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I am submitting herewith a dissertation written by Marek Szapiel entitled "Hypersurfaces of prescribed curvature in hyperbolic space". I have examined the final electronic copy of this dissertation for form and content and recommend that it be accepted in partial fulfillment of the requirements for the degree of Doctor of Philosophy, with a major in Mathematics.

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Hypersurfaces of prescribed curvature in hyperbolic space

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Abstract

In this paper we consider the problem of existence of hypersurfaces with prescribed curvature in hyperbolic space. We use the upper half-space model of hyperbolic space. The hypersurfaces we consider are given as graphs of positive functions on some domain $\Omega \in \mathbb{R}^n$ satisfying equations of form

$$f(A) = f(\kappa_1, \ldots, \kappa_n) = \psi,$$

where A is the second fundamental form of a hypersurface, $f(A)$ is a smooth symmetric function of the eigenvalues of A and ψ is a function of position. If we impose certain conditions on f and ψ, the above equation can be treated as an elliptic, fully non-linear partial differential equation

$$G(D^2u, Du, u) = \psi(x, u).$$

We then derive an existence result for the corresponding Dirichlet problem.

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List of Symbols and Abbreviations

- $\mathbb{R}^n$: Euclidean space with points $(x_1, \ldots, x_n)$, $x_i \in \mathbb{R}$. In this paper $n \geqslant 2$. If $x \in \mathbb{R}^n$, $|x| = (x_1^2 + \cdots + x_n^2)^{1/2}$.

- If $x = (x_1, \ldots, x_n) \in \mathbb{R}^n$ and $y = (y_1, \ldots, y_n) \in \mathbb{R}^n$, denote

$$x \otimes y = \begin{pmatrix} x_1y_1 & x_1y_2 & \ldots & x_1y_n \\ x_2y_1 & x_2y_2 & \ldots & x_2y_n \\ \vdots & & & \vdots \\ x_ny_1 & x_ny_2 & \ldots & x_ny_n \end{pmatrix}.$$

  Note that $x \otimes x$ is positive semi-definite for all $x \in \mathbb{R}^n$.

- ∂S: boundary of set S. $\bar{S}$: closure of S.

- Ω: an open, connected and bounded subset of $\mathbb{R}^n$.

- $S_1 \Subset S_2$: $\bar{S}_1 \subset S_2$ and $\bar{S}_1$ compact.

- If $u : \Omega \subseteq \mathbb{R}^n \to \mathbb{R}$ is a smooth function, then $u_i = D_iu = \partial_i u = \frac{\partial u}{\partial x_i}$, $u_{ij} = D_{ij}u = \partial_{ij}u = \frac{\partial^2 u}{\partial x_i \partial x_j}$, etc. $Du = (u_1, \ldots, u_n)$ is the gradient of u. $|Du| = (u_1^2 + \cdots + u_n^2)^{1/2}$. $D^2u = [u_{ij}]$ is the Hessian matrix of second derivatives of u. We then denote

$$|D^2u| = \|D^2u\|_2 = \max\{|\lambda_1|, \ldots, |\lambda_n|\},$$

  where $\{\lambda_1, \ldots, \lambda_n\}$ is the set of eigenvalues of D^2u. The above relation holds because D^2u is symmetric.

- If $\beta = (\beta_1, \ldots, \beta_n)$, $\beta_i \in \mathbb{N}$, define $|\beta| = \sum_{i=1}^n \beta_i$ and

$$D^\beta u = \frac{\partial^{|\beta|} u}{\partial x_1^{\beta_1} \cdots \partial x_n^{\beta_n}}.$$

  Derivative of u of order k is any derivative $D^\beta u$ with $|\beta| = k$.

- $C^0(\Omega)$: the set of continuous functions on Ω. $C^k(\Omega)$: the set of functions having all derivatives of order $\leqslant k$ continuous in Ω. $C^k(\bar{\Omega})$: the set of functions from $C^k(\Omega)$ all of whose derivatives of order $\leqslant k$ have continuous extension to $\bar{\Omega}$.

- Define for $0 < \alpha \leqslant 1$

$$[f]_{C^\alpha(\Omega)} = \sup_{\substack{x,y\in\Omega \\ x\neq y}} \frac{|f(x) - f(y)|}{|x-y|^\alpha}.$$

  If $[f]_{C^\alpha(\Omega)}$ is finite, we say that f is uniformly Hölder continuous with exponent α in Ω. If f is uniformly Hölder continuous in every compact subset of Ω, we say that f is locally Hölder continuous in Ω.

- $C^{k,\alpha}(\bar{\Omega})$: the space of functions in $C^k(\bar{\Omega})$ whose k-th order partial derivatives are uniformly Hölder continuous with exponent α in Ω.

- $C^{k,\alpha}(\Omega)$: the space of functions in $C^k(\Omega)$ whose k-th order partial derivatives are locally Hölder continuous with exponent α in Ω.

- $C^{k,0}(\Omega) = C^k(\Omega)$ and $C^{k,0}(\bar{\Omega}) = C^k(\bar{\Omega})$.

- Set

$$\begin{aligned}
[u]_{C^{k,0}(\Omega)} &= \sup_{\Omega} |D^k u| = \sup_{|\beta|=k} \sup_{\Omega} |D^\beta u| \\
[u]_{C^{k,\alpha}(\Omega)} &= \sup_{|\beta|=k} [D^\beta u]_{C^\alpha(\Omega)} \\
|u|_{C^k(\bar{\Omega})} &= \sum_{j=0}^{n} [u]_{C^{j,0}(\Omega)} \\
|u|_{C^{k,\alpha}(\bar{\Omega})} &= |u|_{C^k(\bar{\Omega})} + [u]_{C^{k,\alpha}(\Omega)}.
\end{aligned}$$

  The spaces $\left(C^k(\bar{\Omega}), |\cdot|_{C^k(\bar{\Omega})}\right)$ and $\left(C^{k,\alpha}(\bar{\Omega}), |\cdot|_{C^{k,\alpha}(\bar{\Omega})}\right)$ are Banach spaces.

- The *Laplacian* of $u \in C^2(\Omega)$ is defined by

$$\Delta u = \sum_{i=1}^{n} u_{ii}.$$

- We will often omit the limits of summation. In particular,

$$\sum_i \text{ means } \sum_{i=1}^{n}, \quad \sum_{i,j} \text{ means } \sum_{i=1}^{n}\sum_{j=1}^{n}, \quad \text{etc.}$$

1 Introduction

The question of existence of hypersurfaces of prescribed curvature is an important problem in differential geometry. It is often stated as a problem of finding a hypersurface $\mathcal{M}$ satisfying the equation

$$f(\kappa[\mathcal{M}]) = \psi, \tag{1.1}$$

where f is a smooth symmetric function, $\kappa[\mathcal{M}]$ is a vector of principal curvatures of $\mathcal{M}$ and ψ is a function of position (or of position and normal vector). Usually there are also some boundary conditions imposed on $\mathcal{M}$. Hypersurfaces of prescribed mean curvature satisfy (1.1) with $f(\kappa_1, \ldots, \kappa_n) = \kappa_1 + \cdots + \kappa_n$, while hypersurfaces of prescribed Gauss-Kronecker curvature satisfy (1.1) with $f(\kappa_1, \ldots, \kappa_n) = \kappa_1 \cdots \kappa_n$, where $\kappa[\mathcal{M}] = (\kappa_1, \ldots, \kappa_n)$ and ψ is suitably chosen.

The problem (1.1) has been studied in many different settings. Recent developments in the theory of Monge-Ampère equations (see Caffarelli, Nirenberg, Spruck [1], Guan and Spruck [7], Guan [6]) allowed to study the problem of existence of hypersurfaces of constant Gauss-Kronecker curvature (so called K-hypersurfaces). Guan and Spruck [7] and Guan [5] considered the problem of existence of K-hypersurfaces (with $K > 0$) which span a given closed codimension 2 embedded submanifold. The hypersurfaces can be represented as radial graphs over a domain $\Omega \subset S^n$, where S^n is the unit sphere. In [9] Guan and Spruck proved that if Γ is a boundary of a suitable locally convex hypersurface which is C^2 and locally strictly convex along its boundary, then Γ is also a boundary of a locally convex K-hypersurface for any $K \geqslant 0$. Trudinger and Wang also obtained a similar existence result for $K > 0$ in [19].

In [3], Caffarelli, Nirenberg, and Spruck obtained an existence result for a broad class of functions f by studying surfaces that are graphs on a strictly convex domain Ω. Guan and Spruck [10] studied a similar problem without the assumption of convexity of Ω but with more restrictions on f.

In this paper we will consider (1.1) for $\kappa = (\kappa_1, \ldots, \kappa_n)$ being a vector of principal curvatures in hyperbolic space. We will use the upper half-space model, i.e., let

$$\mathbb{H}^{n+1} = \{(x, y) \in \mathbb{R}^n \times \mathbb{R} : \ y > 0\}$$

be equipped with the metric

$$ds^2 = \frac{dx^2 + dy^2}{y^2}.$$

Also denote

$$\partial_\infty \mathbb{H}^{n+1} = \{(x, 0) : \ x \in \mathbb{R}^n\}.$$

In this setting the problem of finding K-hypersurfaces was considered by Rosenberg and Spruck [15] who showed the existence of complete smoothly embedded K-hypersurfaces ($-1 < K < 0$) with the boundary $\Gamma \subset \partial_\infty \mathbb{H}^{n+1}$, where Γ is a smooth $(n-1)$-dimensional submanifold.

Nelli and Spruck [14] studied constant mean curvature surfaces with the asymptotic boundary $\Gamma \subset \partial_\infty \mathbb{H}^{n+1}$ and proved that if Γ is mean convex, then Γ is an asymptotic boundary of a complete hypersurface in $\mathbb{H}^{n+1}$ of constant mean curvature H with $0 < H < 1$, which can be represented as a graph of a $C^{2,\alpha}$ function $u(x)$ over a domain Ω. Guan and Spruck [8] extended this result to $H \in (-1, 1)$ and star shaped domains Ω. In their case the surface can be represented as a radial graph over the upper hemisphere.

We will attempt to prove an existence result that is similar to [3] and [10], but is applied to hyperbolic space $\mathbb{H}^{n+1}$. The hypersurfaces we will consider are given as graphs of positive functions u on some domain $\Omega \subset \partial_\infty \mathbb{H}^{n+1}$ with smooth boundary and $u = \varepsilon$ on $\partial\Omega$, where ε is a positive constant. Then (1.1) will be treated as an elliptic fully non-linear Dirichlet problem that can be investigated using the standard elliptic theory.

Specifically, let

$$\Gamma_n^+ = \{\lambda = (\lambda_1, \ldots, \lambda_n) \in \mathbb{R}^n : \ \lambda_i > 0 \text{ for all } 1 \leqslant i \leqslant n\}$$

and let $f \in C^\infty(\Gamma_n^+) \cap C^0(\bar{\Gamma}_n^+)$ be a symmetric function satisfying the following structure conditions

$$f_i(\lambda) \equiv \frac{\partial f}{\partial \lambda_i}(\lambda) > 0 \text{ in } \Gamma_n^+, \ 1 \leqslant i \leqslant n, \tag{1.2}$$

$$f \text{ is concave}, \tag{1.3}$$

$$f > 0 \text{ in } \Gamma_n^+. \tag{1.4}$$

In addition we assume that for any given constants $\psi_1 \geqslant \psi_0 > 0$ there exists $\sigma_0 > 0$ with

$$\sum_i f_i \lambda_i \geqslant \sigma_0 \text{ on } \left\{\lambda \in \Gamma_n^+ : \ \psi_0 \leqslant f(\lambda) \leqslant \psi_1\right\}. \tag{1.5}$$

Suppose also that for every $c > 0$ and every compact set $E \subset \Gamma_n^+$ there exists $R = R(E, c) > 0$ such that

$$f(\lambda_1, \ldots, \lambda_{n-1}, \lambda_n + R) \geqslant c \qquad \forall \lambda \in E \tag{1.6}$$

and

$$f(R\lambda) \geqslant c \qquad \forall \lambda \in E. \tag{1.7}$$

We also assume that f is normalized so that

$$f(1, 1, \ldots, 1) = 1. \tag{1.8}$$

It is possible to show (see [2]) that functions of form

$$f^{(k)}(\kappa_1, \ldots, \kappa_n) = b_k S_k(\kappa_1, \ldots, \kappa_n)^{1/k}, \tag{1.9}$$

where S_k is the k-th elementary symmetric function given by

$$S_k(\kappa_1, \ldots, \kappa_n) = \sum_{1 \leqslant i_1 < \cdots < i_k \leqslant n} \kappa_{i_1} \cdots \kappa_{i_k}$$

and $b_k = S_k(1, \ldots, 1)^{-1/k}$, satisfy all conditions (1.2)–(1.8). However in this paper we will impose two additional conditions on f that are necessary to prove the *a priori* bounds in sections 2 and 3. Specifically, we assume that

$$f = 0 \quad \text{on } \partial\Gamma_n^+, \tag{1.10}$$

and for any given constants $0 < \psi_0 \leqslant \psi_1$ there exists $A > 0$ such that

$$\sum_i f_i \leqslant A \qquad \text{on } \left\{\lambda \in \Gamma_n^+ : \ \psi_0 \leqslant f(\lambda) \leqslant \psi_1\right\}. \tag{1.11}$$

The conditions (1.10) and (1.11) limit the class of functions that our result applies to. In particular functions $f^{(k)}$ defined in (1.9) do not satisfy (1.10) and (1.11). The condition (1.11) is used only in derivation of the global second derivative bounds and may possibly be dropped if a better method of estimation is found. The condition (1.10) is used in deriving the boundary estimates for second derivatives and is also used for gradient estimates (however in the case of gradient estimates (1.10) can be replaced with some milder condition that $f^{(k)}$ satisfy). The question of whether it is possible to avoid (1.10) and (1.11) will be therefore posed as an open problem.

