

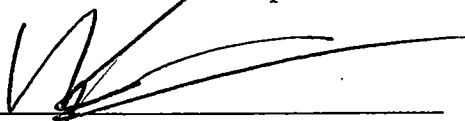
To the Graduate Council:

I am submitting herewith a thesis written by Shabbir Ahmed entitled "Variational Formulations for a Parabolic Partial Differential Equation Describing Variably Saturated Flow." I have examined the final copy of this thesis for form and content and recommend that it be accepted in partial fulfillment of the requirements for the degree of Master of Science, with a major in Mathematics.

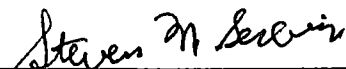


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Interim Vice Provost and
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**VARIATIONAL FORMULATIONS FOR A PARABOLIC PARTIAL
DIFFERENTIAL EQUATION DESCRIBING VARIABLY SATURATED FLOW**

A Thesis

Presented for the

Master of Science

Degree

The University of Tennessee, Knoxville

Shabbir Ahmed

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DEDICATION

This thesis is dedicated to my daughter *Rumana*.

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ABSTRACT

The variably saturated flow in a porous medium is described by a parabolic equation with hydraulic conductivity and soil moisture capacity as the flow characteristics. Variational formulations have been developed to solve the governing parabolic partial differential equation describing variably saturated flow in a porous medium. The finite element method has been used to solve the partial differential equation for a two-dimensional domain. The matrix characteristics and the stability criteria have been investigated to develop the numerical algorithms for solving the partial differential equation to generate time-varying hydraulic heads in the subsurface. A computer program has been written to solve a symmetric positive definite system obtained from the variational formulation for the finite element method. The system of equations is solved using the conjugate gradient method. Analytical solutions have been developed for the simplified governing equation. The finite element solutions were compared with the analytical solutions. A reasonable agreement between finite element and analytical solutions was observed. A variational formulation was also developed using boundary element method. The matrix characteristics and the solution algorithm using boundary element method were investigated.

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1.0 INTRODUCTION

In any numerical method for a partial differential equation, it is required to discretize the given continuous problem to obtain a discrete problem or system of equations with finitely many unknowns that may be solved using an appropriate solution method. In the finite element method, the given partial differential equation is reformulated or transformed to obtain an equivalent variational problem. To solve a parabolic partial differential equation, a similar approach is used to develop an equivalent variational problem. The finite element method is used as a general solution technique for partial differential equations describing water resources systems. Parabolic partial differential equations are solved to describe variations of hydraulic heads in groundwater systems including both unsaturated (above groundwater table) and saturated zones (below groundwater table). The finite element method was used to solve subsurface flow equations (parabolic type) by Neuman (1973), Kuppusamy et al. (1987), Yeh et al. (1993), and Srivastava and Yeh (1993). In the present study, the mathematical and numerical aspects of the finite element method are investigated for solving a parabolic equation describing variably saturated flow in the subsurface.

Recently, researchers have begun to employ the boundary element method to solve problems in porous media flow (Aral and Tang, 1988; Lape and Cheng, 1987; and Fogden et al., 1988; and Subia and Ingber, 1994). In the boundary element method, the governing partial differential equation is transformed into an integral equation on the boundary. A variational finite element formulation of the integral equation is then

obtained to generate a system of algebraic equations. The solutions of the system of equations results in information about the Dirichlet and Neumann boundary conditions which permits the solution of the partial differential equation at any point within the domain. In the present study a variational formulation is obtained based on the boundary integral expression for the parabolic partial differential equation describing variably saturated flow.

The objectives are to develop mathematical formulations and perform numerical analyses for getting solutions for the governing parabolic partial differential equation describing variably saturated flow in the subsurface. In summary, the objectives of this study are as follows:

- to develop a variational formulation for generating a finite element solution algorithm for a parabolic partial differential equation describing variably saturated flow in porous media. The characteristics of the matrices associated with the system of equations for finite element formulations are also investigated.
- to develop a computer program and to compare the finite element results with analytical solutions.
- to develop a variational formulation of the parabolic partial differential equation describing variably saturated flow using the boundary element method. The

characteristics of the matrices associated with the system of equations for boundary element formulations are also investigated.

2. VARIABLY SATURATED FLOW PROCESS

2.1 PHYSICAL SYSTEM

The amount of rainfall that infiltrates through the ground surface results in an unsaturated flow above the groundwater table. The unsaturated flow occurs when the void spaces of the soil matrix remains partially saturated with moisture from the infiltrating water. The portion of the subsurface with unsaturated flow above the groundwater table is known as the unsaturated zone. The flow below the groundwater table is fully saturated because the void spaces of the soil matrix remain completely filled up with water. The saturated portion of the subsurface with a free surface is known as an unconfined aquifer. In unconfined aquifers, the free surface between unsaturated and saturated zones is called the groundwater table. The groundwater table is also known as a phreatic surface. The phreatic surface serves as the upper boundary of an unconfined aquifer. Above the phreatic surface, a capillary fringe exists which is generally neglected in unconfined aquifer studies.

