{i,\alpha} \cap U_{j,\alpha} = \emptyset$; and where, for each i , the collection $\{U_{i,\alpha}\}_{\alpha \in J_i}$ together with U_0 is a chain cover of a subset of X .

In instances in which the number of arms is either undetermined or inconsequential, we will use the phrase **centered chain cover** without specifying the number of arms.

Remark 3.15. Any centered chain cover $\mathcal{U}$ of X with n arms has order 2.

Proof. Let $\mathcal{U}$ be a centered chain cover of the space X with n arms. Note first that by definition the element U_0 of $\mathcal{U}$ has nonempty intersection with n other elements of $\mathcal{U}$, so $\mathcal{U}$ has order at least 2. Let $x \in X$ and suppose that x is an element of the intersection of three distinct sets in $\mathcal{U}$. There are two cases: either U_0 is one of the three sets or it is not. If U_0 is one of the three sets, then we clearly have a contradiction, since U_0 intersects precisely n sets of $\mathcal{U}$, which are pairwise disjoint by definition. If U_0 is not one of the three sets, then we have three sets $U_{i,k}$, $U_{i',k'}$, and U_{i^*,k^*} that have nonempty intersection. By definition, if any two of i, i', i^* are distinct, then the corresponding sets are disjoint; thus $i = i' = i^*$. But then we have three distinct elements $U_{i,k}, U_{i,k'}, U_{i,k^*}$ of a chain cover, so they cannot have nonempty intersection; contradiction. Therefore $\mathcal{U}$ has order at most 2. $\square$

Lemma 3.16. Fix $r \in \mathbb{Z}_+$ and consider the r -ad T_r . Let $\varepsilon > 0$ and let $f : L \rightarrow T_r$ be a continuous function defined on a compact, connected metric space L such that $\text{diam}(f^{-1}(t)) < \varepsilon$ for all $t \in T_r$. Then L has a centered chain cover $\mathcal{U}$ with at most r arms such that $\text{diam}(U_\alpha) < \varepsilon$ for all $U_\alpha \in \mathcal{U}$.

Proof. Let $\varepsilon > 0$. By Corollary 3.7, there exists a $\delta > 0$ such that any open cover of T_r with mesh $\mu \leq \delta$ has the property that the inverse image under f of each set in the cover has diameter less than ε . Now let $m \in \mathbb{Z}_+$ such that $\frac{1}{m} < \delta$. Then we define an open cover $\mathcal{B}$ of T_r with mesh less than δ . First, for $1 \leq i \leq r$, recall that $\theta_i = \frac{2\pi i}{r}$. Set $B_0 = B_{\rho_n}(\langle 0, 0 \rangle, \frac{2}{4m+1}) \cap T$; for $i \in \{1, 2, \dots, r\}$ and $k \in \{1, 2, \dots, 2m-1\}$, set $B_{i,k} = \langle (\frac{k}{2m+1}, \frac{k}{2m+1} + \frac{1}{2m}), \theta_i \rangle$; lastly, for each $i \in \{1, 2, \dots, r\}$, set $B_{i,2m} = \langle (\frac{2m}{2m+1}, 1], \theta_i \rangle$. We note that $\text{diam}(B_0) = \frac{4}{4m+1} < \frac{1}{m}$, and for all k and i , $\text{diam}(B_{i,k}) = \frac{1}{2m} < \frac{1}{m}$. Hence $\text{diam}(B) < \frac{1}{m}$ for all $B \in \mathcal{B}$, so $\mathcal{B}$ has mesh $\mu \leq \frac{1}{m} < \delta$, so $\text{diam}(f^{-1}(B)) < \varepsilon$ for all $B \in \mathcal{B}$.

Claim: $\mathcal{B}$ is a centered chain cover of T_r with r arms.

Proof of Claim: First, we observe that B_0 is the center of $\mathcal{B}$, since it intersects precisely r other sets of $\mathcal{B}$, namely $B_{1,1}, \dots, B_{r,1}$. These intersections are of the form $B_0 \cap B_{i,1} = \langle (\frac{1}{2m+1}, \frac{2}{4m+1}), \theta_i \rangle$. That the sets $B_{i,1}$ are pairwise disjoint follows from the next paragraph.

We have indexed the elements of $\mathcal{B} - \{B_0\}$ as in the definition, where $J_i = J = \{1, 2, \dots, 2m\}$ for each i . Note that, for $i \neq j$, the subsets $\langle (0, 1], \theta_i \rangle$ and $\langle (0, 1], \theta_j \rangle$ of T_r are clearly disjoint; since $B_{i,k} \subset \langle (0, 1], \theta_i \rangle$ and $B_{j,k'} \subset \langle (0, 1], \theta_j \rangle$, we have that $B_{i,k} \cap B_{j,k'} = \emptyset$ for all $k, k' \in J$. Lastly, for each i , the collection $\mathcal{B}_i = \{B_0, B_{i,1}, \dots, B_{i,2m}\}$ forms a chain cover of the subset $A_i = \langle [0, 1], \theta_i \rangle$ of T_r : We note that A_i is homeomorphic to $[0, 1]$; if we consider the image of the cover $\mathcal{B}_i$ under this homeomorphism, we see that it is precisely the chain cover of the unit interval produced in the proof of Lemma 3.12, with one substitution: the initial set, $[0, \frac{1}{2m})$, has been replaced by $[0, \frac{2}{4m+1})$. Since the replacement set is a subset of the original initial set, and this replacement still has nonempty intersection with its immediate successor, we have preserved the properties of a chain cover; therefore $\mathcal{B}_i$ is a chain cover of A_i .

Now note that since L is connected, $f(L)$ is also connected. Hence, for fixed i , if $B_{i,k} \cap f(L) = \emptyset$, then $B_{i,k'} \cap f(L) = \emptyset$ for all $k' \geq k$ in J , since otherwise we would have a separation of $f(L)$ by the obvious sub-collections of $\mathcal{B}$. Furthermore, if $B_0 \cap f(L) = \emptyset$, then $f(L)$ can have nonempty intersection with elements of $\mathcal{B}_i$ (as defined above) for only one i , since we would otherwise again have a separation of $f(L)$ by two of the sub-collections $\mathcal{B}_i$. Let $\mathcal{C} = \{B \in \mathcal{B} : B \cap f(L) \neq \emptyset\}$. We note that $\mathcal{C}$, as a sub-collection of $\mathcal{B}$ that covers $f(L)$, is a centered chain cover of $f(L)$ with t arms, where $t \leq r$. Define a cover $\mathcal{U}$ of L as follows:

For all $B_\alpha \in \mathcal{C}$, set $U_\alpha = f^{-1}(B_\alpha)$ (we use α to indicate that the indices of sets in $\mathcal{C}$ should be carried identically to their pre-images in $\mathcal{U}$, whatever form those indices take). Set $\mathcal{U} = \{U_\alpha\}$.

Claim: For all $U_\alpha, U_\beta \in \mathcal{U}$, $U_\alpha \cap U_\beta \neq \emptyset$ if and only if $B_\alpha \cap B_\beta \neq \emptyset$.

Proof of Claim: Clearly, if $U_\alpha \cap U_\beta = f^{-1}(B_\alpha \cap B_\beta) \neq \emptyset$, then $B_\alpha \cap B_\beta \neq \emptyset$.

Conversely, we observe that for all $U_\alpha \in \mathcal{U}$, $f(L) \cap B_\alpha \neq \emptyset$ by construction of $\mathcal{U}$. Also, if $B_\alpha \cap B_\beta \neq \emptyset$ and both B_α and B_β intersect $f(L)$, then $f(L) \cap (B_\alpha \cap B_\beta) \neq \emptyset$ by connectedness of $f(L)$, since otherwise we would have a separation of $f(L)$. Without loss of generality, we may suppose that B_β is ‘further’ from the center of $\mathcal{B}$ than B_α , i.e., either $B_\alpha = B_0$ and $B_\beta = B_{i,1}$ for some i , or $B_\alpha = B_{i,k}$ and $B_\beta = B_{i,k+1}$ for some i . In either case, say $B_\beta = B_{i,j}$; then the sub-collection $\{B_{i,j}, B_{i,j+1}, \dots, B_{i,2m}\}$ and its complement in $\mathcal{B}$ separate $f(L)$. Note that $U_\alpha \cap U_\beta = f^{-1}(f(L) \cap (B_\alpha \cap B_\beta))$, so if $B_\alpha \cap B_\beta \neq \emptyset$, then $f(L) \cap (B_\alpha \cap B_\beta) \neq \emptyset$ and thus $U_\alpha \cap U_\beta \neq \emptyset$. Therefore $U_\alpha \cap U_\beta \neq \emptyset$ if and only if $B_\alpha \cap B_\beta \neq \emptyset$.

