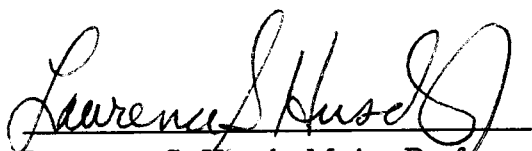


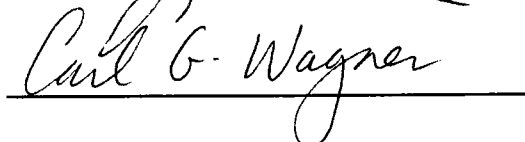
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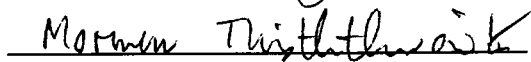
I am submitting herewith a dissertation written by John Wesley Emert entitled "Defining and Distinguishing 'Twisted' Lens Spaces." I have examined the final copy of this dissertation for form and content and recommend that it be accepted in partial fulfillment of the requirements for the degree of Doctor of Philosophy, with a major in Mathematics.


Lawrence S. Husch, Major Professor

We have read this dissertation
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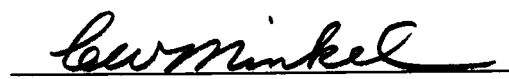








Accepted for the Council:


Vice Provost
and Dean of The Graduate School

DEFINING AND DISTINGUISHING
"TWISTED" LENS SPACES

A Dissertation
Presented for the
Doctor of Philosophy
Degree
The University of Tennessee, Knoxville

John Wesley Emert
August 1989

to my parents,
Howard and Dorthia Emert

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ABSTRACT

The lens space $L_{n,k}$ may be constructed by identifying certain pairs of faces of the suspension of a regular n -gon, so that the cone points are glued together. If the faces are paired together as in the construction of $L_{n,k}$, but the faces are identified with a twist, the resulting complex will be a three-manifold for certain k . Many of these manifolds are shown to be of distinct homotopy type, by an analysis of the number of distinct, irreducible representations of the fundamental group into $SO(3)$.

These manifolds are shown to be Seifert fibered, and are completely distinguished by an analysis of the base orbifold structure. Further, they are shown to be homeomorphic to the cyclic covers of S^3 , branched over the trefoil knot.

This geometric procedure generalizes to construct other classes of manifolds, including certain Brieskorn manifolds—namely, the cyclic coverings of S^3 , branched over an arbitrary torus knot. By similar methods, many of these manifolds are also shown to be distinct. In particular, it is shown that the cyclic covers of S^3 , branched over a fixed torus knot, are distinct.

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INTRODUCTION

In his 1895 paper, *Analysis Situs* [P, §10–13], Henri Poincaré introduces the construction of three-manifolds by the pairwise identification of faces of simply-connected solid polyhedra. Each resulting complex, though not necessarily a manifold, is however a three-manifold off the vertex set. Hence, to determine if an identification scheme produces a manifold, it suffices to determine the link of each vertex in the resulting complex. Poincaré determines these links by computing their Euler characteristic, and hence identifies and distinguishes certain resulting manifolds.

Seifert and Threlfall make use of this method of construction in their 1934 book, *Lehrbuch der Topologie* [S-T, §60]. However, only the lens spaces, the octahedron space, and the Poincaré sphere are given as immediate examples. In other work of Seifert and Threlfall [T-S], a thorough sweep of those manifolds with spherical geometry is made, identifying their fundamental groups. For example, the finite cyclic groups and the dihedral groups are represented by the lens and prism spaces, respectively.

Hans Friedg  [F], a doctoral student at The University of Frankfurt, shows in his 1939 dissertation that the Poincar  sphere is the five-fold cyclic cover of S^3 , branched over the trefoil. He further generalizes the construction of the Poincar  sphere to construct the other homology spheres which are cyclic covers of S^3 , branched over the trefoil.

In 1970, J.F.P. Hudson presented the paper, *Pseudo-Dodekahedral Spaces* [H] at the International Congress, in which he generalized Friedg 's result, using a different proof. The unpublished paper consists of a series of fourteen transparencies and a guide by which the cyclic covers of S^3 , branched over any $\tau(2, p)$ torus knot, may be viewed as a solid 3-manifold with pairwise identified faces. Figure O.1 illustrates the gluing scheme. The construction is, however, different from that of this paper, and does not appear to generalize to arbitrary torus knots. Further, Seifert fibrations of these spaces are not immediately apparent by this construction.

In this paper, the standard lens space construction, described in Section I.1, is first generalized to produce constructions of certain manifolds which I call *twisted lens spaces*. The homology of these manifolds is determined to be cyclic and identical to the homology of the covers of S^3 , branched over the trefoil. Many of these manifolds are distinguished by their fundamental group in Chapter I. This result is shown by counting the number of representations of each fundamental group into $SO(3)$, the Special Orthogonal group on $\mathbf{R}^3$, also thought of as the group of rotations on S^2 . To this end, several results of spherical geometry are shown.

In Chapter II, the twisted lens spaces are shown to have a Seifert fibration, which leads to the realization of these manifolds as the cyclic covers of S^3 , branched over the trefoil knot. By an analysis of their geometry, these manifolds are shown to have a unique Seifert fibration, and hence a unique underlying orbifold structure. These manifolds are then shown to be distinct, as their corresponding orbifolds are all distinct.

In Chapter III, the construction is generalized to construct what I call the *two-rule twisted lens spaces*, by allowing different identification procedures on

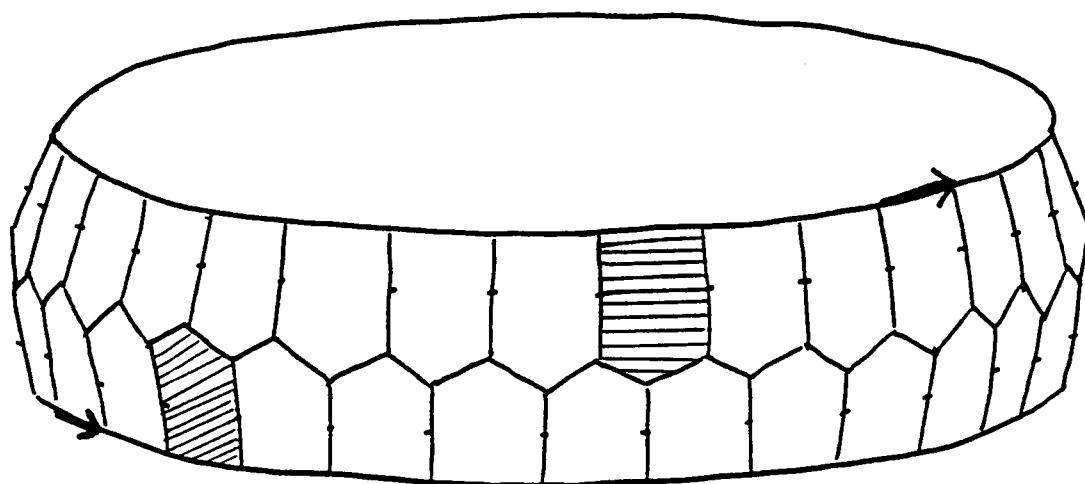


Figure O.1. The Gluing Scheme of Hudson.

alternate faces. This chapter consists primarily of a complete analysis of the possible complexes resulting from this type construction, identifying those constructions which result in manifolds. Among those manifolds which arise in this fashion are connected sums of $L_{3,1}$ and connected sums of $S^1 \times S^2$. An additional class of manifolds arise, and their homology is determined.

Chapter IV further generalizes the construction of lens spaces to consider the identification of faces with an arbitrary number of edges, as well as any number of rules. First, those constructions which result in Seifert fibered manifolds are identified. It is then shown that among these manifolds are all of the Brieskorn manifolds which are branched coverings over a torus **knot**. By methods similar to those of Chapter II, most of these manifolds are shown to be distinct. In particular, all of the cyclic coverings of S^3 , branched over a fixed torus knot, are shown to be distinct. However, some of the Brieskorn manifolds arise more than once, as branched covers over different torus knots. This situation is also discussed in Chapter IV.

The final chapter discusses related work of others, some further results, as well as some anticipated results. In particular, recent work of Peter Scott leads to the realization that the Brieskorn manifolds shown distinct in Chapter IV have in fact distinct fundamental groups.

CHAPTER I

CONSTRUCTING TWISTED LENS SPACES AND DISTINGUISHING THEIR FUNDAMENTAL GROUPS

§1. Lens Spaces.

The *Lens Spaces* $L_{n,k}$ ($k \in \mathbb{Z}_n$) may be formed in the following manner:

A regular n -gon (an n -sided polygon, together with its interior) is suspended, forming a solid whose boundary sphere is composed of $2n$ triangular faces and $3n$ edges. The boundary sphere is separated into upper and lower hemispheres by the boundary of the n -gon. The edges which compose the boundary of the n -gon are oriented, without loss of generality, in the counterclockwise direction, when viewed from the upper cone point. The boundary sphere is oriented so that the induced orientation of the boundary of each upper face agrees with the orientation on the boundary of the n -gon, and the induced orientation of the boundary of each lower face does not agree with the orientation on the boundary of the n -gon. Each upper face is identified in an orientation reversing fashion with the lower face located $\frac{k}{n}2\pi$ around the boundary of the n -gon (in the oriented direction), so that the cone points are identified. The resulting complex is the manifold $L_{n,k}$. (See [Sti, §4.2.8], for example.)

The purpose of this paper is to generalize this particular method of identifying the faces of the suspended n -gon, so as to produce other manifolds. However, the resulting complexes will not always be manifolds. These complexes have however a special structure.

DEFINITION. A *n-pseudo-manifold* is an n -dimensional simplicial complex in which each simplex is a face of at least one n -simplex and the complex is a connected n -manifold off its $(n-2)$ -skeleton. It is an observation of Poincaré [P] that each complex constructed by the identification of faces of a solid polyhedra is a three-pseudo-manifold, and is in fact a manifold off the vertices.

THEOREM I.1.1. A *three-pseudo-manifold* is a *three-manifold* exactly when its Euler characteristic is zero.

This result is shown by Stillwell [Sti, §8.2], for example.

§2. The Construction of Twisted Lens Spaces.

Define the *Twisted Lens Spaces* $L_{n,k,r}$ ($k \in \mathbb{Z}_n$, $r \in \{0, \frac{1}{3}, \frac{2}{3}\}$) as those complexes which are built in the following way:

As in the construction of the lens space $L_{n,k}$, a regular n -gon (an n -sided polygon, together with its interior) is suspended, forming a solid whose boundary sphere is composed of $2n$ triangular faces and $3n$ edges. The boundary sphere is separated into upper and lower hemispheres by the boundary of the n -gon. The edges which compose the boundary of the n -gon are oriented, without loss of generality, in the counterclockwise direction, when viewed from the upper cone point. The boundary sphere is oriented so that the induced orientation of the boundary of each upper face agrees with the orientation on the boundary of the n -gon, and the induced orientation of the boundary of each lower face does not agree with the orientation on the boundary of the n -gon. Each upper face is identified in an orientation reversing fashion with the lower face located $\frac{k}{n}2\pi$ around the boundary of the n -gon (in the oriented direction), so that the upper triangle is twisted r before being glued to the lower triangle. That is, if $r = 0$, the cone points are identified, and $L_{n,k,0} = L_{n,k}$. If $r = \frac{1}{3}$, the upper cone point is identified with

tail of the directed edge on the central n -gon— without loss of generality, the left vertex of the lower triangle. If $r = \frac{2}{3}$, the upper cone point is identified with the head of the directed edge on the central n -gon— without loss of generality, the right vertex of the lower triangle.

REMARK. The space which Seifert and Thelfall call the octahedral space in their book, *Lehrbuch der Topologie* [S-T, §61], is, in this notation, the twisted lens space $L_{4,2,\frac{1}{3}}$.

PROPOSITION I.2.1. *These constructions produce manifolds exactly when $r = 0$, when $r = \frac{1}{3}$ and $k = n - 2$, or when $r = \frac{2}{3}$ and $k = 2$.*

PROOF. As discussed above, $L_{n,k,0} = L_{n,k}$.

We can construct a homeomorphism between $L_{n,k,\frac{1}{3}}$ and $L_{n,n-k,\frac{2}{3}}$ induced by the extension of the homeomorphism between copies of S^1 with different orientations, first suspending to a map on S^2 , and then coning to a map on $B^3 \cong \Sigma_n$.

Therefore, it suffices to consider $r = \frac{1}{3}$.

Enumerate the vertices of the central n -gon as v_1 to v_n , respecting the orientation, and call the upper and lower cone points p and p' , respectively. Name the oriented edge from v_n to v_1 as a_1 , and name the oriented edge from v_{i-1} to v_i as a_i , for $i = 2$ to n . Name the oriented edges from p to v_i and from v_i to p' as a'_i and a''_i , respectively, for $i = 1$ to n .

Then the prescribed gluing identifies the following edges, where the indices are given mod n :

$$a_i \sim a''_{i+k}$$

$$a'_i \sim a''_{i+k-1}$$

$$a'_{i-1} \sim a_{i+k}.$$

By these identifications, each central vertex v_i is identified with each of the suspension points p, p' . Therefore, all of the vertices are identified under the gluing. The complex is a manifold exactly when its Euler characteristic is zero, by Theorem I.1.1. The resulting complex after gluing has one vertex, n faces, and one solid. Hence, the complex is a manifold precisely when it has n edges. Note that the prescribed gluings imply that, for all i ,

$$a_i \sim a''_{i+k} \sim a'_{i+1} \sim a_{i+k+2}.$$

Therefore the $3n$ edges are partitioned into equivalence classes, each containing at least one central, upper, and lower edge. The complex is a manifold precisely when each equivalence class contains exactly three edges. Equivalently, $a_i = a_{i+k+2}$; that is, $k = n - 2, \text{ mod } n$. ■

PROPOSITION I.2.2.

$$\pi_1(L_{n,n-2,\frac{1}{3}}) \cong G_n = \langle x_1, x_2, \dots, x_n \mid x_i x_{i+2} = x_{i+1} \text{ for all } i \in \mathbb{Z}_n \rangle$$

PROOF. The manifold $L_{n,n-2,\frac{1}{3}}$ as constructed above is a CW-complex $\mathcal{C}_n$ with one vertex, n edges, n faces, and one solid. The fundamental group of $\mathcal{C}_n$ is isomorphic to the fundamental group of $\mathcal{C}_n^{(2)}$, the two-skeleton of the CW complex, consisting of the vertex, the edges, and the faces of $\mathcal{C}_n$. Since the gluing map pairs each upper face with a lower face, $\mathcal{C}_n^{(2)}$ is represented by the faces which form the upper hemisphere, with the appropriate edges identified. Specifically, each edge a'_i is identified with a_{i-1} . The edges form a bouquet of n circles, representing the generators of the fundamental group. Let x_i be the generator represented by a_i . Each face represents a relator. The triangle with vertices p, v_{i+1}, v_{i+2} has as boundary $a'_{i+1} a_{i+2} a'^{-1}_{i+2} = a_i a_{i+2} a^{-1}_{i+1}$, and hence represents the relator $x_i x_{i+2} x^{-1}_{i+1}$. ■

LEMMA I.2.3. *Each G_n , for $n > 1$, can be generated by two elements.*

PROOF. Observe that Tietze transformations, applied to the presentation given in Proposition I.2.2, will eliminate the generators $x_n, x_{n-1}, \dots, x_3$, as follows.

$$G_n = \langle x_1, x_2, \dots, x_{n-1}, x_n \mid \\ x_1 x_3 = x_2, x_2 x_4 = x_3, \dots, x_{n-1} x_1 = x_n, x_n x_2 = x_1 \rangle.$$

Note that $x_n = x_1 x_2^{-1}$, so

$$G_n = \langle x_1, x_2, \dots, x_{n-1} \mid \\ x_1 x_3 = x_2, x_2 x_4 = x_3, \dots, \\ x_{n-3} x_{n-1} = x_{n-2}, x_{n-2} x_1 x_2^{-1} = x_{n-1}, x_{n-1} x_1 = x_1 x_2^{-1} \rangle.$$

By the final relator, we have that $x_{n-1} = x_1 x_2^{-1} x_1^{-1}$, so

$$G_n = \langle x_1, x_2, \dots, x_{n-2} \mid \\ x_1 x_3 = x_2, x_2 x_4 = x_3, \dots, \\ x_{n-4} x_{n-2} = x_{n-3}, x_{n-3} x_1 x_2^{-1} x_1^{-1} = x_{n-2}, x_{n-2} x_1 x_2^{-1} = x_1 x_2^{-1} x_1^{-1} \rangle \\ \vdots$$

$$G_n = \langle x_1, x_2 \mid r_1, r_2 \rangle,$$

where r_1 and r_2 are relations composed entirely of the letters x_1 and x_2 . ■

PROPOSITION I.2.4.

$$G_1 = 1$$

$$G_2 = \mathbf{Z}_3$$

$$G_3 = \mathbf{Q}_8, \text{ the quaternion group}$$

$$G_4 = \mathbf{T}_{24}, \text{ the tetrahedral group}$$

$$G_5 = \mathbf{P}_{120}, \text{ the Poincaré group}$$

PROOF. In each case, we apply Tietze transformations, as in the previous Lemma, transforming the presentation given in Proposition I.2.2 to a standard presentation of the group.

CASE 1.

$$\begin{aligned} G_1 &= \langle x_1 \mid x_1^2 = x_1 \rangle \\ &= \langle \mid \rangle. \end{aligned}$$

This is, of course, the trivial group.

CASE 2.

$$\begin{aligned} G_2 &= \langle x_1, x_2 \mid x_1^2 = x_2, x_2^2 = x_1 \rangle \\ &= \langle x_1 \mid x_1^4 = x_1 \rangle \\ &= \langle x_1 \mid x_1^3 = 1 \rangle. \end{aligned}$$

This is a presentation of the group $\mathbf{Z}_3$.

CASE 3.

$$\begin{aligned} G_3 &= \langle x_1, x_2, x_3 \mid x_1 x_3 = x_2, x_2 x_1 = x_3, x_3 x_2 = x_1 \rangle \\ &= \langle x_1, x_2 \mid x_1 x_1 x_2^{-1} = x_2, x_2 x_1 = x_1 x_2^{-1} \rangle \\ &= \langle x_1, x_2 \mid x_1^2 = x_2^2, x_2 x_1 x_2 = x_1 \rangle \\ &= \langle x_1, x_2 \mid x_1^2 = x_2^2, (x_2 x_1)^2 = x_1^2 \rangle. \end{aligned}$$

This is a known presentation of the quaternion group [C-M].

CASE 4.

$$\begin{aligned} G_4 &= \langle x_1, x_2, x_3, x_4 \mid x_1 x_3 = x_2, x_2 x_4 = x_3, x_3 x_1 = x_4, x_4 x_2 = x_1 \rangle \\ &= \langle x_1, x_2, x_3 \mid x_1 x_3 = x_2, x_2 x_1 x_2^{-1} = x_3, x_3 x_1 = x_1 x_2^{-1} \rangle \\ &= \langle x_1, x_2 \mid x_1^2 x_2^{-1} x_1^{-1} = x_2, x_2 x_1 x_2^{-1} = x_1 x_2^{-1} x_1^{-1} \rangle \\ &= \langle x_1, x_2 \mid x_1^2 = x_2 x_1 x_2, (x_2 x_1 x_2) x_2^{-3} (x_2 x_1 x_2) = x_1 \rangle. \end{aligned}$$

By applying the first relator to the second relator, we get

$$G_4 = \langle x_1, x_2 \mid x_1^2 = x_2 x_1 x_2, x_1^2 x_2^{-3} x_1^2 = x_1 \rangle.$$

After multiplying each side of the first relator by x_1 and simplifying the second relator, we get

$$G_4 = \langle x_1, x_2 \mid x_1^3 = x_2^3 = (x_2 x_1)^2 \rangle,$$

which is a known presentation of the tetrahedral group [C-M].

CASE 5.

$$\begin{aligned} G_5 &= \langle x_1, x_2, x_3, x_4, x_5 \mid \\ &\quad x_1 x_3 = x_2, x_2 x_4 = x_3, x_3 x_5 = x_4, x_4 x_1 = x_5, x_5 x_2 = x_1 \rangle \\ &= \langle x_1, x_2, x_3, x_4 \mid \\ &\quad x_1 x_3 = x_2, x_2 x_4 = x_3, x_3 x_1 x_2^{-1} = x_4, x_4 x_1 = x_1 x_2^{-1} \rangle \\ &= \langle x_1, x_2, x_3 \mid \\ &\quad x_1 x_3 = x_2, x_2 x_1 x_2^{-1} x_1^{-1} = x_3, x_3 x_1 x_2^{-1} = x_1 x_2^{-1} x_1^{-1} \rangle \\ &= \langle x_1, x_2 \mid \\ &\quad x_1^2 x_2^{-1} x_1^{-1} x_2 x_1^{-1} = x_2, x_2 x_1 x_2^{-1} x_1^{-1} = x_1 x_2^{-1} x_1^{-1} x_2 x_1^{-1} \rangle. \end{aligned}$$

After simplifying each relator, we get

$$G_5 = \langle x_1, x_2 \mid x_1^2 = x_2 x_1 x_2^{-1} x_1 x_2, x_2^2 = x_1 x_2 x_1^{-1} x_2 x_1 \rangle.$$

After right multiplication by $x_1^{-1} x_2 x_1$, the first relator is changed to

$$G_5 = \langle x_1, x_2 \mid x_1 x_2 x_1 = x_2 x_1 x_2^{-1} (x_1 x_2 x_1^{-1} x_2 x_1), x_2^2 = x_1 x_2 x_1^{-1} x_2 x_1 \rangle,$$

and by applying the second relator to the first, we get

$$G_5 = \langle x_1, x_2 \mid x_1 x_2 x_1 = x_2 x_1 x_2, x_2^2 = x_1 x_2 x_1^{-1} x_2 x_1 \rangle.$$

After left multiplication by x_2 and right multiplication by x_2^2 on the second relator, we get

$$G_5 = \langle x_1, x_2 \mid x_1 x_2 x_1 = x_2 x_1 x_2, x_2^5 = (x_2 x_1 x_2) x_1^{-1} (x_2 x_1 x_2) x_2 \rangle,$$

and by applying the first relator to the second, we get

$$G_5 = \langle x_1, x_2 \mid x_1 x_2 x_1 = x_2 x_1 x_2, x_2^5 = (x_1 x_2 x_1) x_1^{-1} (x_1 x_2 x_1) x_2 \rangle,$$

which can be rewritten as

$$G_5 = \langle x_1, x_2 \mid x_1 x_2 x_1 = x_2 x_1 x_2, x_2^5 = x_1 x_2 x_1 x_2 x_1 x_2 = (x_1 x_2)^3 \rangle.$$

After right multiplication by $x_2 x_1 x_2$ on the first relator, we get

$$G_5 = \langle x_1, x_2 \mid (x_1 x_2)^3 = (x_2 x_1 x_2)^2, x_2^5 = (x_1 x_2)^3 \rangle,$$

which, after applying the second relator to the first, becomes

$$G_5 = \langle x_1, x_2 \mid x_2^5 = [x_2 (x_1 x_2)]^2, x_2^5 = (x_1 x_2)^3 \rangle.$$

By introducing a new generator $z = x_1 x_2$, we get

$$G_5 = \langle x_2, z \mid x_2^5 = (x_2 z)^2 = z^3 \rangle,$$

which is a known presentation of the Poincaré group [C-M]. ■

LEMMA I.2.5.

$$x_i^{x_{i+3}} \equiv x_{i+3}^{-1} x_i x_{i+3} = x_{i+6}$$

PROOF. Since $x_i x_{i+2} = x_{i+1}$ and $x_{i+2} x_{i+4} = x_{i+3}$, it follows that $x_i x_{i+3} = x_i x_{i+2} x_{i+4} = x_{i+1} x_{i+4}$. By induction, $x_i x_{i+3} = x_{i+q} x_{i+3+q}$, for all q . Therefore, $x_i x_{i+3} = x_{i+3} x_{i+6}$, and the result follows. ■

PROPOSITION I.2.6.

$$Ab(G_n) = H_1(L_{n,n-2,1/3}) \cong \begin{cases} 0 & \text{if } (n, 6) = 1 \\ \mathbf{Z}_3 & \text{if } (n, 6) = 2 \\ \mathbf{Z}_2 \oplus \mathbf{Z}_2 & \text{if } (n, 6) = 3 \\ \mathbf{Z} \oplus \mathbf{Z} & \text{if } (n, 6) = 6, \end{cases}$$

where $Ab(G)$ is the Abelianization of the group G , and $(a, b) \equiv \gcd(a, b)$.

PROOF. The Abelianization of G , $Ab(G)$, can be presented by adding the commutators $[x_i, x_j]$, for all $i, j \in \mathbf{Z}_n$, as relators to the presentation of G given in Proposition I.2.2. By Lemma I.2.5, $x_i = x_{i+6}$ for all i , in $Ab(G_n)$. Therefore, $x_i = x_{i+6k}$, for any positive integer k .