Figure 1.1 illustrates a schematic representation of an unconfined aquifer with a phreatic surface in a porous medium. Essentially, a zone of saturation exists below the phreatic surface in which all pores are completely filled with water. An unsaturated zone exists above the phreatic surface in which the pores are partially filled with water. The subsurface having unsaturated and saturated zones can be a heterogeneous porous medium with variable properties. The heterogeneity occurs due to the variations of

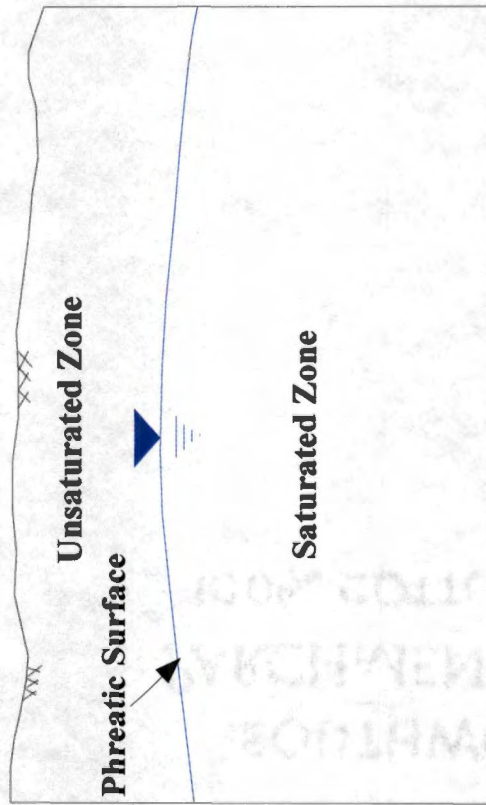


Figure 1.1. Schematic Representation of an Unconfined Aquifer

properties such as hydraulic conductivity and porosity. Strongly heterogeneous porous media with sharp changes in properties can cause discontinuity at the interface between geologic units.

2.2 VARIABLELY SATURATED FLOW EQUATION

Phenomenological and conservation laws can be used to develop a governing equation for describing total head in a porous medium with an unconfined aquifer. According to Darcy's law, the specific discharge through a porous medium can be expressed as:

$$q_s = -K\nabla\phi, \quad (2.1)$$

where

q_s = specific discharge vector (q_{sx}, q_{sy}, q_{sz}),

K = hydraulic conductivity tensor,

ϕ = total head = $\psi + z = p/\gamma + z$,

ψ = pressure head = p/γ ,

p = pressure,

γ = unit weight of water, and

z = elevation from a datum.

In matrix form, Darcy's law can be written as:

$$\begin{bmatrix} q_{sx} \\ q_{sy} \\ q_{sz} \end{bmatrix} = - \begin{bmatrix} K_{xx} & K_{xy} & K_{xz} \\ K_{yx} & K_{yy} & K_{yz} \\ K_{zx} & K_{zy} & K_{zz} \end{bmatrix} \begin{bmatrix} \frac{\partial \phi}{\partial x} \\ \frac{\partial \phi}{\partial y} \\ \frac{\partial \phi}{\partial z} \end{bmatrix} \quad (2.2)$$

The derivation of the continuity equation using a control volume for a porous medium is described in Bear (1979) and Willis and Yeh (1987). The continuity equation can be written as:

$$-\nabla \cdot (\rho q_s) + q = \frac{\partial(\rho S_w \theta_s)}{\partial t}, \quad (2.3)$$

where

ρ = density of groundwater;

q = source/sink;

S_w = degree of saturation = θ/θ_s ;

θ = volumetric moisture content = V_w/V ;

V_w = volume of water in the soil matrix;

V = total volume of the soil; and

θ_s = volumetric saturation moisture content.

For incompressible flow, the density ρ remains constant and does not change with time

i.e. $\frac{\partial \rho}{\partial t} = 0$. Equation (2.3) can be written as:

$$-\nabla \cdot \mathbf{q}_s + q = \frac{\partial(S_w \theta_s)}{\partial t}. \quad (2.4)$$

The right hand side of equation (2.4) describes the rate of change in the amount of water in the void spaces of soil and can be written as (Cooley, 1971 and Neuman, 1973):

$$\frac{\partial(S_w \theta_s)}{\partial t} = S_w S_s \frac{\partial \psi}{\partial t} + \theta_s \frac{\partial S_w}{\partial t}, \quad (2.5)$$

where

S_s = specific storage.

The degree of saturation S_w can be differentiated with respect to time as:

$$\frac{\partial S_w}{\partial t} = \frac{1}{\theta_s} \frac{\partial \theta}{\partial t}. \quad (2.6)$$

The partial derivative of moisture content θ with respect to time can be written as:

$$\frac{\partial \theta}{\partial t} = \frac{\partial \theta}{\partial \psi} \frac{\partial \psi}{\partial t}. \quad (2.7)$$

The term $\frac{\partial \theta}{\partial \psi}$ is known as soil moisture capacity. From equations (2.4), (2.5), (2.6), and (2.7), we get

$$\nabla \cdot (K \nabla \phi) + q = (S_w S_s + C) \frac{\partial \psi}{\partial t}, \quad (2.8)$$

where $C = \frac{\partial \theta}{\partial \psi}$ is the soil moisture capacity. Since $\frac{\partial z}{\partial t} = 0$, the partial derivative of ψ

with respect to t can be replaced by partial derivative of $\phi = \psi + z$ with respect to t .

Therefore, equation (2.8) can be written as:

$$\nabla \cdot (K \nabla \phi) + q = (S_w S_s + C) \frac{\partial \phi}{\partial t}. \quad (2.9)$$

The terms S_w and C are functions of pressure head ψ . Let $E(\psi) = S_w S_s + C$ and equation

(2.9) becomes

$$E(\psi) \frac{\partial \phi}{\partial t} = \nabla \cdot (K \nabla \phi) + q. \quad (2.10)$$

Equation (2.10) is the governing equation for variably saturated flow in a porous medium. The above equation is valid for continuous variations of properties and dependent variable. The first derivative of hydraulic conductivity function must exist i.e. ∇K must exist. The second derivative of total potential ϕ must be defined i.e. $\nabla^2 \phi$ must exist. In a domain Ω , the dependent variable ϕ and hydraulic conductivity K should be:

$$\phi \in C^2(\Omega) \cup C^1(\bar{\Omega}) \text{ and } K \in C^1(\Omega) \cup C^0(\bar{\Omega}). \quad (2.11)$$

In equation (2.9), the degree of saturation $S_w \rightarrow 1$ and $C \rightarrow 0$ below the groundwater table in an aquifer. Above the groundwater table in the unsaturated zone, $S_w < 1$ but $\neq 0$ and $C \neq 0$. Constitutive relationships exist for variables describing the characteristics of the variably saturated porous medium.

