

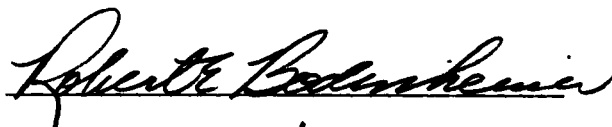
T125
58

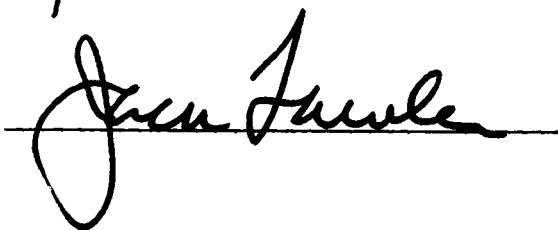
To the Graduate Council:

I am submitting herewith a thesis written by Lang Hong entitled "Techniques of Singular Perturbation for System Applications." I have examined the final copy of this thesis for form and content and recommend that it be accepted in partial fulfillment of the requirements for the degree of Master of Science, with a major in Electrical Engineering.


James C. Hong, Major Professor

We have read this thesis
and recommend its acceptance:





Accepted for the Council:


The Graduate School

TECHNIQUES OF SINGULAR PERTURBATION
FOR SYSTEM APPLICATIONS

A Thesis
Presented for the
Master of Science
Degree
The University of Tennessee, Knoxville

Lang Hong
August 1986

ACKNOWLEDGEMENTS

The author would like to thank Dr. J. C. Hung, the major professor, for his inspiration and guidance throughout this research. Special acknowledgements go to Drs. J. S. Lawler and R. E. Bodenheimer for their participation on the thesis committee and for their invaluable advice.

ABSTRACT

This thesis discusses the techniques of singular perturbation from the engineer's point of view. The technique of singular perturbation is useful for system analysis, order reduction, and system control. An explicit form of a singularly perturbed system is derived from a two-time-scale system by properly scaling. By the quasi-steady state approximation or the L-K transformation, two lower order subsystems, slow and fast mode subsystems, are obtained. The properties of the full order systems, such as stability, controllability, and observability, can be deduced from two lower order systems. Two formulas developed by the author for constructing L and K matrices directly from the eigenvalues are presented. Quasi-steady state approximation and the L-K transformation are evaluated and compared in their applications to order reduction. A modified L transformation is developed here to overcome the existing weaknesses. The effectiveness and merits of three methods for order reduction are shown by the numerical examples.

TABLE OF CONTENTS

CHAPTER	PAGE
1. INTRODUCTION	1
2. SINGULAR PERTURBATION AND TIME SCALE	3
Two-time-scale Systems	3
A Conventional Method of Solving Singularly Perturbed Systems	6
Singular Perturbation and Time Scale	7
3. PROPERTIES OF SINGULARLY PERTURBED SYSTEMS ..	13
Stability	13
Controllability	13
Observability	15
4. TIME SCALE DECOMPOSITIONS	16
Two-time-scale Decomposition Via the L-K Transformation	16
Multi-time-scale Decomposition	25
5. ORDER REDUCTION VIA SINGULAR PERTURBATION ...	28
Order Reduction by the Quasi-steady State Approximation	28
Order Reduction by the L-K Transformation .	34
Order Reduction by the Modified L Transformation	38
Remarks	44

CHAPTER	PAGE
6. CONCLUSIONS	47
LIST OF REFERENCES	49
VITA	53

CHAPTER 1

INTRODUCTION

Singularly perturbed systems due to the presence of some small parameters, such as small time constants, masses, moments of inertia, and parasitic capacitances and inductances, present engineers computational difficulties. The technique of singular perturbation was originally developed to deal with such systems.

In the early days, the singular perturbation appeared in the form of a nearly steady state approximation [1, 8,9,10]. The chief purpose of singular perturbation at that time was to alleviate the high dimensionality and ill conditioning caused by the presence of small parameters. The beauty of nearly steady state approximation is that it keeps the dc gain of the full order system in the slow mode subsystem. This is important to order reduction.

Another technique of singular perturbation, called L-K transformation, was later developed to decompose a high order system into a block diagonal system consisting of two decoupled subsystems which are exact. The L-K transformation is good not only for singularly perturbed systems, but also for any two-time-scale systems [6,13,16,17, 18]. Since two-time-scale systems include singularly perturbed systems, they have been studied thoroughly in

recent years [2,3,4,5,6]. In this thesis , two-time-scale systems are studied primarily.

Today, the technique of singular perurbation have been applied to many areas of control theory including state feedback, output feedback, and filter and observer designs of linear or nonlinear systems. This thesis discusses only the basic theory of the technique of singular perurbation and its application to order reduction.

CHAPTER 2

SINGULAR PERTURBATION AND TIME SCALE

In practice, the physical systems usually involve widely changing dynamics. The simplification of such systems is based on the assumption that during the fast transients, the slow variables remain unchanged and that by the time when their changes become noticeable, the fast transients have already reached the states which are nearly steady states, so called quasi-steady states. The only variables used for short term studies are the fast variables. In the long term studies, the model is formed by the slow variables and the quasi-steady states of the fast variables. As will be shown later, the singularly perturbed system, in fact, is the system involving the fast and slow modes, in which the perturbation is the ratio of the time scales of the slow and fast modes [1].

2-1. Two-time-scale Systems

The time-scale properties of a linear time invariant system can be determined by its spectrum. Generally speaking, a linear time invariant system, such as

$$\begin{bmatrix} \dot{x} \\ \dot{z} \end{bmatrix} = \begin{bmatrix} A & B \\ C & D \end{bmatrix} \begin{bmatrix} x \\ z \end{bmatrix} \quad (2-1)$$

where x is a r -vector and z is an $(n-r)$ -vector, is said to be a two-time-scale system, if it can be transformed into a block-triangular system [2]

$$\begin{bmatrix} \dot{x} \\ \dot{z} \end{bmatrix} = \begin{bmatrix} \hat{A} & B \\ 0 & \hat{D} \end{bmatrix} \begin{bmatrix} x \\ z \end{bmatrix} \quad (2-2)$$

such that

$$\max_i |\lambda_i(\hat{A})| < \min_j |\lambda_j(\hat{D})|. \quad (2-3)$$

The difference between the smallest eigenvalue of $\hat{D}$, $\min_j |\lambda_j(\hat{D})|$, and the largest eigenvalue of $\hat{A}$, $\max_i |\lambda_i(\hat{A})|$, is called the eigenvalue gap [3,4]. If the similar eigenvalue gaps exist in $\hat{A}$ and $\hat{D}$, then the system (2-1) is a multi-time-scale system [5]. The scaling parameter ϵ is defined as

$$\epsilon = \frac{\max_i |\lambda_i(\hat{A})|}{\min_j |\lambda_j(\hat{D})|} \quad (2-4)$$

which is the measurement of the system's time-scale separation [6]. If the parameter ϵ is much smaller than one, i.e. $\epsilon \ll 1$, the two-time-scale system (2-1) is singularly perturbed.

Assume that the vector x in (2-1) is predominately slow and the vector z contains the fast transients, such

that

$$\|\dot{z}\| \gg \|\dot{x}\| . \quad (2-5)$$

A standard singular perturbation model is obtained by multiplying the second equation of system (2-1) with the scaling parameter ϵ

$$\begin{bmatrix} \dot{x} \\ \epsilon \dot{z} \end{bmatrix} = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} \begin{bmatrix} x \\ z \end{bmatrix} \quad (2-6)$$

where $A_{11} = A$, $A_{12} = B$, $A_{21} = \epsilon C$ and $A_{22} = \epsilon D$. A question may be asked at this moment -- for any given systems, how is one to choose the states x and z such that they are predominated by slow and fast modes separately. One approach to identify slow dynamic state variables is to rely upon the physical insights of the systems. For some widely studied dynamic systems there may be an obvious choice of slow and fast state variables. However for large scale systems, such choices may not be so obvious. A permutation matrix is proposed in [6] based on the eigenvector matrix of the system. When the effect of the states on the output is considered, another transformation is presented in [7]. In this thesis, the transformation of state variables is assumed to have been performed, that is, x and z are predominately slow and fast respectively.

2-2. A Conventional Method of Solving Singularly Perturbed Systems.

Consider an implicit form of a singularly perturbed system

$$\dot{x} = f_1(x, z, t) \quad (2-7)$$

$$\dot{z} = f_2(x, z, t) \quad (2-8)$$

where $x(t_0)=x_0$, and $z(t_0)=z_0$. To solve this system, the conventional method ignores the fast transients and only studies the system's quasi-steady state model. That is, the process of studying the system actually begins after the fast transients have reached their quasi-steady states. The quasi-steady state model is derived by setting $\dot{z}$ in (2-8) equal zero

$$\dot{x}_s = f_1(x_s, z_s, t), \quad x_s(t_0)=x_0 \quad (2-9)$$

$$0 = f_2(x_s, z_s, t) \quad (2-10)$$

where the subscript s denotes the quasi-steady states. The quasi-steady state of z can be obtained from (2-10)

$$z_s = \bar{f}_2(x_s, t). \quad (2-11)$$

Substituting (2-11) into (2-9) gives

$$\dot{x}_s = \bar{f}_1(x_s, t), \quad x_s(t_0)=x_0 \quad (2-12)$$

which is a reduced model. The only state variables for long term studies in the reduced model (2-12) is x_s ,

the slow mode. If one examines the process of simplification carefully, he can easily find that an inconsistency exists in this approach [1]. In quasi-steady state, z_s must be constant, as implied by $\dot{z}_s=0$. This assumption, however, is violated by (2-11) which defines z_s as a time varying quantity. Furthermore, the initial condition for z in (2-8) has to be dropped, because there is no freedom to satisfy it. This inconsistency will be removed if the singular perturbation technique is employed.

2-3. Singular Perturbation and Time Scales

A powerful tool for the approximation of a system with different changing modes is the technique of singular perturbation by which the system is studied in the forms of slow and fast subsystems.