The following lemma (partially taken from [2]) will be used in the statement of our main result and so it is placed in this section.

Lemma 1.1. *Suppose that f satisfies* (1.3), (1.4), *and* (1.10). *Then for any $\psi_0 > 0$ there exists a $\delta > 0$ such that*

$$\sum_{i=1}^n \kappa_i \geqslant \delta \text{ in the set } T = \{\kappa \in \Gamma_n^+ : \ f(\kappa) \geqslant \psi_0\}.$$

Moreover if $0 < H_0 < \delta/n$, we have

$$\psi_0 > f(H_0, \ldots, H_0).$$

Proof. It is easy to see that the set T is closed, convex and symmetric in κ. Therefore (by (1.10)) there is a unique point $(b, \ldots, b) \in T$ that is the closest to the origin. Set $\delta = nb$. To show the second part of the lemma let us suppose that $f(H_0, \ldots, H_0) \geqslant \psi_0$. Then $(H_0, \ldots, H_0) \in T$ and therefore $(b, \ldots, b)$ is not the closest point to the origin. Contradiction. □

Let then H_0 be any fixed positive constant such that

$$0 < H_0 < \delta/n, \tag{1.12}$$

where δ is the constant from Lemma 1.1. The upper bound on H_0 depends only on f and ψ_0.

Let $\Omega \subset \mathbb{R}^n$ be a bounded domain with C^∞ boundary. We will associate with Ω the quantity r_Ω defined in the following way. For every $x \in \partial\Omega$ define $r(x)$ to be the least upper bound of radii of closed balls B such that $B \cap \bar{\Omega} = \{x\}$. Then define

$$r_\Omega = \inf_{x \in \partial\Omega} r(x). \tag{1.13}$$

Because $\bar{\Omega}$ is compact and has C^∞ boundary, it follows that $r_\Omega > 0$. If Ω is convex, $r_\Omega = \infty$.

Now let

$$\mathcal{A} = \mathcal{A}(\Omega) = \big\{ u \in C^\infty(\bar{\Omega}) : \ u > 0 \text{ and } \big(\kappa_1[u](x), \ldots, \kappa_n[u](x)\big) \in \Gamma_n^+ \text{ for all } x \in \bar{\Omega} \big\},$$

where $(\kappa_1[u], \ldots, \kappa_n[u])$ denotes the vector of principal hyperbolic curvatures of the graph of u. The elements of $\mathcal{A}$ are called *admissible functions*.

The relationship between the hyperbolic curvatures κ_i and the Euclidean curvatures κ_i^E is given by:

$$\kappa_i = x_{n+1}\kappa_i^E + \frac{1}{w} = x_{n+1}\kappa_i^E + \nu \cdot e_{n+1}, \quad 1 \leqslant i \leqslant n, \tag{1.14}$$

where $w = \sqrt{1 + |Du|^2}$, ν is the unit upper normal to the graph of u and e_{n+1} is the $(n+1)^{\text{st}}$ coordinate vector (see [8]). Since the Euclidean principal curvatures of the graph of $u \in \mathcal{A}$ are the eigenvalues of the matrix $[a_{ij}]$, where

$$a_{ij} = \frac{1}{w} \sum_{k,l=1}^{n} \gamma^{ik} u_{kl} \gamma^{lj}, \qquad \gamma^{ik} = \delta_{ik} - \frac{u_i u_k}{w(1+w)}, \tag{1.15}$$

it follows that the hyperbolic principal curvatures are the eigenvalues of $[h_{ij}]$, where

$$h_{ij} = \frac{1}{w} \sum_{k,l=1}^{n} \gamma^{ik} v_{kl} \gamma^{lj}, \qquad v_{kl} = u u_{kl} + u_k u_l + \delta_{kl}. \tag{1.16}$$

If we set

$$v = \frac{1}{2}u^2(x) + \frac{1}{2}|x - x_0|^2 \tag{1.17}$$

for some $x_0 \in \bar{\Omega}$, it is easy to see that

$$v_{kl}(x) = \frac{\partial^2}{\partial x_k \partial x_l} v(x). \tag{1.18}$$

If $u \in \mathcal{A}$, the matrix $[h_{ij}]$ is positive definite and therefore v is a convex function.

Now let ψ_0 and ψ_1 be two fixed constants satisfying $0 < \psi_0 \leqslant \psi_1 < 1$ and let $\psi \in C^\infty\big(\bar{\Omega} \times (0, \infty)\big)$ be a smooth function such that $\psi_0 \leqslant \psi \leqslant \psi_1$ in $\bar{\Omega} \times (0, \infty)$. We will consider the following Dirichlet problem:

$$\begin{aligned} f(\kappa[u]) &= \psi(x, u) \qquad \text{in } \Omega, \\ u &= \varepsilon \qquad\qquad\quad \text{on } \partial\Omega, \end{aligned} \tag{1.19}$$

where $\varepsilon > 0$ and $u \in \mathcal{A}$. Any solution of (1.19) is a function whose graph is a surface with its hyperbolic curvature being prescribed by the relationship $f(\kappa) = \psi$. Notice that the quantities σ_0 from (1.5), A from (1.11), δ from Lemma 1.1, and H_0 from (1.12) are now fixed as they only depend on f and ψ.

We will sometimes use a different representation of (1.19). Let $\mathfrak{S}^+$ be the set of positive definite $n \times n$ symmetric matrices. We can define a function $F : \mathfrak{S}^+ \to \mathbb{R}$ as

$$F(H) = f(\lambda(H)) \quad \text{for all } H \in \mathfrak{S}^+, \tag{1.20}$$

where $\lambda(H)$ is the vector of eigenvalues of H. The properties of f ensure that F is well defined. Denote

$$F^{ij}(H) = \frac{\partial F}{\partial h_{ij}}(H), \qquad F^{ij,kl}(H) = \frac{\partial^2 F}{\partial h_{ij} h_{kl}}(H), \tag{1.21}$$

where $H = [h_{ij}]$. Then, from [10] and [17], we have

- F is a concave function on $\mathfrak{S}^+$;
- $[F^{ij}(H)]$ is a positive definite matrix with eigenvalues $f_i = \frac{\partial f}{\partial \kappa_i}$;

- when H is diagonal, $F^{ij} = f_i \delta_{ij}$;
- $\sum_{i,j} F^{ij}(H) h_{ij} = \sum_i f_i \kappa_i$;
- $\sum_{i,j} F^{ij} \sum_k h_{ik} h_{kj} = \sum_i f_i \kappa_i^2$,

assuming that $(\kappa_1, \ldots, \kappa_n)$ is the vector of eigenvalues of H. Then (1.19) can be written as

$$
\begin{aligned}
G(D^2u, Du, u) = F([h_{ij}]) = f(\lambda([h_{ij}])) = \psi(x, u) \qquad & \text{in } \Omega, \\
u = \varepsilon \qquad & \text{on } \partial\Omega,
\end{aligned}
\tag{1.22}
$$

where $G(r, p, z)$ is a C^∞ function induced by F defined on a subset of $\mathbb{R}^{n\times n} \times \mathbb{R}^n \times \mathbb{R}$. G is a concave function of r and the operator

$$
\mathcal{G}[u] = \mathcal{G}(D^2u, Du, u, x) = G(D^2u, Du, u) - \psi(x, u) \tag{1.23}
$$

is elliptic with respect to u. We can therefore use the standard elliptic theory to investigate the question of solvability of (1.19).

The most basic method of deriving existence results in non-linear elliptic theory is the *continuity method*. From [4] we have the following theorem (with $\mathcal{G} = \mathcal{G}(r, p, z, x)$ being a non-linear elliptic operator like the one defined in (1.23))

Theorem 1.2. *Let Ω be a bounded domain in $\mathbb{R}^n$ with boundary $\partial\Omega \in C^{2,\alpha}$, $0 < \alpha < 1$, $\mathfrak{U}$ an open subset of the space $C^{2,\alpha}(\bar{\Omega})$ and ϕ a function in $\mathfrak{U}$. Set $E = \{u \in \mathfrak{U} : \mathcal{G}[u] = \sigma\mathcal{G}[\phi]$ for some $\sigma \in [0, 1],\ u = \phi$ on $\partial\Omega\}$ and suppose that*

1. *$\mathcal{G}$ is strictly elliptic in Ω for each $u \in E$;*
2. *$\mathcal{G}_z(D^2u, Du, u, x) \leqslant 0$ for each $u \in E$;*
3. *E is bounded in $C^{2,\alpha}(\bar{\Omega})$;*
4. *$\bar{E} \subset \mathfrak{U}$.*

Then the Dirichlet problem, $\mathcal{G}[u] = 0$ in Ω, $u = \phi$ on $\partial\Omega$ is solvable in $\mathfrak{U}$.

Theorem 1.2 reduces the problem of existence to establishing the *a priori* $C^{2,\alpha}$ bounds for the solutions of the Dirichlet problem (i.e., one needs to show that the condition 3 in the theorem holds). Such bounds will be derived in section 2. Unfortunately in our case we cannot apply Theorem 1.2 directly because the condition 2 is not satisfied. In order to overcome this difficulty we investigate in section 3 a parametrized problem (3.2). We can then apply the continuity method to determine the solvability of (3.2) and this result (together with $C^{k,\alpha}$ estimates) can then be used with Leray-Schauder degree method to deal with the original problem (see section 4). The main result of the paper is the following theorem.

Theorem 1.3. *Suppose that the conditions* (1.2)–(1.8) *together with* (1.10) *and* (1.11) *are satisfied. Then the problem* (1.19) *has a solution* $u \in C^\infty(\bar{\Omega})$ *for any* $\varepsilon > 0$ *provided that* $\varepsilon < r_\Omega H_0$*, where* H_0 *and* r_Ω *are defined in* (1.12) *and* (1.13) *respectively. Note that if* $r_\Omega = \infty$*, the problem* (1.19) *has a solution for all* $\varepsilon > 0$*.*

The conditions (1.10) and (1.11) imposed on f limit the class of problems that Theorem 1.3 can be applied. We can sometimes modify (1.9) using some cutoff functions. For example, let

$$f(\lambda_1, \dots, \lambda_n) = B\left[(\lambda_1 + \cdots + \lambda_n) \cdot \zeta(\lambda_1) \cdots \zeta(\lambda_n)\right]^{1/(n+1)},$$

where $B > 0$ is a scaling constant and $\zeta(t)$ is suitably chosen cutoff function (e.g., $\zeta(t) = 1 - e^{-t}$). Then (1.19) has a solution for the above f.

Another problem is the dependence of the bounds on ε as $\varepsilon \to 0$. Without this dependence we could consider a family of solutions $\{u^\varepsilon \in \mathcal{A} : \ \varepsilon > 0\}$, where u^ε is a solution of (1.19) for a given $\varepsilon > 0$. We would then be able to extract some uniformly convergent sequence u^{ε_n}, where $\varepsilon_n \to 0$, and this way deduce the existence of complete hypersurfaces with prescribed asymptotic behavior at infinity.

The advantage of the methods presented in this paper is that they do not make any assumptions about the domain Ω (other than being smooth and bounded). The condition (1.11) could possibly be dropped with a more careful estimation of $|D^2 u|$ and with no changes made to the rest of the paper. This would permit functions like $f^{(n)}$. The dependence on ε is however much harder to avoid and may require developing new methods to attack this difficulty.

2 A priori estimates

2.1 Preliminary results

In this subsection we will derive the most basic results that will only require conditions (1.2)–(1.4) and (1.8) to hold.