2.3 POROUS MEDIA CHARACTERISTICS AND PARAMETERS

The subsurface porous media system can be conceptualized as consisting of geologic units with varying saturation conditions. The unsaturated moisture condition has impact on the description of the unsaturated hydraulic conductivity function. There exists a number of formulations describing unsaturated flow parameters in the subsurface. A widely accepted description of flow parameters, given by van Genuchten (1980), is used

in formulating media characteristics and flow parameters in the subsurface. van Genuchten derived the unsaturated hydraulic conductivity function that depends on three independent parameters related to soil moisture retention data. In van Genuchten's model, the expressions for water content θ , water capacity ($d\theta/d\psi$) and hydraulic conductivity function are expressed as (Genuchten, 1980):

$$K(\theta) = K_s \left(\frac{\theta - \theta_r}{\theta_s - \theta_r} \right)^{1/2} (1 - [1 - f(\theta)]^m)^2 \text{ for } \theta < \theta_s \text{ and } K(\theta) = K_s \text{ for } \theta = \theta_s, \quad (2.12)$$

$$\theta = \theta_r + \frac{\theta_s - \theta_r}{(1 + (\alpha |\psi|)^n)^m} = f(\psi) \text{ for } \psi < 0 \text{ and } \theta = \theta_s \text{ for } \psi \geq 0, \quad (2.13)$$

$$C = \frac{d\theta}{d\psi} = \alpha(n-1)(1-f(\theta))^m f(\theta)(\theta_s - \theta_r), \quad (2.14)$$

where

θ_r = residual soil moisture content;

$$f(\theta) = \left(\frac{\theta - \theta_r}{\theta_s - \theta_r} \right)^{1/m};$$

$$m = 1 - \frac{1}{n};$$

and α , m , and n are the parameters that depend on soil moisture retention data.

The above relationships indicate the variations of parameters as follows:

$K(\theta) \rightarrow (K_{sx}, K_{sy}, K_{sz})$ as $\theta \rightarrow \theta_s$ and $K(\theta) \rightarrow (0,0,0)$ as $\theta \rightarrow \theta_r$ (i.e. there is an insignificant amount of water to induce flow), and $C \rightarrow 0$ as $\theta \rightarrow \theta_s$ and $C \rightarrow 0$ as $\theta \rightarrow \theta_r$.

The values of S_s and the relationships for the flow parameters $K(\theta)$, S_w , and C for subsurface materials are used in the governing equation (2.10) that describes the spatial and temporal variations of total head in the subsurface.

2.4 DOMAIN DISCONTINUITY

The problem of simulating flow in a porous medium becomes complicated due to the presence of multiple geologic units with sharply varying properties. The governing equation (2.10) is developed based on the assumption that the variations of properties are continuous. In particular, it is required that the gradient of hydraulic conductivity ∇K exists throughout the domain of interest i.e. $K \in C^1(\Omega)$. The subsurface becomes not only heterogeneous but also discontinuous with the presence of sharply varying geologic units. At the points of discontinuities between sharply varying geologic units, the gradient of hydraulic conductivity ∇K does not exist. The governing equation (2.10) can not be defined at points on the interface (line of discontinuity for a two-dimensional problem and surface of discontinuity for a three dimensional problem) due to the

discontinuity of hydraulic conductivity. For this type of discontinuity in a domain, the solution of the governing equation can be obtained by decomposing the domain and applying certain conditions at the interface. Instead of solving for ϕ in the whole domain Ω , the problem can be decomposed into subproblems for each subdomain Ω_i ($i=1,2,\dots,n$) separated by the interfaces $\Gamma_{ij} = \overline{\Omega_i} \cap \overline{\Omega_j}$. Therefore,

$$\Omega = \{\Omega_1 \cup \Omega_2 \cup \Omega_3 \cup \dots \cup \Omega_n\}. \quad (2.15)$$

The two criteria satisfied at the interfaces are:

- (1) The total potential ϕ must be the same at a point on the interface when it is approached from either side; this is known as compatibility criterion.
- (2) The second criterion is obtained from the requirement of the continuity of flux across the interface.

These two criteria for a discontinuous domain can be written as:

Compatibility:

$$\phi_i = \phi_j \text{ on } \Gamma_{ij}. \quad (2.16)$$

Continuity:

$$K_i \frac{\partial \phi_i}{\partial n_i} = K_j \frac{\partial \phi_j}{\partial n_j} \text{ on } \Gamma_{ij}, \quad (2.17)$$

where

ϕ_i = potential corresponding to the geologic unit i at the interface;

ϕ_j = potential corresponding to the geologic unit j at the interface;

K_i = hydraulic conductivity of geologic unit i;

K_j = hydraulic conductivity of geologic unit j;

n_i = unit normal at the interface from geologic unit i; and

n_j = unit normal at the interface from geologic unit j.

2.5 MOVING PHREATIC SURFACE

In an unconfined aquifer, the void spaces above the phreatic surface are partially saturated. The phreatic surface is assumed to be under constant atmospheric pressure with a capillary fringe on it.

The location of the phreatic surface depends on recharge, the hydraulic conductivity of the medium, and vertical and horizontal flows. For this, the phreatic surface is also described as a moving or as a floating internal boundary. The phreatic surface can be defined by satisfying certain criteria. The criteria are as follows:

- (i) the phreatic surface is a material surface,
- (ii) the phreatic surface is under constant atmospheric pressure.

Based on the above criteria, the formulation for the movement of the phreatic surface can be obtained as follows.

The total head η at any point on the phreatic surface can be written as:

$$\eta = z + \frac{p}{\gamma}, \quad (2.18)$$

where

z = elevation of a point on the phreatic surface;

p = pressure on the phreatic surface; and

γ = unit weight of groundwater.