Time Scaling

Assume that t is used for the slow phenomena and that τ is used to describe the fast phenomena defined by

$$\tau = \frac{t - t_0}{\epsilon} \quad (2-13)$$

where t_0 is a particular instant in t , the slow time scale. The equation (2-13) implies that the wider the separation in the time scales, the smaller ϵ will be. On the other hand, the smaller ϵ is, the larger τ will

be for a given interval $(t-t_0)$. In the limit as ϵ tends to zero, even a very short interval in t is 'stretched' to an infinite interval in τ . So by means of ϵ , one can look into the short interval of t in which the fast variable z changes quickly. By setting ϵ to zero, a reduced system, or slow mode system will be obtained. When the fast mode is needed, a fast mode subsystem can be set up by scaling the system in the fast time scale τ .

A Slow Mode Subsystem [8]

With the limit as ϵ approaches zero, the singularly perturbed system written in the explicit form

$$\dot{x} = g_1(x, z, t), \quad x(t_0) = x_0 \quad (2-14)$$

$$\epsilon \dot{z} = g_2(x, z, t), \quad z(t_0) = z_0 \quad (2-15)$$

becomes

$$\dot{x}_s = g_1(x_s, z_s, t), \quad x_s(t_0) = x_0 \quad (2-16)$$

$$0 = g_2(x_s, z_s, t) \quad (2-17)$$

where the subscript s denotes quasi-steady states. Although this is the same form as that in (2-9) and (2-10), their meanings are completely different. The crucial difference is that in (2-17) $\dot{z}_s$ is not equal to zero. It is due to $\epsilon = 0$, therefore $\epsilon \dot{z}_s = 0$, making the left hand side of the equation (2-15) equal to zero. So the inconsistency existing in the conventional method is removed from (2-16)

and (2-17) because here z_s can be a time varying quantity. Solving (2-17) for z_s yields

$$z_s = \bar{g}_2(x_s, t). \quad (2-18)$$

Substituting (2-18) into (2-16) gives a reduced system which is the slow mode subsystem

$$\dot{x}_s = \bar{g}_1(x_s, t), \quad x_s(t_0) = x_0. \quad (2-19)$$

Fast Mode Subsystem [8]

To obtain the fast mode subsystem, (2-14) and (2-15) are rewritten in the fast time scale τ . Substituting t scale in (2-14) and (2-15) with $(\epsilon\tau + t_0)$ gives

$$\dot{x}|_{\tau} = \epsilon g_1(x(\epsilon\tau + t_0), z(\epsilon\tau + t_0), \epsilon\tau + t_0) \quad (2-20)$$

$$\dot{z}|_{\tau} = g_2(x(\epsilon\tau + t_0), z(\epsilon\tau + t_0), \epsilon\tau + t_0) \quad (2-21)$$

where the derivatives of x and z are with respect to the fast time scale τ . Again, examine the limit as ϵ tends to zero. This limiting process results in

$$\dot{x}|_{\tau} = 0 \quad (2-22)$$

which means that $x(\tau)$ is constant in the fast time scale. This exactly coincides with the assumption made before. As ϵ approaches zero, the only variation is the deviation of $z(\tau)$ from its quasi-steady state z_s . In order to obtain a fast component of z , denoted as z_f , the quasi-

steady state z_s should be subtracted from z

$$z_f = z - z_s \quad (2-23)$$

where the subscript f denotes the fast mode. Substituting (2-23) into (2-20) and (2-21) and setting ϵ equal to zero yields

$$\dot{z}_f|_{\tau} = g_2(x(t_0), z_s(t_0) + z_f(\tau), t_0), \quad (2-24)$$

$$z_f(0) = z(t_0) - z_s(t_0)$$

which is the reduced system representing the fast mode, or the fast mode subsystem.

Using (2-19) as the slow subsystem and (2-24) as the fast subsystem, one can expect to approximate x and z by

$$\begin{aligned} x(t) &= x_s(t) + 0(\epsilon) \\ z(t) &= z_s(t) + z_f\left(\frac{t-t_0}{\epsilon}\right) + 0(\epsilon) \end{aligned} \quad (2-25)$$

where z_f is expressed in t time scale instead of τ time scale and $0(\epsilon)$ denotes that the error is the same order of ϵ . The following theorem due to Tihonov [9] establishes the conditions under which (2-25) is valid.

Theorem

Let g_1 and g_2 in (2-14) and (2-15) be twice differentiable functions of x , z and t . Assume that

- i) The equilibrium $z_f = 0$, i.e. $z_f(\tau)|_{\tau \rightarrow \infty} = 0$, of (2-24) is

asymptotically stable and $z_f(0)$ belongs to its domain of attraction.

ii) The eigenvalues of $\partial g / \partial z$ evaluated along $x_s(t)$ and $z_s(t)$ for all t in the interval $[t_0, T]$ have real parts strictly smaller than a fixed negative number.

Then the approximation (2-25) holds for all t in $[t_0, T]$.

Because of assumption ii), the term

$$z_f(\tau) = z_f\left(\frac{t-t_0}{\epsilon}\right) \quad (2-26)$$

decays rapidly in the t time scale. Hence for ϵ sufficiently small, there exists $t_1 > t_0$ such that the approximation

$$\begin{aligned} x(t) &= x_s(t) + 0(\epsilon) \\ z(t) &= z_s(t) + 0(\epsilon) \end{aligned} \quad (2-27)$$

holds for all t in $[t_1, T]$. It is of interest to see that (2-27) is the same result as that obtained by the conventional method from (2-9) and (2-10). In both conventional method and the singular perturbation method, the solution $x(t)$ is the same. The difference exists in the solution $z(t)$ by the term of $z_f(\tau)$. The term $z_f(\tau)$ cannot be obtained by the conventional method. Therefore, if the fast mode $z_f(\tau)$ or the correction is needed, the conventional method will fail.

Applications to Linear Time Invariant Systems

Next, the slow and fast mode subsystems will be derived from the linear time invariant system (2-6). Setting ϵ equal to zero gives quasi-steady states of the system

$$\dot{x}_s = A_{11}x_s + A_{12}z_s, \quad x_s(t_0) = x_0 \quad (2-28)$$

$$0 = A_{21}x_s + A_{22}z_s. \quad (2-29)$$

If A_{22} is not singular, solving the algebraic equation (2-29), one can obtain z_s as

$$z_s = -A_{22}^{-1}A_{21}x_s. \quad (2-30)$$

The slow mode subsystem then is

$$\dot{x}_s = (A_{11} - A_{12}A_{22}^{-1}A_{21})x_s, \quad x_s(t_0) = x_0 \quad (2-31)$$

with the solution

$$x_s = x + O(\epsilon). \quad (2-32)$$

To exhibit the fast mode subsystem, let

$$\begin{aligned} z_f &= z + A_{22}^{-1}A_{21}x_s \\ &= z + A_{22}^{-1}A_{21}x + \bar{O}(\epsilon) \end{aligned} \quad (2-33)$$

where $\bar{O}(\epsilon) = A_{22}^{-1}A_{21}O(\epsilon)$. The substitution of (2-33) into (2-6) yields the fast mode subsystem

$$\epsilon \dot{z}_f = A_{22}z_f + \bar{\bar{O}}(\epsilon) \quad (2-34)$$

where $\bar{\bar{O}}(\epsilon) = \epsilon O(\epsilon) + \epsilon A_{22}^{-1}A_{21}x - A_{22}\bar{O}(\epsilon)$.

CHAPTER 3

PROPERTIES OF SINGULARLY PERTURBED SYSTEMS

In this Chapter, some important properties of singularly perturbed systems, such as stability, controllability and observability, will be examined and deduced from lower order subsystems in separate time scales.

3-1. Stability

Consider a linear time invariant singularly perturbed system

$$\begin{aligned}\dot{x} &= A_{11}x + A_{12}z, & x(t_0) &= x_0 \\ \epsilon \dot{z} &= A_{21}x + A_{22}z, & z(t_0) &= z_0.\end{aligned}\tag{3-1}$$

If the slow mode subsystem (2-31) and the fast mode subsystem (2-34) are asymptotically stable, that is, if

$$\operatorname{Re} \lambda(A_{11} - A_{12}A_{22}^{-1}A_{21}) < 0, \operatorname{Re} \lambda\left(\frac{1}{\epsilon}A_{22}\right) < 0, \tag{3-2}$$

then there exists $\epsilon^* > 0$ such that the original system (3-1) is asymptotically stable for all $\epsilon \in [0, \epsilon^*]$, i.e. for ϵ sufficiently small [4,10]. The similar results for linear time varying systems are given in [11,12].

3-2. Controllability

Let us consider a forced linear time invariant

singularly perturbed system

$$\begin{aligned}\dot{x} &= A_{11}x + A_{12}z + B_1u, \quad x(t_0) = x_0 \\ \epsilon \dot{z} &= A_{21}x + A_{22}z + B_2u, \quad z(t_0) = z_0 \\ y &= C_1x + C_2z\end{aligned}\tag{3-3}$$

where $u(t) \in R^p$ is a control vector and $y(t) \in R^k$ is an output vector. The corresponding slow and fast subsystems are

$$\begin{aligned}\dot{x}_s &= A_{11s}x_s + B_{1s}u, \quad x_s(t_0) = x_0 \\ y_s &= C_{1s}x_s + D_s u,\end{aligned}\tag{3-4}$$

$$\begin{aligned}\epsilon \dot{z}_f &= A_{22}z_f + B_2u, \quad z_f(t_0) = z_0 - z_s(t_0) \\ y_f &= C_2z_f\end{aligned}\tag{3-5}$$

$$\text{where } A_{11s} = A_{11} - A_{12}A_{22}^{-1}A_{21}, \quad B_{1s} = B_1 - A_{12}A_{22}^{-1}B_2,\tag{3-6}$$

$$\text{and } C_{1s} = C_1 - C_2A_{22}^{-1}A_{21}, \quad D_s = -C_2A_{22}^{-1}B_2.$$

The controllability of (3-3) can be determined from (3-4) and (3-5). The controllability follows from the fact

that if A_{22}^{-1} exists and if

$$\text{Rank}[B_{1s} \ A_{11s}B_{1s} \ \dots \ A_{11s}^{(r-1)}B_{1s}] = r,\tag{3-7}$$

$$\text{Rank}[B_2 \ A_{22}B_2 \ \dots \ A_{22}^{(n-r-1)}B_2] = n-r$$

where r and $(n-r)$ are the dimensions of x and z , then there exists $\epsilon^* > 0$ such that the original system (3-3) is controllable for all $\epsilon \in [0, \epsilon^*]$ [13,14,15].