Lemma 2.1. *Suppose that $u \in \mathcal{A}$ satisfies* (1.19)*. Then u does not have a local minimum in Ω. In particular, $u > \varepsilon$ in Ω.*

Proof. Suppose that u has a local minimum at $x_0 \in \Omega$. Then $\kappa_i^E[u](x_0) \geqslant 0$ for all $1 \leqslant i \leqslant n$ and

$$\frac{1}{u}\left[\kappa_i[u](x_0) - \frac{1}{w}\right] \geqslant 0$$

$$\kappa_i[u](x_0) \geqslant 1 \text{ for all } 1 \leqslant i \leqslant n.$$

But this means $f(\kappa[u]) \geqslant f(1, \ldots, 1) = 1 > \psi(x_0, u) = f(\kappa[u])$ – contradiction. □

Lemma 2.2. *Let $u \in \mathcal{A}$ be a solution of* (1.19)*. Then*

$$\varepsilon \leqslant u \leqslant \sqrt{\varepsilon^2 + (\operatorname{diam}\Omega)^2} \text{ in } \bar{\Omega}.$$

Proof. Let v be the function defined in (1.17). Therefore by convexity of v

$$u^2 \leqslant 2v \leqslant 2\sup_{\partial\Omega} v \leqslant \varepsilon^2 + (\operatorname{diam}\Omega)^2.$$

□

Lemma 2.3 below is a type of maximum principle.

Lemma 2.3. *Let $\Omega' \subset \Omega$ be an open domain and let $u, v \in \mathcal{A}(\Omega')$. Suppose also that $u \leqslant v$ on $\partial\Omega'$ and $f(\kappa[v]) < f(\kappa[u])$ in Ω'. If $v - u$ has a local minimum at $x_0 \in \Omega'$, then $v(x_0) \neq u(x_0)$.*

Proof. Since f is a symmetric function, we can assume that the curvatures of u and v are ordered at x_0, i.e.,

$$\kappa_1^E[u] \leqslant \cdots \leqslant \kappa_n^E[u] \text{ and } \kappa_1^E[v] \leqslant \cdots \leqslant \kappa_n^E[v].$$

At x_0 we have:

$$\kappa_i^E[v](x_0) \geqslant \kappa_i^E[u](x_0)$$
$$\frac{1}{v}\left[\kappa_i[v](x_0) - \frac{1}{\sqrt{|Dv|^2+1}}\right] \geqslant \frac{1}{u}\left[\kappa_i[u](x_0) - \frac{1}{\sqrt{|Du|^2+1}}\right].$$

Suppose that $u(x_0) = v(x_0)$. Then (because $Du(x_0) = Dv(x_0)$)

$$\kappa_i[v](x_0) \geqslant \kappa_i[u](x_0) \text{ for all } i,$$

so $f(\kappa[v](x_0)) \geqslant f(\kappa[u](x_0))$ – contradiction. □

2.2 Gradient estimates

Now we will derive an *a priori* global gradient estimate for the solutions of (1.19). The condition (1.11) is not used in this subsection.

Let $u \in \mathcal{A}$ be a solution of (1.19). Denote

$$\Sigma = \{(x_1, \dots, x_n, u(x_1, \dots, x_n)) : \ (x_1, \dots, x_n) \in \Omega\} \tag{2.1}$$

and

$$\Omega_\varepsilon = \{(x_1, \dots, x_n, \varepsilon) : \ (x_1, \dots, x_n) \in \Omega\}. \tag{2.2}$$

Lemma 2.4. *Let $0 < H_1 < 1$ be such that $\psi_1 < f(H_1, \dots, H_1) < f(1, \dots, 1)$ and let $x_0 \in \Omega$. Then*

$$u(x_0) \geqslant \frac{\varepsilon H_1 + \sqrt{\varepsilon^2 + \operatorname{dist}(x_0, \partial\Omega)^2(1-H_1^2)}}{1+H_1} \geqslant \operatorname{dist}(x_0, \partial\Omega)\sqrt{\frac{1-H_1}{1+H_1}}.$$

Proof. Consider a ball B of radius R centered at $(x_0, -H_1 R)$, where $R > 0$. Then $B \cap \{x : \ x_{n+1} > 0\}$ can be represented as a graph whose principal hyperbolic curvatures (with respect to the outer normal) are all equal to H_1. Suppose that initially R is small enough so that the ball B is under the graph of u and also $B \cap \{x : \ x_{n+1} = \varepsilon\} \subset \bar{\Omega}_\varepsilon$. Then we can start increasing R and by Lemma 2.3, B

cannot touch the graph of u inside Ω. Therefore if we take R satisfying

$$(H_1 R + \varepsilon)^2 + \operatorname{dist}(x_0, \partial\Omega)^2 = R^2, \tag{2.3}$$

we get the following estimate:

$$u(x_0) \geqslant (1 - H_1)R.$$

Solving (2.3) for R we get the desired result. □

Lemma 2.5. *Let B be a ball in $\mathbb{R}^{n+1}$ of radius R centered at $a = (a', H_0 R)$ where $a' \notin \bar{\Omega}$ and $\operatorname{dist}(a', \partial\Omega) > \varepsilon/H_0$. If $B \cap \Omega_\varepsilon = \varnothing$, then $B \cap \Sigma = \varnothing$.*

Proof. $B \cap \{x : \ 0 < x_{n+1} < H_0 R\}$ can be represented as a graph whose principal hyperbolic curvatures are all equal to H_0. Suppose that $B \cap \Omega_\varepsilon = \varnothing$ and $B \cap \Sigma \neq \varnothing$. Let us decrease R so that $B \cap \Sigma = \varnothing$. We can then reverse this process and continue increasing R until B touches Σ at some point $(x_0, u(x_0))$. Then B can be locally represented as a graph of some function $v \in C^2$ in a neighborhood $\Omega_{x_0} \subset \Omega$ of x_0. We also have $u(x_0) = v(x_0)$, $v \geqslant u$ on $\partial\Omega_{x_0}$, $f(\kappa[v]) < f(\kappa[u])$ in Ω_{x_0}. But this contradicts Lemma 2.3 (we apply Lemma 2.3 with $\Omega' = \Omega_{x_0}$). □

Let r_Ω be defined by (1.13). Suppose first that $r_\Omega < \infty$ and let $R > 0$ be such that $R^2 = r_\Omega^2 + (H_0 R - \varepsilon)^2$.

Lemma 2.6. *If $\varepsilon < r_\Omega H_0$, then*

$$w \leqslant \frac{R}{H_0 R - \varepsilon} \qquad \text{on } \partial\Omega.$$

Proof. Let $x_0 \in \partial\Omega$ and denote by $\nu(x_0)$ the inner unit normal vector to $\partial\Omega$ at x_0. Consider a ball $B(a, R)$ of radius R centered at $a = (x_0 - r_\Omega \nu, H_0 R)$, where R satisfies the equation $R^2 = r_\Omega^2 + (H_0 R - \varepsilon)^2$. Then by Lemma 2.5 the graph of u cannot touch the ball $B(a, R)$. Near x_0, B can be represented as a graph of a function v such that $u(x_0) = v(x_0)$ and $u(x_0 + t\nu(x_0)) < v(x_0 + t\nu(x_0))$ for all sufficiently small $t > 0$. The result follows from the comparison of the angles that the outer normal vector to $\bar{\Sigma}$ and the inner normal to ∂B form at x_0 with the hyperplane $\{x_{n+1} = \varepsilon\}$. □

If $r_\Omega = \infty$, we can apply the above Lemma to any $r_\Omega > \varepsilon/H_0$ and let $r_\Omega \to \infty$. In that case $R \to \infty$ and we have the inequality:

$$w \leqslant \frac{1}{H_0}.$$

Lemma 2.7. *For any $1 \leqslant i \leqslant n$ and $x \in \Omega$ we have*

$$u|u_i(x)| \leqslant \varepsilon \sup_{\partial\Omega} |u_i| + 2 \operatorname{diam} \Omega.$$

In particular

$$|u_i(x)| \leqslant \sup_{\partial\Omega} |u_i| + \sqrt{\frac{1+H_1}{1-H_1}} \frac{2 \operatorname{diam} \Omega}{\operatorname{dist}(x, \partial\Omega)}. \tag{2.4}$$

Proof. Let $x_0 \in \Omega$ and define v as in (1.17). Then by convexity

$$|v_i(x)| \leqslant \sup_{\partial\Omega} |v_i| \quad \text{in } \Omega.$$

The part (2.4) follows directly from Lemma 2.4. □

Note that the bound obtained in (2.4) depends on H_1, which can cause problems if $H_1 \to 1$. This situation occurs in section 4. However the estimate (2.4) is not used to derive the main result, so the dependence on H_1 can be ignored.

From Lemmas 2.6 and 2.7 we get a global bound

$$|Du| \leqslant C \text{ in } \bar{\Omega},$$

where C depends on ε.

2.3 Second derivative estimates

Let $u \in \mathcal{A}$ be a solution of (1.19). In this subsection we will derive a bound for $|D^2 u|$ in $\bar{\Omega}$. We will accomplish it by first reducing the problem of finding a second derivative bound in $\bar{\Omega}$ to the problem of finding a bound for $|D^2 u|$ on $\partial\Omega$. Then we will use certain properties of the Fréchet derivative of an elliptic operator $\mathcal{G}_0[u] =$

$G(D^2u, Du, u)$ to derive the boundary estimates. The derivation of the bounds for $|D^2u|$ will be the only place in section 2 in which we make use of the condition (1.11) (and all other conditions imposed on f: (1.2)–(1.8), (1.10)).

In order to find a bound for $|D^2u|$ inside Ω, we will estimate the maximum of principal hyperbolic curvatures in Ω. If v is defined in (1.17) and assuming the gradient bound derived in the previous section, it will then follow that $|D^2v|$ is bounded (because v is positive definite). A bound on $|D^2u|$ derived in this way will unfortunately depend on $1/u$. By Lemma 2.4 we can then get either an interior estimate (although not a purely interior estimate), or a global estimate that depends on ε. The method used here is similar to [3, Section 2]. Let

$$\tau = 1/w, \qquad a = \frac{1}{2}\min_{\bar\Omega} \tau. \tag{2.5}$$

Then

$$\frac{1}{\tau - a} \leqslant \frac{1}{a} = 2\max_{\bar\Omega} w.$$

We will estimate

$$M := \sup_{\Omega, i} \frac{(u-\varepsilon)^\beta}{\tau - a}\kappa_i[u](x)$$

for some fixed $\beta \geqslant 0$. M is achieved at some point $x^0 = (x_1^0, \dots, x_n^0) \in \bar\Omega$. If $\beta > 0$, x^0 is an interior point, i.e., $x^0 \in \Omega$. If $\beta = 0$, we can assume that $x^0 \in \Omega$, for otherwise the estimation of M will be reduced to the boundary estimates that are covered later in this subsection. Set

$$W = w(x^0), \qquad U = u(x^0).$$

Then both W and $1/U$ are bounded from above by $C_0/\operatorname{dist}(x^0, \partial\Omega)$ for some $C_0 > 0$ (see Lemmas 2.4 and 2.7). Alternatively we can treat W and $1/U$ as bounded from above by C_0/ε. Now let us choose a new coordinate system with the origin at (x^0, U) and coordinate (orthonormal) vectors $\varepsilon_1, \dots, \varepsilon_{n+1}$ such that:

- the hyperplane spanned by $\varepsilon_1, \dots, \varepsilon_n$ is tangent to the graph of u at x^0;
- ε_1 is the direction of the maximal principal hyperbolic curvature of u at x^0.