Since the pressure at all points on the phreatic surface is assumed to be atmospheric, then

$p = 0$ for all points on the phreatic surface. Therefore, on the phreatic surface:

$$\eta = z \text{ or } z - \eta = 0. \quad (2.19)$$

The Lagrangian form of the material derivative can be written for the phreatic surface as:

$$\frac{D(z - \eta)}{Dt} = 0. \quad (2.20)$$

The material derivative in Lagrangian form can be expressed as conventional Eulerian form as:

$$\frac{D}{Dt} = \frac{\partial}{\partial t} + \mathbf{v} \cdot \nabla, \quad (2.21)$$

where

$\mathbf{v}$ = seepage velocity vector = (v_x, v_y, v_z) .

Therefore, the equation for phreatic surface can be written as:

$$\frac{\partial \eta}{\partial t} = \mathbf{v} \cdot \nabla(z - \eta). \quad (2.22)$$

The seepage velocity $\mathbf{v}$ can be expressed as:

$$\mathbf{v} = \frac{\mathbf{q}_s}{n_e}, \quad (2.23)$$

where

q_s = Darcy's velocity $(q_x, q_y, q_z) = -K\nabla\phi$ and

n_e = effective porosity.

Using equation (2.23) in equation (2.22), the governing equation for the phreatic surface can be obtained as:

$$\frac{\partial\eta}{\partial t} = \frac{1}{n_e} \left[K\nabla\phi \cdot \nabla\eta - K_z \frac{\partial\phi}{\partial z} \right], \quad \eta = z = \phi \text{ on the surface.} \quad (2.24)$$

The governing equation (2.24) can be solved for η to locate the position of the phreatic surface, since $\nabla\phi$ and $\frac{\partial\phi}{\partial z}$ are known quantities obtained from the solution of equation (2.10).

3.0 FINITE ELEMENT FORMULATION

3.1 VARIATIONAL FORMULATION

The governing equation (2.10) can be rewritten with the initial and boundary conditions (Figure 3.1) as follows:

$$\begin{aligned} (D) \quad & E(\psi) \frac{\partial \phi}{\partial t} = \nabla \cdot [K(\psi) \nabla \phi] + q, \quad \text{in } \Omega \times I, \\ & \phi = \phi_d, \quad \text{on } \Gamma_d \times I, \\ & K(\psi) \frac{\partial \phi}{\partial n} = g, \quad \text{on } \Gamma_n \times I, \\ & \phi(x, 0) = \phi^0, \quad \text{in } \Omega. \end{aligned} \tag{3.1}$$

Here Γ_d and Γ_n are the subdivisions of the boundary Γ of a two-dimensional bounded domain Ω , $I = (0, T)$, where T is a given time, and ϕ^0 is a given initial total head.

In equation (3.1), $E(\psi)$ and $K(\psi)$ are nonlinear functions of ψ , and therefore of ϕ . In the present study, $E(\psi)$ and $K(\psi)$ are evaluated assuming that ψ is known. The actual value of ψ will be described later.

It is required to define some function spaces to obtain a variational formulation of the initial boundary value problem (D). Johnson (1990) outlined mathematical principles to

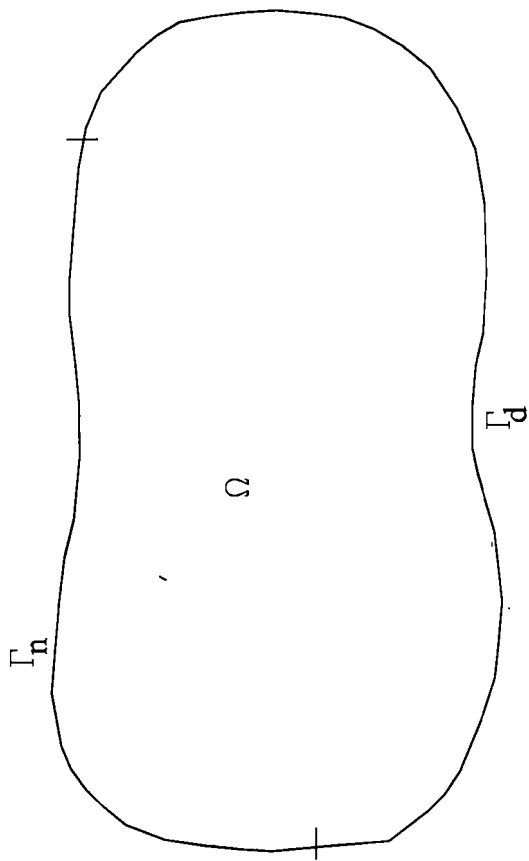


Figure 3.1. A Domain with Boundary Conditions.

develop variational formulations for the initial boundary value problems. Let's define function spaces on $\Omega \subset \mathbb{R}^2$ as follows:

$$L_2(\Omega) = \{v : v \text{ is defined on } \Omega \text{ and } \int_{\Omega} v^2 dx < \infty\},$$

$$H^1(\Omega) = \{v \in L_2(\Omega) : \nabla v \in L_2(\Omega)\},$$

$$V = \{v \in H^1(\Omega) : v = \varphi_d \text{ on } \Gamma_d\},$$

$$V_0 = \{v \in H^1(\Omega) : v = 0 \text{ on } \Gamma_d\}.$$

(3.2)

Here $L_2(\Omega)$, $H^1(\Omega)$, and V^0 are Hilbert spaces, with the appropriate scalar products.