3-3. Observability

Similarly, the observability of the system (3-3) can be guaranteed by those of the slow and fast subsystems. If A_{22} is not singular and if

$$\text{Rank}[C_{1s}^T A_{11s}^T C_{1s}^T \dots (A_{11s}^T)^{(r-1)} C_{1s}^T] = r, \quad (3-8)$$

$$\text{and Rank}[C_2^T A_{22}^T C_2^T \dots (A_{22}^T)^{(n-r-1)} C_2^T] = n-r,$$

then there exists $\epsilon^* > 0$ such that the original system is observable for all $\epsilon \in [0, \epsilon^*]$ [4].

The properties discussed above are important for large scale singularly perturbed or two-time-scale systems because the analyses can be done in the two lower order subsystems. Therefore, the technique of singular perturbation is an alternative method for the analysis of high order systems.

CHAPTER 4

TIME SCALE DECOMPOSITIONS

A singularly perturbed system, or more generally, a two-time-scale system, can be decomposed into two subsystems. Mainly, there are two approaches available: the quasi-steady state approximation and the L-K transformation. The former has been studied in Chapter 2. Now the attention will be focused on the derivations and applications of the L-K transformation and the comparison between the two approaches. At the end of this Chapter, the multi-time-scale decomposition will also be presented.

4-1. Two-time-scale Decomposition Via the L-K Transformation

L-k Transform Matrix

Consider the singularly perturbed system (2-6) again. One can use the following two step transformation to decompose it into two subsystems representing slow and fast modes respectively [6,13,16,17,18]. The first transformation

$$\begin{bmatrix} \mathbf{x} \\ \mathbf{y}_2 \end{bmatrix} = \begin{bmatrix} \mathbf{I} & 0 \\ \mathbf{L} & \mathbf{I} \end{bmatrix} \begin{bmatrix} \mathbf{x} \\ \mathbf{z} \end{bmatrix} \quad (4-1)$$

transforms the system (2-6) into an upper block triangular

system

$$\begin{bmatrix} \dot{x} \\ \dot{y}_2 \end{bmatrix} = \begin{bmatrix} F_1 & A_{12} \\ 0 & F_2 \end{bmatrix} \begin{bmatrix} x \\ y_2 \end{bmatrix} \quad (4-2)$$

where $F_1 = A_{11} - A_{12}L$, $F_2 = A_{22} + LA_{12}$ and L satisfies the algebraic Reccati equation

$$LA_{11} + A_{21} - LA_{12}L - A_{22}L = 0. \quad (4-3)$$

To completely separate the slow and fast subsystems, the second transformation is introduced

$$\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} I & K \\ 0 & I \end{bmatrix} \begin{bmatrix} x \\ y_2 \end{bmatrix} \quad (4-4)$$

where K satisfies

$$KF_2 - F_1K + A_{12} = 0. \quad (4-5)$$

By the second transformation, a block diagonal system will be obtained

$$\begin{bmatrix} \dot{y}_1 \\ \dot{y}_2 \end{bmatrix} = \begin{bmatrix} F_1 & 0 \\ 0 & F_2 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} \quad (4-6)$$

such that $\max_i |\lambda_i(F_1)| < \min_j |\lambda_j(F_2)|$. Combining two

transformations, the L-K transform matrix will finally be obtained

$$\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} I+KL & K \\ L & I \end{bmatrix} \begin{bmatrix} x \\ z \end{bmatrix}. \quad (4-7)$$

The inverse transformation is given by

$$\begin{bmatrix} x \\ z \end{bmatrix} = \begin{bmatrix} I & -K \\ -L & I+LK \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}. \quad (4-8)$$

The decoupled system (4-6) will be very useful when working with large linear systems and will aid in numerical simulation, reduced-order modeling, and control system designs.

Evaluation of the L-K Transformation

One may ask why the system (2-6) should be transformed into a block diagonal form instead of completely diagonal form. The answer follows directly from the following evaluation of the L-K transformation [3].

Merits. i) The physical insights inherent in x and z of system (2-6) are retained in the time scale decoupled state variables y_1 and y_2 . It is obvious that in the transformation (4-7) y_1 and y_2 keep the slow and fast modes of x and z separately. But this is not true for the similar transformation which results in a completely diagonalized system.

ii) Matrices L and K are obtained from algebraic equations

(4-3) and (4-5). The transformation (4-7) and its inverse (4-8) consist of L and K without involving a matrix inverse.

iii) The transformation matrix

$$T = \begin{bmatrix} I+KL & K \\ L & I \end{bmatrix} \quad (4-9)$$

is always real and unique while this cannot always hold for similar transformation.

Demerit. As will be shown later, for current algorithms, the degree of accuracy in calculating L and K matrices and the rate of convergence depend sensitively on the eigenvalue gap.

Algorithms and Formulas for Calculating L and K

Algorithm one for calculating L [17,20]. The iteration is

$$L_{i+1} = A_{22}^{-1}(L_i(A_{11}-A_{12}L_i)+A_{21}), \quad i=1,2,\dots \quad (4-10)$$

with the initial approximation

$$L_0 = A_{22}^{-1}A_{21}. \quad (4-11)$$

- i) Calculate the initial approximation L_0 .
- ii) Substitute L_i into Reccati equation, and evaluate the error matrix R_i

$$R_i = L_i A_{11} + A_{21} - L_i A_{12} L_i - A_{22} L_i. \quad (4-12)$$

Stop if $\|R_i\| < \epsilon$ where ϵ is an arbitrarily small constant.

iii) Solve $A_{22} D_i = R_i$ for D_i and let $L_{i+1} = L_i + D_i$.

iv) Let $i = i+1$ and go to step ii).

The advantage of this algorithm is that it is easy to be implemented. However, as was proved in [17], a criterion involving matrix norms should be satisfied, and the criterion is quite restrictive. Another algorithm was introduced in [6] which is suitable to most two-time-scale systems and has a faster converging rate.

Algorithm two for calculating L [6]. In this algorithm, the right (or left) eigenvectors should be calculated first. Partition the eigenvector matrix M as

$$M = \begin{bmatrix} M_{11} & M_{12} \\ M_{21} & M_{22} \end{bmatrix} \quad (4-13)$$

where the columns of M are the right eigenvectors of the system matrix A, and $M_{11} \in \mathbb{R}^{r \times r}$ and $M_{21} \in \mathbb{R}^{(n-r) \times r}$. the iteration is

$$L_{i+1} = (A_{22} + L_i A_{12})^{-1} (L_i A_{11} + A_{21}) \quad (4-14)$$

with its initial approximation

$$L_0 = -M_{21} M_{11}^{-1}. \quad (4-15)$$

i) Obtain the initial approximation L_0 and set $i=0$.

ii) Evaluate the error matrix R_i

$$R_i = L_i A_{11} + A_{21} - L_i A_{12} L_i - A_{22} L_i \quad (4-16)$$

and stop if $\|R_i\| \leq \epsilon$, where ϵ is an arbitrarily small constant.

iii) Solve $(A_{22} + L_i A_{12}) D_i = R_i$ for D_i and let $L_{i+1} = L_i + D_i$.

iv) Let $i = i + 1$ and go to step ii).

This algorithm requires that L_0 should be close to L . Therefore, a calculation of eigenvectors is needed.

A formula for constructing L . The common demerit of the algorithms above is that the converging rate depends heavily on the eigenvalue gap and the accuracy is determined by the iteration. By the following formula, the L matrix can be constructed directly from the eigenvalues of the system which is realized in the control canonical form. Instead of approximate results, the exact L matrix is obtained by the formula and is independent of the eigenvalue gap.

Let a system be in the control canonical form

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \vdots \\ \dot{x}_n \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 1 & \dots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ -b_1 & -b_2 & -b_3 & \dots & -b_n \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} \quad (4-17)$$

with eigenvalues $\lambda_1, \lambda_2, \dots, \lambda_n$.

If the system (4-17) needs to be transformed into the form

$$\begin{bmatrix} \dot{x} \\ \dot{y}_2 \end{bmatrix} = \begin{bmatrix} F_1 & A_{12} \\ 0 & F_2 \end{bmatrix} \begin{bmatrix} x \\ y_2 \end{bmatrix} \quad (4-18)$$

where $\lambda(F_1) = \lambda_1, \lambda_2, \dots, \lambda_r,$

$\lambda(F_2) = \lambda_{r+1}, \lambda_{r+2}, \dots, \lambda_n,$

the L matrix can be constructed directly as

$$L = \begin{bmatrix} a_{1r} & \dots & a_{13} & a_{12} & a_{11} \\ a_{2r} & \dots & a_{23} & a_{22} & a_{21} \\ a_{3r} & \dots & a_{33} & a_{32} & a_{31} \\ \vdots & & \vdots & \vdots & \vdots \\ a_{(n-r)r} & \dots & a_{(n-r)3} & a_{(n-r)2} & a_{(n-r)1} \end{bmatrix} \quad (4-19)$$

where $a_{11} = -\sum_{i=1}^r \lambda_i, a_{12} = \sum_{\substack{i=1 \\ i < j}}^r \lambda_i \lambda_j, \dots, a_{1r} = (-1)^r \prod_{i=1}^r \lambda_i$

and $a_{ij} = a_{(i-1)(j+1)} - a_{(i-1)1} a_{1j}, 2 \leq i \leq n-r, 1 \leq j \leq r.$

An interesting and important feature of this formula is that it is very convenient to decompose a system into two subsystems representing any modes respectively. For instance, if one wants to decompose a system into the form of (4-18) such that

$$\begin{aligned} \lambda(F_1) &= \lambda_{i+1}, \lambda_{i+2}, \dots, \lambda_{i+r}, \\ \lambda(F_2) &= \lambda_1, \lambda_2, \dots, \lambda_i, \lambda_{i+r+1}, \dots, \lambda_n. \end{aligned} \quad (4-20)$$

only a_{1i} ($i=1,2,\dots,r$) needs to be changed by substituting

$\lambda_1, \lambda_2, \dots, \lambda_r$ with $\lambda_{i+1}, \lambda_{i+2}, \dots, \lambda_{i+r}$. This feature will be utilized in Chapter 5 by a new scheme of order reduction.

