To the Graduate Council:

I am submitting herewith a dissertation written by Paul Thurston entitled " The Topology of 4-Dimensional G-spaces and a Study of 4-Manifolds of Non-Positive Curvature ." I have examined the final copy of this dissertation for form and content and recommend that it be accepted in partial fulfillment of the requirements for the degree of Doctor of Philosophy with a major in Mathematics.

Robert J. Daverman, Major Professor

We have read this dissertation
and recommend its acceptance:

Accepted for the Council:

Associate Vice Chancellor
and Dean of The Graduate School

THE TOPOLOGY OF 4-DIMENSIONAL G-SPACES AND A STUDY OF 4-MANIFOLDS OF NON-POSITIVE CURVATURE

A Dissertation

Presented for the

Doctor of Philosophy Degree

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Paul Thurston

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ABSTRACT

The 40-year-old Busemann G-space conjecture states that all n-dimensional G-spaces (with $n < \infty$) are n-manifolds. I have shown that the conjecture is true when $n = 4$, and the bulk of this paper is dedicated to assembling the elements of metric geometry, algebraic topology, and modern decomposition theory required in the proof.

In his classic text The Geometry of Geodesics, Herbert Busemann set out to create an analogue of differential geometry in metric spaces having apriori no analytic structure. He invented the notion of a geodesic space, or G-space, and his theory has a significant impact in the study Riemannian geometry. Examples of non-Riemannian G-spaces are found by considering certain Finsler spaces and also the Teichmüller moduli space of complex structures on any closed Riemann surface of genus ≥ 2.

In 1955, Busemann established that all 2-dimensional G-spaces are surfaces using purely geometric techniques. In 1968, the polish mathematician Krakus applied a topological 2-sphere recognition theorem of Borsuk to show that the conjecture is true in dimension 3.

The next part of this paper is a study of non-positively curved 4-manifolds. The classic theorem of Cartan-Hadamard states that every n-manifold with non-positive sectional curvature is covered by euclidean space. If the notion of curvature is relaxed to that of Aleksandrov's CAT(ε)-inequalities, Gromov asked in 1981 whether every simply connected, non-positively curved n-manifold is necessarily euclidean. For $n \leq 3$, work of Rolfsen and Brown yield an affirmative solution. For $n \geq 5$, counter-examples

have been constructed by Davis and Januszkiewicz in 1991. When $n = 4$, I am able to establish a partial result in the affirmative, provided the 4-manifold has a single metrically tame point.

Finally I establish a 3-manifold recognition criterion. A compact, finite dimensional metric space is a 3-manifold if and only if it is a homology 3-manifold and it admits an inner metric having curvature bounded above by some positive constant. As there exist non-manifold homology 3-manifolds, the curvature condition can not be discarded.

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CHAPTER 1

Introduction

The interplay between the studies of the geometry and topology of metric spaces is fascinating and the techniques of one of these two disciplines can impact the other with unexpected consequences. The class of metric spaces for which these phenomena are most evident are the intrinsic or inner metric spaces.

1 Geometry on Metric Spaces

Definition 1.1 *Let (X, d) be a metric space. X is said to be an* inner metric space *provided for all x, $y \in X$, $d(x, y) = \inf\{length(c) \mid c : [0, 1] \to X$ is continuous, $c(0) = x$, $c(1) = y\}$.*

There are two consequences of this definition that warrant closer inspection. First in order to insure that the metric is well defined, it must be the case that X is path connected. Second since by convention all metrics have values in $[0, \infty)$, any two points $x, y \in X$ are necessarily joined by a rectifiable path. Any metric space with these two properties is said to be *finitely path connected.*

Note that $\mathbf{R}^n$ equipped with Euclidean distance is a complete inner metric space. Similarly any complete Riemannian manifold (M^n, g), where g denotes the Riemannian metric tensor, is also a complete inner metric space. In both of these examples, the

infimum of the lengths of curves joining x to y is realized by a *geodesic segment*, i.e., an isometry $\gamma : [0, \rho] \to \{M^n, \mathbf{R}^n\}$ such that $\gamma(0) = x, \gamma(\rho) = y$,where $\rho = d(x, y)$. The emphasis here is on the parameterization of the curve.

Of course inner metric spaces need not be manifolds. Given any finite path connected (X, d), one can define the *induced inner metric space* $(X, \tilde{d})$ by defining $\tilde{d} : X \times X \to [0, \infty)$ for all x, y via

$$\tilde{d}(x, y) = \inf \{\text{length}_d(c) \mid \ c \text{ is a path from } x \text{ to } y\}.$$

This construction allows us to produce a wealth of examples of inner metric spaces, but the reader should be warned that (X, d) and $(X, \tilde{d})$ might have different topologies.

In this paper I will study two types of inner metric spaces: Busemann G-spaces and manifolds equipped with metrics of non-positive curvature.

2 Busemann G-spaces

As the field of differential geometry matured, people noticed that many of the results could be established without the assumption of differentiability. Mathematicians such as A.D. Aleksandrov [1], Paul Finsler [47], L.M. Blumenthal [17], and Jacques Tits [89], to name just a few, noticed that the analysis employed on a Riemannian manifold is often directed at understanding the behavior of the geodesics in the manifold. The question which metric geometers soon began to ask was this: What are the minimal hypotheses that a metric space must have so that it has a geometry similar to that of a Riemannian manifold, i.e., what additional assumptions must one make concerning the

geodesics? [1]

Beginning in 1942, Herbert Busemann [26], [27] developed the notion of a G-space as a way of putting a Riemannian-like geometry on a metric space. Further developed by Busemann in [28], the theory of G-spaces continues to be an active area of research. Busemann's problems list found in the appendix of [28] contains several questions which are still open as of this writing.

Definition 1.2 *Let (X, d) be a metric space. X is said to be a G-space provided it satisfies the following Axioms of Busemann:*

1. Menger Convexity *Given distinct points $x, y \in X$, there is a point $z \in X - \{x, y\}$ so that*

$$d(x, z) + d(z, y) = d(x, y).$$

2. Finite Compactness *Every d-bounded infinite set has an accumulation point.*

3. Local Extendibility *To every $w \in X$, there is a positive radius ρ_w, such that for any pair of distinct $x, y \in \operatorname{int} B(w, \rho_w)$, there is $z \in \operatorname{int} B(w, \rho_w) - \{x, y\}$ such that*

$$d(x, y) + d(y, z) = d(x, z).$$

4. Uniqueness of the Extension *Given distinct $x, y \in X$, if there are points $z_1, z_2 \in X$ for which both*

$$d(x, y) + d(y, z_i) = d(x, z_i) \text{ for } i = 1, 2, \text{ and}$$

[1] Modern geometry came into existence, in part, as an attempt to understand Euclid's Five Postulates. In asking this question, geometers had, in some sense, come full circle in their studies.

$$d(y, z_1) = d(y, z_2)$$

hold, then $z_1 = z_2$.

How much of the spirit of Riemannian geometry is captured by the G-space axioms? In particular is G-space theory the study of the geometry of manifolds or perhaps some more general spaces? In his text [28, pages 2 and 49], Busemann conjectured that G-space theory is the study of geometry on manifolds.

Conjecture 1.1 (The Busemann Conjecture) *Every finite dimensional G-space is a topological manifold.*

(Here $\dim X \leq n < \infty$ is covering dimension.) Busemann verified the conjecture when $n = 1, 2$ in his text [28, (9.7)pages 46-47 and (10.4) page 52]. The case $n = 3$ was established in 1968 by Krakus [58] as an application of Borsuk's 2-sphere recognition theorem. Busemann's own prediction concerning the future of the conjecture was amazingly accurate:

> Although this (The Busemann Conjecture) is probably true for any G-space, the proof, if the conjecture is correct, seems quite inaccessible in the present state of topology.[2]

The modern theory of 4-manifolds developed by Michael Freedman and Frank Quinn in [48], [79] and [49] has rendered the 4-dimensional version of the problem accessible. The first part of this paper is devoted to establishing the following result.

[2] page 49, The Geometry of Geodesics, 1955.

Theorem 1.1 *Every 4-dimensional G-space is a 4-manifold.*

Here is an outline for the strategy of the proof.

Step 1. Show that every finite dimensional G-space is a ANR homology manifold. This involves the use of elementary properties of G-spaces together with sheaf-theoretic results of Dydak-Walsh [45].

Step 2. Specialize to $4D$ G-spaces and show that the geometry of these objects allows one to embed arbitrarily small 2-spheres, 3-cells and 3-spheres in such a G-space. The recent result of W.J.R. Mitchell [67] is used in combination with classical 2-manifold recognition theory.

Step 3. Use the geometric objects identified above to show that Quinn's resolution obstruction vanishes on every $4D$ G-space. That is, every $4D$ G-space, X, is the image of a genuine 4-manifold,

$$M^4 \xrightarrow{f} X, \tag{1.1}$$

where f is a continuous surjection such that for all $x \in X$, $f^{-1}(x)$ is a cell-like subset of M^4.

Step 4. Generalize Daverman's theory of singular regular neighborhoods [36] and apply this theory to the space X in question. This step will verify that X has a useful local structure, called semi-nicely spherical.

Step 5. Establish the $4D$ semi-nicely spherical shrinking theorem. This can then be applied to show all maps of the form (1.1) are approximable by homeomorphisms and thus establish the desired result.

3 Manifolds of Non-Positive Curvature

As previously noted, every Riemannian manifold, (M^n, g), naturally admits the structure of a inner metric space. It is a consequence of the Hopf-Rinow Theorem (see [54] or [34, page 146]) that if (M^n, g) is complete, then the natural inner metric is a complete metric and M^n is a G-space. One of the main objectives of classical Riemannian geometry is to characterize manifolds (M^n, g) in terms of geometric invariants such as sectional curvature. Hadamard's theorem (see [51] or [34, page 149]) is one of the earliest of such characterization results.

Theorem 1.2 (Hadamard) *Let (M^n, g) be a complete Riemannian manifold such that for all $p \in M^n$ and for every 2-dimensional subspace $\sigma \in T_p(M)$, the sectional curvature of M, $K(p, \sigma) \leq 0$. Then the exponential map*

$$\exp_p : T_pM \ \rightarrow M$$

is a covering map for all $p \in M$. In particular, if $\pi_1(M^n) = \{1\}$, then $M^n \cong \mathbf{R}^n$.

In light of Hadamard's theorem and in keeping with the theme of this paper, i.e. geometry sans derivatives, one can ask two questions.

Question 1: What is the correct generalization of sectional curvature in the absence of a differentiable structure?

Question 2: (Gromov [50]) Does Hadamard's Theorem generalize ?

A limited answer to question 1 is supplied by triangle comparisons and the notion of curvature bounded from above and/or below. Let $[x, y]$ denote a geodesic segment from x to y.

Definition 1.3 *Let (X, d) be a complete, locally compact inner metric space. Let $M^2(\varepsilon)$ be the unique simply connected space form of constant sectional curvature ε.*

a. *We say X satisfies the $CAT(\varepsilon)-$inequality and write* $\operatorname{curv} X \leq \varepsilon$ *provided for each $p \in X$, there is a neighborhood U_p such that : if T is a geodesic triangle in U_p with vertices x_0, x_1 and x_2, then there is a geodesic triangle $\widetilde{T}$ in $M^2(\varepsilon)$ with the same edge lengths as T such that for every $y \in [x_1, x_2]$, the corresponding point $\widetilde{y} \in [\widetilde{x_1}, \widetilde{x_2}]$ has the property*

$$d(x_0, y) \leq \tilde{d}(\widetilde{x_0}, \widetilde{y}), \tag{1.2}$$

and we impose the condition that the perimeter of T be $\leq \frac{2\pi}{\sqrt{\varepsilon}}$ in case $\varepsilon > 0$.

b. *We write* $\operatorname{curv} X \geq \varepsilon$ *provided for each $p \in X$, there is a neighborhood U_p such that : if T is a geodesic triangle in U_p with vertices x_0, x_1 and x_2, then there is a geodesic triangle $\widetilde{T}$ in $M^2(\varepsilon)$ with the same edge lengths as T such that for every $y \in [x_1, x_2]$, the corresponding point $\widetilde{y} \in [\widetilde{x_1}, \widetilde{x_2}]$ has the property*

$$d(x_0, y) \geq \tilde{d}(\widetilde{x_0}, \widetilde{y}), \tag{1.3}$$

and we impose the condition that the perimeter of T be $\leq \frac{2\pi}{\sqrt{\varepsilon}}$ in case $\varepsilon > 0$.

Remark. Let M be a complete Riemannian manifold. It is a consequence of the Rauch's Theorem (see [34, page 215]) that sectional curvature of M non-positive implies M satisfies $CAT(0)$. Topogonov's Theorem (see [35]) gives that sectional curvature of M non-negative implies $\operatorname{curv} M \geq 0$ in the sense of Definition 1.3 and, moreover, the triangle comparisons hold globally.

Intricately related to the concept of bounded curvature is that of a convex distance function.

Definition 1.4 *Let (X, d) be complete, inner metric. The metric $d : X \times X \to [0, \infty)$ is said to be convex provided for any geodesic segments $\gamma_1, \gamma_2 : [0, \rho] \to X$ with $\gamma_1(0) = p = \gamma_2(0)$, the function $\phi : [0, \rho] \to [0, \infty)$ given by $\phi(t) = d(\gamma_1(t), \gamma_2(t))$ is a convex real-valued function.*

Remark. It is well-known that d convex implies (X, d) contractible. The interested reader is referred to [9, page 64].

It is a consequence of the Cartan-Hadamard-Aleksandrov Theorem (see [3]) that (X, d) simply connected, $\operatorname{curv} X \leq 0$ implies d is convex[3]. Thus question 2 may be re-interpreted as follows.

Question 3: Does there exist a convex metric on any manifold M^n other than $\mathbf{R}^n$? That is, if (M^n, d) is a complete inner metric structure on a topological manifold M^n, and d is convex, can M^n be topologically different from $\mathbf{R}^n$?

Unlike the classical Hadamard Theorem, question 3 is heavily dependent on $n = \dim M$.

$n = 3$: The answer is no. This is essentially due to Rolfsen [82]. Actually it is an easy corollary to Rolfsen's work together with Morton Brown's theorem on manifolds which may be regularly exhausted by cells[20].

$n \geq 5$: The answer is yes. Davis and Januszkiewcz[43] constructed examples in every dimension $n \geq 5$ of manifolds M^n admitting metrics for which $\operatorname{curv} M \leq 0$ and whose

[3] Riemannian manifolds have non-positive sectional curvature if and only if the natural inner metric is locally convex. See Manifolds of Non-Positive Curvature by Ballmann, Gromov and Schroeder,1985

universal covers $\widetilde{M}$ are not simply connected at infinity[4], and hence $\widetilde{M}^n \neq \mathbf{R}^n$. Ancel and Guilbault[5] have shown that the interior of every compact, contractible n-manifold with boundary, $n \geq 5$, admits a metric of non-positive curvature.

$n = 4$: This question is open at the time of this writing. The second part of this paper is devoted to establishing some partial results when the metric has a single tame point.

Definition 1.5 (topological tame point) *Let (M^n, d) be a convex manifold (i.e. a manifold metrized as a complete inner metric space so that the metric d is convex). A point $x \in M$ is said to be topologically tame with respect to d provided for every $r > 0$, the closed metric ball $B_r(x)$ is a manifold-with-boundary whose boundary is the metric sphere $S_r(x)$.*

As an application of Daverman's theory of singular regular neighborhoods, we can obtain an equivalent definition of topologically tame points by requiring only that the metric spheres $S_r(x)$ be codimension 1 submanifolds of M^n.

Another useful notion is that of metric tameness.

Definition 1.6 (metric tame point) *Let (M^n, d) be a convex manifold. Then $x \in M$ is said to be metrically tame with respect to d provided for every $r > 0$, each metric sphere $S_r(x)$ admits a compatible metric ρ_r under which* curv $S_r(x) \leq K_r < \infty$, *where K_r is a constant depending on the radius r.*

Some partial answers to Question 3 are provided by the next two results.

[4]The universal cover of a space of non-positive curvature naturally inherits a metric of non-positive curvature, called the covering metric.

Theorem 1.3 *Let (M^4, d) be a convex 4-manifold having at least one topologically tame point. Then M^4 is topologically equivalent to $\mathbf{R}^4$.*

Theorem 1.4 *Let (M^4, d) be a convex 4-manifold having at least one metrically tame point. Then M^4 is topologically equivalent to $\mathbf{R}^4$.*

Remark. The high dimensional non-euclidean non-positively curved examples of Davis-Januszkiewicz and Ancel-Guilbault have metrically tame points.

Theorem 1.4 follows from Theorem 1.3 and the 3-manifold recognition theorem recorded below. In order to establish Theorem 1.3, it suffices via Freedman [48] to show that M^4 is simply connected at infinity. To this end, it is sufficient to show that, given $x \in M^4$ topologically tame, each metric sphere $S_r(x)$ is bicollared in M^4. For this rather quickly implies that $M^4 - \{x\} \cong S_r(x) \times (0, \infty)$, for every $r > 0$. A moment's reflection then permits us to see that $\pi_1 S_r(x) = \{1\}$ for every $r > 0$ and hence M^4 will then be shown to be simply connected at infinity.

Theorem 1.5 *Let Y be a locally compact, finite dimensional hausdorff space. Then Y is a 3-manifold $\Longleftrightarrow$ the following two conditions are satisfied:*

1. *Y is a homology 3-manifold, i.e., for all $y \in Y$, $H_i(Y, Y - \{y\}; \mathcal{Z}) \cong \begin{cases} 0, & \text{if } i \neq 3 \\ \mathcal{Z}, & \text{if } i = 3 \end{cases}$*
2. *Each component of Y admits a compatible, complete inner metric for which* curv $Y \leq K$ *, for some $K < \infty$.*

Remark. Berestovsky has shown that every finite polyhedron, $\mathbf{P}^n$, admits a metrization that turns it into a 1-domain (cf. [12]). The induced inner metric on each

component yields the same topology on $\mathbf{P}^n$ and has the property that $\operatorname{curv} \mathbf{P}^n \leq 1$. In light of the Bing-Moise 3-manifold triangulation theorems (cf.[13],[69]), the forward implication of Theorem 1.5 is immediate. The reverse implication depends on $n = 3$ and fails in higher dimensions. Simple high-dimensional counter-examples are obtained by suspending triangulated non-trivial homology n-spheres and imposing the Berestovsky 1-domain metric on the suspension. For other applications of the Berestovsky metric, the interested reader is referred to [12] and [5].

CHAPTER 2

Elements of G-space Theory

The purpose of this chapter is to provide the reader with a sufficient understanding of both the classical and modern theories of G-spaces used in the remainder of this paper. No prior familiarity with G-space theory is assumed. By the classical theory, I am referring to Busemann's work in [28], e.g., and the related topics of intrinsic geometry as described by Rinow [83]. I use the phrase 'modern theory' to refer to the study of the topology of finite-dimensional G-spaces as in [58] and continued here. The difference is mainly the point of emphasis. In the classical theory, one looks at various kinds of geometric constraints (curvature restrictions, Desarguian space-form, etc.) and attempts to discover the global properties of the G-space. In the modern theory, one is concerned with the structure of arbitrary G-spaces and brings to bear such tools as algebraic topology and the theory of retracts.

1 Classical Concepts

In order to understand the geometric structure in an arbitrary G-space, it will be necessary to further study Definition 1.2 and identify several important consequences.

Theorem 2.1 *Let (X, d) be a G-space. Then (X, d) is a complete inner metric space which is locally compact.*

Proof. Let $\{x_n\}_{n=1}^{\infty}$ be a Cauchy sequence in X. Given $\epsilon = 1$, there is an index $N > 0$ so that if $n, m \geq N$, then $d(x_n, x_m) < \varepsilon$. Note that $\{x_m\}_{m>N}$ is contained in $B(x_N, \varepsilon)$. Via finite compactness, it follows that $\{x_m\}_{m>N}$ has a limit point $y \in B(x_N, \varepsilon)$ which implies $\{x_n\}_{n=1}^{\infty}$ converges to y.

To verify that (X, d) is an inner metric space, recall that the length, $L(\gamma)$, of a continuous path $\gamma : [a, b] \to X$ is given by

$$L(\gamma) = \sup_P \sum_{j=1}^{n} d\left(\gamma(t_j), \gamma(t_{j+1})\right), \tag{2.1}$$

where P is a partition of $[a, b]$ of the form $a = t_1 < t_2 < \ldots < t_n = b$. Now fix $x, y \in X$. It is an immediate consequence of the triangle inequality that if $\gamma : [a, b] \to X$ is any path such that $\gamma(a) = x$ and $\gamma(b) = y$, then $L(\gamma) \geq d(x, y)$. Hence to show that d is an inner metric, it suffices to construct a path γ for which $L(\gamma) = d(x, y)$.

To this end, choose $r > 0$ sufficiently large so that $x, y \in B(x, r)$.

Existence of Midpoints: Consider the set of points $C = \{z \in X \ | d(x, z) + d(z, y) = d(x, y)\}$. Note that $C \subset B(x, r)$ and, via the continuity of the distance function, if $\{z_k\} \subset C$ is any convergent sequence with $z_k \to z_0$ as $k \to \infty$, then $d(x, z_0) + d(z_0, y) = d(x, y)$. Thus C is closed, bounded and therefore compact, via finite compactness of X. Next define a map $f : C \longrightarrow [0, \infty)$ by $f(z) = \min\{d(x, z), d(y, z)\}$. Then f is continuous and obtains its maximum $\rho = f(m)$. I claim that m is a midpoint for x, y, i.e. $\rho = \frac{1}{2}d(x, y)$. Suppose to the contrary that $\rho \neq \frac{1}{2}d(x, y)$. Without loss of generality, let $\rho = d(x, m) < d(y, m)$ and consider the set

$$K = \{z \in X \setminus \{m\} | d(m, z) + d(z, y) = d(m, y)\} \tag{2.2}$$

{Note: such z are said to be between y and m.} Via definition K is bounded and via Menger convexity, $K \neq \emptyset$. To see that K is compact, it suffices to establish that K is closed. Observe that for all $z \in K$,

$$\begin{aligned} d(x,y) &= d(x,m) + d(m,y) \\ &= d(x,m) + d(m,z) + d(z,y) \\ &\geq d(x,z) + d(z,y) \\ &\geq d(x,y). \end{aligned}$$

Therefore all such z are between x and y, and m is between x and z. Now $z \in K$ implies $f(z) \leq f(m)$, since m maximizes f. Since $f(m) = \rho = d(x,m)$, and since m is between x and z, it must be the case that for all $z \in K$

$$\rho = d(x,m) \leq d(x,z) \tag{2.3}$$

Thus $f(z) = d(z,y)$ and moreover

$$d(z,y) \leq d(x,m) = \rho \tag{2.4}$$

holds for all $z \in K$. Applying 2.4, note that for all $z \in K$

$$\begin{aligned} d(m,z) &= d(x,y) - \{d(x,m) + d(z,y)\} \\ &\geq d(x,y) - 2d(x,m) \\ &= \beta > 0 \end{aligned}$$

where β is independent of the choice of $z \in K$. To see that K is closed, let $\{z_j\} \subset K$ be any convergent sequence in K and let $\lim z_j = w$. Continuity of the distance function guarantees that w satisfies the equation of 2.2. Since $d(m, z_j) \geq \beta > 0$ for all j, we have

that $w \neq m$ and hence $w \in K$. Thus K is closed and bounded and therefore compact, via finite compactness. Consider the function

$$d(m, \cdot): \ K \longrightarrow (0, \infty)$$

This function obtains a positive minimum at some point $z_0 \in K$. Apply Menger convexity and obtain z_1 distinct from and between $\{m, z_0\}$. This implies that $d(m, z_1) < d(m, z_0)$. However $z_1 \in K$ since

$$\begin{aligned} d(m, y) &= d(m, z_0) + d(z_0, y) \\ &= d(m, z_1) + d(z_1, z_0) + d(z_0, y) \\ &\geq d(m, z_1) + d(z_1, y) \\ &\geq d(m, y) \end{aligned}$$

and this contradicts the minimizing property of z_0 on K. It must therefore be the case that $\rho = \frac{1}{2}d(x, y)$, which establishes the existence of midpoints.

Existence of Segments: The goal is to construct an isometry $\gamma : [0, 2\rho] \longrightarrow X$ with $\gamma(0) = x$ and $\gamma(2\rho) = y$. I claim there is a countable dense set $A \subset [0, 2\rho]$ and an isometry $\gamma : A \longrightarrow X$ such that $\{0, 2\rho\} \subset A$, with $\gamma(0) = x, \gamma(2\rho) = y$. This is accomplished by inductively halving the interval $[0, 2\rho]$, and choosing corresponding midpoints. As X is complete, there is a continuous extension of γ to the entire interval, and continuity of the distance function implies that this extension is an isometry. □

Corollary 2.2 *Any two points in a G-space are joined by a geodesic segment.*

Proposition 2.3 (Local uniqueness of joins) *Let (X, d) be a G-space and let $x \in X$. There is a positive radius r_x such that any two points $y, z \in B_{r_x}(x)$ in the closed ball*

are joined by a unique segment in X.

