February 16, 1970

To the Graduate Council:

I am submitting herewith a dissertation written by Charles Robert Wall entitled "Topics Related to the Sum of Unitary Divisors of an Integer." I recommend that it be accepted in partial fulfillment of the requirements for the degree of Doctor of Philosophy, with a major in Mathematics.

Robert M. McConnel
Major Professor

We have read this dissertation
and recommend its acceptance:

J. S. Muskat

Robert Plemmons

J. Rule

D. D. Warren

Accepted for the Council:

Hilton A. Smith
Vice Chancellor for
Graduate Studies and Research

TOPICS RELATED TO THE SUM OF UNITARY DIVISORS OF AN INTEGER

A Dissertation

Presented to

the Graduate Council of

The University of Tennessee

In Partial Fulfillment

of the Requirements for the Degree

Doctor of Philosophy

---

by

Charles Robert Wall

March 1970

ACKNOWLEDGEMENTS

The author is sincerely grateful to his major professor, Dr. Robert M. McConnel, for his patient and helpful advice; to his father and most inspiring teacher, Dr. Basil M. Wall, for having always answered questions with other questions; and especially to his wife, Annette, for her years of patiently reading menus from right to left.

ABSTRACT

A divisor d of n is said to be a unitary divisor if d and n/d are relatively prime. Let $\sigma^*(n)$ be the sum of the unitary divisors of n, and let $\sigma(n)$ be the sum of all the divisors of n. Some of the topics of classical number theory which involve $\sigma(n)$ are investigated with the function σ replaced by σ^*.

An integer n is said to be unitary perfect if $\sigma^*(n) = 2n$; some new results concerning such numbers are presented in Chapter II.

Two integers n and m are unitary amicable if they satisfy $n + m = \sigma^*(n) = \sigma^*(m)$. Several theorems concerning unitary amicable numbers are proved in Chapter II, and an appendix lists 610 pairs of unitary amicable numbers.

Let $D\{X\}$ be the asymptotic density of the set X of integers. It is known that the density function

$$A(x) = D\left\{n : \frac{\sigma(n)}{n} \geqslant x\right\}$$

exists and is continuous for all values of the real variable x. Let ψ be Dedekind's function,

$$\psi(n) = n \prod_{p|n} (1 + p^{-1})$$

with the product over primes p which divide n. In Chapter III the existence and continuity of the density functions

$$B(x) = D\left\{ n : \frac{\psi(n)}{n} \geqslant x \right\}$$

and

$$C(x) = D\left\{ n : \frac{\sigma^{*}(n)}{n} \geqslant x \right\}$$

is proved. In addition, upper and lower bounds are obtained for the functions $B(x)$ and $C(x)$ and, as a result, for $A(x)$.

TABLE OF CONTENTS

CHAPTER		PAGE
I.	INTRODUCTION	1
	Background Material	1
	Unitary Divisors	3
	Dedekind's ψ Function	5
	Statement of Objectives	10
II.	PERFECT, ABUNDANT AND AMICABLE NUMBERS	13
	Background and Definitions	13
	Properties of Multiples	14
	Perfect Numbers	16
	Abundant Numbers and Primitives	18
	Unitary Amicable Numbers	20
III.	DENSITY FUNCTIONS	30
	Background	30
	Existence and Continuity of $B(x)$ and $C(x)$	31
	The Functions $B(x,j,k)$ and $C(x,j,k)$	44
	Estimation Procedure	53
	Estimates for $B(x)$	55
	Estimates for $C(x)$	66
	Estimates for $A(x)$	74
	When is an Abundant Number Unitary Abundant?	82

CHAPTER PAGE

LIST OF REFERENCES 87

APPENDIX 89

VITA 107

LIST OF TABLES

TABLE PAGE

I. Selected Values of d and U Satisfying $\sigma(d)/d = \sigma^*(U)/U$ 23

II. Upper and Lower Bounds for B(x) 59

III. Upper and Lower Bounds for C(x) 69

IV. Upper and Lower Bounds for A(x) 77

CHAPTER I

INTRODUCTION

I. BACKGROUND MATERIAL

An arithmetic function has as its domain the set of positive integers. We say that a non-zero arithmetic function f is multiplicative if $f(mn) = f(m)\,f(n)$ whenever $(m,n) = 1$, i.e., for all relatively prime m and n.

Let $\tau(n)$ and $\sigma(n)$ denote the number and sum, respectively, of the positive divisors of n. It is a standard result (see for example [7], Chapter XVI) that τ and σ are multiplicative, so if

$$n = p_1^{e_1} \cdots p_r^{e_r}$$

is the canonical factorization of n, then

$$\tau(n) = (1 + e_1) \cdots (1 + e_r) ,$$

and

$$\sigma(n) = \prod_{i=1}^{r} (1 + p_i + p_i^2 + \ldots + p_i^{e_i}) .$$

Let $\zeta(s)$ be the Riemann zeta function,

$$\zeta(s) = \sum_{n=1}^{\infty} n^{-s} ,$$

which converges for $\mathrm{Re}\, s > 1$. It will be convenient to use Euler's product for $\zeta(s)$,

$$\zeta(s) = \prod_{p} (1 - p^{-s})^{-1} , \tag{1.1}$$

where the infinite product is taken over all primes p .

If f is an arithmetic function we say that $F(s)$ is its generating Dirichlet series if

$$F(s) = \sum_{n=1}^{\infty} f(n)\, n^{-s} .$$

It is well known (see for example [7], Chapter XVII) that if f is multiplicative, then

$$F(s) = \prod_{p} \left\{1 + f(p)\, p^{-s} + f(p^2)\, p^{-2s} + \ldots\right\} . \tag{1.2}$$

It is a standard result [7] that the generating Dirichlet series for τ is $\zeta^2(s)$, and

$$\sum_{n=1}^{\infty} \sigma(n)\, n^{-s} = \zeta(s)\, \zeta(s-1) . \tag{1.3}$$

II. UNITARY DIVISORS

A divisor d of an integer n is said to be a unitary divisor if $(d,n/d) = 1$; in this case we also say that n is a unitary multiple of d .

Suppose that

$$n = p_1^{e_1} \cdots p_r^{e_r}$$

is the canonical factorization of n . If d is any divisor of n , then

$$d = p_1^{a_1} \cdots p_r^{a_r} ,$$

$$n/d = p_1^{b_1} \cdots p_r^{b_r} ,$$

where $0 \leqslant a_i \leqslant e_i$ and $a_i + b_i = e_i$ for $i = 1, 2, \ldots, r$. If d is a unitary divisor of n , then each a_i is either e_i or 0 . It is clear then that if $\tau^*(n)$ is the number of unitary divisors of n , then

$$\tau^*(n) = 2^r$$

and τ^* is a multiplicative function.

We define

$$\sigma^*(n) = \sum_{\substack{d|n \\ (d,n/d)=1}} d \ .$$

Then $\sigma^*(n)$ is the sum of the unitary divisors of n.

Theorem 1. The σ^* function is multiplicative, and

$$\sigma^*(p_1^{e_1}\cdots p_r^{e_r}) = (1 + p_1^{e_1}) \cdots (1 + p_r^{e_r}) \ , \tag{1.4}$$

where the p_i are distinct primes. In particular, $\sigma^*(n)$ is odd if and only if n is a power of 2.

Proof. Clearly $\sigma^*(p^e) = 1 + p^e$ for all primes p and all positive exponents e. Thus we need only prove either half of the theorem. Suppose $r \geqslant 2$; we write $n = p^e n'$ where $(p,n') = 1$. From the behavior of τ^*, n has twice as many unitary divisors as n' does. If d is a unitary divisor of n', then both d and $p^e d$ are unitary divisors of n. We may thus account for all the unitary divisors of n, so that

$$\sigma^*(n) = (1 + p^e)\, \sigma^*(n') = \sigma^*(p^e)\, \sigma^*(n') \ ,$$

from which the theorem follows by an easy induction argument.

If $\omega(n)$ denotes the number of distinct prime divisors of n, we have already seen that

$$\tau^*(n) = 2^{\omega(n)} .$$

It is known [7], then, that

$$\sum_{n=1}^{\infty} \tau^*(n)\, n^{-s} = \zeta^2(s)/\zeta(2s) .$$

By (1.2) we have

$$\sum_{n=1}^{\infty} \sigma^*(n)\, n^{-s} = \prod_p \left\{1 + \sigma^*(p)\, p^{-s} + \sigma^*(p^2)\, p^{-2s} + \ldots\right\}$$

$$= \prod_p \left\{1 + (1 + p)\, p^{-s} + (1 + p^2)\, p^{-2s} + \ldots\right\}$$

$$= \prod_p \frac{1 - p^{-(2s-1)}}{(1 - p^{-s})(1 - p^{-(s-1)})} .$$

Then by Euler's product (1.1) we have

$$\sum_{n=1}^{\infty} \sigma^*(n)\, n^{-s} = \zeta(s)\,\zeta(s-1)/\zeta(2s-1) . \tag{1.5}$$

III. DEDEKIND'S ψ FUNCTION

We shall have several occasions here to make use of Dedekind's ψ function, which is defined by

$$\psi(n) = n \prod_{p|n} (1 + p^{-1}) , \tag{1.6}$$

where the product is taken over all primes p which divide n. Identity (1.6) should be compared with

$$\varphi(n) = n \prod_{p|n} (1 - p^{-1}) ,$$

where φ denotes, as usual, Euler's totient.

Now, ψ is a multiplicative function, and by (1.6),

$$\psi(p^e) = p^e + p^{e-1} .$$

Theorem 2. For all integers n,

$$\sigma^*(n) \leqslant \psi(n) \leqslant \sigma(n) , \tag{1.7}$$

with either equality in both positions or strict inequality in both positions. Equality occurs if and only if n is squarefree.

Proof. Since

$$1 + p^e \leqslant p^{e-1} + p^e \leqslant 1 + p + \ldots + p^{e-1} + p^e$$

for all $e \geqslant 1$, the theorem is true for all prime powers. As the three functions are all multiplicative, the theorem holds for all integers.

As before, we use (1.2) to conclude that

$$(1.8) \qquad \sum_{n=1}^{\infty} \psi(n)\, n^{-s} = \zeta(s)\,\zeta(s-1)/\zeta(2s) \ .$$

The similarities among the generating Dirichlet series for σ, σ^* and ψ should be noted.

We shall later be examining the ratio $\psi(n)/n$; the following result will be useful.

Theorem 3. If n and m are squarefree and $\psi(n)/n = \psi(m)/m$, then $n = m$.

Proof. If $n = 1$, then clearly $m = n$. Either $m = n$ for all integers n and m satisfying the hypotheses, or else there is a pair n and m which provides a counter-example. If the latter is the case, we may take n and m to be minimal in the sense that they are relatively prime. We write $n = p_1 \cdots p_k$ with $p_1 < \ldots < p_k$ and $m = q_1 \cdots q_j$ with $q_1 < \ldots < q_j$. Then if

$$\frac{1 + p_1}{p_1} \cdot \ldots \cdot \frac{1 + p_k}{p_k} = \frac{\psi(n)}{n} = \frac{\psi(m)}{m} = \frac{1 + q_1}{q_1} \cdot \ldots \cdot \frac{1 + q_j}{q_j} \ ,$$

we have

$$n\,(1 + q_1) \cdots (1 + q_j) = m\,(1 + p_1) \cdots (1 + p_k) \ .$$

Now, $p_k | n$ and $(n,m) = 1$, so

$$p_k \mid (1 + p_1) \cdots (1 + p_{k-1})(1 + p_k) .$$

Thus $p_k | (1 + p_i)$ for some i. But

$$1 + p_1 < 1 + p_2 < \ldots < 1 + p_{k-1} < 1 + p_k .$$

As p_k exceeds each term except possibly $1 + p_{k-1}$ and $1 + p_k$, and $(p_k, 1+p_k) = 1$, we must have $p_k | (1 + p_{k-1})$. Hence, because $1 + p_{k-1} \leq p_k$, we must have $p_k = 1 + p_{k-1}$. Therefore, $k = 2$, $p_1 = 2$, and $p_2 = 3$. A similar argument shows that $j = 2$, $q_1 = 2$, and $q_2 = 3$. Hence $(n,m) = 6$, contradicting the relative primality of n and m.

Corollary. If n and m are squarefree and $\sigma(n)/n = \sigma(m)/m$ or $\sigma^*(n)/n = \sigma^*(m)/m$, then $n = m$.

Lemma. If q is the largest prime dividing m, then $\sigma(m) < q\, \sigma^*(m)$ for all $m > 1$.

Proof. It is known [7] that $\sigma(m)/m < m/\varphi(m)$ for all $m > 1$. Since $\sigma^*(m) > m$ for all $m > 1$, we have

$$\frac{\sigma(m)}{\sigma^*(m)} = \frac{\sigma(m)}{m} \cdot \frac{m}{\sigma^*(m)} < \frac{\sigma(m)}{m} < \frac{m}{\varphi(m)} .$$

But

$$\frac{m}{\varphi(m)} = \prod_{p|m} \frac{p}{p-1} \leq \frac{2}{1} \cdot \frac{3}{2} \cdot \frac{4}{3} \cdot \ldots \cdot \frac{p}{p-1} \cdot \ldots \cdot \frac{q}{q-1} = q \ ,$$

and the lemma is proved.

Theorem 4. For all integers n ,

$$2\psi(n) \geq \sigma(n) + \sigma^*(n) \ . \tag{1.9}$$

Proof. We clearly have equality if n is squarefree.

If n is a prime power, say $n = p^e$, then it is easy to verify that

$$2\psi(p^e) - \sigma(p^e) - \sigma^*(p^e) = (p^e - p)/(p - 1) \geq 0 \ ,$$

so (1.9) holds.

Suppose the theorem is false, and let n be the least integer for which (1.9) fails to hold. By the minimality of n and the multiplicative nature of the three functions, no prime divides n only once. We may take $\omega(n) \geq 2$ since we have already eliminated the case in which n is a prime power. Let p be the largest prime dividing n and write $n = p^a m$ with $(m,p) = 1$, and let q be the largest prime dividing m . Then clearly $q < p$. By the minimal character of n ,

$$2\psi(p^{a-1})\psi(m) \geq \sigma(p^{a-1})\sigma(m) + \sigma^*(p^{a-1})\sigma^*(m)$$

and

$$2\,\psi(p^a)\,\psi(m) < \sigma(p^a)\,\sigma(m) + \sigma^*(p^a)\,\sigma^*(m) \ .$$

Since $a \geqslant 2$, $\psi(p^a) = p\,\psi(p^{a-1})$, so we may multiply the first inequality by p to obtain, from the second,

$$p\,\sigma(p^{a-1})\,\sigma(m) + p\,\sigma^*(p^{a-1})\,\sigma^*(m) < \sigma(p^a)\,\sigma(m) + \sigma^*(p^a)\,\sigma^*(m) \ .$$

Now, $p\,\sigma^*(p^{a-1}) = p^a + p = \sigma^*(p^a) + p - 1$ and $p\,\sigma(p^{a-1}) = \sigma(p^a) - 1$, so it follows that

$$\sigma(m) > (p - 1)\,\sigma^*(m) \ .$$

Thus by the lemma above,

$$p < 1 + \sigma(m)/\sigma^*(m) < 1 + q \leqslant p \ ,$$

an obvious contradiction.

Thus (1.9) holds for all n .

IV. STATEMENT OF OBJECTIVES

There are several topics in classical number theory that may be formulated in terms of the sum of divisors function σ . For example, the Greeks called an integer n <u>perfect</u> if it equals the sum of its proper divisors, a condition equivalent to $\sigma(n) = 2n$. Two integers n and m are said to be <u>amicable</u> if each is equal to

the sum of the proper divisors of the other, i.e., if

$$\sigma(n) = \sigma(m) = n + m \ .$$

Behrend [2] and Davenport [4], among others, investigated the function

$$A(x) = D\left\{n : \frac{\sigma(n)}{n} \geqslant x\right\} \ ,$$

where $D\{X\}$ is the asymptotic density of the set X of integers.

What we shall do in this dissertation is examine these topics with the function σ replaced by σ^* .

Subbarao and Warren [11] have defined and discussed unitary perfect numbers. In Chapter II we shall present some new results and alternate proofs which augment their discussion.

Also in Chapter II we define unitary amicable numbers in a natural way, and we present in an appendix a rather extensive list of such numbers.

It will be seen later that the functions σ and σ^* have markedly different behavior. Dedekind's ψ function is included here as a matter of convenience to indicate the behavior of a function that lies between the two extremes σ and σ^* .

In Chapter III we define, and prove the existence and continuity of, the functions

$$B(x) = D\left\{n : \frac{\psi(n)}{n} \geqslant x\right\}$$

and

$$C(x) = D\left\{n : \frac{\sigma^*(n)}{n} \geqslant x\right\} .$$

It is easy to see by Theorem 2 that

$$C(x) \leqslant B(x) \leqslant A(x) .$$

It will be seen later that lower bounds are relatively easy to obtain for each of the three density functions $A(x)$, $B(x)$ and $C(x)$. Originally, the goal of the research presented in the latter part of Chapter III was to obtain upper bounds for $B(x)$ and use these estimates to obtain upper bounds for $C(x)$. It happens, however, that of the three density functions, $B(x)$ is the easiest to bound. As a result, we use Theorem 4 and the upper bounds for $B(x)$ to obtain meaningful upper bounds for $A(x)$.

CHAPTER II

PERFECT, ABUNDANT AND AMICABLE NUMBERS

I. BACKGROUND AND DEFINITIONS

Euclid defined an integer n to be perfect if it equals the sum of its proper divisors, a requirement equivalent to $\sigma(n) = 2n$. In terms of the σ function, we say that an integer n is abundant if $\sigma(n) > 2n$, and deficient if $\sigma(n) < 2n$.

Let f be σ, ψ, or σ^*. We say that

if $f(n) < 2n$, n is f-<u>deficient</u>;

if $f(n) = 2n$, n is f-<u>perfect</u>;

if $f(n) > 2n$, n is f-<u>abundant</u>.

We shall also from time to time use <u>unitary perfect</u> and <u>unitary abundant</u> instead of σ^*-perfect and σ^*-abundant, respectively.

More generally, we say that n is (x,f)-<u>abundant</u> whenever $f(n) > xn$, and define (x,f)-<u>perfect</u> and (x,f)-<u>deficient</u> numbers in a similar fashion. An (x,f)-<u>nondeficient</u> number is one which is either (x,f)-perfect or (x,f)-abundant; (x,f)-<u>nonabundant</u> numbers are defined analogously. We shall, if convenient, revert to the more restricted names in the case $x = 2$.

II. PROPERTIES OF MULTIPLES

In this section we develop some properties of multiples of (x,f)-abundant numbers. These properties will later be the basis for the definition of primitives.