Claim: $\mathcal{U}$ is a centered chain cover of L with t arms, for some $t \leq r$.

Proof of Claim: We must show two things, namely:

1. $\mathcal{U}$ has a center element that intersects precisely t other elements of $\mathcal{U}$.
2. Using the indexing structure already provided by $\mathcal{C}$, all elements except the center take the form $U_{i,k}$, where $1 \leq i \leq r$, where there are t distinct values of i for some $t \leq r$, and where, for all i , $k \in J'_i \subset J$. Furthermore, if $i \neq j$, then $U_{i,k} \cap U_{j,k'} = \emptyset$ for

all possible values of k and k' , and, for each i , the sub-collection $\mathcal{U}_i = \{U_\alpha : B_\alpha \in \mathcal{B}_i\}$ forms a chain cover of some subset of L .

(1) There are two cases: first, if $B_0 \cap f(L) \neq \emptyset$, then we have an element U_0 of $\mathcal{U}$; we show U_0 is the center of $\mathcal{U}$. Since B_0 is the center of $\mathcal{B}$ and of $\mathcal{C}$, it intersects precisely t elements of $\mathcal{C}$, namely $B_{i,1}$ for t values of i (where $t \leq r$); thus U_0 intersects the corresponding $U_{i,1}$'s, so it intersects precisely t elements of $\mathcal{U}$. Alternatively, if $B_0 \cap f(L) = \emptyset$, then $\mathcal{C}$ contains only elements from one collection $\mathcal{B}_i$, say $B_{i,k}, B_{i,k+1}, \dots, B_{i,k+h}$, and thus $\mathcal{U} = \{U_{i,k}, \dots, U_{i,k+h}\}$; in this case, $U_{i,k}$ is the center; it intersects only the element $U_{i,k+1}$ since $B_{i,k}$ intersects only the element $B_{i,k+1}$ of $\mathcal{C}$.

(2) By choice of $\mathcal{C}$, $\mathcal{C}$ has elements (excluding B_0) $B_{i,k}$ where $1 \leq i \leq r$; there are at most t distinct values of i for some $t \leq r$; and for all i , $k \in J'_i \subset J$. By construction of $\mathcal{U}$ from $\mathcal{C}$ in a manner which preserves the indexing relationship, we have that $\mathcal{U}$ has elements $U_{i,k}$ where $1 \leq i \leq r$; there are t distinct values of i ; and for all i , $k \in J'_i \subset J$. From the previous claim, $U_{i,k} \cap U_{j,k'} \neq \emptyset$ if and only if $B_{i,k} \cap B_{j,k'} \neq \emptyset$, so $i \neq j \implies U_{i,k} \cap U_{j,k'} = \emptyset$ for all possible values of k and k' , since $\mathcal{B}$ is a centered chain cover. Lastly, for each i , the sub-collection $\mathcal{U}_i$, together with U_0 if it exists, must form a chain cover of $f^{-1}(\langle [0, 1], \theta_i \rangle)$: if U_0 exists, then it intersects only the element $U_{i,1}$ of $\mathcal{U}_i$, since B_0 intersects only the element $B_{i,1}$ of $\mathcal{B}_i$; furthermore, $U_{i,k} \cap U_{i,h} \neq \emptyset$ if and only if $B_{i,k} \cap B_{i,h} \neq \emptyset$, which is true if and only if $k = h \pm 1$; hence $\mathcal{U}_i$ is a chain cover of $f^{-1}(\langle [0, 1], \theta_i \rangle)$. It covers this subset of L because $\mathcal{B}_i$ covers $\langle [0, 1], \theta_i \rangle$ and all elements of $\mathcal{B}_i$ that intersect $f(L)$ correspond to elements of $\mathcal{U}_i$ by definition. $\square$

Lemma 3.17. *Let $\varepsilon > 0$ and suppose a compact, connected metric space L has a centered chain cover $\mathcal{U}$ such that $\text{diam}(U_\alpha) < \varepsilon$ for all $U_\alpha \in \mathcal{U}$. Then, for some n -ad T_n , there exists a continuous function $f : L \rightarrow T_n$ such that $\text{diam}(f^{-1}(x)) < \varepsilon$ for all $t \in T_n$.*

The proof is very similar to the proof of Lemma 3.13.

Definition 3.18. Given a space X , an open cover $\mathcal{U}$ of X is a **bicentered chain cover** of X if and only if the following conditions hold:

1. There two elements U_μ and U_ν of $\mathcal{U}$, and a chain cover $\mathcal{U}_0$, collectively called the **center arm** of $\mathcal{U}$, such that:
 - (a) $\mathcal{U}_0 = \{U_{0,1}, \dots, U_{0,k}\}$ for some $k \in \mathbb{Z}_+$, and U_μ and U_ν have nonempty intersection only with the sets $U_{0,1}$ and $U_{0,k}$ of $\mathcal{U}_0$, respectively;
 - (b) U_μ intersects precisely m other pairwise disjoint elements of $\mathcal{U}$, each of which is disjoint from $\mathcal{U}_0$; and
 - (c) U_ν intersects precisely n other pairwise disjoint elements of $\mathcal{U}$, each of which is disjoint from $\mathcal{U}_0$.
2. We can index the other sets in $\mathcal{U}$ as $U_{i,\alpha}$, where:
 - (a) $1 \leq i \leq m+n$ and α is an element of some index set J_i ;
 - (b) those sets $U_{i,1}$ for which $1 \leq i \leq m$ intersect U_μ , and those sets $U_{i,1}$ for which $m+1 \leq i \leq m+n$ intersect U_ν ;
 - (c) $i \neq j \implies U_{i,\alpha} \cap U_{j,\alpha'} = \emptyset$; and
 - (d) for each i , the collection $\{U_{i,\alpha}\}_{\alpha \in J_i}$ together with either U_μ or U_ν is a chain cover of a subset of X .

We note that the cover has precisely $m+n+1$ arms in total, including the center arm.

Lemma 3.19. Fix $m, n \in \mathbb{Z}_+$ and consider the (m, n) -ad $S_{m,n}$. Let $\varepsilon > 0$ and let $f : X \rightarrow S_{m,n}$ be a continuous function defined on a compact, connected metric space X such that $\text{diam}(f^{-1}(s)) < \varepsilon$ for all $s \in S_{m,n}$. Then X has a bicentered chain cover $\mathcal{U}$ with at most $(m+n+1)$ arms such that $\text{diam}(U_\alpha) < \varepsilon$ for all $U_\alpha \in \mathcal{U}$.

Proof. Adapt the proof of Lemma 3.16, generating a bicentered chain cover $\mathcal{B}$ of $S_{m,n}$ and checking that the collection of pre-images of those elements of $\mathcal{B}$ that have nonempty intersection with $f(X)$ is a bicentered chain cover of X . $\square$

Definition 3.20. Given a space X , an open cover $\mathcal{U}$ of a connected space X is a **tree cover** of X if and only if $\mathcal{U}$ has order 2 and contains no cycle $U_1, U_2, \dots, U_n$ such that $U_i \cap U_j \neq \emptyset$ if and only if $|i - j| \leq 1$ or $i = 1$ and $j = n$.

Lemma 3.21. *X is a tree-like space if and only if for all $\varepsilon > 0$, X has a tree cover $\mathcal{U}$ such that $\text{diam}(U) < \varepsilon$ for all $U \in \mathcal{U}$.*

Proof. To show that a tree-like space has the desired cover, we imitate the arguments of Lemmas 3.12, 3.16, and 3.19, applying Corollary 3.7 to the definition of tree-like. To show that a space that has, for all $\varepsilon > 0$, a tree cover $\mathcal{U}$ such that $\text{diam}(U) < \varepsilon$ for all $U \in \mathcal{U}$ is tree-like, we imitate the proof of Lemma 3.13. $\square$

Remark. We note that Bing [2] uses the tree cover condition of Lemma 3.21 as the defining property of a tree-like space; the Lemma shows that our definition is equivalent to Bing's.