Proof. Via Definition1.2, given $x \in X$, there is a $\rho_x > 0$ such that segments with endpoints in $B_{\rho_x}(x)$ may be extended uniquely. Set $r_x = \frac{1}{2}\rho_x$ and let $y, z \in B_{r_x}(x)$. Via Corollary 2.2 above, there is a segment γ joining y to z. Let γ_1, γ_2 be segments joining y to z. There is a third point $w \in B_{\rho_x}(x)$ such that z is between y and w. Necessarily $w \notin \{\mathrm{im}\,\gamma_1, \mathrm{im}\,\gamma_2\}$. Apply Corollary 2.2 to construct a segment γ_3 joining z to w. It is immediate that $L(\gamma_1 \cup \gamma_3) = d(y,w) = L(\gamma_2 \cup \gamma_3)$. Let $-\gamma_3$ be γ_3 with the reverse orientation. Then $-\gamma_3$ has extensions $-\gamma_1 \cup -\gamma_3$ and $-\gamma_2 \cup -\gamma_3$. Via uniqueness of extensions, $-\gamma_1 \cup -\gamma_3 = -\gamma_2 \cup -\gamma_3$, and hence $\gamma_1 = \gamma_2$, as desired. □

One useful feature of G-spaces is the local cone structure. Recall that the cone over a space Y, $\triangle(Y)$, is the space obtained from the quotient map

$$Y \times I \longrightarrow \frac{Y \times I}{\sim} = \triangle(Y)$$

where $\sim$ identifies $Y \times \{0\}$ to a point. In light of Proposition 2.3, it is reasonable to define a similar structure called the *geodesic cone* $x \underset{g}{*} Y$ over a subset Y of some G-space X as the union of the images of all segments joining x to a point $y \in Y$. The next proposition states that locally these two concepts agree in a G-space whenever Y is the boundary of a sufficiently small metric ball.

Proposition 2.4 (Local cones) *Let (X,d) be a G-space and let $x \in X$. There is a positive radius ε_x for which the closed metric ball $B_{\varepsilon_x}(x)$ is homeomorphic to the cone over its boundary. That is,*

$$B_{\varepsilon_x}(x) \cong \triangle\left(S_{\varepsilon_x}(x)\right)$$

where $S_{\varepsilon_x}(x)$ denotes the metric sphere about x.

Proof. Let $\varepsilon_x = \frac{1}{2}r_x$, where r_x is given from Proposition 2.3. By definition, the geodesic cone $x \underset{g}{*} S_{\varepsilon_x}(x)$ is the union of all (images of) segments γ_y joining x to y, where $y \in S_{\varepsilon_x}(x)$. (Note that by Proposition 2.3 and the axiom of uniqueness of extensions, $x \underset{g}{*} S_{\varepsilon_x}(x)$ is a non-singular cone.) It is an immediate consequence of this definition that $x \underset{g}{*} S_{\varepsilon_x}(x) \subset B_{\varepsilon_x}(x)$. To see that $B_{\varepsilon_x}(x) \subset x \underset{g}{*} S_{\varepsilon_x}(x)$, let $z \in B_{\varepsilon_x}(x) \setminus \{x\}$, and let $s = d(x,z)$. If $s = \varepsilon_x$, we are done. Assume that $0 < s < \varepsilon_x$,and let γ_z be the unique segment joining x to z. We aim to show that γ_z may be extended to the boundary $S_{\varepsilon_x}(x)$. To this end, let $\widetilde{\gamma}_z$ be a maximal extension of γ_z in X (which exists via finite compactness) and consider the length of $\widetilde{\gamma}_z$, $L(\widetilde{\gamma}_z)$. Now $L(\widetilde{\gamma}_z) \geq \varepsilon_x = \frac{1}{2}r_x$, for otherwise we may apply local extendibility and extend $\widetilde{\gamma}_z$, which contradicts maximality. Hence $z \in \widetilde{\gamma}_z\,([0,\varepsilon_x]) \subset x \underset{g}{*} S_{\varepsilon_x}(x)$,and therefore $B_{\varepsilon_x}(x) = x \underset{g}{*} S_{\varepsilon_x}(x)$ as sets.

To see that $x \underset{g}{*} S_{\varepsilon_x}(x) \cong \triangle\left(S_{\varepsilon_x}(x)\right)$ consider the map

$$\lambda : \triangle\left(S_{\varepsilon_x}(x)\right) \longrightarrow x \underset{g}{*} S_{\varepsilon_x}(x)$$

defined via the following rule. Each point $[y,t] \in \triangle\left(S_{\varepsilon_x}(x)\right)$ is sent to the unique point $\lambda(y,t) \in x \underset{g}{*} S_{\varepsilon_x}(x)$ on the segment from x to y satisfying $d\left(x, \lambda(y,t)\right) = td(x,y)$. It is clear that λ is a well-defined bijection. It remains to be seen that λ is continuous, and therefore a homeomorphism between compacta. To this end, let$\{[y_n,t_n]\}_{n=1}^{\infty} \subset \triangle\left(S_{\varepsilon_x}(x)\right)$ be a sequence where $[y_n,t_n] \to [y_0,t_0]$ as $n \to \infty$.

Case $t_0 \neq 0$: In this case, $y_n \to y_0$ and $t_n \to t_0$ as $n \to \infty$. Thus $y_n = \lambda(y_n,1) \to \lambda(y_0,1) = y_0$. Let $\left\{\gamma_n : [0,\varepsilon_x] \longrightarrow x \underset{g}{*} S_{\varepsilon_x}(x)\right\}_{n=0}^{\infty}$ be the isometries with $\gamma_n(0) = x$ and

$\gamma_n(\varepsilon_x) = y_n$. This family of isometries is equicontinuous and locally bounded. Hence by the Arzela-Ascoli Theorem, some subsequence uniformly converges to a continuous map

$$\psi : [0, \varepsilon_x] \longrightarrow x \underset{g}{*} S_{\varepsilon_x}(x).$$

Apply semi-continuity of the length function ($L(\psi) \leq \liminf L(\gamma_n)$) and continuity of the distance function to see that ψ is an isometry. It follows via Proposition 2.3 that $\psi = \gamma_0$. Next let $\left\{\lambda(y_{n_j}, t_{n_j})\right\}$ be any convergent subsequence of $\{\lambda(y_n, t_n)\}$ with limit point $w \in x \underset{g}{*} S_{\varepsilon_x}(x)$ (these exist via compactness) and, without loss of generality, suppose that the corresponding subsequence of isometries $\gamma_{n_j} \rightarrow \gamma_0$. Note that $\lambda(y_{n_j}, t_{n_j}) = \gamma_{n_j}(t_{n_j}) \rightarrow \gamma_0(t_0) = w$ as desired. To see that the original sequence $\{\lambda(y_n, t_n)\}$ converges to $w = \gamma_0(t_0)$, use finite compactness.

Case $t_0 = 0$: Use continuity of the distance function to see that every convergent subsequence of $\{\lambda(y_n, t_n)\}$ converges to x and then apply finite compactness to guarantee convergence of $\{\lambda(y_n, t_n)\}$ as in the above case. □

Using the local cone structure, Krakus determined a pair of retraction inequalities {see [58, pages 738-39]} which he used to determine the topology of low dimensional G-spaces. The setup for these inequalities is as follows. Let $x \in X$, let $\varepsilon_1 > 0$ be a cone radius about x, and let $0 < \varepsilon_2 < \frac{\varepsilon_1}{2}$. Use the radial segments to define a continuous retraction $r : B_{\varepsilon_1}(x) - \{x\} \longrightarrow S_{\varepsilon_2}(x)$. Next call a pair of points $a, b \in S_{\varepsilon_2}(x)$ *antipodal* if x lies on the unique segment joining a to b.

Proposition 2.5 (First retraction inequality) *Let $a, b \in S_{\varepsilon_2}(x)$ be non-antipodal points and let y be in the interior of the unique segment joining a to b. Then we have the*

following strict inequality

$$d(y, r(y)) < d(y, a).$$

Proof. Suppose to the contrary that $d(y, r(y)) \geq d(y, a)$ and consider the two cases.

$\underline{d(y, r(y)) = d(y, a)}$: If $r(y)$ is on the segment $[x, y]$, note that via additivity of the length function

$$\begin{aligned} L([x, a] \cup [a, y]) &= L([x, a]) + L([a, y]) \\ &= \varepsilon_2 + L([r(y), y]) \\ &= L([x, r(y)]) + L([r(y), y]) \\ &= d(x, y). \end{aligned}$$

Hence $[x, a] \cup [a, y]$ is the unique segment $[x, y]$, and therefore $a = r(y)$, since $d(x, a) = \varepsilon_2 = d(x, r(y))$. Apply local extendibility and uniqueness of extensions to see that $[x, a] \cup [a, y] \cup [y, b]$ is a segment, which contradicts the choice of $b \in S_{\varepsilon_2}(x)$.

However if $r(y)$ is not on the segment $[x, y]$, then we have

$$\begin{aligned} L([x, y] \cup [y, a]) &= L([x, y]) + L([y, a]) \\ &= L([x, y]) + L(y, r(y)) \\ &= d(x, r(y)) \\ &= \varepsilon_2. \end{aligned}$$

Hence $[x, y] \cup [y, a]$ is the unique segment $[x, a]$ and hence $y \in [x, a]$, which contradicts a, b non-antipodal.

$\underline{d(y, r(y)) > d(y, a)}$: Note that

$$\begin{aligned} \varepsilon_2 = \quad d(x,a) &\leq \quad d(x,y) + d(y,a) \\ &< \quad d(x,y) + d(y, r(y)) \\ &= \qquad d(x, r(y)) \\ &= \qquad \varepsilon_2 \end{aligned}$$

which yields an immediate contradiction. □

Corollary 2.6 (Second Retraction Inequality) *Let $a, b \in S_{\varepsilon_2}(x)$ be non-antipodal points and let y be in the interior of the unique segment joining a to b. Then the following strict inequality holds*

$$d(b, r(y)) < d(a, b).$$

Proof. Apply Proposition 2.5 to get that $d(y, r(y)) < d(y, a)$. Then

$$\begin{aligned} d(b, r(y)) &\leq d(b,y) + d(y, r(y)) \\ &< \quad d(b,y) + d(y,a) \\ &= \qquad d(b,a). \end{aligned}$$

□

Definition 2.1 *A space Y is called homogeneous provided for any two points $x, y \in Y$ there is a homeomorphism of pairs $h : (Y, x) \longrightarrow (Y, y)$.*

Compare the next result with Busemann homogeneity property [28, page 49].

Theorem 2.7 *Every G-space is homogeneous.*

Proof. Let X be a G-space. Via Corollary 2.2, X is path-connected. Fix $x_0 \in X$ and let $A = \{x \in X | \exists \text{ homeomorphism } \phi : (X, x) \longrightarrow (X, x_0)\}$. It suffices to show that $A = X$. Note that $A \neq \emptyset$ since $x_o \in A$

$\underline{A \text{ is open}}$: Let $a \in A$,choose $\varepsilon_a > 0$ a cone radius about a as in Proposition 2.4, and set $\delta = \frac{\varepsilon_a}{4}$. It suffices to show that $\operatorname{int} B_\delta(a) \subset A$. To this end, let $y \in \operatorname{int} B_\delta(a)$ and let $w \in \partial B_\delta(a)$ be the unique boundary point such that y is contained in the interior of the segment joining a to w. Note that via the triangle inequality, $B_{\frac{\varepsilon_a}{2}}(w) \subset \operatorname{int} B_{\varepsilon_a}(a)$. Hence $\frac{\varepsilon_a}{2}$ is a cone radius for w. Let $t_y = d(w, y)$ and let $h : [0, \frac{\varepsilon_a}{2}] \longrightarrow [0, \frac{\varepsilon_a}{2}]$ be a piecewise linear homeomorphism sending $0 \mapsto 0, t_y \mapsto \frac{\varepsilon_a}{4}$, and $\frac{\varepsilon_a}{2} \mapsto \frac{\varepsilon_a}{2}$. Apply Proposition 2.4 to find the homeomorphism $\psi = \lambda^{-1} : B_{\frac{\varepsilon_a}{2}}(w) \longrightarrow \triangle\left(S_{\frac{\varepsilon_a}{2}}(w)\right)$. Define a homeomorphism $\Theta : \triangle\left(S_{\frac{\varepsilon_a}{2}}(w)\right) \longrightarrow \triangle\left(S_{\frac{\varepsilon_a}{2}}(w)\right)$ via $\Theta([z, t]) = [z, h(t)]$. Observe that

$$\phi : B_{\frac{\varepsilon_a}{2}}(w) \longrightarrow B_{\frac{\varepsilon_a}{2}}(w) \quad \text{given by}$$

$$\phi = \psi^{-1} \circ \Theta \circ \psi$$

is a homeomorphism sending y to a and fixing the boundary of $B_{\frac{\varepsilon_a}{2}}(w)$. Extend ϕ to a homeomorphism $\Phi : (X, y) \longrightarrow (X, a)$ by declaring Φ to be the identity on $X - B_{\frac{\varepsilon_a}{2}}(w)$. Then $y \in A$, as desired.

$\underline{A \text{ is closed}}$: Let $\{a_n\}_{n=1}^{\infty} \subset A$ be a convergent sequence such that $\lim_{n \to \infty} a_n = a$. Choose a cone radius $\varepsilon_a > 0$ about a and find $a_n \in \operatorname{int} B_{\frac{\varepsilon_a}{4}}(a)$. Let $w \in \partial B_{\frac{\varepsilon_a}{4}}(a)$ be the unique such point so that a_n is in the interior of the segment joining w to a. Arguing as above, find a homeomorphism $\phi : B_{\frac{\varepsilon_a}{2}}(w) \longrightarrow B_{\frac{\varepsilon_a}{2}}(w)$ fixed on the boundary and sending a_n to a. Extend to a homeomorphism $\Phi : (X, a_n) \longrightarrow (X, a)$ as before. Therefore $a \in A$ as desired.

Since X is connected, it must be the case that $A = X$ and thus X is homogeneous. □

Remark. Busemann asked ([28, page 403]) whether every G-space is locally homogeneous in the sense of Montgomery [70], i.e., can the homogeneity homeomorphisms be chosen so that each is isotopic to the identity. Upon closer inspection of the proof of Theorem 2.7 above, one notices the answer is affirmative, either as a local consideration or as a global one.

2 Modern Concepts

A litmus test for an arbitrary space to be a topological manifold is that it must be a *generalized manifold*, i.e. a finite dimensional ANR homology manifold. It is the aim of this section to establish that every finite dimensional G-space is a generalized manifold.

Definition 2.2 (ANR) *Let Y be a metric space. Y is said to be an Absolute Neighborhood Retract provided for every closed embedding $e : Y \longrightarrow Z$ of Y into a metric space Z, there is an open neighborhood U of the image $e(Y)$ which retracts to $e(Y)$. That is, there is a continuous surjection*

$$r : U \longrightarrow e(Y)$$

with $r(x) = x$ for all $x \in e(Y)$.

The following recognition theorem for ANR's is mainly due to Kuratowski [59] and more general statements of this theorem are found in [37, page 119] and [90, page 237].

Theorem 2.8 *Let Y be a finite dimensional, locally contractible separable metric space. Then Y is an ANR.*

Recall that a space is separable if it has a countable dense subset. For metrizable spaces, the Lindelöf property (every open cover has a countable subcover) implies separability (see [72, page 194]). As an immediate corollary to Theorem 2.8 and finite compactness in G-spaces, we have the following result.

Corollary 2.9 *Every finite dimensional G-space is an ANR.*

Proof. Apply Proposition 2.4 to see that every G-space X is locally contractible. To see that X has the Lindelöf property, note that X is the monotone union of compact metric balls

$$X = \bigcup_{n=1}^{\infty} B_n(x)$$

and hence is Lindelöf. □

Remark. Whether or not every G-space is an ANR is apparently unsettled at the time of this writing. At the heart of the matter is the Busemann finite dimensionality question: Are all G-spaces finite dimensional [28, page 403]? A partial affirmative answer was obtained by Berestovsky [11]. See [29, page 64] for an outline of Berestovsky's work on this problem.

Definition 2.3 (homology manifold) *Let Y be a topological space and let $n \in \mathcal{N}$ be a positive integer. Y is said to be a homology n-manifold provided for every $y \in Y$,*

$$H_i(Y, Y - \{y\}; \mathcal{Z}) \cong H_i(\mathbf{R}^n, \mathbf{R}^n - \{0\}; \mathcal{Z})$$

where $H_i(\cdot, \cdot; \mathcal{Z})$ denotes singular relative homology with integral coefficients.

The sheaf-theoretic approach to homology manifold recognition theory was begun by Bredon [19] and continued in the subsequent work of Bryant [22], and Dydak-Walsh [45]. All three of these results can be applied to the G-space setting to establish the desired conclusion. As a matter of convenience, the reader is referred to [45, page 84] for a proof of the following result.

Theorem 2.10 *Let Y be locally compact, metrizable. Suppose Y satisfies the following properties:*

$\mathbf{P}_1$] *Y is homologically locally connected with respect to singular homology, i.e. given a point $y \in Y$ and a compact neighborhood A_y of y, there is a compact neighborhood $B_y \subset A_y$ such that inclusion induces the trivial homomorphism*

$$i_* : \tilde{H}_*(B_y; \mathcal{Z}) \longrightarrow \tilde{H}_*(A_y; \mathcal{Z}).$$

$\mathbf{P}_2$] *Y has constant integral cohomological dimension $\dim_{\mathcal{Z}} Y = n$, i.e. for each open subset $U \neq \emptyset$, the cohomology with compact supports has the property*

$$H_c^k(U; \mathcal{Z}) \cong 0, \text{ for } k > n$$

and there is some open subset V for which $H_c^n(V; \mathcal{Z}) \neq 0$.

$\mathbf{P}_3$] *For any two points $x, y \in Y$ the local Steenrod homology groups are isomorphic, i.e.*

$$^sH_k(Y, Y - \{y\}; \mathcal{Z}) \cong {}^sH_k(Y, Y - \{x\}; \mathcal{Z}) \text{ for } k = 0, 1, \ldots n.$$

$\mathbf{P}_4$] *Each Steenrod local group $^sH_k(Y, Y - \{y\}; \mathcal{Z})$ is finitely generated.*

Then Y is a homology n-manifold.

It is an immediate consequence of Proposition 2.4 that every G-space satisfies P_1]. If X is a finite dimensional G-space, then by Theorem 2.7, both P_2] and P_3] hold. The next result, which is due to James West [91], is an important property of compact ANR's and will be used to verify P_4].

Theorem 2.11 *(West) Every compact ANR has the homotopy type of a finite polyhedron.*

Theorem 2.12 *Every finite dimensional G-space X satisfies Property 4] of Theorem 2.10, and consequently every such X is a homology manifold.*

Proof. The strategy is to first show that sufficiently small metric spheres in X are compact ANR's and then apply West's Theorem in order to deduce P4]. To this end, let $x \in X$, let ε_1 be a cone radius of x and let $0 < \varepsilon_2 < \frac{\varepsilon_1}{2}$. Fix a point $a \in S_{\varepsilon_2}(x)$ and set

$$Q_\delta(a) = S_{\varepsilon_2}(x) \cap B_\delta(a)$$

where δ is a cone radius about a such that $\delta \leq \varepsilon_2$. Note that no pair of points in $Q_\delta(a)$ is antipodal. Exploiting the cone structure of $B_\delta(a)$, we may naturally define the continuous deformation $H : Q_\delta(a) \times I \longrightarrow B_\delta(a)$. Composing with the retraction of Proposition 2.5, we obtain a deformation

$$r \circ H : Q_\delta(a) \times I \longrightarrow S_{\varepsilon_2}(x).$$

Apply Corollary 2.6 to see that the image of $r \circ H$ is contained in $Q_\delta(a)$ and hence

$$r \circ H : Q_\delta(a) \times I \longrightarrow Q_\delta(a)$$

is a contraction of $Q_\delta(a)$ in itself. Apply Theorem 2.8 and the hypothesis that $\dim X < \infty$ to see that $S_{\varepsilon_2}(x)$ is an ANR, as desired. □

We now extend Definition 2.3 to include a notion of boundary.

Definition 2.4 (homology manifold with boundary) *A space Y is said to be a homology n-manifold with boundary provided Y can be written as the disjoint union of subsets A, B so that*

a. *For all $a \in A$, $H_*(Y, Y - \{a\}; \mathcal{Z}) \cong H_*(\mathbf{R}^n, \mathbf{R}^n - \{0\}; \mathcal{Z})$*

b. *For all $b \in B$, $H_*(Y, Y - \{b\}; \mathcal{Z}) \cong 0$*

As a corollary to Theorem 2.12 and the local cone structure, we obtain the following result.

Corollary 2.13 *Let (X, d) be a G-space, $\dim X = n < \infty$. Then for all sufficiently small $r > 0$ and $x \in X$, we have that*

1. *$B_r(x)$ is a homology n-manifold with boundary $\partial B_r(x) = S_r(x)$*

2. *$S_r(x)$ is a homology $(n-1)$-manifold with empty boundary.*

Proof. Choose $r < \varepsilon_x$ where ε_x is the cone radius about x. Via Theorem 2.12, X is a homology n-manifold. Therefore $\operatorname{int} B_r(x)$ is also a homology n-manifold. Since $\operatorname{int} B_r(x)$ is contractible, it is necessarily orientable (see Dold [44, page 255]) . Let

$z \in S_r(x)$ and apply the cone structure of $B_r(x)$ to see that $B_r(x) - \{z\}$ is contractible. Hence

$$H_*(B_r(x), B_r(x) - \{z\}) = 0$$

which establishes part 1.

For part 2, apply the cone structure to find a homeomorphism

$$e : S_r(x) \times (-1, 1) \longrightarrow U$$

where U is an open set in $\operatorname{int} B_r(x)$ containing $S_r(x)$ and $e(S_r(x) \times \{0\}) = S_r(x)$. Necessarily U is an orientable homology n-manifold. Let $y \in S_r(x)$ and $J = (-1, 1)$. Use duality to obtain that

$$H_*(S_r(x), S_r(x) - \{y\}) \cong H_*(S_r(x) \times J, S_r(x) \times J - \{y \times J\}) \cong \check{H}_c^{n-*}(y \times J).$$

Via [88, page 322], $\check{H}_c^{n-*}(y \times J) \cong \tilde{H}^{n-*}(\mathbf{S}^1) \cong \begin{cases} \mathcal{Z} & , * = n-1 \\ 0 & , \text{otherwise} \end{cases}$ and therefore $S_r(x)$ is a homology $(n-1)$-manifold. □

With this homology data in place, it is relatively straightforward to verify Busemann's conjecture for dimensions ≤ 3.

Corollary 2.14 *Let X be a G-space with $n = \dim X \leq 3$. Then X is a n-manifold.*

Proof. Use the retraction of Proposition 2.5 to see that sufficiently small metric spheres $S_{\varepsilon_2}(x)$ are connected and locally path connected. Therefore such $S_{\varepsilon_2}(x)$ are Peano continua. Utilizing Corollary 2.13, note that such $S_{\varepsilon_2}(x)$ are themselves homology $(n-1)$-manifolds. Now apply classical manifold recognition theory in dimensions 1 and

2 (see [92]) to recognize such $S_{\varepsilon_2}(x)$ as topological manifolds. This now allows us to find, using the local cone/product structure in $B_{\varepsilon_2}(x) - \{x\}$, points in X having manifold neighborhoods. Now apply Theorem 2.7 to see that every point in X has a manifold neighborhood. □

CHAPTER 3

Cell-Like Maps

Modern decomposition theory of manifolds is the study of proper, surjective maps of the form

$$f : M^n \longrightarrow X \tag{3.1}$$

where M^n is a manifold and each fiber of f, $f^{-1}(x)$, is a compact subset of M^n. (Recall that f is said to be proper provided $f^{-1}(C)$ is compact for each compact C.) One of the principal concerns of the theory is to determine when maps of the form (3.1) can be approximated arbitrarily closely by homeomorphisms. If f is approximable by homeomorphisms {or **ABH**}, one is able to immediately determine the topological type of X as being that of M^n.

A necessary condition that a map of the form (3.1) be **ABH** is that each fiber $f^{-1}(x)$ be a *cell-like* set [37, page 35], and such a map is said to be a cell-like map. However this condition is not sufficient and the interested reader may find a plethora of non-**ABH** examples in [37, Section 9 and Chapter VI].

Given a space X, we are naturally led to ask two fundamental questions.

Question A: When does there exist a manifold M^n and a map of the form (3.1) having cell-like fibers?

Question B: Given such a map, when is it **ABH**?

This chapter is devoted to answering Question A when that X is a $4D$ G-space.

1 Cell-Like Sets and Resolutions

Definition 3.1 (Cell-Like Set) *Let C be a compact metric space. C is said to be* cell-like *provided given any embedding $e : C \longrightarrow Y$, where Y is an ANR, and given any open subset $U \subset Y$ such that $e(C) \subset U$, there is a contraction $H : e(C) \times I \longrightarrow U$.*

Examples of cell-like sets include all contractible metric compacta. Non-contractible cell-like examples include the $\sin \frac{1}{x}$ continuum and Whitehead's continuum. See [37, Chapter III] or [90, Chapter 7] for an introduction into this field of study. The survey [68] gives a good account of the current state of topology of cell-like maps and includes an extensive list of open questions in the area.

Definition 3.2 *Let Y be a separable metric space. Y is said to be* resolvable *provided there is a manifold $\mathbf{M}^n$ and a cell-like surjection $f : \mathbf{M}^n \longrightarrow Y$. Such a map is said to be a* resolution *of Y.*

Remark: It is a trivial (but, as we will see, a nonetheless useful) observation that every manifold is resolvable.

If Y is a finite-dimensional separable metric space, then a necessary condition that Y be resolvable is that Y be an ANR homology manifold as defined in Chapter 2 (cf. [37, Section 26]). As we have seen, every finite dimensional G-space is an ANR homology manifold and these spaces are therefore candidates for resolvability. One must proceed carefully as it has recently been shown there are non-resolvable ANR homology

n-manifolds when $n \geq 6$ (cf. [24]), and it is conjectured that such examples exist for all $n \geq 4$.

Definition 3.3 (admissible) *Let Y be a homology n-manifold.*

a. *If $\partial Y = \emptyset$, then Y is said to be admissible to resolution theory provided Y is a locally compact, connected, finite-dimensional, separable, metrizable ANR.*

b. *If $\partial Y \neq \emptyset$, then Y is said to be admissible to resolution theory provided both $Y - \partial Y$ and ∂Y are admissible homology manifolds with empty boundary.*

The basis for all resolution theory is the following theorem due to Quinn [80].