Theorem 1. If n divides m, then $\sigma(m)/m \geq \sigma(n)/n$.

Proof. We note that for p a prime, the ratio $\sigma(p^e)/p^e$ increases as e increases. We write $n = \prod p^{a_p}$ and $m = \prod p^{b_p}$. Since n divides m, $a_p \leq b_p$ for all p.

Corollary. Any multiple of an (x,σ)-nondeficient number is itself (x,σ)-nondeficient.

We have already defined $\omega(n)$ as the number of distinct primes that divide n. We say that a multiple m of n is an ω-multiple of n if $\omega(m) = \omega(n)$. If m is an ω-multiple of n, then it is clear that any prime that divides m also divides n.

Because of identity (1.6) the ratio $\psi(n)/n$ depends not upon the value n, but only upon the set of primes dividing n. If m is an ω-multiple of n, then the respective sets of dividing primes are identical. If m is any multiple of n, then the set of primes that divide n is contained in the corresponding set for m. Hence we have proved the following results:

Theorem 2. If m is an ω-multiple of n, then $\psi(m)/m = \psi(n)/n$.

Theorem 3. If n divides m, then $\psi(m)/m \geq \psi(n)/n$.

Corollary. Any multiple of an (x,ψ)-nondeficient number is itself (x,ψ)-nondeficient.

In the σ^* case the properties of multiples are not quite as nice as the ones above. The extreme cases are given in the following theorem.

Theorem 4. Suppose m is a unitary multiple of n, and that k is an ω-multiple of n. Then

$$\sigma^*(k)/k \leq \sigma^*(n)/n \leq \sigma^*(m)/m .$$

Proof. If $m = nd$ with $(d,n) = 1$, then since $\sigma^*(d) \geq d$, $\sigma^*(m)/m = \sigma^*(n)\,\sigma^*(d)/nd \geq \sigma^*(n)/n$. If k is an ω-multiple of $n = \prod p^{a_p}$, we write $k = \prod p^{b_p}$ with $b_p \geq a_p$ for all p. Hence

$$\sigma^*(k)/k = \prod (1 + p^{-b_p}) \leq \prod (1 + p^{-a_p}) = \sigma^*(n)/n .$$

Corollary. Any unitary multiple of an (x,σ^*)-nondeficient number is itself (x,σ^*)-nondeficient.

III. PERFECT NUMBERS

Euclid showed that $n = 2^{p-1}(2^p - 1)$ is σ-perfect if $2^p - 1$ is a prime. Indeed, it was in this connection that Euclid introduced the concept of prime numbers. Euler proved a partial converse to Euclid's theorem: if an even integer is σ-perfect, then it must be of Euclid's form. The search for even σ-perfect numbers, then, becomes the search for Mersenne primes, those of the form $2^p - 1$. This has been the stimulus for much research in number theory; for example, Lucas's Test is a major result in testing for primality numbers of the form $2^p - 1$. One occasionally sees numbers cited as being the largest known prime: these are nearly always Mersenne primes.

There are no known odd σ-perfect numbers, and there are some reasons to suspect that there are none. Most research on the question of the possible existence of odd σ-perfect numbers has been in the direction of describing the number and type of prime divisors of such integers, if any exist. Another unanswered question about σ-perfect numbers deals with their possible infinitude.

In contrast with the situation with σ-perfect numbers, we can completely characterize the ψ-perfect numbers. We note that the largest squarefree divisor of an integer n is the product of the first powers of the primes dividing n, and is the smallest divisor m such that n is an ω-multiple of m.

Theorem 5. An integer n is ψ-perfect if and only if $n = 2^a 3^b$ with a and b positive integers. In particular, then, there are infinitely many ψ-perfect numbers, each of which is even and has 6 as its largest squarefree divisor.

Proof. Since $\psi(6) = 12$, any ω-multiple of 6 is ψ-perfect by Theorem 2. On the other hand, if n is ψ-perfect and m is the largest squarefree divisor of n, then $\psi(n)/n = \psi(m)/m = \psi(6)/6$ and hence $m = 6$ by Theorem 3 of Chapter I, so n is an ω-multiple of 6.

Subbarao and Warren [11] investigated unitary perfect numbers and reported that the first four are 6, 60, 90 and 87,360. They proved there are no odd unitary perfect numbers; we shall offer an alternate proof of this fact in connection with unitary amicable numbers. Because of Euclid's theorem and Euler's partial converse, coupled with the fact that there are several relatively small Mersenne primes, there is some support for the conjecture that there is an infinite number of σ-perfect numbers. However, unitary perfect numbers are much more complicated, and Subbarao and Warren conjectured that there are only finitely many.

This author, in attempting to understand the difficulty in forming unitary perfect numbers, discovered that the integer $2^{18}3 \cdot 5^4 7 \cdot 11 \cdot 13 \cdot 19 \cdot 37 \cdot 79 \cdot 109 \cdot 157 \cdot 313$ is unitary perfect.

In the theory of numbers, conjecturing that there are only finitely many of a type of objects is at best hazardous. On most questions of possible infinitude, the bulk of evidence rejects the finiteness position. It is the author's opinion that such will be the case with unitary perfect numbers; while these numbers are admittedly very scarce, it is felt that there is probably an infinitude of them.

IV. ABUNDANT NUMBERS AND PRIMITIVES

Because of the corollaries to Theorems 1, 3, and 4 of this chapter, we may define an integer n to be

(a) (x,σ)-primitive if n is (x,σ)-nondeficient and every proper divisor of n is (x,σ)-deficient;

(b) (x,ψ)-primitive if n is (x,ψ)-nondeficient and every proper divisor of n is (x,ψ)-deficient;

(c) (x,σ^*)-primitive if n is (x,σ^*)-nondeficient and every proper unitary divisor of n is (x,σ^*)-deficient.

Let f be σ, ψ, or σ^*. We immediately notice that if $x \leq 1$, then 1 is the only (x,f)-primitive. Also, it is clear that any (x,ψ)-primitive must be squarefree by Theorem 2, and by Theorem 2 of Chapter I, any (x,ψ)-primitive is an (x,f)-primitive. Thus to demonstrate the infinitude of the (x,f)-primitives, and hence that there are infinitely many (x,f)-abundant numbers, we need only examine the (x,ψ)-primitives.

Another question that arises concerning the (x,f)-primitives deals with divisibility. Are all (x,f)-primitives divisible by at least one member of some finite set of primes? We shall answer this question in the negative in the next theorem. First, however, we pause to prove a known result.

Lemma. The product $\prod(1 + p^{-1})$, taken over all primes p, diverges to infinity.

Proof. Since each term in the product exceeds unity, if the product diverges, it diverges to infinity. Suppose the product converges. Then by Euler's product we have

$$\prod_p (1 + p^{-1}) = \prod_p \frac{1 - p^{-2}}{1 - p^{-1}} = \zeta(1)/\zeta(2) = 6\,\zeta(1)/\pi^2 ,$$

which implies the convergence of the harmonic series, an obvious contradiction.

Theorem 6. For any $x > 1$ and any integer $N \geq 1$ there is an infinite sequence of (x,ψ)-primitives which are relatively prime to N and relatively prime in pairs.

Proof. If p is a prime, then $\psi(p)/p = 1 + p^{-1}$. Because any terminal subproduct of the infinite product $\prod(1 + p^{-1})$, taken over the primes, must diverge to infinity, it is clear that we may construct an infinite sequence of pairwise relatively prime (x,ψ)-

primitives by starting with an arbitrary term in the infinite product and multiplying by consecutive terms until we have a finite subproduct which exceeds x ; this finite subproduct is then an (x,ψ)-primitive; the procedure is reiterated to obtain an infinite sequence of primitives. We may insure relative primality to N by starting with any term $1 + p^{-1}$ in the product beyond the term corresponding to the largest prime dividing N .

V. UNITARY AMICABLE NUMBERS

Two integers n and m are said to be amicable if each is equal to the sum of the proper divisors of the other, a condition clearly equivalent to $\sigma(n) = \sigma(m) = n + m$.

We define integers n and m to be unitary amicable if

$$\sigma^*(n) = \sigma^*(m) = n + m .$$

Clearly any unitary perfect number is unitary amicable with itself. Also, if n and m are unitary amicable and $n < m$, then

$$2n < \sigma^*(n) = n + m = \sigma^*(m) < 2m ,$$

so that n is unitary abundant and m is unitary deficient.

In this section we shall discuss some of the properties of unitary amicable numbers. Some of these properties will be of interest in the area of amicable numbers, since there are pairs of

squarefree integers which are both amicable and unitary amicable. In addition, we present in an appendix a list of 610 pairs of unitary amicable numbers.

There are no known amicable pairs in which one member is odd and the other is even, and it has been conjectured that no such pairs exist. One naturally wonders what the situation is with unitary amicable numbers. By Theorem 1 of Chapter I, $\sigma^*(n)$ is odd if and only if n is some power of 2. Suppose n and m are unitary amicable. If $n + m$ were odd, then both n and m would be powers of 2, contradicting that their sum is odd. Thus we have proved the following theorem.

Theorem 7. The two numbers of a unitary amicable pair are either both odd or both even.

Corollary. If there is a pair of amicable numbers having opposite parity, then at least one of the two numbers contains a nontrivial squared factor.

Current lists [1; 3; 5; 6; 9; 10] of amicable numbers support this author's conjecture that if two odd integers are amicable, then they are incongruent modulo 4. We prove a similar result for unitary amicable numbers:

Theorem 8. If n and m are odd and unitary amicable, then

n and m are incongruent modulo 4 .

Proof. Suppose n and m are odd and unitary amicable with $n \equiv m \pmod 4$. Then $n + m \equiv 2 \pmod 4$, so n and m are each prime powers, and by unique factorization must be equal. But then $n = m = p^e$ and

$$\sigma^*(n) = 1 + p^e < 2p^e = n + m ,$$

a contradiction.

Corollary. There are no odd unitary perfect numbers.

In the appendix we list 610 unitary amicable pairs. Most of these pairs are obtained from amicable pairs. Let $N = dn$ and $M = dm$ be amicable, where $(d,n) = (d,m) = 1$, and n and m have no common unitary divisor greater than unity. If N and M are both squarefree, then we immediately have a unitary amicable pair. In most other cases, i.e., except for the miscellaneous pairs under Escott's classification [5], n and m are both squarefree; we seek numbers U such that $(n,U) = (m,U) = 1$ and Un and Um are unitary amicable, i.e.,

$$\sigma(d)/d = \sigma^*(U)/U .$$

The search for such U is greatly simplified by the interchanges listed in Table I.

TABLE I

SELECTED VALUES OF d AND U
SATISFYING $\sigma(d)/d = \sigma^*(U)/U$

d	U
3^3	$2^2 3^3 7$
$3^3 5$	$2 \cdot 3^3 7$
$3^2 5$	$2 \cdot 3^2 5^2$
$3^2 13$	$2 \cdot 3^3$, $2^2 3^3 5$, $2 \cdot 3^4 41$
$5^2 31$	$5^2 7 \cdot 13$
$3^2 5^2 31$	$2 \cdot 3^2 5^2 7$
$3^2 7^2 13 \cdot 19$	$2 \cdot 3^2 5 \cdot 7$, $2 \cdot 3 \cdot 7$
$7^2 19$	$5 \cdot 7^2$

It should be noted that the interchanges in Table I may only be used when conditions of relative primality allow. Since six of the eight substitutions above go from odd numbers to even numbers, one might correctly conclude that most of the examples of unitary amicable numbers are obtained from odd amicable pairs.

In addition to the interchanges of Table I, it should be noted that the following are interchangeable as unitary divisors of U whenever conditions of relative primality permit:

(a) 2 , $2^2 5$, $2^3 3$, $2^4 3 \cdot 17$, $2^5 3 \cdot 11$;

(b) 3 , $3^2 5$, $2^3 3^3 7$, $2^2 3^3 5^3 7^2$, $2^5 3^3 7 \cdot 11$;

(c) 3^3 , $3^4 41$;

(d) 7 , $5^2 7^2 13$, $3^2 5^3 7^2$.

Several other groups of integers could be added to the ones above, but we have listed the more useful ones.

Escott [5] classified amicable numbers by forms, grouping together all amicable numbers $n = Ep_1 \cdots p_a$ and $m = Eq_1 \cdots q_b$, where $(n,m) = E$, the p_i and q_j are distinct primes, and a and b are fixed. Our classification is a modification of Escott's: a typical entry

$$U \ , \ p_1 \cdots p_a \ , \ q_1 \cdots q_b$$

in the listing in the appendix represents the unitary amicable numbers $n = Up_1 \cdots p_a$ and $m = Uq_1 \cdots q_b$, where U is the greatest common unitary divisor of n and m , the p_i and q_j are prime powers, and n is unitary abundant. Allowing prime powers in the place of primes is justified since $\sigma^*(p^e) = 1 + p^e$ for $e = 1, 2, \ldots$; that is, primes and prime powers have the same type image under the σ^* function, while such is not the case with the σ function. Within

each form, the middle entries of the triples are arranged in lexicographical order.

With the five known unitary perfect pairs, we have 615 known pairs of unitary amicable numbers. A study of these examples raises several questions. In the following, suppose $N = dn$ and $M = dm$ are unitary amicable, where $(d,n) = (d,m) = 1$, $n \leq m$, and d is the greatest common unitary divisor of N and M.

Question 1. Given n and m, are there infinitely many choices for d?

We cannot answer Question 1 fully; however, as an approach to a negative answer, we have the following theorem and its corollaries. We remark that Subbarao and Warren [11] proved that for m a fixed positive integer, there are only finitely many unitary perfect numbers that are unitary multiples of 2^m; the following theorem represents an extension of their result to unitary amicable numbers.

Theorem 9. Let a and A be fixed positive integers, and let B be fixed. There are only finitely many pairs of unitary amicable numbers N and M such that $N \leq M$, $2^a \| N$, $2^A \| (N + M)$ and $M/N = B$.

Proof. Suppose the theorem is false; then there is an infinite subset of integers N of the form $N = 2^a CD_i$ with C an odd con-

stant, $(C,D_i) = 1$, and $\omega(D_i) \leq A$. We may then select an infinite subset such that each D_i is composed of the same number of distinct primes. Then we may take the prime powers in D_i to be increasing, so that $\lim_i \sigma^*(D_i)/D_i = 1$.

If $C = 1$, then

$$(1 + 2^a)\,\sigma^*(D_i) = \sigma^*(N) = N + M \geq 2N = 2^{a+1}D_i .$$

Hence $\sigma^*(D_i)/D_i \geq 2^{a+1}/(1 + 2^a) \geq 4/3$, a contradiction.

If $C > 1$, we note that $(N + M)/N$ is constant, so

$$\frac{N + M}{N} = \frac{\sigma^*(N)}{N} = \frac{\sigma^*(2^aCD_i)}{2^aCD_i} \xrightarrow[i\to\infty]{} \frac{\sigma^*(2^aC)}{2^aC} .$$

Hence $(1 + 2^a)\,\sigma^*(C) = \sigma^*(2^aC) = (N + M)2^aC/N = (N + M)/D_i$. Thus $2^A \mid \sigma^*(C)$. But for each i , $2 \mid \sigma^*(D_i)$, so $2^{A+1} \mid \sigma^*(CD_i)$ and hence $2^{A+1} \mid \sigma^*(N) = N + M$, a contradiction.

Corollary. For fixed A , there are only finitely many pairs of unitary amicable numbers N and M such that $2^A \parallel (N + M)$ and M/N is constant.

Corollary. For fixed A , there are only finitely many pairs of unitary amicable numbers N and M such that M/N is constant and $2^{A+1} \nmid (N + M)$.

Question 2. Are there any unitary amicable numbers which are unitary multiples of different powers of 2 ? That is, is there a counter-example to the claim that n and m are always both odd?

We have no known counter-examples. A tedious investigation of special cases shows that if there is a counter-example pair, then each of the numbers is divisible by 2^8 , and the smaller of the two exceeds six million.

Question 3. Assuming a negative answer to Question 2, is it always true that either $n = m$ or $n \not\equiv m \pmod 4$?

Again, we have neither counter-examples nor a proof. One should note, however, the similarity of this question to Theorem 7.

Question 4. Are there any pairs of unitary amicable numbers which share no non-trivial common unitary divisor? That is, can we have $d = 1$?

Question 5. What is the smallest power of 2 which is a unitary divisor of the sum of two odd unitary amicable numbers?

One may show, by considering special cases, that there is no odd unitary abundant number k such that $2^8 \nmid \sigma^*(k)$. However, the eight known pairs of odd unitary amicable pairs are all such that their sums are divisible by 2^{12} . One would like to know if the

exponent 12 can be improved downward.

Question 6. Are there only finitely many choices of N and M such that N + M is a unitary multiple of a given power of 2 ?

This is an extremely difficult problem. The two simplest cases, however, are easily solved: one can show that there are no unitary amicable pairs whose sum is a unitary multiple of 2 , and if $2^2 \| (N + M)$, then N = M and their common value is either 6 or 90 .

Question 7. Are there any pairs of integers which are both amicable and unitary amicable which are not both squarefree?

Finally we shall mention briefly a generalization of perfect numbers and amicable pairs. We say that integers $n_1, n_2, \ldots, n_r$ form a cycle of length r if

$$\sigma^*(n_i) - n_i = n_{i+1} \quad (i = 1, 2, \ldots, r-1) ,$$

$$\sigma^*(n_r) - n_r = n_1 .$$

Then it is easily seen that a unitary perfect number is a cycle of length one, and unitary amicable pairs form cycles of length two.

Clearly, at least one of the n_i of a cycle must be unitary abundant if $r \geqslant 2$. The only cycles, other than unitary perfect and

unitary amicable pairs, which are known to the author are the following:

30 , 42 , 54 (length 3) ;

1482 , 1878 , 1890 , 2142 , 2178 (length 5) ;

2418 , 2958 , 3522 , 3534 , 4146 , 4158 , 3906 , 3774 ,
4434 , 4446 , 3954 , 3966 , 3978 , 3582 (length 14) .

Every other cycle of length not less than three is such that each unitary abundant element of the cycle exceeds 1500 .

CHAPTER III

DENSITY FUNCTIONS

I. BACKGROUND

Let S be a set of integers and let $S(n)$ be the number of elements of S among the first n integers. We define the density $D\{S\}$ of the set S by $D\{S\} = \lim_n S(n)/n$ provided the limit in question exists. In this chapter we shall be investigating the three density functions associated with the (x,f)-abundant numbers, where as in the previous chapter f is σ, ψ or σ^*.

Behrend [2] let $A(x,n)$ be the number of integers $m \leq n$ for which $\sigma(m) \geq xm$, and set

$$A(x) = \lim_{n\to\infty} A(x,n)/n .$$

Davenport [4] proved that $A(x)$ exists and is continuous for all x. It is easily seen, then, that the function $A(x)$ measures the density of the (x,σ)-nondeficient numbers.