4 Inverse Limit Spaces with the Fixed-Point Property

In contrast to the spaces discussed in the results below, which are simply limit spaces of inverse limit sequences of trees with special properties, David Bellamy's example is a complicated creation. He begins the construction with the 6-adic solenoid and a map f on this solenoid that has a single fixed point, namely the infinite tuple $\langle 1, 1, 1, \dots \rangle$. He then removes an arc containing this fixed point and replaces it with a cone over a totally disconnected set, extending the function in such a way that no point of the cone is fixed [1]. This space is carefully constructed by forming multiple quotient spaces and by performing repeated restriction and extension of the function f and the inverse function $\theta(x) = x^{-1}$. A lemma shows the space is tree-like; the construction shows that the space pushes the limits of the tree-like condition, introducing a certain degree of pathology that is not possible within the more restrictive definition of a tree.

The first theorem presented in this section is a known result, but the strategy of the proof is instructive for the proof of the other theorems, so we include this result and its proof as preparation for the more general results.

Theorem 4.1. *Given an inverse limit system $\{[0, 1], f_n\}$ on the unit interval, its limit space $A = \varprojlim \{[0, 1], f_n\}$ has the fixed-point property.*

Proof. Let $\{[0, 1], f_n\}$ be an inverse limit system on the unit interval, and let $A = \varprojlim \{[0, 1], f_n\}$. Note that A is metrizable (as a subspace of $[0, 1]^\omega$) by Lemma 3.4; recall that D denotes this metric on A . Suppose $g : A \rightarrow A$ is a continuous function with no fixed points; choose, by Lemma 3.10, a $\delta > 0$ such that $D(x, g(x)) \geq \delta$ for all $x \in A$. Let $N \in \mathbb{Z}_+$ such that $\frac{1}{N} < \frac{\delta}{2}$ and note that for each $n \geq N$, if π_n denotes the projection map, $\text{diam}(\pi_n^{-1}(x)) < \frac{1}{n} < \frac{\delta}{2}$ for all $x \in [0, 1]$. Thus A satisfies the hypotheses of Lemma 3.12, so we may choose a chain cover $\mathcal{U}$ of A such that $\text{diam}(U_\alpha) < \frac{\delta}{2}$ for all $U_\alpha \in \mathcal{U}$. Then, for $x \in A$, x is an element of U_α for some α , and $g(x) \notin U_\alpha$, by choice of $\mathcal{U}$. Since A is compact by Lemma 3.4, $\mathcal{U}$ has a finite

subcover $\mathcal{V}$. Clearly, since we have obtained $\mathcal{V}$ from $\mathcal{U}$ only by removing elements of $\mathcal{U}$, $\mathcal{V}$ has order at most 2 and elements of $\mathcal{V}$ can only intersect their immediate predecessors and successors from $\mathcal{U}$. Suppose that $U_\alpha \notin \mathcal{V}$ but that U_α has both a predecessor and a successor in $\mathcal{V}$; let U_p be the predecessor of U_α with the largest index less than α in $\mathcal{V}$, and similarly let U_s be the successor of U_α with the smallest index greater than α in $\mathcal{V}$. Then, since $\mathcal{V}$ covers A , the union of U_p and its predecessors and the union of U_s and its successors form a separation of A ; since A is connected by Lemma 3.5, we have a contradiction. Hence if U_α and some successor U_β of U_α are both in $\mathcal{V}$, then all successors of U_α that are predecessors of U_β must also be in $\mathcal{V}$. We index the elements of $\mathcal{V}$ as $V_1, \dots, V_k$ so that these elements follow the same ordering as their indices. Now consider the sets F and B , defined below:

$$\begin{aligned} F &:= \{x \in A : x \in V_i \text{ and } g(x) \in V_{i+j}, \text{ for some } j \leq k-i\}, \text{ and} \\ B &:= \{x \in A : x \in V_i \text{ and } g(x) \in V_{i-j}, \text{ for some } j < i\}. \end{aligned}$$

Note that F and B are nonempty, since $V_1 \subset F$ and $V_k \subset B$. Note also that $F \cap B = \emptyset$ and $F \cup B = A$. Now we show that F and B are open in A .

Let $x_0 \in F$; then $x_0 \in V_i$ for some i and $g(x_0) \in V_{i+j}$ for some $j \geq 2$. (Since $\text{diam}(V_i \cup V_{i+1}) < \frac{\delta}{2} + \frac{\delta}{2} = \delta$, $j \neq 1$.) Let $\varepsilon > 0$ such that $B(g(x_0), \varepsilon) \subset V_{i+j}$. Then by continuity of g , there exists an $\varepsilon' > 0$ such that for all $x \in B(x_0, \varepsilon')$, $g(x) \in B(g(x_0), \varepsilon)$. Thus for all $x \in (V_i \cap B(x_0, \varepsilon'))$, $g(x) \in V_{i+j}$, i.e. $x \in F$. Hence F is open in A . Similarly, B is open in A ; thus $F \cup B$ is a separation of A , contradicting connectedness. Therefore g has a fixed point. $\square$

We now begin to extend the above result in several stages of generality, modifying the argument to suit tree-like spaces with fewer and fewer constraints. In three of the four results that follow, we specify the type of tree to which the space admits a map with arbitrarily small point pre-images; in the final result, however, we place no conditions on the tree itself, but only on the pre-images under such a map of certain vertices of the tree.

Theorem 4.2. *Let $\{T_r, f_n\}$ be an inverse limit system on an r -ad T_r , and let L be its limit space $L = \varprojlim \{T_r, f_n\}$. For each $n \in \mathbb{Z}_+$, let $\pi_n : L \rightarrow T_r$ denote*

projection onto the n^{th} coordinate. Suppose that in each coordinate, the pre-image $\pi_n^{-1}(\mathbf{0})$ of the center point (the origin $\mathbf{0}$) of the r -ad is connected. Then L has the fixed-point property.

Proof. Let $g : L \rightarrow L$ be a continuous function with no fixed points. Choose (by Lemma 3.10) a $\delta' > 0$ such that $D^*(x, g(x)) \geq \delta'$ for all $x \in L$. Let $n \in \mathbb{Z}_+$ such that $\frac{1}{n} < \frac{\delta'}{2}$, and consider $\pi_n^{-1}(\mathbf{0})$. By Lemma 3.3, $\text{diam}(\pi_n^{-1}(\mathbf{0})) < \frac{1}{n} < \frac{\delta'}{2}$, so $g(\pi_n^{-1}(\mathbf{0})) \cap \pi_n^{-1}(\mathbf{0}) = \emptyset$. Recall that A_i denotes the i^{th} arm of T_r . Then, since $\pi_n^{-1}(\mathbf{0})$ is connected (by hypothesis) and $g(\pi_n^{-1}(\mathbf{0})) \cap \pi_n^{-1}(\mathbf{0}) = \emptyset$, $\pi_n(g(\pi_n^{-1}(\mathbf{0}))) \subset (A_i - \mathbf{0})$ for some i , because the sets $A_i - \mathbf{0}$ are precisely the connected components of $T_r - \mathbf{0}$; thus $g(\pi_n^{-1}(\mathbf{0})) \subset \pi_n^{-1}(A_i - \mathbf{0})$ for that i .