CASE 1. If $(n, 6) = 6$, then the presentation of $Ab(G_n)$ described above can be further reduced, using Tietze transformations, by eliminating the generators x_3, x_4, x_5, x_6 and all of the relators except the commutator $[x_1, x_2]$, as follows.

$$Ab(G_n) = \langle x_1, x_2, \dots, x_6 \mid [x_i, x_j], x_i x_{i+2} = x_{i+1} \text{ for all } i, j \in \mathbf{Z}_6 \rangle.$$

Since $x_6 = x_1 x_2^{-1}$, we can replace the relators $[x_i, x_6]$ by the relators $[x_i, x_1 x_2^{-1}]$, $i \in \{1, 2, 3, 4, 5\}$. But since we have relators $[x_i, x_1]$ and $[x_i, x_2]$ included already, these additional relators are superfluous. We replace x_6 by $x_1 x_2^{-1}$ in the remaining relators, and get

$$\begin{aligned} Ab(G_n) = \langle x_1, x_2, \dots, x_5 \mid \\ [x_i, x_j] \text{ for all } i, j \in \{1, 2, 3, 4, 5\}, \\ x_1 x_3 = x_2, x_2 x_4 = x_3, x_3 x_5 = x_4, x_4 x_1 x_2^{-1} = x_5, x_5 = x_2^{-1} \rangle. \end{aligned}$$

Now, since $x_5 = x_2^{-1}$, the relators $[x_i, x_5] = [x_i, x_2^{-1}]$, $i \in \{1, 2, 3, 4\}$ are superfluous. Therefore,

$$\begin{aligned} Ab(G_n) = \langle x_1, x_2, x_3, x_4 \mid \\ [x_i, x_j] \text{ for all } i, j \in \{1, 2, 3, 4\}, \\ x_1 x_3 = x_2, x_2 x_4 = x_3, x_3 x_2^{-1} = x_4, x_4 x_1 = 1 \rangle. \end{aligned}$$

Note that $x_4 = x_1^{-1}$, so the relators $[x_i, x_4] = [x_i, x_1^{-1}]$, $i \in \{1, 2, 3\}$ are superfluous. Therefore,

$$Ab(G_n) = \langle x_1, x_2, x_3 | [x_i, x_j] \text{ for all } i, j \in \{1, 2, 3\}, x_1 x_3 = x_2, x_2 x_1^{-1} = x_3, x_3 x_2^{-1} = x_1^{-1} \rangle.$$

By the third relator, we have $x_3 = x_1^{-1} x_2$, so the relators $[x_i, x_3] = [x_i, x_1^{-1} x_2]$, $i \in \{1, 2, 3\}$ are superfluous. When x_3 is replaced by $x_1^{-1} x_2$ in the remaining relators, we get the presentation

$$\begin{aligned} Ab(G_n) &= \langle x_1, x_2 | [x_1, x_2], x_2 = x_2, x_2 x_1^{-1} = x_1^{-1} x_2 \rangle \\ &= \langle x_1, x_2 | [x_1, x_2] \rangle. \end{aligned}$$

This is a presentation of the group $\mathbf{Z} \oplus \mathbf{Z}$.

CASE 2. If $(n, 6) = 1$, then $n = 6k \pm 1$, for some integer k . Therefore, $i + 6k = i + (6k \pm 1) \mp 1 = i + n \mp 1$, so

$$x_i = x_{i+6k} = x_{i+n \mp 1} = x_{i \mp 1}.$$

It follows that $x_i = x_1$, for all i . Hence, the presentation of $Ab(G_n)$ described above reduces to the trivial group

$$\langle x_1 | x_1^2 = x_1 \rangle = \langle \mid \rangle.$$

CASE 3. If $(n, 6) = 2$, then $n = 6k \pm 2$, for some integer k . Therefore, $i + 6k = i + (6k \pm 2) \mp 2 = i + n \mp 2$, so

$$x_i = x_{i+6k} = x_{i+n \mp 2} = x_{i \mp 2}.$$

It follows that $x_i = x_{i+2}$, for all i . Therefore, the presentation of $Ab(G_n)$ described above reduces to

$$\langle x_1, x_2 | [x_1, x_2], x_1^2 = x_2, x_2^2 = x_1 \rangle = \langle x_1 | x_1^3 = 1 \rangle.$$

CASE 4. If $(n, 6) = 3$, then $n = 6k + 3$, for some integer k . Therefore, $i + 6k = i + (6k + 3) - 3 = i + n - 3$, so

$$x_i = x_{i+6k} = x_{i+n-3} = x_{i-3}.$$

Therefore, the presentation of $Ab(G_n)$ described above reduces to

$$\begin{aligned} \langle x_1, x_2, x_3 | [x_1, x_2], [x_2, x_3], [x_1, x_3], x_1 x_3 = x_2, x_2 x_1 = x_3, x_3 x_2 = x_1 \rangle \\ = \langle x_1, x_2 | [x_1, x_2], x_1^2 = x_2^2 = 1 \rangle. \end{aligned}$$

This is the group $\mathbf{Z}_2 \oplus \mathbf{Z}_2$. ■

PROPOSITION I.2.7. *The group G_n is the fundamental group of the n -fold cyclic cover of S^3 , branched over the trefoil knot, for $n \leq 6$.*

This proposition is shown by Rolfsen [R, §10.D], for example. By Theorem II.5.1, we will show that this result is not unexpected.

§3. Preliminaries Regarding $SO(3)$.

$SO(3)$ is the Special Orthogonal Group on $\mathbf{R}^3$, the group of orientation preserving orthogonal transformations on $\mathbf{R}^3$. $SO(3)$ can be described as the group of 3×3 real matrices A such that $A^t = A^{-1}$ and $\det(A) = 1$. $SO(3)$ can also be described as the group of orientation preserving orthogonal transformations on S^2 ; that is, the group of rotations on S^2 . (Each element A , restricted to the unit sphere $S^2 \subset \mathbf{E}^3$ is an automorphism on S^2 , since $\|x\|_2 = \|Ax\|_2$.) See [B],[B-K].

DEFINITION. Two elements of $SO(3)$ are *distinct* if they are not conjugate by an element of $SO(3)$.

Each nontrivial $\varphi \in SO(3)$, as a rotation on S^2 , fixes exactly two points, a pair of antipodal points. Let $fp(\varphi)$ be the set consisting of these two points.

The Orthogonal Group on $\mathbf{R}^3$, $O(3)$, is the group of (not necessarily orientable) orthogonal transformations on $\mathbf{R}^3$. It can also be described as the group of orthogonal transformations on S^2 , the group of reflections across geodesics and rotations on S^2 . Every reflection is orientation reversing, so the composition of any pair of reflections is orientation preserving, and therefore an element of $SO(3)$.

Let $\rho(g) \in O(3)$ denote the reflection of S^2 about the geodesic $g \subseteq S^2$. Multiplication in G_n is written left to right, but composition in $SO(3)$ and $O(3)$, as functions, will be written right to left.

We will need the following Lemmas.

LEMMA I.3.1. *For any nontrivial $\varphi \in SO(3)$ and any geodesic $g \supset fp(\varphi)$, there exists a unique geodesic $g' \neq g$ such that $\varphi = \rho(g')\rho(g)$. Further, $g' \cap g = fp(\varphi)$.*

PROOF. The geodesic $\varphi(g)$ contains $fp(\varphi)$, but is distinct from g , since φ is nontrivial. Therefore, $\varphi(g) \cap g = fp(\varphi)$, and the geodesics g and $\varphi(g)$ meet at angles α and $\pi - \alpha$, for some $\alpha \in (0, \pi)$. Let $w \subset g$ be a segment whose endpoints are $fp(\varphi)$. Let g' be the geodesic passing through $fp(\varphi)$ which bisects the angle between w and $\varphi(w)$. Note that $\rho(g')\rho(g)(w) = \varphi(w) \equiv w'$ by construction. Since $\rho(g'), \rho(g), \varphi$ are isometries, $\rho(g')\rho(g)(x) = x' = \varphi(x)$, where x, x' are the midpoints of w, w' , respectively. Since $\rho(g')\rho(g)$ fixes $fp(\varphi)$, φ and $\rho(g')\rho(g)$ agree on at least three points. But elements of $SO(3)$ are determined by their action on three points, so $\varphi = \rho(g')\rho(g)$.

Now, suppose g'' is some geodesic such that $\rho(g'')\rho(g) = \varphi$. Then $\rho(g'')(x) = x'$ and $\rho(g'')$ fixes $fp(\varphi)$. Therefore, $\rho(g'')$ and $\rho(g')$ are equal, as they agree on three points. In particular, they are reflections about the same geodesic, $g' = g''$. Therefore, g' is unique. ■

LEMMA I.3.2. Three distinct geodesics on S^2 determine four pair of antipodal, isometric triangles, whose angles are

$$(\alpha, \beta, \gamma), (\bar{\alpha}, \bar{\beta}, \gamma), (\bar{\alpha}, \beta, \bar{\gamma}), (\alpha, \bar{\beta}, \bar{\gamma}),$$

where $\{\alpha, \beta, \gamma\} \subset (0, \pi)$ and $\bar{\delta} = \pi - \delta$ for any angle δ .

PROOF. This Lemma is easily verified by considering Figure I.1. ■

DEFINITIONS. The representation $\phi : G \rightarrow SO(3)$ is *irreducible* if the nontrivial elements of $\phi(G)$ do not all have the same fixed points. Two representations $\phi, \tilde{\phi}$ are *not distinct* if there exists some $\xi \in SO(3)$ such that, for all $\gamma \in G$, $\phi(\gamma)^\xi = \tilde{\phi}(\gamma)$. In compact form, this will be written as $\phi^\xi = \tilde{\phi}$.

§4. Representations of G_n into $SO(3)$.

The goal of this section is to establish the following theorem.

THEOREM I.4.1. Suppose that n is a positive integer not divisible by 3. Then, the number of distinct irreducible representations of G_n into $SO(3)$ is

$$\begin{aligned} \left[\frac{n}{3} \right] & \quad \text{when } n \text{ is even,} \\ \left[\frac{n+1}{3} \right] & \quad \text{when } n \text{ is odd,} \end{aligned}$$

where $[x]$ is the greatest integer less than or equal to x .

To this end, we will show that each distinct irreducible representation gives rise to a collection of equilateral triangles on S^2 of distinct size. Certain mappings of these collections will restrict to mappings of S^1 to itself of distinct degrees. Bounds on the degree will lead to the data found in Theorem I.4.1. First, however, we shall verify the theorem for the trivial cases of $n = 1$ and $n = 2$. Thereafter, we will assume that $n > 2$ within this section.

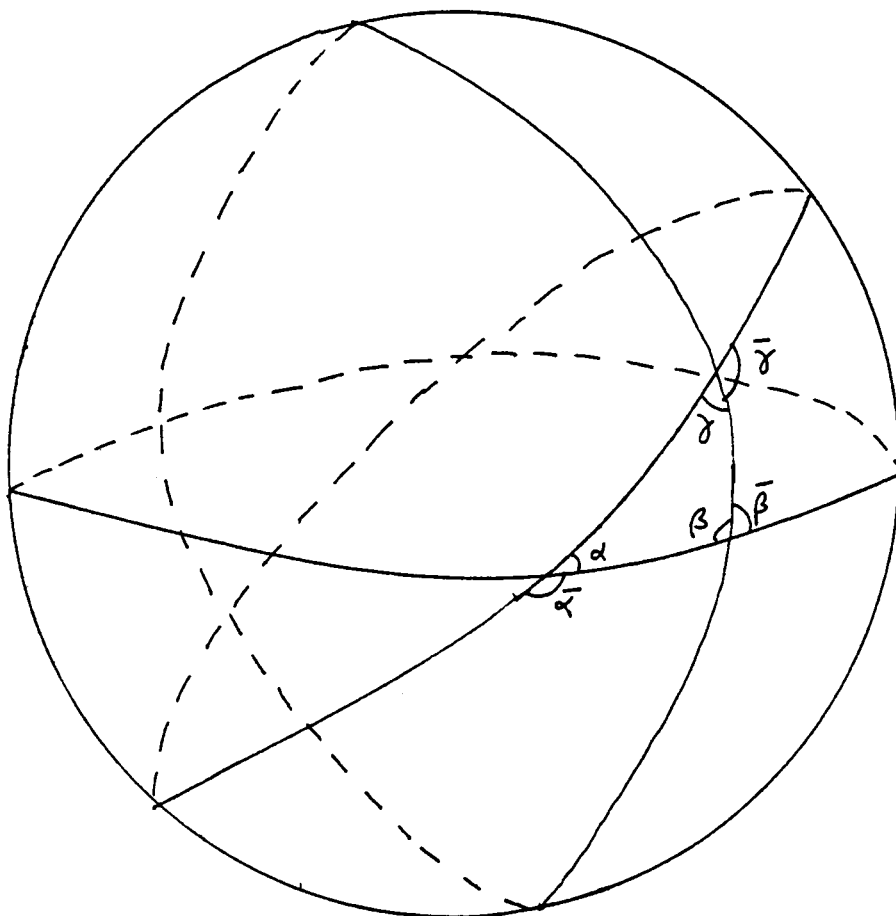


Figure I.1. Three Distinct Geodesics on S^2 .

LEMMA I.4.2. *There are no irreducible representations of G_1 or G_2 into $SO(3)$.*

PROOF. G_1 has no nontrivial elements. G_2 consists only of two nontrivial elements, x and x^2 . Then, for any representation $\phi : G_2 \rightarrow SO(3)$, $\phi(x)^2 = \phi(x^2)$ and $\phi(x)$ share the same fixed points. Hence, no representation of G_2 into $SO(3)$ is irreducible. ■

Suppose that $G_n \xrightarrow{\phi} SO(3)$ is an irreducible representation. By Lemma I.2.3, G_n is generated by two elements x_1 and x_2 . Therefore, the representation ϕ is determined by $\phi(x_1)$ and $\phi(x_2)$, which we shall denote as ψ_1 and ψ_2 , respectively. For any product $yy' \in G$, $\phi(yy') = \phi(y')\phi(y)$, since composition in $SO(3)$ is written in the opposite order.

LEMMA I.4.3. *If ϕ is an irreducible representation, ψ_1 and ψ_2 are distinct and nontrivial. In addition, $fp(\psi_1) \neq fp(\psi_2)$.*

PROOF. If ψ_1 is trivial, then $\phi(G_n)$ is generated by powers of ψ_2 . Therefore, $\phi(G_n)$ consists entirely of rotations fixing $fp(\psi_2)$, and hence is not irreducible.

Similarly, if $fp(\psi_1) = fp(\psi_2)$, then each element of the subgroup generated by ψ_1 and ψ_2 also fixes $fp(\psi_1)$. Hence, ϕ is not irreducible. ■

Let l_1 be the geodesic containing $fp(\psi_1) \cup fp(\psi_2)$. By Lemma I.3.1, there exist geodesics l'_1 and l_2 such that

$$\psi_1 = \rho(l'_1)\rho(l_1)$$

$$\psi_2 = \rho(l_2)\rho(l_1).$$

Note that $l'_1 \neq l_2$, as they meet l_1 in distinct points, namely, $fp(\psi_1)$ and $fp(\psi_2)$, respectively. Define the nontrivial rotation

$$\psi_3 \equiv \psi_2\psi_1^{-1} = \rho(l_2)\rho(l'_1)$$

and note that the fixed points of ψ_1, ψ_2, ψ_3 are all distinct. Indeed, $fp(\psi_3) = l_2 \cap l'_1$.

Further, since $x_3 = x_1^{-1}x_2$,

$$\phi(x_3) = \phi(x_2)\phi(x_1^{-1}) = \psi_2\psi_1^{-1} = \psi_3.$$

Inductively, given a geodesic l_{i-1} and rotations ψ_{i-1} , and ψ_i , where $fp(\psi_i) \neq fp(\psi_{i-1})$, there exist geodesics l'_{i-1} and l_i such that

$$\psi_{i-1} = \rho(l'_{i-1})\rho(l_{i-1})$$

$$\psi_i = \rho(l_i)\rho(l_{i-1}).$$

The geodesics l_i and l'_{i-1} are distinct, since they intersect l_{i-1} in distinct points.

We can therefore define the nontrivial rotation

$$\psi_{i+1} = \psi_i\psi_{i-1}^{-1} = \rho(l_i)\rho(l'_{i-1}).$$

Note that the fixed points of $\psi_{i-1}, \psi_i, \psi_{i+1}$ are all distinct. Indeed, $fp(\psi_{i+1}) = l_i \cap l'_{i-1}$.

LEMMA I.4.4. For all i ,

$$(1) \psi_{n+i} = \psi_i,$$

$$(2) l_{n+i} = l_i \text{ and } l'_{n+i} = l'_i.$$

PROOF.

PART 1. By definition, $\psi_{i+1} = \psi_i\psi_{i-1}^{-1}$, for all $i > 1$. By construction, $\psi_1 = \phi(x_1)$ and $\psi_2 = \phi(x_2)$. Therefore, by induction,

$$\begin{aligned} \psi_{i+1} &= \psi_i\psi_{i-1}^{-1} \\ &= \phi(x_i)\phi(x_{i-1}^{-1}) \\ &= \phi(x_{i-1}^{-1}x_i) \\ &= \phi(x_{i+1}), \end{aligned}$$

for $1 < i < n$.

Further,

$$\psi_{n+1} = \psi_n \psi_{n-1}^{-1} = \phi(x_n) \phi(x_{n-1}^{-1}) = \phi(x_{n-1}^{-1} x_n) = \phi(x_1) = \psi_1.$$

The result then follows by induction:

$$\begin{aligned} \psi_{n+i+1} &= \psi_{n+i} \psi_{n+i-1}^{-1} \\ &= \phi(x_i) \phi(x_{i-1}^{-1}) \\ &= \phi(x_{i+1}) \\ &= \psi_{i+1}, \end{aligned}$$

for $i > 1$.

PART 2. Note that each ψ_i has been expressed as a composition of reflections in each of the following ways, for any positive integer k :

$$\begin{aligned} \psi_i &= \rho(l_{i-1}) \rho(l'_{i-2}) = \rho(l_i) \rho(l_{i-1}) = \rho(l'_i) \rho(l_i) \\ &= \psi_{i+kn} = \rho(l_{i+kn-1}) \rho(l'_{i+kn-2}) = \rho(l_{i+kn}) \rho(l_{i+kn-1}) = \rho(l'_{i+kn}) \rho(l_{i+kn}). \end{aligned}$$

We have that $\psi_i = \rho(l'_i) \rho(l_i) = \rho(l_{i+n}) \rho(l_{i+n-1})$ and $\psi_{i+1} = \rho(l_{i+1}) \rho(l_i) = \rho(l_{i+n+1}) \rho(l_{i+n})$. Therefore, by Lemma I.3.1, the geodesics l_i and l_{i+n} share the four points $fp(\psi_i) \cup fp(\psi_{i+1})$, and are hence identical. The geodesics l'_i and l'_{i+n} are shown to be identical by a similar argument. ■

LEMMA I.4.5. For any i ,

$$\begin{aligned} \rho(l_{i+1})(l_i) &= l'_{i+1} \\ \rho(l_{i+1})(l'_i) &= l_{i+2} \end{aligned}$$

PROOF. Note that, for any point $p \in l_i$,

$$\rho(l'_{i+1}) \rho(l_{i+1})(p) = \psi_{i+1}(p) = \rho(l_{i+1}) \rho(l_i)(p) = \rho(l_{i+1})(p).$$

Since $\rho(l'_{i+1})$ fixes the geodesic $\rho(l_{i+1})(l_i)$ pointwise, $\rho(l_{i+1})(l_i) = l'_{i+1}$. Similarly, $\rho(l_{i+1})(l'_i) = l_{i+2}$, as $\rho(l_{i+2})$ fixes the geodesic $\rho(l_{i+1})(l'_i)$ pointwise. ■

LEMMA I.4.6. *For every i , the geodesics l_i and l'_i meet at the same angles as l_{i+3} and l'_{i+3} .*

PROOF. A reflection about a geodesic preserves the angles between any two geodesics and their images. Therefore, the angles between l_i and l'_i are equal to the angles between

$$\rho(l_{i+3})\rho(l_{i+2})\rho(l_{i+1})(l_i)$$

and

$$\rho(l_{i+3})\rho(l_{i+2})\rho(l_{i+1})(l'_i).$$

However, by Lemma I.4.5, these are the geodesics l_{i+3} and l'_{i+3} . ■

LEMMA I.4.7. *If n is not divisible by 3, then for every i ,*

- (1) *every pair of geodesics l_i, l'_i meet at angles δ and $\bar{\delta}$, for some fixed $\delta \in (0, \pi)$.*
- (2) *the geodesics l_i, l'_i, l_{i+1} bound a pair of equilateral triangles of fixed side length $d \in (0, \frac{2\pi}{3})$ and fixed angle $\delta \in (\frac{\pi}{3}, \pi)$.*

PROOF. Fix i , and note that the geodesics l_i, l'_i, l_{i+1} intersect pairwise in six points, $fp(\psi_i) \cup fp(\psi_{i+1}) \cup fp(\psi_{i+2})$. Choose one of the eight triangles bounded. Its vertices will consist of one point from each of $fp(\psi_i), fp(\psi_{i+1}), fp(\psi_{i+2})$. Label the angles at these vertices α, β, γ , respectively.

By Lemma I.4.6, the pairs of geodesics l_{i+3k}, l'_{i+3k} meet at fixed angles α and $\bar{\alpha}$ for all k . Assuming that $n = 3k \pm 1$ for some k , then $i + 3k = i + (3k \pm 1) \mp 1 = i + n \mp 1$, and therefore l_i, l'_i and $l_{i+n \mp 1}, l'_{i+n \mp 1}$ meet at angles α and $\bar{\alpha}$. However, by Lemma I.4.4, the latter pair of geodesics is $l_{i \mp 1}, l'_{i \mp 1}$. Therefore, for all j , the geodesics l_j and l'_j meet at angles α and $\bar{\alpha}$.

In particular, $\beta = \alpha$ or $\bar{\alpha}$ and $\gamma = \alpha$ or $\bar{\alpha}$.

Note that, by Lemma I.3.2, the geodesics l_i, l'_i, l_{i+1} bound four pair of triangles whose angles are

$$(\alpha, \beta, \gamma), (\bar{\alpha}, \bar{\beta}, \gamma), (\bar{\alpha}, \beta, \bar{\gamma}), (\alpha, \bar{\beta}, \bar{\gamma}),$$

where $\{\alpha, \beta, \gamma\} \subset (0, \pi)$. Since a triangle on S^2 is equilateral if and only if its angles are equal, it suffices to show that one of the four types of triangles listed above has equal angles. Note that

- (1) if $\alpha = \beta = \gamma$, then the triangles whose angles are (α, β, γ) are equilateral.
- (2) if $\alpha = \bar{\beta} = \gamma$, then the triangles whose angles are $(\bar{\alpha}, \beta, \bar{\gamma})$ are equilateral.
- (3) if $\alpha = \beta = \bar{\gamma}$, then the triangles whose angles are $(\bar{\alpha}, \bar{\beta}, \gamma)$ are equilateral.
- (4) if $\alpha = \bar{\beta} = \bar{\gamma}$, then the triangles whose angles are $(\alpha, \bar{\beta}, \bar{\gamma})$ are equilateral.

Set δ to be the angle measure of the equilateral triangle. In the first and last cases, $\delta = \alpha$. In the remaining cases, $\delta = \bar{\alpha}$. Note that, since spherical triangles have angle sum greater than π , δ is greater than $\frac{\pi}{3}$.

By spherical geometry,

$$d = \arccos \left[\frac{\cos \delta + \cos^2 \delta}{\sin^2 \delta} \right].$$

Hence, as δ ranges from $\frac{\pi}{3}$ to π , d ranges from 0 to $\frac{2\pi}{3}$. ■

All eight of the triangles bounded by a triple of geodesics will be equilateral exactly when their angles are all $\frac{\pi}{2}$. Could it be that $\delta = \frac{\pi}{2}$?

LEMMA I.4.8. *Suppose that n is not divisible by 3. Then*

- (1) $\delta \neq \frac{\pi}{2}$.
- (2) *the geodesics l_i, l'_i, l_{i+1} bound exactly one pair of equilateral triangles.*

PROOF. Suppose that $\delta = \frac{\pi}{2}$. We will show that n must be divisible by 3. The second part follows immediately from the first.

Without loss of generality, we may assume that the geodesics l_1, l'_1, l_2 are the equators $y = 0, x = 0, z = 0$, respectively. By Lemma I.4.5, $\rho(l_2)(l_1) = l'_2$. However, since l_2 and l_1 meet at right angles, reflection about l_2 carries l_1 to itself, so $l_1 = l'_2$. Similarly, $\rho(l_3)(l'_2) = l_4 = l_1$, and inductively, $l_i = l_{i+3}$. Also, $\rho(l_2)(l'_1) = l_3$ by Lemma I.4.5, and as before, it follows that $l'_1 = l_3$. Therefore, the geodesics l_1, l_2, l_3 are distinct. By Lemma I.4.4, $l_1 = l_{n+1}$, and therefore, $n + 1 = 3k + 1$, for some integer k . Hence, n is divisible by 3. ■

PROPOSITION I.4.9. *If n is not divisible by 3, then $\bigcup_i fp(\psi_i) \subset S_\kappa(q) \cup S_\kappa(q')$, the union of two circles of some radius κ , centered at antipodal points q and q' .*

PROOF. Each triple l_i, l'_i, l_{i+1} bound a pair of equilateral triangles of fixed angle δ , to be labelled in the following way. Arbitrarily label the equilateral triangles bounded by l_1, l'_1, l_2 as Δ_1 and Δ'_1 . Label their vertices as (p_1, p_2, p_3) and (p'_1, p'_2, p'_3) respectively, where $\{p_i, p'_i\} = fp(\psi_i)$. Continuing inductively for $i = 2$ to $n - 2$, note that $\rho(l_i)(\Delta_{i-1})$ and $\rho(l_i)(\Delta'_{i-1})$ are the equilateral triangles bounded by l_i, l'_i, l_{i+1} , by Lemma I.4.5. Label them Δ_i and Δ'_i , respectively. Label the previously unlabelled vertices in $fg(\psi_{i+2})$ so that Δ_i has vertices p_i, p_{i+1}, p_{i+2} and Δ'_i has vertices p'_i, p'_{i+1}, p'_{i+2} .

Let m_i, m'_i be the midpoints of the arcs $l'_i \cap \Delta_i, l'_i \cap \Delta'_i$, respectively. Let $m(l'_i) = \{m_i, m'_i\}$ denote the midpoint set.

Define g_i to be the geodesic containing the four points $m(l'_i) \cup fp(\psi_{i+1}) = \{m_i, m'_i, p_{i+1}, p'_{i+1}\}$. The geodesic g_i divides Δ_i, Δ'_i into pairs of triangles with side lengths $d, d, \frac{d}{2}$. The isometry $\rho(g_i)$ reflects each across their common side to a triangle with the same side lengths and therefore the same angles, relative to g_i . But this triangle is exactly the paired triangle. Hence, $\rho(g_i)$ must exchange the pair of triangles which correspond to Δ_i , and similarly exchanges the pair which

corresponds to Δ'_i . In particular, $\rho(g_i)$ fixes the geodesic l'_i setwise, yet carries the vertices p_i and p'_i to p_{i+2} and p'_{i+2} , respectively. Therefore, $g_i \perp l'_i$.

Note that g_1 and g_2 are distinct, as they meet l_2 in distinct points. Therefore $g_1 \cap g_2$ is a pair of antipodal points, say $\{q, q'\}$.