More generally, Behrend denoted by $A(x,j,k,n)$ the number of integers of the form $mj \leq n$ with $(m,k) = 1$ and $\sigma(mj) \geq xmj$. We remark that we may as well take k to be squarefree, since if n is the largest squarefree divisor of N, then $(m,N) = 1$ if and only

if $(m,n) = 1$. Clearly $A(x,1,1,n) = A(x,n)$. As before, we define

$$A(x,j,k) = \lim_{n\to\infty} A(x,j,k,n)/n \ ,$$

provided the limit exists. When it is convenient we shall use $A(x)$ instead of $A(x,1,1)$.

By considering ψ instead of σ in the above, we may define $B(x,j,k,n)$, $B(x,j,k)$ and $B(x)$; we substitute σ^* for σ and define $C(x,j,k,n)$, $C(x,j,k)$ and $C(x)$. The definitions of the density functions are only provisional at this point as they depend upon the as yet unproved existence of several limits. In the next two sections we shall establish the existence and continuity of the functions $B(x,j,k)$ and $C(x,j,k)$. The remainder of the chapter will deal with estimates for the three main density functions.

II. EXISTENCE AND CONTINUITY OF $B(x)$ AND $C(x)$

We shall denote by $f\circ g$ the Dirichlet product of arithmetic functions f and g . That is,

$$(f\circ g)(n) = \sum_{d|n} f(d)\, g(n/d) \ .$$

It is well known that if f and g are multiplicative, then $f\circ g$ is also multiplicative. Let ν_k be the multiplicative function defined by $\nu_k(n) = n^k$.

Let μ be the Mobius function, $\mu(1) = 1$, $\mu(n) = (-1)^r$ if n is a product of r distinct primes, and $\mu(n) = 0$ if n is not squarefree. The famed Mobius inversion theorem, a proof of which may be found in any elementary number theory text, states that $f = g \circ \nu_0$ if and only if $g = f \circ \mu$.

Essential to the work of all this chapter is the <u>mean</u> $M\{f\}$ of an arithmetic function f, defined by

$$M\{f\} = \lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} f(n)$$

provided the limit exists. We remark that the density, if existent, of a set of integers is the mean of the characteristic function of that set.

Let f be an arithmetic function, and set $g = f \circ \mu$. It is known that if $M\{f\}$ exists and the series $\sum_1^\infty g(n)/n$ converges, then the two are equal. However, Wintner [12] has shown that the convergence of $\sum_1^\infty g(n)/n$ is neither necessary nor sufficient for the existence of $M\{f\}$. Our procedure for dealing with means is as follows: if $[x]$ denotes the integral part of x, we write

$$\frac{1}{N} \sum_{n=1}^{N} f(n) = \frac{1}{N} \sum_{n=1}^{N} \sum_{d|n} g(d) = \frac{1}{N} \sum_{d=1}^{N} g(d) \left[\frac{N}{d}\right]$$

$$= \sum_{d=1}^{\infty} g(d)/d - \sum_{d=N+1}^{\infty} g(d)/d - \frac{1}{N} \sum_{d=1}^{N} g(d) \left(\frac{N}{d} - \left[\frac{N}{d} \right] \right) .$$

Then if we can show that $\sum_1^{\infty} g(d)/d$ converges and that the last two terms above each approach zero as N increases, we will have

$$M\{f\} = \sum_{d=1}^{\infty} g(d)/d .$$

We shall now apply this technique to the functions $(\sigma^*(n)/n)^z$ and $(\psi(n)/n)^z$ for complex values of z.

Theorem 1. The mean $M\{(\sigma^*(n)/n)^z\}$ exists for all complex numbers z, and

$$M\left\{ \left| \frac{\sigma^*(n)}{n} \right|^z \right\} = \prod_p \left\{ 1 - \frac{1}{p} + \sum_{k=1}^{\infty} \frac{p-1}{p^{k+1}} (1 + p^{-k})^z \right\} ;$$

moreover, the mean is continuous at $z = 0$.

Proof. Let

$$\rho_z(m) = \sum_{d|m} (\sigma^*(d)/d)^z \mu(m/d) .$$

Then ρ_z is multiplicative as μ and σ^* are, and

$$\rho_z(1) = 1 , \quad \rho_z(p) = (1 + p^{-1})^z - 1 ,$$

$$\rho_z(p^e) = (1 + p^{-e})^z - (1 + p^{-e+1})^z \qquad (e \geq 2) .$$

Now, if u and v are real, then

$$u^z - v^z = \int_v^u z\, t^{z-1}\, dt ,$$

so if $\max(u,v) \leq 2$, and $z = x + iy$, then

$$|u^z - v^z| \leq |z|\ 2^{x-1}\ |u - v| .$$

Hence if $e \geq 2$,

$$(3.1) \qquad |\rho_z(p^e)| \leq |z|\ 2^{x-1}\ (p-1)\ p^{-e} \leq |z|\ 2^{x-1}\ p^{-e/2} .$$

Also,

$$|\rho_z(p)| \leq |z|\ 2^{x-1}\ p^{-1} \leq |z|\ 2^{x-1}\ p^{-1/2} ,$$

so, since ρ_z is multiplicative, we have

$$|\rho_z(n)| \leq C^{\omega(n)}\ n^{-1/2} ,$$

where C depends only on z. But

$$\omega(n) = O\left(\frac{\log n}{\log \log n}\right)$$

so we have

$$\rho_z(n) = O(n^{\epsilon - 1/2})$$

for every $\epsilon > 0$, uniformly in z for $|z|$ bounded. In particular,

(3.2) $$\rho_z(n) = O(n^{-1/3}) .$$

By the Mobius inversion theorem we have

$$(\sigma^*(n)/n)^z = \sum_{d|n} \rho_z(d) ,$$

so that if $[t]$ denotes the integral part of t, then

$$\frac{1}{N} \sum_{n=1}^{N} (\sigma^*(n)/n)^z = \frac{1}{N} \sum_{n=1}^{N} \rho_z(n) \, [N/n]$$

$$= \sum_{n=1}^{\infty} \rho_z(n)/n - \sum_{n=1}^{N} \rho_z(n) \left(\frac{1}{n} - \frac{1}{N}\left[\frac{N}{n}\right] \right) - \sum_{n=N+1}^{\infty} \rho_z(n)/n .$$

But

$$\left| \sum_{n=1}^{N} \rho_z(n) \left(\frac{1}{n} - \frac{1}{N}\left[\frac{N}{n}\right] \right) \right| \leq \sum_{n=1}^{N} |\rho_z(n)| \, \frac{1}{N} \left(\frac{N}{n} - \left[\frac{N}{n}\right] \right)$$

$$\leq \frac{1}{N} \sum_{n=1}^{N} |\rho_z(n)| = O\left(\frac{1}{N} \sum_{n=1}^{N} n^{-1/3} \right) = O(N^{-1/3})$$

and

$$\left| \sum_{n=N+1}^{\infty} \rho_z(n)/n \right| \leq \sum_{n=N+1}^{\infty} |\rho_z(n)|/n = O\left(\sum_{n=N+1}^{\infty} n^{-4/3} \right) = O(N^{-1/3}) .$$

Hence

$$\frac{1}{N}\sum_{n=1}^{N}(\sigma^*(n)/n)^z = \sum_{n=1}^{\infty}\rho_z(n)/n + O(N^{-1/3}) ,$$

so that

$$M\{(\sigma^*(n)/n)^z\} = \sum_{n=1}^{\infty}\rho_z(n)\, n^{-1}$$

and the series is absolutely convergent by (3.2).

Since ρ_z is multiplicative, by (1.2) we have

$$\sum_{n=1}^{\infty}\rho_z(n)\, n^{-1} = \prod_p\{1 + \rho_z(p)\, p^{-1} + \rho_z(p^2)\, p^{-2} + \ldots\}$$

$$= \prod_p\{1 - \frac{1}{p} + \sum_{k=1}^{\infty}\frac{p-1}{p^{k+1}}(\sigma^*(p^k)/p^k)^z\}$$

$$= \prod_p\{1 - \frac{1}{p} + \sum_{k=1}^{\infty}\frac{p-1}{p^{k+1}}(1 + p^{-k})^z\} .$$

This completes the proof of Theorem 1.

Theorem 2. The mean $M\{(\psi(n)/n)^z\}$ exists for all complex numbers z, and

$$M\left\{\left|\frac{\psi(n)}{n}\right|^z\right\} = \prod_p\left\{1 - \frac{1}{p} + \frac{1}{p}\left|\frac{1+p}{p}\right|^z\right\} ;$$

moreover, the mean is continuous at $z = 0$.

Proof. The proof of this theorem parallels that of the preceding, with the exception that the upper bound of (3.1) may be replaced by zero since $\psi(p^e)/p^e = \psi(p^{e+1})/p^{e+1}$ for $e \geq 1$. We obtain

$$M\left\{\left|\frac{\psi(n)}{n}\right|^z\right\} = \prod_p \left\{1 - \frac{1}{p} + \sum_{k=1}^{\infty} \frac{p-1}{p^{k+1}} \left|\frac{\psi(p^k)}{p^k}\right|^z\right\}$$

$$= \prod_p \left\{1 - \frac{1}{p} + \frac{1}{p}\left|\frac{1+p}{p}\right|^z \sum_{k=1}^{\infty} \frac{p-1}{p^k}\right\} = \prod_p \left\{1 - \frac{1}{p} + \frac{1}{p}(1 + p^{-1})^z\right\} ,$$

as asserted.

We recall that a distribution function is a non-decreasing left-continuous function f with $f(-\infty) = 0$ and $f(\infty) = 1$. One version of the continuity theorem for Fourier-Stieltjes transforms states that if $\{f_n(w)\}$ is a sequence of distribution functions such that

$$c(y) = \lim_{n\to\infty} \int_{-\infty}^{\infty} e^{iyw}\, df_n(w)$$

exists for all real y and is continuous for $y = 0$, then there is a unique distribution function $f(w)$ such that

$$c(y) = \int_{-\infty}^{\infty} e^{iyw}\, df(w)$$

and $f(w) = \lim_n f_n(w)$ for every w at which $f(w)$ is continuous. According to Kac [8], Wiener showed that

$$\lim_{T\to\infty} \frac{1}{T} \int_0^T |c(y)|^2 \, dy$$

is the sum of the squares of the jumps of $f(w)$. As a result, if this limit is zero, then f is continuous.

We shall describe our approach to establishing the existence and continuity of $B(x)$; a similar approach will be used for $C(x)$. Let $F_N(w)$ be the number of integers $n \leq N$ such that

$$\log \frac{\psi(n)}{n} < w ,$$

and set $f_N(w) = N^{-1} F_N(w)$. Then $f_N(w)$ is a distribution function, and

$$\int_{-\infty}^{\infty} e^{iyw} \, df_N(w) = [\exp (iy \log \frac{\psi(1)}{1}) + \ldots + \exp (iy \log \frac{\psi(N)}{N})]/N$$

so that

$$\lim_{N\to\infty} \int_{-\infty}^{\infty} e^{iyw} \, df_N(w) = M\left\{\left|\frac{\psi(n)}{n}\right|^{iy}\right\} .$$

Then by Theorem 2 and the continuity theorem for Fourier-Stieltjes transforms, there is a distribution function $f(w) = \lim_N f_N(w)$ at

the points of continuity of f, and

$$f(w) = D\left\{n : \log \frac{\psi(n)}{n} < w\right\}.$$

But $B(x) = 1 - f(\log x)$ for $x > 0$. Thus if f exists and is continuous, then $B(x)$ exists and is continuous. To establish the continuity of f, because of Wiener's result we need only show that

$$\lim_{T\to\infty} \frac{1}{T} \int_0^T \left|M\{(\psi(n)/n)^{iy}\}\right|^2 dy = 0 .$$

Lemma. If $\lambda_1, \lambda_2, \ldots, \lambda_k$ are real numbers linearly independent over the rational field, and a_j and b_j are constants for $j = 1, \ldots, k$, then

$$\lim_{T\to\infty} \frac{1}{T} \int_0^T \prod_{j=1}^{k} (a_j + b_j \cos \lambda_j y)\, dy = a_1 a_2 \cdots a_k .$$

Proof. We first note that if α is real, then

$$\lim_{T\to\infty} \frac{1}{T} \int_0^T e^{i\alpha y}\, dy = \begin{cases} 1 & \text{if } \alpha = 0 \\ 0 & \text{if } \alpha \neq 0 , \end{cases} \tag{3.3}$$

for if $\alpha \neq 0$, then

$$\left|\frac{1}{T} \int_0^T e^{i\alpha y}\, dy\right| = \left|\frac{e^{i\alpha T} - 1}{\alpha T}\right| \leq \frac{2}{|\alpha| T} .$$

Let m be any positive integer, and let $\alpha_1, \ldots, \alpha_m$ be real. Then

$$\cos \alpha_1 y \cdots \cos \alpha_m y$$

$$= 2^{-m} (e^{i\alpha_1 y} + e^{-i\alpha_1 y}) \cdots (e^{i\alpha_m y} + e^{-i\alpha_m y})$$

$$= 2^{-m} \sum \exp i [\epsilon_1 \alpha_1 + \ldots + \epsilon_m \alpha_m] y ,$$

where the summation is over all possible choices of $\epsilon_j = \pm 1$ for $1 \leq j \leq m$. If the $\alpha_1, \ldots, \alpha_m$ are now assumed to be linearly independent over the rationals, then no exponent can be zero, and by (3.3) we have

$$\lim_{T\to\infty} \frac{1}{T} \int_0^T \cos \alpha_1 y \cdots \cos \alpha_m y \, dy = 0 .$$

We write

$$\prod_{j=1}^{k} (a_j + b_j \cos \lambda_j y)$$

as a sum of terms, each of which is a product of some (or none) of the a_j and some (or none) of the cosine terms. We may distribute the integral over this finite sum and take limits individually. The only nonzero limit will correspond to the summand $a_1 a_2 \cdots a_k$, that is, to the only term in the sum that contains no cosines. Then

$$\lim_{T\to\infty} \frac{1}{T} \int_0^T \prod_{j=1}^{k} (a_j + b_j \cos \lambda_j y)\, dy$$

$$= \lim_{T\to\infty} \frac{1}{T} \int_0^T a_1 a_2 \cdots a_k \, dy = a_1 a_2 \cdots a_k ,$$

and the lemma is proved.

It is known that the numbers $\log \frac{p+1}{p}$, with p prime, are linearly independent over the rational field. There are several proofs of this fact; one proof may be obtained by a slight modification of the proof of Theorem 3 of Chapter I.

Theorem 3. The function $B(x)$ exists and is continuous for all real x.

Proof. As noted above, we need only show that

$$\lim_{T\to\infty} \frac{1}{T} \int_0^T \left| M\{(\psi(n)/n)^{iy}\} \right|^2 dy = 0 .$$

From Theorem 2 we have

$$M\{(\psi(n)/n)^{iy}\} = \prod_p \left\{ 1 - \frac{1}{p} + \frac{1}{p} \left(\frac{p+1}{p} \right)^{iy} \right\} .$$

Now,

$$(3.4) \qquad \left|1 - \frac{1}{p} + \frac{1}{p}\left(\frac{p+1}{p}\right)^{iy}\right|^2$$

$$= 1 - 2p^{-1} + 2p^{-2} + 2\,\frac{p-1}{p^2}\cos y \log\frac{p+1}{p} .$$

For a later purpose, we remark that

$$(3.5) \qquad \left|1 - \frac{1}{p} + \frac{1}{p}\left(\frac{p+1}{p}\right)^{iy}\right| \leq 1 .$$

By the lemma above,

$$(3.6) \qquad \lim_{T\to\infty} \frac{1}{T}\int_0^T \prod_{p \leq p_k} \left|1 - \frac{1}{p} + \frac{1}{p}\left(\frac{p+1}{p}\right)^{iy}\right|^2 dy = \prod_{p \leq p_k}\left(1 - \frac{2}{p} + \frac{2}{p^2}\right) .$$

Since $\sum(p^{-1} - p^{-2})$ diverges, the product in the right member of (3.6) diverges to zero as k increases without bound.

Theorem 4. The function $C(x)$ exists and is continuous for all real x .

Proof. As before, we need only show that

$$\lim_{T\to\infty} \frac{1}{T}\int_0^T \left|M\{(\sigma^*(n)/n)^{iy}\}\right|^2 dy = 0 .$$

From Theorem 1 we have

$$M\{(\sigma^*(n)/n)^{iy}\} = \prod_p \left\{1 - \frac{1}{p} + \sum_{k=1}^{\infty} \frac{p-1}{p^{k+1}} (1 + p^{-k})^{iy}\right\}$$

$$= \prod_p \left\{1 + \sum_{e=1}^{\infty} p^{-e} \left[(\sigma^*(p^e)/p^e)^{iy} - (\sigma^*(p^{e-1})/p^{e-1})^{iy}\right]\right\} .$$

Now,

$$\left|1 - \frac{1}{p} + \sum_{k=1}^{\infty} \frac{p-1}{p^{k+1}} (1 + p^{-k})^{iy}\right|$$

$$= \left|1 - \frac{1}{p} + \frac{1}{p}\left(\frac{p+1}{p}\right)^{iy} + p^{-2} \sum_{e=0}^{\infty} p^{-e} \left\{\left(\frac{\sigma^*(p^{e+2})}{p^{e+2}}\right)^{iy} - \left(\frac{\sigma^*(p^{e+1})}{p^{e+1}}\right)^{iy}\right\}\right|$$

$$\leqslant \left|1 - \frac{1}{p} + \frac{1}{p}\left(\frac{p+1}{p}\right)^{iy}\right| + p^{-2} \sum_{e=0}^{\infty} 2p^{-e}$$

$$= \left|1 - \frac{1}{p} + \frac{1}{p}\left(\frac{p+1}{p}\right)^{iy}\right| + 2(p-1)/p^3 .$$

Hence by (3.4) and (3.5) we have

$$\left|1 - \frac{1}{p} + \sum_{k=1}^{\infty} \frac{p-1}{p^{k+1}} (1 + p^{-k})^{iy}\right|^2$$

$$\leqslant \left|1 - \frac{1}{p} + \frac{1}{p}\left(\frac{p+1}{p}\right)^{iy}\right|^2 + 4\,\frac{p-1}{p^3}\left|1 - \frac{1}{p} + \frac{1}{p}\left(\frac{p+1}{p}\right)^{iy}\right| + 4(p-1)^2 p^{-6}$$

$$\leq \left(1 - \frac{2}{p} + \frac{6}{p^2} - \frac{4}{p^3} + \frac{4}{p^4} - \frac{8}{p^5} + \frac{4}{p^6}\right) + 2\,\frac{p-1}{p^2}\cos y \log\frac{p+1}{p}\,.$$

Then by the preceding lemma, we have

$$\lim_{T\to\infty} \frac{1}{T}\int_0^T \prod_{p \leq p_k} \left|1 - \frac{1}{p} + \sum_{k=1}^{\infty} \frac{p-1}{p^{k+1}}\,(1 + p^{-k})^{iy}\right|^2 dy$$

$$\leq \prod_{p \leq p_k} (1 - 2p^{-1} + 6p^{-2} - 4p^{-3} + 4p^{-4} - 8p^{-5} + 4p^{-6})$$

and as before the product diverges to zero as k increases without bound.