We will denote this particular value of i by H , so $g(\pi_n^{-1}(\mathbf{0})) \subset \pi_n^{-1}(A_H - \mathbf{0})$. Thus $\pi_n^{-1}(\mathbf{0}) \subset g^{-1}(\pi_n^{-1}(A_H - \mathbf{0}))$; note that $\pi_n^{-1}(\mathbf{0})$ is closed and $g^{-1}(\pi_n^{-1}(A_H - \mathbf{0}))$ is open, so $Y = L - (g^{-1}(\pi_n^{-1}(A_H - \mathbf{0})))$ is closed and disjoint from $\pi_n^{-1}(\mathbf{0})$. Hence $\hat{D}(\pi_n^{-1}(\mathbf{0}), Y) = \delta'' > 0$, so the $\frac{\delta''}{2}$ -neighborhood of $\pi_n^{-1}(\mathbf{0})$ is also contained in $g^{-1}(\pi_n^{-1}(A_H - \mathbf{0}))$. Note that the diameter of this neighborhood is at most $\frac{1}{n} + \delta''$. We set $\delta = \min\{\frac{\delta'}{2}, \frac{1}{n} + \delta''\}$. We obtain (by Lemma 3.16) a centered chain cover $\mathcal{U}$ of L with $\text{diam}(U_\alpha) < \delta$ for all $U_\alpha \in \mathcal{U}$ as the pre-image of a centered chain cover $\mathcal{B}$ on T_r , so U_0 , the center of $\mathcal{U}$, is contained in the $\frac{\delta''}{2}$ -neighborhood of $\pi_n^{-1}(\mathbf{0})$ and thus $U_0 \subset g^{-1}(\pi_n^{-1}(A_H - \mathbf{0}))$. Furthermore, by choice of δ and $\mathcal{U}$, we have that

$$g(U_0) \cap U_0 = \emptyset. \quad (1)$$

Thus $U_0 \cap g^{-1}(U_0) = \emptyset$, so (since $U_0 = \pi_n^{-1}(B_0)$)

$$U_0 \subset g^{-1}(\pi_n^{-1}(A_H - \mathbf{0})) - g^{-1}(U_0) = g^{-1}(\pi_n^{-1}(A_H - B_0)). \quad (2)$$

For reference, we let $m_i = \max(J_i)$, for each i . Now, we obtain (by compactness and connectedness of L) a precise refinement of $\mathcal{U}$ to an open cover $\mathcal{V}$ of L such that for all indices α , $\overline{V_\alpha} \subset U_\alpha$ and $\mathcal{V}$ is also a centered chain cover of L . Let $\overline{\mathcal{V}}$ denote the collection the closures of elements of $\mathcal{V}$.

Observe that, by (1), $g(V_0) \cap V_0 = \emptyset$. So, by (2), $g(V_0) \subset g(U_0) \subset$

$\pi_n^{-1}(A_H - \mathbf{0})$; thus, for some k ,

$$g(V_0) \cap V_{H,k} \neq \emptyset. \quad (3)$$

Now let $x \in \overline{V_{H,m_H}}$; let C denote the component of $L - \overline{V_0}$ containing the point x . Consider the closure $\overline{C}$ of C in L : note that $\overline{C}$ is connected; we will show that $\overline{C}$ intersects $\overline{V_0}$. Since L is compact, connected, and metrizable (by Lemmas 3.4 and 3.5), since $L - \overline{V_0}$ is open, and since C is a component of this open set, $\text{Bd}(L - \overline{V_0})$ contains a limit point of C (see [3], p. 47). But $\text{Bd}(L - \overline{V_0}) = \text{Bd}(\overline{V_0}) \subset \overline{V_0}$, so

$$\overline{C} \cap \overline{V_0} \neq \emptyset. \quad (4)$$

Note also that the set $(V_{H,1} - \overline{V_0}) \cup V_{H,2} \cup \cdots \cup V_{H,m_H}$ and the union of all the sets $V_{i,k}$ with $i \neq H$ form a separation of $L - \overline{V_0}$, so $C \cap V_{i,k} = \emptyset$ for all $i \neq H$. Thus

$$\overline{C} \subset (\overline{V_0} \cup V_{H,1} \cup V_{H,2} \cup \cdots \cup V_{H,m_H}). \quad (5)$$

We now pause to introduce some terminology. There are two definitions:

We say that the function g **increases the index** of $x \in L$ if we have either

1. $x \in \overline{V_0}$ and $g(x) \in V_{H,k}$ for some k , or
2. $x \in V_{H,k}$ and $g(x) \in V_{H,k+h}$, for $h \geq 1$.

We say that the function g **decreases the index** of $x \in L$ if we have either

1. $x \in V_{H,k}$ and $g(x) \in V_{H,k-h}$, for $h \geq 1$,
2. $x \in V_{H,k}$ and $g(x) \in V_{j,k'}$, for $j \neq H$ and some k' or $g(x) \in V_0$, or
3. $x \in \overline{V_0}$ and $g(x) \in V_{j,k'}$, for $j \neq H$ and some k' .

We now define two subsets of $\overline{C}$:

Let $R := \{x \in \overline{C} : g \text{ increases the index of } x\}$, and
let $S := \{x \in \overline{C} : g \text{ decreases the index of } x\}$.

We first show that R and S are both nonempty: Clearly, S is nonempty since $\overline{C} \cap \overline{V_{H,m_H}} \neq \emptyset$ by choice of C , and since $\text{diam}(\overline{V_{H,m_H}}) \leq \delta < \frac{\delta'}{2}$ by choice of $\mathcal{U}$ and $\mathcal{V}$, g moves every point of $\overline{V_{H,m_H}}$ to some other $V_{i,k}$, where either $i = H$ and $k \leq m_H - 2$ or $i \neq H$; by definition, then, g decreases the index

of every point in $C \cap \overline{V_{H,m_H}}$, so S is nonempty.

R is also nonempty: From (3), we have that $g(V_0) \cap V_{H,k} \neq \emptyset$ for some k , so $g(U_0) \cap U_{H,k} \neq \emptyset$ for that k . Since $g(U_0) \subset \pi_n^{-1}(A_H - B_0)$ by (2), $g(\overline{V_0}) \subset g(U_0) \subset \pi_n^{-1}(A_H - B_0) = (V_{H,1} - U_0) \cup V_{H,2} \cup \dots \cup V_{H,m_H}$. Thus g increases the index of all points in $\overline{V_0}$. Furthermore, $\overline{C} \cap \overline{V_0} \neq \emptyset$ by (4), so there are points of $\overline{C}$ whose index is increased by g , i.e., R is nonempty.

From (5) above, we have that $\overline{C} \subset (\overline{V_0} \cup V_{H,1} \cup V_{H,2} \cup \dots \cup V_{H,m_H})$; we note that, since $\text{diam}(V_\alpha) \leq \text{diam}(\overline{V_\alpha}) \leq \text{diam}(U_\alpha) < \delta \leq \frac{\delta'}{2}$ for all α and $D^*(x, g(x)) \geq \delta'$ for all $x \in L$ (and hence all $x \in \overline{C}$), g moves every point of $\overline{C}$ to a different element of $\mathcal{V}$; hence g either increases or decreases the index of every point in $\overline{C}$, so $R \cup S = \overline{C}$. Furthermore, R and S are disjoint by construction. We show that R and S are open in $\overline{C}$:

Let $s \in S$. Then we have several cases:

Case 1: $s \in V_{H,k}$ for some $k \leq m_H$ and $g(s) \in V_{H,k-j}$ for some j .

First, we observe that $j \geq 2$, since $\text{diam}(\overline{V_{H,k}} \cup \overline{V_{H,k+1}}) < \frac{\delta'}{2} + \frac{\delta}{2} = \delta'$. Then we note that $V_{H,k-j} \cap V_{H,k} = \emptyset$. Since $V_{H,k-j}$ is open, there exists an $\varepsilon > 0$ such that $B_{D^*}(g(s), \varepsilon) \subset V_{H,k-j}$. By continuity of g , there exists a $\gamma > 0$ such that for all $x \in \overline{C}$, if $D^*(x, s) < \gamma$, then $D^*(g(x), g(s)) < \varepsilon$; thus for all $x \in (\overline{C} \cap V_{H,k} \cap B_{D^*}(s, \gamma))$, $g(x) \in B_{D^*}(g(s), \varepsilon)$, so $x \in S$. Hence $N = (\overline{C} \cap V_{H,k} \cap B_{D^*}(s, \gamma))$ is a neighborhood of s contained in S .

Case 2.a: $s \in V_{H,k}$ for $k \leq m_H$ and $g(s) \in V_{i,k'}$ for some $i \neq H$, some k' .

We note that $V_{i,k'}$ is open and disjoint from $V_{H,k}$; we proceed as in Case 1 to obtain a neighborhood N of s with $N \subset S$.

Case 2.b: $s \in V_{H,k}$ for $k \leq m_H$ and $g(s) \in V_0$.