LEMMA I.4.10. For all i ,

- (1) $\{q, q'\} \subset g_i$,
- (2) $\rho(g_i)(g_{i-1}) = g_{i+1}$, and
- (3) g_i and g_{i+1} meet at fixed angles ι and $\bar{\iota}$.

PROOF OF LEMMA I.4.10. Inductively assume that $g_{i-1} \cap g_i = \{q, q'\}$. By construction, $\rho(g_i)(fp(\psi_i)) = fp(\psi_{i+2})$. Also, $\rho(g_i)(l'_{i-1}) = l'_{i+1}$, since the angle measure $\angle m_i p_{i+1} p_{i-1} = \angle m_i p_{i+1} p_{i+3} = \frac{3\delta}{2}$. Therefore, $\rho(g_i)(m_{i-1}) = m_{i+1}$, and $\rho(g_i)(m'_{i-1}) = m'_{i+1}$. Since $\rho(g_i)(g_{i-1})$ and g_{i+1} both contain the four points

$$fp(\psi_{i+2}) \cup m(l'_{i+1}),$$

$\rho(g_i)(g_{i-1}) = g_{i+1}$, and the angles between g_i and g_{i-1} are equal in measure to the angles between g_i and g_{i+1} . Since $\rho(g_i)$ fixes $\{q, q'\}$, elements of g_{i-1} , $\{q, q'\} \subset g_{i+1}$, and the lemma is verified.

Since $\rho(g_i)(p_i) = p_{i+2}$ and $\rho(g_i)(q) = q$, $d(p_i, q) = d(p_{i+2}, q) = d(p'_i, q') = d(p'_{i+2}, q')$.

Now, let $\xi \in SO(3)$ be a rotation with angle π , fixing $m(l_2)$, the midpoints of the arcs on l_2 contained in the equilateral triangles Δ_1 and Δ'_1 . The rotation ξ exchanges the triangles (p_2, m_1, p_3) and (p_3, m_2, p_2) , as well as their antipodal triangles. Therefore, $\xi(fp(\psi_2)) = fp(\psi_3)$ and $\xi(m(l'_1)) = m(l'_2)$. It follows that $\xi(g_1) = g_2$.

The set $\{q, q'\} = g_1 \cap g_2$ but $fp(\xi) \neq \{q, q'\}$, so $\xi(q) = q'$ and $\xi(q') = q$. Therefore, $d(p_2, q) = d(p_3, q') = d(p'_3, q)$.

Set $\kappa = d(p_2, q)$. Then, for integers k ranging from 1 to $\frac{n}{2}$,

$$\{p_{2k}, p'_{2k-1}\} \subset S_\kappa(q) \quad \{p_{2k-1}, p'_{2k}\} \subset S_\kappa(q')$$

and the Proposition is shown. ■

Without loss of generality, we may assume $\kappa < \pi/2$. Let e denote the equator between q and q' , the geodesic consisting of those points of distance $\frac{\pi}{2}$ from $\{q, q'\}$.

LEMMA I.4.11. *If n is not divisible by 3, then for all i ,*

- (1) *the geodesic l_i and the equator e meet at the points $m(l_i)$, the midpoints of the segments $l_i \cap \Delta_i$ and $l_i \cap \Delta'_i$.*
- (2) *the segments $l'_i \cap \Delta_i$ and $l'_i \cap \Delta'_i$ do not meet the equator.*

PROOF. Let i be arbitrary.

PART 1. In similar fashion to the previous proposition, let $\xi \in SO(3)$ be a rotation of angle π fixing $m(l_i) = \{\hat{m}_i, \hat{m}'_i\}$. $\xi(q) = q'$, so $d(\hat{m}_i, q) = d(\hat{m}_i, q')$. However, the points $q, q', \hat{m}_i$ lie on a geodesic, and q, q' are antipodal, so $d(\hat{m}_i, q) + d(\hat{m}_i, q') = d(q, q') = \pi$. Therefore, $d(\hat{m}_i, q) = \pi/2$ and $\hat{m}_i \in e$. Similarly, $\hat{m}'_i \in e$.

PART 2. By Lemma I.4.7, d , the length of the segment $l'_i \cap \Delta_i$ is at most $\frac{2\pi}{3}$, so the segment meets the equator e in at most one point. However, $d(p_i, q) = d(p_{i+2}, q) = \kappa$ or $\pi - \kappa$. Therefore, the points p_i and p_{i+2} both lie on the same side of the equator, and hence the segment $l'_i \cap \Delta_i$ does not meet the equator. Similarly, $l'_i \cap \Delta'_i$ does not meet the equator. ■

LEMMA I.4.12. *If n is not divisible by 3, then for each i , $e \cap \Delta_i$ and $e \cap \Delta'_i$ are connected.*

PROOF. By Lemma I.4.11, e intersects the boundary of each Δ_i in exactly two points. Therefore, $e \cap \Delta_i$ is a connected subset of e . The same argument holds for each Δ'_i . ■

Let $e_i = e \cap \Delta_i$ and $e'_i = e \cap \Delta'_i$ be the segments of Lemma I.4.12. Since the Δ_i and Δ'_i are all congruent, it follows that each of the segments e_i and e'_i have the same length. Call this length ζ .

LEMMA I.4.13. *The length $\zeta \in (0, \frac{2}{3}\pi)$, and increases with δ .*

PROOF. Form within Δ_i the equilateral triangle of side length ζ , whose endpoints are the midpoints of the edges of Δ_i . δ is less than π , so Δ_i has area less than that of the hemisphere. Note that as δ approaches π , the area of the inscribed triangle approaches that of Δ_i . Therefore, its perimeter approaches 2π , and ζ approaches $\frac{2}{3}\pi$. Similarly, as δ approaches $\frac{\pi}{3}$, the perimeter of Δ_i approaches zero, and therefore, the perimeter of the inscribed triangle also approaches zero, and ζ approaches zero.

It remains to show the one-to-one relationship between ζ and δ . Clearly, given a triangle Δ_i , the segment e_i can be drawn within the triangle, and the length ζ is determined.

Given ζ , form an equilateral triangle T of side length ζ . At each vertex, draw the geodesic perpendicular to the altitude. The three resulting geodesics determine eight triangles, one of which contains T , call it T' . Since the altitudes of an equilateral triangle bisect the angle and opposite edge, reflection about any altitude a carries T to itself, and therefore exchanges the two vertices of T not contained in a . Further, since reflection about a fixes the edge of T' perpendicular to a , this reflection exchanges the other two edges of T' , and carries T' to itself. As reflections are isometries, we have that T' is equilateral and the altitudes of T bisect the edges of T' . The size of T' is therefore that which yields the given length ζ . ■

LEMMA I.4.14. Suppose n is not divisible by 3. Then distinct $\phi : G_n \rightarrow SO(3)$ give rise to equilateral triangles of distinct size, and conversely.

PROOF. Recall that two representations $\phi, \tilde{\phi}$ are not distinct if and only if there exists some $\xi \in SO(3)$ such that $\tilde{\phi} = \phi^\xi \equiv \xi\phi\xi^{-1}$. Since a representation is determined by the images of the generators x_1, x_2 of G_n , the representations $\phi, \tilde{\phi}$ are not distinct if and only if there exists some $\xi \in SO(3)$ such that $\psi_1^\xi = \tilde{\psi}_1$ and $\psi_2^\xi = \tilde{\psi}_2$.

First assume that the representations are not distinct. Then there exists some $\xi \in SO(3)$ such that $\phi^\xi = \tilde{\phi}$. $(\psi_i)^\xi = \tilde{\psi}_i$, so $\xi(fp(\phi_i)) = fp(\tilde{\phi}_i)$. The geodesics l_i and l'_i are determined by these fixed points, so the rotation ξ takes l_i, l'_i to $\tilde{l}_i, \tilde{l}'_i$, respectively, for all i . Therefore, for any equilateral triangle Δ_i associated with ϕ , the triangle $\xi\Delta_i$ associated with $\tilde{\phi}$ is also equilateral and is congruent to Δ_i .

Conversely, assume that the equilateral triangles of the representations are congruent. Using the previous notation, let $\psi_1 = \phi(x_1)$ and $\psi_2 = \phi(x_2)$, and $\tilde{\psi}_1 = \tilde{\phi}(x_1)$ and $\tilde{\psi}_2 = \tilde{\phi}(x_2)$, and define the geodesics $l_i, l'_i, \tilde{l}_i, \tilde{l}'_i$ as before. Choose one of the two equilateral triangles of ϕ bounded by l_1, l'_1, l_2 as Δ_1 , with vertices p_1, p_2, p_3 . Denote the triangles of $\tilde{\phi}$ bounded by $\tilde{l}_1, \tilde{l}'_1, \tilde{l}_2$ as $\tilde{\Delta}_1$ and $\tilde{\Delta}'_1$, with vertices marked $\tilde{p}_1, \tilde{p}_2, \tilde{p}_3$ and $\tilde{p}'_1, \tilde{p}'_2, \tilde{p}'_3$ as before. The antipodal map, which takes the quadruple $(\tilde{\Delta}_1, \tilde{p}_1, \tilde{p}_2, \tilde{p}_3)$ to the quadruple $(\tilde{\Delta}'_1, \tilde{p}'_1, \tilde{p}'_2, \tilde{p}'_3)$, is orientation reversing. Let ξ be the orthogonal automorphism on S^2 which takes the quadruple $(\Delta_1, p_1, p_2, p_3)$ to $(\tilde{\Delta}_1, \tilde{p}_1, \tilde{p}_2, \tilde{p}_3)$, and let ξ' be the orthogonal automorphism which takes the quadruple $(\Delta_1, p_1, p_2, p_3)$ to $(\tilde{\Delta}'_1, \tilde{p}'_1, \tilde{p}'_2, \tilde{p}'_3)$. The composition $\xi'\xi^{-1}$, which takes the quadruple $(\tilde{\Delta}_1, \tilde{p}_1, \tilde{p}_2, \tilde{p}_3)$ to the quadruple $(\tilde{\Delta}'_1, \tilde{p}'_1, \tilde{p}'_2, \tilde{p}'_3)$, is orientation reversing, so either ξ or ξ' is orientation preserving. Without loss of generality, assume that ξ is orientation preserving.

As ξ takes the geodesics l_1, l'_1, l_2 to $\tilde{l}_1, \tilde{l}'_1, \tilde{l}_2$ respectively, it follows that $\rho(l_1)^\xi = \rho(\tilde{l}_1)$, $\rho(l'_1)^\xi = \rho(\tilde{l}'_1)$, and $\rho(l_2)^\xi = \rho(\tilde{l}_2)$. Therefore, $\psi_1^\xi = \tilde{\psi}_1$ and $\psi_2^\xi = \tilde{\psi}_2$. Since these two images generate the image of the representation, ϕ and $\tilde{\phi}$ are therefore equivalent. ■

By construction, $\rho(l_i)(\Delta_{i-1}) = \Delta_i$ and $\rho(l_i)(\Delta'_{i-1}) = \Delta'_i$ for $i = 2$ to n . The geodesics l_n, l'_n, l_1 , which bound Δ_n , are taken by the reflection $\rho(l_1)$ to the geodesics l'_1, l_2, l_1 respectively. The triangle Δ_n is taken by $\rho(l_1)$ to an equilateral triangle bounded by the geodesics l'_1, l_2, l_1 . Note that under the assumption that n is not divisible by 3, there are exactly two such triangles, by Lemma I.4.8. Is it taken to Δ_1 or Δ'_1 ?

LEMMA I.4.15. *Suppose n is not divisible by 3. Then*

$$\rho(l_1)(\Delta_n) = \begin{cases} \Delta_1 & \text{when } n \text{ is even} \\ \Delta'_1 & \text{when } n \text{ is odd.} \end{cases}$$

PROOF. Let $R = \rho(l_1)\rho(l_n)\rho(l_{n-1})\dots\rho(l_3)\rho(l_2)$.

CASE 1. If n is even, then R is an even number of reflections and is therefore orientation preserving. $R(\Delta_1)$ and Δ_1 have the same orientation, so $R(\Delta_1) = \rho(l_1)(\Delta_n) = \Delta_1$.

CASE 2. If n is odd, then R is orientation reversing, so $R(\Delta_1) = \rho(l_1)(\Delta_n) = \Delta'_1$. ■

For n even, define $\mathcal{A}_n$ to be the CW-complex with the following vertices, edges, and faces. Using cylindrical coordinates $(r, \theta, z) \in \mathbf{R}^3$, fix points

$$t_i = (1, \frac{i}{n}2\pi, (-1)^i),$$

for $i = 1$ to n . Fix edges c_i with endpoints t_i, t_{i+1} , edges c'_i with endpoints t_i, t_{i+2} , and faces V_i , bounded by c_i, c'_i, c_{i+1} , for $i \in \mathbf{Z}_n$. The resulting complex is

an annulus. Let $\epsilon = \mathcal{A}_n \cap \{z = 0\}$ denote the equator of $\mathcal{A}_n$, and let $\epsilon_i = V_i \cap \epsilon$. For n odd, define $\mathcal{A}_n$ as $\mathcal{A}_{2n}$.

There is a natural immersion $\iota : \mathcal{A}_n \rightarrow S^2$, where, for n even, $v_i \rightarrow p_i$, $c_i \rightarrow l_i \cap \Delta_i$, $c'_i \rightarrow l'_i \cap \Delta_i$, $V_i \rightarrow \Delta_i$, and $\epsilon_i \rightarrow e_i$, for $i = 1$ to n . If n is odd, then additionally $v_{i+n} \rightarrow p'_i$, $c_{i+n} \rightarrow l_i \cap \Delta'_i$, $c'_{i+n} \rightarrow l'_i \cap \Delta'_i$, $V_{i+n} \rightarrow \Delta'_i$, and $\epsilon_{i+n} \rightarrow e'_i$, for $i = 1$ to n .

Note that ϵ is mapped onto e . If n is even, the equator ϵ is mapped to n segments, each of length ζ , whose total length $n\zeta$ is an integer multiple of 2π . The degree of the map $\epsilon \rightarrow e$ is therefore $\frac{n\zeta}{2\pi}$. Similarly, if n is odd, then the equator is mapped to $2n$ segments and the degree of the map $\epsilon \rightarrow e$ is $\frac{n\zeta}{\pi}$.

We can therefore define the *winding number of ι* , $\varpi(\iota)$ as the degree of the map $\epsilon \xrightarrow{\iota} e$.

LEMMA I.4.16. Suppose n is not divisible by 3.

- (1) If n is even, then $\varpi(\iota)$ is at most $[\frac{1}{3}n]$.
- (2) If n is odd, then $\varpi(\iota)$ is at most $[\frac{2}{3}n]$, and is odd.

PROOF.

CASE 1. If n is even, then $n\zeta = 2\pi\varpi(\iota)$. However, since $\zeta \in (0, \frac{2}{3}\pi)$ by Lemma I.4.13, $\varpi(\iota) \in (0, \frac{n}{3})$. However, since n is not divisible by 3 by hypothesis, $[\frac{n}{3}] < \frac{n}{3}$, and the result follows.

CASE 2. If n is odd, then $n\zeta = \pi\varpi(\iota)$. Therefore, since $\zeta \in (0, \frac{2}{3}\pi)$ by Lemma I.4.10, $\varpi(\iota)$ is some integer in $(0, \frac{2}{3}n)$.

If $\varpi(\iota)$ is even, then the map $\iota|_\epsilon$ carries antipodal points to the same point. Therefore, $\iota(V_i) \cap \iota(V_{n+1}) \neq \emptyset$. But this contradicts the fact that $\iota(V_i) = \Delta_i$ and $\iota(V_{n+i}) = \Delta'_i$ are disjoint triangles. Therefore, $\varpi(\iota)$ must be odd. Since n is not divisible by 3, $[\frac{2n}{3}] < \frac{2n}{3}$, and the result follows. ■

LEMMA I.4.17.

- (1) If n is even, then each positive integer $\varsigma \leq [\frac{n}{3}]$ is the winding number for a distinct irreducible representation of G_n into $SO(3)$.
- (2) If n is odd, then each odd positive integer $\varsigma \leq [\frac{2n}{3}]$ is the winding number for a distinct irreducible representation of G_n into $SO(3)$.

PROOF. If n even, set $\zeta = \frac{2\pi\varsigma}{n}$. If n is odd, then set $\zeta = \frac{\pi\varsigma}{n}$. By Lemma I.4.13, distinct ζ 's induce distinct δ 's. By Lemma I.4.14, each δ is associated with a distinct representation. ■

PROOF OF THEOREM I.4.1. The case for n even follows immediately from Lemmas I.4.16 and I.4.17. When $n = 6k \pm 1$ for some integer k , $4k - 1$ is the largest odd less than $[\frac{2}{3}n] = [4k \pm \frac{2}{3}]$. There are exactly $2k$ odd integers from 1 to $4k - 1$. Since $2k = [\frac{6k}{3}] = [\frac{6k+2}{3}]$, the result follows. ■

§5. Conclusion.

THEOREM I.5.1. Suppose n, m are distinct integers such that

- (1) n, m are not consecutive odds of the form $6k \pm 1$, for some fixed integer k ,
- (2) if 3 divides both n and m , then $n + m$ is odd.

Then G_n and G_m are distinct groups, and therefore, $L_{n, n-2, \frac{1}{3}}$ and $L_{m, m-2, \frac{1}{3}}$ have distinct fundamental groups.

PROOF. If $n + m$ is odd, or if one is divisible by 3, but not the other, then by Proposition I.2.6, G_n and G_m are distinct.

If n and m are both even, and neither is divisible by 3, then by Theorem I.4.1, they have $[\frac{n}{3}], [\frac{m}{3}]$ distinct representations into $SO(3)$, respectively. Since $[\frac{n}{3}] \neq [\frac{m}{3}]$, the result follows. If n and m are both odd, and neither is divisible

by 3, then by Theorem I.4.1, they have $[\frac{n+1}{3}], [\frac{m+1}{3}]$ distinct representations into $SO(3)$, respectively. Since n and m are nonconsecutive odds, $[\frac{n+1}{3}] \neq [\frac{m+1}{3}]$. ■

CHAPTER II

COMPLETELY DISTINGUISHING THE TWISTED LENS SPACES

In the previous chapter many of the twisted lens spaces $L_{n,n-2,\frac{1}{3}}$ were distinguished by their fundamental groups. In this chapter we will distinguish *all* of the twisted lens spaces, up to homeomorphism, by constructing distinguished Seifert Fibrations for these spaces.

§1. Preliminaries Concerning Seifert Fibrations.

DEFINITIONS. (These definitions are due to [Ste] and [Se].) A decomposition $\mathcal{F}$ of a manifold M into circles is a *foliation* if, for each $p \in M$, there exists a neighborhood N of p and a homeomorphism $\phi : \mathbf{R} \times \mathbf{R}^2 \rightarrow N$ such that, for each $f \in \mathcal{F}$, each component of $f \cap N$ is the image of $\mathbf{R} \times y_0$, for some point $y_0 \in \mathbf{R}^2$.

A *trivial fibered solid torus* $(S^1 \times D^2, \mathcal{F})$ is a solid torus, together with a foliation $\mathcal{F}$ into circle fibers $S^1 \times \{y\}$, for all $y \in D^2$. A *fibered solid torus* $(S^1 \times D^2, \mathcal{F})$ is a solid torus with a foliation by circles for which there exists a fiber-preserving map to a trivial fibered solid torus which is one-to-one on the preimage of $S^1 \times \{0\}$, some fiber $f \in \mathcal{F}$, and q -to-one on the other fibers of $\mathcal{F}$. When $q > 1$, the fiber f is called *exceptional* of degree q .

Such a fibered solid torus can be constructed by cutting a trivial fibered solid torus along a disk $\{x\} \times D^2$ and reattaching after rotating one of the resulting disks $\frac{p}{q}$ revolutions, for some integer p relatively prime to q . Further, the center fiber is exceptional of degree q when, for any fiber $\hat{f}$ in the neighborhood other than the center fiber, and any slice $S = [a, b] \times D^2 \subset S^1 \times D^2$, $\hat{f} \cap S$ consists of

q arcs, each connecting $\{a\} \times \mathbf{D}^2$ with $\{b\} \times \mathbf{D}^2$. This viewpoint will be useful in the following section.

A *Seifert Fibration* $\mathcal{F}$ for a manifold M is a foliation of M into circles, called fibers, so that each $f \in \mathcal{F}$ has a *fibered neighborhood*, a subset of fibers containing f , that can be mapped under a fiber-preserving homeomorphism onto a fibered solid torus, where f is mapped to the center fiber. If f has a fibered neighborhood which is homeomorphic via a fiber-preserving map to the trivial fibered solid torus, then f is called an *ordinary* fiber. Otherwise, f has a fiber neighborhood which is homeomorphic via a fiber-preserving map to a fibered solid torus whose center fiber is exceptional of degree q , in which case f is called an *exceptional* fiber of order q .

THEOREM II.1.1. [Ep] *Any foliation by circles of a compact orientable three-manifold is a Seifert Fibration.*

THEOREM II.1.2. [Sc₂] *Let M and N be compact Seifert fibered three-manifolds, and let $\phi : M \rightarrow N$ be a homeomorphism. Then ϕ is homotopic to a fiber-preserving homeomorphism unless one of the following occurs.*

- (1) M is covered by $\mathbf{S}^3$, $\mathbf{S}^2 \times \mathbf{R}$, or $\mathbf{S}^1 \times \mathbf{S}^1 \times \mathbf{S}^1$.
- (2) M is $\mathbf{S}^1 \times \mathbf{D}^2$ or an I -bundle over the torus or Klein bottle.

§2. Building a Seifert Fibration for $L_{n,n-2,\frac{1}{3}}$.

Let P_n be a regular n -gon centered at the origin and lying in the xy -plane. Let Σ_n be the suspension of P_n , using the points $(0,0,1)$ and $(0,0,-1)$. Label the vertices of P_n consecutively as $\{v_1, v_2, v_3, \dots, v_n\}$. Let a_1 denote the directed segment from v_n to v_1 and, for $i = 2$ to n , let a_i be the directed segment from v_{i-1} to v_i . For each i from 1 to n , let a'_i and a''_i denote the directed arcs from

$(0,0,1)$ to v_i and from v_i to $(0,0,-1)$, respectively. Let Δ_i and Δ'_i denote the faces of Σ_n determined by vertices v_{i-1} , v_i , and $(0,0,1)$ or $(0,0,-1)$, respectively.

Consider the decomposition

$$\mathcal{F}'_n = \{l_{(x,y)} \cap \Sigma_n \mid (x,y) \in \mathbf{R}^2\}$$

of Σ_n , where $l_{(x,y)}$ is the line $\{(x,y,t), t \in \mathbf{R}\}$. $\mathcal{F}'_n$ decomposes Σ_n into arcs which are perpendicular to P_n and points in $bd P_n$.

Let $g_n : \Sigma_n \rightarrow L_{n,n-2,\frac{1}{3}}$ be the gluing map which identifies the boundary faces of Σ_n . More precisely, the face centers are identified, the appropriate edges are identified linearly, and the identification is extended linearly over the faces. Therefore, if the faces are subdivided by starring from their center points, the map would identify the paired subfaces linearly.

PROPOSITION II.2.1. *Under g_n , $\mathcal{F}'_n$ induces a decomposition $\mathcal{F}_n$ of $L_{n,n-2,\frac{1}{3}}$ into circles.*

PROOF. The map g_n identifies all of the vertices, pairs of points which lie in the face interiors, and triples of points which lie in the edge interiors.

Under g_n , segments f'_1 and f'_2 will form a connected subset of some element $f \in \mathcal{F}_n$ if they have endpoints which are identified under g_n . Segments whose boundaries lie in the face interiors connect together to form one-manifolds, as g_n identifies pairs of those points which lie in the face interiors.

Of the points which lie in edge interiors, g_n identifies triples consisting of points from an upper, a central, and a lower edge. The central point is a singleton $f'_0 \in \mathcal{F}'_n$, and the other two points are endpoints of segments f'_1 and f'_2 . The identification of the triple transforms f'_0, f'_1 , and f'_2 into a one-manifold at the glued point.

The segments of $\mathcal{F}'_n$ which contain vertices consist of the center fiber with boundary $\{(0, 0, 1), (0, 0, -1)\}$, together with the vertices which lie on the boundary of P_n . When these points are identified, the resulting element of $\mathcal{F}_n$ is a simple closed curve.

The elements of $\mathcal{F}_n$ are one-manifolds without boundary. To verify that these elements are circles, it remains to show that only finitely many $f' \in \mathcal{F}'_n$ are mapped into each $f \in \mathcal{F}_n$. By construction, each point of Δ_i lies in the same element of $\mathcal{F}_n$ as the point in Δ_{i-2} located by twisting the face $\frac{1}{3}$ revolution, counterclockwise. Dependent upon $\gcd(n, 6)$, the interior of each face contains at most three points which lie in a given element of $\mathcal{F}_n$. The boundary of each face contains at most six such points. Hence, finitely many elements of $\mathcal{F}'_n$ are mapped to each $f \in \mathcal{F}_n$. ■

PROPOSITION II.2.2. $\mathcal{F}_n$ is a foliation of $L_{n, n-2, \frac{1}{3}}$.

PROOF. Choose $p \in L_{n, n-2, \frac{1}{3}}$.

CASE 1. If $g_n^{-1}(p) \in \text{int } \Sigma_n$, then the desired neighborhood N of p can be constructed using the product structure of $\mathcal{F}'_n$.

CASE 2. Suppose $g_n^{-1}(p) = \{p_1, p_2\} \in \text{int } \Delta_i \cup \text{int } \Delta'_{i-2}$, with $p_1 \in \text{int } \Delta_i$ and $p_2 \in \text{int } \Delta'_{i-2}$. Homeomorphisms ϕ_1, ϕ_2 and disjoint neighborhoods N_1, N_2 of p_1, p_2 in Σ_n can be constructed, using the product structure of $\mathcal{F}'_n$, so that, for $i = 1$ and 2 :

- (1) $\phi_i : \mathbf{R}_+ \times \mathbf{R}^2 \rightarrow N_i$,
- (2) each component of $f' \cap N_i$ is the image of $\mathbf{R}_+ \times \{y\}$, for some $y \in \mathbf{R}^2$,
- (3) $g_n \phi_1 = g_n \phi_2$ on $\{0\} \times \mathbf{R}^2$.