III. THE FUNCTIONS $B(x,j,k)$ AND $C(x,j,k)$

In this section we shall prove the existence and continuity of $B(x,j,k)$ and $C(x,j,k)$. We shall first obtain this result for the special case in which every prime that divides j also divides k. The general case for all j and k will follow from the special case.

We define the character function $\chi_k(n)$ by

$$\chi_k(n) = \begin{cases} 1 & \text{if } (n,k) = 1 \\ 0 & \text{if } (n,k) > 1 \end{cases}$$

and remark that for each k, χ_k is completely multiplicative; that

is, $\chi_k(n)\chi_k(m) = \chi_k(nm)$ for all integers n and m, whether relatively prime or not.

Theorem 5. Let j and k be integers such that every prime divisor of j also divides k. Let f be a multiplicative function and let $g = f \circ \mu$. Set

$$\sum_{n=1}^{\infty} g(n)\, n^{-s} = \prod_{p} G(p,s) .$$

Suppose that $M\{f\}$ exists, that for all primes p, $G(p,1) \neq 0$, and that $\sum g(n)/n$ converges. Suppose also that

$$\sum_{n=1}^{N} |g(n)| = o(N)$$

and that

$$\sum_{n=N+1}^{\infty} |g(n)/n| = o(1) .$$

Then $M\{f(nj)\chi_k(n)\}$ exists and

$$M\{f(nj)\chi_k(n)\} = \frac{f(j)}{j} \cdot \frac{\varphi(k)}{k} \cdot \frac{M\{f\}}{\prod_{p|k} G(p,1)} .$$

Proof. We shall first prove the result for $j = 1$. Let $[x]$

denote the integral part of x. Then

$$\sum_{n=1}^{N} f(n)\chi_k(n) = \sum_{n=1}^{N} \chi_k(n) \sum_{d|n} g(d)$$

$$= \sum_{d=1}^{N} g(d)\left\{\chi_k(d) + \chi_k(2d) + \ldots + \chi_k(d[N/d])\right\}$$

$$= \sum_{d=1}^{N} g(d)\chi_k(d)\left\{\chi_k(1) + \ldots + \chi_k([N/d])\right\}$$

by the complete multiplicativity of the χ_k function. Now, the term $\chi_k(1) + \ldots + \chi_k([N/d])$ inside the summation is the number of integers not exceeding $[N/d]$ which are relatively prime to k. This number is $[N/d]\,\varphi(k)/k + O(1)$. Thus

$$\sum_{n=1}^{N} f(n)\chi_k(n) = \sum_{d=1}^{N} g(d)\chi_k(d)\left\{\frac{\varphi(k)}{k}\left[\frac{N}{d}\right] + O(1)\right\}$$

$$= \frac{\varphi(k)}{k} \sum_{d=1}^{N} g(d)\chi_k(d)\left[\frac{N}{d}\right] + O(o(N)) .$$

Hence

$$\frac{1}{N}\sum_{n=1}^{N} f(n)\chi_k(n) = \frac{\varphi(k)}{k}\,\frac{1}{N}\sum_{d=1}^{N} g(d)\chi_k(d)\,[N/d] + o(1)$$

$$= \frac{\varphi(k)}{k} \left\{ \sum_{d=1}^{\infty} \frac{g(d)\chi_k(d)}{d} - \frac{1}{N} \sum_{d=1}^{N} g(d)\chi_k(d) \left(\frac{N}{d} - \left[\frac{N}{d} \right] \right) \right.$$

$$\left. - \sum_{d=N+1}^{\infty} \frac{g(d)\chi_k(d)}{d} \right\} + o(1) \ .$$

But

$$\left| \sum_{d=1}^{N} g(d)\chi_k(d) \left(\frac{N}{d} - \left[\frac{N}{d} \right] \right) \right| \leq \sum_{d=1}^{N} |g(d)\chi_k(d)| = o(N)$$

and

$$\left| \sum_{d=N+1}^{\infty} \frac{g(d)\chi_k(d)}{d} \right| \leq \sum_{d=N+1}^{\infty} |g(d)/d| = o(1) \ ,$$

so that

$$M\{f\chi_k\} = \frac{\varphi(k)}{k} \sum_{d=1}^{\infty} \frac{g(d)\chi_k(d)}{d} = \frac{\varphi(k)}{k} \prod_{p \nmid k} G(p,1)$$

$$= \frac{\varphi(k)}{k} \prod_{p} G(p,1) \Big/ \prod_{p|k} G(p,1) = \frac{\varphi(k)}{k} \cdot \frac{M\{f\}}{\prod_{p|k} G(p,1)} \ .$$

Now, since every prime that divides j also divides k, either $\chi_k(n) = 0$ or $(n,j) = 1$. Since f is multiplicative, we have

$$M\{f(nj)\chi_k(n)\} = \lim_{N\to\infty} \frac{1}{N} \sum_{nj \leq N} f(nj)\chi_k(n)$$

$$= \lim_{N\to\infty} \frac{1}{N} \sum_{nj \leq N} f(j)\, f(n)\chi_k(n) = \lim_{N\to\infty} \frac{f(j)}{j} \cdot \frac{1}{N/j} \sum_{n \leq N/j} f(n)\chi_k(n)$$

$$= \frac{f(j)}{j} M\{f(n)\chi_k(n)\} = \frac{f(j)}{j} \cdot \frac{\varphi(k)}{k} \cdot \frac{M\{f\}}{\prod_{p|k} G(p,1)} .$$

Corollary. For all j and k such that every prime divisor of j also divides k, the functions $B(x,j,k)$ and $C(x,j,k)$ exist and are continuous for all x.

Proof. One can verify that Theorem 1 and Theorem 2 may be obtained from Theorem 5 by considering the appropriate functions and setting $j = k = 1$. Furthermore, one can carry through the proof of Theorem 3 for $M\{(\psi(nj)/nj)^{iy}\chi_k(n)\}$ and the proof of Theorem 4 for $M\{(\sigma^*(nj)/nj)^{iy}\chi_k(n)\}$ with essentially no change, and obtain the corollary.

Let f be any multiplicative function with $f(n) \geq 1$ for all choices of n. For example, $f(n)$ might be $\sigma(n)/n$, $\psi(n)/n$ or $\sigma^*(n)/n$. As before, we define the density function $F(x,j,k)$, if existent, of f, and we write $F(x,1,1) = F(x)$. We define the inverse image set

$$F^{-1}(x,j,k) = \{n : j|n, (n/j,k) = 1 \text{ and } f(n) \geq x\} .$$

If $F(x,j,k)$ exists, then clearly

$$F(x,j,k) = D\left\{F^{-1}(x,j,k)\right\} .$$

Suppose, as in the previous remarks, that $F(x,j,k)$ exists if every prime that divides j also divides k. We now wish to show that this result is sufficient to obtain the existence of $F(x,j,k)$ for all j and k.

Suppose that p is a prime which divides neither j nor k. Since any multiple of p is a unitary multiple of some power of p, we have

$$F^{-1}(x,p^m j,k) = \bigcup_{i=m}^{\infty} F^{-1}(x,p^i j,pk)$$

and we remark that the union is over pairwise disjoint sets. Since density distributes over disjoint unions, we have

$$F(x,p^m j,k) = \sum_{i=m}^{\infty} F(x,p^i j,pk) . \tag{3.7}$$

We remark that it is easy to prove the convergence of the series (3.7).

It is clear that we could repeat the procedure above and express any $F(x,j,k)$ as a series of terms $F(x,j',k')$ where each j' is an ω-multiple of j, and k' is the largest squarefree divisor of jk. But then any prime that divides such a j' also

divides k' and we know, by the corollary to Theorem 5, that $F(x,j',k')$ exists. Thus we have the following result.

Theorem 6. For all integers j and k, the functions $B(x,j,k)$ and $C(x,j,k)$ exist and are continuous for all x.

If X is a set of integers and m is an integer we define

$$mX = \{m\}X = \{mx : x \in X\} .$$

If $D\{X\}$ exists, then it is easy to show that

$$D\{mX\} = \frac{1}{m} D\{X\} .$$

Using this observation it is easy to prove the following result.

Theorem 7. If every prime that divides m also divides k, and $(m,j) = 1$, then

$$m F^{-1}(x,j,k) = F^{-1}(xf(m),mj,k) ,$$

so that

$$F(x,mj,k) = \frac{1}{m} F(x/f(m),j,k) ,$$

and in particular

$$F(x,m,k) = \frac{1}{m} F(x/f(m),1,k) .$$

Throughout the remainder of this chapter let p_1 , p_2 , ... be the primes in order. We define $P_m = p_1 \cdots p_m$ as the product of the first m primes with $P_0 = 1$. Then

$$F^{-1}(x,1,1) = F^{-1}(x,2,1) \cup F^{-1}(x,1,2)$$

$$= F^{-1}(x,2,1) \cup F^{-1}(x,3,2) \cup F^{-1}(x,1,6)$$

$$= F^{-1}(x,2,1) \cup F^{-1}(x,3,2) \cup F^{-1}(x,5,6) \cup F^{-1}(x,1,30)$$

$$= \ldots = F^{-1}(x,1,P_m) \cup \bigcup_{i=1}^{m} F^{-1}(x,p_i,P_{i-1}) = \ldots .$$

As the sets $F^{-1}(x,p_i,P_{i-1})$ are pairwise disjoint and, for $i \leq m$, each is disjoint from $F^{-1}(x,1,P_m)$, we have

$$F(x) = F(x,1,1) = F(x,2,1) + F(x,1,2)$$

$$= F(x,2,1) + F(x,3,2) + F(x,1,6)$$

$$= F(x,2,1) + F(x,3,2) + F(x,5,6) + F(x,1,30)$$

$$= \ldots = F(x,1,P_m) + \sum_{i=1}^{m} F(x,p_i,P_{i-1}) = \ldots .$$

If we allow m to increase without bound, we obtain

$$F(x) = \sum_{i=1}^{\infty} F(x,p_i,P_{i-1}) \ .$$

Since

$$F(x,p_i,P_{i-1}) = \sum_{j=1}^{\infty} F(x,p_i^j,P_i) \ ,$$

by Theorem 7, we have

$$F(x,p_i,P_{i-1}) = \sum_{j=1}^{\infty} p_i^{-j} F(x/f(p_i^j),1,P_i) \ .$$

Hence

$$F(x) = F(x,1,P_m) + \sum_{i=1}^{m} \sum_{j=1}^{\infty} p_i^{-j} F(x/f(p_i^j),1,P_i)$$

for $m = 1, 2, \ldots$, and

$$F(x) = \sum_{i=1}^{\infty} \sum_{j=1}^{\infty} p_i^{-j} F(x/f(p_i^j),1,P_i) \ .$$

Finally, we note that because of (1.7) and (1.9) we clearly have

$$C^{-1}(x,j,k) \leq B^{-1}(x,j,k) \leq A^{-1}(x,j,k) \leq B^{-1}(\tfrac{x+1}{2},j,k)$$

so that

$$(3.8) \qquad C(x,j,k) \leq B(x,j,k) \leq A(x,j,k) \leq B(\frac{x+1}{2},j,k) \ .$$

IV. ESTIMATION PROCEDURE

The following three sections are devoted to obtaining upper and lower bounds for the three density functions $A(x)$, $B(x)$ and $C(x)$ at selected values of x between 1 and 3 . For $B(x)$ and $C(x)$, we compute bounds at intervals of 0.005 for x between 1 and 1.1 , 0.01 for x between 1.1 and 2 , 0.02 for x between 2 and 2.1 , and 0.05 for x between 2.1 and 3 . The function $A(x)$ is bounded at the same points for $x \geq 1.1$ and at intervals of 0.01 for x between 1 and 1.1 .

Our purpose is not to obtain upper and lower bounds that are so close together that the functional values will be pinpointed, but rather to obtain estimates that will indicate, in some intuitional way, the behavior of the functions.

It should be noted that one may obtain lower bounds for the three functions by making use of primitives. The procedure amounts to computing the density of the integers that are divisible by one or more of a finite number of integers. However, more sophisticated techniques are needed to obtain upper bounds.

Let f be a multiplicative function with $f(n) \geq 1$ for all

n . As before, $f(n)$ could be $\sigma(n)/n$, $\psi(n)/n$ or $\sigma^*(n)/n$. Suppose that k is a squarefree number and that j is an integer such that each prime that divides j also divides k .

We denote by $F(x,j,k,n)$ the number of integers of the form $mj \leq n$ with $(m,k) = 1$ for which $f(mj) \geq x$. We let $G(x,j,k,n)$ be the number of such integers $mj \leq n$ for which $f(mj) < x$. Let $F(x,j,k) = \lim_n F(x,j,k,n)/n$. We remark that if $(m,k) = 1$, then $(m,j) = 1$, so that if $f(j) \geq x$, the mj are all such that $f(mj) \geq x$ and hence $F(x,j,k) = \varphi(k)/jk$.

Suppose now that $f(j) < x$. Then either $f(mj) \geq x$ or $f(j) \leq f(mj) < x$. Hence

$$\sum_{mj \leq n} f(mj)\chi_k(m) \geq x\, F(x,j,k,n) + f(j)\, G(x,j,k,n) .$$

Since

$$F(x,j,k,n) + G(x,j,k,n) = \frac{n\,\varphi(k)}{jk} + O(1) ,$$

we have

$$F(x,j,k,n)/n \leq \frac{1}{x - f(j)} \left(\frac{1}{n} \sum_{mj \leq n} f(mj)\chi_k(m) - \frac{f(j)\,\varphi(k)}{jk} + O(1/n) \right) .$$

Then by Theorem 5 we have

$$F(x,j,k) \leq \frac{1}{x - f(j)} \left(M\{f(mj)\chi_k(m)\} - \frac{f(j)\ \varphi(k)}{jk} \right)$$

$$= \frac{1}{x - f(j)} \cdot \frac{f(j)}{j} \left(M\{f\chi_k\} - \varphi(k)/k \right)$$

$$= \frac{f(j)}{x - f(j)} \cdot \frac{1}{j} M$$

where $M = M\{f\chi_k\} - \varphi(k)/k$.

Finally, for x close to $f(j)$ we will find it convenient to use the trivial estimate

$$F(x,j,k) \leq \varphi(k)/jk \ .$$

These two inequalities involving $F(x,j,k)$ will provide our upper bounds.

V. ESTIMATES FOR $B(x)$

From the previous section we have

$$B(x,j,k) \leq \frac{\psi(j)/j}{x - \psi(j)/j} \cdot \frac{M}{j} \ ,$$

where

$$M = \frac{\varphi(k)}{k} \left(\frac{15}{\pi^2} \prod_{p|k} \frac{p^2}{1 + p^2} - 1 \right) \ .$$

Let k be squarefree and let j be a divisor of k, say $j = q_1 \cdots q_m$ with the q_i distinct primes. Then if $\psi(j)/j < x$,

$$S_j = \sum_{i=1}^{m} \sum_{e_i=1}^{\infty} B(x, q_1^{e_1} \cdots q_m^{e_m}, k)$$

$$\leq \frac{M\,\psi(j)/j}{x - \psi(j)/j} \cdot \frac{1}{q_1 - 1} \cdot \cdots \cdot \frac{1}{q_m - 1} = \frac{\psi(j)/j}{x - \psi(j)/j} \cdot \frac{M}{\varphi(j)} .$$

We obtain upper bounds for $B(x)$ by summing the estimates S_j above over all j dividing k for which $\psi(j)/j < x$, and adding this sum to a bound for those divisors j of k for which $\psi(j)/j \geq x$.

To obtain the bounds presented in this section, we use the value $k = P_7 = 2\cdot3\cdot5\cdot7\cdot11\cdot13\cdot17$. Then $M < 0.00237672$. There are 64 odd divisors j of k; the corresponding S_j were bounded as above. The upper bounds were computed to six decimal places, summed and truncated to four decimal places. This led to upper bounds for $B(x,1,2)$. After refinement, described below, of these bounds, the identity

$$B(x,2,1) = B(\tfrac{2}{3}x,1,2) ,$$

obtained from Theorem 7, was used to obtain the upper bounds which are presented, truncated to three decimal places, in the latter part of this section. Except as noted below for $x = 2$, the value $k = P_8 = 2\cdot3\cdot5\cdot7\cdot11\cdot13\cdot17\cdot19$ was used to compute the lower bounds.

To obtain more precisely the density of the ψ-abundant numbers we revise upward our lower bound for $B(2)$. The following even numbers are ψ-nondeficient: $2\cdot3$, $2\cdot5\cdot7$, $2\cdot7\cdot11\cdot13$, $2\cdot5\cdot17\cdot19$, $2\cdot5\cdot11p$ $(13 \leqslant p \leqslant 53)$, $2\cdot5\cdot13p$ $(17 \leqslant p \leqslant 31)$. A routine computation based on these primitives yields

$$B(2,2,1) > 0.17985 .$$

The following odd numbers are ψ-abundant: $3\cdot5\cdot7\cdot11p$ $(13 \leqslant p \leqslant 383)$, $3\cdot5\cdot7\cdot13p$ $(17 \leqslant p \leqslant 61)$. Using these primitives we conclude that

$$B(2,1,2) > 0.00058 .$$

Hence

$$B(2) > 0.17985 + 0.00058 > 0.1804 .$$

Preliminary investigation showed that the behavior of $B(x)$ for x close to 1 is crucial to the behavior at other points. We pause now to describe our procedure for estimating $B(x)$ for x close to 1 .

Suppose $0 < x < 1/p_n$. Then if $(m,P_n) > 1$, $\psi(m)/m \geqslant 1 + x$. Hence

$$B(1 + x) \geqslant 1 - \varphi(P_n)/P_n .$$

We set

$$M = \frac{\varphi(P_n)}{P_n}\left(\frac{15}{\pi^2}\prod_{i=1}^{n}\frac{p_i^2}{1+p_i^2} - 1\right) .$$

Then as before we have

$$1 - \varphi(P_n)/P_n \leq B(1+x) \leq 1 - \varphi(P_n)/P_n + M/x$$

if $0 < x < 1/p_n$. This procedure was used to obtain upper bounds for $17 \leq p_n \leq 67$. Use was also made of the fact that the upper bound curves, obtained as above, are concave upward. These refinements were incorporated into the upper bounds for $B(x,1,2)$.