Then there is a neighborhood of x^0 in which the graph of u can be represented as a graph of a function $y \to \tilde u(y)$ in the new coordinate system. Let $a_i = \varepsilon_i \cdot e_{n+1}$ (e_{n+1}

is the $n+1$ vector of the old coordinate system). It is easy to see that

$$a_{n+1} > 0 \tag{2.6}$$

$$e_{n+1} = \sum_{i=0}^{n+1} a_i \varepsilon_i \tag{2.7}$$

$$\sum_{k=1}^{n} x_k e_k + u(x) e_{n+1} = \sum_{k=1}^{n} y_k \varepsilon_k + \tilde{u}(y) \varepsilon_{n+1} + \sum_{k=1}^{n} x_k^0 e_k + U e_{n+1} \tag{2.8}$$

$$u(x) = \sum_{k=1}^{n} a_k y_k + a_{n+1} \tilde{u}(y) + U. \tag{2.9}$$

The hyperbolic principal curvatures of $\tilde{u}$ are the eigenvalues of $[h_{ij}]$ where

$$h_{ij} = \frac{1}{\tilde{w}} \sum_{k,l} \tilde{\gamma}^{ik} \tilde{v}_{kl} \tilde{\gamma}^{lj}$$

with

$$\tilde{w} = \sqrt{|D\tilde{u}|^2 + 1} \tag{2.10}$$

$$\tilde{\gamma}^{st} = \delta_{st} - \frac{\tilde{u}_s \tilde{u}_t}{\tilde{w}(1+\tilde{w})} \tag{2.11}$$

$$\tilde{v}_{st} = \left(\sum_{k=1}^{n} a_k y_k + a_{n+1} \tilde{u} + U \right) \tilde{u}_{st} + \left(a_{n+1} - \sum_{k=1}^{n} a_k \tilde{u}_k \right) \left(\delta_{st} + \tilde{u}_s \tilde{u}_t \right).$$

At the origin $h_{ij} = \tilde{v}_{ij}$ and therefore ε_1 is an eigenvector of $[\tilde{v}_{ij}]$. This gives $\tilde{v}_{1j} = 0$ for $j > 1$. By rotating the $\varepsilon_2, \ldots, \varepsilon_n$, we diagonalize $[\tilde{v}_{ij}]$ at 0.

Let $y = (y_1, \ldots, y_n)$ be a point in the domain of $\tilde{u}$ and let $\zeta = (\zeta_1, \ldots, \zeta_{n+1})$ be a unit vector that is tangent to the graph of $\tilde{u}$ at y. Then we can express the Euclidean normal curvature of the graph of $\tilde{u}$ in the direction ζ as (see [18])

$$\kappa_\zeta^E = \frac{1}{\tilde{w}} \sum_{i,j=1}^{n} \tilde{u}_{ij} \zeta_i \zeta_j.$$

For $\zeta^1 = \frac{1}{\sqrt{1+\tilde{u}_1^2}}\varepsilon_1 - \frac{\tilde{u}_1}{\sqrt{1+\tilde{u}_1^2}}\varepsilon_{n+1}$ we get

$$\kappa^E_{\zeta^1} = \frac{\tilde{u}_{11}}{(1+\tilde{u}_1^2)\tilde{w}}.$$

The quantity $\kappa^E_{\zeta^1}$ is the Euclidean normal curvature in the y_1 direction. Now using (1.14) we can easily derive the formula for the hyperbolic normal curvature in the y_1 direction:

$$\kappa_{\zeta^1} = \frac{\tilde{v}_{11}}{(1+\tilde{u}_1^2)\tilde{w}}.$$

Then the function

$$f(y) = \frac{\left(\sum_{k=1}^n a_k y_k + a_{n+1}\tilde{u} + U - \varepsilon\right)^\beta}{\tau - a}\frac{\tilde{v}_{11}}{(1+\tilde{u}_1^2)\tilde{w}}, \tag{2.12}$$

has its maximum at $y = 0$ and $f(0) = M$. $\log f$ also assumes the maximum at the origin, and so its gradient vanishes:

$$\beta\frac{a_i}{U-\varepsilon} - \frac{\tau_i}{\tau - a} + \frac{\tilde{v}_{11,i}}{\tilde{v}_{11}} = 0 \text{ for all } 1 \leqslant i \leqslant n. \tag{2.13}$$

Also the Hessian matrix is negative definite, therefore (for all $1 \leqslant i \leqslant n$):

$$\frac{\beta a_{n+1}\tilde{u}_{ii}}{U-\varepsilon} - \frac{\beta a_i^2}{(U-\varepsilon)^2} - \left(\frac{\tau_i}{\tau - a}\right)_i - \tilde{u}_{ii}^2 - 2\tilde{u}_{1i}^2 + \frac{\tilde{v}_{11,ii}}{\tilde{v}_{11}} - \frac{\tilde{v}_{11,i}^2}{\tilde{v}_{11}^2} \leqslant 0. \tag{2.14}$$

At the origin we also have

$$\tau = \frac{1}{W} = a_{n+1} \tag{2.15}$$

$$\tau_i = \frac{\partial \tau}{\partial y_i} = -\sum_j a_j \tilde{u}_{ij} = -a_i\tilde{u}_{ii} \tag{2.16}$$

$$\tau_{ii} = \frac{\partial^2 \tau}{\partial y_i^2} = -\frac{\sum_k \tilde{u}_{ki}^2}{W} - \sum_j a_j \tilde{u}_{iij} = -a_{n+1}\tilde{u}_{ii}^2 - \sum_{j=1}^n a_j \tilde{u}_{iij} \tag{2.17}$$

$$h_{ii} = \tilde{v}_{ii} \tag{2.18}$$

$$h_{ii,j} = \frac{\partial h_{ii}}{\partial y_j} = \tilde{v}_{ii,j} \tag{2.19}$$

$$h_{ii,11} = \frac{\partial^2 h_{ii}}{\partial y_1^2} = \tilde{v}_{ii,11} - \tilde{u}_{11}^2 \tilde{v}_{ii} - 2\delta_{i1}\tilde{u}_{11}^2 \tilde{v}_{11}. \tag{2.20}$$

Now it remains to find the derivatives of $\tilde{v}_{ij}$:

$$\begin{aligned}
\tilde{v}_{ij}(0) &= U\tilde{u}_{ij}(0) + \delta_{ij} a_{n+1} \\
\tilde{v}_{ij,k}(0) &= a_k \tilde{u}_{ij} + U\tilde{u}_{ijk} - \delta_{ij} a_k \tilde{u}_{kk} \\
\tilde{v}_{ij,kl}(0) &= a_{n+1}\tilde{u}_{kl}\tilde{u}_{ij} + a_k \tilde{u}_{ijl} + a_l \tilde{u}_{ijk} + U\tilde{u}_{ijkl} \\
&\qquad - \delta_{ij} \sum_s a_s \tilde{u}_{skl} + a_{n+1} \left(\tilde{u}_{ik}\tilde{u}_{jl} + \tilde{u}_{il}\tilde{u}_{jk} \right).
\end{aligned}$$

Therefore

$$\tilde{v}_{11,ii}(0) - \tilde{v}_{ii,11}(0) = 2a_i \tilde{u}_{11i} - 2a_1 \tilde{u}_{ii1} - \sum_s a_s \tilde{u}_{sii} + \sum_s a_s \tilde{u}_{s11}. \tag{2.21}$$

Using the above equalities we have:

$$\begin{aligned}
\sum_i f_i h_{ii,11} &= \sum_i f_i \left(\tilde{v}_{ii,11} - \tilde{u}_{11}^2 \tilde{v}_{ii} \right) - 2f_1 \tilde{v}_{11} \tilde{u}_{11}^2 \\
&= \sum_i f_i \left(\tilde{v}_{11,ii} - \tilde{u}_{11}^2 \tilde{v}_{ii} \right) - 2f_1 \tilde{v}_{11} \tilde{u}_{11}^2 - 2\sum_i a_i f_i \tilde{u}_{11i} \\
&\qquad + 2a_1 \sum_i f_i \tilde{u}_{ii1} + \sum_{i,s} a_s f_i \tilde{u}_{sii} - \sum_{i,s} a_s \tilde{u}_{s11} f_i.
\end{aligned}$$

Define (at the origin)

$$A = \sum_i f_i, \qquad B = \sum_i f_i v_{ii} = \sum_i f_i \kappa_i, \qquad D = \sum_{j=1}^n a_j^2. \tag{2.22}$$

Then B is bounded from below by a positive constant σ_0 (see Condition (1.5)), and $D \leqslant 1$. We then have

$$\begin{aligned}
\sum_i f_i h_{ii,11} \leqslant \tilde{v}_{11} \sum_i f_i \Bigg(& \frac{\tilde{v}_{11,i}^2}{\tilde{v}_{11}^2} + \left(\frac{\tau_i}{\tau - a} \right)_i + \tilde{u}_{ii}^2 \\
& + \frac{\beta a_i^2}{(U-\varepsilon)^2} - \frac{\beta a_{n+1} \tilde{u}_{ii}}{U - \varepsilon} - \frac{\tilde{u}_{11}^2}{\tilde{v}_{11}} \tilde{v}_{ii} \Bigg)
\end{aligned}$$

$$
\begin{aligned}
&\quad + 2a_1 \sum_i f_i \tilde{u}_{ii1} - 2 \sum_i a_i f_i \tilde{u}_{11i} + \sum_{i,s} a_s f_i \tilde{u}_{iis} - A \sum_s a_s \tilde{u}_{s11} \\
&= \tilde{v}_{11} \sum_i f_i \left(\left(\frac{\tau_i}{\tau - a} \right)^2 + \left(\frac{\tau_i}{\tau - a} \right)_i - 2\beta \frac{a_i}{U - \varepsilon} \frac{\tau_i}{\tau - a} + \tilde{u}_{ii}^2 \right. \\
&\quad \left. + \frac{\beta(\beta+1)a_i^2}{(U-\varepsilon)^2} - \frac{\beta a_{n+1} \tilde{u}_{ii}}{U - \varepsilon} - \frac{\tilde{u}_{11}^2}{\tilde{v}_{11}} \tilde{v}_{ii} \right) \\
&\quad + 2a_1 \sum_i f_i \tilde{u}_{ii1} - 2 \sum_i a_i f_i \tilde{u}_{11i} + \sum_{i,s} a_s f_i \tilde{u}_{iis} - A \sum_s a_s \tilde{u}_{s11} \\
&= \tilde{v}_{11} \sum_i f_i \left(\frac{-a_{n+1} \tilde{u}_{ii}^2 - \sum_j a_j \tilde{u}_{jii}}{\tau - a} + 2\beta \frac{a_i}{U - \varepsilon} \frac{a_i \tilde{u}_{ii}}{\tau - a} + \tilde{u}_{ii}^2 \right. \\
&\quad \left. + \frac{\beta(\beta+1)a_i^2}{(U-\varepsilon)^2} - \frac{\beta a_{n+1} \tilde{u}_{ii}}{U - \varepsilon} - \frac{\tilde{u}_{11}^2}{\tilde{v}_{11}} \tilde{v}_{ii} \right) \\
&\quad + 2a_1 \sum_i f_i \tilde{u}_{ii1} - 2 \sum_i a_i f_i \tilde{u}_{11i} + \sum_{i,s} a_s f_i \tilde{u}_{iis} - A \sum_s a_s \tilde{u}_{s11}
\end{aligned}
$$

Now denote (for $1 \leqslant j \leqslant n$)

$$
\tilde{\psi}_j(y) = \frac{\partial}{\partial y_j} \psi(x, u(x)), \qquad \tilde{\psi}_{ij}(y) = \frac{\partial^2}{\partial y_i \partial y_j} \psi(x, u(x)).
$$

It is easy to see that $|\tilde{\psi}_j(0)| \leqslant C_1$ and $|\tilde{\psi}_{11}(0)| \leqslant C_1(1 + \tilde{v}_{11})$ for some C_1 dependent on $1/U$. We will now differentiate the equation

$$
F([h_{ij}]) = \psi(x, u(x))
$$

with respect to y_k:

$$
\sum_{i,j} F^{ij} h_{ij,k} = \tilde{\psi}_k.
$$

By concavity of F we have

$$
\tilde{\psi}_{11} \leqslant \sum_{i,j} F^{ij} h_{ij,11}.
$$

Then (at the origin)

$$\sum_i f_i v_{ii,k} = \tilde{\psi}_k \tag{2.23}$$

$$\sum_i f_i h_{ii,11} \geqslant \tilde{\psi}_{11}. \tag{2.24}$$

and

$$\sum_i f_i \tilde{u}_{ii} = \frac{B - Aa_{n+1}}{U}$$

$$\sum_i f_i \tilde{u}_{iis} = \frac{\tilde{\psi}_s}{U} + \frac{Aa_s\tilde{v}_{ss} - a_s B}{U^2}$$

$$\sum_s a_s \tilde{u}_{11s} = -\frac{\tilde{v}_{11}\sum_s a_s\tilde{v}_{ss}}{U^2(\tau - a)} + \frac{aD\tilde{v}_{11}}{U^2(\tau - a)} - \frac{\beta D\tilde{v}_{11}}{U(U-\varepsilon)} + \frac{\sum_s a_s^2\tilde{v}_{ss}}{U^2}.$$