Algorithm three for calculating K [6,17,19]. When L has been obtained, F_1 and F_2 can be calculated before the algorithm begins

$$F_1 = A_{11} - A_{12}L, \text{ and } F_2 = A_{22} + LA_{12}.$$

- i) Set $K_0 = 0$, $R_0 = A_{12}$ and $i = 0$.
- ii) Solve $D_i F_2 = R_i$ for D_i and let $K_{i+1} = K_i - D_i$.
- iii) Evaluate $R_{i+1} = K_{i+1} F_2 - F_1 K_{i+1} + A_{12}$ and stop if $\|R_{i+1}\| \leq \epsilon$ where ϵ is an arbitrarily small constant.
- iv) Set $i=i+1$ and go to ii).

A formula for constructing K. Like the previous formula, matrix K can be constructed directly from the eigenvalues of the system provided the system is in the control canonical form.

$$K = [K_1 \ K_2 \ \dots \ K_r]^T \quad (4-21)$$

where $K_i = \Delta^{i-1} M^{-1} D$. The matrices in (4-21) are defined as

$$\Delta = \begin{bmatrix} a_{11} & a_{21} & a_{31} & \dots & a_{(n-r)1} - b_{r+1} \\ 1 & 0 & 0 & \dots & -b_{r+2} \\ 0 & 1 & 0 & \dots & -b_{r+3} \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & \dots & -b_n \end{bmatrix} \quad (4-22)$$

$$M = [a_{1r}I + a_{1(r-1)}\Delta + \dots + a_{11}\Delta^{r-1} + \Delta^r] \quad (4-23)$$

where a_{i1} and a_{1j} ($i=1,2,\dots,n-r$ and $j=1,2,\dots,r$) are the same as those in the formula (4-19) , and

$$D = [-1 \ 0 \ 0 \ \dots \ 0]^T \quad (4-24)$$

is an $(n-r)$ -vector.

Comparison Between the Quasi-steady State Approximation and the L-K Transformation

In the quasi-steady state approximation, the subsystems were obtained based on the fact that the small parameter ϵ was set to zero. Since ϵ is not zero, the subsystems obtained are with error. Furthermore, if the separation of the time scales is small, that is , the parameter ϵ is large, the error will be considerably large. However, the two subsystems (4-6) obtained by the L-K transformation are exact. Therefore, the L-K transformation is good for composite control in which the controls are separately designed for the slow and fast subsystems [21,22,23].

It is of interest to note that for single input single output systems, the subsystems obtained by the L-K transformation are the same as those obtained by the partial fraction method. However, for multi-input multi-output systems, it is convenient to use L-K transformation

to get the slow and fast subsystems.

4-2. Multi-time Scale Decomposition

The decomposition of multi-time scale systems can be achieved by repeatedly utilizing two-time-scale decompositions. For convenience, a three-time-scale system is discussed here

$$\begin{bmatrix} \dot{x} \\ \dot{y} \\ \dot{z} \end{bmatrix} = \begin{bmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} \quad (4-25)$$

where $x \in R^{r1}$, $y \in R^{r2}$, and $z \in R^{(n-r1-r2)}$. The state variables x , y and z are predominated by slow, medium and fast modes respectively. The system can be decomposed into slow, medium and fast subsystems through the following two -step transformations.

The first transformation is introduced

$$\begin{bmatrix} \bar{y}_1 \\ \bar{y}_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} I+K_1L_1 & K_1L_2 & K_1 \\ K_2L_1 & I+K_2L_2 & K_2 \\ L_1 & L_2 & I \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} \quad (4-26)$$

which reduces (4-25) to

$$\begin{bmatrix} \dot{\bar{y}}_1 \\ \dot{\bar{y}}_2 \\ \dot{y}_3 \end{bmatrix} = \begin{bmatrix} F_{11} & F_{12} & 0 \\ F_{21} & F_{22} & 0 \\ 0 & 0 & F_3 \end{bmatrix} \begin{bmatrix} \bar{y}_1 \\ \bar{y}_2 \\ y_3 \end{bmatrix} \quad (4-27)$$

where

$$\begin{bmatrix} F_{11} & F_{12} \\ F_{21} & F_{22} \end{bmatrix} = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} - \begin{bmatrix} A_{13} \\ A_{23} \end{bmatrix} [L_1 \quad L_2] \quad (4-28)$$

$$F_3 = A_{33} + [L_1 \quad L_2] \begin{bmatrix} A_{13} \\ A_{23} \end{bmatrix} \quad (4-29)$$

and $[L_1 \quad L_2]$ and $[K_1 \quad K_2]^T$ satisfy the following equations

$$\begin{aligned} [L_1 \quad L_2] \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} + [A_{31} \quad A_{32}] - A_{33}[L_1 \quad L_2] \\ - [L_1 \quad L_2] \begin{bmatrix} A_{13} \\ A_{23} \end{bmatrix} [L_1 \quad L_2] = 0 \end{aligned} \quad (4-30)$$

$$\begin{bmatrix} K_1 \\ K_2 \end{bmatrix} F_3 - \begin{bmatrix} F_{11} & F_{12} \\ F_{21} & F_{22} \end{bmatrix} \begin{bmatrix} K_1 \\ K_2 \end{bmatrix} + \begin{bmatrix} A_{13} \\ A_{23} \end{bmatrix} = 0. \quad (4-31)$$

The second transformation

$$\begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} I+K_3L_3 & K_3 & 0 \\ L_3 & I & 0 \\ 0 & 0 & I \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} \quad (4-32)$$

decomposes (4-27) to

$$\begin{bmatrix} \dot{y}_1 \\ \dot{y}_2 \\ \dot{y}_3 \end{bmatrix} = \begin{bmatrix} F_1 & 0 & 0 \\ 0 & F_2 & 0 \\ 0 & 0 & F_3 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} \quad (4-33)$$

where $F_1 = F_{11} - F_{12}L_3$, $F_2 = F_{22} + L_3F_{12}$ and L_3 and K_3 satisfy

$$L_3F_{11} + F_{21} - L_3F_{12}L_3 - F_{22}L_3 = 0, \quad (4-34)$$

$$K_3F_2 - F_1K_3 + F_{12} = 0. \quad (4-35)$$

The system (4-33) is a decoupled system consisting of slow, medium and fast mode subsystems. Similarly, the L-K transformation can be applied to more time scale systems having the same merits described in the two-time-scale decomposition.

CHAPTER 5

ORDER REDUCTION VIA SINGULAR PERTURBATION

Order reduction is one of the most important applications of the technique of singular perturbation. The quasi-steady state approximation and the L-k transformation are the two approaches available for order reduction. Order reduction is achieved by keeping the slow mode subsystem and neglecting the fast mode subsystem after decomposition.

5-1. Order Reduction By the Quasi-steady State Approximation [8,10,24,25]

Examine the following linear time invariant system

$$\begin{bmatrix} \dot{x} \\ \dot{z} \end{bmatrix} = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} \begin{bmatrix} x \\ z \end{bmatrix} + \begin{bmatrix} B_1 \\ B_2 \end{bmatrix} u \quad (5-1)$$

$$y = [C_1 \quad C_2] \begin{bmatrix} x \\ z \end{bmatrix}$$

By the quasi-steady state approximation described in Chapter 2, it can be transformed into the slow and fast mode subsystems. Keeping the slow mode subsystem, the reduced system is

$$\dot{x}_s = (A_{11} - A_{12}A_{22}^{-1}A_{21})x_s + (B_1 - A_{12}A_{22}^{-1}B_2)u \quad (5-2)$$

$$y_s = (C_1 - C_2 A_{22}^{-1} A_{21}) x_s - C_2 A_{22}^{-1} B_2 u \quad (5-3)$$

provided that A_{22} is not singular. Next, the proof that the reduced system retains the steady state gain of the original system will be given. The steady state gain of the reduced system (5-2) and (5-3) is

$$G_{r|dc} = -(C_1 - C_2 A_{22}^{-1} A_{21}) (A_{11} - A_{12} A_{22}^{-1} A_{21})^{-1} (B_1 - A_{12} A_{22}^{-1} B_2) - C_2 A_{22}^{-1} B_2. \quad (5-4)$$

The steady state gain of the full order system (5-1) is

$$\begin{aligned} G_{|dc} &= -[C_1 \quad C_2] \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}^{-1} \begin{bmatrix} B_1 \\ B_2 \end{bmatrix} \\ &= -(C_1 - C_2 A_{22}^{-1} A_{21}) (A_{11} - A_{12} A_{22}^{-1} A_{21})^{-1} (B_1 - A_{12} A_{22}^{-1} B_2) - C_2 A_{22}^{-1} B_2. \end{aligned} \quad (5-5)$$

The equation (5-5) shows that the steady state gain of the full order system is kept in the reduced system. Now, two typical numerical examples are given to test the effectiveness of the quasi-steady state approximation.