Theorem 3.1 (Quinn's local index)Let $n \geq 1$ *and let Y be an admissible homology n-manifold . There is a 'local index' $i(Y) \in (1 + 8\mathcal{Z})$ defined, which has the following properties.*

1. *For every non-empty, open $U \subset Y$, $i(U) = i(Y)$.*

2. *If $\partial Y \neq \emptyset$, then $i(\partial Y) = i(Y)$.*

3. *If X is admissible, then $i(X \times Y) = i(X) \times i(Y)$*

4. *If $\dim Y \geq 5$, or if $\dim Y = 4$ and ∂Y is either empty or a 3-manifold, then Y is resolvable if and only if $i(Y) = 1$.*

There are some immediate and striking corollaries to Theorem 3.1 and the seemingly innocuous remark following Definition 3.2.

Corollary 3.2 *Let Y be an admissible homology n-manifold, $n \geq 4$. Assume $\partial Y = \emptyset$.*

a. *If there is a single point $y \in Y$ having a manifold neighborhood $U_y \subset Y$, then $i(Y) = 1$ and Y is resolvable.*

b. *If Y is polyhedral, then $i(Y) = 1$ and Y is resolvable.*

Proof. For part a], since U_y is a manifold, $\mathrm{i}(U) = 1$. Apply Property #1 of the local index to see that $\mathrm{i}(Y) = 1$. Part b] follows after applying part a] to any point y lying in the interior of a top dimensional simplex. □

Corollary 3.3 *Let Y be an admissible homology n-manifold, $n \geq 4$. Assume $\partial Y = \emptyset$. If there is an embedding $e : \mathbf{N}^{n-1} \longrightarrow Y$ of an topological $(n-1)-$manifold $\mathbf{N}^{n-1}$ such that $e(\mathbf{N}^{n-1})$ locally separates Y, then $i(Y) = 1$ and Y is resolvable.*

Proof. Find a neighborhood $W \subset Y$ and an open cell $D^{n-1} \subset e(\mathbf{N})$ for which $W - D^{n-1}$ is the disjoint union of two open subsets W_1 and W_2 whose common boundary is D^{n-1}. Set $X = W_1 \cup D^{n-1}$ and note that X is an admissible homology manifold whose boundary is the open $n-1$ cell D^{n-1}. Let $Z = X \bigcup_{D^{n-1}} \{D^{n-1} \times [0,1)\}$ be the adjunction space formed by attaching an open collar to ∂X. Observe that for every open set U contained in the collar, $\mathrm{i}(U) = 1$. Via Property #1 of the local index, $1 = \mathrm{i}(U) = \mathrm{i}(Z) = \mathrm{i}(W_1) = \mathrm{i}(Y)$, and hence Y is resolvable . □

2 Resolving 4D G-spaces

The overall strategy to verify resolvability of a $4D$ G-space X is to construct a locally separating embedding $e : D^3 \longrightarrow X$ and apply Corollary 3.3 above. Recall the definition of $Q_\delta(a) = S_{\varepsilon_2}(x) \bigcap B(a)$, for suitably chosen ε_2 and δ, and consider the frontier of $Q_\delta(a)$, $K_\delta(a)$, given by $K_\delta(a) = S_{\varepsilon_2}(x) \cap S_\delta(a)$. It is precisely $K_\delta(a)$ which we wish to analyze using the following recent theorem of Mitchell [67]

Theorem 3.4 (Mitchell) *Let Y be a locally compact, first countable homology n-manifold with boundary. Assume that* $\dim_{\mathcal{Z}} Y < \infty$. *Then either*

1. *∂Y is empty, or*

2. *∂Y is a homology $(n-1)$-manifold with empty boundary. Moreover if $\partial Y \neq \emptyset$, then ∂Y is cohomologically locally connected over $\mathcal{Z}$.*

We say that a space Y is *cohomologically locally connected over $\mathcal{Z}$* provided for every $y \in Y$ and every neighborhood U_y, there is a sub-neighborhood V_y so that inclusion induces the trivial homomorphism $i^* : \tilde{H}^*(U_y; \mathcal{Z}) \longrightarrow \tilde{H}^*(V_y; \mathcal{Z})$. (Compare with P1] of Theorem 2.10.) We now apply Mitchell's Theorem to analyze the topology of $\partial Q_\delta(a)$.

Proposition 3.5 *Let X be a G-space, with* $\dim X = n < \infty$, *let $x \in X$ and let $Q_\delta(a)$ be as in the proof of Theorem 2.12. Then $Q_\delta(a)$ has the following properties.*

1. *$Q_\delta(a)$ is a homology $(n-1)$-manifold whose boundary $\partial Q_\delta(a) = K_\delta(a)$.*

2. *$K_\delta(a)$ is a homology $(n-2)$-manifold .*

3. *$K_\delta(a)$ is cohomologically locally connected over $\mathcal{Z}$.*

Proof. Via Corollary 2.13, $S_{\epsilon_2}(x)$ is a homology $(n-1)$-manifold (with no boundary). Thus $\operatorname{int} Q_\delta(a) = S_{\epsilon_2}(x) \bigcap \operatorname{int} B_\delta(a)$ is a homology manifold with empty boundary. Recall the contraction $r \circ H : Q_\delta(a) \times I \longrightarrow Q_\delta(a)$ from page 25 and, applying the second retraction inequality, note that for each $w \in K_\delta(a)$, the restriction $r \circ H| : Q_\delta(a) - \{w\} \times I \longrightarrow Q_\delta(a) - \{w\}$ is a contraction of $Q_\delta(a) - \{w\}$. Therefore the local homology groups $H_*(Q_\delta(a), Q_\delta(a) - \{w\}; \mathcal{Z})$ vanish which implies $\partial Q_\delta(a) = K_\delta(a)$. By Mitchell's Theorem, $K_\delta(a)$ is a homology $(n-2)$-manifold which is cohomologically locally connected over $\mathcal{Z}$. □

At the core of the proof of Theorem 1.1 is the following recognition result.

Proposition 3.6 *Let $n = 4$ and let $K_\delta(a) = \partial Q_\delta(a)$ be as above. Then $\partial Q_\delta(a)$ is topologically equivalent to the 2-sphere, S^2.*

Proof. Via Proposition 3.5 above, $\partial Q_\delta(a)$ is a homology 2-manifold. Since $Q_\delta(a)$ is contractible, it is orientable over $\mathcal{Z}$ {see [4]}. Via the proof of Theorem 2.12, $Q_\delta(a) - \partial Q_\delta(a)$ is also contractible. Apply Lefschetz Duality (cf. White [93, Thm. 4.3]) to see that

$$H_*(\partial Q_\delta(a)) \cong H_*(\mathbf{S}^2) \tag{3.2}$$

which gives that $\partial Q_\delta(a)$ is connected (among other things). Apply the third conclusion of Mitchell's theorem and [53, page 348] to see that $\partial Q_\delta(a)$ is locally connected. Since $\partial Q_\delta(a)$ is compact, it is necessarily a Peano continuum. Classical 2-manifold recognition theory (see [92]) guarantees that every homology 2-manifold which is a Peano continuum is a genuine 2-manifold. From the homology data in (3.2), $\partial Q_\delta(a)$ is seen to have the

homology of S^2. Now apply [62, page 26] to recognize $\partial Q_\delta(a)$ as S^2. □

Corollary 3.7 *Every 4-dimensional G-space is resolvable.*

Proof. Given X^4 a G-space, let $\partial Q_\delta(a)$ be as above and consider the (non-singular) geodesic cone $x \underset{g}{*} \partial Q_\delta(a) \subset B_{\varepsilon_2}(x)$. Via Proposition 3.6 and the cone structure of $B_{\varepsilon_2}(x)$, we are able to recognize $x \underset{g}{*} \partial Q_\delta(a)$ as being a genuine 3-cell. Moreover since $\partial Q_\delta(a)$ separates $S_{\varepsilon_2}(x)$, duality guarantees that $x \underset{g}{*} \partial Q_\delta(a)$ separates $B_{\varepsilon_2}(x)$. Now apply Corollary 3.3 to see that $\mathrm{i}(X) = 1$ and hence X^4 is resolvable by Quinn's Theorem 3.1. □

Remark. Notice that in the proof of the above corollary, we have a choice of either of two 3-cells having the separation property: $D^3_+ = x \underset{g}{*} \partial Q_\delta(a)$ and $D^3_- = a \underset{g}{*} \partial Q_\delta(a)$. That is, coning over $\partial Q_\delta(a)$ from either x or a yields an embedded 3-cell in X. It is an immediate consequence of the second retraction inequality that these two 3-cells D^3_+ and D^3_- meet only in their common boundary $\partial Q_\delta(a) \cong S^2$. Therefore $D^3_+ \cup D^3_-$ is a topological 3-sphere embedded in X which separates. Consider the unique segment $[x, a]$ and let y be in the interior of this segment. Clearly a component of $X - D^3_+ \cup D^3_-$ is a neighborhood of y and the boundary of this neighborhood is an embedded 3-sphere.

A space X each of whose points have arbitrarily small neighborhoods with n-sphere boundaries is said to be *locally n-spherical.* The next result establishes that each $4D$ G-space has this local structure.

Proposition 3.8 *Let X be a 4-dimensional G-space. Then X is locally 3-spherical.*

Proof. Let $y \in X$ and let W_y be an arbitrary open neighborhood of y in X. The goal is to produce an open neighborhood $V_y \subset W_y$ having a 3-sphere boundary. To this end, choose ε such that $0 < \varepsilon < \frac{\varepsilon_y}{2}$ and with the property $B_\varepsilon(y) \subset W_y$ (here ε_y denotes the cone radius about y). Choose a point $x \in \operatorname{int} B_\varepsilon(y)$ so that

$$y \in \operatorname{int} B_r(x) \subset B_r(x) \subset \operatorname{int} B_\varepsilon(y)$$

for some $0 < r < \frac{\varepsilon}{2}$. Note that the unique segment $[x, y]$ has a natural extension to the boundary $S_r(x)$ and this extension meets the boundary in a single endpoint which we denote by a.

Arguing as in Corollary 3.7, we see that both $D^3_+ = x \underset{g}{*} \partial Q_r(a)$ and $D^3_- = a \underset{g}{*} \partial Q_r(a)$ are 3-cells whose union forms a 3-sphere Σ^3. We claim that Σ^3 is contained in $\operatorname{int} B_\varepsilon(y)$, as Σ^3 is contained in $B_r(x) \cup B_r(a)$. This can be obtained as follows. By our choice of r and x, we have $B_r(x) \subset \operatorname{int} B_\varepsilon(y)$. Moreover for any $w \in B_r(a)$, we have

$$\begin{aligned} d(w,y) &\leq d(w,a) + d(a,y) \\ &\leq r + d(a,x) \\ &\leq r + r \\ &< 2r \\ &< \varepsilon \end{aligned}$$

and therefore $B_r(a) \subset \operatorname{int} B_\varepsilon(y)$, yielding $\Sigma^3 \subset \operatorname{int} B_\varepsilon(y)$. Via duality and contractibility of $\operatorname{int} B_\varepsilon(y)$, we have $\tilde{H}_0\left(\operatorname{int} B_\varepsilon(y) - \Sigma^3; \mathcal{Z}\right) \cong H^3\left(\Sigma^3; \mathcal{Z}\right) \cong \mathcal{Z}$. Hence Σ^3 separates $\operatorname{int} B_\varepsilon(y)$ into two components having Σ^3 as the common boundary (see Raymond [81, Thm. 2, page 12]).

By our choice or r, neither x nor a is an element of $\partial Q_r(a)$. An application of uniqueness of joins reveals that $\text{int}[x,a] \not\subset \Sigma^3$. In particular $y \notin \Sigma^3$ and the desired open neighborhood V_y is the component of $\text{int}\, B_\varepsilon(y) - \Sigma^3$ containing y. $\square$

It was shown in Corollary 2.14 that all G-spaces of dimension ≤ 3 are locally spherical. However, in this low dimensional setting, these local spheres are both tamely embedded and compatible with the local cone structure in the sense that coning over these spheres yields embedded cells. Neither of these features is readily apparent in dimension 4 and the next chapter is devoted to constructing local spheres which, while not completely tame, are sufficiently well-behaved for our purpose.

CHAPTER 4

Locally Spherical Spaces

In the previous chapter, we saw every $4D$ G-space X is in fact resolvable and locally spherical in the sense that points have arbitrarily small neighborhoods whose boundaries are topological 3-spheres. It is precisely this local spherical structure which will allow us to establish that every cell-like surjection $f : M^4 \longrightarrow X^4$ of a manifold onto the $4D$ G-space is **ABH** and thus establish Theorem 1.1.

Let $f : M^n \longrightarrow Y$ be a cell-like surjection of an n-manifold onto a locally spherical space Y. In the absence of tameness conditions on the $(n-1)$-spheres in Y, deciding the **ABH** question for f is difficult, and indeed f need not always have this property. Armentrout [7] has constructed a non-**ABH** example when $n = 3$ and the general question remains unsolved for $n \geq 4$ as of this writing. The goal of this chapter is to establish a certain tameness criterion on the local 3-spheres in a $4D$ G-space, called *semi-niceness*.

1 Singular Regular Neighborhoods: The Manifold Case

Let $\Sigma^{n-1} \subset \mathbf{R}^n$ be an embedded codimension 1 sphere. We say that Σ^{n-1} is *flatly embedded* (or simply *flat*) provided there is a homeomorphism $h : \mathbf{R}^n \longrightarrow \mathbf{R}^n$ sending Σ^{n-1} to the standard sphere $\mathbf{S}^{n-1}$. Via Morton Brown's solution to the generalized Schoenflies conjecture [21], Σ^{n-1} is flat precisely when it is *bi-collared*, i.e., when it has

a neighborhood W parameterized by a homeomorphism

$$\phi : \mathbf{S}^{n-1} \times (-1,1) \longrightarrow W$$

such that $\phi(\mathbf{S}^{n-1} \times \{0\}) = \Sigma^{n-1}$.

More generally, a codimension 1 submanifold $N^{n-1} \subset M^n$ is said to be *locally flat* in M^n provided each point $x \in N^{n-1}$ is contained in a sufficiently small, bi-collared cell D^{n-1}. One of the principal tools for detecting local flatness is the following theorem due to Bing [14] for $n = 3$, Freedman and Quinn [49, Thm. 9.3A] for $n = 4$, and independently to Daverman [40] and Černavskiĭ [33] for $n \geq 5$.

Theorem 4.1 *Let $N^{n-1} \subset M^n$. Then N^{n-1} is locally flat in M^{n-1} if and only if N^{n-1} is 1-LCC embedded in M^n.*

A subspace $X \subset Y$ is said to be *1-LCC* at the point $x \in X$ provided for every open neighborhood $U_x \subset Y$ there is a subneighborhood $V_x \subset U_x$ so that every map of the form $\phi : \mathbf{S}^1 \longrightarrow V_x - X$ can be extended to a map $\Phi : \mathbf{D}^2 \longrightarrow U_x - X$. X is 1-LCC embedded if it is 1-LCC at each of its points. It is an elementary exercise that each flatly embedded sphere $\Sigma^{n-1} \subset \mathbf{R}^n$, $n \geq 3$, is 1-LCC embedded in $\mathbf{R}^n$.

Detecting the 1-LCC condition is a difficult task in general, but strategies exist in the presence of certain types of deformations. In particular, the notion of a *singular regular neighborhood* due to Hempel [52], and later studied by Cannon [32] and Daverman [36] has shown to be particularly useful in detecting 1-LCC-ness in the case that $N^{n-1} \subset M^n$ is a codimension 1 submanifold.

Definition 4.1 *Let $X \subset Y$ be a subspace separating Y into two components U and V.*

a. *X has a singular regular neighborhood in U provided there is a homotopy*

$$H : X \times I \longrightarrow \overline{U}$$

such that $H_0 = 1_X$ and $H_t(x) \in U$ for all $t > 0$ and for all $x \in X$.

b. *Given $z \in X$, X is said to be locally singularly collared at z from U provided there is a neighborhood $W_z \subset X$ and a homotopy $H : W_z \times I \longrightarrow \overline{U}$ such that $H_0 = 1_{W_z}$ and $H_t(x) \in U$ for all $t > 0$ and for all $x \in W_z$.*

c. *Given $z \in X$, X is said to be locally singularly bi-collared at z provided it is locally singularly collared at z from both U and V.*

Theorem 4.2 (Daverman [36]) *Let $N^{n-1} \subset M^n$ be a submanifold separating the manifold M^n into two components U and V.*

1. *If $z \in N^{n-1}$ and N^{n-1} is locally singularly collared at z from U, then N^{n-1} is 1-LCC in $\overline{U}$ at z.*
2. *If $z \in N^{n-1}$ and N^{n-1} is locally singularly bi-collared at z, then N^{n-1} is 1-LCC in M^n at z.*
3. *If N^{n-1} has a singular regular neighborhood in both U and V, then N^{n-1} is 1-LCC embedded in M^n.*

Our aim is to generalize Daverman's Theorem so that it applies to the local spheres in a $4D$ G-space. Using the notation of Theorem 2.12, recall the contraction

$$r \circ H : Q_\delta(a) \times I \longrightarrow Q_\delta(a).$$

As an application of the second retraction inequality, note that the restriction of this map $r \circ H| : \partial Q_\delta(a) \times I \longrightarrow Q_\delta(a)$ gives rise to a singular regular neighborhood of $\partial Q_\delta(a)$ in $Q_\delta(a)$. We would like to conclude from Daverman's Theorem that the 2-sphere $\partial Q_\delta(a)$ is 1-LCC embedded in $Q_\delta(a)$. However Daverman's Theorem does not immediately apply as $S_{\varepsilon_2}(x)$, although separated by $\partial Q(a)$, is not necessarily a topological manifold. All is not lost, for $S_{\varepsilon_2}(x)$ is a homology manifold and it retains enough of the homological properties of a manifold to allow Daverman's Theorem to be extended to this setting.

2 Algebraic Preliminaries

Let $\check{H}_c^*(Y;\mathcal{Z})$ denote Čech cohomology with compact supports and integral coefficients for the space Y. For the definition of this group see Spanier [88, page 327 or 358]. An important fact to note is that $\check{H}_c^*(\cdot;\mathcal{Z})$ is a contravariant functor from the category of topological spaces and proper maps to the category of Abelian groups and homomorphisms. Moreover $\check{H}_c^*(\cdot;\mathcal{Z})$ is a proper homotopy functor in the sense that if $G : Y \times I \longrightarrow X$ is a proper map then G_0 and G_1 induce the same homomorphism between $\check{H}_c^*(X;\mathcal{Z})$ and $\check{H}_c^*(Y;\mathcal{Z})$.

Definition 4.2 (orientable) *Let Y be an admissible homology n-manifold with $\partial Y = \emptyset$ (as defined on page 31). Y is said to orientable provided if $U \subset Y$ is any connected open subset with $\overline{U}$ compact, then inclusion $j : U \longrightarrow Y$ induces an isomorphism $j^* : \check{H}_c^n(U;\mathcal{Z}) \longrightarrow \check{H}_c^n(Y;\mathcal{Z})$. Here j^* is the homomorphism in the exact sequence of the*

pair $(\dot{Y}, U)$

$$\cdots \to \check{H}_c^{n-1}(Y-U;\mathcal{Z}) \longrightarrow \check{H}_c^n(U;\mathcal{Z}) \xrightarrow{j^*} \check{H}_c^n(Y;\mathcal{Z}) \to \cdots$$

Proposition 4.3 *Let Y be an admissible homology n-manifold, $\partial Y = \emptyset$.*

1. *Y is orientable if and only if $\check{H}_c^n(Y;\mathcal{Z}) \cong \mathcal{Z}$.*
2. *Y orientable implies every non-empty connected open subset $U \subset Y$ is orientable.*
3. *Y orientable implies $\check{H}_c^n(A;\mathcal{Z}) \cong 0$ for every proper closed subset $A \underset{\neq}{\subset} Y$.*
4. *Y orientable implies every covering $\tilde{Y}$ of Y has orientable components.*

Proof. See Wilder [92], Raymond [81], or Dold [44, Section VIII.2]. □

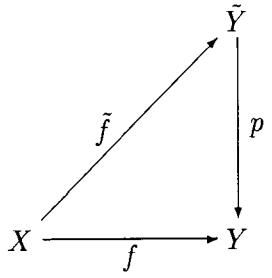
Let $[X]$ and $[Y]$ denote generators of $\check{H}_c^n(X;\mathcal{Z})$ and $\check{H}_c^n(Y;\mathcal{Z})$ respectively where X and Y are orientable admissible homology n-manifolds. The *degree* of a proper map $f : X \longrightarrow Y$ is the integer $\deg f$ such that $f^*([Y]) = \deg f\ [X]$. This integer is unique up to sign.

The following result, which is due to Olum (see [75], [46]) in the manifold setting, is a key ingredient in Daverman's Theorem and its generalization.

Theorem 4.4 *Let $f : X \longrightarrow Y$ be a proper map between orientable admissible homology n-manifolds such that* $\deg f = \pm 1$. *Then f induces an epimorphism of fundamental groups*

$$f_\# : \pi_1 X \longrightarrow \pi_1 Y$$

Consider the following commutative diagram



(4.1)

where $p : \tilde{Y} \longrightarrow Y$ is the covering map associated to the subgroup $f_{\#}(\pi_1 X) \subset \pi_1 Y$. Note that f proper implies $\tilde{f}$ is proper. Let $j = [\pi_1 Y : f_{\#}(\pi_1 X)]$ be the index of $f_{\#}(\pi_1 X)$ in $\pi_1 Y$. It is well known (see [72, page 342]) that p is a j-fold covering.

Lemma 4.5 *$j = \infty$ implies* $\deg f = 0$.

Proof. Let $y \in Y$ and set $K = f^{-1}(y)$. Since f is proper, K is a compact subset of X and via commutativity of (4.1), $\tilde{f}(K)$ is a compact subset of $p^{-1}(y)$. Therefore $\tilde{f}(K)$ is a finite collection of points $\{\tilde{y}_1, \ldots, \tilde{y}_k\}$. Consider the pairs $(X, X - K)$ and $(Y, Y - \{y\})$.

Let $\mathcal{C}_Y$ denote the collection of all compact subsets Y . Any compact subset of Y is contained in a compact, connected subset, for we may first cover such a set by a finite number of connected open sets having compact closure, then join the elements of this finite cover by arcs. This says that the collection of continuum of Y, $\mathcal{K}_Y$, is *cofinal* in $\mathcal{C}_Y$ in the sense of Dold [44, Def.VIII.5.16]. This implies that

$$\check{H}^n_c(Y) = \varinjlim \left\{ \check{H}^n(Y, Y - C) \,|C \in \mathcal{C}_Y \right\} = \varinjlim \left\{ \check{H}^n(Y, Y - C') \,|C' \in \mathcal{K}_Y \right\}.$$

Apply universal coefficients to see that

$$\check{H}^n(Y, Y - C') \cong H^n(Y, Y - C') \cong \mathrm{Hom}\,(H_n(Y, Y - C'), \mathcal{Z}) \oplus \mathrm{Ext}\,(H_{n-1}(Y, Y - C'), \mathcal{Z}).$$

Orientability implies that $\mathrm{Ext}\,(H_{n-1}(Y, Y-C'), \mathcal{Z}) \cong 0$, $H_n(Y, Y-C') \cong \mathcal{Z}$. (See [44, Section VIII.3].) These isomorphisms are compatible with inclusion and hence we obtain an isomorphism

$$\alpha_Y : \check{H}^n(Y, Y-\{y\}) \longrightarrow \check{H}^n_c(Y).$$

as $\{y\} \in \mathcal{K}_Y$. Naturality yields a map α_X and a commutative diagram

$$\begin{array}{ccc} \check{H}^n_c(X) & \xleftarrow{\alpha_X} & \check{H}^n(X, X-K) \\ \uparrow{\scriptstyle f^*} & & \uparrow{\scriptstyle f^*} \\ \check{H}^n_c(Y) & \xleftarrow[\cong]{} & \check{H}^n(Y, Y-\{y\})) \end{array} \tag{4.2}$$

Using diagrams (4.2) and (4.1), we obtain the following commutative diagram

$$\begin{array}{ccccc} \check{H}^n_c(X) & \xleftarrow{\alpha_X} & \check{H}^n(X, X-K) & & \\ \uparrow{\scriptstyle f^*} & & \uparrow{\scriptstyle f^*} & \nwarrow{\scriptstyle \tilde{f}^*} & \\ \check{H}^n_c(Y) & \xleftarrow[\cong]{} & \check{H}^n(Y, Y-\{y\}) & \xrightarrow[p^*]{} & \check{H}^n\left(\tilde{Y}, \tilde{Y}-p^{-1}(y)\right) \end{array} \tag{4.3}$$

To see that $\deg f = 0$, it suffices to show that $\alpha_X \circ \tilde{f}^* : \check{H}^n\left(\tilde{Y}, \tilde{Y}-p^{-1}(y)\right) \longrightarrow \check{H}^n_c(X)$ is the zero homomorphism. Use excision to produce the isomorphism

$$\begin{gathered} \check{H}^n\left(\tilde{Y}, \tilde{Y}-p^{-1}(y)\right) \cong \\ \bigoplus_{i=1}^{k} \check{H}^n(\tilde{Y}, \tilde{Y}-\{\tilde{y}_i\}) \oplus \check{H}^n\left(\tilde{Y}, \left(\tilde{Y}-p^{-1}(y)\right) \cup \{\tilde{y}_1, \ldots, \tilde{y}_k\}\right), \end{gathered} \tag{4.4}$$

allowing us to identify $\check{H}^n\left(\tilde{Y}, \tilde{Y} - p^{-1}(y)\right)$ with the right hand side of (4.4). Consider the restriction $\tilde{f}^*| : \check{H}^n\left(\tilde{Y}, \tilde{Y} - p^{-1}(y) \bigcup \{\tilde{y}_1, \ldots, \tilde{y}_k\}\right) \longrightarrow \check{H}^n(X, X - K)$. There is a commutative triangle

$$\begin{array}{ccc} \check{H}^n\left(\tilde{Y}, \tilde{Y} - p^{-1}(y) \bigcup \{\tilde{y}_1, \ldots, \tilde{y}_k\}\right) & \xrightarrow{\tilde{f}^*|} & \check{H}^n(X, X - K) \\ & {\scriptstyle \tilde{f}^*} \searrow \quad \nearrow {\scriptstyle inc.^*} & \\ & \check{H}^n(X, X) = 0 & \end{array}$$

which implies that this restriction of $\tilde{f}^*$ is zero.