We have

$$\begin{aligned}\sum_i f_i h_{ii,11} \leqslant -\frac{\tilde{u}_{11}^2}{\tilde{v}_{11}^2}B\tilde{v}_{11}^2 + \tilde{v}_{11}\Bigg[&\frac{2\beta(B - A\tau)}{U(U-\varepsilon)(\tau - a)} + \frac{\beta(\beta+1)A}{(U-\varepsilon)^2} \\ &- \frac{\beta\tau(B - A\tau)}{U(U-\varepsilon)} - \frac{\sum_j a_j\tilde{\psi}_j}{U(\tau - a)} + \frac{DB}{U(\tau - a)} + \frac{\beta AD}{U(U-\varepsilon)} + \frac{2Aa_1^2}{U^2} \\ &+ \frac{2B}{U^2(\tau - a)} + \frac{AB}{U(U-\varepsilon)} + \frac{A}{U^2}\Bigg] \\ &+ \frac{2a_1\tilde{\psi}_1}{U} - \frac{2a_1^2 B}{U^2} + \frac{\sum_j a_j\tilde{\psi}_j}{U}.\end{aligned}$$

Assume first that

$$\frac{\tilde{u}_{11}^2}{\tilde{v}_{11}^2} \geqslant \frac{1}{4U^2}. \tag{2.25}$$

Then

$$\frac{U^2}{B}\sum_i f_i h_{ii,11} \leqslant -\frac{\tilde{v}_{11}^2}{4} + \tilde{v}_{11}\Bigg[\frac{2\beta U}{(U-\varepsilon)(\tau - a)} + \frac{\beta(\beta+1)AU^2}{(U-\varepsilon)^2 B} + \frac{\beta AU\tau^2}{B(U-\varepsilon)}$$

$$
\begin{aligned}
&- \frac{U\sum_j a_j\tilde{\psi}_j}{B(\tau-a)} + \frac{UD}{\tau-a} + \frac{\beta ADU}{B(U-\varepsilon)} + \frac{2Aa_1^2}{B} + \frac{2}{\tau-a} \\
&+\frac{AU}{U-\varepsilon} + \frac{A}{B}\Big] \\
&+ \frac{2Ua_1\tilde{\psi}_1}{B} - 2a_1^2 + \frac{U\sum_j a_j\tilde{\psi}_j}{B}.
\end{aligned}
$$

Now by (2.24) we get the following quadratic inequality:

$$
\frac{\tilde{v}_{11}^2}{4} + M_1\tilde{v}_{11} \leqslant M_2, \tag{2.26}
$$

where the constants M_1 and M_2 depend on $\inf B$ and $\operatorname{dist}(x^0, \partial\Omega)$ and in addition M_1 depends on $\sup A$. Therefore we have

$$
\tilde{v}_{11} \leqslant C\left(\sup A, \operatorname{dist}(x^0, \partial\Omega)\right). \tag{2.27}
$$

If (2.25) is false, (2.27) still holds, because then

$$
\begin{gathered}
\left|U\frac{\tilde{u}_{11}}{\tilde{v}_{11}}\right| < \frac{1}{2} \\
-\frac{1}{2} < 1 - \frac{\tau}{\tilde{v}_{11}} < \frac{1}{2} \\
\tilde{v}_{11} < 2\tau.
\end{gathered}
$$

The above reasoning leads to the following two lemmas

Lemma 2.8. *Suppose that $u \in \mathcal{A}$ is a solution of (1.19). Then for any $\Omega' \Subset \Omega$,*

$$
|D^2u| \leqslant C \qquad \text{in } \Omega',
$$

where C depends on $\inf_{x\in\Omega'} \operatorname{dist}(x, \partial\Omega)$ and does not depend on ε as $\varepsilon \to 0$.

If we set $\beta = 0$ in the formula for M, we will get

Lemma 2.9. *Suppose that $u \in \mathcal{A}$ is a solution of (1.19). If M achieves its maximum inside Ω then*

$$
|D^2u| \leqslant C,
$$

for some C depending on ε.

Before we proceed with the estimation of $|D^2u|$ on the boundary we need to state the following two lemmas.

Lemma 2.10. *Consider the $n \times n$ symmetric matrix*

$$M = \begin{pmatrix} d_1 & 0 & 0 & \dots & 0 & a_1 \\ 0 & d_2 & 0 & \dots & 0 & a_2 \\ \vdots & & \ddots & & & \vdots \\ 0 & & & d_{n-2} & 0 & a_{n-2} \\ 0 & 0 & \dots & 0 & d_{n-1} & a_{n-1} \\ a_1 & \dots & \dots & a_{n-2} & a_{n-1} & a \end{pmatrix} \tag{2.28}$$

with $d_1, \dots, d_{n-1}$ fixed, $|a|$ tending to infinity and

$$|a_i| \leqslant C, \qquad 1 \leqslant i \leqslant n-1.$$

Then the eigenvalues $\lambda_1, \dots, \lambda_n$ behave like

$$\begin{aligned} \lambda_\alpha &= d_\alpha + o(1), \qquad 1 \leqslant \alpha \leqslant n-1 \\ \lambda_n &= a\big(1 + O(1/a)\big), \end{aligned}$$

where the $o(1)$ and $O(1/a)$ are uniform — depending only on $d_1, \dots, d_{n-1}$ and C.

Proof. The proof is given in [2, Lemma 1.2]. □

Lemma 2.11.

$$\lim_{R\to\infty} G(r + Rq \otimes q, p, z) = \infty, \qquad \forall q \in \mathbb{R}^n,\ q \neq 0,\ \forall (r,p,z) \in \operatorname{domain}(G),\ z > 0.$$

Proof. If $r' \in \mathfrak{S}^+$ is a positive definite symmetric matrix in $\mathbb{R}^{n\times n}$ define

$$\hat{G}(r', p) = F([a_{ij}]),$$

where

$$a_{ij} = \frac{1}{w}\sum_{k,l}\gamma^{ik}r'_{kl}\gamma^{lj},$$
$$w = \sqrt{1+|p|^2}, \qquad \gamma^{ik} = \delta_{ik} - \frac{p_i p_k}{w(1+w)}.$$

Then it is easy to see that $G(r,p,z) = \hat{G}(zr + I + p\otimes p, p)$ and $G(r + Rq\otimes q, p, z) = \hat{G}(r' + R'q\otimes q, p)$, where $r' = zr + I + p\otimes p$ and $R' = zR$. By [10, Lemma 2.2],

$$\lim_{R'\to\infty}\hat{G}(r' + R'q\otimes q, p) = \infty,$$

which gives us the desired conclusion. □

Now we are ready to estimate $|D^2u|$ on the boundary $\partial\Omega$. The method used here is very similar to [10]. Let $x_0 \in \partial\Omega$ be an arbitrary boundary point and let

$$g_{ij} = \delta_{ij} + u_i u_j \qquad \gamma_{ij} = \delta_{ij} + \frac{u_i u_j}{1+w}. \tag{2.29}$$

Then $[\gamma_{ij}] = [\gamma^{ij}]^{-1}$ and $g_{ij} = \sum_k \gamma_{ik}\gamma_{kj}$. Also for $\delta > 0$ denote

$$B_\delta = B_\delta(x_0) = \{x \in \mathbb{R}^n : \ |x - x_0| < \delta\}.$$

Since $[g_{ij}]$ is positive definite, there is a constant $\beta > 0$ such that

$$[g_{ij}] \geqslant 4\beta u I \qquad \text{in } \bar{\Omega}, \tag{2.30}$$

where I is the identity matrix. For example we can take

$$\beta = \frac{1}{4\sqrt{\varepsilon^2 + (\operatorname{diam}\Omega)^2}}.$$

This is due to the fact that all eigenvalues of $[g_{ij}]$ are bigger than or equal to 1 and due to Lemma 2.2.

Let $C_0 > 0$ be an arbitrary constant. As in (1.22) denote

$$G\left(D^2u, Du, u\right) = F\big([h_{ij}]\big) = f\left(\lambda\big([h_{ij}]\big)\right), \tag{2.31}$$

where h_{ij} is defined by (1.16) and $\lambda\big([h_{ij}]\big)$ denotes the vector of eigenvalues of $[h_{ij}]$. Let a_{ij} be defined in (1.15). Then the Euclidean curvatures $\kappa_i^E[u]$ are the eigenvalues of $[a_{ij}]$.

By concavity of f and the fact that $f(0, \dots, 0) = 0$ we have

$$\sum_i f_i \kappa_i \leqslant f(\kappa[u]) \leqslant \psi_1. \tag{2.32}$$

Let F^{ij} be defined by (1.21). We have

$$\begin{aligned} G^{st} = \frac{\partial G}{\partial u_{st}} &= \frac{u}{w} \sum_{i,j} F^{ij} \gamma^{is} \gamma^{tj} \\ \sum_{i,j} F^{ij} h_{ij} &= \sum_i f_i \kappa_i \\ \sum_{k,l} G^{kl} u_{kl} &= \sum_i f_i \kappa_i - \frac{1}{w} \sum_i F^{ii}. \end{aligned} \tag{2.33}$$

Therefore

$$\frac{u}{w^3} \sum_i F^{ii} \leqslant \sum_i G^{ii} \leqslant \frac{u}{w} \sum_i F^{ii}. \tag{2.34}$$

Now

$$\begin{aligned} G^s = \frac{\partial G}{\partial u_s} = &-\frac{u_s}{w^2} \sum_i f_i \kappa_i - \frac{2}{w} \sum_{i,j,k} F^{ij} h_{ik} \left(\frac{w u_k \gamma^{sj} + u_j \gamma^{ks}}{1 + w} \right) \\ &+ \frac{2}{w^2} \sum_{i,j} F^{ij} u_i \gamma^{sj} \end{aligned}$$

$$\sum_s G^s u_s = -\frac{w^2 - 1}{w^2} \sum_i f_i \kappa_i - \frac{2}{w^2} \sum_{i,j,k} F^{ij} h_{ik} u_k u_j + \frac{2}{w^3} \sum_{i,j} F^{ij} u_i u_j. \tag{2.35}$$

So (by (1.11))

$$\sum_s G^s u_s \leqslant C_1.$$

Denote $L' = \sum_{s,t} G^{st}\partial_{st} + \sum_s G^s \partial_s$ and let $d(x) = \operatorname{dist}(x, \partial\Omega)$. Now for any $N > 0$ choose $\delta_1 > 0$ small enough so that

$$2N\delta_1 D^2 d \geqslant -\beta I \qquad \text{in } B_{\delta_1} \cap \Omega.$$

Then we have (in $B_{\delta_1} \cap \Omega$)

$$D^2(Nd^2) - 2\beta I \geqslant 2NDd \otimes Dd - 3\beta I. \tag{2.36}$$

Therefore (by (2.30)) the expression $G(D^2(Nd^2) - 2\beta I, Du, u)$ is well defined and the eigenvalues of

$$[g_{ij}] - 3\beta u I$$

lie in a compact subset of Γ_n^+. Using (2.36) together with Lemma 2.11 we obtain the following result.

Lemma 2.12. *For any $C > 0$ there is an $N > 0$ big enough and $\delta_2 > 0$ small enough so that*

$$G(D^2(Nd^2) - 2\beta I, Du, u) \geqslant C \qquad \text{in } B_{\delta_2} \cap \Omega.$$

The value of N depends on ε as $\varepsilon \to 0$.

By the concavity of $G(r, \cdot, \cdot)$,

$$\begin{aligned} L'(u - \varepsilon - Nd^2) + 2\beta \sum_i G^{ii} &\\ &= \sum_{i,j} G^{ij}(u - Nd^2)_{ij} + 2\beta \sum_i G^{ii} + \sum_i G^i (u - Nd^2)_i \\ &\leqslant G(D^2 u, Du, u) - G(D^2(Nd^2) - 2\beta I, Du, u) + \sum_i G^i u_i \\ &\qquad - 2Nd \sum_i G^i d_i \end{aligned}$$

in $B_{\delta_2}(x_0)\cap\Omega$ where $0<\delta_2\leqslant\delta_1$ small enough (but dependent on N). By Lemma 2.12 we can choose N big enough (and δ_2 small enough) so

$$G(D^2(Nd^2)-2\beta I,Du,u)\geqslant G(D^2u,Du,u)+C_1+C_0+2.$$

Then

$$L'(u-\varepsilon-Nd^2)\leqslant -C_0-2\beta\sum_i G^{ii}+(2Nd\sum_i|G^i|-2).$$

Now take $0<\delta_3\leqslant\delta_2$ small enough so that $2N\delta_3\sum_i|G^i|\leqslant 1$ in $B_{\delta_3}(x_0)\cap\Omega$ and choose $t>0$ such that $t\sum_i|G^i|\leqslant 1$ and $t|D^2d|\leqslant\beta$ in $B_{\delta_3}\cap\Omega$. Then

$$L'(u-\varepsilon+td-Nd^2)\leqslant -C_0-\beta\sum_i G^{ii}.$$

Now take $0<\delta\leqslant\min\{\delta_3,\varepsilon\}$ such that $u-\varepsilon+td-Nd^2\geqslant 0$ on $\partial(B_\delta(x_0)\cap\Omega)$. Then we have the following

Lemma 2.13. *Let $x_0\in\partial\Omega$. For any constant $C_0>0$ there exist positive constants t, δ sufficiently small and N sufficiently large such that the function $\phi=u-\varepsilon+td-Nd^2$ satisfies*

$$\begin{cases} L'\phi\leqslant -C_0-\beta\sum_i G^{ii} & \text{in } \Omega\cap B_\delta,\\ \phi\geqslant 0 & \text{on } \partial(\Omega\cap B_\delta),\end{cases}$$

where N depends on ε as $\varepsilon\to 0$.