Our upper and lower bounds for $B(x)$ are given in Table II. Because of the behavior of $B(x)$, the spread between upper and lower bounds is not always an accurate indicator of how accurate are our estimates: the wider spreads between upper and lower bounds generally occur when both bounds are dropping sharply. The bounds in Table II are illustrated in Figure I.

TABLE II

UPPER AND LOWER BOUNDS FOR B(x)

x	B(x) Lower Bound	B(x) Upper Bound
1.00	1.000	1.000
1.005	0.870	0.940
1.01	0.870	0.906
1.015	0.868	0.894
1.02	0.861	0.888
1.025	0.851	0.884
1.03	0.847	0.880
1.035	0.836	0.876
1.04	0.836	0.872
1.045	0.828	0.869
1.05	0.828	0.865
1.055	0.819	0.863
1.06	0.808	0.860
1.065	0.808	0.860
1.07	0.808	0.860
1.075	0.808	0.860
1.08	0.794	0.850
1.085	0.794	0.850
1.09	0.794	0.850
1.095	0.777	0.848
1.10	0.777	0.831
1.11	0.777	0.820
1.12	0.776	0.812
1.13	0.776	0.807
1.14	0.776	0.803
1.15	0.745	0.800
1.16	0.744	0.794
1.17	0.744	0.783
1.18	0.743	0.776
1.19	0.743	0.772

TABLE II (continued)

x	B(x) Lower Bound	B(x) Upper Bound
1.20	0.743	0.769
1.21	0.717	0.769
1.22	0.711	0.758
1.23	0.696	0.745
1.24	0.694	0.737
1.25	0.691	0.731
1.26	0.691	0.727
1.27	0.689	0.723
1.28	0.686	0.720
1.29	0.686	0.718
1.30	0.682	0.715
1.31	0.679	0.709
1.32	0.679	0.706
1.33	0.679	0.704
1.34	0.634	0.701
1.35	0.592	0.686
1.36	0.592	0.667
1.37	0.591	0.649
1.38	0.584	0.639
1.39	0.584	0.628
1.40	0.584	0.622
1.41	0.578	0.618
1.42	0.573	0.614
1.43	0.573	0.611
1.44	0.570	0.607
1.45	0.565	0.604
1.46	0.561	0.600
1.47	0.561	0.597
1.48	0.556	0.592
1.49	0.556	0.590
1.50	0.555	0.585
1.51	0.425	0.515
1.52	0.425	0.479
1.53	0.408	0.468
1.54	0.390	0.461

TABLE II (continued)

x	B(x) Lower Bound	B(x) Upper Bound
1.55	0.381	0.450
1.56	0.375	0.441
1.57	0.367	0.434
1.58	0.357	0.427
1.59	0.347	0.422
1.60	0.347	0.421
1.61	0.335	0.420
1.62	0.317	0.409
1.63	0.309	0.401
1.64	0.292	0.396
1.65	0.291	0.373
1.66	0.291	0.371
1.67	0.289	0.358
1.68	0.289	0.348
1.69	0.288	0.347
1.70	0.285	0.340
1.71	0.285	0.335
1.72	0.255	0.334
1.73	0.252	0.330
1.74	0.252	0.323
1.75	0.250	0.322
1.76	0.248	0.310
1.77	0.248	0.302
1.78	0.248	0.301
1.79	0.248	0.296
1.80	0.248	0.291
1.81	0.222	0.291
1.82	0.216	0.290
1.83	0.214	0.279
1.84	0.200	0.279
1.85	0.196	0.265
1.86	0.196	0.256
1.87	0.193	0.255
1.88	0.193	0.248
1.89	0.190	0.240

TABLE II (continued)

x	B(x) Lower Bound	B(x) Upper Bound
1.90	0.190	0.239
1.91	0.187	0.239
1.92	0.187	0.235
1.93	0.187	0.235
1.94	0.183	0.232
1.95	0.183	0.229
1.96	0.180	0.229
1.97	0.180	0.222
1.98	0.180	0.219
1.99	0.180	0.219
2.00	0.180	0.216
2.02	0.092	0.213
2.04	0.092	0.178
2.06	0.084	0.160
2.08	0.084	0.149
2.10	0.084	0.132
2.15	0.070	0.120
2.20	0.061	0.109
2.25	0.055	0.094
2.30	0.039	0.088
2.35	0.038	0.077
2.40	0.038	0.068
2.45	0.015	0.058
2.50	0.012	0.047
2.55	0.009	0.040
2.60	0.007	0.036
2.65	0.005	0.033
2.70	0.005	0.029
2.75	0.003	0.027
2.80	0.001	0.024
2.85	0.001	0.021
2.90	0.000	0.020
2.95	0.000	0.019
3.00	0.000	0.018

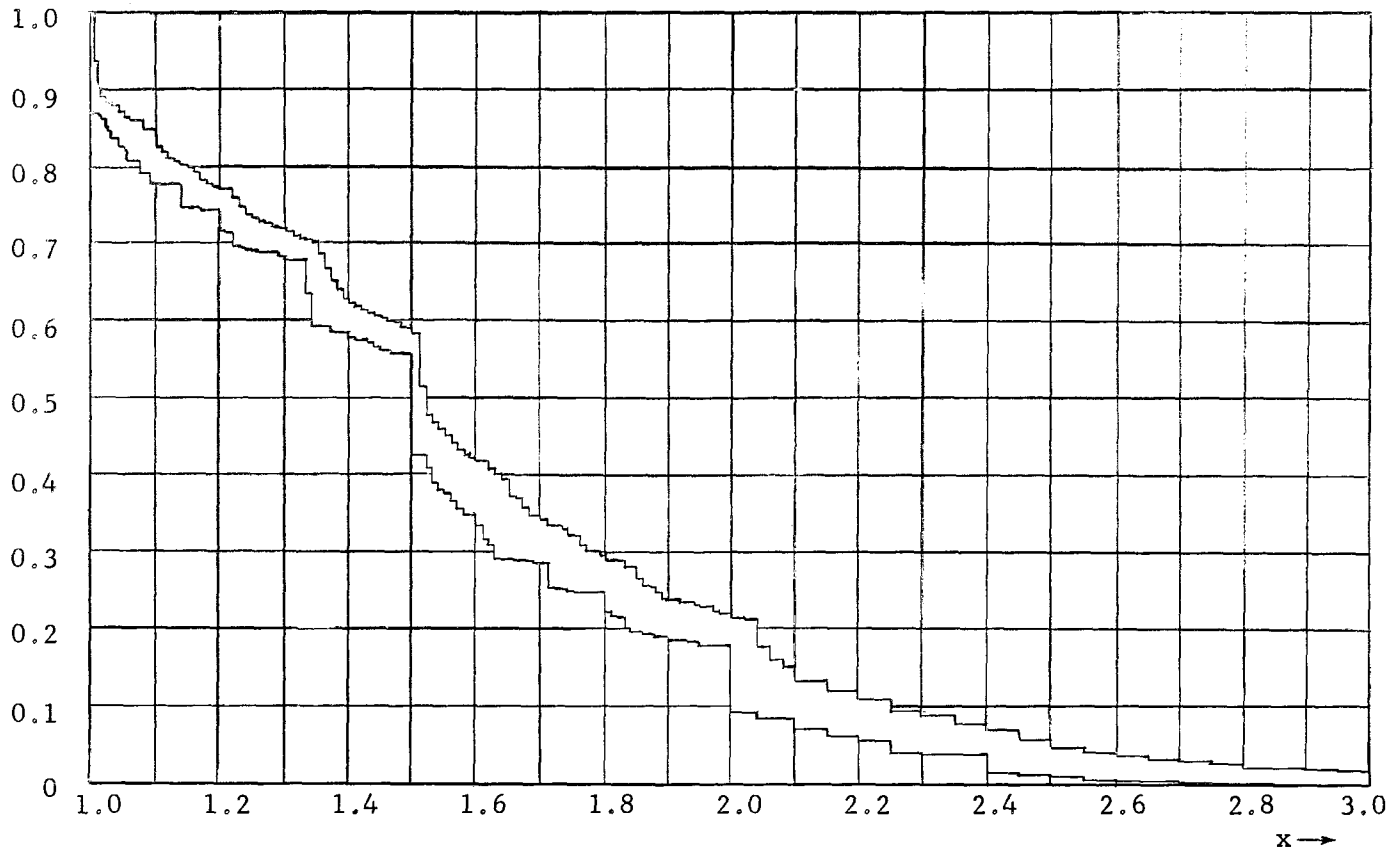

FIGURE I

UPPER AND LOWER BOUNDS FOR B(x)

It would appear from Table II and Figure I that

$$\lim_{x \to 0^+} \frac{1}{1 - B(1 + x)} = 0 .$$

We prove a weaker statement, namely that $B(1 + x) < 1 - x$ for x small and positive. Even this weaker result, however, will imply that $B(x)$ cannot possess a derivative at $x = 1$. As a result, we may use Theorem 7 to conclude that $B(x)$ does not possess a derivative at any point x for which there is an integer n such that $x = \psi(n)/n$.

Theorem 8. For $0 < x \leq 0.28$, $B(1 + x) < 1 - x$.

Proof. We claim that if $p_n \geq 13$, then

$$\frac{15}{\pi^2} \prod_{p < p_n} \frac{p^2}{1 + p^2} = \prod_{p \geq p_n} (1 + p^{-2}) < 1 + 2/3p_n .$$

If $x \geq 13$, then $\frac{20}{11}(x - 2) \geq \frac{3}{2} x$. Also, $\exp x \leq 1 + 11x/10$ if $0 \leq x \leq 1/7$. Now let $p_n = 2m + 1 \geq 13$. Then

$$\log \prod_{p \geq p_n} (1 + p^{-2}) = \sum_{p \geq p_n} \log (1 + p^{-2}) \leq \sum_{p \geq p_n} p^{-2}$$

$$\leq \sum_{i=m}^{\infty} (2i + 1)^{-2} < \int_m^{\infty} (2t - 1)^{-2} \, dt = \frac{1}{2(2m - 1)} = \frac{1}{2(p_n - 2)} .$$

Hence

$$\prod_{p \geqslant p_n} (1 + p^{-2}) \leqslant \exp\left(\frac{1}{2(p_n - 2)}\right) \leqslant 1 + \frac{11}{10 \cdot 2(p_n - 2)} \leqslant 1 + \frac{1}{3p_n/2}$$

since $p_n \geqslant 13$, and the claim is proved.

Now, if $n \geqslant 7$, then

$$1 - \frac{2}{2} \cdot \frac{3}{4} \cdot \frac{4}{6} \cdot \frac{7}{10} \cdot \ldots \cdot \frac{p_{n-1}}{p_n - 1} \geqslant 1 - \frac{2}{2} \cdot \frac{3}{4} \cdot \frac{5}{6} \cdot \frac{7}{10} \cdot \frac{11}{12} \cdot \frac{13}{16} > 2/3 .$$

Hence if $p_n \geqslant 17$, i.e., if $n \geqslant 7$, then

$$\frac{1}{p_{n+1}} \left(1 - \frac{2}{2} \cdot \frac{3}{4} \cdot \frac{5}{6} \cdot \frac{7}{10} \cdot \ldots \cdot \frac{p_{n-1}}{p_n - 1}\right) > \frac{2}{3p_{n+1}} > \frac{15}{\pi^2} \prod_{i=1}^{n} \frac{p_i^2}{1 + p_i^2} - 1 .$$

Suppose $p_{n+1}^{-1} \leqslant x < p_n^{-1} \leqslant 1/17$. Then

$$B(1 + x) < 1 - \frac{\varphi(P_n)}{P_n} + \frac{1}{x} \frac{\varphi(P_n)}{P_n} \left(\frac{15}{\pi^2} \prod_{i=1}^{n} \frac{p_i^2}{1 + p_i^2} - 1\right)$$

$$\leqslant 1 - \frac{\varphi(P_n)}{P_n} \left(1 - p_{n+1} \left(\frac{15}{\pi^2} \prod_{i=1}^{n} \frac{p_i^2}{1 + p_i^2} - 1\right)\right)$$

$$< 1 - \frac{\varphi(P_n)}{P_n} \cdot \frac{2}{2} \cdot \frac{3}{4} \cdot \frac{5}{6} \cdot \frac{7}{10} \cdot \ldots \cdot \frac{p_{n-1}}{p_n - 1}$$

$$= 1 - \frac{\varphi(P_n)}{P_n} \cdot \frac{2}{1} \cdot \frac{3}{2} \cdot \frac{5}{4} \cdot \frac{7}{6} \cdot \ldots \cdot \frac{p_{n-1}}{p_{n-1} - 1} \cdot \frac{p_n}{p_n - 1} \cdot \frac{1}{p_n} = 1 - 1/p_n .$$

Thus $B(1 + x) < 1 - x$ if $0 < x < 1/17$. From our previously obtained estimates we may extend this result to $0 < x \leq 0.28$.

VI. ESTIMATES FOR C(x)

Using $k = P_6 = 2\cdot3\cdot5\cdot7\cdot11\cdot13$ we compute upper and lower bounds for $C(x,1,30)$, utilizing previous techniques and

$$M = \frac{\varphi(k)}{k}\left\{\frac{\zeta(2)}{\zeta(3)}\prod_{p|k}\frac{p^2+p}{p^2+p+1} - 1\right\} < 0.00309579 .$$

This was done by visualizing an infinite cube of lattice points with the correspondence

$$(a,b,c) \longleftrightarrow 7^a 11^b 13^c .$$

The ratios $\sigma^*(n)/n$ were computed to two decimal places for all $n = 7^a 11^b 13^c$ $(a,b,c \geq 0)$. The inequality

$$C(x,1,30) \leq B(x,1,30)$$

improved the upper bounds in most cases.

Because

$$C(x,1,6) = C(x,5,6) + C(x,1,30)$$

$$= C(x,1,30) + \frac{1}{5}C(\frac{5}{6}x,1,30) + \frac{1}{25}C(\frac{25}{26}x,1,30) + \frac{1}{125}C(\frac{125}{126}x,1,30) + \ldots$$

and because $C(x,j,k)$ is monotone, we have

$$1.01\ C(x,1,30) + 0.2\ C(\tfrac{5}{6}x,1,30) + 0.04\ C(\tfrac{25}{26}x,1,30)$$

$$\leqslant C(x,1,6)$$

$$\leqslant C(x,1,30) + 0.2\ C(\tfrac{5}{6}x,1,30) + 0.04\ C(\tfrac{25}{26}x,1,30)$$

$$+ 0.01\ C(\tfrac{125}{126}x,1,30)\ .$$

These inequalities were used to obtain bounds for $C(x,1,6)$. Bounds for $C(x,1,2)$ were achieved in much the same way, using

$$\tfrac{163}{162}\ C(x,1,6) + \tfrac{1}{3}\ C(\tfrac{3}{4}x,1,6) + \tfrac{1}{9}\ C(\tfrac{9}{10}x,1,6)$$

$$+ \tfrac{1}{27}\ C(\tfrac{27}{28}x,1,6) + \tfrac{1}{81}\ C(\tfrac{81}{82}x,1,6)$$

$$\leqslant C(x,1,2)$$

$$\leqslant C(x,1,6) + \tfrac{1}{3}\ C(\tfrac{3}{4}x,1,6) + \tfrac{1}{9}\ C(\tfrac{9}{10}x,1,6) + \tfrac{1}{27}\ C(\tfrac{27}{28}x,1,6)$$

$$+ \tfrac{1}{81}\ C(\tfrac{81}{82}x,1,6) + \tfrac{1}{162}\ C(\tfrac{243}{244}x,1,6)\ .$$

The function $C(x,2,1)$ was bounded by making use of

$$32\ C(\tfrac{2}{3}x,1,2) + 16\ C(\tfrac{4}{5}x,1,2) + 8\ C(\tfrac{8}{9}x,1,2) + 4\ C(\tfrac{16}{17}x,1,2)$$

$$+ 2\ C(\tfrac{32}{33}x,1,2) + C(\tfrac{64}{65}x,1,2) + C(x,1,2)$$

$$\leqslant 64\ C(x,2,1)$$

$$\leq 32\ C(\tfrac{2}{3}x,1,2) + 16\ C(\tfrac{4}{5}x,1,2) + 8\ C(\tfrac{8}{9}x,1,2) + 4\ C(\tfrac{16}{17}x,1,2)$$

$$+ 2\ C(\tfrac{32}{33}x,1,2) + C(\tfrac{64}{65}x,1,2) + C(\tfrac{128}{129}x,1,2)\ .$$

All bounds to this point were computed to four decimal places. Finally, we use

$$C(x) = C(x,1,2) + C(x,2,1)$$

and truncate the bounds to three decimal places to obtain the bounds given in Table III. These bounds are illustrated by Figure II.

We remark that, in particular, the density of the unitary abundant numbers is between 0.0674 and 0.1055, exclusive.