Next fix $i \in \{1, \ldots, k\}$ and consider the composition

$$\check{H}^n\left(\tilde{Y}, \tilde{Y} - \{\tilde{y}_i\}\right) \xrightarrow{\tilde{f}^*} \check{H}^n(X, X - K) \xrightarrow{\alpha_X} \check{H}^n_c(X).$$

To see that this map is zero, let $z \in p^{-1}(y) - \{\tilde{y}_1, \ldots, \tilde{y}_k\}$ and consider the following diagram

$$\begin{array}{ccccc} \check{H}^n(X, X - K) & \xrightarrow{\alpha_X} & \check{H}^n_c(X) & \xleftarrow{\beta_X} & \check{H}^n(X, X) = 0 \\ \uparrow {\scriptstyle \tilde{f}^*} & & \uparrow {\scriptstyle \tilde{f}^*} & & \uparrow {\scriptstyle \tilde{f}^*} \\ \check{H}^n\left(\tilde{Y}, \tilde{Y} - \{\tilde{y}_i\}\right) & \xrightarrow[\cong]{} & \check{H}^n_c(\tilde{Y}) & \xleftarrow[\cong]{} & \check{H}^n\left(\tilde{Y}, \tilde{Y} - \{z\}\right) \end{array}$$

This establishes that f^* is the zero homomorphism and thus $\deg f = 0$, as desired. □

Now under the hypothesis of Theorem 4.4, $\deg f \neq 0$. Hence, by the Lemma 4.5 above, j is finite and therefore p is a proper map. The result will immediately follow

from functorial properties and the following lemma which is stated under the hypothesis of Theorem 4.4.

Lemma 4.6 *Let $p : \tilde{Y} \longrightarrow Y$ be a $j-$fold cover where j is finite. Then* $\deg p = j$.

Proof. Since Y is orientable, we may orient Y and $\tilde{Y}$ consistently by insisting that p be orientation preserving over some evenly covered, connected, open subset $U \subset Y$ where $\overline{U}$ is compact. Specifically, $p^{-1}(U) = \bigsqcup_{i=1}^{j} \tilde{U}_i$ where the restriction $p| : \tilde{U}_i \longrightarrow U$ is a homeomorphism and we choose the orientation on $\tilde{Y}$ so that p is orientation preserving on each $\tilde{U}_i$. We compute the degree of p by considering (as in [44, page 268]) the induced map of pairs

$$p_* : H_n\left(\tilde{Y}, \tilde{Y} - p^{-1}(\overline{U})\right) \longrightarrow H_n\left(Y, Y - \overline{U}\right).$$

Using excision and the compatible orientations, we have that p_* is equivalent to the homomorphism $\bigoplus_{i=1}^{j} \mathcal{Z} \longrightarrow \mathcal{Z}$ which maps a $j-$tuple $(n_1, \ldots, n_j) \mapsto \sum_{i=1}^{j} n_i$. Therefore $(1, \ldots, 1) \mapsto j$, and hence $\deg p = j$, as desired. □

Proof of Theorem 4.4. Functorial properties and commutativity of (4.1) imply that

$$\deg f = \pm 1 = \deg p \ \deg \tilde{f} = j \deg \tilde{f}$$

and hence $j = 1$, which implies that $f_{\#} : \pi_1 X \longrightarrow \pi_1 Y$ is surjective. □

3 Singular Regular Neighborhoods: A General Case

The goal of this section is to establish the following generalization of Daverman's Theorem.

Theorem 4.7 *Let Σ be a compact, orientable admissible homology n-manifold. Assume that Σ is a homology sphere, i.e. $H_*(\Sigma) \cong H_*(\mathbf{S}^n)$. Let $N^{n-1} \subset \Sigma$ be a compact topological $(n-1)$-manifold which separates Σ into two components U and V. If N is locally singularly collared in U, then N is 1-LCC imbedded in $\overline{U}$.*

The proof of Theorem 4.7 closely parallels Daverman's argument in [36]. As in [36], we begin with the following lemma.

Lemma 4.8 *Let D^{n-1} be an $(n-1)$-cell in N^{n-1} and $f : D \times I \longrightarrow \overline{U}$ is a map such that $f(d,0) = d$ and $f(d,t) \in U$ for all $d \in D$ and $t \in I$. Then there is a connected open subset $W \subset \overline{U}$ satisfying*

$$\operatorname{int} D \subset W \subset f(D \times I) \tag{4.5}$$

Proof. Let $\lambda : N^{n-1} \longrightarrow [0,1]$ be a Urysohn function such that $\lambda^{-1}(0) = N - \operatorname{int} D$. Define a homotopy $G : N \times I \longrightarrow N \cup f(D \times I)$ by the rule

$$G(x,t) = \begin{cases} f(x, \lambda(x)t) & , \ x \in D \\ x & , \ x \notin D \end{cases}$$

and notice that G satisfies three conditions:

Condition 1. $G_0(x) = x$, for all $x \in N$

Condition 2. $G_t(x) = x$, for all $x \in N - \operatorname{int} D$

Condition 3. $G_t(x) \cap N = \emptyset$, for all $t > 0$ and all $x \in \operatorname{int} D$.

Our first aim is to show that $G_1(N)$ separates Σ. To this end, note that via conditions 1-3 above it is possible to write $G_1(N) = (N - \operatorname{int} D) \cup G_1(D)$ where $(N - \operatorname{int} D) \cap$

$G_1(D) = S^{n-2} = \partial D$. Consider the inclusion induced homomorphisms of homology

$$i_* : H_{n-2}(S^{n-2}) \longrightarrow H_{n-2}(N - \operatorname{int} D)$$

$$j_* : H_{n-2}(S^{n-2}) \longrightarrow H_{n-2}(G_1(D)).$$

In each case a generating cycle of $H_{n-2}(S^{n-2})$ is sent to a bounding cycle and hence i_*, and j_* are the zero homomorphisms. This implies that the induced maps of Čech cohomology i^* and j^* are trivial as well. Consider the Mayer-Vietoris sequence

$$\cdots \rightarrow \check{H}^{n-2}(N - \operatorname{int} D) \oplus \check{H}^{n-2}(G_1(D)) \stackrel{i^*+j^*}{\longrightarrow} \check{H}^{n-2}(S^{n-2}) \stackrel{\psi}{\rightarrow} \check{H}^{n-1}(G_1(N)) \rightarrow \cdots \tag{4.6}$$

Since $i^* + j^*$ is trivial and the sequence is exact, $\check{H}^{n-1}(S^{n-2}) \stackrel{\psi}{\rightarrow} \check{H}^{n-2}(G_1(N))$ is a monomorphism. In particular we have that $\check{H}^{n-1}(G_1(N)) \neq 0$.

Since Σ is a homology sphere, it is orientable and hence by generalized Alexander duality we have that

$$H_1(\Sigma, \Sigma - G_1(N)) \cong \check{H}^{n-1}(G_1(N)) \neq 0.$$

Upon considering the long exact sequence of the pair $(\Sigma, \Sigma - G_1(N))$, we have

$$0 = H_1(\Sigma) \rightarrow H_1(\Sigma, \Sigma - G_1(N)) \stackrel{\cong}{\longrightarrow} \tilde{H}_0(\Sigma - G_1(N)) \rightarrow \tilde{H}_0(\Sigma) = 0$$

and thus $G_1(N)$ separates Σ.

Letting V denote the component of $\Sigma - G_1(N)$ containing $\operatorname{int} D$, we set $W = \overline{U} \cap V$. To see that W is the desired open subset of $\overline{U}$, suppose to the contrary that there is point

$$w \in W - f(D \times I) \subset W - G(N \times I).$$

Let $\mathbf{S}^{n-1}$ be the standard $(n-1)$-sphere in $\mathbf{R}^n$, and express $\mathbf{S}^{n-1}$ as the union of its upper and lower hemispheres E_+ and E_-. Let $\theta : E_+ \longrightarrow E_-$ be a homeomorphism which is the identity on the common boundary $\mathbf{S}^{n-2}$. Name a homeomorphism $h : E_+ \longrightarrow D$. We define a map $\phi : \mathbf{S}^{n-1} \longrightarrow \Sigma$ via the rule

$$\phi(x) = \begin{cases} h(x) & \text{, for } x \in E_+ \\ G_1 \circ h \circ \theta(x) & \text{, for } x \in E_- \end{cases}$$

and, via a Mayer-Vietoris analysis as above, $\check{H}^{n-1}(\operatorname{im}\phi) \neq 0$ which implies that $\operatorname{im}\phi$ separates Σ. Enumerate the components of $\Sigma - \operatorname{im}\phi$, $\Lambda_1, \Lambda_2, \cdots$ and assume the labeling is such that $w \in \Lambda_1$. Choose a connected open neighborhood Ω_w such that $\Omega_w \subset \overline{\Omega_w} \subset \Lambda_1 - f(D \times I)$. There are two cases to consider.

Case 1. $\Sigma - f(D \times I) \subset \Lambda_1$.

We claim there is a $j \geq 2$ so that $W^* = \overline{\Lambda}_j - G_1(D)$ satisfies (4.5) above. To this end, note that $\Sigma - f(D \times I) \subset \Lambda_1$ implies that

$$\Lambda_2 \cup \Lambda_3 \cup \cdots \subset \Sigma - \Lambda_1 \subset f(D \times I)$$

and hence the union $\Lambda_2 \cup \Lambda_3 \cup \cdots$ is an open set contained in $f(D \times I)$. Let $x \in \operatorname{int} D$ and let F^{n-1} be a subdisc of $\operatorname{int} D$ with $x \in \operatorname{int} F$. Let $\mathcal{O}_x$ be a connected open subset so that $\mathcal{O}_x \subset \Sigma - G_1(D)$ and $\operatorname{int} F$ separates $\mathcal{O}_x$ into two components A_1 and A_2. Note that A_1 and A_2 are subsets of $\Sigma - \operatorname{im}\phi$.

We aim to show that A_1 and A_2 lie in different components of $\Sigma - \operatorname{im}\phi$. Consider

the following commutative diagram

$$\begin{array}{ccc} H_0(\mathcal{O}_x - \operatorname{int} F) & \xrightarrow{\cong} & \check{H}_c^n(\mathcal{O}_x - \operatorname{int} F) \\ \Big\downarrow \kappa & & \Big\downarrow \beta \\ H_0(\Sigma - \operatorname{im}\phi) & \xrightarrow[\cong]{} & \check{H}_c^n(\Sigma - \operatorname{im}\phi) \end{array} \tag{4.7}$$

where κ and β are induced by inclusion and the horizontal maps are duality isomorphisms. It suffices to show that κ is a monomorphism. (Note: Here we are using the natural identification $\check{H}_c^*(A - B) \cong \check{H}_c^*(A, B)$ where B is a closed subset of A and A is a locally compact Hausdorff space. See [88, Thm. 6.7.15, page 322].) Via the commutativity of (4.7), we need only show that β is injective. Consider the exact sequence of the pair $(\operatorname{im}\phi, \operatorname{int} F)$

$$\cdots \to \check{H}_c^{n-1}(\operatorname{int} F) \xrightarrow{\alpha} \check{H}_c^{n-1}(\operatorname{im}\phi) \xrightarrow{\omega} \check{H}_c^{n-1}(\operatorname{im}\phi - \operatorname{int} F) \to 0 \tag{4.8}$$

To see that α is a monomorphism, consider the induced exact sequence

$$0 \to \ker\alpha \longrightarrow \check{H}_c^{n-1}(\operatorname{int} F) \longrightarrow \operatorname{im}\alpha \to 0 \tag{4.9}$$

Now $\operatorname{im}\alpha \subset \check{H}_c^{n-1}(\operatorname{im}\phi) \cong \check{H}_0(\Sigma - \operatorname{im}\phi)$ which implies $\operatorname{im}\alpha$ is free abelian. Hence (4.9) is split exact, i.e., $\mathcal{Z} \cong \check{H}_c^{n-1}(\operatorname{int} F) \cong \ker\alpha \oplus \operatorname{im}\alpha$, and therefore one of $\ker\alpha$, $\operatorname{im}\alpha$ must be trivial. Suppose that $\operatorname{im}\alpha = 0$. Then (4.8) becomes

$$0 \to \check{H}_c^{n-1}(\operatorname{im}\phi) \xrightarrow{\omega} \check{H}_c^{n-1}(\operatorname{im}\phi - \operatorname{int} F) \to 0 \tag{4.10}$$

revealing ω to be an isomorphism. Arguing as in (4.6), we have that Mayer-Vietoris sequence has the form

$$0 \to \check{H}^{n-2}\left(\mathbf{S}^{n-2}\right) \to \check{H}^{n-1}(\operatorname{im}\phi) \to \check{H}^{n-1}(F) \oplus \check{H}^{n-1}(G_1(D) \cup (D - \operatorname{int} F)) \to 0 \tag{4.11}$$

which is split exact, since

$$\check{H}^{n-1}(G_1(D) \cup (D - \operatorname{int} F)) \cong \check{H}^{n-1}(G_1(D)) \cong \check{H}_c^{n-1}(\operatorname{im}\phi - \operatorname{int} F)$$

and $\check{H}_c^{n-1}(\operatorname{im}\phi - \operatorname{int} F)$ is free abelian via (4.10) above. Via the split exactness of (4.11), we have $\check{H}^{n-1}(\operatorname{im}\phi) \cong \check{H}^{n-2}(\mathbf{S}^{n-2}) \oplus \check{H}_c^{n-1}(\operatorname{im}\phi - \operatorname{int} F) \ncong \check{H}_c^{n-1}(\operatorname{im}\phi - \operatorname{int} F)$ which contradicts the isomorphism of (4.10). Thus $\operatorname{im}\alpha \neq 0$ which implies $\ker\alpha = 0$, and hence α is a monomorphism.

Via Definition 4.2, the inclusion induced map $\gamma : \check{H}_c^n(\mathcal{O}_x) \to \check{H}_c^n(\Sigma)$ is an isomorphism. Hence we have the following commutative ladder with exact rows

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \check{H}_c^{n-1}(\operatorname{int} F) & \xrightarrow{\xi} & \check{H}_c^n(\mathcal{O}_x - \operatorname{int} F) & \longrightarrow & \check{H}_c^n(\mathcal{O}_x) & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle \alpha} & & \downarrow{\scriptstyle \beta} & & \downarrow{\scriptstyle \gamma} & & \\
0 & \longrightarrow & \check{H}^{n-1}(\operatorname{im}\phi) & \longrightarrow & \check{H}_c^n(\Sigma - \operatorname{im}\phi) & \longrightarrow & \check{H}_c^n(\Sigma) & \longrightarrow & 0
\end{array} \tag{4.12}$$

(Note: The bottom row is short exact since Σ is a homology sphere. The short exactness of the top row can be ascertained via considering $\check{H}_c^{n-1}(\operatorname{int} F) \cong \ker\xi \oplus \operatorname{im}\xi$.) As α and γ are monomorphisms, the Short Five-Lemma ([56, Lemma 4.1.7, page 176]) yields β to

be a monomorphism, and hence κ is a monomorphism. Thus A_1 and A_2 lie in different components of $\Sigma - \operatorname{im}\phi$, as claimed.

Let $j \geq 2$ be such that one of the A_i lies in Λ_j. Set $W^* = \overline{\Lambda}_j - G_1(D)$. W^* is an open subset of $\overline{U}$ and

$$\operatorname{int} D \subset W^* \subset f(D \times I) \subset \overline{U}$$

as desired.

Case 2. $\Sigma - f(D \times I) \not\subset \Lambda_1$.

There is a point $z \in \Sigma - f(D \times I)$ so that w, z are in different components of $\Sigma - \operatorname{im}\phi$, say Λ_1 and Λ_2. Select a connected open neighborhood Ω_z so that $\Omega_z \subset \overline{\Omega_z} \subset \Lambda_2 - f(D \times I)$ and let $\ddot{\Sigma} = \Sigma - \{ \Omega_w \bigsqcup \Omega_z \}$. We claim that inclusion induces an isomorphism onto a direct summand

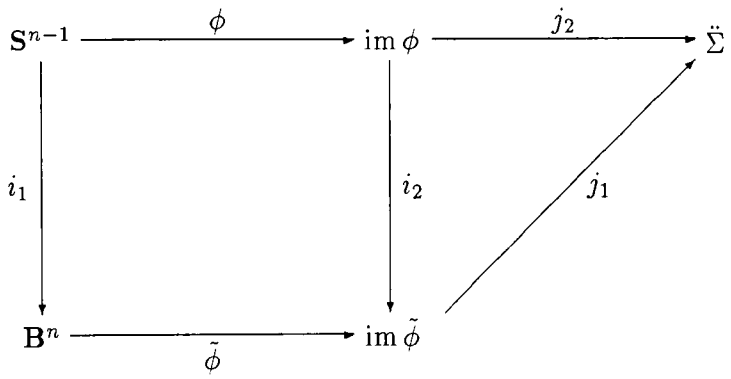
$$j_2^* : H^{n-1}(\ddot{\Sigma}) \cong \mathcal{Z} \longrightarrow H^{n-1}(\operatorname{im}\phi) \cong \oplus\mathcal{Z}.$$

This is obtained by considering the diagram

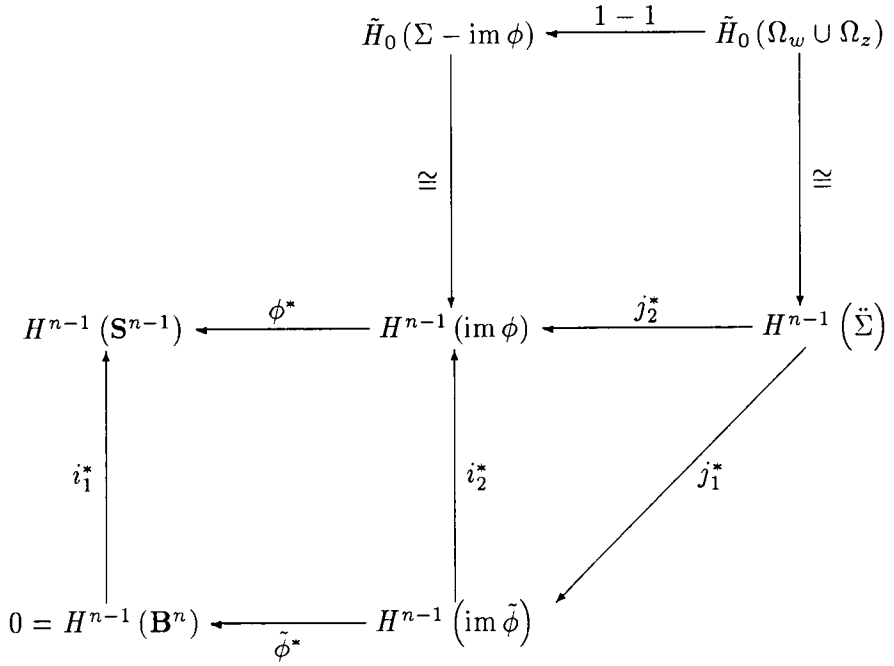
$$\begin{array}{ccc} \tilde{H}_0(\Omega_w \cup \Omega_z) & \xrightarrow{1-1} & \tilde{H}_0(\Sigma - \operatorname{im}\phi) \\ \cong \downarrow & & \downarrow \cong \\ H^{n-1}\left(\ddot{\Sigma}\right) & \xrightarrow[j_2^*]{} & H^{n-1}(\operatorname{im}\phi) \end{array}$$

where the vertical maps are duality isomorphisms and the horizontal maps are inclusion induced.

Next consider the following commutative diagram



where the map $\tilde{\phi} : \mathbf{B}^n \to \ddot{\Sigma}$ is the natural extension of ϕ obtained from the homotopy G and i_s, j_s are inclusion. This induces the following commutative diagram on cohomology



(4.13)

which yields that the composition $\phi^* \circ j_2^*$ is the zero homomorphism.

Next we show that ϕ^* is surjective. Consider the commutative ladder induced by

the Mayer-Vietoris sequences

$$\begin{array}{ccccccccc}
\cdots \xrightarrow{0} & H^{n-2}(\mathbf{S}^{n-2}) & \xrightarrow{1-1} & H^{n-1}(\operatorname{im}\phi) & \longrightarrow & H^{n-1}(D)\oplus H^{n-1}(G_1(D)) & \longrightarrow 0 \\
& \downarrow \cong & & \downarrow \phi^* & & \downarrow \phi^*\oplus\phi^* & \\
\cdots \xrightarrow[0]{} & H^{n-2}(\mathbf{S}^{n-2}) & \xrightarrow[\cong]{} & H^{n-1}(\mathbf{S}^{n-1}) & \longrightarrow & H^{n-1}(E_+)\oplus H^{n-1}(E_-) & \longrightarrow 0
\end{array}$$

An easy diagram chase shows that ϕ^* is surjective.

Now suppose that $\operatorname{im}\phi$ separates Σ into precisely two components. Then $H^{n-1}(\operatorname{im}\phi) \cong \mathcal{Z}$ and thus $\phi^* \circ j_2^*$ is an isomorphism, which contradicts $\phi^* \circ j_2^* = 0$. For the general case, note that $\operatorname{im}\phi$ union any number of the components of $\Sigma - \operatorname{im}\phi$ is a compact subset of Σ (for its complement is open in Σ). Let $\Phi = \operatorname{im}\phi \bigcup_{j\geq 3} \Lambda_j$ and observe that diagram (4.13) becomes

$$\begin{array}{ccccc}
 & & \tilde{H}_0(\Sigma-\Phi) & \xleftarrow{\cong} & \tilde{H}_0(\Omega_w\cup\Omega_z) \\
 & & \downarrow\cong & & \downarrow\cong \\
H^{n-1}(\mathbf{S}^{n-1}) & \xleftarrow{\cong^*} & H^{n-1}(\Phi) & \xleftarrow{\cong} & H^{n-1}\left(\ddot{\Sigma}\right) \\
\uparrow i_1^* & & \uparrow i_2^* & \swarrow j_1^* & \\
0 = H^{n-1}(\mathbf{B}^n) & \xleftarrow[\tilde{\phi}^*]{} & H^{n-1}\left(\operatorname{im}\tilde{\phi}\cup\Phi\right) & &
\end{array} \tag{4.14}$$

which yields the obvious contradiction. □

Proof of Theorem 4.7. Fix a point $p \in N$ and $\varepsilon > 0$ such that the $\varepsilon-$neighborhood about $p, B_\varepsilon(p)$, is contained in a given open neighborhood about p, U_1. Our aim is to find a connected open neighborhood V_1 of p in $\overline{U}$ such that inclusion induces the trivial homomorphism

$$\pi_1(V_1 - N) \xrightarrow{inc_\#} \pi_1(U_1 - N).$$

By hypothesis on N, there is disc $D^{n-1} \subset N$ containing p in its interior and a homotopy

$$f : D^{n-1} \times I \longrightarrow B_\varepsilon(p) \text{ where}$$

$$f_0 = \text{Identity}$$

$$f(D^{n-1} \times (0,1]) \subset U$$

where U is the prescribed component of $\Sigma - N$.

Apply Lemma 4.8 above to find a connected open set $W \subset \overline{U}$ satisfying

$$\operatorname{int} D \subset W \subset f(D \times I) \subset B_\varepsilon(p).$$

Let Y be the component of $f^{-1}(W)$ containing $\operatorname{int} D \times \{0\}$ and consider the restriction $f|_Y$. This restriction is a proper map, for if $K \subset f(Y) \subset f(D \times I)$ is compact set, then necessarily $f^{-1}(K)$ is a closed subset of $D \times I$, and hence a compact subset of $D \times I$. Since Y is an open subset of $D \times I$ containing $f^{-1}(K)$, this restricted map is proper.

Let $W_U = W \cap U$, $Y_U = Y \cap f^{-1}(U)$ and $f_U = f|_{Y_U}$. We aim to show that

$$f_U : Y_U \longrightarrow W_U$$

has degree ± 1. We have the following commutative diagram {see Massey [63, Thm. 2,

page 20]}

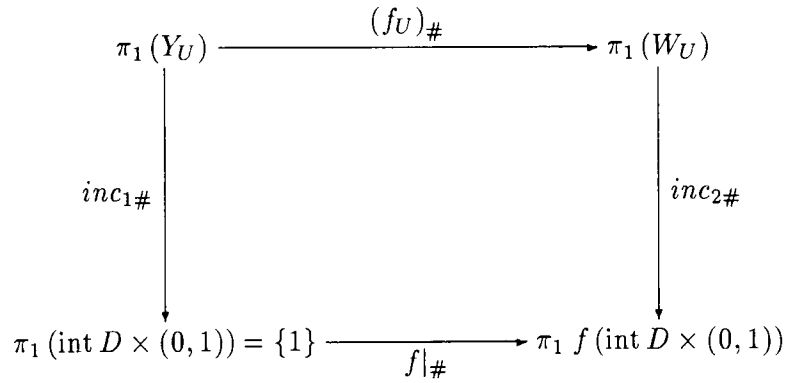
$$\begin{array}{ccccc}
\check{H}_c^{n-1}(\text{int}\, D \times \{0\}) & \xrightarrow{d^*} & \check{H}_c^n(Y_U) & \longrightarrow & \check{H}_c^n(Y) \cong 0 \\
\uparrow{\scriptstyle f^*| = 1^*} & & \uparrow{\scriptstyle f_U^*} & & \uparrow \\
\check{H}_c^{n-1}(\text{int}\, D) & \xrightarrow[d^*]{} & \check{H}_c^n(W_U) & \longrightarrow & \check{H}_c^n(W) \cong 0
\end{array}$$

The fact that $\check{H}_c^n(Y) \cong 0 \cong \check{H}_c^n(W)$ can been seen as follows. Using Raymond [81] we may double Y and W along their respective boundaries to obtain oriented homology manifolds with empty boundary. Notice that Y and W are each proper closed subsets of these doubles. Now apply conclusion #3 of Proposition 4.3. Using conclusion #2 of Proposition 4.3, we deduce that each of Y_U and W_U are orientable homology manifolds. The diagram can therefore be simplified to the form

$$\begin{array}{ccccc}
\mathcal{Z} & \xrightarrow{d^*} & \mathcal{Z} & \longrightarrow 0 \\
\uparrow{\scriptstyle \cong} & & \uparrow{\scriptstyle f_U^*} & \\
\mathcal{Z} & \xrightarrow[d^*]{} & \mathcal{Z} & \longrightarrow 0
\end{array}$$

The maps d^* are epimorphisms and hence isomorphisms of $\mathcal{Z}$. This implies that the map f_U has degree ± 1 and, by Theorem 4.4, $(f_U)_{\#} : \pi_1(Y_U) \longrightarrow \pi_1(W_U)$ is an

epimorphism. Consider the commutative square



Note that $(f_U)_\#$ surjective and $\pi_1(\text{int } D \times (0,1)) = \{1\}$ implies that $\text{inc}_{2\#}$ is the trivial homomorphism, which in turn implies that $\pi_1(W_U) \to \pi_1(f(\text{int } D \times (0,1))) \to \pi_1(U_1 - N)$ is trivial, as desired. □

Next, we establish the following 1-LCC sum theorem.