We now have

Lemma 2.14. *Let $h\in C^2(\overline{\Omega\cap B_\delta})$ where B_δ is centered at the origin which is on $\partial\Omega$. Suppose h satisfies $h\leqslant C_0|x|^2$ on $(\partial\Omega)\cap B_\delta$ for some $C_0>0$, $h(0)=0$ and*

$$-Lh\leqslant C_1\left(1+\sum_i G^{ii}\right)\qquad \text{in } \Omega\cap B_\delta,$$

where $L=\sum_{s,t}G^{st}\delta_{st}+\sum_s G^s\delta_s+G_u$, $G_u=\frac{\partial G}{\partial u}$. Then $h_n(0)\leqslant C$ for some $C>0$ dependent on ε as $\varepsilon\to 0$.

Proof. Let us first notice that

$$G_u = \frac{\partial G}{\partial u} = \sum_{i,j} F^{ij} a_{ij}, \quad \text{so } G_u u = \sum_i f_i \kappa_i - \frac{1}{w} \sum_i F^{ii} \leqslant C_2. \tag{2.37}$$

Therefore

$$-\frac{w^2}{u^2} \sum_i G^{ii} \leqslant G_u \leqslant \frac{C_2}{u}.$$

We can assume that δ is sufficiently small so that

$$|h(x)| \leqslant \varepsilon^4$$

holds in $\Omega \cap B_\delta$. Then we have

$$G_u h \leqslant C_3 \left(1 + \sum_i G^{ii}\right)$$

and

$$-L'h \leqslant C_4 \left(1 + \sum_i G^{ii}\right)$$

in $B_\delta \cap \Omega$. By Lemma 2.13, $Av + B|x|^2 - h \geqslant 0$ on $\partial(\Omega \cap B_\delta)$ and

$$L'(Av + B|x|^2 - h) \leqslant 0 \qquad \text{in } \Omega \cap B_\delta$$

when $A >> B$ are both large. Thus $Av + B|x|^2 - h \geqslant 0$ in $\overline{\Omega \cap B_\delta}$ by the maximum principle. Consequently,

$$Av_n(0) - h_n(0) = D_n(Av + B|x|^2 - h)(0) \geqslant 0$$

since $Av + B|x|^2 - h = 0$ at the origin. □

Note that Lemma 2.14 remains true if L is replaced by L'.

Lemma 2.15. *Let $1 \leqslant i < j \leqslant n$. Then*

$$L(x_i u_j - x_j u_i) = x_i \psi_j - x_j \psi_i + \psi_{n+1}(x_i u_j - x_j u_i).$$

Proof. For any $\theta \in \mathbb{R}$, let us introduce a new coordinate system: $y_1 = x_1, \ldots, y_{i-1} = x_{i-1}, y_i = x_i \cos\theta - x_j \sin\theta, y_{i+1} = x_{i+1}, \ldots, y_{j-1} = x_{j-1}, y_j = x_i \sin\theta + x_j \cos\theta, y_{j+1} = x_{j+1}, \ldots, y_n = x_n$. Then, since the hyperbolic curvatures of the graph of u do not change under the above transformation, we have

$$G(D^2u(y), Du(y), u(y)) = \psi(y, u(y))$$

for all θ and all $y \in \Omega$. We therefore have:

$$\begin{aligned}
\frac{\partial}{\partial\theta} G(D^2u, Du, u) &= L\left(\frac{\partial u}{\partial\theta}\right) \\
&= L\big(u_i(-x_i \sin\theta - x_j \cos\theta) + u_j(x_i \cos\theta - x_j \sin\theta)\big) \\
\frac{\partial}{\partial\theta}\psi(y) &= \psi_i(-x_i \sin\theta - x_j \cos\theta) + \psi_j(x_i \cos\theta - x_j \sin\theta) \\
&\quad + \psi_{n+1}\big(u_i(-x_i \sin\theta - x_j \cos\theta) + u_j(x_i \cos\theta - x_j \sin\theta)\big).
\end{aligned}$$

Setting $\theta = 0$ we get

$$L(x_i u_j - x_j u_i) = x_i \psi_j - x_j \psi_i + \psi_{n+1}(x_i u_j - x_j u_i).$$

□

Consider now any fixed point of $\partial\Omega$. We may assume it to be the origin of $\mathbb{R}^n$ and choose the coordinates so that the positive x_n axis is the interior normal to $\partial\Omega$ at 0. Near the origin, $\partial\Omega$ can be represented as a graph

$$x_n = \rho(x') = \frac{1}{2} \sum_{\alpha,\beta<n} B_{\alpha\beta} x_\alpha x_\beta + O(|x'|^3), \qquad x' = (x_1, \ldots, x_{n-1}).$$

Since u is constant on $\partial\Omega$, it follows that

$$|u_{\alpha\beta}(0)| \leqslant C, \qquad \alpha, \beta < n.$$

Next, for fixed $\alpha < n$ consider the operator

$$T = \partial_\alpha + \sum_{\beta<n} B_{\alpha\beta}(x_\beta \partial_n - x_n \partial_\beta).$$

Then $L(Tu) = T\psi(x,u)$ by Lemma 2.15. It follows that

$$|L(Tu)| \leqslant C.$$

Moreover, since $u - \varepsilon = 0$ on $\partial\Omega$, near the origin we have

$$|T(u-\varepsilon)| \leqslant C|x|^2 \text{ on } \partial\Omega$$

(this is because $\partial_s T(u-\varepsilon) = 0$ for $s < n$ at the origin). Applying Lemma 2.14 to $h = \pm T(u-\varepsilon)$, it follows that

$$|u_{\alpha n}(0)| \leqslant C.$$

Now it remains to show that

$$|u_{\nu\nu}| \leqslant C \text{ on } \partial\Omega$$

where ν is the unit interior normal vector to $\partial\Omega$.

We first prove that

$$M := \min_{x\in\partial\Omega} \min_{\xi\in T_x(\partial\Omega),\ |\xi|=1} \sum_{i,j} v_{ij}(x)\xi_i\xi_j \geqslant c_0 \tag{2.38}$$

for some $c_0 > 0$, where $T_x(\partial\Omega)$ denotes the tangent space of $\partial\Omega$ at $x \in \partial\Omega$ and v is defined by (1.17). Let σ be a smooth defining function of Ω, i.e.

$$\Omega = \{x \in \mathbb{R}^n :\ \sigma(x) < 0\},\ \partial\Omega = \{x \in \mathbb{R}^n :\ \sigma(x) = 0\},\ \text{and } |D\sigma| = 1 \text{ on } \partial\Omega.$$

Let $\underline{v} = \frac{1}{2}\varepsilon^2 + \frac{1}{2}|x - x_0|^2$ with x_0 being the same as in the definition of v. Since $v - \underline{v} = 0$ on $\partial\Omega$ and $v - \underline{v} > 0$ in Ω, we see that $v - \underline{v} = \eta\sigma$ for some function $\eta \leqslant 0$. Note that $D\sigma = -\nu$ on $\partial\Omega$ where ν is the interior unit normal to $\partial\Omega$. We

have $\eta = -(v - \underline{v})_\nu$ on $\partial\Omega$ and

$$\sum_{i,j} v_{ij}\xi_i\xi_j = \sum_{i,j} \underline{v}_{ij}\xi_i\xi_j - (v - \underline{v})_\nu \sum_{i,j} \sigma_{ij}\xi_i\xi_j \text{ on } \partial\Omega$$

for any tangent vector field $\xi = (\xi_1, \ldots, \xi_n)$ to $\partial\Omega$. We may choose coordinates in $\mathbb{R}^n$ such that M is achieved at the origin with $\xi = (1, 0, \ldots, 0)$ and $e_n = \nu(0)$ (i.e. the n-th coordinate vector is $\nu(0)$). Thus

$$M = v_{11}(0) = \underline{v}_{11}(0) - (v - \underline{v})_\nu(0)\sigma_{11}(0) = 1 - (v - \underline{v})_\nu(0).$$

We may assume

$$(v - \underline{v})_\nu(0)\sigma_{11}(0) > \frac{1}{2}\underline{v}_{11}(0) = \frac{1}{2}$$

for otherwise we have $M \geqslant \frac{1}{2}$ and we are done. Let $\zeta = (\zeta_1, \ldots, \zeta_n)$ be defined as

$$\begin{aligned}
\zeta_1 &= -\sigma_n \left(\sigma_1^2 + \sigma_n^2\right)^{-1/2} \\
\zeta_j &= 0, \qquad 2 \leqslant j \leqslant n-1 \\
\zeta_n &= \sigma_1 \left(\sigma_1^2 + \sigma_n^2\right)^{-1/2}.
\end{aligned}$$

By the continuity of $\sum_{i,j} \sigma_{ij}\zeta_i\zeta_j$ and the fact that $0 \leqslant (v - \underline{v})_\nu \leqslant C$ on $\partial\Omega$, there exists $c_1 > 0$ and $\delta > 0$ such that in $\Omega \cap B_\delta(0)$ we have

$$\sum_{i,j} \sigma_{ij}\zeta_i\zeta_j(x) \geqslant \frac{1}{2}\sum_{i,j} \sigma_{ij}\zeta_i\zeta_j(0) \geqslant \frac{\sigma_{11}(0)}{2} > \frac{1}{4(v - \underline{v})_\nu(0)} \geqslant c_1.$$

Thus the function

$$\Phi = \frac{\sum_{i,j} \underline{v}_{ij}\zeta_i\zeta_j - M}{\sum_{i,j} \sigma_{ij}\zeta_i\zeta_j}$$

is smooth and bounded in $\Omega \cap B_\delta(0)$. We also have

$$\sum_{i,j} \underline{v}_{ij}\zeta_i\zeta_j + (D(v - \underline{v}) \cdot D\sigma) \sum_{i,j} \sigma_{ij}\zeta_i\zeta_j = \sum_{i,j} v_{ij}\zeta_i\zeta_j \geqslant M,$$

therefore

$$\Phi + D(v - \underline{v}) \cdot D\sigma \geqslant 0 \text{ on } (\partial\Omega) \cap B_\delta(0).$$

By (2.34), (2.33), (2.35), and (2.37), $L\Phi$, $-L(D\underline{v}\cdot D\sigma)$ are bounded by

$$C\left(\sum_i G^{ii}+1\right)$$

for some C that depends on ε. In addition, since $Lu\leqslant C$ (by (2.33), (2.35), and (2.37)), we have $L(Dv)\leqslant C\,(\sum_i G^{ii}+1)$. Therefore also:

$$L(Dv\cdot D\sigma)\leqslant C\left(\sum_i G^{ii}+1\right).$$

Let $h=-(\Phi+D(v-\underline{v})\cdot D\sigma)$. Applying Lemma 2.14 to h we get:

$$-\Phi_n(0)+(v-\underline{v})_{nn}=h_n(0)\leqslant C,$$

which shows that $|D^2v|$ is bounded at the origin. By (1.16) the principal hyperbolic curvatures of u are bounded at the origin. Since $f(\kappa[u])\geqslant\psi_0>0$ in $\bar{\Omega}$ and $f=0$ on $\partial\Gamma_n^+$, the principal curvatures at the origin admit a uniform positive lower bound. This in turn yields a positive lower bound for the eigenvalues of $D^2v(0)$ which implies (2.38).

Now let $x_0\in\partial\Omega$ be an arbitrary point on the boundary. If $d_1\leqslant\cdots\leqslant d_{n-1}$ are the eigenvalues of $[v_{\alpha\beta}(x_0)]$ $(1\leqslant\alpha<n,\,1\leqslant\beta<n)$, we see that

$$c_0\leqslant d_i\leqslant C. \tag{2.39}$$

Suppose that $v_{nn}(x_0)$ can be arbitrarily large. In order to apply Lemma 2.10 we need to rotate coordinates $(x_1,\dots,x_{n-1})$ keeping the direction x_n fixed so that the matrix $[v_{ij}(x_0)]$ has the form (2.28). According to the lemma the eigenvalues $\kappa_1,\dots,\kappa_n$ behave like

$$\begin{aligned}\kappa_\alpha&=\frac{1}{w}d_\alpha+o(1),\qquad \alpha<n,\\ \kappa_n&=\frac{1}{w^3}v_{nn}(x_0)\left(1+O\left(\frac{1}{v_{nn}(x_0)}\right)\right)\end{aligned}$$

as $v_{nn}(x_0) \to \infty$. It follows then that κ_n is arbitrary large while $(\kappa_1, \dots, \kappa_{n-1}, 1)$ lie in a compact subset E of Γ_n^+. This contradicts (1.6) and the fact that $f(\kappa_1, \dots, \kappa_n) \leqslant \psi_1$.