Example 5-1: Consider the following transfer function

$$G_1(s) = \frac{(s+1)(s+2)}{(s+3)(s+4)(s+5)} = \frac{s^2 + 3s + 2}{s^3 + 12s^2 + 47s + 60} \quad (5-6)$$

realized in the control canonical form

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -60 & -47 & -12 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} u \quad (5-7)$$

$$y = \begin{bmatrix} 2 & 3 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}.$$

The reduced system is

$$\begin{bmatrix} \dot{x}_{1s} \\ \dot{x}_{2s} \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -5 & -3.92 \end{bmatrix} \begin{bmatrix} x_{1s} \\ x_{2s} \end{bmatrix} + \begin{bmatrix} 0 \\ 0.083 \end{bmatrix} u \quad (5-8)$$

$$y_s = \begin{bmatrix} -3 & -0.92 \end{bmatrix} \begin{bmatrix} x_{1s} \\ x_{2s} \end{bmatrix} + 0.083u.$$

In the form of the transfer function, the reduced system is

$$G_{1s}(s) = \frac{0.083s^2 + 0.666s + 0.167}{s^2 + 3.92s + 5} \quad (5-9)$$

The step input response of the reduced system (5-9) is shown in Figure 5-1.

Example 5-2: Consider another system

$$G_2(s) = \frac{(s+3)(s+4)}{(s+1)(s+2)(s+20)} = \frac{s^2 + 7s + 12}{s^3 + 23s^2 + 62s + 40} \quad (5-10)$$

with a state variable model

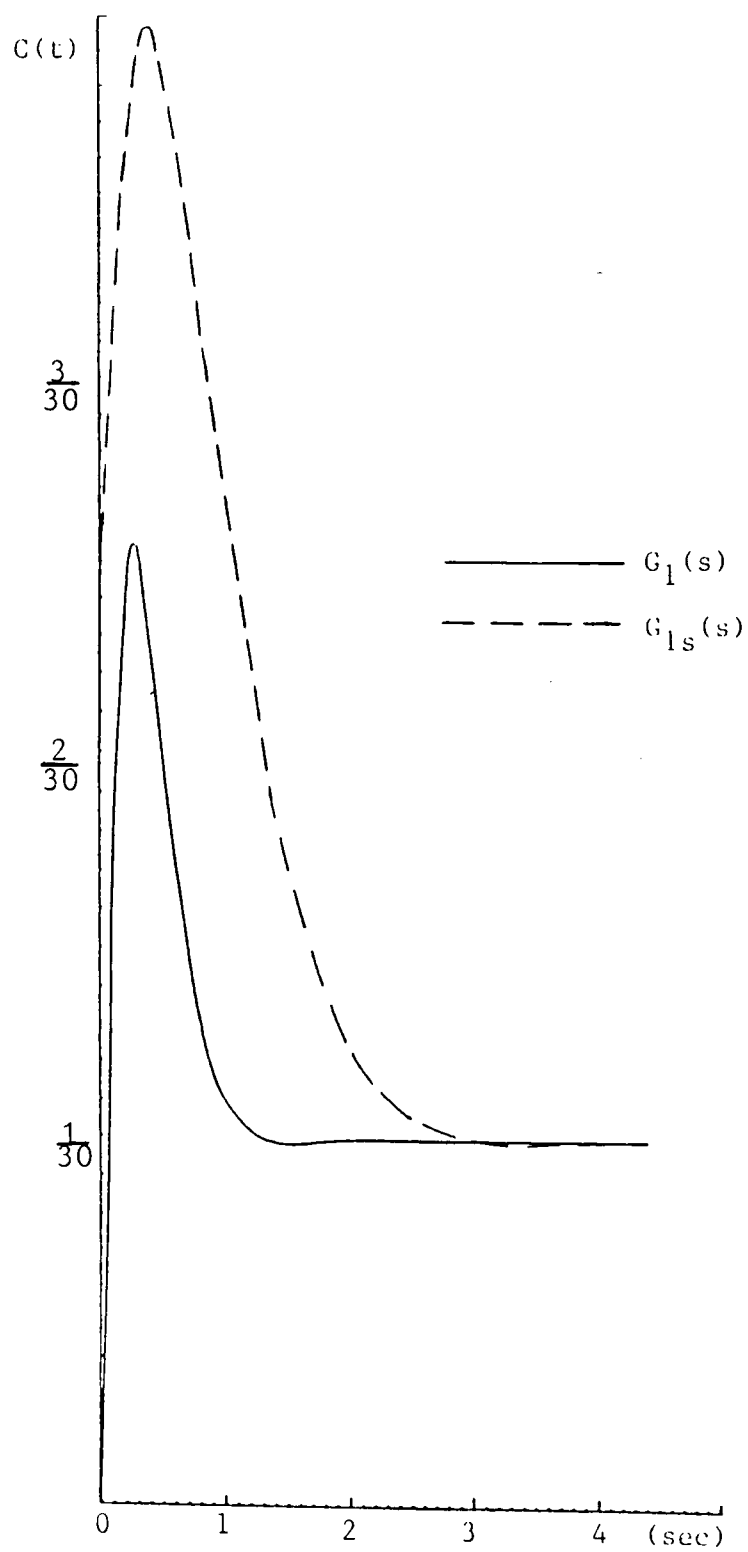


Figure 5-1. Step responses of $G_1(s)$ and $G_{1s}(s)$

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -40 & -62 & -23 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} u \quad (5-11)$$

$$y = [12 \quad 7 \quad 1] \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}.$$

The reduced system is

$$\begin{bmatrix} \dot{x}_{1s} \\ \dot{x}_{2s} \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -1.739 & -2.696 \end{bmatrix} \begin{bmatrix} x_{1s} \\ x_{2s} \end{bmatrix} + \begin{bmatrix} 0 \\ 0.0435 \end{bmatrix} u \quad (5-12)$$

$$y_s = [10.261 \quad 4.304] \begin{bmatrix} x_{1s} \\ x_{2s} \end{bmatrix} + 0.0435u,$$

$$G_{2s}(s) = \frac{0.0435s^2 + 0.304s + 0.522}{s^2 + 2.696s + 1.739} \quad (5-13)$$

The step input response of the reduced system is depicted in Figure 5-2.

Comparing Figure 5-1 and Figure 5-2, one can reach the conclusions:

- i) The steady state gains of the full order systems are retained in the reduced systems.
- ii) Since there is no dominant mode in system one, $G_1(s)$, there is a large error in the reduced system, referred to in Figure 5-1. However in system two, $G_2(s)$, since the gap of eigenvalues

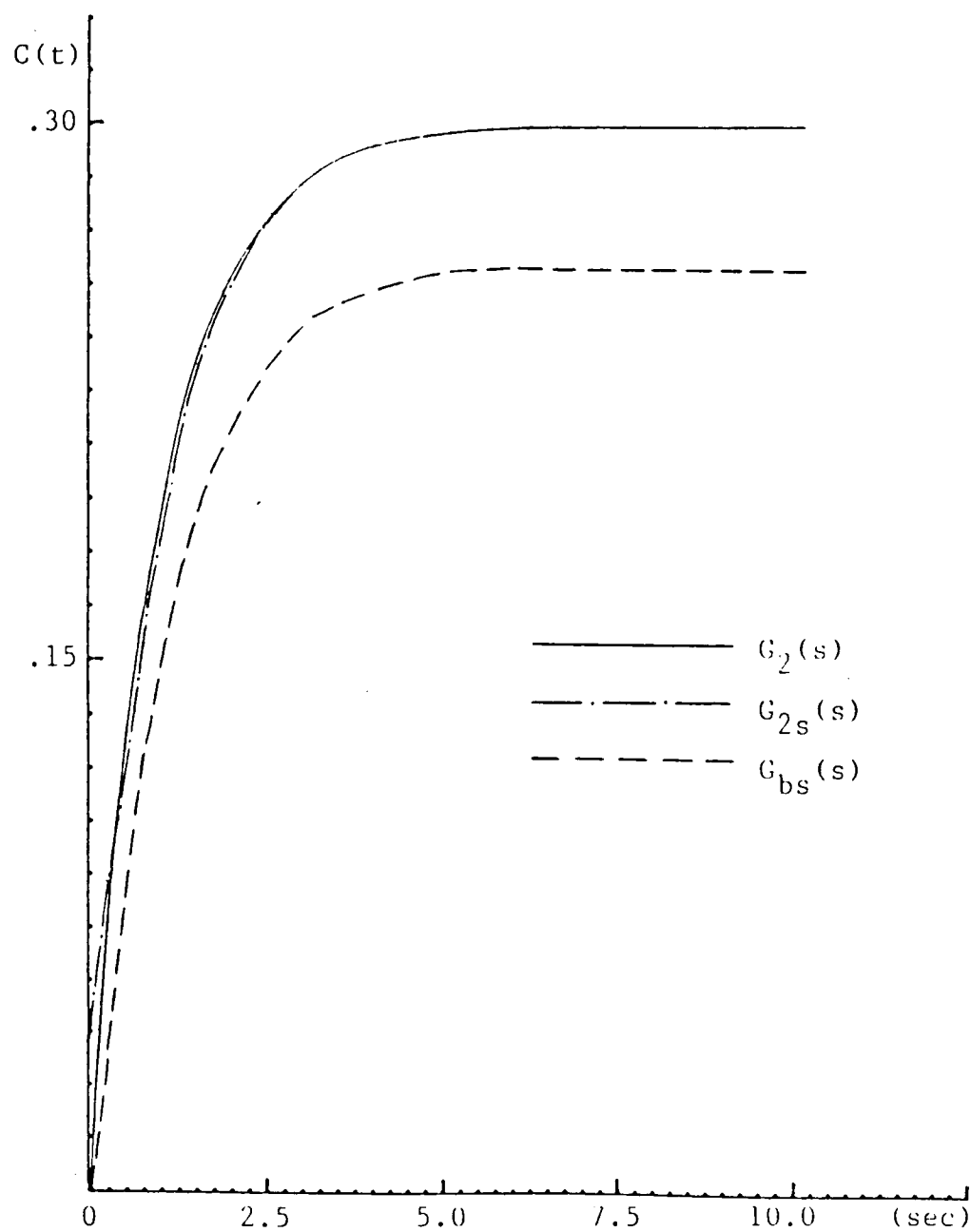


Figure 5-2. Step reponses of $G_2(s)$, $G_{2s}(s)$ and $G_{bs}(s)$

between $\lambda_1=-1$, $\lambda_2=-2$ and $\lambda_3=-20$ is big, there is little error in the reduced system, referred to in Figure 5-2.