Let φ_d be extended to all of Ω . Set $\varphi = \varphi_d + u$ in equation (3.1) to obtain homogeneous Dirichlet boundary conditions as follows:

$$\begin{aligned} (D) \quad & E(\psi) \frac{\partial u}{\partial t} = \nabla \cdot [K(\psi) \nabla u] + \tilde{q}, \quad \text{in } \Omega \times I, \\ & u = 0, \quad \text{on } \Gamma_d \times I, \\ & K(\psi) \frac{\partial u}{\partial n} = g, \quad \text{on } \Gamma_n \times I, \\ & u(x, 0) = u^0, \quad \text{in } \Omega, \end{aligned} \tag{3.3a}$$

where

$$\tilde{q} = q + \nabla \cdot (K(\psi) \nabla \varphi_d) - E(\psi) \frac{\partial \varphi_d}{\partial t}.$$

Here $u(t) \in V^0$.

Without loss of generality, we can take $\varphi = 0$ on Γ_d , and use (D) with $\varphi_d = 0$.

Multiply equation (3.1) by the test function $v \in V$ and integrate over Ω to obtain:

$$\int_{\Omega} E \frac{\partial \varphi}{\partial t} v \, d\Omega = \int_{\Omega} \nabla \cdot [K \nabla \varphi] v \, d\Omega + \int_{\Omega} q v \, d\Omega. \quad (3.3b)$$

Using Green's formula, the first integral on the right hand side can be written as:

$$\int_{\Omega} \nabla \cdot [K \nabla \varphi] v \, d\Omega = \int_{\Gamma} K \frac{\partial \varphi}{\partial n} v \, d\Gamma - \int_{\Omega} K \nabla \varphi \cdot \nabla v \, d\Omega. \quad (3.4)$$

Substituting (3.4), the integral equation (3.3b) can be written as:

$$\int_{\Omega} E \frac{\partial \varphi}{\partial t} v \, d\Omega = \int_{\Gamma} K \frac{\partial \varphi}{\partial n} v \, d\Gamma - \int_{\Omega} K \nabla \varphi \cdot \nabla v \, d\Omega + \int_{\Omega} q v \, d\Omega. \quad (3.5)$$

Therefore, the variational formulation is defined as follows:

To find $\varphi(t) \in V^0$ such that

$$(V) \begin{cases} E\dot{\varphi}(t), v \Big) + a(\psi; \varphi(t), v) = (q(t), v) + \langle g(t), v \rangle_{\Gamma_n} & \forall v \in V^0, t \in I, \\ \varphi(\cdot, 0) = \varphi^0, \end{cases} \quad (3.6)$$

where

$$a(\psi; v, w) = \int_{\Omega} K(\psi) \nabla v \cdot \nabla w \, d\Omega,$$

$$\left(E\dot{\varphi}(t), v \right) = \int_{\Omega} E \dot{\varphi}(t) v \, d\Omega$$

$$(q(t), v) = \int_{\Omega} q(t) v \, d\Omega,$$

$$\langle g(t), v \rangle_{\Gamma_n} = \int_{\Gamma_n} g(t) v \, d\Gamma.$$

Equation (3.6) gives a variational formulation of (D). Therefore, φ is a solution of the problem (D) given by the variational problem (V) as given in equation (3.6). The variational formulation (V) in equation (3.6) can also be rewritten as:

$$a(\psi; \varphi, v) - (h(t), v) - \langle g, v \rangle_{\Gamma_n} = 0, \quad (3.7a)$$

$$\text{or, } a(\psi; \varphi, v) = (h(t), v) + \langle g, v \rangle_{\Gamma_n}, \quad (3.7b)$$

where

$$(h(t), v) = \left(q(t) - E\dot{\varphi}(t), v \right)$$

The variational formulation automatically produces a solution which satisfies the conditions (2.16) and (2.17) for a discontinuous domain.

3.2 FORMULATIONS IN A FINITE DIMENSIONAL SPACE

Let us consider a triangulation of the domain Ω i.e. subdivide Ω into a set T_h of nonoverlapping triangles K_i ($i=1,2, \dots, m$) (Figure 3.2) such that

$$\Omega = \bigcup_{K \in T_h} K = K_1 \cup K_2 \cup \dots \cup K_m$$

Let V_h^0 be a finite dimensional subspace of the space V^0 consisting of piecewise linear functions. Let us define $P_1(K)$ as the space of linear functions on K , i.e.

$$P_1(K) = \{ v : v \text{ is a polynomial of degree } \leq 1 \text{ on } K \}$$

Here $V_h^0 = \{ v \in V^0 : v|_K \in P_1(K) \text{ and } v \text{ is continuous at the nodes, } \forall K \in T_h \}$. Let the

basis functions for $P_1(K)$ of the finite dimensional subspace V_h^0 of V^0 be

$$\{L_1, L_2, L_3, \dots, L_M\}.$$

For $K \in T_h$, define $h_K =$ the longest side of K with $\rho_K =$ the diameter of the circle

inscribed in K . Let $h = \max_{K \in T_h} h_K$.

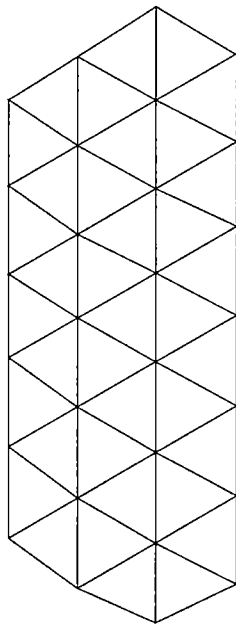


Figure 3.2. Triangulation of a Domain.

Consider a family of triangulations $\{T_h\}$ with a constant $\beta > 0$ independent of h , such that

$$\frac{\rho_k}{h_k} \geq \beta \quad \forall K \in T_h, \quad \forall h. \quad (3.8)$$

This condition implies that the triangles $K \in T_h$ are not allowed to be arbitrarily thin, or equivalently, the angles of the triangles K are not allowed to be arbitrarily small. The constant β gives a measure of the smallest angle in any $K \in T_h$ for any $T_h \in \{T_h\}$.