Therefore we have the following result.

Lemma 2.16. *Suppose $u \in \mathcal{A}$ is a solution of* (1.19)*. If the conditions* (1.2)–(1.8)*,* (1.10)*, and* (1.11) *are satisfied, we have a global bound*

$$|D^2 u| \leqslant C$$

where C depends on ε and $\sup |Du|$, $\sup |u|$.

Now we can combine Lemmas 2.1, 2.7, and 2.16 to get

Lemma 2.17. *Suppose that $u \in \mathcal{A}$ is a solution of* (1.19)*.If the conditions* (1.2)–(1.8)*,* (1.10)*, and* (1.11) *are satisfied, we have a global bound*

$$|u|_{C^{2,\alpha}(\bar{\Omega})} \leqslant C,$$

where C depends on ε and $0 < \alpha < 1$.

Proof. This result follows from [12] or from [4, Theorem 17.26]. Indeed from (1.7) we get

$$\lim_{t \to \infty} f(t, \dots, t) = \infty.$$

By concavity of f and by (1.5) we have (for any $\lambda \in \Gamma_n^+$ and $t > 0$)

$$f(t, \dots, t) \leqslant f(\lambda) + \sum_i f_i(\lambda)(t - \lambda_i) \leqslant f(\lambda) + t \sum_i f_i(\lambda).$$

By fixing t large enough we get the following inequality:

$$\sum_i f_i(\lambda) \geqslant \sigma_1 \text{ on } \left\{\lambda \in \Gamma_n^+ : \ \psi_0 \leqslant f(\lambda) \leqslant \psi_1\right\}, \tag{2.40}$$

where σ_1 is a positive constant that depends on ψ_0 and ψ_1 (and therefore is fixed for fixed ψ). The operator $\mathcal{G}[u]$ from (1.23) is not uniformly elliptic. However by (2.40), (1.11), and by Lemma 2.16, we can treat $\mathcal{G}[u]$ as an operator that is uniformly elliptic with respect to u (i.e., it is uniformly elliptic on a compact set

$\big\{(D^2u(x), Du(x), u(x)) : \ x \in \bar{\Omega}\big\}$). Then we can use Theorem 17.26 from [4] directly. □

Now if $u \in \mathcal{A}$ is a solution of (1.19), its derivative $u_s = \frac{\partial u}{\partial x_s}$ satisfies

$$Lu_s = \psi_s + \psi_{n+1}u_s,$$

where L is defined in Lemma 2.14. We can treat L as a linear elliptic operator that is uniformly elliptic with respect to u. Also its coefficients are in $C^{0,\alpha}(\bar{\Omega})$. By Schauder theory (see for example [4]) it follows that $|u_s|_{C^{2,\alpha}(\bar{\Omega})}$ is bounded for all $1 \leqslant s \leqslant n$. Therefore $|u|_{C^{3,\alpha}(\bar{\Omega})}$ is also bounded. Repeating this argument $k-2$ times we get a priori bounds for higher order derivatives:

$$|u|_{C^{k,\alpha}(\bar{\Omega})} \leqslant C_k, \qquad k \geqslant 2.$$

3 Parametrized curvature equation

In this section we will investigate a parametrized version of problem (1.19). That is in place of operator $\mathcal{G}(D^2u, Du, u, x)$ (defined in (1.23)) we will consider operators of form $\mathcal{G}(D^2u, Du, \eta, x)$ for some fixed functions η. Such operators can be used with the continuity method (Theorem 1.2) to derive an existence result.

Let H_0 be defined by (1.12) and let $r = r_\Omega$ (where r_Ω is defined by (1.13)). If $r_\Omega = \infty$, take r to be any positive number such that $H_0 r > \varepsilon$. Now let

$$R = \frac{-\varepsilon H_0 + \sqrt{\varepsilon^2 + r^2(1 - H_0^2)}}{1 - H_0^2} \qquad \text{(so that } R^2 = r^2 + (H_0 R - \varepsilon)^2\text{)}$$

and define

$$\varphi(t) = H_0 R - \sqrt{R^2 - (t + r)^2} \text{ for } 0 \leqslant t \leqslant R - r.$$

For a positive constant δ denote

$$\Omega_\delta = \{x \in \Omega : \ 0 < \operatorname{dist}(x, \partial\Omega) < \delta\}.$$

Now let

$$\mathcal{C} = \big\{\zeta \in C^5(\bar{\Omega}) : \ \zeta > -\varepsilon/2 \text{ in } \Omega, \ \zeta = 0 \text{ on } \partial\Omega, \\ \text{and } \zeta(x) < \varphi(\operatorname{dist}(x, \partial\Omega)) - \varepsilon \text{ in } \Omega_{R-r}\big\}.$$

Then $\mathcal{C}$ is an open convex set in a Banach space $C_0^5(\bar{\Omega}) = \{\zeta \in C^5(\bar{\Omega}) : \ \zeta = 0 \text{ on } \partial\Omega\}$. Note that any solution of (1.19) must be of form $u = \zeta + \varepsilon$ for some $\zeta \in \mathcal{C}$. This fact follows from section 2 (especially Lemmas 2.1, 2.5, and the regularity result from Schauder theory).

For a fixed constant $K > 0$ define

$$\mathcal{C}_K = \{\zeta \in \mathcal{C} : \ |\zeta|_{C^5} < K\}.$$

For $\zeta \in \bar{\mathcal{C}}_K$ we will consider the following Dirichlet problem:

$$\begin{aligned} G(D^2u, Du, \eta) &= \psi(x, \eta) \qquad \text{in } \Omega, \\ u &= \varepsilon \qquad\qquad\quad \text{on } \partial\Omega, \end{aligned} \tag{3.1}$$

where G is defined in (2.31) and $\eta = \zeta + \varepsilon$. The problem (3.1) can be written as

$$\begin{aligned} f(\kappa[u, \eta]) &= \psi(x, \eta) \qquad \text{in } \Omega, \\ u &= \varepsilon \qquad\qquad\quad \text{on } \partial\Omega, \end{aligned} \tag{3.2}$$

where $\kappa[u, \eta] = (\kappa_1[u, \eta], \ldots, \kappa_n[u, \eta])$ is given by

$$\kappa_i[u, \eta] = \eta \kappa_i^E[u] + \frac{1}{w}. \tag{3.3}$$

$\kappa_i[u, \eta]$ are eigenvalues of the matrix $[h_{ij}]$, where

$$h_{ij} = \frac{1}{w} \sum_{k,l} \gamma^{ik} v_{kl} \gamma^{lj}, \qquad v_{kl} = \eta u_{kl} + u_k u_l + \delta_{kl}. \tag{3.4}$$

Let

$$\mathcal{A} = \mathcal{A}(\Omega, \eta) = \left\{ u \in C^\infty(\bar{\Omega}) : \ u > 0 \text{ and } \kappa[u, \eta] \in \Gamma_n^+ \text{ for all } x \in \Omega \right\}.$$

We will now derive *a priori* bounds for the solutions of (3.2). Fortunately many arguments from section 2 can be repeated here with slight modifications. Only the proofs of Lemmas 3.3 and 3.5 differ significantly from their analogues in section 2.

Lemma 3.1. *Suppose that $u \in \mathcal{A}$ satisfies (3.2). Then u does not have a local minimum in Ω. In particular $u > \varepsilon$ in Ω.*

Proof. The proof is similar to the proof of Lemma 2.1. □

The following lemma is a type of maximum principle.

Lemma 3.2. *Let $\Omega' \subseteq \Omega$ be an open domain and let $u, v \in \mathcal{A}(\Omega')$. Suppose also that $u \leqslant v$ on $\partial\Omega'$ and $f(\kappa[v, \eta]) < f(\kappa[u, \eta])$ in Ω'. Then $u < v$ in Ω'.*

Proof. Suppose that $u \geqslant v$ at some point in Ω'. Then $v - u$ assumes a non-positive local minimum at a point $x_0 \in \Omega'$. Since f is symmetric we can assume that

$$\kappa_1^E[u] \leqslant \cdots \leqslant \kappa_n^E[u] \text{ and } \kappa_1^E[v] \leqslant \cdots \leqslant \kappa_n^E[v].$$

At x_0 we have

$$\kappa_i^E[v](x_0) \geqslant \kappa_i^E[u](x_0)$$
$$\frac{1}{\eta}\left[\kappa_i[v](x_0) - \frac{1}{\sqrt{|Dv|^2+1}}\right] \geqslant \frac{1}{\eta}\left[\kappa_i[u](x_0) - \frac{1}{\sqrt{|Du|^2+1}}\right].$$

Then (because $Du(x_0) = Dv(x_0)$)

$$\kappa_i[v](x_0) \geqslant \kappa_i[u](x_0) \text{ for all } i,$$

so $f(\kappa[v](x_0)) \geqslant f(\kappa[u](x_0))$ – contradiction. □

Lemma 3.3. *Let $u \in \mathcal{A}$ be a solution of* (3.2)*. Then*

$$\varepsilon \leqslant u(x) \leqslant C \text{ in } \bar{\Omega},$$

where C does not depend on ε as $\varepsilon \to 0$.

Proof. Consider the hyperplane $g(x) = \frac{1}{H_0}\sum_i x_i + C_0$, where $C_0 \geqslant 0$ is chosen so that $g(x) \geqslant \varepsilon$ on $\partial\Omega$. Since $\kappa_i[g,\eta] \leqslant H_0$ for $1 \leqslant i \leqslant n$, we have

$$f(\kappa[g,\eta]) < f(\kappa[u,\eta]).$$

By the maximum principle (Lemma 3.2), we get $u(x) \leqslant g(x)$ in $\bar{\Omega}$. □

Using the notation introduced in (2.1) and (2.2) we have

Lemma 3.4. *Let $x_0 \in \partial\Omega$ and let $\nu(x_0)$ be the inner unit normal vector to $\partial\Omega$. Consider a ball B of radius R centered at $(x_0 - r\nu(x_0), H_0 R)$. Then $B \cap \Sigma = \varnothing$.*

Proof. By the definition of r and R, $B \cap \Omega_\varepsilon = \varnothing$. Suppose $B \cap \Sigma \neq \varnothing$. Let us decrease R so that $B \cap \Sigma = \varnothing$. We can then reverse this process and continue

increasing R until B touches Σ at some point $(x_1, u(x_1))$. Then B can be locally represented as a graph of some function $v \in C^2$ in a neighborhood Ω_{x_1} of x_1. We also have $u(x_1) = v(x_1)$, $v \geqslant u$ on $\partial\Omega_{x_1}$, $f(\kappa[v, \eta]) < f(\kappa[u, \eta])$ in Ω_{x_1}. The last inequality is the consequence of the fact that $\eta(x) \leqslant \varphi(\mathrm{dist}(x, \partial\Omega)) \leqslant v(x)$ on Ω_{R-r}. But this contradicts Lemma 3.2. □

Using the above lemma it is easy to see that the Lemma 2.6 holds true for the equation (3.2).

Lemma 3.5. *Let $u \in \mathcal{A}$ be a solution of (3.2) and suppose that $rH_0 > \varepsilon$. Then*

$$|Du| \leqslant C \qquad \text{in } \bar{\Omega}$$

for some C depending on ε.