Therefore, the homeomorphism $\phi : \mathbf{R} \times \mathbf{R}^2 \rightarrow g_n(N_1 \cup N_2)$, where

$$\phi(x_1, x_2, x_3) = \begin{cases} g_n \phi_1(x_1, x_2, x_3) & \text{on } \mathbf{R}_+ \times \mathbf{R}^2 \\ g_n \phi_2(-x_1, x_2, x_3) & \text{on } \mathbf{R}_- \times \mathbf{R}^2, \end{cases}$$

is the desired homeomorphism. Here $\mathbf{R}_+ = [0, \infty)$ and $\mathbf{R}_- = (-\infty, 0]$.

CASE 3. If, for some i , $g_n^{-1}(p) \in \text{int } a_i \cup \text{int } a'_{i+1} \cup \text{int } a''_{i-2}$, let p_0, p_1 , and p_2 be these respective preimage points. Homeomorphisms ϕ_0, ϕ_1, ϕ_2 and disjoint neighborhoods N_0, N_1, N_2 of p_0, p_1, p_2 in Σ_n can be constructed, using the product structure of $\mathcal{F}'_n$, so that, for $i = 1$ and 2 :

$$(1) \phi_i : \mathbf{R}_+ \times \mathbf{R}^2 \rightarrow N_i,$$

$$(2) \text{ each component of } f' \cap N_i \text{ is the image of } \mathbf{R}_+ \times \{y\}, \text{ for some } y \in \mathbf{R}^2,$$

and, in addition:

$$(3) \phi_0 : \cup\{[-r, r] \times \mathbf{R} \times \{r\} \mid r \in \mathbf{R}_+\} \rightarrow N_0,$$

$$(4) \text{ each component of } f' \cap \phi_0 \text{ is the image of } [-r, r] \times \{y\} \times \{r\}, \text{ for some } r \in \mathbf{R}_+, y \in \mathbf{R},$$

$$(5) g_n \phi_1 = g_n \phi_2 \text{ on } \{0\} \times \mathbf{R} \times \mathbf{R}_-,$$

$$(6) g_n \phi_1(0, y, r) = g_n \phi_0(r, y, r), \text{ for all } r \in \mathbf{R}_+, y \in \mathbf{R}, \text{ and}$$

$$(7) g_n \phi_2(0, y, r) = g_n \phi_0(-r, y, r), \text{ for all } r \in \mathbf{R}_+, y \in \mathbf{R}.$$

Therefore, the homeomorphism $\phi : \mathbf{R} \times \mathbf{R}^2 \rightarrow g_n(N_1 \cup N_2)$, shows that $\mathcal{F}_n$ is a foliation at p , where, as illustrated in Figure II.1, ϕ is defined as:

$$\phi(x_1, x_2, x_3) = \begin{cases} g_n \phi_1(x_1, x_2, x_3) & \text{on } \mathbf{R}_+ \times \mathbf{R} \times \mathbf{R}_- \\ g_n \phi_2(-x_1, x_2, x_3) & \text{on } \mathbf{R}_- \times \mathbf{R} \times \mathbf{R}_- \\ g_n \phi_1(x_1 - x_3, x_2, x_3) & \text{on } [r, \infty) \times \mathbf{R} \times \{r\}, \text{ for } r \in \mathbf{R}_+ \\ g_n \phi_2(-x_1 - x_3, x_2, x_3) & \text{on } [-r, -\infty) \times \mathbf{R} \times \{r\}, \text{ for } r \in \mathbf{R}_+ \\ g_n \phi_0(x_1, x_2, x_3) & \text{on } [-r, r] \times \mathbf{R} \times \{r\}, \text{ for } r \in \mathbf{R}_+. \end{cases}$$

CASE 4. Suppose p is the image of the vertices under g_n .

Consider as a fibered neighborhood of $(0, 0, 1)$ the pyramid $\Sigma_n \cap \{(x, y, z) \mid z \geq \frac{9}{10}\}$. Each of the n horizontal segments which comprise $bd \Sigma_n \cap \{(x, y, z) \mid z = \frac{9}{10}\}$, as a subset of an upper face Δ_i , is identified with the underlying point set of a one-complex (specifically, two segments and three vertices) within the lower face Δ'_{i-2} , near the vertex v_{i-3} . Following the foliation of Σ_n , this set is carried to a set in Δ_{i-2} near v_{i-3} , which is identified with a similar point set in Δ'_{i-4} , near

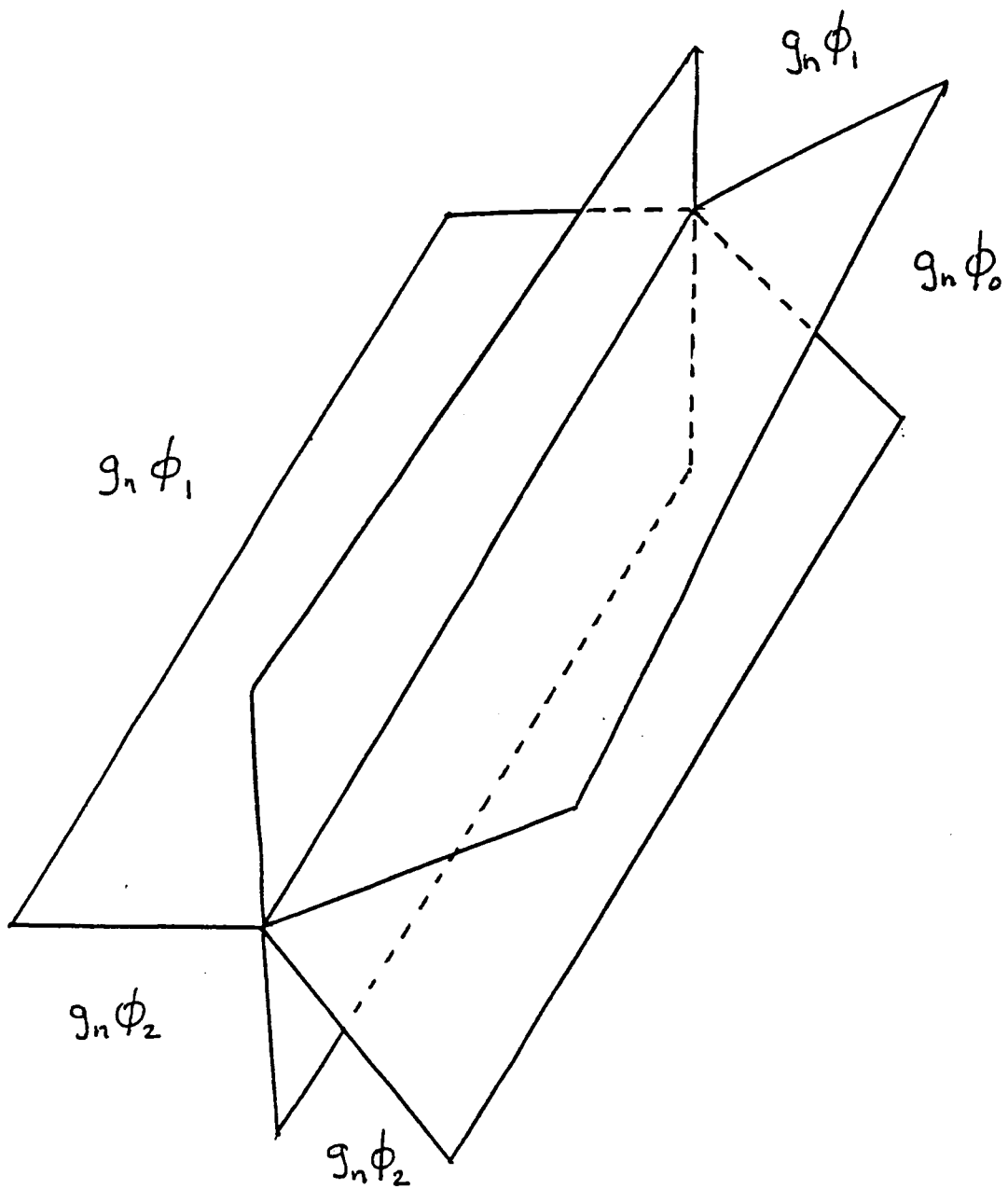


Figure II.1. The Composition of ϕ in Proposition II.2.2, Case 3.

v_{i-4} . Following the foliation of Σ_n , this set is carried to a set in Δ_{i-4} , which is identified with the segment $\Delta'_{i-6} \cap \{(x, y, z) \mid z = -\frac{9}{10}\}$, contained within the boundary of the pyramid $\Sigma_n \cap \{(x, y, z) \mid z \leq -\frac{9}{10}\}$, a fiber neighborhood of $(0, 0, -1)$.

For each vertex v_i , consider the fibered neighborhood N'_i whose link consists of those fibers which meet the nearby underlying point sets. By construction, these n neighborhoods, together with the two pyramid neighborhoods, fit together when identified by g_n to form a fibered neighborhood of the vertex in $L_{n, n-2, \frac{1}{3}}$, as shown in Figure II.2. Note that a typical fiber passes vertically through the upper pyramid, through two fibered neighborhoods N'_i , and vertically through the lower pyramid. The gluing of these neighborhoods can be achieved so that each fiber is vertical, as shown. ■

COROLLARY II.2.3. $\mathcal{F}_n$ is a Seifert Fibration of $L_{n, n-2, \frac{1}{3}}$.

PROOF. This result follows immediately from Proposition II.2.2 and Theorem II.1.1. ■

PROPOSITION II.2.4. The fibers of $\mathcal{F}_n$ which do not pass through the vertices, the center of a face, or the center of an edge of Σ_n are ordinary.

PROOF.

CASE 1. Suppose $f \in \mathcal{F}_n$ meets the image of the boundary faces of Σ_n in the image of interior points, but not the center points of the faces. The fiber f contains as subsets the images of certain segments, consecutively labelled as $f'_i \in \mathcal{F}'_n$. Construct small, connected, disjoint fibered neighborhoods N'_i of these f'_i 's in Σ_n , so that each boundary cap (a component of $N'_i \cap bd \Sigma_n$) is a small set missing the center and boundary of the face, lying within a sector based at the face center of angle $\frac{2}{3}\pi$. Further assume that N'_i 's are chosen so that the upper boundary cap

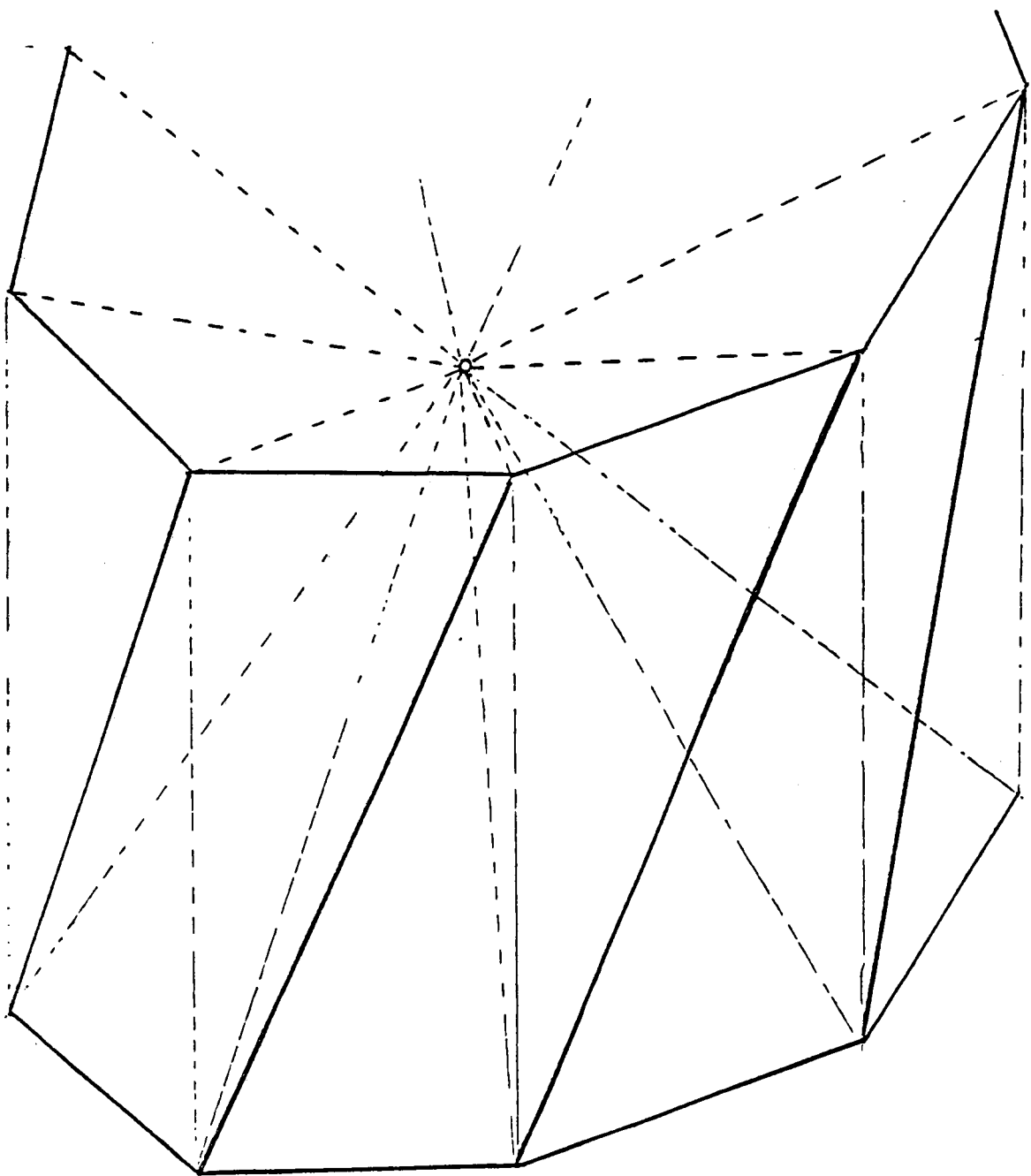


Figure II.2. The Fibered Neighborhood of Proposition II.2.2, Case 4.

of N'_i and the lower boundary cap of N'_{i+1} are identified by g_n , so that the image of $\cup_i N'_i$ is a fibered neighborhood N of f .

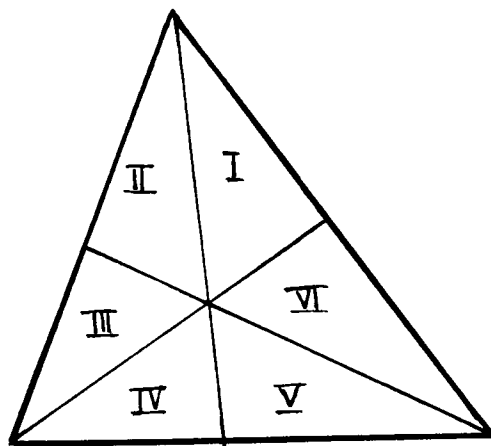
Choose an arbitrary fiber $\hat{f}$ in N other than f . If n is divisible by 3, then $\hat{f} \cap g_n(\Delta_i)$ is a singleton or the empty set, for all i . Hence, each $g_n^{-1}(\hat{f}) \cap N'_i$ consists of one segment.

Otherwise, each fiber of $\mathcal{F}_n$ which meets a particular face interior passes through three points of its interior. The triod of arcs from these points to the center point is such that all angles are of size $\frac{2}{3}\pi$. Hence, by assumption, at most one of any such trio of points lies in any one boundary cap. Therefore, each $g_n^{-1}(\hat{f}) \cap N'_i$ consists of one segment.

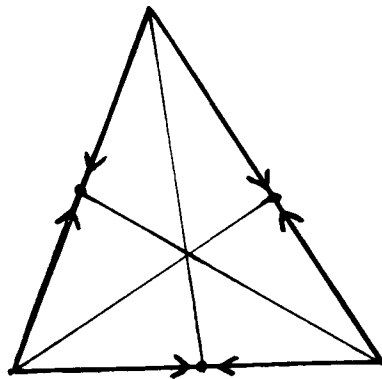
In either case there is hence a fiber-preserving homeomorphism from N to the trivial fibered solid torus, constructed by sending each N_i to a slice of the solid torus.

CASE 2. Suppose $f \in \mathcal{F}_n$ passes through edge interiors of Σ_n , but does not pass through the edge centers. The fiber f contains as subsets the images of certain elements of $\mathcal{F}'_n$, consecutively labelled as f'_i . Construct small, connected, disjoint fibered neighborhoods N'_i of these f'_i 's in Σ_n as in the proof of Case 1, so that each boundary cap is a small set missing the centers and boundary points of the edges, and also missing the centers of the faces.

Partition each upper face into six regions by starring from the center point, and number them from I to VI, as shown in Figure II.3a. Each point of Δ_i is identified with the point of Δ_{i-2} located by twisting $\frac{1}{3}$ revolution about the center point, carrying even numbered regions to even numbered, and odd numbered regions to odd numbered. Hence, no fiber f can pass through the interior of both even and odd numbered regions.



a. The Partition of Δ_i into Regions I to VI.



b. The Partition of $bd \Delta_i$ into Six Segments.

Figure II.3. The Partition of Δ_i .

Similarly, suppose the boundary of each face is subdivided at its center, forming six segments, directed toward the edge centers, as shown in Figure II.3b. Any coset of segments identified under g_n are so identified that their orientation is consistent. Therefore, no fiber f passes through a given segment in two different points.

By assumption, each boundary cap lies in the interior of the union of two regions of consecutive faces, one marked by an even number, the other by an odd, together with a segment. Any fiber $\hat{f} \in N$ falls in one of three classes. The arcs $g_n^{-1}(\hat{f})$ pierce even regions, odd regions, or segments. In each case, there is only one point in each boundary cap through which $g_n^{-1}(\hat{f})$ may pass. Hence, for all i , $g_n^{-1}(\hat{f}) \cap N'_i$ consists of one arc. There is hence a fiber-preserving homeomorphism from N to the trivial fibered solid torus, constructed by sending each N_i to a slice or wedge of the solid torus. ■

PROPOSITION II.2.5. *The fiber $f \in \mathcal{F}_n$ whose preimage passes through the vertices of Σ_n is singular of order $\frac{n}{(n,6)}$.*

PROOF. Let S be the smallest fibered neighborhood of the center fiber of Σ_n which contains the pyramid neighborhoods of $(0,0,1)$ and $(0,0,-1)$ in Σ_n formed in Proposition II.2.2, Case 4. The neighborhoods N'_i of the vertices v_i , also formed in the previous proposition, together with the right polygonal cylinder S , are identified by g_n to form a fibered neighborhood for the fiber f . It remains to determine the number of components of $\hat{f} \cap g_n(S)$, for any ordinary fiber $\hat{f} \in \mathcal{F}_n$.

Recall that g_n identifies each point in Δ_i near the upper suspension point $(0,0,1)$ to a point in a lower face Δ'_{i-2} near the vertex v_{i-3} . It identifies each point in Δ_{i-2} near v_{i-3} to a point in Δ'_{i-4} near the vertex v_{i-3} . It identifies each point in Δ_{i-4} near the vertex v_{i-3} to a point in Δ'_{i-6} near the suspension point $(0,0,-1)$. Therefore, each fiber in N which pierces the image of Δ_i traverses

then the image of two short segments in $g_n(N_{i-3})$ and $g_n(N_{i-4})$. Then it passes through $g_n(S)$ again and pierces the image of Δ_{i-6} , near the image of $(0, 0, 1)$. Therefore, $\hat{f} \cap g_n(S)$ consists of $\frac{n}{(n,6)}$ segments, and the result is shown. ■

PROPOSITION II.2.6.

- (1) *If n is divisible by 3, then the fibers of $\mathcal{F}_n$ which pass through the center of the faces of Σ_n are ordinary.*
- (2) *If $(n, 6) = 2$, then two fibers of $\mathcal{F}_n$ pass through the centers of the faces of Σ_n , each of order 3.*
- (3) *If $(n, 6) = 1$, then one fiber of $\mathcal{F}_n$ of order 3 passes through the centers of the faces of Σ_n .*

PROOF. Continuing in the fashion of Proposition II.2.4, form small fibered neighborhoods $N'_i \subset \Sigma_n$ of the fibers f'_i passing through the center of Δ_i , formed so that the upper boundary cap of N'_i is identified with the lower boundary cap of N'_{i-2} by g_n . Therefore, all of the face centers will lie on a single fiber if n is odd, and if n is even, the centers of the even-numbered faces and the odd-numbered faces will each lie on a fiber. It suffices to determine the number of components of $\hat{f} \cap g_n(N'_1)$, for any ordinary fiber $\hat{f} \in \mathcal{F}_n$ in the resulting fibered neighborhood. Since g_n attaches each face Δ_i to Δ'_{i-2} after twisting $\frac{1}{3}$ revolution about the center point, each fiber $\hat{f}$ pierces each face Δ_i one time if n is divisible by 3, and three times otherwise. The result then follows. ■

PROPOSITION II.2.7.

- (1) *If n is even, the fibers of $\mathcal{F}_n$ which pass through the centers of the edges of Σ_n are ordinary.*
- (2) *If $(n, 6) = 3$, three fibers of $\mathcal{F}_n$ pass through the centers of the edges of Σ_n , each of order 2.*

- (3) If $(n, 6) = 1$, one fiber of $\mathcal{F}_n$ of order 2 passes through the centers of the edges of Σ_n .

PROOF. Let S_i be the smallest fibered neighborhood of the fiber $f'_i \in \mathcal{F}'$ passing through the center of the edge a'_i in Σ_n , which contains those neighborhoods of the points $bd f'_i$ formed in Proposition II.2.2, Case 3. These cylinders S_i , together with the fibered neighborhoods N'_i of the midpoints of a_i , also formed in the previous proposition, are identified by g_n to form fibered neighborhoods for the fibers of $\mathcal{F}$ which contain the image of the center of an edge.

The edge a'_i , as part of Δ_{i+1} , is identified with a_{i-1} , as part of Δ'_{i-1} . As part of Δ_{i-1} , the edge a_{i-1} is identified with the edge a''_{i-3} . Therefore, each fiber f which passes through the image of the midpoint of a'_i then passes through the image of N'_{i-1} and next meets the image of the upper hemisphere in the image of the midpoint of a'_{i-3} . Therefore, the edge centers lie on one of three fibers if n is divisible by 3. Else, all of the edge centers lie on a single fiber.

It remains to determine the number of components of $\hat{f} \cap g_n(S'_i)$, for any fiber $\hat{f} \in \mathcal{F}_n$ in a fibered neighborhood other than the center fiber. Without loss of generality we may assume that $\hat{f}$ passes through the face interiors, since all of the fibers in the neighborhood except the center fiber have been shown to be ordinary.

Assuming that each boundary cap is small, each cap lies within the union of two sectors of angle $\frac{2}{3}\pi$, centered at the centers of adjacent faces. Therefore, the fiber $\hat{f}$ can pass only once through each half of the boundary cap. After $\hat{f}$ pierces the image of Δ_i , it next pierces the image of Δ_{i-2} . Therefore, $\hat{f}$ will only pierce the even or odd faces if n is even, and will pierce all of the faces if n is odd. Therefore, $\hat{f} \cap N_i$ will consist of two components exactly when n is odd, and the result follows. ■

THEOREM II.2.8. *The orders of the exceptional fibers of the Seifert fibration $\mathcal{F}_n$ of $L_{n,n-2,\frac{1}{3}}$ are:*

$$n, 2, 3 \quad \text{when } (n, 6) = 1$$

$$\frac{n}{2}, 3, 3 \quad \text{when } (n, 6) = 2$$

$$\frac{n}{3}, 2, 2, 2 \quad \text{when } (n, 6) = 3$$

$$\frac{n}{6} \quad \text{when } (n, 6) = 6.$$

PROOF. This result follows from the preceding propositions. ■

§3. Preliminaries Concerning Orbifolds.

DEFINITION. Recall that an n -manifold is a Hausdorff, paracompact space which is locally homeomorphic to $\mathbf{R}^n$. An n -orbifold is a Hausdorff, paracompact space which is locally homeomorphic to a quotient space of $\mathbf{R}^n$ under a finite group action. Following Thurston [T], an orbifold $\mathcal{O}$ formally consists of a Hausdorff space $X_{\mathcal{O}}$, with some additional structure. $X_{\mathcal{O}}$ is covered by a collection $\{U_i\}$ of open sets, closed under finite intersections. To each U_i is associated a finite subgroup Γ_i of $O(n)$, an action of Γ_i on an open neighborhood $\tilde{U}_i \subset \mathbf{R}^n$, and a homeomorphism $\phi_i : \tilde{U}_i/\Gamma_i \rightarrow U_i$. Whenever $U_i \subset U_j$, there is an injective homomorphism $f_{ij} : \Gamma_i \hookrightarrow \Gamma_j$, and an embedding $\tilde{\phi}_{ij} : \tilde{U}_i \hookrightarrow \tilde{U}_j$, equivariant with respect to f_{ij} (that is, for $\gamma \in \Gamma_i$, $\tilde{\phi}_{ij}(\gamma x) = f_{ij}(\gamma)\tilde{\phi}_{ij}(x)$), such that the diagram below commutes.

$$\begin{array}{ccc}
\tilde{U}_i & \xrightarrow{\tilde{\phi}_{ij}} & \tilde{U}_j \\
\downarrow & & \downarrow \\
\tilde{U}_i/\Gamma_i & \xrightarrow{\phi_{ij}} & \tilde{U}_j/f_{ij}\Gamma_i \\
& & \downarrow \\
& & \tilde{U}_j/\Gamma_j \\
\phi_i \downarrow & & \phi_j \downarrow \\
U_i & \xrightarrow{\text{inclusion}} & U_j
\end{array}$$

The covering $\{U_i\}$ is not an intrinsic part of the structure of an orbifold; two coverings give rise to the same orbifold if they can be combined to give a larger covering which still satisfies the conditions above.

The *singular set* $\Sigma(\mathcal{O})$ consists of those points $x \in X_{\mathcal{O}}$ for which, for any covering $\mathcal{U}$ and any element $U_i \in \mathcal{U}$ containing x , the corresponding subgroup Γ_i is nontrivial.

Note that every manifold is an orbifold with empty singular set.