First, we observe that $k \geq 2$, since $\text{diam}(\overline{V_0} \cup \overline{V_{H,1}}) < \frac{\delta'}{2} + \frac{\delta}{2} = \delta'$. Then V_0 is open and disjoint from $V_{H,k}$; we again proceed as in Case 1 to obtain a neighborhood N of s with $N \subset S$.

Case 3: $s \in \overline{V_0}$ and $g(s) \in V_{i,k'}$ for some $i \neq H$ and some k' .

As shown above, g increases the index of all points in $\overline{V_0}$. Hence there is no $s \in \overline{V_0}$ such that $g(s) \in V_{i,k'}$ for some $i \neq H$ and some k' .

In all applicable cases, we obtain an open set N of $\overline{C}$ such that $s \in N \subset S$; therefore S is open in $\overline{C}$.

The proof that R is open in $\overline{C}$ has three cases, the first two of which are nearly identical to Cases 1 and 2 above, replacing $k - j$ and $k - 1$ with $k + j$

and $k + 1$, respectively. We handle the last case here:

Let $r \in R$. If $r \in \overline{V_0}$ and $g(r) \in V_{H,k}$ for some k , then $r \in U_0$ and $g(r) \in U_{H,k}$. Furthermore, $k \geq 2$, since $\text{diam}(U_0 \cup U_{H,1}) < \frac{\delta'}{2} + \frac{\delta'}{2} = \delta'$. For $k \geq 2$, $U_{H,k}$ is disjoint from U_0 , so $V_{H,k}$ is also disjoint from U_0 . Since $V_{H,k}$ is open, there exists an $\varepsilon > 0$ such that $B_{D^*}(g(r), \varepsilon) \subset V_{H,k}$. By continuity of g , there exists a $\gamma > 0$ such that for all $x \in \overline{C}$, if $D^*(x, r) < \gamma$, then $D^*(g(x), g(r)) < \varepsilon$; thus for all $x \in (\overline{C} \cap U_0 \cap B_{D^*}(r, \gamma))$, $g(x) \in B_{D^*}(g(r), \varepsilon)$, so $x \in R$. Hence $N = (\overline{C} \cap U_0 \cap B_{D^*}(r, \gamma))$ is a neighborhood of r contained in R .

In all cases, we obtain an open set N of $\overline{C}$ such that $r \in N \subset R$; therefore R is open in $\overline{C}$.

Thus R and S form a separation of $\overline{C}$, contradicting connectedness. Therefore the continuous function g must have a fixed point. $\square$

Theorem 4.3. *For each $n \in \mathbb{Z}_+$, let T_n be an n -ad, i.e., $T_1 = [0, 1]$, $T_2 = [0, 1] \vee [0, 1]$, T_3 is the triad, and so forth, so that T_n has n arms and T_{n+1} has $n + 1$ arms, for all n . For each $n \in \mathbb{Z}_+$, let $f_{n+1} : T_{n+1} \rightarrow T_n$ be any continuous function. Consider the inverse limit sequence $\{T_n, f_n\}$, and let $\Lambda = \varprojlim \{T_n, f_n\}$ denote its limit space. For each $n \in \mathbb{Z}_+$, let $\pi_n : \Lambda \rightarrow T_n$ denote projection onto the n^{th} coordinate. Suppose that in each coordinate, the inverse image $\pi_n^{-1}(\mathbf{0})$ of the center point (the origin $\mathbf{0}$) of the n -ad is connected. Then Λ has the fixed-point property.*

Proof. Copy the proof of Theorem 4.2 above, replacing L with Λ , T_r with T_n , and r with n throughout. $\square$

Remark. The argument of Theorem 4.2 in fact applies to a more general case with purely notational adjustments: any compact, connected metric space (X, ρ) that admits for each $\varepsilon > 0$ a continuous function $f : X \rightarrow T_n$ for some n -ad such that the pre-image $f^{-1}(\mathbf{0})$ is connected has the fixed-point property.

Theorem 4.4. *Let X be an inverse limit of compact, connected subspaces of $\mathbb{R}^K$, let $S_{m,n}$ be an (m,n) -ad, and suppose that, for each $\varepsilon > 0$, there exists a function $h : X \rightarrow S_{m,n}$ (for some m and n) such that $\text{diam}(h^{-1}(s)) < \varepsilon$ for all $s \in S_{m,n}$, and such that $h^{-1}(\langle 0, 0 \rangle)$ and $h^{-1}(\langle 1, 0 \rangle)$ are connected. Then X has the fixed-point property.*

Proof. Let $X = \varprojlim \{X_n, f_n\}$, where X_n is a compact, connected subspace of $\mathbb{R}^K$ for each $n \in \mathbb{Z}_+$, and suppose that, for each $\varepsilon > 0$, there exists a function $h : X \rightarrow S_{m,n}$ (for some m and n) such that $\text{diam}(h^{-1}(s)) < \varepsilon$ for all $s \in S_{m,n}$ and $h^{-1}(\langle 0, 0 \rangle)$ and $h^{-1}(\langle 1, 0 \rangle)$ are connected. Let $g : X \rightarrow X$ be a continuous function and suppose it has no fixed points. Choose (by Lemma 3.10) a $\delta' > 0$ such that $D^*(x, g(x)) \geq \delta'$ for all $x \in X$. Choose h such that $\text{diam}(h^{-1}(s)) < \frac{\delta'}{2}$ for all $s \in S_{m,n}$. Then $\text{diam}(h^{-1}(\langle 0, 0 \rangle)) < \frac{\delta'}{2}$ and $\text{diam}(h^{-1}(\langle 1, 0 \rangle)) < \frac{\delta'}{2}$, so $g(h^{-1}(\langle 0, 0 \rangle)) \cap h^{-1}(\langle 0, 0 \rangle) = \emptyset$ and $g(h^{-1}(\langle 1, 0 \rangle)) \cap h^{-1}(\langle 1, 0 \rangle) = \emptyset$.

Recall (from Definition 2.9) that A_i denotes the i^{th} arm of $S_{m,n}$, where A_0 is the shared arm, where $A_1, \dots, A_m$ are the arms emanating from $\langle 0, 0 \rangle$, and where $A_{m+1}, \dots, A_{m+n}$ are the arms emanating from $\langle 1, 0 \rangle$.

Then, since $h^{-1}(\langle 0, 0 \rangle)$ is connected (by hypothesis) and disjoint from $g(h^{-1}(\langle 0, 0 \rangle))$, $h(g(h^{-1}(\langle 0, 0 \rangle)))$ is contained in some connected component of $S_{m,n} - \{\langle 0, 0 \rangle\}$.

These connected components are precisely the sets $(A_i - \langle 0, 0 \rangle)$ for $1 \leq i \leq m$ and the set $\mathcal{A}_N = (A_0 - \langle 0, 0 \rangle) \cup A_{m+1} \cup A_{m+2} \cup \dots \cup A_{m+n}$. Similarly, since $h^{-1}(\langle 1, 0 \rangle)$ is connected (by hypothesis) and disjoint from $g(h^{-1}(\langle 1, 0 \rangle))$, $h(g(h^{-1}(\langle 1, 0 \rangle)))$ is contained in some connected component of $S_{m,n} - \{\langle 1, 0 \rangle\}$. These connected components are precisely the sets $(A_i - \langle 1, 0 \rangle)$ for $m+1 \leq i \leq m+n$ and the set $\mathcal{A}_M = (A_0 - \langle 1, 0 \rangle) \cup A_1 \cup A_2 \cup \dots \cup A_m$.

There are three cases:

Case 1: If $h(g(h^{-1}(\langle 0, 0 \rangle))) \subset (A_i - \langle 0, 0 \rangle)$ for some $1 \leq i \leq m$, then we have $g(h^{-1}(\langle 0, 0 \rangle)) \subset h^{-1}(A_i - \langle 0, 0 \rangle)$ for that i . Our situation is now nearly identical to that established in the second paragraph of the proof of Theorem 4.2; since we have confined $g(h^{-1}(\langle 0, 0 \rangle))$ to the pre-image under h of the $(m+1)$ -ad portion of $S_{m,n}$, we may proceed as in the proof of Theorem 4.2, using a bicentered chain cover rather than a centered chain cover (and adjusting the notation as necessary).