Theorem 4.9 (1-LCC sum) *Let Y^{n+1} be an oriented, admissible homology $(n+1)$-manifold ($\partial Y = \emptyset$) and let Σ^n be an n-sphere embedded in Y. Assume that Σ^n is the union of two n-cells $\Sigma = D_1^n \cup D_2^n$ so that*

1. *D_1^n and D_2^n meet only in their common boundary.*
2. *D_1^n and D_2^n are each 1-LCC embedded in Z, where Z is the closure of one of the components of $Y - \Sigma$.*

Then Σ^n is 1-LCC embedded in Z.

Proof. Let $a \in \Sigma^n$ and $U_a \subset Z$ be an open neighborhood of a. Choose closed subdiscs $E_i \subset D_i$ satisfying

1. $E_1 \cup E_2$ is neighborhood of a in Σ

2. $E_1 \cup E_2 \subset U_a$.

We choose an open neighborhood $W_a \subset Z$ of a such that

3. $W_a \cap \Sigma \subset \mathrm{int}_\Sigma\,(E_1 \cup E_2)$

4. $W_a \subset U_a$.

Since D_1 is 1-LCC embedded in Z, there is a subneighborhood $W_a^* \subset W_a$ so that loops in $W_a^* - D_1$ are contractible in $W_a - D_1$. Consider a loop of the form

$$\phi : \mathbf{S}^1 \longrightarrow W_a^* - \Sigma \subset W_a^* - D_1.$$

By hypothesis, this extends to a map $\phi : \mathbf{B}^2 \longrightarrow W_a - D_1$.

Next, choose a closed subdisc $F^n \subset \mathrm{int}\, E_2$ so that $\phi(\mathbf{B}^2) \cap \Sigma \subset F$. Note that $\phi : \mathbf{B}^2 \longrightarrow \phi(\mathbf{B}^2) \cup F^n \subset W_a - D_1$ and that $\phi(\mathbf{B}^2) \cap \Sigma \subset \mathrm{int}\, F^n \subset E_2$. Observe that $\mathrm{int}\, F^n$ is 1-LCC in Z since $F \subset D_2$.We claim that $\mathrm{int}\, F^n$ is 0-LCC in Z. To this end let $p \in \mathrm{int}\, F^n$ and let $V_p \subset Y$ be an open neighborhood. Choose a subdisc $D_p \subset F^n$ so that $p \in \mathrm{int}\, D_p \subset D_p \subset V_p$ and choose a connected open subset of V_p, V_p^* containing p so that $\mathrm{int}\, D_p$ separates V_p^* into two path components $\mathcal{C}_1$ and $\mathcal{C}_2$ (apply Raymond [81, Thm. 2]). Assume the notation so that $\mathcal{C}_1 \subset Z$. Then $\overline{\mathcal{C}_1}$ is an neighborhood of p in Z and, moreover, every map $f : \{0,1\} \to \overline{\mathcal{C}_1} - F^n = \mathcal{C}_1$ extends to a map $f : [0,1] \to \mathcal{C}_1$ as $\mathcal{C}_1$ is path connected.

The argument given by Cannon in [31, 2C.2, page 34] now applies to our setting, and we obtain a map $\psi : \mathbf{B}^2 \longrightarrow W_a - \Sigma$ with $\psi = \phi$ on $\mathbf{S}^1$. This provides a contraction of the loop ϕ in $W_a - \Sigma \subset U_a - \Sigma$ as desired. □

We now to turn to applications of this theory for 4-dimensional G-spaces. We say that Y is *semi-nicely locally* $(n-1)$*-spherical* provided each point $y \in Y$ has arbitrarily small neighborhoods V whose boundary ∂V is topologically S^{n-1} and 1-LCC embedded in $\overline{V}$. Y is said to be *nicely locally* $(n-1)$*-spherical* provided there exists arbitrarily small V as before such that ∂V is 1-LCC embedded *in all of* Y. We have the following immediate corollary to Theorem 4.7.

Corollary 4.10 *Let X be a 4D Busemann G-space and $x \in X$.*

a. *For each $r > 0$ such that r is less than the cone radius ε_x about x, the metric sphere $S_r(x)$ is semi-nicely locally 2-spherical.*

b. *X is semi-nicely locally 3-spherical.*

Proof. a] Recall the notation from page 25. Given $b \in S_r(x)$ choose $0 < \delta < 2r$ and consider $S^2 \cong \partial Q_\delta(b) \subset Q_\delta(b)$. Applying the second retraction inequality, we note the deformation $r \circ H : \partial Q_\delta(b) \times I \longrightarrow Q_\delta(b)$ is a singular collar. Since X is a homology manifold, $S_r(x)$ is a homology sphere and thus Theorem 4.7 implies that $\partial Q_\delta(b)$ is 1-LCC embedded in $Q_\delta(b)$. Hence $S_r(x)$ is semi-nicely locally 2−spherical.

b] Let $V = \operatorname{int} B_{\varepsilon_x}(x)$ be the contractible open neighborhood of x in X and choose $\varepsilon > 0$ so that $\varepsilon < \frac{\varepsilon_x}{4}$. Note that the closed metric ball $B_\varepsilon(x)$ is contained in V. Fix $b \in S_\varepsilon(x)$ and consider the metric ball $B_\varepsilon(b)$. Note that $B_\varepsilon(b) \subset B_{\varepsilon_x}(x)$ and therefore $B_\varepsilon(b)$ is a cone. Now $x \in S_\varepsilon(b)$ and via a] we have that $\partial Q_\varepsilon(x)$ is 1-LCC embedded in $Q_\varepsilon(x)$. Via symmetry in this construction also have that $\partial Q_\varepsilon(b)$ is 1-LCC in $Q_\varepsilon(b)$, and $\partial Q_\varepsilon(x) = \partial Q_\varepsilon(b)$. Arguing as in Proposition 3.8, we have that the union of geodesic

cones is a topological 3-sphere Σ^3, where

$$\Sigma^3 = x \underset{g}{*} \partial Q_\varepsilon(x) \bigcup b \underset{g}{*} \partial Q_\varepsilon(b).$$

Since $\check{H}^3_c(\Sigma^3) \cong \mathcal{Z} \neq 0$ and $H_1(\operatorname{int} B_{\varepsilon_x}(x)) = 0$, we may argue as in Lemma 4.8 to see that Σ^3 separates $V = \operatorname{int} B_{\varepsilon_x}(x)$ into two components W_1, W_2. Applying Raymond [81] or Ancel [4], we have that Σ^3 is the common boundary of W_1 and W_2. Let W_1 be the component of $V - \Sigma^3$ which contains the interior of the unique segment $[x, b]$. Arguing as in the proof of Theorem 4.9, we have that Σ^3 is 0-LCC embedded in $\overline{W_1}$, and we claim that Σ^3 is 1-LCC embedded in $\overline{W_1}$.

In order to verify this claim, first consider the 3-cell $x \underset{g}{*} \partial Q_\varepsilon(x)$. For each parameter $t \in (0, d(x,b)]$,we may naturally identify a homeomorphic copy of $Q_\varepsilon(x)$, $Q_\varepsilon(x) \times \{t\}$, in $\overline{W_1}$ by considering the cone structure. Thus given a point z on the 3-cell $x \underset{g}{*} \partial Q_\varepsilon(x)$ and an open subset $U \subset \overline{W_1}$ containing z, let $t = d(x,z)$ and consider the following three cases.

Case 1. $0 < t < \varepsilon$.

Set $U_0 = U \cap Q_\varepsilon(x) \times \{t\}$ and find a connected open neighborhood V_0 of z in $Q_\varepsilon(x) \times \{t\}$ so that inclusion induces the trivial homomorphism

$$i_\# : \pi_1(V_0 - \partial Q_\varepsilon(x) \times \{t\}) \longrightarrow \pi_1(U_0 - \partial Q_\varepsilon(x) \times \{t\}).$$

Use the cone structure to find $V_1 \subset V_0$ and $\delta > 0$ so that $V_1 \times (t-\delta, t+\delta) \subset U$. That this inclusion induces a trivial homomorphism of fundamental groups follows immediately

from the commutativity of the diagram below.

$$\begin{array}{ccc} \pi_1\left(V_0 - \partial Q_\varepsilon(x) \times \{t\}\right) & \xrightarrow{i_\# = 0} & \pi_1\left(U_0 - \partial Q_\varepsilon(x) \times \{t\}\right) \\ \downarrow \cong & & \downarrow \\ \pi_1\left(V_0 \times (t-\delta, t+\delta) - x \underset{g}{*} \partial Q_\varepsilon(x)\right) & \longrightarrow & \pi_1\left(U - x \underset{g}{*} \partial Q_\varepsilon(x)\right) \end{array} \tag{4.15}$$

Case 2. $t = 0$.

The situation is somewhat easier in the case that $x = z$. In this case any neighborhood U of x in $\overline{W_1}$ must contain $Q_\varepsilon(x) \times \{t\}$, for some $t > 0$, and hence U contains $Q_\varepsilon(x) \times (0, t)$. Since $\operatorname{int} Q_\varepsilon(x)$ is contractible, $\pi_1(\operatorname{int} Q_\varepsilon(x) \times (0,t)) \cong \{1\}$, yielding the trivial homomorphism upon considering the inclusion $Q_\varepsilon(x) \times (0,t) - x \underset{g}{*} \partial Q_\varepsilon(x) \subset U - x \underset{g}{*} \partial Q_\varepsilon(x)$. This establishes Case 2.

Case 3. $t = \varepsilon$.

Choose an open neighborhood $\mathcal{O}_z \subset \overline{W_1}$ so that $\mathcal{O}_z \subset \overline{\mathcal{O}_z} \subset U$ and such that the intersection $\mathcal{O}_z \cap b \underset{g}{*} \partial Q_\varepsilon(x)$ is a subdisc F^3. Set $U_0 = \mathcal{B}_z \cap Q_\varepsilon(b)$, where $\mathcal{B}_z$ is an open neighborhood of z in X such that $\mathcal{B}_z \cap \overline{W_1} = \mathcal{O}_z$. Using the cone structure, we may find a neighborhood $V_0 \times \{t\} \subset U_0$ of z in $Q_\varepsilon(b)$ and a $\delta > 0$, where $\delta < \varepsilon < \varepsilon_x$, so that

$$V_0 \times (t-\delta, t+\delta) \bigcap \overline{W_1} \subset \mathcal{O}_z.$$

Since $\partial Q_\varepsilon(x)$ is 1-LCC in $Q_\varepsilon(b)$, there is an open neighborhood $V_1 \times \{t\}$ of z in $V_0 \times \{t\}$ so that loops in $V_1 \times \{t\} - \partial Q_\varepsilon(x)$ contract in $V_0 \times \{t\} - \partial Q_\varepsilon(x)$. We set

$$\mathcal{V} = V_1 \times (t-\delta, t+\delta) \cap \mathcal{O}_z.$$

Consider a loop $\phi : \mathbf{S}^1 \longrightarrow \mathcal{V}-x \underset{g}{*} \partial Q_\varepsilon(x)$. Since Σ^3 is 0-LCC in $\overline{W_1}$, we may apply Cannon ([32, 2C.2, page 34]) to homotopically adjust ϕ in $\mathcal{V}-x \underset{g}{*} \partial Q_\varepsilon(x)$ so that $\operatorname{im}\phi$ misses Σ^3. Use the product structure to define a map $\psi : \mathbf{S}^1 \times [0,1] \rightarrow (V_1 - \partial Q_\varepsilon(x)) \times (t-\delta, t+\delta)$ so that $\psi_0 = \phi$ and $\operatorname{im}\psi_1 \subset (V_1 - \partial Q_\varepsilon(x)) \times \{t\}$. Then ψ_1 extends to a map of a disc into $V_0 - \partial Q_\varepsilon(x)$ and therefore ψ extends to a map $\psi : \mathbf{B}^2 \rightarrow V_0 \times (t-\delta, t+\delta)$ where $\psi = \phi$ on $\mathbf{S}^1$. In particular $\psi(\mathbf{S}^1)$ misses F^3.

Apply the *disc-snipping* technique of Burgess ([25, Lemma 1, page 82]) to obtain a map $\lambda : \mathbf{B}^2 \rightarrow V_0 \times (t-\delta, t+\delta)$ so that

i. $\lambda = \phi$ on $\mathbf{S}^1$

ii. $\operatorname{im}\lambda \subset \operatorname{im}\psi \cup \operatorname{int} F^3$

iii. $\operatorname{im}\lambda - F^3$ is connected.

In particular, $\operatorname{im}\lambda$ lies $\mathcal{O}_z - \left(x \underset{g}{*} \partial Q_\varepsilon(x)\right)$ and hence λ is the desired contraction of ϕ. Thus $x \underset{g}{*} \partial Q_\varepsilon(x)$ is 1-LCC embedded in $\overline{W_1}$.

By a symmetric argument, we have that $b \underset{g}{*} \partial Q_\varepsilon(b)$ is also 1-LCC embedded in $\overline{W_1}$. Apply the 1-LCC sum theorem (with $Y = V$ orientable) to see that the union

$$\Sigma^3 = x \underset{g}{*} \partial Q_\varepsilon(x) \bigcup b \underset{g}{*} \partial Q_\varepsilon(b)$$

is 1-LCC embedded in $\overline{W_1}$, as asserted.

The final step in the argument is to assure that the semi-nice sphere structure in X is truly local. One simple way to do this is to make a slight modification in the above construction. Specifically given y and a neighborhood $W \subset X$, first choose an $\varepsilon < \varepsilon_y$

and then find a point $x \in intB_\varepsilon(y)$ so that

$$y \in intB_\delta(x) \subset intB_\varepsilon(y)$$

for some $0 < \delta < \frac{\varepsilon}{4}$ Consider the unique segment $[x, y]$. This segment extends uniquely to the boundary $S_\delta(x)$ and we call this endpoint b. Now proceed with the symmetric construction described above using these points x and b. □

CHAPTER 5

The 4D Semi-Nicely Spherical Shrinking Theorem

In the previous chapters, we have discovered that each 4D G-space X is both resolvable and semi-nicely locally 3-spherical. It is the aim of this chapter to establish that these two properties are sufficient to recognize X as a topological 4-manifold, and thus prove Theorem 1.1. This chapter constitutes step 5 of the strategy outlined on page 5.

1 Spherical Shrinking: A Historical Overview

A natural setting in which one encounters a locally spherical space occurring as a decomposition space is when considering proper surjective maps between manifolds

$$\pi : M^n \longrightarrow N^n. \tag{5.1}$$

If such a map is **ABH**, then fibers of π are not only cell-like, but have the additional feature of being cellular.

Definition 5.1 *Let C be a compact subset of an n-manifold M^n. C is said to be cellular in M provided there is a sequence of n-cells $\{D_i^n\}$ such that $D_{i+1} \subset \operatorname{int} D_i^n$ and*

$$C = \bigcap_{i=1}^{\infty} D_i^n.$$

If C is cellular in M^n, then it follows immediately [90, page 314] that C is a cell-like set. This next result distinguishes cellular sets from among all cell-like sets in manifolds

of sufficiently high dimension and is due to McMillan [64] for $n \geq 5$, and Freedman [48] for $n = 4$.

Theorem 5.1 (cellularity criterion) *Let C be a cell-like subset contained in a manifold $M^n, n \geq 4$. Then C is cellular in M^n if and only if*

(†) *For each open neighborhood U of C in M, there is a neighborhood V of C in M, so that every map $\phi : S^1 \to V \setminus C$ extends to a map $\phi : D^2 \to U \setminus C$.*

Remark. When $n = 3$ and C is a cell-like subset of $\mathbf{S}^3$, McMillan [64] established that C is cellular in $\mathbf{S}^3$ if and only if C satifies condition (†) in $\mathbf{S}^3$. The general statement for 3-manifolds implies the Poincaré's conjecture (see [64, page 330] and [37, page 147])

.

The concept of cellularity comes into play quite naturally in the study of cell-like maps between manifolds. If π is a cell-like map of the form (5.1), it is known (see [37, page 147]) that the fibers of π satisfy condition (†) in M^n and hence π is a cellular map provided $n \geq 4$. A similar result can be established when the image is assumed to be locally spherical.

Lemma 5.2 *Let $\pi : M^n \longrightarrow Y$ be a cell-like map of a manifold onto a locally spherical space, where $n \geq 3$. Then the fibers of π satisfy condition (†) and hence if $n \geq 4$, π is a cellular map.*

Proof. Apply Corollary 18.5C of [37, page 146]. It suffices to show that each point $y \in Y$ is 1-LCC embedded in Y. Given a neighborhood U_y in Y, use the fact that Y

is an ANR (see [37, page 129]) to find a sub-neighborhood W_y which contracts in U_y. Consider a map $\phi : \mathbf{S}^1 \to W_y - \{y\}$. Since W_y contracts in U_y, ϕ naturally extends to a map

$$\Phi : D^2 \longrightarrow U_y.$$

Next find V_y such that

$$y \in V_y \subset \overline{V_y} \subset W_y \subset U_y$$

and so that $\partial V_y \cong (n-1)$-sphere and $\phi(\mathbf{S}^1) \cap V_y = \emptyset$. Since ∂V_y is an ANR, there is a pair of subneighborhoods of W_y, $\mathcal{O}_2 \subset \mathcal{O}_1$, with $y \notin \mathcal{O}_1$ and there is a homotopy $H : \mathcal{O}_2 \times I \to \mathcal{O}_1$ so that $H_0 = 1_{\mathcal{O}_2}$ and H_1 is a retraction of $\mathcal{O}_2$ onto ∂V_y (see [90, Thm. 5.1.1, page 191] and [37, Thm. 8, page 117]).

Since W_y contracts in U_y, ϕ naturally extends to a map

$$\Phi : D^2 \longrightarrow U_y.$$

Construct a compact 2-manifold with boundary $F^2 \subset D^2$ so that

$$\Phi^{-1}(\partial V_y) \subset \operatorname{int} F^2 \subset F^2 \subset \Phi^{-1}(\mathcal{O}_2) \subset \operatorname{int} D^2.$$

Proceed inductively on the inner most boundary components of $F^2, \mathcal{C}_1, \ldots, \mathcal{C}_k$. Given a boundary component $\mathcal{C}_i$, redefine Φ in small collar neighborhood of $\mathcal{C}_i$ in $\mathcal{T}_i$, where $\mathcal{T}_i$ is the component of $D^2 - \mathcal{C}_i$ missing ∂D^2, via defining the map $\Phi_i : \mathcal{C}_i \times [0,1] \longrightarrow \mathcal{O}_1$ where

$$\Phi_i(x,t) = H\left(\Phi_{i-1}(x), t\right).$$

Make the obvious extension of Φ_i to $D^2 - \mathcal{T}_i$. Note that $\Phi_i(\cdot, 1)$ maps the loop $\mathcal{C}_i \times \{1\}$ to ∂V_y which is simply connected by hypothesis. Therefore Φ_i extends to a map Φ_i :

$T_i \to \mathcal{O}_1$ such that $\Phi_i|_{C_j} = \Phi_{i-1}|_{C_j}$,and $\Phi_i|_{\partial D^2} = \Phi|_{\partial D^2}$. The map Φ_k maps D^2 into $W_y - \{y\}$ and hence the loop ϕ contracts in $U_y - \{y\}$. □

Local spherical structures and the notion of cellularity were exploited by Moore [71], Roberts and Steenrod [84], Armentrout [8], Siebenmann [87] and Quinn [79] to establish this next foundational result.

Theorem 5.3 *Let $\pi : M^n \longrightarrow N^n$ be a proper, continuous surjection between n-manifolds.*

a. *If $n \neq 3$ and π is cell-like, then π is* **ABH.**

b. *If $n = 3$ and π is assumed to be cellular , then π is* **ABH.**

The additional assumption of cellularity when $n = 3$ is crucial and the necessity of the hypothesis is intricately related to Poincare's Conjecture (see [37, page 147] or [68]).

We turn our attention to the generalizations of Theorem 5.3 in high dimensions ($n \geq 4$) as these will be used to establish the $4D$ semi-nicely spherical shrinking theorem. Cannon ([31]) established the next result which he dubbed *weak taming.*

Proposition 5.4 (Cannon's weak taming) *Let $\pi : M^n \longrightarrow Y$ be cell-like, where $n \geq 5$. Assume there is an embedded $(n-1)$-sphere $\Sigma^{n-1} \subset Y$ satisfying*

1. *The restriction $\pi| : M - \pi^{-1}(\Sigma) \longrightarrow Y - \Sigma$ is 1 to 1*

2. *Σ is 1-LCC embedded in $\dot{Y}$*

Then π is **ABH.**

Cannon's weak taming theorem holds $n = 3$ provided π is a cellular map. The case $n = 4$ also holds and is easily established as a consequence of Theorem 5.3 and the following result of Freedman-Quinn [49, Thm.9.3A].

Theorem 5.5 (Freedman-Quinn 1-LCC flattening) *Let Y be as admissible homology 4-manifold ($\partial Y = \emptyset$) and let $\Sigma \subset Y$ be a closed subset satisfying*

1. *$Y - \Sigma$ is a topological 4-manifold*
2. *$\Sigma \times \mathbf{R}^1$ is a topological manifold*
3. *$\dim \Sigma = 3$*
4. *Σ is 1-LCC embedded in Y.*

Then there is a normal bundle for Σ in Y and , moreover, if Σ is a 3-manifold, then Y is a 4-manifold and Σ is locally flatly embedded in Y.

We remark that this theorem has high dimensional analogues and the interested reader is referred to [49, page 138] for a history of this result.

As an application of weak taming, Cannon went on to establish the following spherical taming theorem, which we state using the terminology introduced in the previous chapter (see page 59).

Theorem 5.6 (Cannon's strong taming) *Let $\pi : M^n \longrightarrow Y$ be cell-like, where $n \geq 5$. If Y is nicely locally $(n-1)$-spherical, then π is* **ABH.**

Moreover, strong taming in arbitrary dimension n is equivalent to the all three of the following:

1. *weak taming holds in dimension n,*

2. *Theorem 5.1 holds in dimension n, and*

3. *Theorem 5.3 holds in the cellular setting for this n.*

As noted above, Freedman [48, page 373] established that the cellularity criterion holds when $n = 4$. Combining this with Theorem 5.5, we see that Cannon's strong taming theorem holds in dimension 4.

The situation for Y semi-nicely spherical was settled by Daverman [39] in the case that $n \geq 5$.

Theorem 5.7 (Daverman's semi-nicely shrinking, n ≥ 5) *Let* $\pi : M^n \longrightarrow Y$ *be cell-like, where* $n \geq 5$. *If* Y *is semi-nicely locally* $(n-1)$*-spherical, then* π *is* **ABH.**

Daverman's technique for establishing Theorem 5.7 relied on the very high dimensional ($n \geq 5$) cell-like approximation theorem of Edwards which, as of this writing, has no analogue in dimension 4.

2 The 4D Shrinking Theorem

The goal of this section is to establish the $4D$ analogue of Daverman's semi-nicely shrinking theorem. This is the last remaining step in the proof of Theorem 1.1.

Theorem 5.8 (semi-nicely shrinking, $n = 4$) *Let* M^4 *be* 4*-manifold and let*

$$\pi : M^4 \longrightarrow X$$

be a cell-like resolution of a semi-nicely locally 3*-spherical space* X*. Then* π *is* **ABH** *and hence* X *is a* 4*-manifold.*

Proof. For simplicity, assume that M^4 is compact. Let $CE(M^4, X)$ denote the collection of cell-like maps of M^4 onto X. Recall that $CE(M^4, X)$ is a complete metric space, when equipped with the sup-norm metric (see [37, page 133]). For each positive η, let $\mathcal{O}_\eta \subset CE(M^4, X)$ be the collection of η-maps, i.e. cell-like maps such that $\operatorname{diam} f^{-1}(x) < \eta$ for all $x \in X$. The aim is to verify that $\mathcal{O}_\eta$ is a dense open subset of $CE(M^4, X)$ for all $\eta > 0$, and thus obtain the result as a consequence of the Baire Category Theorem [72, page 294].

To this end fix $\eta, \varepsilon > 0$, and $\pi \in CE(M^4, X)$.