THEOREM II.3.1. [T] *The singular set of a two-dimensional orbifold has three types of local models:*

- (1) *Mirrors: $\mathbf{R}^2/\mathbf{Z}_2$, where $\mathbf{Z}_2$ acts by reflection.*
- (2) *Elliptic points of order n : $\mathbf{R}^2/\mathbf{Z}_n$, with $\mathbf{Z}_n$ acting by rotations.*
- (3) *Corner reflectors of order n : $\mathbf{R}^2/\mathbf{D}_n$, where $\mathbf{D}_n$ is the dihedral group of order $2n$.*

PROOF. These are the only finite subgroups of $O(2)$. ■

NOTATION. A two-dimensional orbifold with underlying manifold point set $X_{\mathcal{O}}$, cone points of orders $p_1, p_2, \dots, p_{\alpha}$ and corner reflectors of orders $q_1, q_2, \dots, q_{\beta}$ is denoted by

$$\mathcal{O} = (X_{\mathcal{O}}; p_1, p_2, \dots, p_{\alpha}; q_1, q_2, \dots, q_{\beta}).$$

DEFINITION. The *Euler Characteristic* of an orbifold $\mathcal{O}$, where

$$\mathcal{O} = (X_{\mathcal{O}}; p_1, p_2, \dots, p_{\alpha}; q_1, q_2, \dots, q_{\beta}),$$

is given by the formula

$$\chi(\mathcal{O}) = \chi(X_{\mathcal{O}}) - \sum_{i=1}^{\alpha} \left(1 - \frac{1}{p_i}\right) - \frac{1}{2} \sum_{i=1}^{\beta} \left(1 - \frac{1}{q_i}\right).$$

Note that the Euler characteristic of a manifold, thought of as an orbifold with no singularity points, is by this definition the normal Euler characteristic.

PROPOSITION II.3.2. [Sc₂, p. 430] *The quotient space $\mathcal{O}$ of a Seifert fibered space M , obtained by identifying each fiber to a point, is a surface with a natural orbifold structure. In particular, each exceptional fiber of order n maps to a cone point of order n , and each ordinary fiber maps to an ordinary point.*

THEOREM II.3.3. [Sc₂, pp. 476–478] *Each Seifert fibered space M admits a geometric structure, which is unique to the underlying manifold. In particular,*

- (1) *if $\chi(\mathcal{O}) > 0$, M has the geometry $\mathbf{S}^2 \times \mathbf{R}$ or $\mathbf{S}^3$,*
- (2) *if $\chi(\mathcal{O}) = 0$, M has the geometry $\mathbf{E}^3$ or $\mathbf{Nil}$,*
- (3) *if $\chi(\mathcal{O}) < 0$, M has the geometry $\mathbf{H}^2 \times \mathbf{R}$ or $\widetilde{\mathbf{SL}_2\mathbf{R}}$.*

In each case, M has the first geometry exactly when $e(\mathcal{F}) = 0$, where $e(\mathcal{F}_n)$ is the Euler number of any Seifert bundle $\mathcal{F}$ of M .

This is a principal result of Peter Scott's paper, "The Geometries of 3-Manifolds."

COROLLARY II.3.4. *If $\chi(\mathcal{O}) < 0$, then the Seifert fibration of M is unique, up to fiber-preserving homeomorphism. Therefore, the base orbifold associated with M is unique.*

PROOF. This corollary follows immediately from Theorems II.1.2 and II.3.3. ■

§4. Determining Orbifolds for each $L_{n,n-2,\frac{1}{3}}$.

The goal of this section is to prove the following theorem.

THEOREM II.4.1. *The manifolds $L_{n,n-2,\frac{1}{3}}$, $n \in \mathbb{Z}_+$, are all distinct.*

Define $\mathcal{O}_n$ to be the base orbifold of the Seifert fibration $\mathcal{F}_n$ of $L_{n,n-2,\frac{1}{3}}$. Let X_n be the underlying point set of $\mathcal{O}_n$. Denote the quotient maps which carry fibers to points as $\pi_{\mathcal{F}_n} : L_{n,n-2,\frac{1}{3}} \rightarrow X_n$ and $\pi_{\mathcal{F}'_n} : \Sigma_n \rightarrow \hat{P}_n$, the upper hemisphere. Let $g'_n : \hat{P}_n \rightarrow X_n$ be the map which identifies points which map to the same fiber $f \in \mathcal{F}_n$.

The following lemma then follows, by definition.

LEMMA II.4.2. *The following diagram commutes:*

$$\begin{array}{ccc} \Sigma_n & \xrightarrow{\pi_{\mathcal{F}'_n}} & \hat{P}_n \\ g_n \downarrow & & g'_n \downarrow \\ L_{n,n-2,\frac{1}{3}} & \xrightarrow{\pi_{\mathcal{F}_n}} & X_n \end{array}$$

PROPOSITION II.4.3.

- (1) *If $n = 6k$, then $X_n = T^2$, the two-torus.*
- (2) *If $n \neq 6k$, then $X_n = S^2$, the two-sphere.*

PROOF. Each fiber which passes through the face interiors passes at least once through $\Delta_1 \cup \Delta_2$. Each fiber which passes through the one-skeleton passes at least once through $a_1 \cup a_2 \cup a'_1$. Hence, $X_n = \Delta_1 \cup \Delta_2 / g'_n$. Note that $a_1 \sim a'_2$ and $a_2 \sim a'_0{}^{-1}$ under g'_n .

CASE 1. If $(n, 6) = 6$, then each fiber passes once through the interior of Δ_1 , Δ_2 , a_1 , a_2 , a'_1 , or passes through the vertices. Hence, the only identification when passing from $\Delta_1 \cup \Delta_2$ to X_n is the identification of the boundary in orientation-reversing fashion, as shown in Figure II.4a. The resulting surface is clearly T^2 , either by a computation of the Euler characteristic, or directly from the figure.

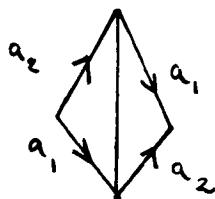
CASE 2. If $(n, 6) = 3$, then each fiber which passes through the interior of Δ_1 also passes through the interior of Δ_2 . However, those which pass through edge interiors will pass through one of a_1 , a'_0 , or a'_1 . In this case, each edge is identified with its inverse, so these three edges can be bisected into a pair of edges which are folded together, as shown in Figure II.4b. The resulting surface has Euler characteristic equal to 2, and hence is S^2 .

CASE 3. If $(n, 6) = 2$, then each fiber passes three times through the interior of either Δ_1 or Δ_2 , or else passes once through each of the edges. The resulting surface can thus be represented by a'_1 , together with a third of the faces Δ_1 and Δ_2 , identified as shown in Figure II.4c. The resulting surface has Euler characteristic equal to 2, and hence is S^2 .

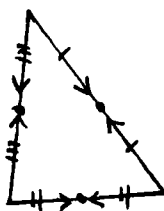
CASE 4. If $(n, 6) = 1$, then each fiber passes three times through each face interior, twice through each edge, or through the vertices. Therefore, the resulting surface can be represented by a third of Δ_1 , identified as shown in Figure II.4d. The resulting surface has Euler characteristic equal to 2, and hence is S^2 . ■

THEOREM II.4.4.

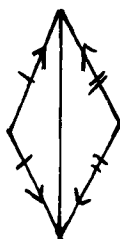
$$\mathcal{O}_n = \begin{cases} (S^2; n, 2, 3), & \text{if } (n, 6) = 1 \\ (S^2; \frac{n}{2}, 3, 3), & \text{if } (n, 6) = 2 \\ (S^2; \frac{n}{3}, 2, 2, 2), & \text{if } (n, 6) = 3 \\ (T^2; \frac{n}{6}), & \text{if } (n, 6) = 6. \end{cases}$$



a. The Surface X_n of Proposition II.4.3, Case 1.



b. The Surface X_n of Proposition II.4.3, Case 2.



c. The Surface X_n of Proposition II.4.3, Case 3.



d. The Surface X_n of Proposition II.4.3, Case 4.

Figure II.4. The Surfaces of Proposition II.4.3.

PROOF. By Proposition II.3.2, each cone point of $\mathcal{O}_n$ of order q corresponds to an exceptional fiber of $\mathcal{F}_n$ of order q . The result then follows from Theorem II.2.8, together with Proposition II.4.3. ■

COROLLARY II.4.5.

$$\chi(\mathcal{O}_n) = \begin{cases} -\frac{1}{6} + \frac{1}{n} & \text{if } (n, 6) = 1 \\ -\frac{1}{3} + \frac{2}{n} & \text{if } (n, 6) = 2 \\ -\frac{1}{2} + \frac{3}{n} & \text{if } (n, 6) = 3 \\ -1 + \frac{6}{n} & \text{if } (n, 6) = 6. \end{cases}$$

PROOF. The result follows from the previous theorem, using the definition of the Euler characteristic for orbifolds given in the preceding section. ■

PROPOSITION II.4.6. *When $n > 6$, there is a unique Seifert fibration for $L_{n,n-2,\frac{1}{3}}$, up to fiber-preserving homeomorphism.*

PROOF. By Corollary II.4.5, the Euler characteristic is negative for all $n > 6$. By Corollary II.3.4, it follows that the Seifert fibration of $L_{n,n-2,\frac{1}{3}}$ is unique, up to fiber-preserving homeomorphism. ■

PROOF OF THEOREM II.4.1.

For $n \leq 6$, the manifolds $L_{n,n-2,\frac{1}{3}}$ are all distinct, for their fundamental groups are distinct, by Proposition I.2.4. By Theorem II.3.3, these manifolds are distinct from those $L_{n,n-2,\frac{1}{3}}$ where $n > 6$, as they have different geometries.

By Proposition II.4.6, we know that, for $n > 6$, the orbifold structure on each $L_{n,n-2,\frac{1}{3}}$ is unique. However, by Theorem II.4.4, it is clear that these orbifolds are distinct. Therefore, the manifolds $L_{n,n-2,\frac{1}{3}}$, for $n > 6$, are all distinct, and the result follows. ■

§5. $L_{n,n-2,\frac{1}{3}}$ as Cyclic Branched Covers of S^3 .

In this section we will show that the twisted lens spaces are homeomorphic to the cyclic covers of S^3 , branched over the torus knot $\tau(2, 3)$, also known as the trefoil knot. Precisely, we will show the following theorem.

THEOREM II.5.1. $L_{n,n-2,\frac{1}{3}}$ is the n -to-1 cyclic cover of S^3 , branched over the trefoil.

To this end the following result of Seifert will be of use.

PROPOSITION II.5.2. [Se, §11] Suppose that $(M, \mathcal{F})$ is a homology 3-sphere, together with a Seifert fibration having at most two exceptional fibers, of orders p and q , where $(p, q) = 1$. Then $M = S^3$ and the ordinary fibers of $\mathcal{F}$ are $\tau(p, q)$ torus knots.

PROOF OF THEOREM II.5.1. By Proposition I.2.4, the manifold $L_{1,0,\frac{1}{3}}$ is a homotopy sphere. Further, by Theorem II.2.8, the Seifert fibering of this manifold derived by the vertical fibering of the solid Σ_1 has exactly two exceptional fibers, of orders 2 and 3. Hence, by Proposition II.5.2, each ordinary fiber is a $\tau(2, 3)$ torus knot. In particular, the fiber of $L_{1,0,\frac{1}{3}}$ formed by the center fiber of Σ_1 is a trefoil.

Σ_n is the union of n wedges, each the join of a_i with the center fiber. The solid Σ_n is an n -fold cyclic cover of Σ_1 , branched over the center fiber, via the map which identifies these wedges.

Therefore, we have the following diagram, where g_i denotes the gluing maps.

$$\begin{array}{ccc} \Sigma_n & \xrightarrow{g_n} & L_{n,n-2,\frac{1}{3}} \\ \pi_n \downarrow & & \pi'_n \downarrow \\ \Sigma_1 & \xrightarrow{g_1} & L_{1,0,\frac{1}{3}} \end{array}$$

We claim that the covering map π_n induces a map π'_n on the twisted lens spaces. Since the gluing maps g_i are one-to-one on the interior of Σ_i , we certainly have $\pi'_n : g_n(\text{int } \Sigma_n) \rightarrow g_1(\text{int } \Sigma_1)$. However, for each point p in $g_n(\text{bd } \Sigma_n)$, $\pi_n g_n^{-1}(p)$ consists of two points which are identified under g_1 . Therefore, we can extend π'_n as claimed.

The deck transformation group on Σ_n is Z_n . These actions consist of rotations, and are generated by the rotation of $\frac{1}{n}$ revolution, carrying each wedge to the adjacent wedge, counterclockwise. Such rotations preserve the gluing pattern. That is, each pair of points which is identified under g_n is carried to another such pair of points. Therefore, the group of deck transformations on Σ_n induces an isomorphic group of deck transformations on $L_{n,n-2,\frac{2}{3}}$.

Further, the center branching fiber of Σ_n is carried to the branching set of $L_{n,n-2,\frac{1}{3}}$, which is mapped by π'_n to the image of the center fiber of Σ_1 . This fiber in $L_{1,0,\frac{1}{3}}$ is the trefoil. Therefore, each $L_{n,n-2,\frac{1}{3}}$ is an n -fold cyclic cover of $L_{1,0,\frac{1}{3}} = S^3$, branched over a trefoil knot. ■

In Chapter IV, the gluing scheme which produces the twisted lens spaces is generalized to a schema which constructs manifolds which are similarly recognizable as the cyclic covers of S^3 , branched over an arbitrary torus knot, using Proposition II.5.2.

COROLLARY II.5.3. *The cyclic covers of S^3 , branched over the trefoil, are distinct.*

PROOF. The result follows immediately from Theorems II.4.1 and II.5.1. ■

Corollary II.5.3 is a special case of Theorem IV.6.6. In the subsequent chapter it is shown that for an arbitrary torus knot $\tau(p, q)$, the cyclic covers of S^3 , branched over $\tau(p, q)$, are all distinct.

By the work of Chapter I, a stronger statement can be made when considering many pairs of cyclic coverings, branched over the trefoil.

COROLLARY II.5.4. *Suppose n, m are distinct integers such that*

- (1) n, m are not consecutive odds of the form $6k \pm 1$, for some fixed integer k ,*
- (2) if 3 divides both n and m , then $n + m$ is odd.*

Then the n -to-one and m -to-one cyclic covers of S^3 , branched over the trefoil, have distinct fundamental groups.

PROOF. The result follows immediately from Theorems I.5.1 and II.5.1. ■

CHAPTER III

THE TWO-RULE TWISTED LENS SPACES

§1. The construction of Two-Rule Twisted Lens Spaces.

In this chapter we generalize the construction of twisted lens spaces to allow separate identification rules for the even and odd faces of Σ_n , labelled as before. Here n is even, k and l , elements of $\mathbf{Z}_n$, are both odd or both even, and r and s are each elements of $\{\frac{1}{3}, \frac{2}{3}, 0\}$. The complex $L_{n,(k,l),(r,s)}$ is constructed by identifying each even numbered upper face with the lower face located by turning $\frac{k}{n}$ rotation, after twisting r , and similarly identifying each odd numbered face with the lower face located by turning $\frac{l}{n}$ rotation, after twisting s . Note that when $k = l$ and $r = s$, $L_{n,(k,l),(r,s)}$ is the complex $L_{n,k,r}$ of the previous chapter.

§2. Determining the Manifolds among the Two-Rule Twisted Lens Spaces.

We wish to classify these complexes by their twisting data (r, s) . By renumbering the faces, we have that $L_{n,(k,l),(r,s)}$ and $L_{n,(l,k),(s,r)}$ are homeomorphic. As before in Section I.2, we can construct a homeomorphism between $L_{n,(k,l),(r,s)}$ and $L_{n,(-k,-l),(-r,-s)}$ induced by the extension of the homeomorphism between copies of S^1 with different orientations, first suspending to a map on S^2 , and then coning to a map on $B^3 \cong \Sigma_n$. Therefore, it suffices to consider

$$(r, s) \in \{(0, 0), (0, \frac{1}{3}), (\frac{1}{3}, \frac{1}{3}), (\frac{1}{3}, \frac{2}{3})\}.$$

We will consider each of these cases separately. In addition, we will consider the cases k, l even and odd separately.

NOTATION. In each case, the directed edge pointing from v_{i-1} to v_i is denoted a_i . The directed edge from the upper cone point to v_i is denoted a'_i , and the directed edge from v_i to the lower cone point is denoted a''_i .

CASE 1. $L_{n,(k,l),(0,0)}$; k, l even.

This construction requires the following equivalences, where i is even:

$$\begin{aligned} a'_i &\sim a''_{i+l}^{-1} \sim a'_{i+l-k} \\ a'_{i+1} &\sim a''_{i+1+l}^{-1} \sim a'_{i+1+l-k} \\ a_i &\sim a_{i+k} \\ a_{i+1} &\sim a_{i+1+l}. \end{aligned}$$

Here, if $k = l$, then the complex is $L_{n,k,0}$, the ordinary lens space $L_{n,k}$. We claim that this complex is a manifold only when $k = l$.

Without loss of generality, we may assume that $0 \leq k < l < n$. If the complex is a manifold, then the Euler characteristic must be zero. In each case, the complex has n faces and one solid. By the equivalence data given above, we find that the complex has V vertices and E edges, as given below. The complex is a manifold if and only if $E - V = n - 1$.

$$\begin{aligned} V &= 1 + \gcd(n, k, l) \\ E &= \frac{1}{2}\gcd(n, k) + \frac{1}{2}\gcd(n, l) + \gcd(n, l - k). \end{aligned}$$

Indeed, the vertices of the complex may be counted as follows. The two poles are identified, and are not identified with any other vertex. The remaining vertices are identified so that, for all j ,

$$v_j \sim v_{j+k} \sim v_{j+l}.$$

Therefore, the even numbered and the odd numbered vertices are each partitioned into $\frac{1}{2}gcd(n, k, l)$ equivalence classes.

The even and odd numbered lateral edges of the complex are partitioned into $\frac{1}{2}gcd(n, k)$ and $\frac{1}{2}gcd(n, l)$ equivalence classes, respectively. The even and odd vertical edges are each partitioned into $\frac{1}{2}gcd(n, l - k)$ equivalence classes.

SUBCASE 1. Suppose $0 = k < l < n$. Here $gcd(n, k) = n$ and $gcd(n, k, l) = gcd(n, l) \leq \frac{n}{2}$, so

$$\begin{aligned}
 n - 1 &= E - V \\
 &= [\frac{1}{2}gcd(n, k) + \frac{1}{2}gcd(n, l) + gcd(n, l - k)] - [1 + gcd(n, k, l)] \\
 &= [\frac{n}{2} + \frac{1}{2}gcd(n, l) + gcd(n, l)] - [1 + gcd(n, l)] \\
 &= \frac{n}{2} - 1 + \frac{1}{2}gcd(n, l) \\
 &\leq \frac{3}{4}n - 1.
 \end{aligned}$$

Contradiction.

SUBCASE 2. Suppose $0 < k < l < n$. Here $gcd(n, k), gcd(n, l), gcd(n, l - k) \leq \frac{n}{2}$, and $gcd(n, k, l) \geq 2$. Then

$$\begin{aligned}
 n - 1 &= E - V \\
 &= [\frac{1}{2}gcd(n, k) + \frac{1}{2}gcd(n, l) + gcd(n, l - k)] - [1 + gcd(n, k, l)] \\
 &\leq [\frac{n}{4} + \frac{n}{4} + \frac{n}{2}] - [1 + 2] \\
 &= n - 3.
 \end{aligned}$$

Contradiction.

CASE 2. $L_{n,(k,l),(0,0)}; k, l$ odd.

This construction requires the following equivalences, where i is even:

$$\begin{aligned} a'_i &\sim a''_{i+l}^{-1} \sim a'_{i+l-k} \\ a'_{i+1} &\sim a''_{i+1+l}^{-1} \sim a'_{i+1+l-k} \\ a_i &\sim a_{i+k} \sim a_{i+k+l}. \end{aligned}$$

Here, if $k = l$, then the complex is $L_{n,k,0}$, the ordinary lens space $L_{n,k}$. We claim that this complex is a manifold only when $k = l$.

We determine the number of vertices V and edges E to be as given below.

$$\begin{aligned} V &= 1 + \gcd(n, l, k) \\ E &= \frac{1}{2}\gcd(n, k+l) + \gcd(n, k-l). \end{aligned}$$

Indeed, as before, the poles are identified, and are not identified with any other vertex. The remaining vertices are identified so that, for all j ,

$$v_j \sim v_{j+l} \sim v_{j+k}.$$

The even and odd lateral edges are paired by construction, and form $\frac{1}{2}\gcd(n, k+l)$ equivalence classes. The even and odd numbered vertical edges are each partitioned into $\frac{1}{2}\gcd(n, l-k)$ equivalence classes.

Assuming $k \neq l$, note that $\gcd(n, l-k) \leq \frac{n}{2}$, $\gcd(n, k+l) \leq n$, and $V \geq 2$, so

$$\begin{aligned} n-1 &= E - V \\ &= [\frac{1}{2}\gcd(n, k+l) + \gcd(n, k-l)] - 2 \\ &\leq [\frac{n}{2} + \frac{n}{2}] - 2 \\ &= n-2. \end{aligned}$$

Contradiction.

CASE 3. $L_{n,(k,l),(\frac{1}{3},\frac{1}{3})}; k, l$ even.

This gluing construction identifies the following edges, where i is even:

$$\begin{aligned} a_i &\sim a''_{i+k} & a_{i+1} &\sim a''_{i+1+l} \\ a'_i &\sim a''_{i-1+k} & a'_{i+1} &\sim a''_{i+1} \\ a'_{i-1} &\sim a_{i+k} & a'_i &\sim a_{i+1+l}. \end{aligned}$$

This construction requires the following equivalences, where i is even:

$$\begin{aligned} a_i &\sim a''_{i+k} \sim a'_{i+1+k-l} \sim a_{i+2+2k-l} \\ a_{i+1} &\sim a''_{i+1+l} \sim a'_{i+2-k+l} \sim a_{i+3-k+2l}. \end{aligned}$$

Note that all of the vertices are identified, so $V = 1$. Therefore, the complex is a manifold if and only if $E = n$. Since the $3n$ edges lie in cosets each containing at least three edges, there are at most n edges in the resulting complex. In particular, there are exactly n edges when

$$2 + 2k = l \text{ and } 2 + 2l = k \pmod{n}.$$

Therefore, $6 + 3l = 3(2 + l) = 0 \pmod{n}$. Hence, if 3 does not divide n , then $l = n - 2$; if 3 divides n , then $2 + l = 0 \pmod{\frac{n}{3}}$.

SUBCASE 1. If $l = n - 2$, then $k = l$ and the manifold is $L_{n,(n-2,n-2),(\frac{1}{3},\frac{1}{3})} = L_{n,n-2,\frac{1}{3}}$, a twisted lens space discussed in the previous chapter.

SUBCASE 2. If $3|n$ and $2 + l = 0 \pmod{\frac{n}{3}}$, then

$$(k, l) \in \{(n-2, n-2), (\frac{n}{3}-2, \frac{2}{3}n-2), (\frac{2}{3}n-2, \frac{n}{3}-2)\}.$$

The case $(k, l) = (n-2, n-2)$ has been considered. Further, since $L_{n,(k,l),(\frac{1}{3},\frac{1}{3})}$ and $L_{n,(l,k),(\frac{1}{3},\frac{1}{3})}$ are homeomorphic, it suffices to investigate the manifolds

$$L_{n,(\frac{n}{3}-2,\frac{2}{3}n-2),(\frac{1}{3},\frac{1}{3})}, \quad \text{where } 6|n.$$

CASE 4. $L_{n,(k,l),(\frac{1}{3},\frac{1}{3})}; k, l$ odd.

This gluing construction identifies the following edges, where i is even:

$$\begin{aligned} a_i &\sim a''_{i+k} & a_{i+1} &\sim a''_{i+1+l} \\ a'_i &\sim a''_{i-1+k} & a'_{i+1} &\sim a''_{i+l} \\ a'_{i-1} &\sim a_{i+k} & a'_i &\sim a_{i+1+l}. \end{aligned}$$

This construction requires the following equivalences, where i is even:

$$a_i \sim a''_{i+k} \sim a'_{i+1+k-l} \sim a_{i+2+2k-l} \sim a''_{i+2+2k} \sim a'_{i+3+k} \sim a_{i+4+k+l}.$$

Therefore, the number of edges of the complex is at most $\frac{n}{2}$. All of the vertices are identified, so $E - V < n - 1$. Hence, no manifolds arise in this case.

CASE 5. $L_{n,(k,l),(0,\frac{1}{3})}; k, l$ even.

This gluing construction identifies the following edges, where i is even:

$$\begin{aligned} a_i &\sim a_{i+k} & a_{i+1} &\sim a''_{i+1+l} \\ a'_i &\sim a''_{i+k}^{-1} & a'_{i+1} &\sim a''_{i+l} \\ a'_{i-1} &\sim a''_{i-1+k}^{-1} & a'_i &\sim a_{i+1+l}. \end{aligned}$$

This construction requires the following equivalences, where i is even:

$$\begin{aligned} a_i &\sim a_{i+k} \\ a_{i+1} &\sim a''_{i+1+l} \sim a'^{-1}_{i+1-k+l} \sim a''^{-1}_{i-k+2l} \sim a'_{i-2k+2l} \sim a_{i+1-2k+3l}. \end{aligned}$$

All of the vertices are identified, so $V = 1$. Therefore, the complex is a manifold if and only if $E = n$. There are at most n edges in the complex, and there are exactly n edges when

$$k = 0 \text{ and } 2k = 3l \pmod{n}.$$

Therefore, $3l = 0 \pmod n$. If 3 does not divide n , then $l = 0$; if 3 divides n , then $l \in \{0, \frac{n}{3}, \frac{2}{3}n\}$. Therefore, it suffices to investigate the following manifolds:

$$L_{n,(0,0),(0,\frac{1}{3})}, \text{ where } 2|n$$

$$L_{n,(0,\frac{n}{3}),(0,\frac{1}{3})} \text{ and } L_{n,(0,\frac{2}{3}n),(0,\frac{1}{3})}, \text{ where } 6|n.$$

CASE 6. $L_{n,(k,l),(0,\frac{1}{3})}; k, l$ odd.

This gluing construction identifies the following edges, where i is even:

$$\begin{aligned} a_i &\sim a_{i+k} & a_{i+1} &\sim a''_{i+1+l} \\ a'_i &\sim a''_{i+k}^{-1} & a'_{i+1} &\sim a''_{i+l} \\ a'_{i-1} &\sim a''_{i-1+k}^{-1} & a'_i &\sim a_{i+1+l}. \end{aligned}$$

This construction requires the following equivalences, where i is even:

$$a_i \sim a_{i+k} \sim a''_{i+k+l} \sim a'_{i+l}^{-1} \sim a''_{i-1+2l}^{-1} \sim a'_{i-1-k+2l} \sim a_{i-k+3l}.$$

Therefore, the number of edges of the complex is at most $\frac{n}{2}$. All of the vertices are identified, so $E - V < n - 1$. Hence, no manifolds arise in this case.