For $\Omega \subset \mathbb{R}^2$ and V_h^0 as described, the semi-discrete analogue of the variational formulation (V) can be written as

To find $\phi_h(t) \in V_h^0$ such that

$$(V) \quad \begin{cases} E\phi_h(t), v + a(\psi; \phi_h(t), v) = (q(t), v) + (g(t), v) & \forall v \in V_h^0, t \in I, \\ (\phi_h(\cdot, 0), v) = (\phi_h^0, v) \end{cases} \quad (3.9)$$

Let us represent $\phi_h(t)$ using shape functions $L_i(x)$ as

$$\phi_h(x, t) = \sum_{i=1}^M \xi_i(t) L_i(x), \quad \xi_i(t) \in \mathbb{R}.$$

Now taking $v=L_j$, we get

$$\sum_{i=1}^M \dot{\xi}_i(t) b(\psi; L_i, L_j) + \sum_{i=1}^M \xi_i(t) a(\psi; L_i, L_j) = (f(t), L_j),$$

$$\sum_{i=1}^M \xi_i(0) (L_i, L_j) = (\phi^0, L_j),$$

where

$$b(\psi; L_i, L_j) = \int_{\Omega} E(\psi) L_i L_j \, d\Omega,$$

$$a(\psi; L_i, L_j) = \int_{\Omega} K(\psi) \nabla L_i \cdot \nabla L_j \, d\Omega,$$

$$(f(t), L_j) = (q(t), L_j) + \langle g(t), L_j \rangle = \int_{\Omega} q(t) L_j \, d\Omega + \int_{\Gamma_n} g(t) L_j \, d\Gamma.$$

In matrix form, we have

$$B \dot{\xi}(t) + A \xi(t) = F(t),$$

$$B \xi(0) = \phi^0,$$

where

$$B = (b_{ij}),$$

$$b_{ij} = \int_{\Omega} E(\psi) L_i L_j \, d\Omega,$$

$$A = (a_{ij}),$$

$$a_{ij} = \int_{\Omega} K(\psi) \nabla L_i \cdot \nabla L_j \, d\Omega,$$

$$F = (f_i),$$

$$f_i = \int_{\Omega} q L_i \, d\Omega + \int_{\Gamma_n} g L_i \, d\Gamma,$$

$$\phi_i^0 = (\phi^0, L_i)$$

The matrices B and A contain functions depending on ϕ causing nonlinearity. It is assumed that the linearization will be done through some approximation. Introducing the Cholesky decomposition of B as $P^T P$ and the new variable $\eta = P\xi$, the above problem can be written in a simpler form as

$$\begin{aligned}\dot{\eta}(t) + \bar{A}\eta(t) &= \ell(t), \\ \eta(0) &= \eta^0,\end{aligned}\tag{3.10}$$

where

$$\begin{aligned}\bar{A} &= P^{-T} A P^{-1}, \\ \ell(t) &= P^{-T} F, \\ \eta^0 &= P^{-T} \phi^0.\end{aligned}$$

The solution of the initial value problem described by the ordinary differential equation (3.10) can be written as

$$\eta(t) = \eta_0 e^{-\bar{A}t} + \int_0^t e^{-\bar{A}(t-s)} \ell(s) ds, \quad t \in I.\tag{3.11}$$

The fully discretized version can be derived using the Backward Euler and Crank-Nicolson methods.

Let $0 = t_0 < t_1 < t_2 < \dots < t_N = T$ denote a partitioning of I so that $I_n = (t_{n-1}, t_n)$. Let $k_n = t_n - t_{n-1}$ be the time step size. Let the values of $E(\psi)$ and $K(\psi)$ be evaluated based on the values of ψ at previous time step $n-1$. By using backward Euler method, the semi-discrete variational problem in equation (3.9) can be written as

To find $\phi_h^n \in V_h^0$, $n = 1, 2, \dots, N$ such that

$$\begin{aligned} \left(E \frac{\phi_h^n - \phi_h^{n-1}}{k_n}, v \right) + a(\psi; \phi_h^n, v) &= (q(t_n), v) + \langle g(t_n), v \rangle \quad \forall v \in V_h^0, \\ (\phi_h(\cdot, 0), v) &= (\phi^0, v), \quad \forall v \in V_h^0. \end{aligned} \quad (3.12)$$

Using the backward Euler method, the derivative $\phi(t_n)$ has been replaced by the difference quotient $\frac{\phi_h^n - \phi_h^{n-1}}{k_n}$ with the discretization error $O(k_n)$. The above formulation can also be written as

$$\begin{aligned} (E(\phi_h^n - \phi_h^{n-1}), v) + k_n a(\psi; \phi_h^n, v) &= k_n (q(t_n), v) + k_n \langle g(t_n), v \rangle \quad \forall v \in V_h^0, \\ (\phi_h(\cdot, 0), v) &= (\phi^0, v), \quad \forall v \in V_h^0. \end{aligned} \quad (3.13)$$

In a matrix form, equation (3.13) can be written as:

$$(B^{n-1} + k_n A^{n-1}) \xi^n = B^{n-1} \xi^{n-1} \pm k_n F(t_n), \quad (3.14)$$

where

$$B = (b_{ij}),$$

$$b_{ij} = \int_{\Omega} E(\psi^{n-1}) L_i L_j \, d\Omega,$$

$$A = (a_{ij}),$$

$$a_{ij} = \int_{\Omega} K(\psi^{n-1}) \nabla L_i \cdot \nabla L_j \, d\Omega,$$

$$F = (f_i),$$

$$f_i = \int_{\Omega} q L_i \, d\Omega + \int_{\Gamma_n} g L_i \, d\Gamma.$$

Another approach is to use Crank-Nicolson method. In the Crank-Nicolson method, an

average time-discretization is used. The time derivative $\frac{1}{2}(\dot{\phi}_h(t_n) + \dot{\phi}_h(t_{n-1}))$ is replaced

by $\frac{\phi_h^n - \phi_h^{n-1}}{k_n}$ with a corresponding error $O(k_n^2)$. Therefore, using the average time-

discretization approach, the formulation can be written as

To find $\phi_h^n \in V_h^0$ of $\phi(., t_n)$, $n=1, 2, \dots, N$ such that

$$\left(E \left(\frac{\varphi_h^n - \varphi_h^{n-1}}{k_n} \right), v \right) + a \left(\psi, \frac{\varphi_h^n + \varphi_h^{n-1}}{2}, v \right) = \left(\frac{q(t_n) + q(t_{n-1})}{2}, v \right) + \left\langle \frac{g(t_n) + g(t_{n-1})}{2}, v \right\rangle_{\Gamma_n},$$

$$\forall v \in V_h^0,$$

$$(\varphi_h^0, v) = (\varphi^0, v), \forall v \in V_h^0.$$
(3.15)