Part 1. Open cover manipulations

Applying Lemma 5.2 above, we may find open cover of M^4 by open 4−cells and a $\lambda > 0$ such that if $A \subset X$ and $\operatorname{diam} A < \lambda$, then $\pi^{-1}(A)$ is contained in the interior of a 4-cell $D^4 \in \mathcal{D}$. Here $\mathcal{D}$ is an open cover of M^4 by 4-cells so that π is a $\mathcal{D}-$map. Cover X by open balls of radius $< \min\{\frac{\lambda}{2}, \frac{\varepsilon}{20}\}$. Since $\dim X = 4$, there is a refinement of this cover, say Ω, whose order is ≤ 5 [72, page 302]. Furthermore, Ω may be partitioned into 5 subcollections $\Omega_0, \ldots, \Omega_4$ so that each Ω_i consists of a finite number of open sets having pair-wise disjoint compact closures [90, page 144]. Next, using the local spherical structure of X, refine Ω by a finite open cover $\{V_i\}_{i=1}^k$ where each $\partial V_i \cong S^3$ is semi-nicely embedded in $\overline{V_i}$. For emphasis, note that $k = \operatorname{card}\left\{\overline{V_i}\right\}_{i=1}^k$ and $\operatorname{diam} \overline{V_i} < \frac{\lambda}{2}$. Choose a decreasing sequence $\lambda = \lambda_0 > \lambda_1 > \cdots > \lambda_k > \frac{1}{2}\lambda$. By imposing an

ordering of the V_i, we may select an integral indexing

$$0 = \omega_0 < \omega_1 < \cdots < \omega_4 = \operatorname{card} \Omega$$

so that if $t \in \{1, \ldots k\}$ with the property that $\omega_i < t \leq \omega_{i+1}$, then there is an element $W_t \in \Omega_{i+1}$ so that

$$\overline{V_t} \subset W_t.$$

Technical point: For each $i = 1, 2, \ldots, k$, specify a sequence of closed proper shrinkings $\mathcal{A}^{(i)} = \left\{\mathcal{A}^{(i)}(\overline{V_m})\right\}_{m=1}^{k}$ of the cover $\left\{\overline{V_i}\right\}$ so that each refinement again covers X.

Part 2. $\mathcal{O}_\eta$ is dense in $CE(M^4, X)$

The aim of this step is to show that there is a $g_\eta \in \mathcal{O}_\eta$ so that $d(\pi, g_\eta) < \varepsilon$. We claim that there is a finite sequence $\{f_i\}_{i=0}^{k} \subset CE(M^4, X)$ satisfying the following criterion:

$\mathbf{C}_0$. $\pi = f_0$

$\mathbf{C}_1$. $d(f_{\omega_i - 1}, f_{\omega_i}) < \frac{\varepsilon}{5}$, for $i = 1, 2, 3, 4$.

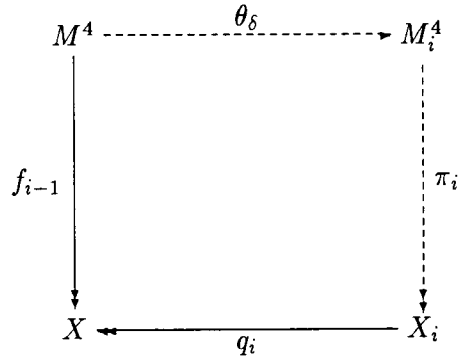
$\mathbf{C}_2$. For each $i \geq 1$ and for each $j = 1, \ldots i - 1$, we have that f_i is an η-map over $\mathcal{A}^{(i)}(\overline{V_j})$

$\mathbf{C}_3$. f_i is an η-map over V_i

$\mathbf{C}_4$. For each $i \geq 1$ and to each $A \subset X$ with $\operatorname{diam} A < \lambda_i$, there corresponds $D^4 \in \mathcal{D}$ such that

$$f_i^{-1}(A) \subset \operatorname{int} D^4.$$

The construction of these f_i proceeds by induction on $i = 0, 1, 2, \ldots, k$. Set $f_0 = \pi$. Given f_{i-1} satisfying C_0 - C_4, consider the following diagram.



(5.2)

The space X_i is the adjunction space

$$X_i = X - V_i \bigcup_{\partial V_i} \partial V_i \times [0,1] \bigcup_{\partial V_i \times \{0\}} V_i \times \{0\}$$

where the natural identification $\partial V_i \times \{1\} = \partial V_i$ is made. As an application of the Vietoris-Begle mapping Theorem (see [88, page 344]), or the argument given in [37, page 191, Cor.26.1A], one can show that X is an admissible homology manifold.[1] Now apply Raymond [81, Thm. 1.] to see that X_i is itself admissible. Recalling that $\partial V_i \cong S^3$, note that the Quinn local index $i(X_i) = 1$, since $\partial V_i \times (0,1)$ is a manifold. Therefore the map $\pi_i : M_i^4 \longrightarrow X_i$ is a cell-like resolution whose existence is ensured via Theorem 3.1.

The map $q_i : X_i \longrightarrow X$ is the natural map obtained the collapsing each of the collar lines of $\partial V_i \times [0,1]$ to a point. The fibers of q_i over $\partial V_i \times [0,1]$ are either points or arcs, and therefore q_i is a proper, cell-like surjection. Applying [90, Cor. 7.1.9], we see that the composition $q_i \circ \pi_i : M_i^4 \longrightarrow X$ is a resolution of X. Applying Quinn [79, Thm.

[1] It is an immediate consequence of the local spherical structure that X is finite dimensional.

2.6.1], for each $\delta > 0$, we may select a homeomorphism

$$\theta_\delta : M^4 \longrightarrow M_i^4$$

which makes diagram (5.2) $\delta-$commutative. We choose $\delta = \delta(i)$ to be sufficiently less than the minimum of $\frac{\varepsilon}{2^{i+1}}$ and the prescribed controllers, $v_1^i, \ldots, v_3^i$ which are described below.

The controllers v_1^i For each $j = 0, 1, 2, 3, 4$ consider Ω_j. By construction Ω_j consists of finitely many open sets with pair-wise disjoint compact closures. Hence there is a $0 < \rho_j < \frac{\varepsilon}{10}$ such that for each distinct pair $E_1, E_2 \in \Omega_j$, $N(\overline{E_1}, \rho_j) \cap N(\overline{E_2}, \rho_j) = \emptyset$. Let $\rho = \min \rho_j$ and choose a sequence of positive numbers $\{v_1^i\}_{i=1}^k$ so that $\sum_{i=1}^{k} v_1^i < \rho$.

The controller v_2^i For each $j = 1, \ldots, i-1$ we insist that $q_i \circ \pi_i \circ \theta_i$ be an η-map over $\mathcal{A}^{(i)}(\overline{V_j})$. This may be accomplished as follows. Let $j \in \{1, \ldots, i-1\}$ and let

$$\alpha_j = \inf \left\{ d\left(f_{i-1}(x), f_{i-1}(y)\right) | x, y \in f_{i-1}^{-1}\left(\mathcal{A}^{i-1}(\overline{V_j})\right), d(x,y) \geq \eta \right\}.$$

Now $\alpha_j > 0$ follows from the compactness of $f_{i-1}^{-1}\left(\mathcal{A}^{i-1}(\overline{V_j})\right)$ and the hypothesis that $f_{i-1}$is an η-map over $\mathcal{A}^{i-1}(\overline{V_j})$. Let $\beta_j > 0$ be such that if $\phi : M^4 \longrightarrow X$ is β_j-close to f_{i-1} , then

$$\phi^{-1}\left(\mathcal{A}^{(i)}(\overline{V_j})\right) \subset f_{i-1}^{-1}\left(\mathcal{A}^{(i-1)}(\overline{V_{i-1}})\right).$$

Now set $v_2^i = \frac{1}{2} \min_{1 \leq j \leq i-1} \{\{\alpha_j\}, \{\beta_j\}\}$. If ϕ is v_2^i-close to f_{i-1}, then for each $z \in \mathcal{A}^{(i)}(\overline{V_j})$ consider x, y in the fiber $\phi^{-1}(z) \subset f_{i-1}^{-1}\left(\mathcal{A}^{(i-1)}(\overline{V_j})\right)$. Note that

$$\begin{aligned} d\left(f_{i-1}(x), f_{i-1}(y)\right) &\leq d\left(f_{i-1}(x), \phi(x)\right) + d\left(f_{i-1}(y), \phi(y)\right) \\ &< 2v_2^i \\ &\leq \alpha_j \end{aligned}$$

and thus, by definition of α_j, $d(x,y) < \eta$

The controller v_3^i This controller is chosen so that if $d(f_{i-1},\phi) < v_3^i$, then $A \subset X$ with

$$\operatorname{diam} A < \lambda_i$$

implies there is a $D^4 \in \mathcal{D}$ such that $f_i^{-1}(A) \subset \operatorname{int} D^4$. The existence of such a v_3^i is established in [55, page 62].

Let $\delta(i) = \dfrac{\min\{v_1^i, v_2^i, v_3^i, \frac{\epsilon}{2^{i+1}}\}}{5}$ and choose a homeomorphism $\theta_{\delta(i)}$ so that diagram (5.2) $\delta(i)$-commutes.

Lemma 5.9 *The map $\pi_i \circ \theta_{\delta(i)}$ is approximable by CE maps $\mu : M^4 \longrightarrow X_i$ such that*

$$\mu \text{ is 1-1 over } \partial V_i \times \{0\}.$$

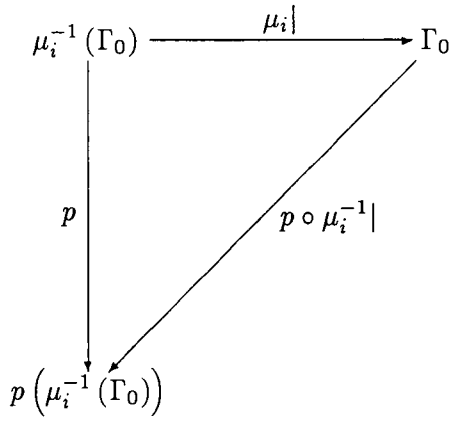
Proof. Let $\mu_i = \pi_i \circ \theta_\delta$, and let $\Gamma_0 = \partial V_i \times \{0\}$. We begin by letting

$$G(\Gamma_0) = \left\{\mu_i^{-1}(x) | x \in \Gamma_0\right\} \cup \text{ singletons of } M^4 - \mu_i^{-1}(\Gamma_0)$$

be the induced upper semi-continuous decomposition of M^4. Name the quotient map

$$p : M^4 \longrightarrow \frac{M^4}{G(\Gamma_0)}$$

and consider the folowing commutative diagram



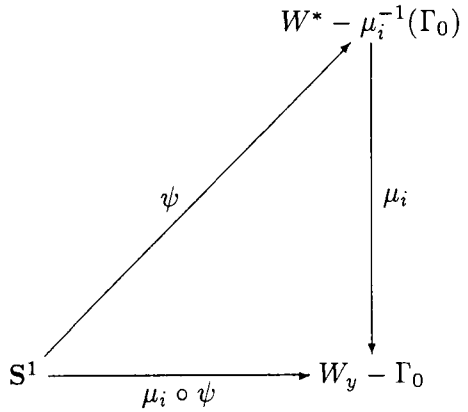
(5.3)

It is easy to see that $p \circ \mu_i^{-1}|$ is a well-defined, continuous bijection between compacta and therefore an embedding of Γ_0 in $\frac{M^4}{G(\Gamma_0)}$. Let Γ_1 denote $p\left(\mu_i^{-1}(\Gamma_0)\right)$.

We claim that Γ_1 is 1-LCC embedded in $\frac{M^4}{G(\Gamma_0)}$. To this end, let $x \in \Gamma_1$ and let U_x be a neighborhood of x in $\frac{M^4}{G(\Gamma_0)}$. Note that $p^{-1}(U_x)$ is an open neighborhood of $p^{-1}(x)$ and, by defintion of p, there is a $y \in \Gamma_0$ so that $\mu_i^{-1}(y) = p^{-1}(x)$. Therefore we may find a μ_i-saturated $U^* \subset p^{-1}(U_x)$ so that $\mu_i(U^*)$ is an open neighborhood of $\mu_i\left(p^{-1}(x)\right) = y$.

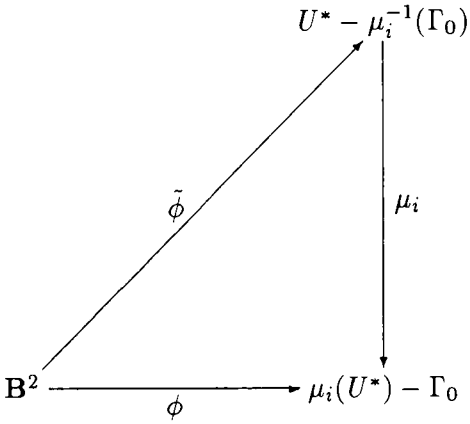
Applying the local semi-nice hypothesis together with the collar on Γ_0 in X_i, we obtain that Γ_0 is 1-LCC in X_i. Select $W_y \subset \mu_i(U^*)$ so that loops of the form $\phi : \mathbf{S}^1 \longrightarrow W_y - \Gamma_0$ contract in $\mu_i(U^*) - \Gamma_0$. Set $W^* = \mu_i^{-1}(W_y)$. We claim that loops in $W^* - \mu_i^{-1}(\Gamma_0)$ contract in $U^* - \mu_i^{-1}(\Gamma_0)$. Given a loop $\psi : \mathbf{S}^1 \longrightarrow W^* - \mu_i^{-1}(\Gamma_0)$, we

have the following commutative diagram



(5.4)

Now $\mu_i \circ \psi$ extends to a map $\phi : \mathbf{B}^2 \longrightarrow \mu_i(U^*) - \Gamma_0$. Apply approximate CE lifting (see [37, Thm.16.7]) to obtain an approximately commutative diagram

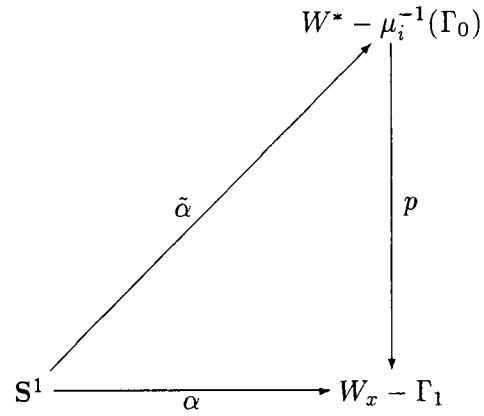


(5.5)

so that $\tilde{\phi} = \psi$ on $\mathbf{S}^1$.

Now set $W_x = p(W^*)$. Note that $W_x \subset U_x$ is an open neighborhood of x since W^* is μ_i-saturated and hence p-saturated. To establish that Γ_1 is 1-LCC in $\dfrac{M^4}{G(\Gamma_0)}$ at x, we need only show that loops in $W_x - \Gamma_1$ contract in $U_x - \Gamma_1$. Given a loop $\alpha : \mathbf{S}^1 \to W_x - \Gamma_1$,

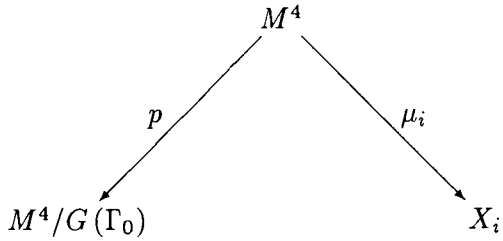
we apply approximate CE lifting to obtain an approximately commutative triangle



(5.6)

so that $p \circ \tilde{\alpha}$ and α are homotopic in $W_x - \Gamma_1$ (see [37, Thm. 16.8, page 127]). The map $\tilde{\alpha}$ extends to a map $\beta : \mathbf{B}^2 \to U^* - \mu_i^{-1}(\Gamma_0)$. In particular $p \circ \tilde{\alpha}$ is null-homotopic in $U_x - \Gamma_1$ which implies that α is null-homotopic in $U_x - \Gamma_1$. Thus Γ_1 is 1-LCC embedded in $\dfrac{M^4}{G(\Gamma_0)}$.

Finally, consider the diagram



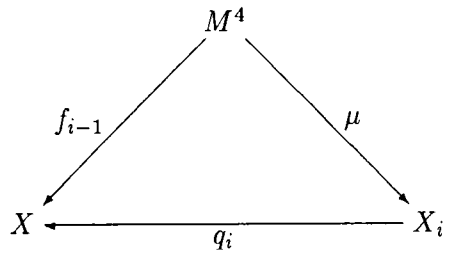
(5.7)

As an application of Cannon's weak taming (Theorem 5.4), we obtain that the quotient map p is approximable by homeomorphisms $h : M^4 \longrightarrow \dfrac{M^4}{G(\Gamma_0)}$. It is easy to see that maps of the form

$$\mu_i \circ p^{-1} \circ h : M^4 \longrightarrow X_i \tag{5.8}$$

are approximations to $\pi_i \circ \theta_\delta$ and have the property of being 1-1 over $\Gamma_0 = \partial V_i \times \{0\}$, as desired. □

Consider the following diagram



(5.9)

where μ satisfies the conclusion of Lemma 5.9. In particular, the triangle (5.9) is approximately commutative. This next result constitutes the inductive step.

Lemma 5.10 (radial shrinking) *Given $f_{i-1} : M^4 \longrightarrow X$ and $\delta(i) > 0$ as above, there is a CE map $f_i : M^4 \longrightarrow X$ such that*

a. *For all $x \in X - \overline{V_i}$, f_i and f_{i-1} are $\delta(i)$-close*

b. *f_i is an η-map over V_i*

c. *For each $j = 1, \ldots i-1$, we have that f_i is an η-map over $\mathcal{A}^{(i)}(\overline{V_j})$*

Proof. (Note: This argument will exploit Price's technique of radial shrinking and the reader is referred to [78], or [37, section 8].) Given $\delta(i) > 0$ as above, choose q_i and μ as in (5.9) so that $d(f_{i-1}, q_i \circ \mu) < \delta(i)$ and that $\xi = q_i \circ \mu$ is an η-map over $\mathcal{A}^{(i)}(\overline{V_j})$ for all $j = 1, \ldots, i-1$. Furthermore, we may assume that $\xi^{-1}\left(\overline{V_i}\right)$ is contained in int D^4, for some $D^4 \in \mathcal{D}$. Next, let $\Sigma_0 = \mu^{-1}\left(\partial V_i \times \{0\}\right)$ be the 3-sphere embedded in M^4.

Step1. Σ_0 bounds a 4-cell B_0 in M^4.

A straightfoward application of CE lifting techniques (see Lemma 5.9 above), gives that Σ_0 is 1-LCC embedded in M^4. Apply Theorem 4.1 to see that Σ_0 is flat in int $D^4$and hence, via Brown [20], bounds a 4-cell B_0.

Step 2. Improving ξ over V_i.

We begin by observing that $\mu^{-1}(\overline{V_i}) = B_0$, since $\mu^{-1}(\overline{V_i})$ is the closure of the compact component of int $D^4 - \Sigma_0$. Specify a homeomorphism $h : B_0 \longrightarrow \mathbf{B}^4$, where $\mathbf{B}^4$ is the standard round 4-cell. Via uniform continuity, we may choose $\hat{\delta} > 0$ so that $A \subset \mathbf{B}^4$ with diam $A < 4\hat{\delta}$ implies diam $h^{-1}(A) < \eta$. Let $h(G)$ represent the decomposition of $\mathbf{B}^4$ given by $\left\{h\left(\mu^{-1}(x)\right) | x \in \overline{V_i} \subset X_i\right\}$.

Let n be a positive integer chosen so that $\frac{1}{n} < \hat{\delta}$. We claim that since no non-degenerate element g of $h(G)$ meets $\partial\mathbf{B}^4 = \mathbf{S}^3_1$, there is a radius $0 < 1 - \delta_0 < 1$ so that any $g \in h(G)$ meeting the annulus $[\mathbf{S}^3_{1-\delta_0}, \mathbf{S}^3_1]$ is such that its radial projection to $\mathbf{S}^3_1$ has diameter less than $2\hat{\delta}$. This can be seen as follows. Via [37, Prop. 7.3], the set given by

$$\mathcal{K}_{\geq\beta} = \bigcup \{g \in h(G) | \operatorname{diam} g \geq \beta\}$$

is a closed subset of $\mathbf{B}^4$ for each $\beta > 0$. Necessarily $\mathcal{K}_{\geq\beta}$ is disjoint from $\mathbf{S}^3_1$ and thus we may choose an annulus $[\mathbf{S}^3_{1-\alpha}, \mathbf{S}^3_1]$ such that $[\mathbf{S}^3_{1-\alpha}, \mathbf{S}^3_1] \cap \mathcal{K}_{\geq\beta} = \emptyset$, where α depends on β. Now apply uniform continuity of the radial projection. Similarly, we may obtain a sequence of radii $1 - \delta_0 < 1 - \delta_1 < \cdots < 1 - \delta_n < 1$ so that if g intersects $[\mathbf{S}^3_{1-\delta_{i+1}}, \mathbf{S}^3_1]$, then $g \subset \text{int}[\mathbf{S}^3_{1-\delta_i}, \mathbf{S}^3_1]$.

Next use the cone structure of $\mathbf{B}^4$ to define the natural homeomorphism

$$\theta : \mathbf{B}^4 \longrightarrow \mathbf{B}^4$$

satisfying the following properties:

i. $\theta(0) = 0$

ii. $\theta = 1$, on $\mathbf{S}^3_1$

iii. $\theta\left([\mathbf{S}^3_{1-\delta_i}, \mathbf{S}^3_{1-\delta_{i+1}}]\right) = [\mathbf{S}^3_{\frac{i}{n}}, \mathbf{S}^3_{\frac{i+1}{n}}]$, for each $i = 1, 2, \ldots, n-2$.

Now consider a non-degenerate $g \in h(G)$. If $g \subset \mathbf{B}^4_{1-\delta_1}$, or if $g \cap [\mathbf{S}^3_{1-\delta_0}, \mathbf{S}^3_1] \neq \emptyset$, (in which case $g \subset [\mathbf{S}^3_{1-\delta_1}, \mathbf{S}^3_1]$) we have, by definition, $\theta(g) \subset \mathbf{B}^4_{\frac{1}{n}}$ and therefore $\operatorname{diam} \theta(g) \leq \frac{1}{n} < \hat{\delta}$. Otherwise there is an $i \geq 2$ for which $g \subset [\mathbf{S}^3_{1-\delta_{i-1}}, \mathbf{S}^3_{1-\delta_{i+1}}]$. Then $\theta(g)$ is contained in the intersection

$$0 * \operatorname{rad}(g) \cap [\mathbf{S}^3_{\frac{i-1}{n}}, \mathbf{S}^3_{\frac{i+1}{n}}]$$

where $0 * \operatorname{rad}(g)$ denotes the cone over the radial projection of g to $\mathbf{S}^3_1$ from the origin. Therefore $\operatorname{diam} \theta(g) \leq \operatorname{diam} 0 * \operatorname{rad}(g) \cap [\mathbf{S}^3_{\frac{i-1}{n}}, \mathbf{S}^3_{\frac{i+1}{n}}] < 4\hat{\delta}$.

Next, let $h_1 : M^4 \longrightarrow M^4$ be the homeomorphism given by

iv. $h_1 = 1$ on $M^4 - \operatorname{int} B_0$

v. $h_1 = h^{-1} \circ \theta \circ h$ on B_0

and note that for all $x \in \overline{V_i} \subset X_i$, $\operatorname{diam} h_1 \mu^{-1}(x) < \eta$. Let $f_i = q_i \circ \mu \circ h_1 : M^4 \longrightarrow X$ and observe that f_i is an η-map over $V_i \subset X$.

We claim that f_i is an η-map over $\mathcal{A}^{(i)}(\overline{V_j})$, where $j \in \{1, \ldots, i-1\}$. If none of the $\mathcal{A}^{(i)}(\overline{V_j})$ intersect ∂V_i, then f_i continues to be an η-map over $\mathcal{A}^{(i)}(\overline{V_j})$. On the other hand if $\mathcal{A}^{(i)}(\overline{V_j}) \cap \partial V_i \neq \emptyset$, recall that ξ is an η-map over $\mathcal{A}^{(i)}(\overline{V_j})$. Hence over the intersection $\mathcal{A}^{(i)}(\overline{V_j}) \cap V_i$, f_i continues to be an η-map. Now if $x \in \mathcal{A}^{(i)}(\overline{V_j}) \cap \partial V_i$, note that

$$f_i^{-1}(x) = (q_i \circ \mu \circ h_1)^{-1}(x) = h_1^{-1}(\mu^{-1}(q_i^{-1}(x))) = \mu^{-1}(q_i^{-1}(x)) = \xi^{-1}(x)$$

since $h_1 = 1$ on $M^4 - \operatorname{int} B_0$.

Thus f_i continues to be an η-map over $\mathcal{A}^{(i)}(\overline{V_j})$. This ends step 2 and the proof of Lemma 5.10 is now complete. □

Proof of the main result. Given $f_{i-1} : M^4 \longrightarrow X$ and $\delta(i) > 0$ as above, we construct f_i via Lemma 5.10 above. Note that f_i satisfies conditions C_2, C_3 and C_4. For condition C_1, fix ω_s, ω_{s+1},where $s \in \{0,1,2,3\}$ and let $x \in X$. We have two cases.

Case 1. There exists j where $\omega_s < j \leq \omega_{s+1}$ so that $f_{\omega_s}(x) \in \overline{V_j}$.

By our construction of Ω_{s+1} there is a $U \in \Omega_{s+1}$ for which $\overline{V_j} \subset U$. Via conclusion a] of Lemma 5.10 and our choice of the controllers v_1^i, we have for $\omega_s < t \leq \omega_{s+1}$, the subsequent images of x, $f_t(x)$ are contained in $N(U, \frac{\varepsilon}{20})$. Thus $d\left(f_{\omega_s}(x), f_{\omega_{s+1}}(x)\right) < \frac{\varepsilon}{10}$.

Case 2. There does not exist j where $\omega_s < j \leq \omega_{s+1}$ so that $f_{\omega_s}(x) \in \overline{V_j}$.