TABLE III

UPPER AND LOWER BOUNDS FOR C(x)

x	C(x) Lower Bound	C(x) Upper Bound
1.00	1.000	1.000
1.005	0.817	0.940
1.01	0.815	0.905
1.015	0.812	0.892
1.02	0.810	0.881
1.025	0.809	0.879
1.03	0.809	0.874
1.035	0.804	0.869
1.04	0.797	0.864
1.045	0.790	0.856
1.05	0.790	0.849
1.055	0.780	0.844
1.06	0.769	0.836
1.065	0.757	0.835
1.07	0.757	0.822
1.075	0.756	0.819
1.08	0.742	0.812
1.085	0.741	0.811
1.09	0.741	0.805
1.095	0.721	0.804
1.10	0.720	0.797
1.11	0.719	0.785
1.12	0.698	0.766
1.13	0.673	0.750
1.14	0.671	0.730
1.15	0.641	0.724
1.16	0.638	0.720
1.17	0.636	0.705
1.18	0.630	0.693
1.19	0.627	0.685

TABLE III (continued)

x	C(x) Lower Bound	C(x) Upper Bound
1.20	0.624	0.679
1.21	0.583	0.661
1.22	0.578	0.647
1.23	0.575	0.640
1.24	0.569	0.634
1.25	0.564	0.626
1.26	0.519	0.603
1.27	0.514	0.586
1.28	0.508	0.578
1.29	0.504	0.571
1.30	0.499	0.564
1.31	0.496	0.558
1.32	0.487	0.552
1.33	0.487	0.547
1.34	0.421	0.520
1.35	0.417	0.501
1.36	0.410	0.491
1.37	0.397	0.483
1.38	0.394	0.475
1.39	0.391	0.469
1.40	0.386	0.459
1.41	0.379	0.454
1.42	0.373	0.444
1.43	0.365	0.436
1.44	0.360	0.430
1.45	0.359	0.428
1.46	0.353	0.425
1.47	0.352	0.417
1.48	0.350	0.409
1.49	0.349	0.407
1.50	0.348	0.401
1.51	0.240	0.364
1.52	0.237	0.337
1.53	0.227	0.320
1.54	0.225	0.317

TABLE III (continued)

x	C(x) Lower Bound	C(x) Upper Bound
1.55	0.225	0.312
1.56	0.221	0.303
1.57	0.219	0.298
1.58	0.209	0.289
1.59	0.203	0.280
1.60	0.203	0.276
1.61	0.189	0.266
1.62	0.180	0.260
1.63	0.180	0.255
1.64	0.169	0.250
1.65	0.167	0.244
1.66	0.166	0.242
1.67	0.140	0.234
1.68	0.140	0.218
1.69	0.139	0.213
1.70	0.137	0.202
1.71	0.135	0.196
1.72	0.119	0.194
1.73	0.117	0.192
1.74	0.117	0.188
1.75	0.115	0.187
1.76	0.112	0.179
1.77	0.112	0.172
1.78	0.111	0.170
1.79	0.109	0.165
1.80	0.109	0.162
1.81	0.088	0.160
1.82	0.085	0.151
1.83	0.083	0.143
1.84	0.083	0.141
1.85	0.080	0.137
1.86	0.080	0.135
1.87	0.079	0.134
1.88	0.079	0.129
1.89	0.079	0.126

TABLE III (continued)

x	C(x) Lower Bound	C(x) Upper Bound
1.90	0.077	0.124
1.91	0.072	0.122
1.92	0.072	0.119
1.93	0.072	0.118
1.94	0.071	0.116
1.95	0.071	0.113
1.96	0.070	0.112
1.97	0.069	0.111
1.98	0.069	0.108
1.99	0.068	0.107
2.00	0.067	0.106
2.02	0.032	0.093
2.04	0.032	0.078
2.06	0.028	0.075
2.08	0.026	0.072
2.10	0.026	0.068
2.15	0.020	0.059
2.20	0.016	0.055
2.25	0.016	0.047
2.30	0.010	0.043
2.35	0.010	0.038
2.40	0.010	0.034
2.45	0.003	0.029
2.50	0.003	0.027
2.55	0.002	0.024
2.60	0.001	0.022
2.65	0.001	0.021
2.70	0.001	0.019
2.75	0.000	0.018
2.80	0.000	0.017
2.85	0.000	0.016
2.90	0.000	0.015
2.95	0.000	0.014
3.00	0.000	0.014

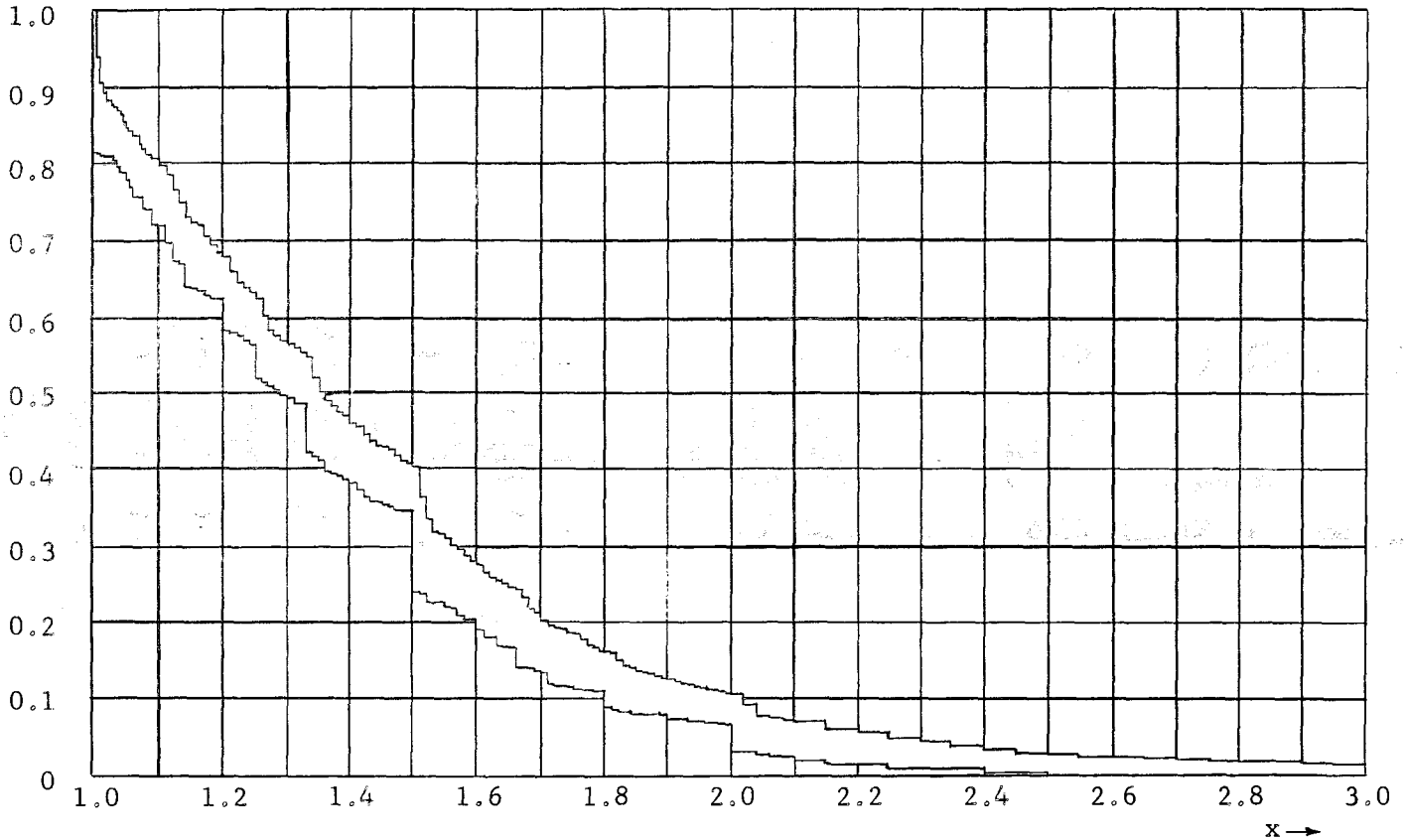

FIGURE II

UPPER AND LOWER BOUNDS FOR C(x)

VII. ESTIMATES FOR A(x)

As in the previous section, we use $k = P_6 = 2\cdot3\cdot5\cdot7\cdot11\cdot13$ to obtain upper and lower bounds for $A(x,1,30)$. In this case we have

$$M = \frac{\varphi(k)}{k}\left\{\frac{\pi^2}{6}\prod_{p|k}\frac{p^2-1}{p^2} - 1\right\} < 0.00320287 .$$

The inequality

$$B(x,1,30) \leq A(x,1,30) \leq B(\tfrac{x+1}{2},1,30)$$

was used to improve the bounds in a few cases.

As before, we obtain bounds for $A(x,1,2)$ by successive use of the inequalities

$$A(x,1,30) + 0.2\ A(\tfrac{5}{6}x,1,30) + 0.04\ A(\tfrac{25}{31}x,1,30) + 0.01\ A(\tfrac{125}{156}x,1,30)$$

$$\leq A(x,1,6)$$

$$\leq A(x,1,30) + 0.2\ A(\tfrac{5}{6}x,1,30) + 0.04\ A(\tfrac{25}{31}x,1,30) + 0.01\ A(\tfrac{4}{5}x,1,30)$$

and

$$A(x,1,6) + \frac{1}{3}\,A(\frac{3}{4}x,1,6) + \frac{1}{9}\,A(\frac{9}{13}x,1,6) + \frac{1}{27}\,A(\frac{27}{40}x,1,6)$$
$$+ \frac{1}{81}\,A(\frac{81}{121}x,1,6) + \frac{1}{162}\,A(\frac{243}{364}x,1,6)$$

$$\leqslant A(x,1,2)$$

$$\leqslant A(x,1,6) + \frac{1}{3}\,A(\frac{3}{4}x,1,6) + \frac{1}{9}\,A(\frac{9}{13}x,1,6) + \frac{1}{27}\,A(\frac{27}{40}x,1,6)$$
$$+ \frac{1}{81}\,A(\frac{81}{121}x,1,6) + \frac{1}{162}\,A(\frac{2}{3}x,1,6) \ .$$

The function $A(x,2,1)$ was bounded by use of the inequality

$$32\,A(\frac{2}{3}x,1,2) + 16\,A(\frac{4}{7}x,1,2) + 8\,A(\frac{8}{15}x,1,2)$$
$$+ 4\,A(\frac{16}{31}x,1,2) + 2\,A(\frac{32}{63}x,1,2) + 2\,A(\frac{64}{127}x,1,2)$$

$$\leqslant 64\,A(x,2,1)$$

$$\leqslant 32\,A(\frac{2}{3}x,1,2) + 16\,A(\frac{4}{7}x,1,2) + 8\,A(\frac{8}{15}x,1,2)$$
$$+ 4\,A(\frac{16}{31}x,1,2) + 2\,A(\frac{32}{63}x,1,2) + A(\frac{64}{127}x,1,2) + A(\frac{x}{2},1,2) \ .$$

As in the previous section, all bounds to this point were computed to four decimal places.

We improve the lower bound for $A(2)$ by considering the following primitives: $2\cdot3$, $2^2 5$, $2\cdot5\cdot7$, $2\cdot5\cdot11p$ $(13 \leqslant p \leqslant 41)$, $2\cdot5\cdot13p$ $(17 \leqslant p \leqslant 31)$, $2\cdot5\cdot17\cdot19$, $2\cdot7\cdot11\cdot13$, $2^2 7$, $2^2 11\cdot13$, $2^2 11\cdot17$, $2^2 11\cdot19$, $2^3 11$, $2^3 13$, $2^3 17p$ $(19 \leqslant p \leqslant 41)$, $2^3 19p$ $(23 \leqslant p \leqslant 41)$, $2^3 23p$ $(29 \leqslant p \leqslant 41)$, $2^3 29\cdot31$, $2^4 p$ $(17 \leqslant p \leqslant 31)$, $2^5 37$,

$2^5 41$, $2 \cdot 5^2 11$ and $2 \cdot 5^2 13$. Hence $A(2,2,1) > 0.2433$, so that

$$0.244 < A(2) < 0.301 .$$

This represents a slight improvement over the bounds which Behrend [2] gave, namely 0.241 and 0.314 .

Finally, we use the identity

$$A(x) = A(x,1,2) + A(x,2,1)$$

and truncate the bounds to three decimal places to obtain the bounds given in Table IV. These bounds are illustrated by Figure III.

The author has been unable to find in the literature any previous upper bounds for the function $A(x)$ other than those given by Behrend [2] for $x = 6/5$, $4/3$, $3/2$, 2 , 3 , 4 , 5 , 6 , 7 , 8 , 10 and 20 . In particular, it seems that in the past no attention has been paid to the behavior of $A(x)$ for values of x close to 1 . We shall see later that, as was the case with $B(x)$, the behavior of $A(x)$ close to $x = 1$ is repeated close to any value of x that is actually assumed by one of the ratios $\sigma(n)/n$. Thus at least the first four points at which Behrend obtained estimates are, in a sense, atypical. To be fair, however, we should not confuse Behrend's goals with our own: he was primarily concerned with estimating $A(2)$; the other estimates he gave should be regarded as parenthetical comments, not as indications of the general behavior of $A(x)$.

TABLE IV

UPPER AND LOWER BOUNDS FOR A(x)

x	A(x) Lower Bound	A(x) Upper Bound
1.00	1.000	1.000
1.01	0.870	0.934
1.02	0.861	0.906
1.03	0.847	0.894
1.04	0.836	0.888
1.05	0.828	0.873
1.06	0.808	0.862
1.07	0.808	0.860
1.08	0.794	0.850
1.09	0.794	0.850
1.10	0.777	0.839
1.11	0.777	0.832
1.12	0.776	0.825
1.13	0.776	0.816
1.14	0.776	0.810
1.15	0.745	0.807
1.16	0.745	0.803
1.17	0.744	0.799
1.18	0.743	0.789
1.19	0.743	0.783
1.20	0.743	0.777
1.21	0.717	0.773
1.22	0.711	0.758
1.23	0.708	0.746
1.24	0.704	0.744
1.25	0.693	0.741
1.26	0.692	0.734
1.27	0.689	0.730
1.28	0.687	0.725
1.29	0.686	0.721

TABLE IV (continued)

x	A(x) Lower Bound	A(x) Upper Bound
1.30	0.684	0.720
1.31	0.679	0.716
1.32	0.679	0.713
1.33	0.679	0.712
1.34	0.656	0.710
1.35	0.632	0.686
1.36	0.631	0.673
1.37	0.620	0.672
1.38	0.616	0.666
1.39	0.614	0.664
1.40	0.613	0.657
1.41	0.606	0.655
1.42	0.605	0.649
1.43	0.600	0.645
1.44	0.600	0.641
1.45	0.585	0.641
1.46	0.578	0.630
1.47	0.578	0.627
1.48	0.576	0.620
1.49	0.570	0.619
1.50	0.567	0.613
1.51	0.500	0.609
1.52	0.497	0.569
1.53	0.482	0.553
1.54	0.474	0.550
1.55	0.467	0.541
1.56	0.466	0.535
1.57	0.462	0.535
1.58	0.451	0.522
1.59	0.449	0.513
1.60	0.448	0.511
1.61	0.432	0.506
1.62	0.430	0.500
1.63	0.429	0.495
1.64	0.421	0.487

TABLE IV (continued)

x	A(x) Lower Bound	A(x) Upper Bound
1.65	0.416	0.483
1.66	0.412	0.482
1.67	0.411	0.476
1.68	0.411	0.470
1.69	0.408	0.469
1.70	0.408	0.463
1.71	0.406	0.457
1.72	0.391	0.454
1.73	0.391	0.452
1.74	0.386	0.449
1.75	0.385	0.447
1.76	0.352	0.444
1.77	0.349	0.420
1.78	0.348	0.417
1.79	0.344	0.406
1.80	0.342	0.402
1.81	0.325	0.399
1.82	0.321	0.393
1.83	0.318	0.384
1.84	0.311	0.379
1.85	0.309	0.372
1.86	0.309	0.368
1.87	0.303	0.367
1.88	0.282	0.362
1.89	0.282	0.357
1.90	0.278	0.347
1.91	0.274	0.343
1.92	0.272	0.336
1.93	0.270	0.334
1.94	0.260	0.331
1.95	0.260	0.326
1.96	0.256	0.320
1.97	0.248	0.316
1.98	0.247	0.309
1.99	0.244	0.305

TABLE IV (continued)

x	A(x) Lower Bound	A(x) Upper Bound
2.00	0.244	0.301
2.02	0.207	0.295
2.04	0.204	0.272
2.06	0.193	0.267
2.08	0.191	0.259
2.10	0.189	0.248
2.15	0.168	0.232
2.20	0.152	0.216
2.25	0.141	0.198
2.30	0.127	0.184
2.35	0.109	0.174
2.40	0.100	0.153
2.45	0.091	0.141
2.50	0.079	0.133
2.55	0.068	0.120
2.60	0.058	0.110
2.65	0.051	0.100
2.70	0.044	0.089
2.75	0.038	0.083
2.80	0.033	0.075
2.85	0.026	0.069
2.90	0.022	0.063
2.95	0.019	0.057
3.00	0.017	0.052

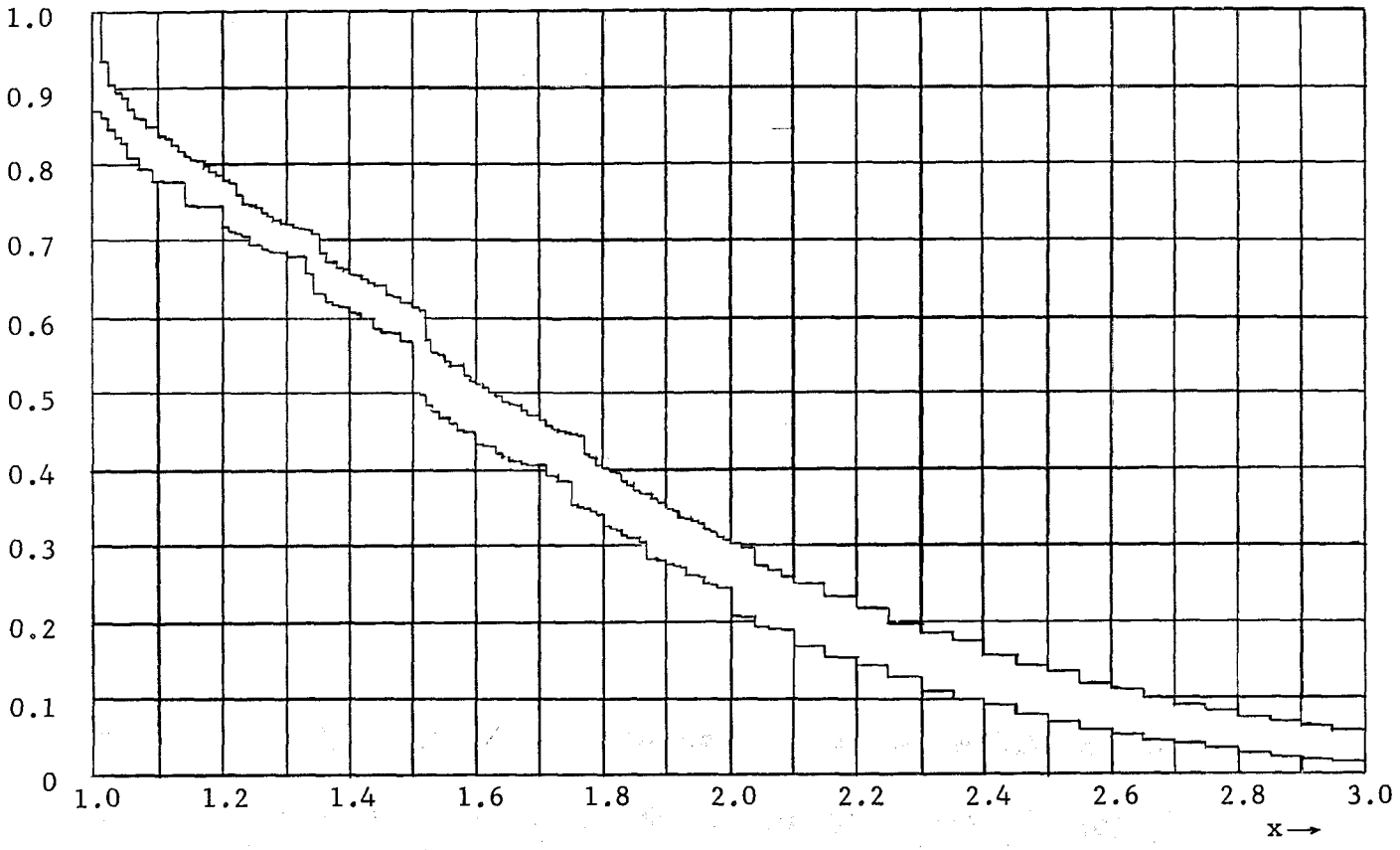

FIGURE III

UPPER AND LOWER BOUNDS FOR A(x)

Our work with density functions allows us to conclude an apparently new result on the possible existence of odd σ-perfect numbers. By (3.8) we have

$$A(x,j,k) \leqslant B(\frac{x+1}{2},j,k) \ .$$

Combining this result with Theorem 8, we see that

$$A(1 + x) \leqslant 1 - x/2 \ ,$$

so that $A(x)$ is not differentiable at $x = 1$. It follows from the series expansions of Section III that if there exists an integer n with $\sigma(n)/n = x$, then the derivative $A'(x)$ does not exist. One may extend this notion to the functions $A(x,1,k)$ to conclude that if there is an integer n with $\sigma(n)/n = x$ and $(n,k) = 1$, then $A'(x,1,k)$ does not exist. Thus we obtain the following result.