Using the representation of $\varphi_h(t)$ as before and taking $v=L_j$, the following system of equations is obtained

$$\left(B^{n-1} + \frac{k_n}{2} A^{n-1} \right) \xi^n = \left(B^{n-1} - \frac{k_n}{2} A^{n-1} \right) \xi^{n-1} + \frac{1}{2} k_n (F(t_n) + F(t_{n-1})).$$
(3.16)

The elements of matrices A and B are given before.

3.3 TEST FUNCTIONS

Let $N_i, i=1,2,3, \dots, M$ be the nodes of T_h . Given the basis functions $L_i, i = 1,2,\dots,M$,

we can define the interpolant $L_h \in V_h^0$ such that

$$L_i(N_j) = L_h(N_j), \quad i = 1,2,3,\dots,M.$$
(3.17)

Consider a triangle $K \in T_h$, then the function space $P_1(K)$ can be written as:

$$P_1(K) = \left\{ v : v(x) = \sum_{0 \leq i+j \leq 1} c_{ij} x_1^i x_2^j \text{ for } x \in K, c_{ij} \in \mathbb{R} \right\}. \quad (3.18)$$

V_h^0 is considered as the space of piecewise continuous linear functions as

$$V_h^0 = \left\{ v \in C^0(\overline{\Omega}) : v|_K \in P_1(K), \forall K \in T_h, v = 0 \text{ on } \Gamma_d \right\}. \quad (3.19)$$

The functions in V_h^0 are described by choosing the node points of T_h . Therefore, when $K \in T_h$ is a triangle, the degrees of freedom for K will be the values at the vertices a^i , $i = 1, 2, 3$. Given the values α_i , $i=1, 2, 3$, $\exists$ a uniquely determined function $v \in P_1(K)$ such that

$$v(a^i) = \alpha_i. \quad (3.20)$$

Therefore, the nodal basis functions L_i ($i = 1, 2, 3$) for $P_1(K)$ associated with the degrees of freedom can be determined such that

$$L_i(a^j) = \delta_{ij} = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{if } i \neq j \end{cases} \text{ for } i, j = 1, 2, 3. \quad (3.21)$$

Therefore, a function $v(x) \in P_1(K)$ then has the form

$$v(x) = \sum_{i=1}^3 v(a^i) L_i(x). \quad (3.22)$$

Therefore, three special cases of $(\alpha_1, \alpha_2, \alpha_3)$ given by $(1,0,0)$, $(0,1,0)$, and $(0,0,1)$ can be used to solve for $L_i(x)$ where

$$L_i(x) = c_{1i}x_1 + c_{2i}x_2 + c_{3i}. \quad (3.23)$$

Using the conditions for $(\alpha_1, \alpha_2, \alpha_3)$ in equation (3.20), the coefficients c_{1i} , c_{2i} , c_{3i} can be uniquely determined. Using the uniquely determined coefficients, the basis functions $L_i(x) \in K$ ($i=1,2,3$) can be obtained as follows.

$$\begin{aligned} L_1(x) &= \frac{1}{2A_T} [x_{12}x_{23} - x_{13}x_{22} + (x_{22} - x_{23})x_1 + (x_{13} - x_{12})x_2], \\ L_2(x) &= \frac{1}{2A_T} [x_{13}x_{21} - x_{11}x_{23} + (x_{23} - x_{21})x_1 + (x_{11} - x_{13})x_2], \\ L_3(x) &= \frac{1}{2A_T} [x_{11}x_{21} - x_{22}x_{21} + (x_{21} - x_{22})x_1 + (x_{12} - x_{11})x_2], \end{aligned} \quad (3.24)$$

where

A_T = area of triangle.

To evaluate the integrals for a_{ij} , b_{ij} ($i,j=1,2,3$) in equation (3.14), it is required to evaluate

$\nabla L_i(x)$ ($i=1,2,3$). The partial derivatives $\frac{\partial L_1}{\partial x_1}$, $\frac{\partial L_2}{\partial x_1}$, $\frac{\partial L_3}{\partial x_1}$, $\frac{\partial L_1}{\partial x_2}$, $\frac{\partial L_2}{\partial x_2}$ and $\frac{\partial L_3}{\partial x_2}$ can be

obtained by partial differentiation of L_1 , L_2 , and L_3 . The integrals in equation (3.14) are evaluated numerically. A three-point numerical quadrature formula is used as follows:

$$a_{ij} = \sum_{m=1}^3 (K(\psi) [\nabla L_i \cdot \nabla L_j])_m \omega_m A_e,$$

$$b_{ij} = \sum_{m=1}^3 (F(\psi) [L_i L_j])_m \omega_m A_e,$$

$$f_i = \sum_{m=1}^{n_{pe}} (qL_i)_m \omega_m A_e + \sum_{m=1}^{n_{pb}} (gL_i)_m \omega_m L_e,$$

where

n_{pe} = number of integration points in a triangular element in the domain;

ω_m = weight at the integration point m ;

A_e = area of element in the domain;

n_{pb} = number of integration points in an element along the boundary;

L_e = length of element along the boundary.