- iii) The reduced systems do not keep the slow mode eigenvalues of the full order systems. For instance, the slow mode eigenvalues of system two are $\lambda_1=-1$ and $\lambda_2=-2$, while those of the reduced system, $G_{2s}(s)$, are $\lambda_{1s}=-1.069$ and $\lambda_{2s}=-1.626$.

5-2 Order Reduction By the L-K Transformation [3,7,18,26]

Consider again the system (5-1) which is rewritten below for convenience.

$$\begin{bmatrix} \dot{x} \\ \dot{z} \end{bmatrix} = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} \begin{bmatrix} x \\ z \end{bmatrix} + \begin{bmatrix} B_1 \\ B_2 \end{bmatrix} u \quad (5-14)$$

$$y = [C_1 \quad C_2] \begin{bmatrix} x \\ z \end{bmatrix}$$

where $x \in R^r$, $z \in R^{(n-r)}$ and u and y are single input, single output signals. By the L-K transformation

$$\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} I+KL & K \\ L & I \end{bmatrix} \begin{bmatrix} x \\ z \end{bmatrix}, \quad (5-15)$$

two decoupled subsystems can be obtained

$$\dot{y}_1 = F_1 y_1 + \bar{B}_1 u \quad (5-16)$$

$$\dot{y}_2 = F_2 y_2 + \bar{B}_2 u \quad (5-17)$$

$$y = \bar{C}_1 y_1 + \bar{C}_2 y_2 \quad (5-18)$$

where $F_1 = A_{11} - A_{12}L$, $F_2 = A_{22} + LA_{12}$,

$$\bar{B}_1 = B_1 + KLB_1 + KB_2, \quad \bar{B}_2 = LB_1 + B_2,$$

and $\bar{C}_1 = C_1 - C_2L$, $\bar{C}_2 = -C_1K + C_2 + C_2LK$.

Neglecting the fast subsystem (5-17) gives

$$\dot{y}_1 = F_1 y_1 + \bar{B}_1 u \quad (5-19)$$

$$y = \bar{C}_1 y_1$$

which is a reduced system.

Example 5-3: Consider the same system as that in Example 5-1

$$G_a(s) = \frac{s^2 + 3s + 2}{s^3 + 12s^2 + 47s + 60} \quad (5-20)$$

which is realized in the control canonical form

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -60 & -47 & -12 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} u \quad (5-21)$$

$$y = [2 \quad 3 \quad 1] \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}.$$

The L-K transform matrix

$$\begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} -5 & -3.5 & -0.5 \\ 30 & 18.5 & 2.5 \\ 12 & 7 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \quad (5-22)$$

results in the reduced system

$$\begin{bmatrix} \dot{y}_1 \\ \dot{y}_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -12 & -7 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} + \begin{bmatrix} -0.5 \\ 2.5 \end{bmatrix} u \quad (5-23)$$

$$y = [-10 \quad -4] \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}$$

with the transfer function being

$$G_{as}(s) = \frac{-5s-14}{s^2+7s+12} \quad (5-24)$$

Since the output will go negative when a step input is applied, the step response of $G_{as}(s)$ is not depicted.

Example 5-4: The system in Example 5-2 is studied here again

$$G_b(s) = \frac{s^2+7s+12}{s^3+23s^2+62s+40} \quad (5-25)$$

The state variable model is the same as that in (5-11).

The L-K transform matrix

$$\begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} 0.994 & -8.77 \times 10^{-3} & -2.92 \times 10^{-3} \\ 0.117 & 1.175 & 0.0585 \\ 2 & 3 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \quad (5-26)$$

reduces the full order system (5-11) to

$$\begin{bmatrix} \dot{y}_1 \\ \dot{y}_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -2 & -3 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} + \begin{bmatrix} -2.92 \times 10^{-3} \\ 0.0585 \end{bmatrix} u \quad (5-27)$$

$$y = [10 \quad 4] \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}$$

which has a transfer function

$$G_{bs}(s) = \frac{0.205s + 0.521}{s^2 + 3s + 2} \quad (5-28)$$

The step input response is shown in Figure 5-2.

Observing Figure 5-2, one can see that:

- i) The steady state gain of the full order systems are not fully kept in the reduced systems. The smaller the eigenvalue gap of the system is, the more steady state gain will be lost.
- ii) The eigenvalues of the reduced systems are the same as the slow mode eigenvalues of the full order systems.
- iii) So far, the quasi-steady state approximation is preferred to the L-K transformation for order

reduction, especially, when the eigenvalue gap is small.

5-3 Order Reduction By the Modified L Transformation

As discussed before, the quasi-steady state approximation has some advantages over the L-K transformation. However, if the system is not sufficiently singularly perturbed, the quasi-steady state approximation is not applicable either. Recall that the eigenvalues of the reduced systems obtained by the quasi-steady state approximation are changed. If the medium mode of the system is dropped and the slow and fast modes are adjusted to compensate for the loss of the medium mode, the reduced system which fits the original system's output will be achieved. The material in this section deviates slightly from the concept of singular perturbation, because in singular perturbation it does not make sense to drop the medium mode instead of the fast mode. A modified L transformation is developed here for order reduction by utilizing the concept of strong and weak subsystems.

A full order model is organized as follows

$$\begin{bmatrix} \dot{\bar{x}} \\ \dot{\bar{z}} \end{bmatrix} = \begin{bmatrix} A_w & \bar{A}_{12} \\ \bar{A}_{21} & A_s \end{bmatrix} \begin{bmatrix} \bar{x} \\ \bar{z} \end{bmatrix} + \begin{bmatrix} B_w \\ B_s \end{bmatrix} u \quad (5-29)$$

$$y = \begin{bmatrix} C_w & C_s \end{bmatrix} \begin{bmatrix} \bar{x} \\ \bar{z} \end{bmatrix}$$

where the subscripts w and s indicate the weak and strong subsystems. The system (5-29) is said to have strong and weak subsystems if the impulse response matrix of the strong subsystem, $C_s e^{A_s t} B_s$, is close to that of the full order model, or the impulse response matrix of the weak subsystem, $\|C_w e^{A_w t} B_w\|$, is fairly small. Consider the two-time-scale system (5-14) again. It can be transformed into strong and weak subsystems (5-29) such that the strong subsystem contains slow and fast modes while the weak subsystem contains only the medium mode. Eliminating the weak subsystem and substituting its steady state component

$$\bar{x}_w = -A_w^{-1} \bar{A}_{12} \bar{z}_s - A_w^{-1} B_w u \quad (5-30)$$

into the strong subsystem

$$\begin{aligned} \dot{\bar{z}}_s &= \bar{A}_{21} \bar{x}_w + A_s \bar{z}_s + B_s u \\ y_s &= C_w \bar{x}_w + C_s \bar{z}_s \end{aligned} \quad (5-31)$$

gives the reduced system

$$\begin{aligned} \dot{\bar{z}}_s &= \bar{A}_s \bar{z}_s + \bar{B}_s u \\ y_s &= \bar{C}_s \bar{z}_s + \bar{D}_s u \end{aligned} \quad (5-32)$$

where $\bar{A}_s = A_s - \bar{A}_{21} A_w^{-1} \bar{A}_{12}$, $\bar{B}_s = B_s - \bar{A}_{21} A_w^{-1} B_w$,

$$\bar{C}_s = C_s - C_w A_w^{-1} \bar{A}_{12}, \text{ and } \bar{D}_s = -C_w A_w^{-1} B_w.$$

As noted in Chapter 4, when the system is in the control

canonical form, the L matrix can be directly constructed from the eigenvalues required to be grouped. Here the L matrix is constructed by the medium mode eigenvalues, so that two subsystems will contain the slow and fast and the medium modes separately. However, a perturbation to the L matrix should be introduced to adjust the slow and fast mode eigenvalues shifting towards smaller and larger directions. The modified L matrix is

$$\bar{L} = \begin{bmatrix} \bar{a}_{1r} & \cdots & \bar{a}_{13} & \bar{a}_{12} & \bar{a}_{11} \\ \bar{a}_{2r} & \cdots & \bar{a}_{23} & \bar{a}_{22} & \bar{a}_{21} \\ \bar{a}_{3r} & \cdots & \bar{a}_{33} & \bar{a}_{32} & \bar{a}_{31} \\ \vdots & & \vdots & \vdots & \vdots \\ \bar{a}_{(n-r)r} & \cdots & \bar{a}_{(n-r)3} & \bar{a}_{(n-r)2} & \bar{a}_{(n-r)1} \end{bmatrix} \quad (5-33)$$

where $\bar{a}_{11} = -\sum_{i=1}^r (\lambda_i + \Delta L)$, $\bar{a}_{12} = \sum_{\substack{i=1 \\ i < j}}^r (\lambda_i + \Delta L)(\lambda_j + \Delta L)$,

$$\bar{a}_{1r} = (-1)^r \prod_{i=1}^r (\lambda_i + \Delta L), \quad \text{and}$$

$$\bar{a}_{ij} = \bar{a}_{(i-1)(j+1)} - \bar{a}_{(i-1)1} \bar{a}_{1j} \quad (2 \leq i \leq n-r, 1 \leq j \leq r),$$

here $\lambda_1, \lambda_2, \dots, \lambda_r$ are the medium mode eigenvalues. ΔL can be determined from the weak system condition described later. By the modified L matrix, the system (5-14) is transformed into system (5-29), where

$$A_w = A_{11} - A_{12} \bar{L}, \quad \bar{A}_{12} = A_{12}$$

$$\bar{A}_{21} = \bar{L} A_{11} + A_{21} - \bar{L} A_{12} \bar{L} - A_{22} \bar{L},$$

$$A_s = A_{22} + \bar{L}A_{12}, \quad B_w = B_1, \quad B_s = \bar{L}B_1 + B_2$$

$$\text{and} \quad C_w = C_1 - C_2\bar{L}, \quad C_s = C_2.$$

In order to ensure that the subsystem (A_w, B_w, C_w) is a weak system, C_w is set equal to zero. Therefore, the impulse response matrix $C_w e^{A_w t} B_w$ will equal zero. For single input, single output systems, ΔL can be determined from

$$C_w = 0. \quad (5-34)$$

The equation (5-34) can also be reached by making the reduced system (5-32) match the Markov parameter J_0 [29]. So the reduced system not only retains the steady state gain but also keeps the initial transient response of the full order model.