Theorem 9. The existence of $A'(2,1,2)$ is a sufficient condition for the nonexistence of odd σ-perfect numbers.

VIII. WHEN IS AN ABUNDANT NUMBER UNITARY ABUNDANT?

We want to know under what conditions an abundant number is unitary abundant. Since any unitary abundant number is abundant, what we are in fact trying to do is to characterize all the unitary abundant numbers, which is at best a highly improbable task. However,

we do know that an integer which is both abundant and squarefree is unitary abundant, and this fact might provide a convenient starting point for an attack on the question that concerns us here. In terms of density, what portion of the unitary abundant numbers are squarefree? Can we say anything about unitary abundant numbers that are not squarefree? In this section we present some answers to these questions.

If n is not squarefree we may associate with n the real number x_n defined by

$$x_n = \frac{\psi(n) - \sigma^*(n)}{\sigma(n) - \sigma^*(n)} .$$

By (1.7) and (1.9) we know that $0.5 \leqslant x_n \leqslant 1$ for all n. If n is abundant, we may define the real number y_n by

$$y_n = \frac{\psi(n) - 2n}{\sigma(n) - 2n} .$$

If positive, the number y_n compares, in a sense, the extent to which n is ψ-abundant with the extent to which n is σ-abundant. The following result provides a connection between x_n and y_n.

Theorem 10. Let n be a σ-abundant number. Then n is unitary abundant if and only if one of the following two conditions holds:

(a) n is squarefree; or

(b) n is not squarefree and

$$\frac{\psi(n) - 2n}{\sigma(n) - 2n} > \frac{\psi(n) - \sigma^*(n)}{\sigma(n) - \sigma^*(n)} . \tag{3.9}$$

Proof. If n is squarefree, $\sigma(n) = \sigma^*(n)$ and part (a) is trivial.

If n is not squarefree, then $\sigma^*(n) < \psi(n) < \sigma(n)$ by Theorem 2 of Chapter I, and (3.9) holds if and only if

$$\psi(n)\,\sigma(n) - 2n\,\sigma(n) - \psi(n)\,\sigma^*(n) + 2n\,\sigma^*(n)$$

$$= [\psi(n) - 2n][\sigma(n) - \sigma^*(n)]$$

$$> [\psi(n) - \sigma^*(n)][\sigma(n) - 2n]$$

$$= \psi(n)\,\sigma(n) - 2n\,\psi(n) - \sigma^*(n)\,\sigma(n) + 2n\,\sigma^*(n) ,$$

which is equivalent to

$$\sigma^*(n)\,[\sigma(n) - \psi(n)] > 2n\,[\sigma(n) - \psi(n)] ,$$

i.e., $\sigma^*(n) > 2n$.

In order to determine, in terms of density, how many of the unitary abundant numbers are squarefree, we shall need a preliminary result.

Theorem 11. The density of those numbers which are squarefree and divisible by a given squarefree integer k is

$$\frac{6}{\pi^2} \prod_{p|k} \frac{1}{p+1} .$$

Proof. We shall first set up the necessary machinery to use Theorem 5.

Let $f(n) = |\mu(n)|$ and $g = f \circ \mu$. Then

$$\sum_{n=1}^{\infty} f(n)\, n^{-s} = \zeta(s) / \zeta(2s) ,$$

and

$$\sum_{n=1}^{\infty} g(n)\, n^{-s} = 1/\zeta(2s) = \prod_{p} (1 - p^{-2s}) .$$

One checks that $g(n) \neq 0$ if and only if n is the square of a squarefree integer, and that $|g(n)|$ is either 0 or 1 for all n. Then

$$\sum_{n=1}^{N} |g(n)| = \frac{6}{\pi^2} \sqrt{N} + O(\sqrt{N}) = o(N) .$$

By Theorem 5 we have

$$M\{f\chi_k\} = \frac{\varphi(k)}{k} M\{f\} \Big/ \prod_{p|k} (1 - p^{-2})$$

$$= \frac{6}{\pi^2} \prod_{p|k} \frac{p - 1}{p} \cdot \frac{p^2}{p^2 - 1} = \frac{6}{\pi^2} \prod_{p|k} \frac{p}{p + 1} ,$$

and the theorem follows immediately.

Using Theorem 11 and the primitives $2\cdot3$, $2\cdot5\cdot7$, $2\cdot5\cdot11\cdot13$, $2\cdot5\cdot11\cdot17$, $2\cdot5\cdot11\cdot19$, $2\cdot5\cdot13\cdot17$, $2\cdot5\cdot13\cdot19$, $2\cdot5\cdot17\cdot19$ and $2\cdot7\cdot11\cdot13$, we obtain 0.0544 as a lower bound for the density of the squarefree abundant (and hence unitary abundant) integers. The estimates obtained earlier showed that $0.0674 < C(2) < 0.1055$. Hence in terms of density, at least half of all unitary abundant numbers are squarefree.

LIST OF REFERENCES

LIST OF REFERENCES

1. J. Alanen, O. Ore and J. Stemple, "Systematic computations on amicable numbers," Math. Comp. 21 (1967), 242-245.

2. F. Behrend, "Über numeri abundantes, II," Preuss. Akad. Wiss. Sitzungsber. 6 (1933), 280-293.

3. P. Bratley and J. McKay, "More amicable numbers," Math. Comp. 22 (1968), 677-678.

4. H. Davenport, "Über numeri abundantes," Preuss. Akad. Wiss. Sitzungsber. 26/29 (1933), 830-837.

5. E. B. Escott, "Amicable numbers," Scripta Math. 12 (1946), 61-72.

6. M. Garcia, "New amicable pairs," Scripta Math. 23 (1957), 167-171.

7. G. H. Hardy and E. M. Wright, An Introduction to the Theory of Numbers, 4th ed., Oxford University Press, New York, 1960.

8. M. Kac, Statistical Independence in Probability, Analysis and Number Theory, Carus Mathematical Monograph 12, John Wiley and Sons, New York, 1959.

9. E. J. Lee, "Amicable numbers and the bilinear diophantine equation," Math. Comp. 22 (1968), 181-187.

10. P. Poulet, "Forty-three new couples of amicable numbers," Scripta Math. 14 (1948), 77.

11. M. V. Subbarao and L. J. Warren, "Unitary perfect numbers," Canad. Math. Bull. 9 (1966), 147-153.

12. A. Wintner, Eratosthenian Averages, Waverly Press, Baltimore, 1943.

APPENDIX

APPENDIX

We recall from Chapter II that two integers n and m are said to be unitary amicable if

$$\sigma^*(n) = \sigma^*(m) = n + m .$$

Any unitary perfect number is unitary amicable with itself; we list here 610 pairs of non-perfect unitary amicable numbers.

Escott [5] classified amicable numbers by forms, grouping together all amicable numbers $n = Ep_1 \cdots p_a$ and $m = Eq_1 \cdots q_b$, where $(n,m) = E$, the p_i and q_j are distinct primes, and a and b are fixed. Our classification is a modification of Escott's: a typical entry

$$U , p_1 \cdots p_a , q_1 \cdots q_b$$

in the listing here represents the pair of unitary amicable numbers $n = Up_1 \cdots p_a$ and $m = Uq_1 \cdots q_b$, where U is the greatest common unitary divisor of n and m, the p_i and q_j are prime powers, and n is unitary abundant. Allowing prime powers in the place of primes is justified since

$$\sigma^*(p^e) = 1 + p^e$$

for $e = 1, 2, \ldots$; that is, primes and prime powers have the same type image under the σ^* function, while such is not the case with the σ function. Within each form, the middle entries of the triples are arranged in lexicographical order.

For the sake of brevity the abbreviations below are used; each subscript indicates, for accounting purposes, the number of values that the abbreviation represents.

$A_2 = 3^3$ or $3^4 41$

$B_2 = 2$ or $2^3 3$

$C_2 = 11$ or $2^2 11^2 31 \cdot 61$

$D_2 = 2$ or $2^2 5$

$E_2 = 3$ or $3^2 5$

$A_3 = 3^2 5^2 13$, $3^3 5$ or $3^4 5 \cdot 41$

$B_3 = 2$, $2^3 3$ or $2^4 3 \cdot 17$

$C_3 = 2$, $2^3 3$ or $2^5 3 \cdot 11$

$A_4 = 2 \cdot 3 \cdot 7$, $2 \cdot 3 \cdot 5^2 7^2 13$, $2 \cdot 3^2 5 \cdot 7$ or $2^2 3 \cdot 5 \cdot 7$

$B_4 = 2$, $2^3 3$, $2^4 3 \cdot 17$ or $2^5 3 \cdot 11$

$A_6 = 2 \cdot 3^3 5^2 7^2 13$, $2 \cdot 3^3 7$, $2 \cdot 3^3 5^2 7^2 13 \cdot 41$, $2 \cdot 3^4 7 \cdot 41$, $2^2 3^3 5 \cdot 7$ or $2^2 3^4 5 \cdot 7 \cdot 41$

$A_7 = 2 \cdot 3^3 5^2 7^2 13$, $2 \cdot 3^3 7$, $2 \cdot 3^3 5^2 7^2 13 \cdot 41$, $2 \cdot 3^4 7 \cdot 41$, $2^2 3^3 5 \cdot 7$, $2^2 3^4 5 \cdot 7 \cdot 41$ or $2^2 3^2 5^2 7 \cdot 13$

$B_7 = 2 \cdot 3^2 5^3 7^2$, $2 \cdot 7$, $2^2 5 \cdot 7$, $2^3 3 \cdot 7$, $2^3 3^2 5 \cdot 7$, $2^4 3 \cdot 7 \cdot 17$ or $2^4 3^2 5 \cdot 7 \cdot 17$

Because the abbreviations tend to obscure the actual values, each of the eight pairs of odd unitary amicable numbers in the following list is indicated by an asterisk.

FORM U,pq,r ..... 35 PAIRS

$2 \cdot 7 \cdot A_2$, $5 \cdot 17$, 107

$2 \cdot 3^3 7 \cdot 41 \cdot 163$, $5 \cdot 977$, 5867

$2 \cdot 3^2 5^2 19 \cdot 37$, $7 \cdot 887$, 7103

$2 \cdot A_3$, $11 \cdot 19$, 239

$7 \cdot 13 \cdot 1013 \cdot A_2 D_2$, $11 \cdot 4051$, 48623

$5 \cdot 7^2 23 \cdot B_3$, $11 \cdot 13523$, 162287

$7 \cdot 79 \cdot U$, $17 \cdot 7109$, 127979 ; $U = 2 \cdot 3^2 5^3$, $5^2 13 \cdot C_3$

$2 \cdot 19 \cdot A_3$, $29 \cdot 569$, 17099

$29 \cdot A_4$, $41 \cdot 173$, 7307

$2 \cdot 3^2 5^2 7$, $53 \cdot 1889$, 102059

$7 \cdot 23 \cdot U$, $83 \cdot 1931$, 162287 ; $U = 2 \cdot 3^2 5$, $3 \cdot D_2$

$2\cdot5^2 7\cdot13\cdot A_2$, 149·449 , 67499

$5\cdot11\cdot61\cdot B_3$, 239·161039 , 38649599

$2\cdot3^2 5^2 7\cdot107$, 3209·4493 , 14425739

FORM U,pq,rs ..... 121 PAIRS

D_2 , 3·19 , $3^2 7$

$2\cdot3^3 7\cdot41$, 5·4591 , 163·167

$2\cdot7\cdot37\ A_2$, 5·14207 , 191·443

$2\cdot3^2 5^2 19$, 7·227 , 37·47

$2\cdot3^2 5^2 31$, 7·929 , 11·619

$5\cdot31\cdot B_4$, 7·30689 , 59·4091

$5\cdot107\cdot1069\cdot2137\cdot25643\cdot C_3$, 7·5538887 , 17·2461727

$2\cdot A_3$, 11·199 , 29·79

A_4 , 11·10499 , 89·1399

$2\cdot41\cdot U$, 11·2686319 , 223·143909 ; $U = 3^3 5$, $3^2 5^2 13$

$2^6 3\cdot5\cdot11\cdot13$, 17·263 , 43·107

$2\cdot79\cdot157\cdot A_3$, 17·5023 , 23·3767

$2 \cdot A_3$, $19 \cdot 47$, $29 \cdot 31$

$5^2 7 \cdot 13 \cdot B_4$, $19 \cdot 359$, $47 \cdot 149$

$11 \cdot A_6$, $23 \cdot 659$, $79 \cdot 197$

$5 \cdot 13 \cdot B_4$, $23 \cdot 2339$, $53 \cdot 1039$

$19 \cdot 37 \cdot 73 \cdot A_7$, $23 \cdot 6569$, $107 \cdot 1459$

$11 \cdot A_6$, $23 \cdot 7523$, $53 \cdot 3343$

$2 \cdot 19 \cdot A_3$, $29 \cdot 44687$, $1063 \cdot 1259$

$2 \cdot 19 \cdot A_3$, $31 \cdot 184337$, $263 \cdot 22343$

$2 \cdot 19 \cdot A_3$, $37 \cdot 1583$, $227 \cdot 263$

$2 \cdot 79 \cdot A_3$, $37 \cdot 22751$, $11 \cdot 72047$

$5 \cdot 11 \cdot 229 \cdot B_3$, $43 \cdot 494639$, $197 \cdot 109919$

U , $47 \cdot 65519$, $1663 \cdot 1889$; $U = 7^2 A_5$, $7 \cdot B_4$, $5 \cdot 7 \cdot V$,

where $V = 2^2$, $2^3 3^2$, $2^4 3^2 17$, $2^5 3^2 11$

$5 \cdot 11 \cdot B_3$, $53 \cdot 1759$, $59 \cdot 1583$

$2 \cdot 3^2 5^2 7$, $59 \cdot 708959$, $421 \cdot 100799$

$17 \cdot 31 \cdot 61 \cdot A_7$, $67 \cdot 4391$, $101 \cdot 2927$

$7 \cdot 11 \cdot 43 \cdot U$, $67 \cdot 874619$, $1289 \cdot 46103$; $U = 2 \cdot 3^2 5^2 13$, $A_2 D_2$

$2 \cdot 3^2 5^2 7$, $71 \cdot 4339$, $239 \cdot 1301$

$2 \cdot 3^2 5^2 7$, $71 \cdot 5879$, $223 \cdot 1889$

$17 \cdot 19 \cdot A_7$, $79 \cdot 3229$, $199 \cdot 1291$

$2^6 3 \cdot 5 \cdot 11 \cdot 13 \cdot 23$, $131 \cdot 36988691$, $3041 \cdot 1605031$

$2 \cdot 5^2 7 \cdot 13 \cdot A_2$, $191 \cdot 589049$, $271 \cdot 415799$

* $3 \cdot 5 \cdot 7 \cdot C_2$, $233 \cdot 1019479$, $1091 \cdot 218459$

* $3 \cdot 5 \cdot 7 \cdot C_2$, $293 \cdot 5279$, $1231 \cdot 1259$

* $3 \cdot 5 \cdot 7 \cdot C_2$, $347 \cdot 23099$, $449 \cdot 17863$

* $3 \cdot 5 \cdot 7 \cdot C_2$, $503 \cdot 1319$, $769 \cdot 863$

FORM U,pqr,s ..... 7 PAIRS

$5 \cdot 929 \cdot B_3$, $7 \cdot 11 \cdot 5573$, 535103

A_4 , $11 \cdot 79 \cdot 2029$, 1948799

FORM U,pq,rst ..... 18 PAIRS

$5 \cdot B_4$, $7 \cdot 21599$, $19 \cdot 47 \cdot 179$

$5 \cdot B_4$, $7 \cdot 60659$, $23 \cdot 29 \cdot 673$

$2 \cdot A_3$, $11 \cdot 258299$, $29 \cdot 59 \cdot 1721$

$19 \cdot 61 \cdot 853 \cdot U$, $11 \cdot 3889679$, $17 \cdot 37 \cdot 68239$; $U = 7 \cdot B_2$,

$5^2 7^2 13 \cdot B_2$, $2 \cdot 3^2 5^3 7^2$, $2^2 5 \cdot 7$, $2^3 3^2 5 \cdot 7$

FORM U,pqr,st ..... 317 PAIRS

$2 \cdot 7 \cdot A_2$, $5 \cdot 17 \cdot 1187$, $131 \cdot 971$

$2 \cdot 11 \cdot 17 \cdot 373 \cdot A_2$, $5 \cdot 47 \cdot 2237$, $71 \cdot 8951$

$7 \cdot 17 \cdot C_3$, $5 \cdot 47 \cdot 173501$, $1223 \cdot 40823$

$7 \cdot 17 \cdot C_3$, $5 \cdot 47 \cdot 33195287$, $1181 \cdot 8088191$

$7 \cdot 17 \cdot C_3$, $5 \cdot 101 \cdot 797$, $113 \cdot 4283$

$2 \cdot 7 \cdot 31 \cdot A_2$, $5 \cdot 4493 \cdot 6287$, $23 \cdot 7064567$

$2 \cdot 3^2 5^2$, $7 \cdot 11 \cdot 29$, $31 \cdot 89$

$5 \cdot B_3$, $7 \cdot 11 \cdot 11369$, $757 \cdot 1439$

$5 \cdot B_4$, $7 \cdot 19 \cdot 107$, $47 \cdot 359$

$5 \cdot 67 \cdot B_4$, $7 \cdot 59 \cdot 401$, $31 \cdot 6029$

$2 \cdot A_3$, $7 \cdot 1949 \cdot 12239$, $127 \cdot 2983499$

$5 \cdot B_3$, $11 \cdot 13 \cdot 809$, $19 \cdot 6803$

$13 \cdot U$, $11 \cdot 17 \cdot 1039$, $53 \cdot 4159$; $U = 2 \cdot 3^2 5^3 7^2$, $7 \cdot D_2$, $2^3 7 \cdot E_2$