Case 2: If $h(g(h^{-1}(\langle 0, 0 \rangle))) \subset \mathcal{A}_N$ and $h(g(h^{-1}(\langle 1, 0 \rangle))) \subset (A_i - \langle 1, 0 \rangle)$ for some $m + 1 \leq i \leq m + n$, then we concern ourselves with $g(h^{-1}(\langle 1, 0 \rangle))$: we have $g(h^{-1}(\langle 1, 0 \rangle)) \subset h^{-1}(A_i - \langle 1, 0 \rangle)$ for that i . Similarly to Case 1, we have confined $g(h^{-1}(\langle 1, 0 \rangle))$ to the pre-image under h of the $(n + 1)$ -ad portion of $S_{m,n}$, so the rest of the proof follows as a modification of the proof of Theorem 4.2.

Case 3: $h(g(h^{-1}(\langle 0, 0 \rangle))) \subset \mathcal{A}_N$ and $h(g(h^{-1}(\langle 1, 0 \rangle))) \subset \mathcal{A}_M$. Since $h(g(h^{-1}(\langle 0, 0 \rangle))) \subset \mathcal{A}_N$, $g(h^{-1}(\langle 0, 0 \rangle)) \subset h^{-1}(\mathcal{A}_N)$. Thus $h^{-1}(\langle 0, 0 \rangle) \subset g^{-1}(h^{-1}(\mathcal{A}_N))$; note that $h^{-1}(\langle 0, 0 \rangle)$ is closed and $g^{-1}(h^{-1}(\mathcal{A}_N))$ is open, so $Y = X - (g^{-1}(h^{-1}(\mathcal{A}_N)))$ is closed and disjoint from $h^{-1}(\langle 0, 0 \rangle)$. Hence $\delta'' = \hat{D}(h^{-1}(\langle 0, 0 \rangle), Y) > 0$, so the $\frac{\delta''}{2}$ -neighborhood of $h^{-1}(\langle 0, 0 \rangle)$ is also contained in $g^{-1}(h^{-1}(\mathcal{A}_N))$. Note that the diameter of this neighborhood is at most $\frac{\delta'}{2} + \delta''$. Similarly, since $h(g(h^{-1}(\langle 1, 0 \rangle))) \subset \mathcal{A}_M$, $g(h^{-1}(\langle 1, 0 \rangle)) \subset h^{-1}(\mathcal{A}_M)$. Thus $h^{-1}(\langle 1, 0 \rangle) \subset g^{-1}(h^{-1}(\mathcal{A}_M))$; note that $h^{-1}(\langle 1, 0 \rangle)$ is closed and $g^{-1}(h^{-1}(\mathcal{A}_M))$ is open, so $Y' = X - (g^{-1}(h^{-1}(\mathcal{A}_M)))$ is closed and disjoint from $h^{-1}(\langle 1, 0 \rangle)$. Hence $\delta''' = \hat{D}(h^{-1}(\langle 1, 0 \rangle), Y') > 0$, so the $\frac{\delta'''}{2}$ -neighborhood of $h^{-1}(\langle 1, 0 \rangle)$ is also contained in $g^{-1}(h^{-1}(\mathcal{A}_M))$. Note that the diameter of this neighborhood is at most $\frac{\delta'}{2} + \delta'''$.

Set $\delta = \min \left\{ \frac{\delta'}{2}, \left(\frac{\delta'}{2} + \delta'' \right), \left(\frac{\delta'}{2} + \delta''' \right) \right\} = \frac{\delta'}{2}$.

Then we obtain (by Lemma 3.19) a bicentered chain cover $\mathcal{U}$ of X (with sets indexed as in Definition 3.18) with $\text{diam}(U_\alpha) < \delta$ for all $U_\alpha \in \mathcal{U}$ as the pre-image of a bicentered chain cover $\mathcal{B}$ on $S_{m,n}$, so U_μ is contained in the $\frac{\delta''}{2}$ -neighborhood of $h^{-1}(\langle 0, 0 \rangle)$ and thus $U_\mu \subset g^{-1}(h^{-1}(\mathcal{A}_N))$, and similarly U_ν is contained in the $\frac{\delta'''}{2}$ -neighborhood of $h^{-1}(\langle 1, 0 \rangle)$, so $U_\nu \subset g^{-1}(h^{-1}(\mathcal{A}_M))$. Furthermore, by choice of δ and $\mathcal{U}$, we have:

$$g(U_\mu) \cap U_\mu = \emptyset \quad (6)$$

$$g(U_\nu) \cap U_\nu = \emptyset \quad (7)$$

Thus $U_\mu \cap g^{-1}(U_\mu) = \emptyset$ and $U_\nu \cap g^{-1}(U_\nu) = \emptyset$, so we have

$$U_\mu \subset g^{-1}(h^{-1}(\mathcal{A}_N)) - g^{-1}(U_\mu) = g^{-1}(h^{-1}(\mathcal{A}_N - B_\mu)) \text{ and} \quad (8)$$

$$U_\nu \subset g^{-1}(h^{-1}(\mathcal{A}_M)) - g^{-1}(U_\nu) = g^{-1}(h^{-1}(\mathcal{A}_M - B_\nu)) \quad (9)$$

since $U_\mu = h^{-1}(B_\mu)$ and $U_\nu = h^{-1}(B_\nu)$.

For reference, we let $m_i = \max(J_i)$, for each i .

Now, we obtain (by compactness and connectedness of X) a precise refinement of $\mathcal{U}$ to an open cover $\mathcal{V}$ of X such that for all indices α , $\overline{V_\alpha} \subset U_\alpha$ and $\mathcal{V}$ is also a bicentered chain cover of X . Let $\overline{\mathcal{V}}$ denote the collection of the closures of elements of $\mathcal{V}$.

Observe that, by (6), $g(V_\mu) \cap V_\mu = \emptyset$. Also, by (8),

$$g(V_\mu) \subset g(\overline{V_\mu}) \subset g(U_\mu) \subset h^{-1}(\mathcal{A}_N - B_\mu). \quad (10)$$

Similarly, by (7), $g(V_\nu) \cap V_\nu = \emptyset$, and, by (9),

$$g(V_\nu) \subset g(\overline{V_\nu}) \subset g(U_\nu) \subset h^{-1}(\mathcal{A}_M - B_\nu). \quad (11)$$

Now let $x \in V_\nu$; let E denote the component of $X - (\overline{V_\mu})$ containing the point x . Consider the closure $\overline{E}$ of E in X : note that $\overline{E}$ is connected; we will show that $\overline{E}$ intersects $\overline{V_\mu}$. Since X is compact, connected, and metrizable (by Lemmas 3.4 and 3.5), since $X - (\overline{V_\mu})$ is open, and since E is a component of this open set, $\text{Bd}(X - (\overline{V_\mu}))$ contains a limit point of E (see [3], p. 47). But $\text{Bd}(X - (\overline{V_\mu})) = \text{Bd}(\overline{V_\mu}) \subset \overline{V_\mu}$, so $\overline{E} \cap \overline{V_\mu} \neq \emptyset$.

By choice of x , we also have

$$\overline{E} \cap \overline{V_\nu} \neq \emptyset. \quad (12)$$

Note also that the sets $\bigcup_{1 \leq i \leq m} V_{i,k} - \overline{V_\mu}$ and $\bigcup_{1 \leq i \leq n} V_{m+i,k} \cup (h^{-1}(A_0) - \overline{V_\mu})$ form a separation of $X - \overline{V_\mu}$, so $E \cap V_{i,k} = \emptyset$ for all $1 \leq i \leq m$, and thus $\overline{E} \cap (V_{i,k} - \overline{V_\mu}) = \emptyset$ for all $1 \leq i \leq m$.