Example 5-5: The system considered here is still the same as that in Example 5-1

$$G_c(s) = \frac{s^2 + 3s + 2}{s^3 + 12s^2 + 47s + 60} \quad (5-35)$$

whose state variable model is shown in (5-7). The modified L matrix is constructed from the medium mode eigenvalue $\lambda_2 = -4$

$$\bar{L} = \begin{bmatrix} 4 - \Delta L \\ -(-4 + \Delta L)^2 \end{bmatrix} \quad (5-36)$$

which transforms the system (5-35) or (5-7) into

$$\begin{bmatrix} \dot{x} \\ \dot{z}_1 \\ \dot{z}_2 \end{bmatrix} = \begin{bmatrix} -4+\Delta L & 1 & 0 \\ 0 & 4-\Delta L & 1 \\ -\Delta L^3+\Delta L & -47-(-4+\Delta L)^2 & -12 \end{bmatrix} \begin{bmatrix} x \\ z_1 \\ z_2 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} u \quad (5-37)$$

$$y = [\Delta L^2 - 5\Delta L + 6 \quad 3 \quad 1] \begin{bmatrix} x \\ z_1 \\ z_2 \end{bmatrix}.$$

Then solve the following equation for ΔL

$$C_w = \Delta L^2 - 5\Delta L + 6 = 0. \quad (5-38)$$

Substituting $\Delta L=2$ into (5-37) and the steady state component of the weak subsystem into the strong subsystem yields the reduced system

$$\begin{bmatrix} \dot{z}_{1s} \\ \dot{z}_{2s} \end{bmatrix} = \begin{bmatrix} 2 & 1 \\ -54 & -12 \end{bmatrix} \begin{bmatrix} z_{1s} \\ z_{2s} \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u \quad (5-39)$$

$$y_s = [3 \quad 1] \begin{bmatrix} z_{1s} \\ z_{2s} \end{bmatrix}.$$

The corresponding transfer function is

$$G_{cs}(s) = \frac{s+1}{s^2+10s+30} \quad (5-40)$$

whose step input response is shown in Figure 5-3.

Example 5-6: For the following system

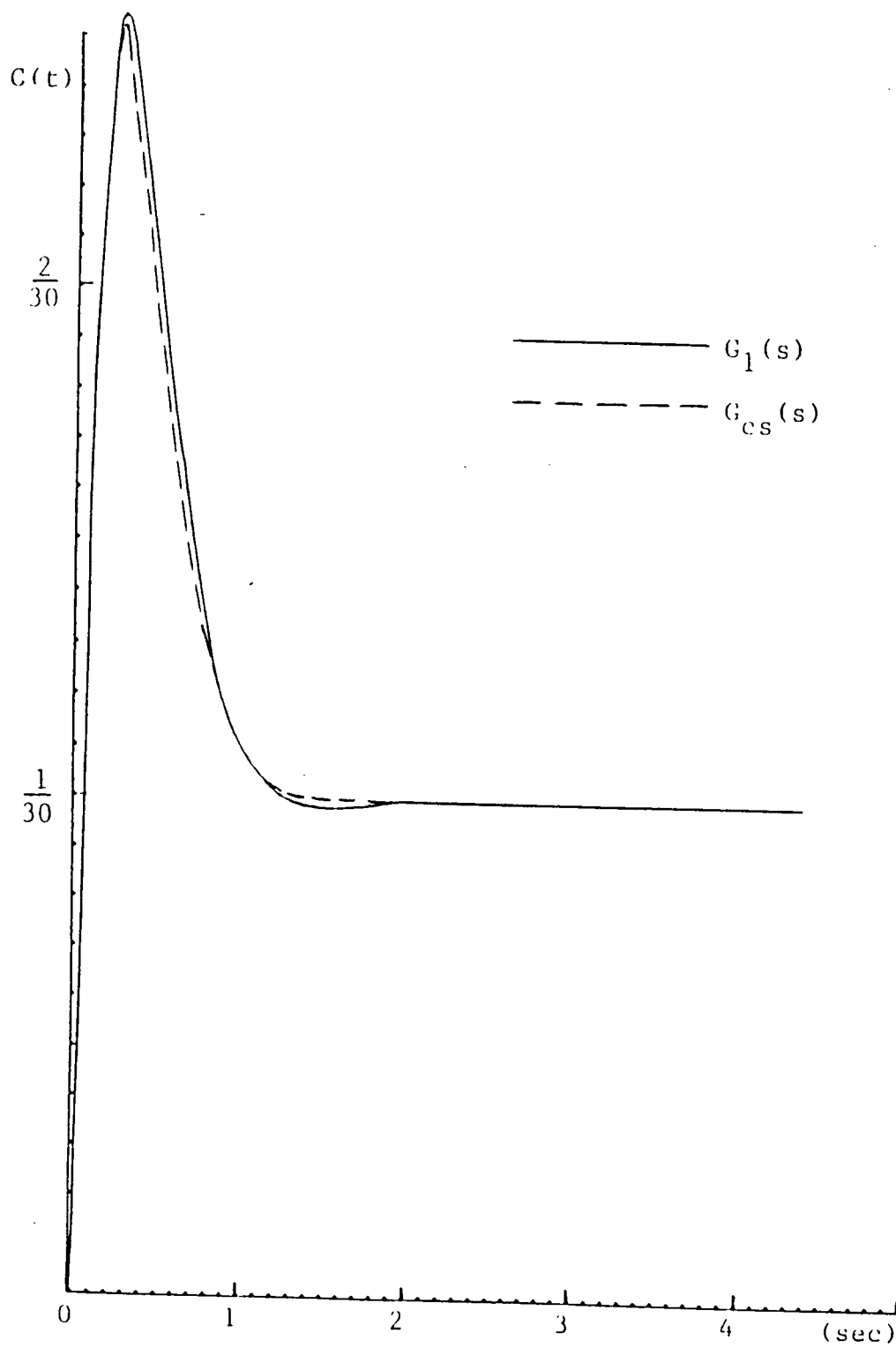


Figure 5-3. Step responses of $G_1(s)$ and $G_{cs}(s)$

$$G_d(s) = \frac{s^2 + 7s + 12}{s^3 + 23s^2 + 62s + 40} \quad (5-41)$$

which was considered in Example 5-2 the modified L matrix is

$$\bar{L} = \begin{bmatrix} 2 - \Delta L \\ -(-2 + \Delta L)^2 \end{bmatrix}. \quad (5-42)$$

Similarly, by the L transformation and by substituting the steady state component of the weak subsystem into the strong subsystem, the reduced system is obtained

$$\begin{bmatrix} \dot{z}_{1s} \\ \dot{z}_{2s} \end{bmatrix} = \begin{bmatrix} 3 & 1 \\ -82.33 & -23 \end{bmatrix} \begin{bmatrix} z_{1s} \\ z_{2s} \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u \quad (5-43)$$

$$y = \begin{bmatrix} 7 & 1 \end{bmatrix} \begin{bmatrix} z_{1s} \\ z_{2s} \end{bmatrix}$$

which has a transfer function

$$G_{ds}(s) = \frac{s+4}{s^2 + 20s + 13.33}. \quad (5-44)$$

The step input response is shown in Figure 5-4.

5-4 Remarks

Compared to the L-K transformation, the quasi-steady state approximation has a remarkable advantage, that is, it keeps the steady state gains of the full order