CASE 7. $L_{n,(k,l),(\frac{1}{3},\frac{2}{3})}; k, l$ even.

This gluing construction identifies the following edges, where i is even:

$$\begin{aligned} a_i &\sim a''_{i+k} & a_{i+1} &\sim a''_{i+l}^{-1} \\ a'_i &\sim a''_{i-1+k} & a'_{i+1} &\sim a_{i+1+l}^{-1} \\ a'_{i-1} &\sim a_{i+k} & a'_i &\sim a''_{i+1+l}. \end{aligned}$$

This construction requires the following equivalences, where i is even:

$$\begin{aligned} a_i &\sim a''_{i+k} \sim a_{i+1+k-l}^{-1} \sim a'_{i+1+k-2l} \sim a_{i+2+2k-2l} \\ a'_i &\sim a''_{i-1+k} \sim a'_{i-2+k-l}. \end{aligned}$$

All of the vertices are identified, so $V = 1$. Therefore, the complex is a manifold if and only if $E = n$. There are at most n edges in the complex, and there are exactly n edges when

$$2 + 2k = 2l \text{ and } k = 2 + l \pmod{n}.$$

Therefore, $6 = 0 \pmod{n}$, and hence n divides 6. Since n is even by hypothesis, $n = 2$ or 6.

If $n = 2$, then $(k, l) = (0, 0)$. If $n = 6$, then $k = 2 + l \pmod{6}$, and therefore

$$(k, l) \in \{(2, 0), (4, 2), (0, 4)\}.$$

In this case, only the four complexes with the above data are manifolds.

CASE 8. $L_{n, (k, l), (\frac{1}{3}, \frac{2}{3})}; k, l$ odd.

This gluing construction identifies the following edges, where i is even:

$$\begin{aligned} a_i &\sim a''_{i+k} & a_{i+1} &\sim a''_{i+l}^{-1} \\ a'_i &\sim a''_{i-1+k} & a'_{i+1} &\sim a_{i+1+l}^{-1} \\ a'_{i-1} &\sim a_{i+k} & a'_i &\sim a''_{i+1+l}. \end{aligned}$$

This construction requires the following equivalences, where i is even:

$$\begin{aligned} a_i &\sim a''_{i+k} \sim a_{i+1+k-l}^{-1} \sim a'_{i-l}^{-1} \sim a_i \\ a'_i &\sim a''_{i-1+k} \sim a'_{i-2+k-l}. \end{aligned}$$

The tails of the edges which are identified by each equivalence relation above are identified in that:

$$\begin{aligned} v_{i-1} &\sim v_{i+k} \sim v_{i+1+k-l} \sim v_{i-l} \sim v_{i-1} \\ v_i &\sim p \sim p', \end{aligned}$$

where p and p' are the cone points of the suspension. All of the even vertices are identified under the gluing, and none of the odd vertices are identified with an even vertex. Hence, there are at least two vertices in the resulting complex. However, there are at most n edges. Therefore, $E - V \leq n - 2$, so none of these complexes are manifolds.

We have just shown the following proposition.

PROPOSITION III.2.1. *Suppose n is even. Of the complexes $L_{n,(k,l),(r,s)}$, only the following are manifolds, up to renumbering and re-orientation.*

- (1) $L_{n,(k,k),(0,0)} = L_{n,k}$,
- (2) $L_{n,(n-2,n-2),(\frac{1}{3},\frac{1}{3})} = L_{n,n-2,\frac{1}{3}}$,
- (3) $L_{n,(\frac{n}{3}-2,\frac{2}{3}n-2),(\frac{1}{3},\frac{1}{3})}$, where $6|n$,
- (4) $L_{n,(0,0),(0,\frac{1}{3})}$,
- (5) $L_{n,(0,\frac{n}{3}),(0,\frac{1}{3})}$, where $6|n$,
- (6) $L_{n,(0,\frac{2}{3}n),(0,\frac{1}{3})}$, where $6|n$,
- (7) $L_{2,(0,0),(\frac{1}{3},\frac{2}{3})}$,
- (8) $L_{6,(k,l),(\frac{1}{3},\frac{2}{3})}$, where $(k,l) \in \{(2,0), (4,2), (0,4)\}$.

The lens spaces are included above as the first item, and the (one-rule) twisted lens spaces of the previous chapter are included as the second item. However, no claim is made that the manifolds of this list are distinct.

PROPOSITION III.2.2.

- (1) $\pi_1(L_{n,(0,0),(0,\frac{1}{3})}) = *_{\frac{n}{2}}\mathbb{Z}_3$, for n even.
- (2) $\pi_1(L_{n,(0,\frac{n}{3}),(0,\frac{1}{3})}) = \pi_1(L_{n,(0,\frac{2}{3}n),(0,\frac{1}{3})}) = *_{\frac{n}{3}}\mathbb{Z}$, for n divisible by 6.

PROOF. As in the previous chapter, each coset of edges, represented by the edge a_i , corresponds to the generator x_i of a presentation of the fundamental group,

and the faces of the upper hemisphere correspond to relators of the presentation, after using the equivalence relations described in the proof of Proposition III.2.1, Case 5.

CASE 1. Under the identification of edges, the boundaries of the upper faces are, for i even,

$$\begin{aligned} bd \Delta_i &= a'_{i-1}{}^{-1} a'_i a_i^{-1} = a_{i-1} a_{i+1} a_i^{-1} \\ bd \Delta_{i+1} &= a'_i a_{i+1} a'_{i+1}{}^{-1} = a_{i+1}^3. \end{aligned}$$

Hence, the fundamental group has the following presentation:

$$\langle x_1, x_2, \dots, x_n | x_{i+1}^3, x_{i-1} x_{i+1} x_i^{-1}, \text{ for all even } i \in \mathbb{Z}_n \rangle.$$

However, this presentation may be transformed using Tietze transformations, by omitting the even indexed generators and the second type of relators, leading to the following presentation:

$$\langle x_1, x_3, \dots, x_{n-1} | x_i^3, \text{ for all odd } i \in \mathbb{Z}_n \rangle.$$

This group is the $\frac{n}{2}$ -fold free product of $\mathbb{Z}_3$.

CASE 2. First consider the case $l = \frac{n}{3}$. Under the identification of edges, the boundaries of the upper faces are, for i even,

$$\begin{aligned} bd \Delta_i &= a'_{i-1}{}^{-1} a'_i a_i^{-1} = a_{i-1+\frac{2}{3}n} a_{i+1+\frac{n}{3}} a_i^{-1} \\ bd \Delta_{i+1} &= a'_i a_{i+1} a'_{i+1}{}^{-1} = a_{i+1+\frac{n}{3}} a_{i+1} a_{i+1+\frac{2}{3}n}. \end{aligned}$$

Hence, the fundamental group has the following presentation:

$$\langle x_1, x_2, \dots, x_n | x_{i+1-\frac{n}{3}} x_{i-1+\frac{n}{3}} x_i^{-1}, x_{i+1} x_{i+1-\frac{n}{3}} x_{i+1+\frac{n}{3}}, \text{ for all even } i \in \mathbb{Z}_n \rangle.$$

However, this presentation may be transformed using Tietze transformations, omitting the even indexed generators and the first type of relators. Note also that each relator of the second type is listed three times. These transformations therefore lead to the following presentation:

$$\langle x_1, x_3, \dots, x_{n-1} \mid x_i x_{i-\frac{n}{3}} x_{i+\frac{n}{3}}, \text{ for all odd } i \in [1, \frac{n}{3} - 1] \rangle.$$

Now, by using Tietze transformations, we can eliminate the remaining relators and the final third of the generators, yielding the following presentation:

$$\langle x_1, x_3, \dots, x_{\frac{2}{3}n-1} \mid \quad \rangle.$$

This group is the $\frac{n}{3}$ -fold free product of $\mathbf{Z}$. The case for $l = \frac{2}{3}n$ follows in exactly the same way, yielding the same fundamental group after Tietze transformations. ■

In fact, the manifolds of the previous proposition can be seen geometrically as connected sums of either the lens space $L_{3,1}$ or $S^2 \times S^1$. Cone from the bisector of each even indexed edge a_i to the center arc joining the poles p, p' , forming disks. After gluing, each pair of adjacent disks forms a two-sphere. Push slightly to make these disks disjoint. In the first case, the resulting complementary regions consist of S^3 with $\frac{n}{2}$ three-balls removed, and $\frac{n}{2}$ complementary regions, each a copy of the lens space $L_{3,1}$ with a three-ball removed. In the second case, the resulting complementary regions consist of S^3 with $\frac{n}{2}$ three-balls removed, and $\frac{n}{6}$ complementary regions, each a copy of $S^2 \times S^1$, with three three-balls removed. The identification of two common boundary components of each $S^2 \times S^1 \setminus \text{three balls}$ with $S^3 \setminus \frac{n}{2}$ balls produces another copy of $S^2 \times S^1$.

PROPOSITION III.2.3.

$$H_1(L_{n,(\frac{n}{3}-2, \frac{2}{3}n-2),(\frac{1}{3}, \frac{1}{3})}) = \begin{cases} \mathbf{Z} \oplus \mathbf{Z} & \text{when } (n, 18) = 18 \\ \mathbf{Z}_9 \oplus \mathbf{Z}_3 & \text{when } (n, 18) = 6. \end{cases}$$

PROOF. A presentation of the fundamental group is found as before, induced by the resulting two-skeleton. Under the identification of edges, in the proof of Proposition III.2.1, Case 3, the boundaries of the upper faces are, for i even,

$$\begin{aligned} bd \Delta_i &= a_i a'_i{}^{-1} a'_{i-1} = a_i a_{i-1+\frac{2}{3}n}^{-1} a_{i-2+\frac{n}{3}} \\ bd \Delta_{i+1} &= a_{i+1} a'_{i+1}{}^{-1} a'_i = a_{i+1} a_{i+\frac{n}{3}}^{-1} a_{i-1+\frac{2}{3}n}. \end{aligned}$$

By increasing each index of the first type relator by $2 + \frac{2}{3}n$ and each index of the second type relator by $2 + \frac{1}{3}n$, the fundamental group has the following presentation:

$$\langle x_1 x_2, \dots, x_n \mid x_i x_{i+2+\frac{2}{3}n} x_{i+1+\frac{n}{3}}^{-1}, x_{i+1} x_{i+3+\frac{n}{3}} x_{i+2+\frac{2}{3}n}^{-1}, \text{ for all even } i \in \mathbb{Z}_n \rangle.$$

For even i , we have, by the first relator, that

$$x_{i+2+\frac{2}{3}n} x_{i+4+\frac{n}{3}} x_{i+3}^{-1} = 1,$$

and hence

$$x_i x_{i+3} = x_i x_{i+2-\frac{n}{3}} x_{i+4+\frac{n}{3}} = x_{i+1+\frac{n}{3}} x_{i+4+\frac{n}{3}}.$$

Similarly, by the second relator, we have that, for all even i ,

$$x_{i+3+\frac{n}{3}} x_{i+5+\frac{2}{3}n} x_{i+4}^{-1} = 1,$$

and hence

$$x_{i+1} x_{i+4} = x_{i+1} x_{i+3+\frac{n}{3}} x_{i+5-\frac{n}{3}} = x_{i+2-\frac{n}{3}} x_{i+5-\frac{n}{3}}.$$

Therefore, by these two facts together, we have that, for i even,

$$x_i x_{i+3} = x_{i+1+\frac{n}{3}} x_{i+4+\frac{n}{3}} = x_{i+2} x_{i+5}.$$

It follows by induction that, all even i and any integer κ ,

$$x_i x_{i+3} = x_{i+2\kappa} x_{i+3+2\kappa}.$$

However, since

$$x_{i+1}x_{i+4} = x_{i-\frac{n}{3}}x_{i+3-\frac{n}{3}} = x_i x_{i+3},$$

for all even i , it follows that, for all $j, j' \in \mathbb{Z}_n$,

$$x_j x_{j+3} = x_{j'} x_{j'+3}.$$

In particular, for all j , $x_j x_{j+3} = x_{j+3} x_{j+6}$; that is,

$$x_j^{x_{j+3}} = x_{j+3}^{-1} x_j x_{j+3} = x_{j+6}.$$

Therefore, under abelianization, by appending the relators $[x_j, x_{j'}]$, for all $j, j' \in \mathbb{Z}_n$, we have that $x_j = x_{j+6}$, for all j . How does this fact affect the presentation of the abelianized group? The situation falls into two cases.

CASE 1. $n = 18k$. Then the homology group has as presentation

$$\langle x_1, x_2, \dots, x_6 \mid [x_i, x_j], x_i x_{i+2} x_{i+1}^{-1}, \text{ for all } i, j \in \mathbb{Z}_6 \rangle.$$

By Proposition I.2.6, we know that this group is $\mathbb{Z} \oplus \mathbb{Z}$.

CASE 2. $n = 18k \pm 6$. If $n = 18k - 6$ for some integer k , then $\frac{n}{3} = 4 \pmod{6}$ and the homology group has as presentation

$$\langle x_1, x_2, \dots, x_6 \mid [x_j, x_{j'}], x_i x_{i+4} x_{i+5}^{-1}, x_{i+1}^2 x_{i+4}^{-1}, \text{ for all } j, j', \text{ for even } i \in \mathbb{Z}_6 \rangle.$$

If $n = 18k + 6$, the group has this presentation after increasing the index of each x_i by one.

By Tietze transformations, we can eliminate x_2, x_4, x_6 , and the second type of relators. By further Tietze transformations, another generator and relator are eliminated, resulting in the following presentation:

$$\langle x_1, x_3 \mid [x_1, x_3], x_1^3 x_3^6, x_1^6 x_3^3 \rangle.$$

By further Tietze transformations, including the addition of generator x' and relator $x_1x_3x'^{-1}$, we achieve the presentation

$$\langle x_1, x' | [x_1, x'], x_1^9, x'^3 \rangle,$$

and the proposition is shown. ■

CHAPTER IV

GENERALIZED TWISTED LENS SPACES

§1. A Motivation for Generalized Twisted Lens Spaces.

In Section II.5, the twisted lens spaces were determined to be homeomorphic to the cyclic covers of S^3 which are branched over the trefoil knot. In this chapter we generalize the previous gluing schema to produce, among other manifolds, the cyclic covers of S^3 which are branched over an arbitrary torus knot. As before, this construction will readily yield information about these generalized twisted lens spaces (GTLS's).

§2. The Construction of GTLS's.

To construct the complex $L_{n,k,r}^{(p)}$, where $p \geq 3$ and $n \geq 1$ are integers, $k \in \mathbb{Z}_n$, and $r \in \{0, \frac{1}{p}, \frac{2}{p}, \dots, \frac{p-1}{p}\}$, we identify the boundary faces of Σ_n in the following way. Each lateral edge $a_i \subset P_n \subset \Sigma_n$ is subdivided equally into $p - 2$ edges, consecutively labelled $a_i^{[1]}, a_i^{[2]}, \dots, a_i^{[p-2]}$. These superscripts will be denoted mod p . Call the vertices v_i the *original* vertices, and the vertices added by the subdivision of lateral edges the *new* vertices. Each face of Σ_n by this subdivision becomes p -sided. Each face may be subdivided into p triangles by coning from the center point. To generalize the twisting of faces to this situation, we define the twisting of a face by $\frac{q}{p}$ to be the map which takes each triangle linearly to the triangle located q positions in the counterclockwise direction.

The complex $L_{n,k,r}^{(p)}$ is formed by twisting each upper face r and identifying it with the lower face located by turning $\frac{k}{n}$ along the lateral circle. See Figure IV.1.

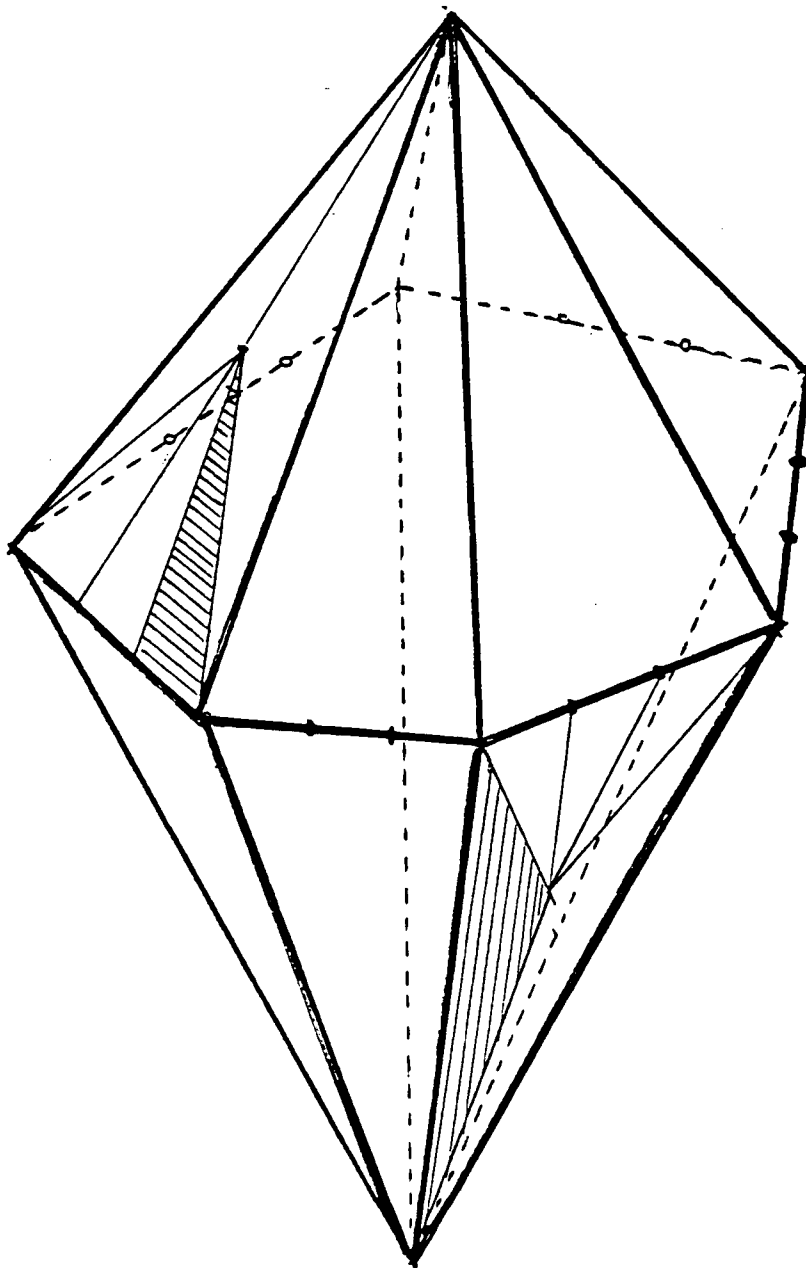


Figure IV.1. The Construction of $L_{6,2,\frac{2}{5}}^{(5)}$.

The construction of $L_{n,(k,l),(r,s)}^{(p,q)}$ is a similar generalization of the two rule twisted lens spaces of Chapter III. As in Chapter III, assume that n is even. The lateral edges of Σ_n are alternately subdivided into $p - 2$ and $q - 2$ edges, labelled as before, so that the even and odd numbered faces of Σ_n become p -gons and q -gons, respectively. Each even numbered upper face is identified, after twisting r , with the lower face located by turning $\frac{k}{n}$ about the lateral circle. Each odd numbered face is identified, after twisting s , with the lower face located by turning $\frac{l}{n}$ about the lateral circle. See Figure IV.2.

We must assume that k and l are even if $p \neq q$, as only equal-sided faces of Σ_n may be glued together.

The two-rule GTLS's can be further generalized to m -rule GTLS's. Let the complex $L_{n,(k_1,\dots,k_m),(r_1,\dots,r_m)}^{(p_1,\dots,p_m)}$ be constructed from Σ_n in the following way, assuming that m divides n . Subdivide each lateral edge into sufficient pieces so that for $i \in \mathbb{Z}_m$ and arbitrary γ , the faces $\Delta_{i+\gamma m}$ of Σ_n are p_i -gons. Identify each upper face $\Delta_{i+\gamma m}$ of Σ_n , twisted r_i revolution, with the lower face located by turning $\frac{k_i}{n}$ about the lateral circle. Here we must restrict the k_i to those values which will result in the identification of equally-sided faces. In particular, if we assume that all of the p_i are distinct, then all of the k_i must be divisible by m .

§3. Determining the Manifold GTLS's.

Consider those $L_{n,k,r}^{(p)}$ where $\gcd(p, pr) \equiv (p, pr) = 1$. Here each coset of edges identified under gluing consists of p -tuples of edges: an upper edge, a lower edge, and one of each type of lateral edge $a_*^{[i]}$, for $i = 1$ to $n - 2$. In particular, the following edge identifications are achieved, for all $i \in \mathbb{Z}_n$:

$$\begin{aligned} a'_{i-1} &\sim a_{i+k}^{[pr]} & a_i^{[-pr-1]} &\sim a''_{i+k} \\ a'_i &\sim a_{i+k}^{[pr-1]-1} & a_i^{[-pr]} &\sim a''_{i+k-1} \\ a_i^{[j]} &\sim a_{i+k}^{[j+pr]}, & \text{for all other } j, \end{aligned}$$

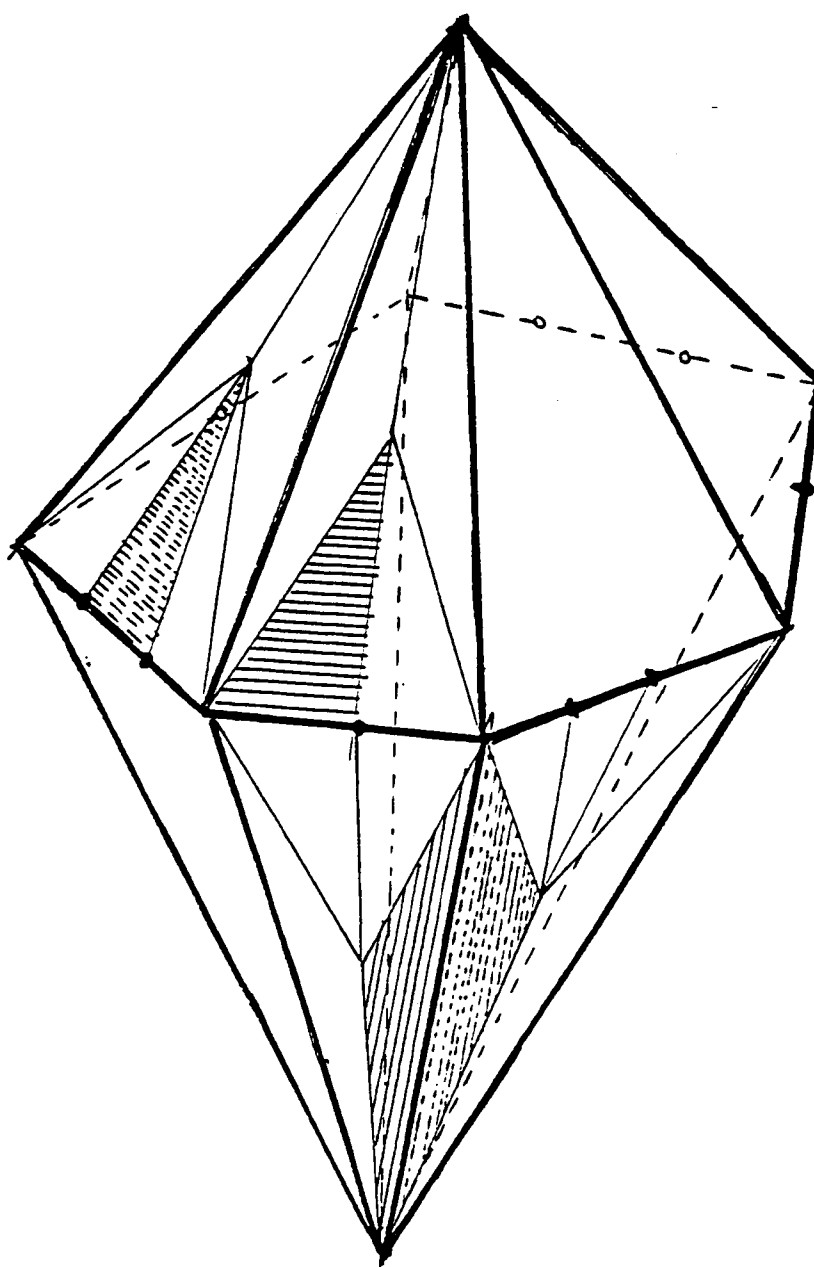


Figure IV.2. The Construction of $L_{6,(2,0),(\frac{3}{5},\frac{2}{4})}^{(5,4)}$.

with the exceptions that

$$a'_{i-1} \sim a''_{i+k} \text{ if } pr = p-1$$

$$a'_i \sim a''_{i+k-1} \text{ if } pr = 1.$$

Since $(p, pr) = 1$, there exists a unique positive integer $\alpha < p$ such that $\alpha pr = p-1 \pmod p$.

LEMMA IV.3.1. Suppose that $L_{n,k,r}^{(p)}$ is constructed as above, assuming that $(p, pr) = 1$. Further assume that α is defined as in the previous paragraph. Then the following edge equivalence holds, for all i :

$$a'_i \sim a''_{i+\alpha k+1} \sim a'_{i+(2\alpha-p)k+2}.$$

In fact, for each directed edge of type a'_* in a coset of directed edges identified under the construction, there are $\alpha-1$ directed edges of type $a_*^{[*]}$, $p-\alpha-1$ directed edges of type $a_*^{[*]-1}$, and a directed edge of type a''_* .