Now let $y \in (\overline{E} \cap \overline{V_\nu})$, and let C denote the component of $F = \overline{E} - (\overline{V_\nu})$ containing the point y . Consider the closure $\overline{C}$ of C in $\overline{E}$: note that $\overline{C}$ is connected; we will show that $\overline{C}$ intersects $\overline{V_\nu}$. Note that $\overline{E}$ is compact because it is closed in the compact space X , that it is connected (as shown above), and that it is Hausdorff as a subspace of the Hausdorff space X . Furthermore, F is an open subset of $\overline{E}$ by (12). Now, since $\overline{E}$ is a compact,

connected Hausdorff space, since F is open, and since C is a component of this open set, $\text{Bd}(F)$ contains a limit point of C (see [3], p. 47). But $\text{Bd}(F) \subset \text{Bd}(\overline{V}_\nu) \subset \overline{V}_\nu$, so

$$\overline{C} \cap \overline{V}_\nu \neq \emptyset. \quad (13)$$

Furthermore, by choice of y ,

$$\overline{C} \cap \overline{V}_\mu \neq \emptyset. \quad (14)$$

Note also that, since the sets $F \cap (h^{-1}(A_0) - \overline{V}_\nu)$ and $F \cap \bigcup_{1 \leq i \leq n} V_{m+i,k} - (\overline{V}_\nu)$ form a separation of F , $C \cap V_{i,k} = \emptyset$ for all $m+1 \leq i \leq m+n$, and thus $\overline{C} \cap (V_{i,k} - \overline{V}_\nu) = \emptyset$ for all $m+1 \leq i \leq m+n$; since $\overline{C} \subset \overline{E}$, we also have $\overline{C} \cap (V_{i,k} - \overline{V}_\mu) = \emptyset$ for all $1 \leq i \leq m$. Hence

$$\overline{C} \subset (\overline{V}_\mu \cup V_{0,1} \cup V_{0,2} \cup \cdots \cup V_{0,m_0} \cup \overline{V}_\nu). \quad (15)$$

We now pause to introduce some terminology. There are two definitions:

We say that the function g **moves x forward** if we have either

1. $x \in \overline{V}_\mu$ and either
 - (a) $g(x) \in V_\nu$ or
 - (b) $g(x) \in V_{i,k}$ for $i = 0$ or $m+1 \leq i \leq m+n$; or
2. $x \in V_{0,k}$ for some k and either
 - (a) $g(x) \in V_\nu$,
 - (b) $g(x) \in V_{0,k+h}$ for some $h \geq 1$, or
 - (c) $g(x) \in V_{i,k'}$ for $m+1 \leq i \leq m+n$ and some k' .

We say that the function g **moves x backward** if we have either

1. $x \in \overline{V}_\nu$ and either
 - (a) $g(x) \in V_\mu$ or
 - (b) $g(x) \in V_{i,k}$ for $i = 0$ or $1 \leq i \leq m$; or
2. $x \in V_{0,k}$ for some k and either
 - (a) $g(x) \in V_\mu$,
 - (b) $g(x) \in V_{0,k-h}$ for some $h \geq 1$, or
 - (c) $g(x) \in V_{i,k'}$ for $1 \leq i \leq m$ and some k' .

We now define two subsets of $\overline{C}$:

Let $\mathcal{F} := \{x \in \overline{C} : g \text{ moves } x \text{ forward}\}$, and
let $\mathcal{B} := \{x \in \overline{C} : g \text{ moves } x \text{ backward}\}$.

We first show that $\mathcal{F}$ and $\mathcal{B}$ are both nonempty:

Recall that, by (10), $g(\overline{V}_\mu) \subset h^{-1}(\mathcal{A}_N - B_\mu)$ and that $\mathcal{A}_N = (A_0 - \langle 0, 0 \rangle) \cup A_{m+1} \cup A_{m+2} \cup \dots \cup A_{m+n}$; since $h^{-1}(A_i)$ is covered by the sets $V_{i,1}, \dots, V_{i,m_i}$ for each i and since $U_\mu = h^{-1}(B_\mu)$,

$$h^{-1}(\mathcal{A}_N - B_\mu) = (V_{0,1} - U_\mu) \cup V_{0,2} \cup \dots \cup V_{0,m_0} \cup V_\nu \cup \left(\bigcup_{1 \leq i \leq n} V_{m+i,k} \right).$$

Thus, for each $x \in \overline{V}_\mu$, $g(x) \in V_\nu$ or $g(x) \in V_{i,k}$ for $i = 0$ or $m+1 \leq i \leq m+n$; therefore g moves each x in $\overline{V}_\mu$ forward; since $\overline{C} \cap \overline{V}_\mu \neq \emptyset$ by (14), $\mathcal{F}$ is nonempty.

Similarly, by (11), we have that, for each $x \in \overline{V}_\nu$, $g(x) \in V_\mu$ or $g(x) \in V_{i,k}$ for $i = 0$ or $1 \leq i \leq m$, so g moves each x in $\overline{V}_\nu$ backward. Since $\overline{C} \cap \overline{V}_\nu \neq \emptyset$ by (13), $\mathcal{B}$ is also nonempty.

From (15) above, we have that $\overline{C} \subset (\overline{V}_\mu \cup V_{0,1} \cup V_{0,2} \cup \dots \cup V_{0,m_0} \cup \overline{V}_\nu)$; we note that, since $\text{diam}(V_\alpha) < \delta \leq \frac{\delta'}{2}$ for all α and $D^*(x, g(x)) \geq \delta$ for all $x \in X$ (and hence all $x \in \overline{C}$), g moves every point of $\overline{C}$ to a different element of $\mathcal{V}$; hence g moves every point of $\overline{C}$ either forward or backward, so $\mathcal{F} \cup \mathcal{B} = \overline{C}$. Furthermore, $\mathcal{F}$ and $\mathcal{B}$ are disjoint by construction. We show that $\mathcal{F}$ and $\mathcal{B}$ are open in $\overline{C}$:

Let $x \in \mathcal{F}$. Then we have several cases:

Case 3.1.a: $x \in \overline{V}_\mu$ and $g(x) \in V_\nu$.

Since $\mathcal{V}$ is a refinement of $\mathcal{U}$, $x \in U_\mu$ and $g(x) \in U_\nu$; since $\mathcal{U}$ is a bicentered chain cover, $U_\mu \cap U_\nu = \emptyset$, and hence $U_\mu \cap V_\nu = \emptyset$. Since V_ν is open, there exists an $\varepsilon > 0$ such that $B_{D^*}(g(x), \varepsilon) \subset V_\nu$. By continuity of g , there exists a $\gamma > 0$ such that for all $y \in \overline{C}$, if $D^*(y, x) < \gamma$, then $D^*(g(y), g(x)) < \varepsilon$; thus for all $y \in (\overline{C} \cap U_\mu \cap B_{D^*}(x, \gamma))$, $g(y) \in B_{D^*}(g(x), \varepsilon)$, so $y \in \mathcal{F}$. Hence $N = (\overline{C} \cap U_\mu \cap B_{D^*}(x, \gamma))$ is a neighborhood of x contained in $\mathcal{F}$.

Case 3.1.b: $x \in \overline{V_\mu}$ and $g(x) \in V_{i,k}$ for $i = 0$ or $m + 1 \leq i \leq m + n$.

Since $\mathcal{V}$ is a refinement of $\mathcal{U}$, $x \in U_\mu$ and $g(x) \in U_{i,k}$ for $i = 0$ or $m + 1 \leq i \leq m + n$; if $m + 1 \leq i \leq m + n$, $U_\mu \cap U_{i,k} = \emptyset$, and hence $U_\mu \cap V_{i,k} = \emptyset$. If $i = 0$, then $k \geq 2$, since $\text{diam}(U_\mu \cup U_{0,1}) < \frac{\delta'}{2} + \frac{\delta'}{2} = \delta'$. For $k \geq 2$, $U_{0,k}$ is disjoint from U_μ , so $V_{0,k}$ is also disjoint from U_μ . In either case, we have $V_{i,k} \cap U_\mu = \emptyset$. Note also that both $V_{i,k}$ and U_μ are open, and proceed as in Case 3.1.a to obtain an open set N of $\overline{C}$ such that $x \in N \subset \mathcal{F}$.

Case 3.2.a: $x \in V_{0,k}$ for some k and $g(x) \in V_\nu$.

We note that $k \leq m_0 - 1$, since $\text{diam}(V_{0,m_0} \cup V_\nu) < \frac{\delta'}{2} + \frac{\delta'}{2} = \delta'$. For $k \leq m_0 - 1$, $V_{0,k}$ is disjoint from V_ν . Note also that both $V_{0,k}$ and V_ν are open, and proceed as in Case 3.1.a to obtain an open set N of $\overline{C}$ such that $x \in N \subset \mathcal{F}$.