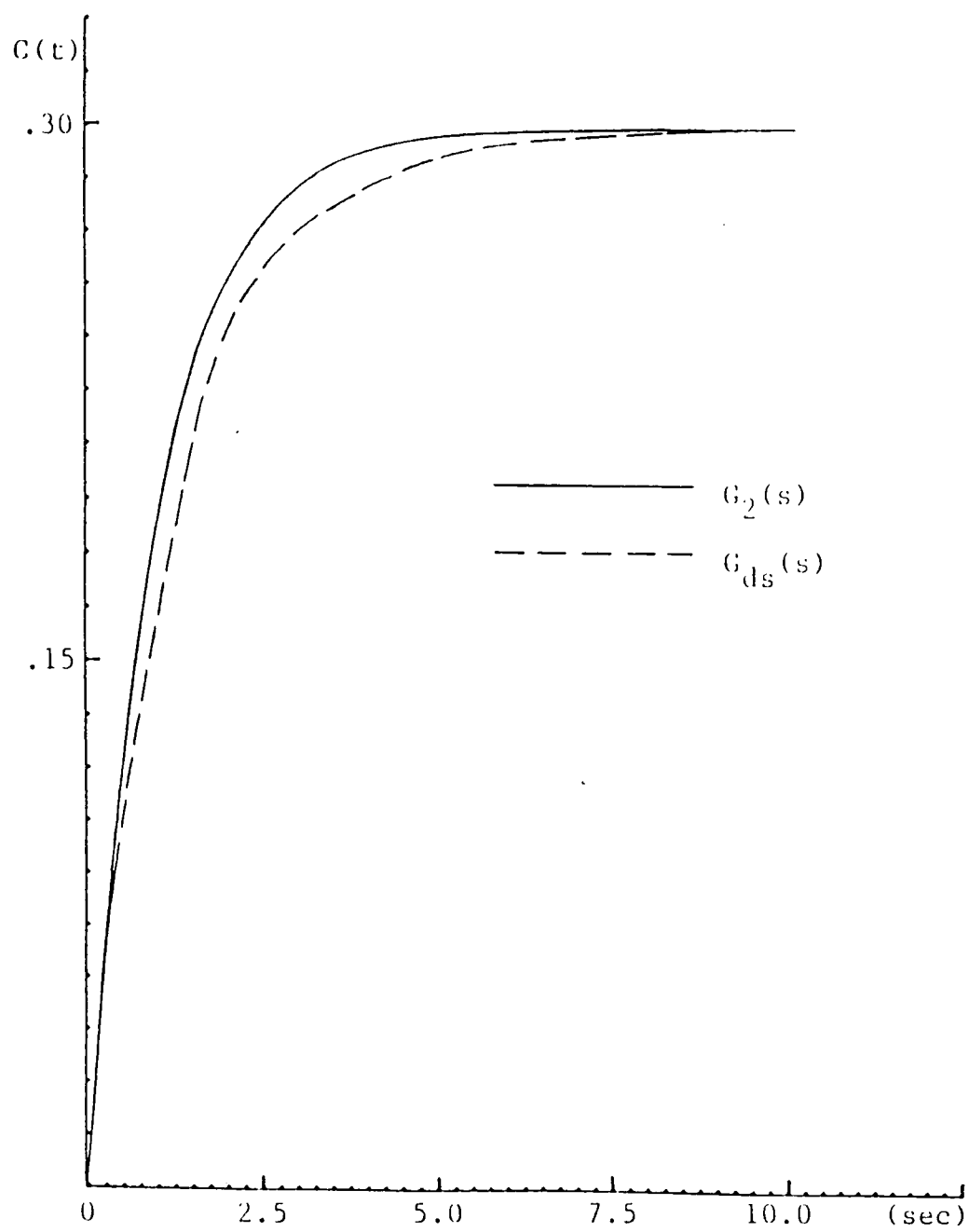


Figure 5-4. Step responses of $G_2(s)$ and $G_{ds}(s)$

systems. However, there are two weaknesses of the quasi-steady state approximation: i) The error of this approximation nearly proportional to the parameter ϵ . When the parameter ϵ is not sufficiently small, as in the case of Example 5-1, the error is considerably large. ii) When a step input is applied to the reduced system obtained by the quasi-steady state approximation, the response has a 'jump' at $t=0$. This is very harmful to some systems which require smooth responses.

The modified L transformation is introduced to overcome the weaknesses of the quasi-steady state approximation. One can see from Example 5-3 and Example 5-4 that the modified L transformation is good for either the systems with large parameters ϵ or the ones with small parameters ϵ . Therefore, it is suitable for any two-time-scale systems. Furthermore, this transformation gives smooth step responses and keeps the steady state gains of the full order systems. The demerit of the modified L transformation, however, is that in solving sufficiently singularly perturbed systems, the difficulty in computation still exists, because this transformation retains the slow and fast modes of the full order systems.

CHAPTER 6

CONCLUSIONS

In general, a singular perturbation technique decomposes a singularly perturbed system or a two-time-scale system into two lower order subsystems, slow and fast subsystems. The quasi-steady state approximation and the L-K transformation are available in the singular perturbation technique.

In the system analysis, stability, controllability and observability are studied. It is shown that if the slow and fast subsystems are asymptotically stable, the full order system is also asymptotically stable. It is also shown that the full order system's controllability and observability can follow from those of the slow and fast subsystems. Therefore, the singular perturbation technique is an alternative method for the analysis of high order systems.

Singular perturbation can be applied to order reduction. It is pointed out and shown by numerical examples that the L-K transformation always loses some dc component, or steady state gain, of the full order system. Therefore, as far as order reduction is concerned, the quasi-steady state approximation and the modified L transformation are preferable. The error in the

quasi-steady state approximation depends on the parameter ϵ . The step responses of the reduced systems are not smooth at $t=0$. These weaknesses are overcome by the modified L transformation which still keeps all the advantages of the L-K transformation. But the modified L transformation will meet an obstacle in computation when the system studied is sufficiently singularly perturbed.

By decomposing a high order system into two lower order subsystems, the composite control, consisting of slow and fast controls, is introduced. Since the L-K transformation yields two exact subsystems, it is recommended for the composite control, especially when the system is not sufficiently singularly perturbed.

These studies can be extended to the multi-time-scale systems, and similar results will be expected.

LIST OF REFERENCES

LIST OF REFERENCES

1. Chow, J. H. (ed)(1982)," Time-scale Modeling of Dynamic Networks with Application to Power Systems," Lecture Notes in Control Information Science 46, Springer, Berlin.
2. Chow, J. H. and P. V. Kokotovic (1976), " Eigenvalue Placement in Two Time Scale Systems," Proc. of IFAC Symposium on Large Scale Systems, Unine Italy, pp. 321-326.
3. Anderson, L. R. (1979)," Decoupling and Reduced Order Modeling of Two-time-scale Control Systems," Doctoral Dissertation, University of Arizona, Tucson.
4. Saksena, V. R., J. O'Reilly and P. V. Kokotovic (1984)," Singular Perturbations and Time-scale Methods in Control Theory: Survey 1976-1983," Automatica, vol. 20, pp.273-293.
5. Coderch, M., A. S. Willsky, S. S. Sastry and D. A. Castanon (1983)," Hierarchical Aggregation of Linear Systems with Multiple Time Scales," IEEE Trans. Auto. Control, AC-28, pp. 1017-1030.
6. Anderson, L. R. (1978)," Decoupling of Two-time-scale Linear Systems," Proc. of JACC, pp. 153-163.
7. Litz, L. and H. Roth (1981)," State Decomposition for Singular Perturbation Order Reduction-- A Model Approach," Int. J. Control, 34, pp. 937-954.
8. Kokotovic, P. V. (1972)," A Control Engineer's Introduction to Singular Perturbations," Singular Perturbations: Order Reduction in Control System Design, New York, pp. 1-12.
9. Tihonov, A. V. (1952)," Systems of Differential Equation Containing A Small Parameter Multiplying the Derivative," Math-Sbornik (Moscow), vol. 73, pp. 575-586.
10. Kokotovic, P. V., R. E. O'Malley, Jr and P. Sannuti (1976)," Singular Perturbations and Order Reduction in Control Theory-- An Overview," Automatica, vol. 12 pp. 123-132.

11. Wilde, R. R. and P. V. Kokotovic (1972)," Stability of Singularly Perturbed Systems and Networks with Parasitics," IEEE Trans. Auto. Control, AC-17, pp. 245-246.
12. Javid, S. H. (1978)," Uniform Asymptotic Stability of Linear Time Varying Singularly Perturbed Systems," J. Franklin Inst., 305, pp. 27-37.
13. Kokotovic, P. V. and A. H. Haddad (1975)," Controllability and Time-Optimal Control of Systems with Slow and Fast Modes," IEEE Trans. Auto. Control, AC-20, pp. 111-113.
14. Sannuti, P. (1977)," On the Controllability of Singularly Perturbed Systems," IEEE Trans. Auto. Control, AC-22, pp. 622-624.
15. O'Reilly, J. (1979)," Two Time Scale Feedback Stabilization of Linear Time Varying Singularly Perturbed Systems," J. Franklin Inst., 308, pp. 465-474.
16. Chang, K. W. (1972)," Singular Perturbations of A General Value Problem," SIAM J. Math. Anal., vol. 3, pp. 520-526.
17. Kokotovic, P. V. (1975)," A Riccati Equation for Block-Diagonalization of Ill-Conditioned Systems," IEEE Trans. Auto. Control, AC-20, pp. 812-814.
18. O'Malley, R. E. and R. L. Anderson (1982)," Time-Scale Decoupling and Order Reduction for Linear Time-Varying Systems," Optimal Control Meth. Appl., 3, pp. 133.
19. Winkelman, J. R., J. H. Chow, J. J. Allemong and P. V. Kokotovic (1980)," Multi-time-scale Analysis of A Power System," Automatica, vol. 16, pp. 35-43.
20. Kokotovic, P. V., J. J. Allemong, J. R. Winkelman and J. H. Chow (1980)," Singular Perturbation and Iterative Separation of Time Scales," Automatica, vol. 16, pp. 23-32.
21. Chow, J. H. and P. V. Kokotovic (1976)," A Decomposition of Near-Optimal Regulators for Systems with Slow and Fast Modes," IEEE Trans. Auto. Control, AC-21, pp. 701-705.

22. Kokotovic, P. V. and J. H. Chow (1983), "Composite Feedback Control of Nonlinear Singularly Perturbed Systems," Singular Perturbations in Systems and Control, Springer, New York.
23. Philips, R. G. (1980), "A Two Stage Design of Linear Feedback Control," IEEE Trans. Auto. Control, AC-26, pp. 1220-1222.
24. Kokotovic, P. V. and P. Sannuti (1968), "Singular Perturbation Method for Reducing the Model Order in Optimal Control Design," IEEE Trans. Auto. Control, AC-13, pp. 377-384.
25. Gardner, Jr, B. F. and J. B. Cruz, Jr (1980), "Lower Order Control for Systems with Fast and Slow Modes," Automatica, vol. 16, pp. 211-213.
26. Fernando, K. V. and H. Nicholson (1982), "Singular Perturbational Model Reduction of Balanced Systems," IEEE Trans. Auto. Control, AC-27, pp. 466-468.
27. Moore, B. C. (1981), "Principal Component Analysis in Linear Systems: Controllability, Observability, and Model Reduction," IEEE Trans. Auto. Control, AC-26, pp. 17-31.
28. Moore, B. C. (1978), "Singular Value Analysis of Linear Systems," Proc. of IEEE Conf. Dec. Control, pp. 66-73.
29. Kung, S. (1979), "A New Identification and Model Reduction Via Singular Value Decompositions," Proc. 12th Ann. Asilomar Conf. Circuit Syst. Compu., pp. 705-714.

VITA

Lang Hong was born in Fujian, P. R. China on December 3, 1956. He graduated from the Second High School of Fuzhou in China in August 1974. From September 1974 to January 1978, he worked in a factory as a mechanic. In February 1978, he entered The University of Fuzhou and in February 1982 he received a Bachelor of Science degree in Electrical Engineering. Then he worked in The University of Fuzhou as a teaching assistant until August 1984. In the Fall of 1984 he began his graduate study towards Master's and Ph. D's degrees. During the course of study he worked as a teaching assistant for the Department of Electrical Engineering. He received the degree of Master of Science with a major in the Electrical Engineering in June 1986.

The author is a student member of IEEE.