$23 \cdot A_6$, $11 \cdot 19 \cdot 367$, $79 \cdot 1103$

$2 \cdot A_3$, $11 \cdot 19 \cdot 1409$, $449 \cdot 751$

$2 \cdot A_3$, $11 \cdot 23 \cdot 239$, $179 \cdot 383$

$5 \cdot 79 \cdot B_2$, $11 \cdot 23 \cdot 7109$, $17 \cdot 113759$

$2 \cdot A_3$, $11 \cdot 31 \cdot 89$, $127 \cdot 269$

$5 \cdot 19 \cdot B_3$, $11 \cdot 41 \cdot 34154399$, $9629 \cdot 1787519$

$2 \cdot A_3$, $11 \cdot 47 \cdot 59$, $71 \cdot 479$

$5 \cdot 19 \cdot B_3$, $11 \cdot 47 \cdot 27739$, $359 \cdot 44383$

$5 \cdot 19 \cdot B_3$, $11 \cdot 59 \cdot 15199$, $151 \cdot 71999$

$5 \cdot 19 \cdot B_3$, $11 \cdot 61 \cdot 538649$, $139 \cdot 2862539$

A_4 , $11 \cdot 83 \cdot 5711$, $2351 \cdot 2447$

A_4 , $11 \cdot 83 \cdot 38821$, $1061 \cdot 36847$

A_4 , $11 \cdot 83 \cdot 63439$, $1039 \cdot 61487$

A_4 , $11 \cdot 83 \cdot 83591$, $1031 \cdot 81647$

$5 \cdot 17 \cdot B_2$, $11 \cdot 89 \cdot 227629$, $1699 \cdot 144611$

$5 \cdot 17 \cdot B_2$, $11 \cdot 101 \cdot 1889$, $1427 \cdot 1619$

$13 \cdot U$, $11 \cdot 103 \cdot 149$, $17 \cdot 10399$; $U = 2 \cdot 3^2 5^3 7^2$, $7 \cdot D_2$, $2^3 7 \cdot E_2$

A_4 , $11 \cdot 359 \cdot 9463$, $107 \cdot 378559$

$7 \cdot 13 \cdot A_2 D_2$, $11 \cdot 467 \cdot 33569$, $10529 \cdot 17903$

$7 \cdot 13 \cdot A_2 D_2$, $11 \cdot 467 \cdot 60779$, $7019 \cdot 48623$

$7 \cdot 13 \cdot A_2 D_2$, $11 \cdot 467 \cdot 488239$, $5743 \cdot 477359$

$5 \cdot 7^2 C_3$, $13 \cdot 17 \cdot 41$, $97 \cdot 107$

$3^3 7 \cdot 23 \cdot 229 \cdot 457 \cdot D_2$, $13 \cdot 17 \cdot 2741$, $107 \cdot 6397$

$7 \cdot 113 \cdot A_2 D_2$, $13 \cdot 17 \cdot 6553$, $11 \cdot 137633$

$11 \cdot B_7$, $13 \cdot 19 \cdot 10889$, $83 \cdot 36299$

$5 \cdot 17 \cdot C_3$, $13 \cdot 41 \cdot 23459$, $3331 \cdot 4139$

$5 \cdot 17 \cdot C_3$, $13 \cdot 47 \cdot 2549$, $359 \cdot 4759$

$11 \cdot B_7$, $13 \cdot 71 \cdot 241$, $23 \cdot 10163$

$7 \cdot 11 \cdot 43 \cdot U$, $13 \cdot 131 \cdot 1289$, $139 \cdot 17027$; $U = D_2$, $V \cdot E_2$,

where $V = 2^3$, $2^4 17$

* $3 \cdot 5 \cdot 7 \cdot 23$, $13 \cdot 137 \cdot 149$, $139 \cdot 2069$

A_4 , $17 \cdot 23 \cdot 1335949$, $3079 \cdot 187379$

$2^6 3 \cdot 5 \cdot 11 \cdot 13$, $17 \cdot 29 \cdot 16631$, $263 \cdot 34019$

* $3 \cdot 5 \cdot 7 \cdot 23$, $17 \cdot 41 \cdot 229$, $107 \cdot 1609$

$2 \cdot 307 \cdot U$, $17 \cdot 41 \cdot 613$, $107 \cdot 4297$; $U = 7 \cdot E_2$, $3 \cdot 5^2 7^2 13$

$2 \cdot A_3$, $17 \cdot 43 \cdot 149$, $19 \cdot 5939$

$7 \cdot U$, $17 \cdot 109 \cdot 149$, $107 \cdot 2749$; $U = 2 \cdot 3^2 5^3$, $5^2 13 \cdot C_3$

$83 \cdot A_4$, $17 \cdot 149 \cdot 829$, $107 \cdot 20749$

$5 \cdot 13 \cdot C_3$, $17 \cdot 179 \cdot 5381$, $2339 \cdot 7451$

$5 \cdot 13 \cdot C_3$, $17 \cdot 197 \cdot 2339$, $1619 \cdot 5147$

$11 \cdot A_6$, $17 \cdot 197 \cdot 21059$, $7019 \cdot 10691$

$11 \cdot A_6$, $17 \cdot 197 \cdot 49139$, $4211 \cdot 41579$

$11 \cdot A_6$, $17 \cdot 197 \cdot 135089$, $3761 \cdot 127979$

$11 \cdot A_6$, $17 \cdot 197 \cdot 1379069$, $3581 \cdot 1372139$

$7 \cdot 13 \cdot 181 \cdot U$, $17 \cdot 229 \cdot 1447$, $11 \cdot 499599$; $U = D_2$, $2^3 E_2$

$2^6 3 \cdot 5 \cdot 13 \cdot 23 \cdot 53$, $17 \cdot 2437 \cdot 5179$, $11 \cdot 18943259$

$7 \cdot U$, $19 \cdot 59 \cdot 599$, $79 \cdot 8999$; $U = 5^2 13 \cdot B_3$, $2 \cdot 3^2 5^3$,

$2^5 3 \cdot 5^2 11 \cdot 13$

$5 \cdot 13 \cdot B_4$, $19 \cdot 83 \cdot 129011$, $5039 \cdot 43003$

$5 \cdot 13 \cdot B_4$, $19 \cdot 89 \cdot 70979$, $1039 \cdot 122849$

$11 \cdot A_6$, $19 \cdot 89 \cdot 910909$, $38219 \cdot 42899$

$11 \cdot A_6$, $19 \cdot 89 \cdot 1205819$, $26729 \cdot 81199$

$11 \cdot A_6$, $19 \cdot 89 \cdot 1590467$, $24097 \cdot 118799$

$2 \cdot A_3$, $23 \cdot 29 \cdot 97$, $19 \cdot 3527$

$2 \cdot 41 \cdot U$, $23 \cdot 29 \cdot 3361$, $71 \cdot 33619$; $U = 3^2 5^2 13$, $3^3 5$

$7 \cdot U$, $29 \cdot 41 \cdot 59$, $179 \cdot 419$; $U = 5^2 13 \cdot C_3$, $2 \cdot 3^2 5^3$,

$2^4 3 \cdot 5^2 13 \cdot 17$

$5 \cdot 19 \cdot 37 \cdot B_4$, $29 \cdot 73 \cdot 491$, $179 \cdot 6067$

$2 \cdot 19 \cdot A_3$, $29 \cdot 569 \cdot 113021$, $28349 \cdot 68171$

$2 \cdot 19 \cdot A_3$, $29 \cdot 569 \cdot 117779$, $27179 \cdot 74099$

$2 \cdot 19 \cdot A_3$, $29 \cdot 569 \cdot 125113$, $25849 \cdot 82763$

$2 \cdot 19 \cdot A_3$, $29 \cdot 569 \cdot 152459$, $23099 \cdot 112859$

$2 \cdot 19 \cdot A_3$, $29 \cdot 569 \cdot 289381$, $19531 \cdot 253349$

$2 \cdot 19 \cdot A_3$, $37 \cdot 113 \cdot 28499$, $7219 \cdot 17099$

$2 \cdot 19 \cdot A_3$, $37 \cdot 113 \cdot 255587$, $4483 \cdot 246923$

$2 \cdot 19 \cdot A_3$, $37 \cdot 113 \cdot 1165187$, $4363 \cdot 1156643$

$2^6 3 \cdot 5 \cdot 13 \cdot 19 \cdot 37$, $41 \cdot 1109 \cdot 11369$, $11 \cdot 44172449$

$2 \cdot 3^2 5^2 7$, $53 \cdot 1889 \cdot 886463$, $139967 \cdot 646379$

$2 \cdot 3^2 5^2 7$, $53 \cdot 1889 \cdot 1411829$, $121013 \cdot 1190699$

$2 \cdot 3^2 5^2 7$, $53 \cdot 1931 \cdot 198769$, $81971 \cdot 252979$

$2 \cdot 3^2 5^2 7$, $53 \cdot 1931 \cdot 211319$, $77279 \cdot 285281$

$2 \cdot 3^2 5^2 7$, $53 \cdot 2099 \cdot 49633$, $26891 \cdot 209299$

$2 \cdot 3^2 5^2 7$, $53 \cdot 4073 \cdot 42239$, $3779 \cdot 2458367$

$2 \cdot 3^2 5^2 7$, $59 \cdot 419 \cdot 147377$, $68207 \cdot 54449$

$2 \cdot 3^2 5^2 7$, $59 \cdot 419 \cdot 170741$, $40949 \cdot 105071$

$2 \cdot 3^2 5^2 7$, $59 \cdot 419 \cdot 182159$, $38639 \cdot 118799$

$2 \cdot 3^2 5^2 7$, $59 \cdot 419 \cdot 233279$, $33599 \cdot 174959$

$2 \cdot 3^2 5^2 7$, $59 \cdot 419 \cdot 244199$, $32999 \cdot 186479$

$2\cdot 3^2 5^2 7$, $59\cdot 419\cdot 636473$, $27449\cdot 584303$

$2\cdot 3^2 5^2 7$, $59\cdot 419\cdot 1274249$, $26249\cdot 1223279$

$2\cdot 3^2 5^2 7$, $59\cdot 419\cdot 2011129$, $25849\cdot 1960559$

$2\cdot 3^2 5^2 7$, $59\cdot 419\cdot 5316959$, $25439\cdot 5266799$

$2\cdot 3^2 5^2 7$, $59\cdot 461\cdot 9337$, $8819\cdot 29347$

$2\cdot 3^2 5^2 7$, $83\cdot 139\cdot 78539$, $19403\cdot 47599$

$2\cdot 3^2 5^2 7$, $83\cdot 139\cdot 93683$, $16879\cdot 65267$

$2\cdot 3^2 5^2 7$, $83\cdot 139\cdot 108863$, $15679\cdot 81647$

$2\cdot 3^2 5^2 7$, $83\cdot 139\cdot 5742623$, $11807\cdot 5719279$

$2\cdot 3^2 5^2 7$, $83\cdot 149\cdot 5807$, $2879\cdot 25409$

$2\cdot 3^2 5^2 7$, $83\cdot 149\cdot 42767$, $1889\cdot 285119$

$2\cdot 3^2 5^2 7$, $97\cdot 113\cdot 25849$, $12539\cdot 23029$

$2\cdot 5^2 7\cdot 13\cdot A_2$, $139\cdot 569\cdot 6594659$, $287279\cdot 1831849$

$2\cdot 5^2 7\cdot 13\cdot A_2$, $149\cdot 449\cdot 395039$, $148139\cdot 179999$

$2\cdot 5^2 7\cdot 13\cdot A_2$, $149\cdot 449\cdot 438899$, $115499\cdot 256499$

$2\cdot 5^2 7\cdot 13\cdot A_2$, $149\cdot 449\cdot 521399$, $98999\cdot 355499$

$2 \cdot 5^2 7 \cdot 13 \cdot A_2$, 149·449·603899 , 91499·445499

$2 \cdot 5^2 7 \cdot 13 \cdot A_2$, 149·449·1076399 , 77999·931499

$2 \cdot 5^2 7 \cdot 13 \cdot A_2$, 149·449·1558619 , 74219·1417499

$2 \cdot 5^2 7 \cdot 13 \cdot A_2$, 149·449·1707947 , 73547·1567499

$2 \cdot 5^2 7 \cdot 13 \cdot A_2$, 149·449·2366099 , 71699·2227499

$2 \cdot 5^2 7 \cdot 13 \cdot A_2$, 149·449·6203411 , 69011·6067499

$2 \cdot 5^2 7 \cdot 13 \cdot A_2$, 149·461·291857 , 14699·1375901

FORM U,pqrs,t ..... 7 PAIRS

* 3·5·7 , 11·13·37·3779 , 24131519

$2 \cdot A_3$, 11·19·211·14699 , 747935999

$2 \cdot A_3$, 11·23·79·1051 , 24238079

FORM U,pqr,stu ..... 62 PAIRS

* $3 \cdot 5 \cdot C_2$, 7·17·439 , 23·43·59

$2 \cdot 3^2 5^2$, 7·19·2663 , 11·73·479

$5 \cdot B_4$, 7·19·7127 , 71·79·197

$5 \cdot B_4$, 7·89·359 , 23·59·179

$5 \cdot C_3$, $7 \cdot 107 \cdot 719$, $17 \cdot 179 \cdot 191$

$5 \cdot B_4$, $7 \cdot 131 \cdot 2339$, $19 \cdot 53 \cdot 2287$

$5 \cdot B_4$, $7 \cdot 163 \cdot 449$, $19 \cdot 59 \cdot 491$

$5 \cdot B_4$, $7 \cdot 863 \cdot 2579$, $23 \cdot 29 \cdot 24767$

A_7 , $11 \cdot 17 \cdot 227$, $23 \cdot 37 \cdot 53$

$5 \cdot B_2$, $11 \cdot 19 \cdot 115877$, $17 \cdot 61 \cdot 24919$

$5 \cdot B_2$, $11 \cdot 41 \cdot 239$, $17 \cdot 29 \cdot 223$

$2 \cdot A_3$, $11 \cdot 59 \cdot 644999$, $29 \cdot 719 \cdot 21499$

$5 \cdot 17 \cdot B_2$, $11 \cdot 101 \cdot 3659$, $17 \cdot 719 \cdot 6221$

$11 \cdot B_7$, $13 \cdot 43 \cdot 13499$, $29 \cdot 359 \cdot 769$

$11 \cdot U$, $13 \cdot 191 \cdot 5939$, $19 \cdot 307 \cdot 2591$; $U = 2 \cdot 3^2 5^3 7^2$,

$7 \cdot B_3$, $5 \cdot 7 \cdot V$, where $V = 2^2$, $2^4 3^2 17$

$11 \cdot U$, $17 \cdot 149 \cdot 3079$, $19 \cdot 53 \cdot 7699$; $U = 7 \cdot B_2$, $2^2 5 \cdot 7$,

$2 \cdot 7^2 V$, where $V = 3^2 5^3$, $5^2 13$

$2^6 3 \cdot 5 \cdot 11 \cdot 13$, $17 \cdot 107 \cdot 1038311$, $37 \cdot 4751 \cdot 11177$

$2^6 3 \cdot 5 \cdot 11 \cdot 13$, $19 \cdot 1259 \cdot 2969$, $29 \cdot 149 \cdot 16631$

FORM U,pqrs,tu ..... 11 PAIRS

$7 \cdot C_3$, $5 \cdot 13 \cdot 17 \cdot 293$, $71 \cdot 6173$

$5 \cdot B_3$, $7 \cdot 11 \cdot 929 \cdot 953$, $2879 \cdot 29573$

$5 \cdot B_3$, $7 \cdot 11 \cdot 929 \cdot 1019$, $2447 \cdot 37199$

$5 \cdot B_2$, $11 \cdot 17 \cdot 19 \cdot 47$, $239 \cdot 863$

FORM U,pqrs,tuv ..... 30 PAIRS

$7 \cdot B_3$, $5 \cdot 11 \cdot 97 \cdot 26212247$, $23 \cdot 6803 \cdot 1132627$

$11 \cdot B_3$, $5 \cdot 23 \cdot 43 \cdot 67$, $7 \cdot 197 \cdot 271$

$2 \cdot 3^2 5^2$, $7 \cdot 19 \cdot 23 \cdot 71$, $31 \cdot 79 \cdot 107$

B_7 , $11 \cdot 13 \cdot 29 \cdot 47$, $19 \cdot 23 \cdot 503$

$2^6 3 \cdot 5 \cdot 13$, $11 \cdot 17 \cdot 19 \cdot 47$, $31 \cdot 71 \cdot 89$

$5 \cdot B_3$, $11 \cdot 19 \cdot 41 \cdot 103$, $31 \cdot 179 \cdot 181$

$5 \cdot B_2$, $11 \cdot 19 \cdot 89 \cdot 383$, $17 \cdot 359 \cdot 1279$

$5 \cdot B_2$, $11 \cdot 23 \cdot 79 \cdot 7109$, $17 \cdot 79 \cdot 113759$

$5 \cdot B_3$, $13 \cdot 23 \cdot 139 \cdot 63737$, $11 \cdot 5879 \cdot 42491$

$5 \cdot B_2$, $17 \cdot 19 \cdot 71 \cdot 90149$, $11 \cdot 647 \cdot 300499$

$2^6 3 \cdot 5 \cdot 13 \cdot 19$, $17 \cdot 97 \cdot 773 \cdot 7789$, $11 \cdot 113 \cdot 7774829$

$2^6 3 \cdot 5 \cdot 13 \cdot 19$, $23 \cdot 37 \cdot 569 \cdot 121469$, $11 \cdot 151 \cdot 34618949$

$2 \cdot 3^3 5^2 7$, $29 \cdot 41 \cdot 43 \cdot 59$, $19 \cdot 131 \cdot 1259$

FORM U,pqrs,tuvw ..... 2 PAIRS

$17 \cdot B_2$, $5 \cdot 23 \cdot 1223 \cdot 72901$, $7 \cdot 11 \cdot 67 \cdot 1968353$

VITA

Charles Robert Wall was born in Palestine, Texas, on March 16, 1941. He attended public schools in Huntsville and Fort Worth, Texas. He received his B.A. (1963) and M.A. (1964) degrees from Texas Christian University.

After completing military service, he entered the Graduate School at The University of Tennessee in September, 1966, and received the Doctor of Philosophy degree with a major in Mathematics in March, 1970.

Mr. Wall is married and the father of two daughters.