Case 3.2.b: $x \in V_{0,k}$ for some k and $g(x) \in V_{0,k+h}$ for some $h \geq 1$.

We note that $h \geq 2$ (and hence $k \leq m_0 - 2$), since $\text{diam}(V_{0,k} \cup V_{0,k+1}) < \frac{\delta'}{2} + \frac{\delta'}{2} = \delta'$. For $h \geq 2$, $V_{0,k}$ is disjoint from $V_{0,k+h}$. Note also that both $V_{0,k}$ and $V_{0,k+h}$ are open, and proceed as in Case 3.1.a to obtain an open set N of $\overline{C}$ such that $x \in N \subset \mathcal{F}$.

Case 3.2.c: $x \in V_{0,k}$ for some k and $g(x) \in V_{i,k'}$ for $m + 1 \leq i \leq m + n$ and some k' .

Note that, since $i \neq 0$, $V_{0,k} \cap V_{i,k'} = \emptyset$. Note also that both $V_{0,k}$ and $V_{i,k'}$ are open, and proceed as in Case 3.1.a to obtain an open set N of $\overline{C}$ such that $x \in N \subset \mathcal{F}$.

In all cases, we obtain an open set N of $\overline{C}$ such that $x \in N \subset \mathcal{F}$; hence $\mathcal{F}$ is open in $\overline{C}$.

The proof that $\mathcal{B}$ is open in $\overline{C}$ is nearly identical to the proof for $\mathcal{F}$, with adaptations that reflect the symmetry between the definitions of ‘moving forward’ and ‘moving backward’. Therefore $\mathcal{F}$ and $\mathcal{B}$ are disjoint, nonempty, open subsets of $\overline{C}$ whose union is $\overline{C}$, i.e., $\mathcal{F}$ and $\mathcal{B}$ separate $\overline{C}$. But $\overline{C}$ is connected, so we have a contradiction. Hence the continuous function $g : X \rightarrow X$ must have a fixed point. $\square$

Theorem 4.5. *Let (X, ρ) be a compact, connected metric space and suppose that, for each $\varepsilon > 0$, there exists a continuous function $f : X \rightarrow T$ for some tree T such that $\text{diam}(f^{-1}(t)) < \varepsilon$ for all $t \in T$, and such that the pre-image of each vertex of order at least 3 is connected. Then X has the fixed-point property.*

Proof. The proof of this theorem is similar in nature to that of the previous result, but we must prove that the situations are sufficiently similar before re-using the argument.

Without loss of generality, we may assume that the tree T is a series-reduced tree (meaning it has no vertices of order 2): if T is not series-reduced, then it is homeomorphic (via some homeomorphism h) to a series-reduced tree T^* , and so we may take $(h \circ f) : X \rightarrow T^*$ as our function with the desired properties. We may also assume that T has at least three vertices of order 3 or greater, since otherwise we have a case previously addressed in one of the earlier Theorems.

Now, we establish some convenient notation and terminology: in the tree T , we will refer to vertices of order at least 3 as “major” vertices and to vertices of order 1 as “minor” vertices belonging to the major vertex with which they share an edge. Let $v_1, \dots, v_n$ be the major vertices, and let $w_{i,k}$ be the minor vertices, where i is the index of the major vertex with which the vertex shares an edge. We will call edges which have a minor vertex as an endpoint “arms” and edges between two major vertices “ribs”. We will denote these respectively as follows: the arm $a_{i,k}$ has endpoints $w_{i,k}$ and v_i , and the rib $r_{i,j}$ has endpoints v_i and v_j . (Note that edges include their endpoints and are thus closed sets.) We also define the “minor subtree” M_i to be the union of all the arms $a_{i,k}$ that have the vertex v_i as an endpoint. Lastly, if two vertices u and v share an edge $e_{u,v}$, then let $T_{u,v}$ be the union of $e_{u,v} - \{v\}$ with all those points $t \in T$ such that any path from v to t must pass through u .

Now let $g : X \rightarrow X$ be a continuous function and suppose it has no fixed points. Choose (by Lemma 3.10) a $\delta' > 0$ such that $\rho(x, g(x)) \geq \delta'$ for all $x \in X$, and choose a function f such that $\text{diam}(f^{-1}(t)) < \frac{\delta'}{2}$ for all $t \in T$. Then $\text{diam}(f^{-1}(v_i)) < \frac{\delta'}{2}$ for each $1 \leq i \leq n$. Thus for each i , $g(f^{-1}(v_i)) \cap f^{-1}(v_i) = \emptyset$. Then, since $f^{-1}(v_i)$ is connected by hypothesis

and disjoint from $g(f^{-1}(v_i))$, $f(g(f^{-1}(v_i)))$ is contained in some connected component of $T - \{v_i\}$. These connected components are precisely the sets $(a_{i,k} - v_i)$ for all possible k , and the sets T_{v_j, v_i} for all vertices v_j with which v_i shares an edge. There are two cases:

Case 1: If $f(g(f^{-1}(v_i))) \subset (a_{i,k} - v_i)$ for some i and k , then we proceed as in the proof of Theorem 4.2, where a suitable neighborhood of $f^{-1}(v_i)$ plays the role of the cover element U_0 and the set $f^{-1}(a_{i,k})$ the role of the set $\pi_n^{-1}(A_H)$. The argument should be altered, of course, to obtain a tree cover of X of appropriate mesh rather than a centered chain cover.

Case 2: If, on the other hand, $f(g(f^{-1}(v_i))) \subset T - M_i$ for all i , then we show that we have a situation to which the argument of the proof of Theorem 4.4 (specifically Case 3) applies.

We need another new term: If v and u share the edge $e_{v,u}$, we shall say that v **moves toward** u under the function g if $f(g(f^{-1}(v))) \subset T_{u,v}$, where $T_{u,v}$ is as defined above. We note that by choice of δ' and f , every vertex of T must move toward some other vertex of T . Since (by assumption) no major vertex v_i moves toward $w_{i,k}$ for any of its minor vertices $w_{i,k}$, we have that all major vertices must move toward some other major vertex with which they share an edge.

Claim: There are two major vertices v_i and v_j of T that move toward each other under the function g :

Proof of Claim: First, note that removing a vertex of order one and the interior of the edge containing it from a tree still leaves a tree; thus the subgraph R of T consisting of the union of the ribs (which includes all the major vertices-) is a tree, since we obtain this subgraph by removing all of the minor vertices and arm interiors from T . Note also that any tree with three or more vertices must have at least one vertex of order two or greater; applying this to R , we see that some major vertex of T must share an edge with at least two other major vertices. We also note that any tree must have some vertex of order one, so again applying this to R , we have that some major vertex of T shares an edge with only one other major vertex of T . Now we proceed by induction:

We begin by checking the case that T has exactly three major vertices.

Suppose, without loss of generality, that v_2 shares an edge with both v_1 and v_3 . Then, since v_1 must move toward some major vertex with which it shares an edge, and likewise for v_3 , both v_1 and v_3 must move toward v_2 . Then v_2 must move toward either v_1 or v_3 , so we have a pair (v_1 and v_2 or v_2 and v_3) of vertices that move toward each other under the function g .

Now the induction step: Suppose that for any tree T with n major vertices, there are two major vertices of T that move toward each other under g . Let T have $n + 1$ major vertices. We note that T has a vertex v_i that shares an edge only with some vertex v_j , so v_i must move toward v_j . If v_j moves toward v_i , we're done. If v_j does not move toward v_i , then we note that $T - (M_i - \{v_i\})$ is a tree with n major vertices, so two of these n vertices must move toward one another under the restriction of g to $T - T_{v_i}$. In either case, we have obtained a pair of major vertices of T that move toward one another under g .

Now we note that our situation, namely $f^{-1}(v_i) \subset T_{v_j}$ and $f^{-1}(v_j) \subset T_{v_i}$, is precisely analogous to the situation of Case 3 of the proof of Theorem 4.4, where $h(g(h^{-1}(\langle 0, 0 \rangle))) \subset \mathcal{A}_N$ and $h(g(h^{-1}(\langle 1, 0 \rangle))) \subset \mathcal{A}_M$; we apply the argument from this case, obtaining a tree cover of X of appropriate mesh rather than a bicentered chain cover (and making necessary adjustments in notation). $\square$

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Vita

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