PROOF. The first equivalence is immediate when $pr = p-1$; the second immediate when $pr = 1$, since $i + \alpha k - (p-\alpha)k + 2 = i + (2\alpha-p)k + 2$. Otherwise, we have that

$$\begin{aligned} a'_i &\sim a_{i+1+k}^{[pr]} \sim a_{i+1+2k}^{[2pr]} \sim \dots \\ &\sim a_{i+1+(\alpha-1)k}^{[(\alpha-1)pr]}. \end{aligned}$$

However, $(\alpha-1)pr = -1 - pr \pmod p$, so

$$\begin{aligned} a'_i &\sim a''_{i+1+\alpha k} \\ &\sim a_{i+2+(\alpha-1)k}^{[-pr]-1} \sim a_{i+2+(\alpha-2)k}^{[-2pr]-1} \sim \dots \\ &\sim a_{i+2+[\alpha-(p-\alpha-1)]k}^{[-(p-\alpha-1)pr]-1}. \end{aligned}$$

Since $-(p - \alpha - 1)pr = \alpha pr + pr = -1 + pr \pmod{p}$, we have that

$$a'_i \sim a'_{i+(2\alpha-p)k+2}.$$

■

Hence, the cone points $(0, 0, 1)$ and $(0, 0, -1)$ are identified with each of the original vertices v_i . The endpoints of all of the lower and upper edges are identified as one under the gluing, and thus the endpoints of all the edges are identified under the gluing, the resulting complex having exactly one vertex.

There being exactly one vertex, n faces, and one solid in the complex $L_{n,k,r}^{(p)}$, the complex is a manifold if and only if it contains exactly n edges. Equivalently, each coset of identified edges must consist of exactly one representative of each of the p types of edges. In particular, each coset must consist of exactly one upper edge. Hence, the resulting complex will be a manifold exactly when

$$(p - 2\alpha)k = 2 \pmod{n}.$$

We therefore have shown the following proposition.

PROPOSITION IV.3.2. *Suppose that $k \in \mathbb{Z}_n$ and $0 < r < 1$ chosen so that pr is an integer and $(pr, p) = 1$. Then the complex $L_{n,k,r}^{(p)}$ has the following properties:*

- (1) *The complex has exactly one vertex.*
- (2) *The complex is a manifold precisely when*

$$(p - 2\alpha)k = 2 \pmod{n}, \text{ where } \alpha pr = p - 1 \pmod{p}.$$

Note that the above condition on k will not always be possible, for there exist triples p, n, r , where $(p - 2\alpha, n) \geq 3$. In general situations where $(pr, p) > 1$, the resulting complex could have more than one vertex, and hence is more difficult to analyze.

A similar result holds for the second type of GTLS's constructed above.

PROPOSITION IV.3.3. Suppose that $(pr, p) = (qs, q) = 1$ and k, l, n even. Then the complex $L_{n, (k, l), (r, s)}^{(p, q)}$ has the following properties:

- (1) The complex has exactly one vertex.
- (2) The complex is a manifold precisely when

$$l = \frac{-2p}{\alpha\beta - (p - \alpha)(q - \beta)} \quad \text{and} \quad k = \frac{-2q}{\alpha\beta - (p - \alpha)(q - \beta)},$$

where $\alpha pr = p - 1 \pmod p$ and $\beta qs = q - 1 \pmod q$.

- (3) Given p and q , there exists a manifold-producing pair (k, l) precisely when $\alpha\beta - (p - \alpha)(q - \beta)$ divides $\gcd(p, q)$.

The proposition is false if k and l are odd. As a counterexample, consider the complex $L_{2, (1, 1), (\frac{1}{3}, \frac{2}{3})}$, which has two vertices. See Case 8 of Proposition III.1.2. (Note that k and l can be odd only when $p = q$.)

PROOF. In this case the following edge identifications are achieved, for i even.

$$\begin{aligned} a'_{i-1} &\sim a_{i+k}^{[pr]} & a'_i &\sim a_{i+l+1}^{[qs]} \\ a'_i &\sim a_{i+k}^{[pr-1]-1} & a'_{i+1} &\sim a_{i+l+1}^{[qs-1]-1} \\ a_i^{[-pr-1]} &\sim a''_{i+k} & a_{i+1}^{[-sq-1]} &\sim a''_{i+l+1} \\ a_i^{[-pr]} &\sim a''_{i+k-1} & a_{i+1}^{[-sq]} &\sim a''_{i+l} \\ a_i^{[j]} &\sim a_{i+k}^{[j+pr]}, \text{ for all other } j, & a_{i+1}^{[j]} &\sim a_{i+l+1}^{[j+qs]}, \text{ for all other } j, \end{aligned}$$

with the exceptions that

$$\begin{aligned} a'_i &\sim a''_{i+k-1} \text{ if } pr = 1 & a'_{i+1} &\sim a''_{i+l} \text{ if } sq = 1 \\ a'_{i-1} &\sim a''_{i+k} \text{ if } pr = p - 1 & a'_i &\sim a''_{i+l+1} \text{ if } sq = q - 1. \end{aligned}$$

Since $(pr, p) = (qs, q) = 1$, there are unique positive integers $\alpha < p$ and $\beta < q$ so that $\alpha pr = p - 1 \pmod p$ and $\beta qs = q - 1 \pmod q$.

LEMMA IV.3.4. For all even $i \in \mathbb{Z}_n$,

$$a'_{i-1} \sim a''_{i+\alpha k} \sim a'_{i+\alpha k-(q-\beta)l+1}$$

$$a'_i \sim a''_{i+\beta l+1} \sim a'_{i+\beta l-(p-\alpha)k+2}.$$

In particular,

- (1) for each odd indexed upper directed edge a'_{i-1} in a coset, there is an even indexed lower directed edge, $\beta - 1$ even indexed directed edges of type $a_*^{[*]}$, and $p - \alpha - 1$ odd indexed directed edges of type $a_*^{[*]-1}$.
- (2) for each even indexed upper directed edge a'_i in a coset, there is an odd indexed lower directed edge, $\alpha - 1$ odd indexed directed edges of type $a_*^{[*]}$, and $q - \beta - 1$ even indexed directed edges of type $a_*^{[*]-1}$.

PROOF OF LEMMA IV.3.4. The proof is similar to that of Lemma IV.3.1. Observe that $a'_{i-1} \sim a''_{i+k}$ if $pr = p - 1$, and $a''_{i+\alpha k} \sim a'_{i+1+\alpha k-l}$ if $qs = 1$. Else,

$$\begin{aligned} a'_{i-1} &\sim a_{i+k}^{[pr]} \sim a_{i+2k}^{[2pr]} \sim \dots \\ &\sim a''_{i+\alpha k} \\ &\sim a_{i+1+\alpha k-l}^{[-qs]-1} \sim a_{i+1+\alpha k-2l}^{[-2qs]-1} \sim \dots \\ &\sim a'_{i+\alpha k-(q-\beta)l+1}. \end{aligned}$$

The second part follows in the same manner.

Continuing with the proof of the proposition, note that by the lemma, there are at most n edges in the resulting complex. The cone points $(0, 0, 1)$ and $(0, 0, -1)$ are identified with each of the original vertices v_i . The endpoints of all of the lower and upper edges are identified as one under the gluing, and thus the endpoints of all the edges are identified under the gluing, the resulting complex having exactly one vertex.

Since there are n faces and one solid in the resulting complex, the complex is a manifold when it has exactly n edges. Hence, each coset contains exactly one upper edge. Hence, we must have that

$$\alpha k + \beta l + 2 = pk$$

$$\alpha k + \beta l + 2 = ql.$$

The result of Part 2 then follows by algebra.

To conclude the proof, recall that k and l are even by hypothesis, and hence, by Part 2, we have that

$$\frac{p}{\alpha\beta - (p - \alpha)(q - \beta)} \text{ and } \frac{q}{\alpha\beta - (p - \alpha)(q - \beta)}$$

are integers precisely when the complex is a manifold. Hence, $\alpha\beta - (p - \alpha)(q - \beta)$ divides $\gcd(p, q)$ if and only if the complex is a manifold for some pair (k, l) . ■

We can likewise show the following proposition for the general m -rule GTLS's.

PROPOSITION IV.3.5. *The complex $L_{n, (k_1, \dots, k_m), (r_1, \dots, r_m)}^{(p_1, \dots, p_m)}$ has exactly one vertex when m divides each k_i and $(p_i r_i, p_i) = 1$ for all $i \in \mathbb{Z}_m$.*

PROOF. As before, each upper edge is identified with some lower edge. Therefore, the vertices $(0, 0, 1)$ and $(0, 0, -1)$ are identified with each of the original vertices v_i . In addition, since each edge is identified with some upper edge and some lower edge, it follows that all of the vertices are identified under the gluing, the resulting complex having exactly one vertex. ■

With the omission of these hypotheses, the proposition need not be true. For example, recall that a lens space has two vertices, and consider the following result.

PROPOSITION IV.3.6. *The complex $L_{n, (k_1, \dots, k_m), (0, \dots, 0)}^{(p_1, \dots, p_m)}$ is a manifold exactly when all of the k_i are equal, in which case the resulting manifold is L_{n, k_1} .*

PROOF. First note that since there is no twisting, we may assume that $p_i = 3$, for all i .

Since each complex is a pseudo-manifold, the resulting complex is a manifold exactly when the link of each vertex is S^2 . Here, the suspension points $(0, 0, 1)$ and $(0, 0, -1)$ are identified by the gluing, and are identified with no other vertex. Hence, the link of this resulting vertex is formed by gluing the links L and L' of $(0, 0, 1)$ and $(0, 0, -1)$ in Σ_n along their boundaries by pairing edges, one from each link. Label the edges of L and L' so that $L \cap \Delta_i = \epsilon_i$ and $L' \cap \Delta'_i = \epsilon'_i$, and assume that ϵ_n is identified with ϵ'_k . Now, ϵ_1 must be identified with ϵ'_{k+1} . Else, the resulting surface will have nonzero genus. Continuing inductively, we find that, for all i , ϵ_i and ϵ_{i+k} are identified, and hence, $k_i = k$ for all i . ■

§4. Determining the Seifert Fibered GTLS's.

The goal of this section is to prove the following Theorem.

THEOREM IV.4.1. *Suppose that $(p_i r_i, p_i) = 1$ for all i , m divides each k_i , and m divides n . If the complex*

$$L_{n, (k_1, \dots, k_m), (r_1, \dots, r_m)}^{(p_1, \dots, p_m)}$$

is a manifold, then it is Seifert fibered.

PROOF. By the arguments given to prove Proposition II.2.2, we have that the complex has an induced foliation in a neighborhood of every point, except possibly the vertices.

By Proposition IV.3.5, the complex has exactly one vertex. Assuming that the complex is a manifold, this vertex has a ball neighborhood. Therefore, by Theorem II.1.1, it suffices to show that some such neighborhood has an foliation induced by the vertical foliation of Σ_n .

Consider as a fibered neighborhood of $(0, 0, 1)$ the intersection

$$\Sigma_n \cap \{(x, y, z) \mid z \geq \frac{9}{10}\}.$$

This neighborhood is a pyramid with regular n -gon base. The n horizontal edges of this neighborhood, as subsets of the upper faces, are identified with piecewise linear segments within the lower faces. (By the definition of twisting, each such edge maps to the underlying point set of a complex consisting of two segments and three vertices.) They are located near certain lateral vertices, since $(p_i r_i, p_i) = 1$ for all i . Following the foliation of Σ_n , these segments go to edges in the upper hemisphere. These new edges are identified with segments in the lower hemisphere, near different lateral vertices. Continue this process until the segments are near the vertex $(0, 0, -1)$. These segments are the horizontal edges on the boundary of the fibered neighborhood

$$\Sigma_n \cap \{(x, y, z) \mid z \leq -\frac{9}{10}\}.$$

Form fibered neighborhoods of the lateral vertices by intersecting Σ_n with planes perpendicular to $z = 0$, so that the nearby segments are edges in the boundary. See Figure IV.3.

The fibered neighborhoods of the new vertices have four triangular faces in $bd \Sigma_n$; the neighborhoods of the original lateral vertices have eight triangular faces in $bd \Sigma_n$; the neighborhoods of $(0, 0, 1)$ and $(0, 0, -1)$ each have n triangular faces in $bd \Sigma_n$. By their construction, these neighborhoods fit together to form a neighborhood for the vertex in the image space by the pairwise identification of the triangular faces. See Figure IV.4.

A typical fiber passes vertically through the upper pyramid, and then vertically through the neighborhoods of certain lateral vertices, and passes on through the

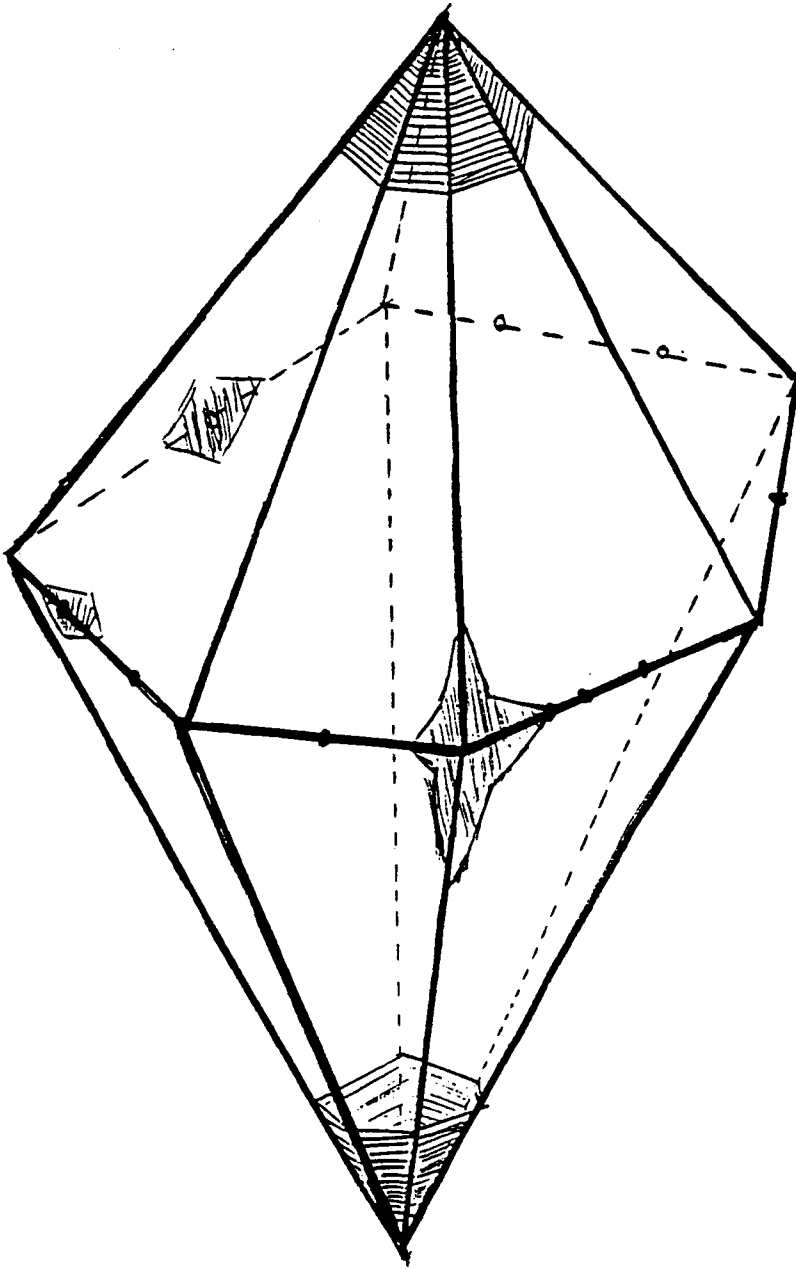


Figure IV.3. The Fibered Neighborhoods for the Vertices of Σ_n .

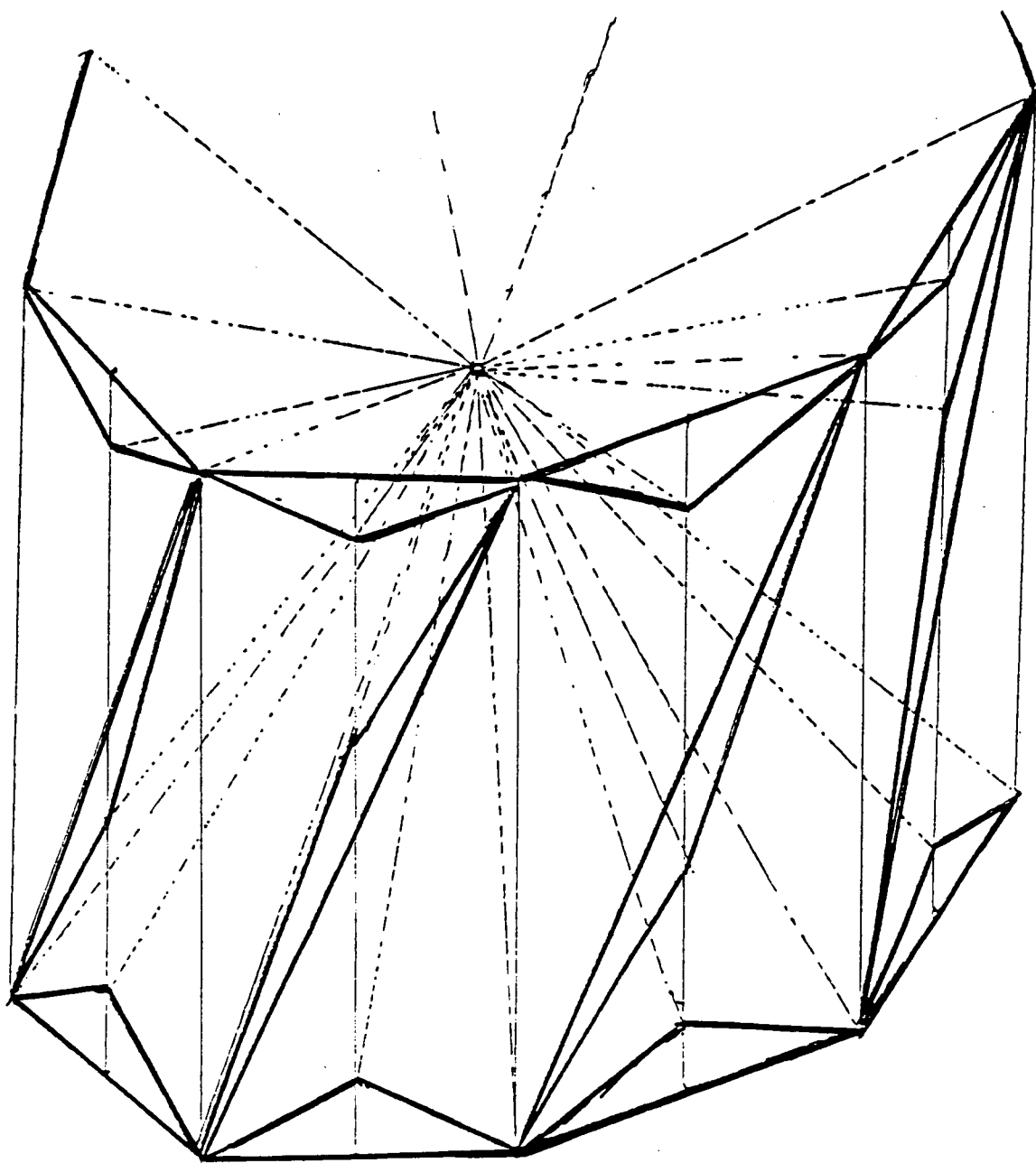


Figure IV.4. The Image of the Fibered Neighborhoods in $L_{n,(k_1,\dots,k_m),(r_1,\dots,r_m)}^{(p_1,\dots,p_m)}$.

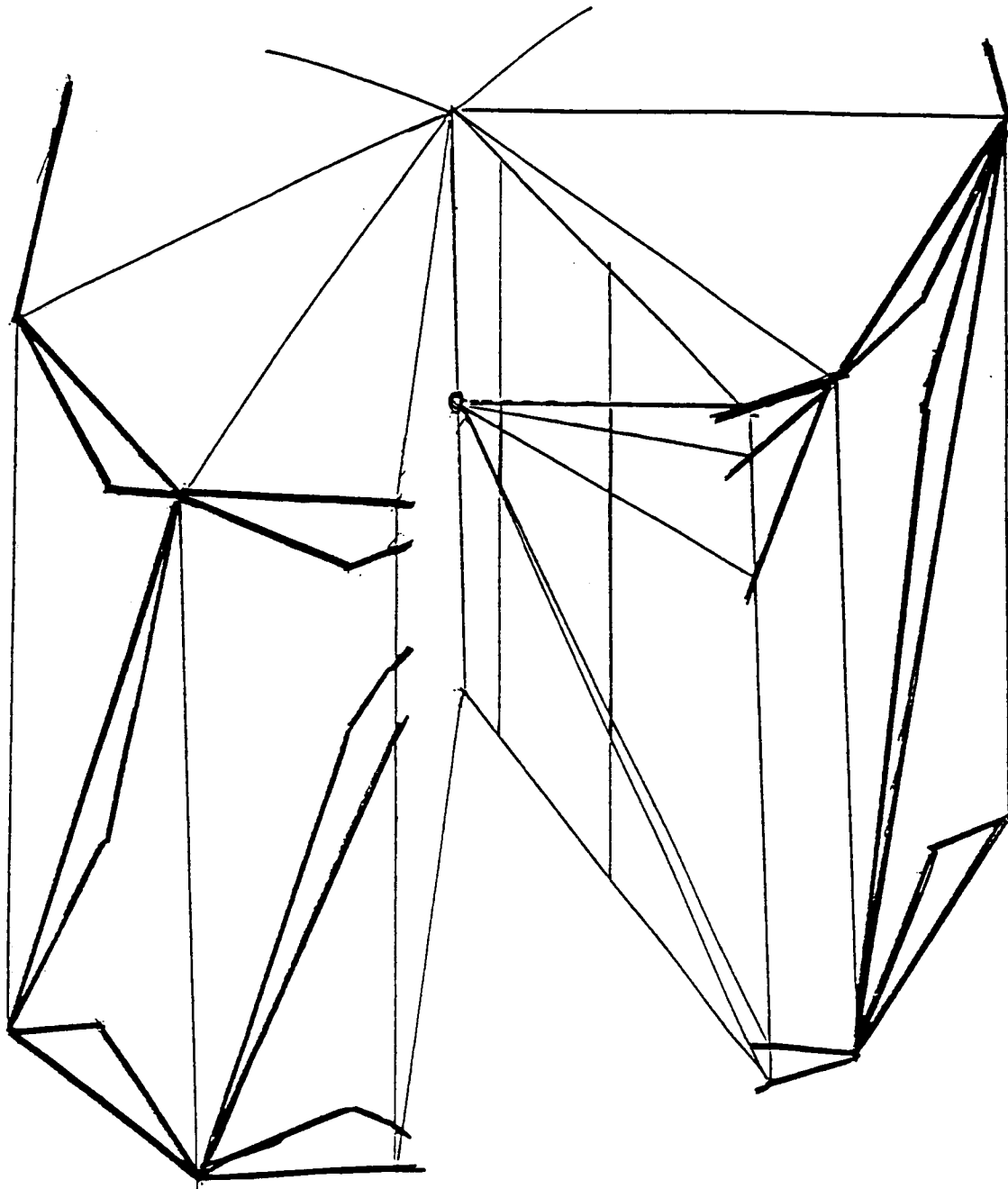


Figure IV.5. The Passage of a Typical Fiber.

lower pyramid. The fibering is in fact represented by the fibering on the portion of the boundary of this ball which lies on the neighborhoods of the lateral vertices. See Figure IV.5.

Since this neighborhood has as an induced vertical fibration, we have shown that the complex is foliated at the vertex, and hence is Seifert fibered. ■

§5. Recognizing Brieskorn Manifolds among the GTLS's.

DEFINITION. [Mil] A *Brieskorn manifold* is a cyclic cover of S^3 which is branched over a torus knot or link.

By Theorem II.5.1 we know that the twisted lens spaces are cyclic covers of S^3 which are branched over the trefoil and hence are Brieskorn manifolds. The purpose of this section is to show that all of the Brieskorn manifolds which branch over a torus **knot** are included among the one- and two-rule GTLS's constructed in Section 2.

PROPOSITION IV.5.1. *Up to renumbering of the faces by changing the starting point or the direction of numbering, we have the following results.*

- (1) Suppose p is odd and r chosen so that $L_{1,0,r}^{(p)}$ is a manifold with exactly one vertex. It is then a homology 3-sphere exactly when $r = \frac{2}{p}$. This homology sphere is in fact a homotopy sphere.
- (2) Suppose that $(p, q) = 1$ and (r, s) chosen so that $L_{2,(0,0),(r,s)}^{(p,q)}$ is a manifold with exactly one vertex. It is then a homology 3-sphere when $(pr, p) = (qs, q) = 1$.

PROOF. In each case, the fundamental group of the resulting manifold is carried by its 2-skeleton. As in the previous work, the edges of the resulting complex represent generators, the faces relators of a presentation of the fundamental group.

CASE 1. The manifold has one face, one solid, and, by hypothesis, one vertex. Hence, since its Euler characteristic must be zero, it must have exactly one edge, with corresponding generator x , and the p -sided face must correspond to the trivial relator x or x^{-1} . As all of the edges of the face must be identified, the rotation r must be chosen so that $(pr, p) = 1$, and hence there exists a unique $\alpha \in \mathbb{Z}_p$ so that $\alpha pr = p - 1 \pmod{p}$.

We have therefore the identification of directed edges as described in Lemma IV.3.1, identifying the upper and lower directed edges with $\alpha - 1$ of the directed edges of type $a_0^{[*]}$ and the inverse of $p - \alpha - 1$ such directed edges. Hence the upper face represents the relator $x^{1+(\alpha-1)-(p-\alpha-1)-1} = x^{2\alpha-p} = 1$. We want this exponent to be 1 or $-1 \pmod{p}$. If $p - 2\alpha = 1 \pmod{p}$, then $\alpha = \frac{p-1}{2}$, and hence $pr = 2$. On the other hand, if $2\alpha - p = 1 \pmod{p}$, then $\alpha = \frac{p+1}{2}$. Note that $\frac{p+1}{2}pr = p - 1 = -(p + 1) \pmod{p}$ is satisfied by $pr = -2$. By allowing a change in the direction of the numbering of the faces, we may assume that $r = \frac{2}{p}$.

CASE 2. Here we have two faces, one solid, and, by hypothesis, one vertex. Therefore, this manifold must have exactly two edges, which represent generators x and y .