Proof. We will obtain a bound for

$$z = |Du(x)|e^{4u(x)/\varepsilon} \qquad \text{for } x \in \bar{\Omega}.$$

If z achieves its maximum on $\partial\Omega$, we are done by Lemma 2.6. Suppose therefore that z achieves its maximum at a point $x_0 \in \Omega$. By rotating the coordinates $(x_1, \ldots, x_n)$ we may assume that at x_0 we have

$$|Du| = u_1 > 0, \qquad u_\alpha = 0 \text{ for } \alpha > 1.$$

Then $\log u_1 + \frac{4}{\varepsilon}u$ also takes its maximum at x_0. Therefore

$$\frac{u_{1i}}{u_1} + \frac{4}{\varepsilon}u_i = 0,$$

so $u_{11} = -\frac{4}{\varepsilon}u_1^2$ and $u_{1\alpha} = 0$ for $\alpha > 1$. Now since $u \in \mathcal{A}$,

$$\eta u_{11} + u_1^2 + 1 > 0$$
$$\left(1 - \frac{4}{\varepsilon}\eta\right) u_1^2 + 1 > 0.$$

Since $\eta \geqslant \varepsilon/2$, we have $1 - \frac{4}{\varepsilon}\eta \leqslant -1$ and thus

$$-u_1^2 + 1 \geqslant \left(1 - \frac{4}{\varepsilon}\eta\right) u_1^2 + 1 > 0.$$

The above inequality gives $u_1 < 1$ and the bound for z is:

$$z \leqslant \exp\left[4u(x_0)/\varepsilon\right] \leqslant \exp\left[\frac{4}{\varepsilon} \sup_{\Omega} u\right].$$

We have then the following bound for $|Du|$:

$$|Du| \leqslant \exp\left[\frac{4}{\varepsilon}\left(\sup_{\Omega} u - u\right)\right].$$

□

Now we will need to estimate the second derivatives of u. The following lemma is an analogue of Lemma 2.16

Lemma 3.6. *Suppose $u \in \mathcal{A}$ is a solution of* (3.2)*. If the condition* (1.11) *is satisfied, we have a global bound*

$$|D^2 u| \leqslant C$$

where C depends on ε, $\sup |Du|$, $\sup |u|$, and K.

Proof. The proof is similar to the proof of Lemma 2.16. Let τ and a be defined by (2.5). We will estimate

$$M := \sup_{\Omega, i} \frac{1}{\tau - a} \kappa_i[u, \eta](x).$$

Suppose first that M is achieved at some point $x^0 = (x_1^0, \ldots, x_n^0) \in \Omega$ and set

$$W = w(x^0), \qquad U = u(x^0).$$

Now as in subsection 2.3 let us choose a new coordinate system with the origin at (x^0, U) and coordinate (orthonormal) vectors $\varepsilon_1, \ldots, \varepsilon_{n+1}$ such that:

- the hyperplane spanned by $\varepsilon_1, \ldots, \varepsilon_n$ is tangent to the graph of u at x^0;

- ε_1 is the direction of the maximal principal curvature $\kappa[u,\eta]$ at x^0.

Then there is a neighborhood of x^0 in which the graph of u can be represented as a graph of a function $y \to \tilde{u}(y)$ in the new coordinate system. It is also easy to see that (2.6)–(2.7) still hold. $\kappa_i[u,\eta]$ are eigenvalues of $[h_{ij}]$ defined by

$$h_{ij} = \frac{1}{\tilde{w}} \sum_{k,l} \tilde{\gamma}^{ik} \tilde{v}_{kl} \tilde{\gamma}^{lj}$$

where $\tilde{w}$ and $\tilde{\gamma}^{ik}$ are given by (2.10) and (2.11) while $\tilde{v}_{st}$ is given by

$$\tilde{v}_{st}(y) = \eta(x)\tilde{u}_{st}(y) + \left(a_{n+1} - \sum_{k=1}^{n} a_k \tilde{u}_k \right) (\delta_{st} + \tilde{u}_s \tilde{u}_t). \tag{3.5}$$

By rotating the $\varepsilon_2, \ldots, \varepsilon_n$ we diagonalize $[\tilde{v}_{ij}]$ at $y = 0$. The curvature of $\tilde{u}$ in the ε_1 direction is

$$\frac{\tilde{v}_{11}}{(1+\tilde{u}_1^2)\,\tilde{w}}.$$

Therefore the function

$$f(y) = \frac{1}{\tau - a} \frac{\tilde{v}_{11}}{(1+\tilde{u}_1^2)\,\tilde{w}}$$

has its maximum at $y = 0$ and $f(0) = M$. $\log f$ also assumes the maximum at the origin, so we get (2.13) and (2.14) with $\beta = 0$. Also it is easy to see that (2.15)–(2.20) hold. Now it remains to calculate the derivatives of $\tilde{v}_{ij}$:

$$\begin{aligned}
\tilde{v}_{ij}(0) &= \eta(x^0)\tilde{u}_{ij} + \delta_{ij} a_{n+1} \\
\tilde{v}_{ij,k}(0) &= \tilde{\eta}_k \tilde{u}_{ij} + \eta \tilde{u}_{ijk} - \delta_{ij} a_k \tilde{u}_{kk} \\
\tilde{v}_{ij,kl}(0) &= \tilde{\eta}_{kl} \tilde{u}_{ij} + \tilde{\eta}_k \tilde{u}_{ijl} + \tilde{\eta}_l \tilde{u}_{ijk} + \eta \tilde{u}_{ijkl} - \delta_{ij} \sum_s a_s \tilde{u}_{skl} \\
&\quad + a_{n+1}(\tilde{u}_{ik}\tilde{u}_{jl} + \tilde{u}_{il}\tilde{u}_{jl}),
\end{aligned}$$

where

$$\tilde{\eta}_k(y) = \frac{\partial}{\partial y_k}\eta(x), \qquad \tilde{\eta}_{kl}(y) = \frac{\partial^2}{\partial y_k \partial y_l}\eta(x).$$

Therefore

$$\tilde{v}_{11,ii}(0) - \tilde{v}_{ii,11}(0) = \tilde{\eta}_{ii}\tilde{u}_{11} - \tilde{\eta}_{11}\tilde{u}_{ii} + 2\tilde{\eta}_i\tilde{u}_{11i} - 2\tilde{\eta}_1\tilde{u}_{ii1} + \sum_s a_s\tilde{u}_{s11} - \sum_s a_s\tilde{u}_{sii}.$$

Now using the same method as in subsection 2.3 we arrive at an inequality

$$\frac{\tilde{v}_{11}^2}{4} + M_1\tilde{v}_{11} \leqslant M_2,$$

where M_1 and M_2 depend on K, ε, and $\sup A$.

Now suppose that M is achieved at a point $x^0 \in \partial\Omega$. Then we can estimate M by establishing bounds for D^2u on the boundary. We use the same method here as in subsection 2.3. Let $L' = \sum_{s,t} G^{st}\partial_{st} + \sum_s G^s\partial_s$. Then Lemma 2.13 holds with $\beta = \frac{1}{4K}$, while Lemma 2.14 holds with L replaced by L'. Note that (2.34) now becomes

$$\frac{\eta}{w^3}\sum_i F^{ii} \leqslant \sum_i G^{ii} \leqslant \frac{\eta}{w}\sum_i F^{ii}. \tag{3.6}$$

We also have the following analogue of Lemma 2.15:

Lemma 3.7. *Let $1 \leqslant i < j \leqslant n$. Then*

$$L'(x_iu_j - x_ju_i) = x_i\psi_j - x_j\psi_i + (\psi_{n+1} - G_u)(x_i\eta_j - x_j\eta_i).$$

Then the proof proceeds as in subsection 2.3 except that in place of L we have L' and the equality $L'(Tu) = T\psi(x,u)$ does not hold. However we still have

$$|L'(Tu)| \leqslant C\left(1 + \sum_i G^{ii}\right),$$

where C depends on ε. This happens because

$$G_u\eta = \sum_i f_i\kappa_i - \frac{1}{w}\sum_i f_i,$$

and from (3.6)

$$\sum_i f_i \leqslant \frac{w^3}{\eta} \sum_i G^{ii}.$$

In order to show that $|u_{\nu\nu}| \leqslant C$ we use the same method as in subsection 2.3 taking $L = L'$ and v as defined by (1.17). We can do this because on $\partial\Omega$ we have

$$\eta u_{kl} + u_k u_l + \delta_{kl} = u u_{kl} + u_k u_l + \delta_{kl}.$$

□

As in section 2 we get

Lemma 3.8. *Suppose that $u \in \mathcal{A}$ is a solution of* (3.2)*. If the conditions* (1.2)–(1.8)*,* (1.10)*, and* (1.11) *are satisfied, we have a global bound*

$$|u|_{C^{2,\alpha}(\bar{\Omega})} \leqslant C,$$

where C depends on ε, K, and α is a constant such that $0 < \alpha < 1$.

By Schauder theory we get the following global estimate:

$$|u|_{C^{k,\alpha}} \leqslant C_k, \qquad k \geqslant 2.$$

Now we will apply the continuity method to the following problem

$$\begin{aligned} f(\kappa[u,\eta]) &= \psi^t(x,\eta) && \text{in } \Omega, \\ u &= \varepsilon && \text{on } \partial\Omega, \end{aligned} \tag{3.7}$$

where $0 \leqslant t \leqslant 1$ and $\psi^t = t\psi + 1 - t$. Note that for the above family of problems, $|\psi^t|_{C^2(\bar{\Omega})}$ is uniformly bounded for all $0 \leqslant t \leqslant 1$ and ψ_0 can be chosen so that it does not depend on t. On the other hand, $\psi_1 \to 1$ as $t \to 0$. Due to the fact that the conditions (1.5) and (1.11) are satisfied with $\psi_1 = 1$, we can conclude that all solutions of (3.7) are uniformly bounded for $t > 0$. Note also that for $t = 0$ and any $\zeta \in \bar{\mathcal{C}}_K$, (3.7) has a unique solution (this solution is $u \equiv \varepsilon$).

The continuity method will then give us the unique solution for $t = 1$. Therefore

Lemma 3.9. *The problem* (3.2) *has a unique solution for any* $\zeta \in \bar{\mathcal{C}}_K$ *and for any* $\varepsilon > 0$ *provided that* $\varepsilon < r_\Omega H_0$. *Moreover*

$$|u|_{C^{k,\alpha}(\bar{\Omega})} \leqslant C, \qquad k \geqslant 2,$$

where C *depends on* K, k, *and* ε, *but is independent of the choice of* ζ.

4 The main result

Proof of Theorem 1.3. For $0 \leqslant t \leqslant 1$, consider the following Dirichlet problem

$$\begin{aligned} G(D^2u, Du, u) &= t\psi(x,u) + 1 - t \qquad \text{in } \Omega, \\ u &= \varepsilon \qquad\qquad\qquad\qquad \text{on } \partial\Omega. \end{aligned} \tag{4.1}$$

As shown in section 2, any solution of (4.1) satisfies the *a priori* bound

$$|u|_{C^5(\bar{\Omega})} \leqslant C \tag{4.2}$$

independent of t (see also the discussion following the problem (3.7)). Moreover we claim that $u - \varepsilon \notin \partial\mathcal{C}$. Indeed if $u - \varepsilon \in \partial\mathcal{C}$ then either

1. $u(x_0) = \varepsilon/2$ for some $x_0 \in \Omega$,
2. $u(x_0) = \varphi(\operatorname{dist}(x_0, \partial\Omega))$ for some $x_0 \in \Omega_{R-r}$.

The first case is impossible by Lemma 2.1. The second case is impossible by Lemma 2.5. By (4.2) we can choose $K > 0$ sufficiently large so that (4.1) has no solution u such that $u - \varepsilon \in \partial\mathcal{C}_K$. Now for $0 \leqslant t \leqslant 1$, $\zeta \in \bar{\mathcal{C}}_K$, and $\eta = \zeta + \varepsilon$ consider the following Dirichlet problem

$$\begin{aligned} G(D^2u, Du, \eta) &= t\psi(x,\eta) + 1 - t \qquad \text{in } \Omega \\ u &= \varepsilon \qquad\qquad\qquad\qquad \text{on } \partial\Omega. \end{aligned} \tag{4.3}$$

Then by Lemma 3.9 there exists a unique solution $u^t \in \mathcal{A}(\Omega, \eta)$ of (4.3) for each $t \in (0, 1]$. For $t = 0$ this solution is $u^0 \equiv \varepsilon$ (regardless of ζ). From the elliptic theory the map $T^t : \bar{\mathcal{C}}_K \to \mathcal{C}$ defined by

$$T^t\zeta = u^t - \varepsilon$$

is completely continuous. On the other hand, there are no solutions of

$$T^t\zeta = \zeta \tag{4.4}$$

on the boundary of $\mathcal{C}_K$. Therefore the degree

$$\deg(I - T^t, \mathcal{C}_K, 0) = \gamma$$

is well defined and independent of t. For $t = 0$,

$$T^0\zeta \equiv 0 \quad \text{for all } \zeta \in \bar{\mathcal{C}}_K.$$

Since 0 is the only singular point of $I = I - T^0$ we have (see [11])

$$\gamma = \operatorname{ind}(0, I - T^0) = 1.$$

Consequently $\deg(I - T^t, \mathcal{C}_K, 0) = 1$ and (4.4) has a solution $\zeta^t \in \mathcal{C}_K$ for all $0 \leqslant t \leqslant 1$. The function $u^1 = \zeta^1 + \varepsilon$ is then a solution of (1.19). □

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Vita

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