If all of the subsequent images of x are also not contained in one of the $\overline{V_j}$, then by our choice of the $\delta(i)$, we have that $d\left(f_{\omega_s}(x), f_{\omega_{s+1}}(x)\right) < \frac{\varepsilon}{5} \sum_{i=1}^{k} \frac{1}{2^{i+1}} < \frac{\varepsilon}{10}$. Otherwise let $\omega_s < t \leq \omega_{s+1}$ be the first instance where the subsequent image of x lies in some $\overline{V_j}$. By our choice of the $\delta(i)$, we have that $d\left(f_{\omega_s}(x), f_t(x)\right) < \frac{\varepsilon}{5} \sum_{i=1}^{k} \frac{1}{2^{i+1}} < \frac{\varepsilon}{10}$. Moreover, arguing as in case 1] above, we have that there is a $U \in \Omega_{s+1}$ so that $\{f_t(x), f_{t+1}(x), \ldots, f_{\omega_{s+1}}(x)\} \subset N\left(U, \frac{\varepsilon}{20}\right)$. This implies $d\left(f_t(x), f_{\omega_{s+1}}(x)\right) < \frac{\varepsilon}{10}$ and an application of the triangle inequality yields $d\left(f_{\omega_s}(x), f_{\omega_{s+1}}(x)\right) < \frac{\varepsilon}{5}$.

Finally, consider $f_k : M^4 \longrightarrow X$. Condition C_1 immediately gives that f_k and π are ε-close. Moreover f_k is an η-map over $\bigcup \mathcal{A}^{(k)}\left(\overline{V_i}\right)$. As this union covers X, we have that $\mathcal{O}_\eta$ is a dense open subset of $\mathrm{CE}(M^4, X)$, for all $\eta > 0$. Applying Baire's theorem, the intersection $\bigcap \mathcal{O}_{\frac{1}{n}}$ is again dense in $\mathrm{CE}(M^4, X)$. However $\bigcap \mathcal{O}_{\frac{1}{n}}$ consists entirely of homeomorphisms and therefore π is **ABH.** In particular, X is a 4-manifold,

as desired. □

3 On the Structure of Metric Balls in 4D G-spaces

As a consequence of Theorem 5.8 above, every $4D$ Busemann G-space is a topological manifold. Since the cone structure of small metric balls $B_r(x)$ played a vital role in establishing Theorem 5.8, one is led to ask whether $B_r(x)$ is a genuine 4-cell. This next result shows that there is at most one decomposition-theoretic obstruction to answering this question in the affirmative.

Theorem 5.11 *Let X^4 be a G-space, let $x \in X$ and let $r > 0$ be less than the cone radius of x. Then $B_r(x)$ is homeomorphic to the 4-cell $\mathbf{B}^4$ if and only if $S_r(x)$ has a cell-like resolution.*

Remark. Recall that for sufficiently small r, the metric sphere $S_r(x)$ is an admissible homology 3-manifold ($\partial S_r(x) = \emptyset$). Although Quinn's local index does not apply to homology 3-manifolds, every known admissible homology 3-manifold has a cell-like resolution and it is a long standing conjecture that such resolutions always exist for $n = 3$, provided Poincaré's Conjecture is true.

Proof. Note that the forward implication is trivial. For the reverse, suppose $S_r(x)$ is resolvable and consider a cell-like surjection

$$\pi : M^3 \longrightarrow S_r(x).$$

Via Proposition 3.6, $S_r(x)$ is locally 2-spherical and thus, via Lemma 5.2, each of the fibers of π satisfy condition (†) of Theorem 5.1 as cell-like subsets of M^3. Although these fibers may not be cellular in M^3, we may apply Daverman's cellular improvement (see [42, page 238]) to obtain a manifold N^3 and a *cellular* map

$$f : N^3 \longrightarrow S_r(x).$$

We claim that f is **ABH.** To this end, it suffices to show that $S_r(x)$ is nicely locally 2-spherical for then Cannon's strong taming theorem ($n = 3$) yields the desired approximation by homeomorphisms. Fix $b \in S_r(x)$ and let $\{Q_\varepsilon(b)\}_{0<\varepsilon<2r}$ be the contractible closed neighborhoods of b in $S_r(x)$ as described in Theorem 2.12. Using the techniques developed by Bryant in [23], we deduce that all but a countable number of the 2-spheres $\partial Q_\varepsilon(b)$ are 1-LCC in $S_r(x)$ (recall that Corollary 4.10 yields that each $\partial Q_\varepsilon(b)$ is 1-LCC in $Q_\varepsilon(b)$). Therefore $S_r(x)$ is nicely locally 2-spherical and thus, via Cannon's Theorem 5.6, f is **ABH**. In particular, $S_r(x)$ is a 3-manifold

Next we claim that $S_r(x)$ is homeomorphic to the 3-sphere. This can be seen as follows. Fix a pair of antipodal points $b_1, b_2 \in S_r(x)$. Since $S_r(x)$ is a homology 3-sphere, we have $H_1\left(S_r(x) - \{b_j\}\right) \cong 0$, for $j = 1, 2$. Applying a Mayer-Vietoris argument to the triple

$$(S_r(x), S_r(x) - \{b_1\}, S_r(x) - \{b_1\})$$

we have that $H_1\left(S_r(x) - \{b_1, b_2\}; \mathcal{Z}\right) \cong 0$,and hence, via [38, Cor.3.5, page 190], we obtain that the partition $G = \{\partial Q_\varepsilon(b_1) |\ 0 < \varepsilon < 2r\}$ is upper semi-continuous. As an application of the Realization Lemma (see [37, page 11]), we see that the quotient space

is homeomorphic to the open interval

$$\frac{S_r(x) - \{b_1, b_2\}}{G} \cong (0, 2r).$$

Since $S_r(x) - \{b_1, b_2\}$ is a 3-manifold, we may apply [38, Cor.7.4, page 202] and obtain a homeomorphism

$$S_r(x) - \{b_1, b_2\} \cong \partial Q_\varepsilon(b_1) \times (-1, 1)$$

for each $0 < \varepsilon < 2r$. In particular $S_r(x) - \{b_1, b_2\} \cong \mathbf{S}^2 \times (-1, 1)$ and therefore $S_r(x) \cong \mathbf{S}^3$.

Name a homeomorphism $h : \mathbf{S}^3 \longrightarrow S_r(x)$ and exploit the cone structures of both $\mathbf{B}^4$ and $B_r(x)$ to extend h naturally to a homeomorphism $H : \mathbf{B}^4 \longrightarrow B_r(x)$. □

CHAPTER 6

4-Manifolds of Non-Positive Curvature

In contrast to a Busemann G-space where segments do not bifurcate, spaces X such that curv $X \leq 0$ allow for the possibility of bifurcating segments. It is precisely these bifurcations that give rise both to an interesting topology on X and to (non-trivial) applications of decomposition theory in the study of this topology.

1 Length Spaces of Non-Positive Curvature

Following Gromov [50], we make the following definition.

Definition 6.1 (length space) *A metric space (X, d) is said to be a length space provided it is a complete, locally compact inner metric space.*

As verified in Chapter 2, any Menger convex, finitely compact metric space (Y, d) is a length space (see Theorem 2.1). The aim of the next result is to establish the converse which will be needed later.

Proposition 6.1 *Let (X, d) be a length space. Then (X, d) is Menger convex and finitely compact.*

Proof. Fix $x \in X$ and a neighborhood U_x of having compact closure. Then for all $r > 0$ sufficiently small, the closed metric ball $B_r(x)$ is compact. Fix such an r and

choose t such that

$$0 < t < 2t < r.$$

The first step is to show that $B_t(x)$ is radially Menger convex for each $x \in X$ (note that finite compactness is immediate). Let $y \in B_t(x) - \{x\}$ and consider the family of paths $\mathcal{C}$ joining x to y in X. By definition,

$$d(x, y) = \inf_{\gamma \in \mathcal{C}} \text{Length}(\gamma)$$

hence we may reduce to the case that images of all $\gamma \in \mathcal{C}$ are contained in $B_r(x)$. Using compactness we may select a path g_y joining x to y and such that $\text{Length}(g_y) = d(x, y)$. Upon reparameterizing g_y with respect to arc length, we obtain a segment $[x, y]$. Therefore $B_t(x)$ is (radially) Menger convex . The above argument is easily improved via the compactness of $B_r(x)$ to show that $B_r(x)$ is itself (radially) Menger convex.

It thus suffices to show that for each positive real number $s > 0$, the metric ball $B_s(x)$ is compact. To this end, let $\mathcal{A} = \{r > 0 | B_r(x) \text{ is compact}\}$ and assume $\mathcal{A}$ is bounded above. Let $r_0 = \sup \mathcal{A}$. the second step is to show that $r_0 \in \mathcal{A}$. Note that by definition of r_0, any point $z \in \text{int}\, B_{r_0}(x)$ can be joined by a radial segment to x. Fix $z \in \partial B_{r_0}(x)$. Applying local compactness of X, we may choose

$$0 < \varepsilon_1 < 2\varepsilon_1 < \varepsilon$$

so that $B_\varepsilon(z)$ is compact. Choose a sequence $\{r_i\}_{i=1}^{\infty} \subset \mathcal{A}$ so that $r_i \to r_0$ as $\to \infty$. Note that the union $B_{r_i}(x) \cup B_{\varepsilon_1}(z)$ is compact for all i. For each i, we may choose, via compactness, $y_i \in B_{r_i}(x)$ so that $d(z, B_{r_i}(x)) = d(z, y_i)$. Necessarily this implies that $d(y_i, z) \to 0$ and hence $y_i \to z$. Note that there is an index $N > 0$ such that $i \geq N$ implies

the segments $[x, y_i] \subset B_N(x) \cup B_{\varepsilon_1}(z)$ which is compact. Hence there is a convergent subsequence of $\{[x, y_{i_k}]\}$ whose limit is the desired segment $[x, z]$. This implies that $B_{r_0}(x)$ is radially Menger convex. To see that $B_{r_0}(x)$ is compact, consider a sequence of points $\{z_n\} \subset B_{r_0}(x)$. Choose radial segments $[x, z_n]$ and choose $y_{2,n} \in [x, z_n]$so that

$$d(x, y_{2,n}) = \frac{1}{2} d(x, z_n).$$

Necessarily the sequence $\{y_{2,n}\} \subset B_{\frac{r_0}{2}}(x)$ which is compact via definition of r_0. By passing to subsequences if necessary, we may assume that $y_{2,n} \to y_0$ as $n \to \infty$. By using a standard diagonalization procedure and the fact that each $[x, z_n]$ is an isometry, we conclude that, after passing to a subsequence, $[x, z_n)$ is uniformly convergent. A straightforward application of the triangle inequality reveals that $\{z_n\}$ is a Cauchy sequence. Since X is complete, this sequence converges to some z_0 and continuity of the distance function insures that $z_0 \in B_{r_0}(x)$.

Finally to see that $\mathcal{A} = (0, \infty)$, note that compactness of $B_{r_0}(x)$ implies there is a compact thickening of $B_{r_0}(x)$ obtained from a finite cover of compact metric balls having radii $< r_0$. $\square$

As we have just seen, each length space X shares two common features with Busemann G-spaces. If we further assume X has curv $X \leq 0$ and is a homology manifold, we can recover the axiom of local prolongability.

Proposition 6.2 *Let (M^n, d) be a homology manifold $(\partial M = \emptyset)$ equipped with a length space metric such that* curv $M^n \leq 0$. *Then the segments of M^n are locally extendible.*

Proof. The curvature condition yields, upon an application of [2, page 19], that the segments of M^n depend continuously on their endpoints. First consider the case that M^n is simply connected. Via [3, Theorem 1], $\pi_1 M^n = \{1\}$, curv $M^n \leq 0$ implies that (M^n, d) is convex in the sense of Definition 1.4 (see page 8). In particular every pair of points in M^n is joined by a unique segment, and every metric ball $B_r(x) \subset M^n$ contracts radially to its center.

Now suppose there is $x \in M^n$ such that for every $\varepsilon > 0$, there exist $y, z \in \text{int}\, B_\varepsilon(x)$ whose unique segment $[y, z]$ has no extension in $B_\varepsilon(x)$. Let $r = d(y, z)$ and consider $B_{2r}(z) - \{y\}$. The natural geodesic contraction of $B_{2r}(z)$ restricts to a contraction of $B_{2r}(z) - \{y\}$ in itself. Since M is a homology manifold, we discover, upon considering the long exact homology sequence of the pair $(B_{2r}(z), B_{2r}(z) - \{y\})$, that $B_{2r}(z) - \{y\}$ has the homology of an $(n-1)$-sphere, which contradicts contractibility.

In the case that $\pi_1 M^n \neq \{1\}$, curv $M^n \leq 0$ implies that M^n is locally contractible and hence has a universal cover $\widetilde{M^n}$. Applying the proof of [3, Thm. 4], we may equip $\widetilde{M^n}$ with a metric $\tilde{d}$ so that $\left(\widetilde{M^n}, \tilde{d}\right)$ is a length space, curv $\widetilde{M^n} \leq 0$ and the covering projection $p : \widetilde{M^n} \longrightarrow M^n$ is a local isometry. Local extendibility in M^n is now immediate. □

Remark. The axiom of local prolongability in a length space X is equivalent to the notion of *geodesic completeness*, i.e., each local isometry $\gamma : [a, b] \to X$ extends to a local isometry $\gamma : (-\infty, \infty) \to X$. Proposition 6.2 may restated to say every non-positively curved homology manifold with empty boundary is geodesically complete.

As an immediate corollary to Proposition 6.2 and the work of Berestovky (see [10]

or [2, Thm. 12.2, page33]) and Nikoleav (see [74] or [2, Thm. 14.1, page 49]) we have the following result.

Corollary 6.3 *Let M^n be an admissible homology manifold ($\partial M = \emptyset$) equipped with a length space metric such that $K_1 \leq \operatorname{curv} M^n \leq K_2$ for real constants $K_1 \leq K_2$, then M^n is a topological manifold. Moreover, M^n has the structure of a smooth manifold.*

Proof. The hypothesis on M^n imply geodesic completeness. The total bound on curvature yields, via [10], that M^n is a manifold. The smooth structure follows from [74]. □

Remark. Corollary 6.3 has been extended in the recent work of Plaut by replacing the upper curvature bound with geodesic completeness (see [76, Thm. F, page 517]) and by allowing for the possibility of manifolds with boundary (see [77, Thm. A, page 334]).

Define a *ray* in a length space X to be an isometry $\rho : [0, \infty) \to X$. For the sake of brevity, a space is said to be *non-positively curved* provided it is a length space satisfying the CAT(0)−inequality.

The following is an immediate corollary to Proposition 6.2.

Corollary 6.4 *Let (M^n, d) be a simply connected, non-positively curved homology manifold. Then each segment in M extends to a ray and M is contractible (to any of its points).*

An important concept associated with non-positively curved spaces is that of *strong convexity* (see [82],[18]).

Definition 6.2 (strongly convex) *Let (X, d) be a metric space. A subset $A \subset X$ is said to be strongly convex provided given any two points $a, b \in A$, there is a unique segment $[a, b]$ joining a to b and this segment is contained in A.*

As was observed in the proof of Proposition 6.2, $\pi_1 X = \{1\}$ and $\operatorname{curv} X \leq 0$ implies that X is strongly convex. The next result asserts that this property is shared by the metric balls in such a space.

Proposition 6.5 *Let (X, d) be simply connected, non-positively curved. Then the following are true for each $x \in X$ and $r > 0$.*

1. $\operatorname{int} B_r(x)$ *is strongly convex.*
2. $B_r(x)$ *is strongly convex.*
3. *Let $y, z \in \partial B_r(x)$. Then* $\operatorname{int}[y, z]$ *is contained in* $\operatorname{int} B_r(x)$.

Proof. Unique segments follows from the hypotheses on X. Applying [3] and [9, Chapter 10, Thm. 7], we have that (X, d) satisfies CAT(0) globally. Thus given $y, z \in \partial B_r(x)$, let T be the geodesic triangle $[x, y] \cup [y, z] \cup [z, x]$ and let T' be the corresponding comparison triangle in $\mathbf{R}^2$. Fix $w \in \operatorname{int}[y, z]$ and let $w' \in [y', z']$ be the corresponding point on T'. Letting $t = d(x', w')$ and applying the law of cosines in $\mathbf{R}^2$, we have

$$t^2 = 2r^2 - 2r^2 \cos\theta < r^2$$

where θ is the angle between $[x', z']$ and $[z', y']$. Thus $d(x, w) < r$, which establishes 2] and 3]. A similar argument establishes 1]. □.

Remark. Metric balls in G-spaces need not be strongly convex, even when joins are globally unique, as was shown by Busemann and Phadke in [30].

2 Topology of Geodesics

The goal of this section is to understand the contraction along rays one naturally obtains in a simply connected, non-positively curved homology manifold.

Definition 6.3 (geodesic contraction) *Let X^n be a simply connected, non-positively curved homology manifold ($\partial X = \emptyset$). For each $r > 0$, name the map given in Corollary 6.4*

$$c_r : X^n - B_r(x) \to S_r(x)$$

i.e., c_r is the map sending a point $y \in X - B_r(x)$ to the point on the unique segment $[x, y]$ which is distance r away from x.

Note that c_r is a continuous deformation retraction $\forall\ r > 0$ and that, for $p > r$, the geodesic contraction can be thought of as a map between concentric metric spheres

$$c_r : S_p(x) \to S_r(x).$$

In a sense, all of the topology of X is contained in the fibers of the map c_r, for, if c_r has trivial fibers for all r, then X is easily seen to be the cone over any $S_r(x)$ from x. When X is a simply connected, non-positively curved 4-manifold, this has special implications for its structure, as we will later see. In particular, such a 4-manifold is necessarily the cone over a simply connected 3-manifold, and hence $\mathbf{R}^4$.

Let X^n be as in Definition 6.3 above and assume X is finite dimensional. As a consequence of Proposition 6.5, we have that X is an admissible homology manifold. It is immediate that, given $x \in X$ and $r > 0, \operatorname{int} B_r(x)$ is also admissible. As a corollary to Mitchell's Theorem [67], we have the following result.

Proposition 6.6 *For each $r > 0$, the closed metric ball $B_r(x)$ is a homology n-manifold whose boundary*

$$\partial B_r(x) = S_r(x)$$

is a homology $(n-1)-$manifold.

Proof. Given $y \in S_r(x)$, it suffices to show that $B_r(x) - \{y]$ is contractible, for then

$$\tilde{H}_*\left(B_r(x), B_r(x) - \{y\}\right) = 0.$$

An application of Mitchell's Theorem will then yield the desired result.

To see that $B_r(x) - \{y\}$ is contractible, note that Proposition 6.5 implies that $B_r(x)$ is strongly convex, hence contractible. The restriction of the natural geodesic contraction of $B_r(x)$ to $B_r(x) - \{y\}$ clearly yields a contraction of $B_r(x) - \{y\}$ in itself. □

The following duality theorem will be used to analyze the topology of the fibers of the maps c_r.

Proposition 6.7 (Lefschetz-Alexander Duality) *Let B be a compact,contractible admissible homology n-manifold with boundary ∂B. Let $A \subset B$ be a closed subset and set*

$$\partial A = A \cap \partial B.$$

Then for all $i = 0, 1, \ldots n$,

$$\tilde{H}_i(B - A; G) \cong \tilde{H}^{n-i-1}(A, \partial A; G) \tag{6.1}$$

with singular homology , Čech cohomology and coefficients in a $\mathcal{Z}-$*module* G.

Proof. Consider the double $D(B, \partial B) = \partial(B \times [0,1])$. Applying [81], we have that $D(B, \partial B)$ is an oriented, admissible homology $n-$manifold with empty boundary and a straightforward Mayer-Vietoris argument coupled with Lefschetz duality as in [93, page 339] yields that $D(B, \partial B)$ is a simply connected homology sphere, denoted as Σ.

Let $T \subset \Sigma$ be the closed set $T = A \times \{0\} \cup \partial A \times [0,1] \cup B \times \{1\}$. Via Alexander duality for homology spheres (see [73, page 424]) we have

$$\tilde{H}_i(\Sigma - T; G) \cong \tilde{H}^{n-i-1}(T; G). \tag{6.2}$$

Consider the long exact cohomology sequence of the pair (T, Y), where $Y = \partial A \times [0,1] \cup B \times \{1\}$. We have that

$$\cdots \to \tilde{H}^{n-i-1}(Y; G) \to \tilde{H}^{n-i}(T, Y; G) \xrightarrow{\cong} \tilde{H}^{n-i}(T; G) \to \tilde{H}^{n-i}(Y; G) \to \cdots$$

since B is contractible. Excision yields that $\tilde{H}^{n-i}(T, Y; G) \cong \tilde{H}^{n-i-1}(A, \partial A; G)$ so that (6.2) becomes

$$\tilde{H}_i(\Sigma - T; G) \cong \tilde{H}^{n-i-1}(A, \partial A; G). \tag{6.3}$$

Note that $\Sigma - T$ deformation retracts to $B - A$, hence (6.3) becomes (6.1), completing the proof. □

For the remainder, all cohomology groups are Čech cohomology and homology is singular homology. If unspecified, the coefficients are assumed to be the integers.

Definition 6.4 (G-acyclic) *Let G be an abelian group.*

i. *A compact, hausdorff space A is said to be G-acyclic provided $\tilde{H}^*(A;G) \cong 0$.*

ii. *A proper surjective map $f : Y \longrightarrow Z$ between locally compact Hausdorff spaces is said to be G-acyclic provided $f^{-1}(y)$ is G-acyclic for all $y \in Y$.*

If the fibers of the maps c_r are all points, then X has a very simple structure, that of a cone. This next result asserts that the fibers of c_r, if not points, at least have the cohomology of points.

Corollary 6.8 *Let X^n be a finite dimensional, simply connected, non-positively curved homology n-manifold and let G be a $\mathcal{Z}$-module. For each $x \in X$ and for all $p > r > 0$, the geodesic contraction*

$$c_r : S_p(x) \to S_r(x)$$

is an G-acyclic map between homology $(n-1)$-manifolds.

Proof. Fix $z \in S_r(x)$ and consider the compactum $c_r^{-1}(z)$. In $B_p(x)$ let $z *_g c_r^{-1}(z)$ denote the singular geodesic cone of $c_r^{-1}(z)$ from z. Via Proposition 6.5, it is easy to see that

$$B_p(x) - z *_g c_r^{-1}(z)$$

is contractible in itself (simply restrict the natural geodesic contraction). Applications of Proposition 6.6 and Proposition 6.7 (with $A = z *_g c_r^{-1}(z)$) yield

$$0 \cong \tilde{H}_i(B_p(x) - z *_g c_r^{-1}(z); G) \cong \tilde{H}^{n-i-1}(z *_g c_r^{-1}(z), c_r^{-1}(z); G).$$

Now use the contractibility of $z *_g c_r^{-1}(z)$ and the long exact cohomology sequence of the pair $(z *_g c_r^{-1}(z), c_r^{-1}(z))$ to see that $\tilde{H}^*(c_r^{-1}(z)) \cong 0$,as desired. □

3 Acyclic Maps of (4−1)-Manifolds

This section is essentially a survey of the study of acyclic maps of 3-manifolds initiated by McMillan (see [65],[66]) and continued by Wright (see [94] and [61]), Row [85] and others.

For each integer p, let $\mathcal{Z}_p$ denote $\frac{\mathcal{Z}}{p\mathcal{Z}}$, if p is a positive prime and $\mathcal{Z}$ otherwise.

Definition 6.5 (McMillan's strong acyclicity) *Let M^n be a manifold and p, j be integers. A compact, connected set $X \subset M^n$ satisfies $uv^j(\mathcal{Z}_p)$ if for any neighborhood $U \supset X$ in M there is a neighborhood $V \supset X$ in U so that inclusion induces the trivial homomorphism of homology groups*

$$i_* : \tilde{H}_k(V; \mathcal{Z}_p) \to \tilde{H}_k(U; \mathcal{Z}_p)$$

for all $k = 0, 1, \ldots, j$.

McMillan showed in [66] that property $uv^j(\mathcal{Z}_2)$ in particular had important implications regarding maps defined on 3-manifolds. He showed, under suitable hypothesis on the range, that a map defined on a 3-manifold all of whose fibers satisfy $uv^3(\mathcal{Z}_2)$ is almost cell-like. McMillan's work was extended by Wright in the following result.

Theorem 6.9 (Wright, [94],[61]) *Let M^3, N^3 be manifolds and consider*

$$f : M^3 \to N^3$$

a proper, continuous surjection. If each fiber $f^{-1}(y) \subset M^3$ *is* $uv^3(\mathcal{Z}_2)$*, then the collection of points*

$$C = \{y \in \mathbb{N}^3 \mid \ f^{-1}(y) \text{ not cellular in } M^3\}$$

is a locally finite subset of M^3*.*

Remark. This example, due to McMillan, shows that Wright's Theorem may not be extended to dimensions larger than 3. Choose an n-manifold Σ^n that is a homology sphere so that $\pi_1\Sigma^n \neq \{1\}$. As in [49, page 146], Σ bounds a compact, contractible $(n+1)$-manifold N^{n+1}. Via the solution to Poincare's conjecture when $n+1 \geq 4$, the double $D(N, \Sigma)$ is topologically an $(n+1)$-sphere. Find a bi-collared neighborhood $U \cong \Sigma \times [-1, 1] \subset D(N, \Sigma)$. Choose an open n-cell $\operatorname{int} D^n \subset \Sigma$ and consider the quotient map

$$f : D(N, \Sigma) \longrightarrow \frac{D(N, \Sigma)}{G_f}$$

where the non-degenerate elements of G_f are the components of $D(N, \Sigma) - \Sigma \times (-1, 1)$ together with the sets $(\Sigma - \operatorname{int} D^n) \times \{t\}$, for $t \in [1-, 1]$. Note that these sets appear to be points when viewed homologically. The image of f is easily seen to be the suspension of an $n-$sphere and hence an $(n+1)-$sphere, however the sets $(\Sigma - \operatorname{int} D^n) \times \{t\}$ are not cellular in $D(N, \Sigma)$. For cellularity implies $\pi_1(\Sigma - \operatorname{int} D^n) \cong \{1\}$ (see [37, page 41]) and an application of Van Kampen's then reveals Σ to be simply connected.