By assumption, $(pr, p) = (qs, q) = 1$, so we have α and β defined as before. By Lemma IV.3.4, we have that a'_0 is identified with a''_1 , $\alpha - 1$ edges of the form $a_1^{[*]}$, and $q - \beta - 1$ edges of the form $a_0^{[*]-1}$. Also, a'_1 is identified with a''_0 , $\beta - 1$ edges of the form $a_0^{[*]}$, and $p - \alpha - 1$ edges of the form $a_1^{[*]-1}$.

Therefore, the homology of this manifold is presented by

$$G = \langle x, y \mid [x, y], \alpha x - (p - \alpha)y, \beta y - (q - \beta)x \rangle.$$

By Tietze, we may append the relators

$$\beta\{\alpha x - (p - \alpha)y\} \text{ and } \alpha\{\beta y - (q - \beta)x\}$$

without changing the group G . However, note that

$$\beta\{\alpha x - (p - \alpha)y\} = \alpha\beta x - (p - \alpha)(q - \beta)x = \{\alpha\beta - (p - \alpha)(q - \beta)\}x$$

$$\alpha\{\beta y - (q - \beta)x\} = \alpha\beta y - (q - \beta)(p - \alpha)y = \{\alpha\beta - (p - \alpha)(q - \beta)\}y.$$

By Proposition IV.3.3, $\alpha\beta - (p - \alpha)(q - \beta)$ divides $\gcd(p, q)$, which is 1, by hypothesis. Hence, $\alpha\beta - (p - \alpha)(q - \beta) = \pm 1$, and the group is therefore trivial. ■

DEFINITION. We will say that (r, s) satisfies $\star$ when $(pr, p) = (qs, q) = 1$ and $\alpha\beta - (p - \alpha)(q - \beta) = \gcd(p, q)$, where $\alpha pr = p - 1 \pmod{p}$ and $\beta qs = q - 1 \pmod{q}$. Note that if (r, s) satisfies $\star$, then there exist k and l so that the resulting complex is a manifold, by Proposition IV.3.3.

PROPOSITION IV.5.2.

- (1) If p is odd, the complex $L_{n, k, \frac{2}{p}}^{(p)}$ is a manifold exactly when $k = 2$.
- (2) Assuming that n is even, $(p, q) = 1$, and (r, s) satisfies $\star$, the complex $L_{n, (k, l), (r, s)}^{(p, q)}$ is a manifold exactly when $k = -2q$ and $l = -2p$.

PROOF. The result follows from Propositions IV.3.2 and IV.3.3. ■

Let $\mathcal{F}_n^{(p)}$ denote the Seifert fibration of $L_{n, 2, \frac{2}{p}}^{(p)}$, as given in Theorem IV.4.1, induced by the vertical fibering of Σ_n . For n even and p and q satisfying $\gcd(p, q) = 1$ similarly define $\mathcal{F}_n^{(p, q)}$ to be the induced fibration of $L_{n, (-2q, -2p), (r, s)}^{(p, q)}$, where (r, s) satisfies $\star$.

LEMMA IV.5.3.

- (1) The only fibers of $\mathcal{F}_n^{(p)}$ which are possibly exceptional are those which pass through the vertices, edge centers, or face centers.
- (2) The only fibers of $\mathcal{F}_{2n}^{(p, q)}$ which are possibly exceptional are those which pass through the vertices or face centers.

PROOF. This generalization of Proposition II.2.4 follows by an identical proof, together with the observation in Case 2 that, by assuming $p \neq q$, no directed edge is identified with its inverse, and hence no exceptional fiber passes through edge centers. ■

PROPOSITION IV.5.4.

- (1) Assume that p is odd and $(n, 2p) = p'q'$, where $p'|p$ and $q'|2$. Then the possibly exceptional fibers of $\mathcal{F}_n^{(p)}$ are of orders

$$\frac{n}{p'q'}, \quad \overbrace{\frac{p}{p'}, \dots, \frac{p}{p'}}^{q' \text{ times}}, \quad \overbrace{\frac{2}{q'}, \dots, \frac{2}{q'}}^{p' \text{ times}}.$$

- (2) Assume that $(p, q) = 1$ and $(n, pq) = p'q'$, where $p'|p$ and $q'|q$. Then the possibly exceptional fibers of $\mathcal{F}_{2n}^{(p,q)}$ are of orders

$$\frac{n}{p'q'}, \quad \overbrace{\frac{p}{p'}, \dots, \frac{p}{p'}}^{q' \text{ times}}, \quad \overbrace{\frac{q}{q'}, \dots, \frac{q}{q'}}^{p' \text{ times}}.$$

PROOF. The proof is similar to that of the special case, Theorem II.2.8.

CASE 1. All of the face centers lie on one fiber $f \in \mathcal{F}_n^{(p)}$ when n is odd. When n is even, the centers of the even and odd numbered face centers lie on separate fibers. Consider a small fibered neighborhood N for such a fiber and an arbitrary fiber $\hat{f} \subset N$ other than f . It remains to determine the number of components of $\hat{f} \cap N'$, where N' is any slice of N . Let N' be such that its inverse image is a fibered neighborhood of f which meets some boundary face Δ_i through which f passes. By construction, each fiber which passes through the image of Δ_i next passes through the image of Δ_{i+2} , then Δ_{i+4} , and eventually passes again through the image of Δ_i n steps later if n is odd [or $\frac{n}{2}$ steps, if n is even]. However, with each step, a twisting of $\frac{2}{p}$ occurs. Therefore, the fiber returns to Δ_i with no

twisting after $lcm(n, p) = \frac{np}{p'}$ [or $lcm(\frac{n}{2}, p) = \frac{np}{2p'}$] steps. Therefore, each ordinary fiber $\hat{f}$ passes through Δ_i $\frac{p}{p'}$ times, in either case.

As in Proposition II.2.7, the fibers which pass through the edge centers are exceptional exactly when n is odd, in which case they are of order 2. It remains to determine the number of such fibers. Under the gluing, each such fiber which passes upward through the image of edge a'_i then passes through the images of $a_{i+3}^{[2]}, a_{i+5}^{[4]}, \dots, a_{i+p-2}^{[p-3]}$, followed by the image of a''_{i+p} . It then passes through the image of a'_{i+p} . Hence, there are $gcd(n, p) = p'$ such fibers.

Consider a small fibered neighborhood N for the fiber $f \in \mathcal{F}_n^{(p)}$ whose preimage passes through the vertices. Consider any other fiber $\hat{f} \subset N$, and let $N' \subset N$ denote the image of a small neighborhood of the segment f' passing from $(0, 0, -1)$ to $(0, 0, 1)$. It remains to determine the number of components of $\hat{f} \cap N'$. A fiber which passes through the image of Δ_i , near $(0, 0, 1)$ next passes near $p-1$ new and original lateral vertices, turning by $\frac{2}{n}$ each time, before passing through the image of Δ_{i+2p} , near $(0, 0, 1)$. Therefore, the fiber will pass through N' with every p -th step. The fiber will return to the starting point after $\frac{1}{2}lcm(n, 2p) = \frac{pn}{p'q'}$ steps. Therefore, there are $\frac{n}{p'q'}$ components of $\hat{f} \cap N'$.

CASE 2. The fibers which pass through the centers of the even faces are all of the same order. As before, a small fibered neighborhood N can be constructed about such a fiber, and an arbitrary ordinary fiber $\hat{f} \in N$ selected. It suffices to determine the number of times $\hat{f}$ passes through the image of some even indexed face Δ_i . After $\frac{1}{2q}lcm(2n, 2q) = \frac{n}{q'}$ steps, the fiber passes again through Δ_i , but possibly with twist. Recall that $(pr, p) = 1$, so the face Δ_i returns, with no twisting, after some multiple of p steps; hence, it returns pointwise after $lcm(\frac{n}{q'}, p) = \frac{1}{q'}lcm(n, p) = \frac{np}{p'q'}$ steps. Therefore, each ordinary fiber passes through Δ_i $\frac{p}{p'}$ times.

We must now determine the number of such singular fibers. Since $k = -q$, the fiber which passes through the image of the center of Δ_i next passes through the image of Δ_{i-q} . Hence, there are $\gcd(n, q) = q'$ such fibers.

By a similar argument we determine that there are p' fibers, passing through the image of the centers of the odd faces, each of order $\frac{q}{q'}$.

It remains to determine the order of the fiber whose preimage passes through the vertices. An ordinary fiber $\hat{f}$ whose preimage passes through Δ_i near the vertex $(0, 0, 1)$, for some even i , next passes by $p - 1$ new or original lateral vertices, turning $-\frac{q}{n}$ each time, before passing through the image of Δ_{i-pq} , near the vertex $(0, 0, 1)$. Therefore, the fiber $\hat{f}$ passes near the vertex $(0, 0, 1)$ at every p -th step. The fiber returns to the starting point after $\frac{1}{pq} \text{lcm}(n, pq) = \frac{n}{p'q'}$ sets of p steps. Therefore, the preimage of any such ordinary fiber passes by $(0, 0, 1)$ a total of $\frac{n}{p'q'}$ times. ■

COROLLARY IV.5.5. *Each of the manifolds $L_{1,0,\frac{2}{p}}^{(p)}$ and $L_{2,(0,0),(r,s)}^{(p,q)}$, where $(p, q) = 1$ and (r, s) satisfies $\star$, has exactly two exceptional fibers, of orders p and 2 and of orders p and q , respectively.*

PROOF. The result is a special case of the preceding theorem. ■

PROPOSITION IV.5.6. *The manifolds $L_{1,0,\frac{2}{p}}^{(p)}$, where p is odd, and $L_{2,(0,0),(r,s)}^{(p,q)}$, where $(p, q) = 1$ and (r, s) satisfies $\star$, are each homeomorphic to S^3 . Further, the ordinary fibers are $\tau(p, 2)$ or $\tau(p, q)$ torus knots, respectively.*

PROOF. Each space is a homology sphere by Proposition IV.5.1. By Corollary IV.5.5, there are two exceptional fibers of orders p and 2 and orders p and q , respectively. Therefore, by Proposition II.5.2, each space is homeomorphic to S^3 , and the ordinary fibers are torus knots $\tau(2, p)$ and $\tau(p, q)$, respectively. ■

THEOREM IV.5.7.

- (1) The manifold $L_{n,2,\frac{2}{p}}^{(p)}$ is the n -fold cyclic covering of S^3 , branched over the $\tau(p, 2)$ torus knot, when p is odd.
- (2) The manifold $L_{2n,(-2q,-2p),(r,s)}^{(p,q)}$ is the n -fold cyclic covering of S^3 , branched over the $\tau(p, q)$ torus knot, assuming that $(p, q) = 1$ and (r, s) satisfies $\star$.

PROOF. This result is a generalization of Theorem II.5.1, and its proof follows in like manner. ■

§6. Underlying Orbifolds of GTLS's.

In this section we generalize the results of Section II.4, in order to distinguish many of the Brieskorn manifolds constructed in this chapter.

Let $\mathcal{O}_n^{(p)}$ and $\mathcal{O}_{2n}^{(p,q)}$ denote the orbifolds achieved by shrinking to a point each fiber of $\mathcal{F}_n^{(p)}$ and $\mathcal{F}_{2n}^{(p,q)}$, respectively. Further, let $X_n^{(p)}$ and $X_{2n}^{(p,q)}$ denote the respective underlying surfaces of the orbifolds. Let T_m denote the orientable two-manifold of genus m .

PROPOSITION IV.6.1.

- (1) Assume that p is odd and $(n, 2p) = p'q'$, where $p'|p$ and $q'|2$. Then the surface

$$X_n^{(p)} = \begin{cases} T_{\frac{p'-1}{2}} & \text{when } p' > 1 \text{ and } q' = 2, \\ S^2 & \text{otherwise.} \end{cases}$$

- (2) Assume that $(p, q) = 1$ and $(n, pq) = p'q'$, where $p'|p$ and $q'|q$. Then the surface

$$X_{2n}^{(p,q)} = \begin{cases} T_{\frac{(p'-1)(q'-1)}{2}} & \text{when } p' > 1 \text{ and } q' > 1, \\ S^2 & \text{otherwise.} \end{cases}$$

PROOF.

CASE 1. Under the identification of fibers to points, the n upper hemisphere faces map to two faces exactly when $q' = 2$. By the proof of Proposition IV.5.4, each face "returns" to itself setwise after n steps when n is odd, $[\frac{n}{2}]$ steps when n is even], and returns to itself pointwise after $lcm(n, p) = \frac{np}{p'}$ steps when n is odd [$lcm(\frac{n}{2}, p) = \frac{np}{2p'}$ steps when n is even]. Each resulting face has $gcd(n, p) = p'$ [or $gcd(\frac{n}{2}, p) = p'$] edges.

If $q' = 2$, then X_n is the image of $\Delta_1 \cup \Delta_2$. Each face, under the identification map, has p' edges, including the shared edge a'_1 . The union $\Delta_1 \cup \Delta_2$ is therefore a disk with $2p' - 2$ boundary edges, paired by the identification map. Place an orientation on $bd \Delta_1 \cup \Delta_2$. Note that each a'_i is identified with $a_i^{-1'}$ as the fibers of $\mathcal{F}'_i$ are mapped to the upper hemisphere. Hence, by the proof of Lemma IV.3.1, each pair of identified edges consists of an edge of $\{a_n, a_1^{[*]}\}$ and an edge of $\{a_2, a_2^{[*]-1}\}$. (Here $k = 2$, $pr = 2$, and $\alpha = \frac{p-1}{2}$.) Hence, each pair is glued in an orientation reversing fashion, resulting in a closed, orientable surface. Further, the vertices of $\Delta_1 \cup \Delta_2$ are identified, since in fact all of the vertices of Σ_n are identified.

Therefore, the resulting orientable surface X_n has two faces, p' edges, and one vertex, and has Euler characteristic $3 - p'$. This surface is therefore $T_{\frac{p'-1}{2}}$. Note that if $p' = 1$, then the surface is of genus zero—it is S^2 .

If $q' = 1$, then X_n can be represented by the image of Δ_1 , and any ordinary fiber which passes through an edge passes through that edge twice. Hence, each edge can be subdivided at its center, exchanging an edge for two subedges, which are identified, and an additional vertex. These edge-vertex pairs may be removed from the count. After such changes, the resulting surface has no edges, one vertex, and one face. Its Euler characteristic is 2, and hence, the surface is S^2 .

CASE 2. In this case, the n even indexed faces of the upper hemisphere map to $(n, q) = q'$ faces. After $\frac{n}{q'}$ steps each p -sided face returns to itself setwise, though perhaps not pointwise. However, since $(pr, p) = 1$, the face will return to itself pointwise after some multiple of p times. Therefore, each resulting face has $(\frac{n}{q'}, p) = \frac{1}{q'}(n, pq') = \frac{1}{q'}(n, pq) = p'$ edges.

Similarly, the n odd indexed faces map to $p' q'$ -sided faces. The resulting odd and even indexed faces each have $p' q'$ edges, which are paired under the identification. Orient the boundary of each resulting face consistent with the lateral edges $a_i^{[*]}$. By Lemma IV.3.4, each edge of an even indexed face is paired with an edge of an odd indexed face, in orientation reversing fashion, and hence the resulting closed surface is orientable. This surface has $p' q'$ edges, one vertex, as the vertices are identified as before, and $p' + q'$ faces. Therefore, the surface has Euler characteristic $1 - p' q' + p' + q'$. The orientable surface has genus $1 - \frac{1 - p' q' + p' + q'}{2} = \frac{(p' - 1)(q' - 1)}{2}$, and hence is $T_{\frac{(p' - 1)(q' - 1)}{2}}$. ■

PROPOSITION IV.6.2.

- $$(1) \mathcal{O}_n^{(p)} = (T_{\frac{(p' - 1)(q' - 1)}{2}}; \frac{n}{p' q'}, \overbrace{\frac{p}{p'}, \dots, \frac{p}{p'}}^{q' \text{ times}}, \overbrace{\frac{2}{q'}, \dots, \frac{2}{q'}}^{p' \text{ times}}),$$
- where $(n, 2p) = p' q'$, assuming that $p' | p$ and $q' | 2$.
- $$(2) \mathcal{O}_{2n}^{(p, q)} = (T_{\frac{(p' - 1)(q' - 1)}{2}}; \frac{n}{p' q'}, \overbrace{\frac{p}{p'}, \dots, \frac{p}{p'}}^{q' \text{ times}}, \overbrace{\frac{q}{q'}, \dots, \frac{q}{q'}}^{p' \text{ times}}),$$
- where $(n, pq) = p' q'$, assuming that $p' | p$ and $q' | q$.

PROOF. This result follows immediately from Propositions IV.5.4 and IV.6.1. ■

COROLLARY IV.6.3.

- $$(1) \chi(\mathcal{O}_n^{(p)}) = p' q' (\frac{1}{n} + \frac{1}{p} - \frac{1}{2}),$$
- where $(n, 2p) = p' q'$, assuming that $p' | p$ and $q' | 2$.

$$(2) \chi(\mathcal{O}_{2n}^{(p,q)}) = p'q'(\frac{1}{n} + \frac{1}{p} + \frac{1}{q} - 1),$$

where $(n, pq) = p'q'$, assuming that $p'|p$ and $q'|q$.

PROPOSITION IV.6.4.

(1) Assuming that p is odd, the manifolds $L_{n,2,\frac{2}{p}}^{(p)}$ have unique Seifert fibrations, except possibly when $\frac{1}{n} + \frac{1}{p} + \frac{1}{2} \geq 1$. Specifically, the Seifert fibration is unique, except possibly when:

$$(a) n \leq 2,$$

$$(b) n = 3 \text{ and } p = 5, \text{ or}$$

$$(c) n \leq 6 \text{ and } p = 3.$$

(2) Assuming that $(p, q) = 1$ and (r, s) satisfies $\star$, the manifolds $L_{2n,(-q,-p),(r,s)}^{(p,q)}$ have unique Seifert fibrations, except possibly when $\frac{1}{n} + \frac{1}{p} + \frac{1}{q} \geq 1$. Specifically, the Seifert fibration is unique, except possibly when:

$$(a) n = 1, \text{ or}$$

$$(b) n = 2 \text{ and } \{p, q\} \in \{\{3, 4\}, \{3, 5\}\}.$$

PROOF. This result follows from the preceding corollary, together with Corollary II.3.4.

LEMMA IV.6.5. The manifolds

$$L_{2,0,\frac{2}{p}}^{(p)}, \text{ for } p \text{ odd, and}$$

$$L_{n,2,\frac{2}{3}}^{(3)}, \text{ for } n \in \{3, 4, 5, 6\},$$

are all distinct.

PROOF. In the case of $L_{2,0,\frac{2}{p}}^{(p)}$, the identified edges represent generators x and y in the fundamental group. As before, the faces represent relators, and hence a presentation of its fundamental group is given by

$$\begin{aligned} & \langle x, y \mid (xy^{-1})^{\frac{p-1}{2}} y^{-1}, (yx^{-1})^{\frac{(p-1)}{2}} x^{-1} \rangle \\ &= \langle x, y \mid (xy^{-1})^{\frac{p-1}{2}} y^{-1}, xy \rangle \\ &= \langle y \mid y^{-(p-1)} y^{-1} \rangle \\ &= \langle y \mid y^p \rangle = \mathbf{Z}_p \end{aligned}$$

The two-fold cover of S^3 , branched over the $\tau(2, p)$ knot, is in fact $L_{p,1}$ [B-Z, §12].

The fundamental groups of the remaining manifolds are determined in Proposition I.2.4 to be $\mathbf{Q}_8$, $\mathbf{T}_{24}$, $\mathbf{P}_{120}$, and G_6 , an infinite group. These manifolds are distinct, as they have distinct fundamental groups. ■

THEOREM IV.6.6. *Among the Brieskorn manifolds which arise as coverings of S^3 , branched over a torus knot, the n -to-1 cyclic branched covering over $\tau(p, q)$ and the $\hat{n}$ -to-1 cyclic branched covering over $\tau(\hat{p}, \hat{q})$ are distinct, unless:*

- (1) $n = \hat{n} = 1$, in which case the resulting manifold is S^3 .
- (2) the sets $\{n, p, q\} = \{\hat{n}, \hat{p}, \hat{q}\}$ consist of coprime integers, in which case the resulting manifold is a homology sphere.
- (3) the indices are such that $(p-1)(q-1) = (\hat{p}-1)(\hat{q}-1)$ and $\frac{n}{pq} = \frac{\hat{n}}{\hat{p}\hat{q}}$ is integer. In the case that this fraction is 1, the resulting manifold is a circle bundle over the surface $T_{\frac{(p-1)(q-1)}{2}}$.
- (4) the indices are such that

$$\gcd(n, p) = \gcd(\hat{n}, \hat{p}) \text{ and } \gcd(n, q) = \gcd(\hat{n}, \hat{q}).$$

In this case, the orbifold data is identical, and hence the manifolds are possibly identical.

PROOF. For those pairs not excluded the orbifold data given in Proposition IV.6.2 are distinct, and hence by Proposition IV.6.4 and Proposition IV.6.5, the branched covers are distinct.

The comment of Case 1 is a restatement of Proposition IV.5.6. The comment of Case 2 is a result of Seifert [Se, §12]. The comment of Case 3 is a result of Milnor [Mil], together with Proposition IV.6.1. ■

In particular, we have the following result.

COROLLARY IV.6.7. *The cyclic branched covers of S^3 , branched over a fixed arbitrary torus knot $\tau(p, q)$, are all distinct.*

CHAPTER V

EPILOGUE

Although the technique of constructing manifolds by the pairwise identification of faces of a simply-connected solid polyhedra is not new, this method presents an alternative model for many spaces, in addition to the classical lens spaces. The small (non-Haken) Seifert fibered spaces are of particular interest to those investigating essential laminations. Further, it is anticipated that this technique can be shown to include constructions of **all** Brieskorn manifolds.

Jerome Minkus [Min] recently found constructions which yield the cyclic covers of S^3 , branched over arbitrary two-bridge knots, including the torus knots $\tau(2, p)$. His construction, though a generalization of the ordinary lens space construction, is different from that of this paper. However, presentations of the fundamental group of each space are immediately accessible using his construction, in a manner similar to that of Chapter I of this paper. Minkus uses these presentations to produce formulae for the homology of the branched covers and Alexander polynomial of each knot.

The fundamental group presentations of both Minkus and this paper are cyclically presented. These one relator type groups have generated some degree of interest. See for example [Em]. One class of examples introduced by Conway $[C_1], [C_2]$, the Fibonacci Groups, is of the form:

$$F_n = \langle x_1, x_2, \dots, x_n \mid x_i x_{i+1} = x_{i+2}, \text{ for all } i \rangle.$$

They are often generalized to the form:

$$F_{n,r,k} = \langle x_1, x_2, \dots, x_m \mid x_i x_{i+1} \dots x_{i+r} = x_{i+r+k}, \text{ for all } i \rangle.$$

It is interesting that G_n , the fundamental group of $L_{n,n-2,\frac{1}{3}}$, is isomorphic to $F_{n,1,\frac{n-1}{2}}$, when n is odd.

Much work has been done in investigation of the Fibonacci groups. See [J-W-W] for a complete discussion. However, the question of identification has been considered primarily among those groups determined to be finite.

The periodicity of $H_1(L_{n,n-2,\frac{1}{3}})$ determined in Chapter I is known for cyclic branched coverings over not only the trefoil, but all torus knots [B-Z, §9]. This more general case can be shown using methods similar to those of Chapter I, or by constructing the relator matrix as a ribbon matrix with a nonzero corner, performing row reductions to reduce it to a block diagonal matrix whose upper block is triangular and whose lower block is the matrix corresponding to a lower-indexed branched cover.

The minimal genus of a Heegard splitting of the branched covers over the trefoil is known to be two. However, the presentation of the fundamental group given in Chapter I yields another proof. In short, it can be shown that a generator which appears one time in exactly three relators can, together with one of the relators, be removed by disk trading. Generally, certain generators will, after this transformation, appear more than three times among the resulting relators. However, in the case of the G_n , these presentations can be achieved so that all but two generators appear exactly three times among the relators. The result follows by induction.

This paper only considered manifolds produced by the two-rule method. Even among them, many manifolds were glanced over. The manifolds

$$L_{n,(\frac{n}{3}-2, \frac{2n}{3}-2),(\frac{1}{3}, \frac{1}{3})},$$

for n divisible by 6, are also branched coverings over the trefoil. It is conjectured that they have as deck transformation group $D_{\frac{n}{2}}$. In Chapter IV, the interest shifted to those manifolds which are cyclic branched covers over S^3 . Manifolds which are branched covers of RP^3 and other quotient spaces of S^3 can also be constructed, leading to the conjecture that all Brieskorn manifolds can be constructed in the manner of Chapter IV.

Further, I conjecture that the method of Chapter I which shows that many of the coverings of S^3 , branched over the trefoil, are of distinct homotopy type will generalize to the case of $\tau(2, p)$ torus knots, by considering representations of the fundamental groups into $SO(p)$. Possible extensions to the general case are not clear at this time.

However, there is another way to attack this question. A recent result of Scott [Sc₁] states that among the closed, irreducible Seifert fibered spaces with infinite fundamental group, distinct manifolds have distinct fundamental groups. Using this result together with an analysis of the finite fundamental groups, we have that distinct GTLS's are of distinct homotopy type.

I have determined that among the complexes $L_{n,(k_1, k_2, k_3),(\frac{1}{3}, \frac{1}{3}, \frac{1}{3})}$, no manifolds arise, except for the one-rule manifolds of Chapter I. I have also determined that among those m -rule complexes where all the twisting coefficients are $\frac{1}{3}$ and the turning coefficients are equal, mod m , the complex is a manifold only when m is divisible by six.

The Brieskorn manifolds considered in Chapter IV all have non-zero Euler number. This result follows immediately from the Seifert invariants for all but the

pq -to-one cover, branched over the $\tau(p, q)$ knot, where $\gcd(p, q) = 1$, by a formula for the Euler number [Sc₂, p. 437]. In these remaining cases, the result follows from the fact that the only orientable Euclidean three-manifold which is a circle bundle over a surface with nonzero genus is the three-torus $T^3 = S^1 \times S^1 \times S^1$ [Sc₂, p. 447], a space whose fundamental group cannot be presented using only two generators.

I conjecture that all of the Seifert fibered manifolds which arise from any m -rule construction will have nonzero Euler number. In particular, I conjecture that T^3 cannot arise by this type construction. By a result of Hirsch and Neumann [H-N], T^3 cannot be a cyclic branched covering of S^3 .

This simple generalization of the lens space construction still holds many mysteries within. I hope to discover more of them.

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