The next result due to McMillan [66] yields a useful method of detecting property $uv^1(\mathcal{Z}_p)$ for compact, connected subset of orientable 3-manifolds which will be exploited.

Theorem 6.10 (McMillan's Acyclicity Criterion). *Let M^3 be an orientable, pl-manifold. Let X be a continuum in M^3. Let p be either a positive prime or zero. Then:*

i. *X is $uv^1(\mathcal{Z}_p) \iff$ $\exists$ a neighborhood $U \supset X$ in M^3 satisfying*

a. *Each polyhedral simple closed curve $J \in U$ is $\mathcal{Z}_p-$homologous to zero in M^3.*

b. *Each polyhedral simple closed curve $J \subset U - X$ is $\mathcal{Z}_p-$homologous to zero in $M^3 - X$*

ii. *If X is $uv^1(\mathcal{Z}_p)$, then X is $uv^3(\mathcal{Z}_p) \Leftrightarrow$ there is a connected neighborhood $U \supset X$ satisfying*

c. *Each polyhedral 2-sphere $\Sigma^2 \subset U$ separates U.*

d. *For all $p, q \in U - X$,there is an open $V \supset X$ in U such that no polyhedral 2-sphere $\Sigma^2 \subset V$ separates p from q in U.*

We conclude this section by recording a result of McMillan [66] which relates property $uv^1(\mathcal{Z}_p)$ and the notion of cell-likeness.

Theorem 6.11 (Detecting cell-likeness). *Let $f : M^n \to Y$ be a monotone mapping of an n-manifold onto a non-degenerate Hausdorff space such that each fiber of f satisfies $uv^1(\mathcal{Z}_p)$, where p is a positive prime or zero. Suppose there is a $y \in Y$ with the following property:*

(†) *For each neighborhood $U \supset f^{-1}(y)$ in M there is an open neighborhood $V \supset f^{-1}(y)$ such that each loop $\alpha : S^1 \to V$ has null homotopic image $S^1 \xrightarrow{\alpha} V \xrightarrow{f} f(U)$.*

Then if $f^{-1}(y)$ embeds in $\mathbf{R}^3$, $f^{-1}(y)$ has property UV^1.

Remark: Theorem 6.11 detects cell-like sets in the following manner. Assume that $n \geq 3$ and Y is locally contractible (which then automatically yields condition †) . Set $X = f^{-1}(y)$ where and assume X is $\mathcal{Z}$-acyclic and X embeds in $\mathbf{R}^3$. Theorem 6.11 implies that X has property UV^1. Via a theorem of Lacher (see [60] or [37, page 125]), X acyclic and UV^1 implies X is UV^n for all n and hence cell-like, by [37, Cor. 16. 4A, page 125].

4 A 3-Manifold Recognition Theorem

The goal of this section is to establish Theorem 1.5 (see page 10) and the techniques developed in this section will be generalized in the section to follow. An important concept (see [37, page 232]) is that of a *sliced decomposition.*

Definition 6.6 (sliced decomposition, sliced maps)

1. *A decomposition $\mathcal{D}$ of $X \times \mathbf{R}^1$ is said to be sliced if each element of $\mathcal{D}$ is contained in some slice $X \times \{t\}$, where $t \in \mathbf{R}^1$.*
2. *A map $f : X \times \mathbf{R}^1 \longrightarrow Y$ is said to be sliced provided each fiber $f^{-1}(y)$ is contained in some slice $X \times \{t\}$, where $t \in \mathbf{R}^1$.*
3. *If $f : X \times \mathbf{R}^1 \longrightarrow Y$ is sliced, the level maps of f are the maps induced by restricting f to $X \times \{t\}$, where $t \in \mathbf{R}^1$.*

The following result is due to Daverman and Preston [41, 1980].

Theorem 6.12 (Daverman-Preston sliced shrinking) *Let $f : X \times \mathbf{R}^1 \longrightarrow Y$ be a proper, continuous sliced surjection. If each of the level maps of f are* **ABH**, *then f is* **ABH**.

We now turn to the task of establishing a 3-manifold recognition theorem.

Proof. (Theorem 1.5) As we noted earlier, it suffices to establish the reverse implication. To this end, let (Y, d) be a finite dimensional, locally compact, homology 3-manifold metrized as a length space with $\operatorname{curv} Y \leq K < \infty$. Note that by Proposition 6.1, Y is separable. Apply [9, page 193] to see that for each $y \in Y$, there is an $0 < p < \frac{\pi}{\sqrt{K}}$ for which the closed metric ball $B_p(x)$ is strongly convex. Via [9, page 190], segments in $B_p(x)$ depend continuously on their endpoints. Hence the radial contraction provides a contraction of $B_p(x)$. Applying Mitchell's Theorem, we observe that for each $0 < t \leq p$, the metric sphere $S_t(x)$ is a compact, connected, locally connected homology 2−manifold which bounds $B_t(x)$. Arguing as in Corollary 2.13, we have that $S_t(x)$ is a 2-sphere, for all $0 < t \leq p$.

Consider the restriction of the radial contraction

$$c_r : S_t(x) \longrightarrow S_r(x), \text{ for } t > r > 0.$$

Apply Corollary 6.8 to see that the fibers of this map are $\mathcal{Z}$-acyclic, and in particular the fibers of c_r fail to separate the 2-sphere $S_t(x)$. Arguing as in [37, Cor.15.4C, page 122], we see that each of the maps c_r is cellular, and hence **ABH** via Theorem 5.3.

Define a sliced map $f : S_p(x) \times (0, p) \longrightarrow \operatorname{int} B_p(x) - \{x\}$ via the rule $f(y, t) = c_t(y)$. The level maps of f are the maps $c_r : S_p(x) \longrightarrow S_r(x)$, which are **ABH.** Apply the Daverman-Preston Theorem to see that $\operatorname{int} B_p(x) - \{x\}$ is homeomorphic to $\mathbf{S}^2 \times (0, p)$ and hence $\operatorname{int} B_p(x) \cong \mathbf{R}^3$. Thus Y is a 3-manifold. □

There are two main applications of Theorem1.5. For the first application, recall the respective definitions of *tame points* given on page 9.

Corollary 6.13 *Let M^4 be a simply connected, non-positively curved 4-manifold. If $x \in M^4$ is a metrically tame point, then it is topologically tame.*

In light of this result, we say that a simply connected, non-positively curved 4-manifold is *tame at x* provided it is either metrically or topologically tame at x. Such an x is said to be a *tame point* of the manifold.

The second application regards the nature of cone constructions. Following [2, Def.4.3.1], we make the following definition.

Definition 6.7 (K-cone) *Let (Ω, α) be a metric space and let $K \leq 0$ be given. The K-cone over Ω is the space obtained as the image of the quotient map*

$$q : \Omega \times [0, \infty) \longrightarrow \frac{\Omega \times [0, \infty)}{\Omega \times \{0\}}$$

(i.e. the open cone $\overset{\circ}{\triangle}\,\Omega$) and equipped with the following cone metric, *ρ_k.*

Case K = 0: *For $v, w \in \Omega$, $t, s \in [0, \infty)$ define $\rho_0([x,t],[y,s])$ via*

$$\rho_K([v,t],[w,s]) = \begin{cases} \sqrt{t^2 + s^2 - 2st\cos\alpha(v,w)} & , \text{if } \alpha(v,w) \leq \pi \\ t + s & , \text{if } \alpha(v,w) > \pi \end{cases}$$

Case $K < 0$: *For $v, w \in \Omega$, $t, s \in [0, \infty)$ define $\rho_K([x,t],[y,s])$ via*

$$\begin{cases} \frac{1}{K}\cosh^{-1}\left(\cosh Kt \cosh Ks - \sinh Kt \sinh Ks \cos\alpha(v,w)\right) & , \text{if } \alpha(v,w) \leq \pi \\ t + s & , \text{if } \alpha(v,w) > \pi \end{cases}$$

It is helpful to think of the K-cone as a "vector" space where each "vector" has a direction component coming from the metric space (Ω, α), and a magnitude determined by $[0, \infty)$.

The following result is due to Berestovsky (see [2, Lemma 4.1], or [12]).

Proposition 6.14 (Berestovsky) *Let* (Ω, α) *be a compact length space. Then* $\operatorname{curv} \Omega \leq 1$ *if and only if the* K*-cone over* $\Omega, (\overset{\circ}{\triangle}\, \Omega, \rho_k)$ *has curvature* $\leq K$.

Here is the second application of the 3-manifold recognition theorem.

Corollary 6.15 *Let* (Ω, α) *be a compact length space and* $K \leq 0$. *If the* K*-cone over* Ω *yields a non-positively curved* 4*-manifold* M^4, *then* $M^4 \cong \mathbf{R}^4$ *and* Ω *is a simply connected* 3*-manifold.*

Proof. Let ω denote the cone point in M^4. Apply Mitchell's Theorem and Proposition 6.14 to see that Ω is a homology 3-manifold with $\operatorname{curv} \Omega \leq 1$. Now apply Theorem 1.5 to see that Ω is a 3-manifold. Ω is simply connected since M^4 is contractible and hence M^4 is simply connected at ∞. Now apply [48] □

Remark. In the topological category, it is possible to obtain $\mathbf{R}^4$ from the cone over a non-manifold [37, page 206]. For example let A be a non-cellular arc in $\mathbf{S}^3$ and consider the space Y obtained from crushing A to a point. Via a result due to Andrews and Curtis [6], the open cone over Y is topologically $\mathbf{R}^4$. As a consequence of Corollary 6.15, we see that Y may not be given the geometric structure of a length space having $\operatorname{curv} Y \leq 1$.

5 Metric Spheres about a Tame Point are Bi-Collared

As noted on page 10, in order to establish Theorem 1.3 and Theorem 1.4, we first show that the metric spheres about a tame point are bi-collared (i.e. locally flatly em-

bedded). To this end, let M^4 be a simply connected, non-positively curved 4−manifold and let $x \in M^4$ be a (topologically or metrically) tame point.

Paraphrasing, this next result shows that the metric spheres are collared on the inside.

Lemma 6.16 *M^4 as above, $x \in M^4$ tame.*

1. *Each metric sphere $S_r(x)$ is 1-LCC embedded in the closed metric ball $B_r(x)$.*
2. *Each closed metric ball $B_r(x)$ is a 4-manifold with boundary $S_r(x)$.*

Proof. For 1], note that the geodesic contraction provides a singular regular neighborhood for $S_r(x)$ in $B_r(x)$. Now apply Daverman's Theorem on singular regular neighborhoods (Theorem 4.2). We obtain 2] , after doubling $B_r(x)$ along $S_r(x)$, as an application of Freedman-Quinn 1-LCC flattening (Theorem 5.5). □

To verify that the $S_r(x)$ are collared on the outside, we will exploit McMillan's theory of acyclic maps. Let us proceed with the following Lemma.

Lemma 6.17 *Fix $p > r > 0$. Then each fiber of $c_r : S_p^3(x) \to S_r^3(x)$ is $uv^3(\mathcal{Z}_2)$ in $S_p^3(x)$.*

Proof. Since M^4 is a 4-manifold, each 3-manifold $S_r(x)$ necessarily has the homology of a 3-sphere and is therefore orientable. Apply Corollary 6.8 and conclude that the fibers of c_r are G−acyclic for every abelian group G. Hence $c_r^{-1}(z)$ is compact and connected for all $z \in S_r^3(x)$. The result will follow as a consequence of McMillan's acyclicity criterion (Theorem 6.10).

To see that $c_r^{-1}(z)$ is $uv^1(\mathcal{Z}_2)$ in $S_p^3(x)$, note that via universal coefficients

$$H_1(S_p^3(x);\mathcal{Z}_2) \cong H_1\left(S_p^3(x)\right) \otimes \mathcal{Z}_2 \ \oplus \ \mathrm{Tor}\,(\mathcal{Z},\mathcal{Z}_2) \cong 0 \oplus 0$$

and via duality and the $\mathcal{Z}_2-$acyclicity of $c_r^{-1}(z)$ we have

$$H_1\left(S_p^3(x) - c_r^{-1}(z);\mathcal{Z}_2\right) \cong H^2\left(c_r^{-1}(z);\mathcal{Z}_2\right) \cong 0.$$

As an application of Theorem 6.10, $c_r^{-1}(z)$ is $uv^1(\mathcal{Z}_2)$ in $S_p^3(x)$.

To see that $c_r^{-1}(z)$ is $uv^3(\mathcal{Z}_2)$ in $S_p^3(x)$, let Σ^2 be a polyhedral 2-sphere in $S_p^3(x)$. Via duality

$$H_0\left(S_p^3(x) - \Sigma^2\right) \cong H^3\left(S_p^3(x),\Sigma^2\right) \neq 0$$

and thus every such Σ^2 separates. Finally, given $y_0, y_1 \in S_p^3(x) - c_r^{-1}(z)$, choose an open neighborhood V of $c_r^{-1}(z)$ in $S_p^3(x) - \{c_r^{-1}(y_0), c_r^{-1}(y_1)\}$ by setting $V = c_r^{-1}(\mathrm{int}\, D^3)$, where $\mathrm{int}\, D^3$ is a sufficiently small 3-cell neighborhood of z in $S_r(x)$. Apply the Vietoris-Begle mapping theorem ([88, Thm. 6.9.18]) and conclude that the restriction of c_r induces isomorphisms on compactly supported cohomology

$$c_r^* : H_c^*\left(\mathrm{int}\, D^3\right) \xrightarrow{\cong} H_c^*(V).$$

Via duality, $0 \cong H_c^1(V) \cong H_2(V)$. No polyhedral 2-sphere $\Sigma^2 \subset V$ separates y_0 from y_1 in $S_p^3(x)$ since Σ^2 represents a bounding 2-cycle in V and hence in $S_p^3(x) - \{y_0, y_1\}$. Via Theorem 6.10, $c_r^{-1}(z)$ is $uv^3(\mathcal{Z}_2)$ in $S_p^3(x)$. $\square$

We are now in position to establish the main result of this section.

Theorem 6.18 *Let M^4 be a simply connected, non-positively curved $4-$manifold which is tame at $x \in M^4$. Then for each $r > 0$, the metric sphere $S_r^3(x)$ is bicollared in M^4.*

Proof. First we show that $S^3_r(x)$ is 1-LCC embedded in $M^4 - \operatorname{int} B_r(x)$. To this end, let $r > 0$ be given. Choose $p > r$ and consider the geodesic contraction

$$c_{p,r} : S^3_p(x) \to S^3_r(x)$$

Apply Lemma 6.17 and Wright's theorem to see that the set of points

$$\Upsilon = \{z \mid c^{-1}_{p,r}(z) \text{ is not cellular in } S^3_p(x)\}$$

is finite. Fix $z_0 \in \mathbf{S}^3_r(x) - \Upsilon$ and choose an open 3-cell $\operatorname{int} D^3$ about z_0 missing Υ. Then the restriction of the geodesic contraction

$$c_{p,r} \mid : c^{-1}_{p,r}(\operatorname{int} D^3) \to \operatorname{int} D^3$$

is a cellular map between 3-manifolds. Via Theorem 5.3, $c_{p,r} \mid$ is approximable by homeomorphisms. In particular, $c^{-1}_{p,r}(\operatorname{int} D^3) \cong \mathbf{R}^3$.

Consider the open annular region, $A_{rp}(\operatorname{int} D^3)$ given by

$$A_{rp}(\operatorname{int} D^3) = \{y \in B_p(x) \mid t = d(x,y) \in (r,p) \text{ and } c_{p,t}(y) \in \operatorname{int} D^3\}.$$

Define a sliced map $f : c^{-1}_{p,r}(\operatorname{int} D^3) \times (r,p) \longrightarrow A_{rp}(\operatorname{int} D^3)$ via

$$f(y,t) = c_{p,t}(y).$$

We claim that f is **ABH.** In light of Daverman-Preston sliced shrinking, it suffices to show that each of the level maps of f is **ABH.** For each $t \in (r,p)$, note that the t-level map of f is simply the surjective map given by

$$f| = c_{p,t} : c^{-1}_{p,r}(\operatorname{int} D^3) \subset S_p(x) \longrightarrow c^{-1}_{t,r}(\operatorname{int} D^3) \subset S_t(x).$$

Hence each level map is an acyclic map between 3-manifolds whose fibers satisfy $uv^3(\mathcal{Z}_2)$, via Lemma 6.17. Applying Theorem 6.11 (and the remark which immediately follow), we see that this level map is a cell-like map between 3-manifolds. As the domain of each of the level maps is topologically $\mathbf{R}^3$, we have, via McMillan's cellularity criterion, that each level map is a cellular map between 3-manifolds. Now apply Theorem 5.3 to see that each level map is **ABH** and hence f is itself **ABH**. Fix an approximating homeomorphism

$$h : c_{p,r}^{-1}(\operatorname{int} D^3) \times (r,p) \longrightarrow A_{rp}(\operatorname{int} D^3). \tag{6.4}$$

Let $y \in \operatorname{int} D^3 \subset S_r(x)$. Next we claim that $S_r(x)$ is 1-LCC in $M^4 - \operatorname{int} B_r(x)$ at the point y. To see this, let $U_y \subset M^4 - \operatorname{int} B_r(x)$ be given. Using the homeomorphism of (6.4), choose a neighborhood $V_y \subset U_y$ so that $V_y - S_r(x) \cong \operatorname{int} D^3 \times (a,b)$. Since $\pi_1(V_y - S_r(x)) \cong \{1\}$, the metric sphere is 1-LCC at y as claimed.

The above argument shows that $S_r(x) - \Upsilon$ is 1-LCC in $M^4 - \operatorname{int} B_r(x)$, where Υ is a finite set. Apply 1-LCC flattening to get that $S_r(x) - \Upsilon$ is locally collared in $M^4 - \operatorname{int} B_r(x)$. However by Kirby's flattening theorem (see [57] or [86, Section 4.9]) the set of points at which $S_r(x)$ is not locally collared contains no isolated points. Therefore $S_r(x)$ is collared in $M^4 - \operatorname{int} B_r(x)$.

Applying Lemma 6.16, we have that $S_r(x)$ is bi-collared in M^4, as desired. □

The final step in the proof of Theorems 1.3 and 1.4 is the next result which is established as an application of Daverman's *stretching bi-collars* technique (see [38, page 191]).

Theorem 6.19 *Let M^4 be a simply connected, non-positively curved 4−manifold which*

is tame at $x \in M^4$ *and fix* $r > 0$.

a] $M^4 - \{x\} \cong S_r(x) \times (0, \infty)$

b] $M^4 \cong \mathbf{R}^4$

Proof. a] Consider the map $q : M^4 - \{x\} \longrightarrow \mathbf{R}^1$ given by

$$y \longmapsto d(x, y).$$

The fibers of q are precisely the bicollared metric spheres. Choose a embedding

$$\phi : S_r(x) \times [r, r + \varepsilon] \longrightarrow M^4 - \operatorname{int} B_r(x) \text{ for some } \varepsilon > 0.$$

Let $\mathcal{H}(M_+)$ be the set of all self homeomorphisms of $M^4 - \operatorname{int} B_r(x)$ having compact support and which are the identity on some fixed neighborhood U of $S_r(x)$. Set

$$A = \{t \in [r, \infty)\} \text{there exists } h \in \mathcal{H}(M_+) \ni \ q^{-1}([r, t]) \subset h(\operatorname{im} \phi)\}.$$

It is immediate that A is connected. Using the identity map 1_{M_+} and the collar ϕ, we see that A is non-empty.

Claim: $A = [r, \infty)$.

Reasons: Suppose that A were bounded above and let $p = \sup A$. Let $M^4_+ = M^4 - \operatorname{int} B_r(x)$ and name an embedding

$$\psi : S_p(x) \times [p - \delta, p + \delta] \longrightarrow M_+, \text{ for some } \delta > 0$$

parameterized so that $q^{-1}\left([p - \frac{\delta}{2}, p + \frac{\delta}{2}]\right) \subset \operatorname{im} \psi$. Choose an element $h \in \mathcal{H}(M_+)$ such that

$$q^{-1}\left([r, p - \frac{\delta}{2}]\right) \subset h(\operatorname{im} \phi).$$

Define a homeomorphism $\theta \in \mathcal{H}(M_+)$ in the following manner. First find a piecewise linear homeomorphism $\beta : [p-\delta, p+\delta] \to [p-\delta, p+\delta]$ fixed at the end points and sending $p - \frac{\delta}{2}$ to $p + \frac{\delta}{2}$. Now define $\theta : \operatorname{im}\psi \longrightarrow \operatorname{im}\psi$ via $\theta(\psi(y,t)) = \psi(y, \beta(t))$ and extend θ via the identity to M^4_+.

Finally observe that $q^{-1}([r, p+\gamma]) \subset \theta \circ h(\operatorname{im}\phi)$ for some $\gamma > 0$, contradicting the assumption that A was bounded.

A symmetric argument involving $M^4_- = B_r(x) - \{x\}$ in place of M^4_+ yields a monotone union

$$\overset{\bullet}{M}{}^4 = \bigcup_{n=1}^{\infty} \Phi^n(S_r(x) \times [-n, n]),$$

where $\overset{\bullet}{M}{}^4 = M^4 - \{x\}$, and each $\Phi^n \in \mathcal{H}\left(\overset{\bullet}{M}{}^4\right)$ satisfies

B_1. $\Phi^n(S_r(x) \times 0) = S_r(x)$

B_2. $\Phi^n = 1$, on U

B_3. $\Phi^n(S_r(x) \times [-n, n]) \subset \Phi^{n+1}(S_r(x) \times (-n-1, n+1))$

B_4. $\Phi^{n+1}(S_r(x) \times \{\pm n\}) \subset \overset{\bullet}{M}{}^4 - \Phi^n(S_r(x) \times [-n, n])$

It is a corollary to the techniques developed by Brown in [20, 1961] that

$$\overset{\bullet}{M}{}^4 \cong S_r(x) \times (-\infty, \infty).$$

For completeness sake, the details follow.

Given $n \geq 1$, let S_n denote $\Phi^n(S_r(x) \times \{n\})$, let $S_0 = \Phi({}^1S_r(x) \times \{0\})$and let $[S_n, \Phi^{n+1}(S_n)]$ be the obvious annular region bounded by S_n and$\Phi^{n+1}(S_n)$. Via B_4 we have

$$M^4_+ = [S_0, S_1] \cup [S_1, \Phi^2(S_1)] \cup \cdots \cup [S_n, \Phi^{n+1}(S_n)] \cup \cdots$$

and hence it suffices to show that each annular region $[S_n, S_{n+1}]$ is homeomorphic to $S_r(x) \times [n, n+1]$. Trivially we have that $[S_0, S_1]$ is a product.

Fix $n \geq 1$, and consider the restriction $\Phi^{n+1}| \, S_r(x) \times [-n-1, n+1]$. Denote the image of the restriction $\Phi^{n+1}|$ by $Q^{n+1}$and note that, via B_3, we have that $S_n \subset \operatorname{int} Q^{n+1}$. Let $f : S_n \times [0,1] \to \operatorname{int} Q^{n+1}$ be a collar on S_n satisfying

1. $f(S_n \times \{0\}) = Sn$
2. $f(S_n \times [0,1]) \bigcap \Phi^{n+1}(S_n) = \emptyset$
3. $f(S_n \times [0,1]) \bigcap C_n = \emptyset$,

where C_n is the bounded component of $M^4 - S_n$. Use the product structure of Q^{n+1} to define a homeomorphism $g : Q^{n+1} \longrightarrow Q^{n+1}$ satisfying

4. $g(S_n) \subset U$
5. $g \circ f(S_n \times \{\frac{1}{2}\}) = S_n$
6. g is the identity on $\Phi^{n+1}(S_n)$.

Finally, consider the composition $g^{-1} \circ \Phi^{n+1} \circ g \circ f : S_n \times [0, \frac{1}{2}] \longrightarrow Q^{n+1}$. Observe that

$$\begin{aligned} g^{-1} \circ \Phi^{n+1} \circ g \circ f \left(S_n \times [0, \tfrac{1}{2}]\right) &= g^{-1} \circ \Phi^{n+1} \circ g \left(\left[S_n, f(S_n \times \{\tfrac{1}{2}\}\right]\right) \\ &= g^{-1} \circ \Phi^{n+1} ([g(S_n), S_n]) \\ &= g^{-1} ([g(S_n), \Phi^{n+1}(S_n)]) \\ &= [S_n, \Phi^{n+1}(S_n)] \end{aligned}$$

which shows that the annular region $[S_n, \Phi^{n+1}(S_n)]$ is a product and hence $M^4_+ \cong S_r(x) \times [r, \infty)$. The symmetry in the argument reveals that $M^4 - \{x\} \cong S_r(x) \times (-\infty, \infty)$.

b] Use general position in a neighborhood of x to see that

$$\{1\} \cong \pi_1\left(M^4\right) \cong \pi_1\left(M^4 - \{x\}\right) \cong \pi_1\left(S_r(x) \times (-\infty, \infty)\right) \cong \pi_1\left(S_r(x)\right).$$

It follows that M^4 is simply connected at infinity. Applying Freedman's characterization of $\mathbf{R}^4$ ([48, Cor.1.2]), we have M^4 contractible implies $M^4 \cong \mathbf{R}^4$. $\square$

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VITA

Paul Donald Thurston was born June 5, 1964 in El Paso, Texas. After six months in Texas, the Thurston family began a series of moves to locales including Las Cruces, New Mexico, Boston, Massachusetts, Kwajalein Island (a U.S. Trust Territory of the Pacific), and Greensboro, North Carolina. The wandering Thurstons finally settled in Cabot, Arkansas in April, 1974.

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Paul came to Knoxville as a graduate student in August, 1985 and began studying analysis at the Department of Mathematics. However the lure of topology was too seductive and Paul switched to that field of study in the fall of 1989.

Paul married Marilyn McPherson on March 18, 1989. They have a son, Richard, who was born October 3, 1992.

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