3.4 BILINEAR FORM AND MATRIX ANALYSIS

It is necessary to prove certain conditions associated with the variational problem. The bilinear form $a(.,.)$ defines the stiffness matrix A . The conditions associated with the bilinear form and the right hand side in equation (3.7b), are as follows, if K is symmetric, bounded and positive definite:

- (i) $a(.,.)$ is symmetric.
- (ii) $a(.,.)$ is continuous i.e. $\exists \gamma > 0$ such that $|a(v, w)| \leq \gamma \|v\|_v \|w\|_v \forall v, w \in V^0$.
- (iii) $a(.,.)$ is v -elliptic i.e. $\exists \alpha > 0$ such that $a(v, v) \geq \alpha \|v\|_v^2 \forall v \in V^0$.
- (iv) The right hand side in equation (3.19) should be continuous. Let $\ell(v) = (h, v) + \langle g, v \rangle$. $\ell(v)$ is continuous i.e. $\exists \Lambda > 0$ such that $|\ell(v)| \leq \Lambda \|v\|_v \forall v \in V^0$.

Proof (i)

$$\text{Here } a(\psi; v, w) = \int_{\Omega} K(\psi) \nabla v \cdot \nabla w \, d\Omega$$

$$\text{Now } a(\psi; w, v) = \int_{\Omega} K(\psi) \nabla w \cdot \nabla v \, d\Omega$$

Since $K \nabla v \cdot \nabla w = K \nabla w \cdot \nabla v$, therefore, $a(\psi; v, w) = a(\psi; w, v)$. This implies that $a(.,.,.)$ is symmetric.

Proof (ii)

Let's introduce the scalar products and norm in V^0 .

$$(v, w) = \int_{\Omega} K(\psi) \nabla v \cdot \nabla w \, d\Omega,$$

$$\|v\| = \left(\int_{\Omega} K(\psi) \nabla v \cdot \nabla v \, d\Omega \right)^{1/2}.$$

$$\text{Here } a(\psi; v, w) = \int_{\Omega} K(\psi) \nabla v \cdot \nabla w \, d\Omega.$$

$$\text{Now } a(\psi; v, v) = \int_{\Omega} K(\psi) \nabla v \cdot \nabla v \, d\Omega = \|v\|^2.$$

By Cauchy's inequality

$$a(\psi; v, w) \leq a(\psi; v, v)^{1/2} a(\psi; w, w)^{1/2},$$

which can be written as

$$a(\psi; v, w) \leq \|v\| \|w\|.$$

Therefore, $a(\psi; v, w)$ is continuous.

Proof (iii)

$$\text{As before, } a(\psi; v, v) = \int_{\Omega} K(\psi) \nabla v \cdot \nabla v \, d\Omega = \|v\|^2.$$

This implies that $a(\psi; v, v)$ is V -elliptic.

Proof (iv)

Here, $\ell(v) = (h, v) + \langle g, v \rangle$.

By Cauchy's inequality and the Poincare inequality, there is a constant C such that

$$|(h, v)| \leq C \|h\|_{L_2(\Omega)} \|v\|, \text{ and}$$

$$|\langle g, v \rangle| \leq C \|g\|_{L_2(\Gamma)} \|v\|.$$

Therefore,

$$|\ell(v)| \leq C (\|h\|_{L_2(\Omega)} + \|g\|_{L_2(\Gamma)}) \|v\|$$

$$\text{or, } |\ell(v)| \leq \Lambda \|v\|.$$

Therefore, $\ell(v)$ is continuous with $\Lambda = C(\|h\|_{L_2(\Omega)} + \|g\|_{L_2(\Gamma)})$.

From this, the matrix $B + k_n A$ is symmetric and positive definite.

$$\text{Here } b(\psi; v, w) = \int_{\Omega} E(\psi) v w \, d\Omega.$$

$$\text{Now } b(\psi; w, v) = \int_{\Omega} E(\psi) w v \, d\Omega.$$

Since $vw = wv$, therefore, $b(\psi; v, w) = b(\psi; w, v)$. This implies that $b(., ., .)$ is symmetric.

$b(\psi; v, v)$ can be written as

$$b(\psi; v, v) = b\left(\psi; \sum_{i=1}^m \eta_i \phi_i, \sum_{j=1}^m \eta_j \phi_j = \sum_{i,j=1}^m \eta_i b(\psi; \phi_i, \phi_j) \eta_j\right) = \eta \cdot B \eta.$$

$$\text{Also, } b(\psi; v, v) = \int_{\Omega} E(\psi) v^2 d\Omega. \quad (3.25)$$

From Poincare's inequality, there exists a constant $C > 0$ such that

$$\int_{\Omega} v^2 d\Omega \leq C \int_{\Omega} |\nabla v|^2 d\Omega \quad \forall v \in V^0.$$

Since $E(\psi) > 0$, therefore,

$$\int_{\Omega} E(\psi) v^2 d\Omega \leq C \int_{\Omega} E(\psi) |\nabla v|^2 d\Omega > 0, \quad \forall v \in V^0. \quad (3.26)$$

From equations (3.25) and (3.26)

$$\eta \cdot B\eta = b(\psi; v, v) > 0 \Rightarrow B \text{ is positive definite.}$$

B is proved to be symmetric earlier, so, B is symmetric positive definite.

By the proofs of (i), (ii), and (iii), matrix A is symmetric and positive definite. Therefore, $B + k_n A$ is symmetric positive definite.

3.5 STABILITY ANALYSIS

A basic stability estimate for the semi-discrete variational problem (V) is obtained by taking $v = \varphi_h^n$ in equation (3.13) to yield:

$$E(\psi) \|\varphi_h^n\|^2 - E(\psi)(\varphi_h^n, \varphi_h^{n-1}) + a(\psi; \varphi_h^n, \varphi_h^n) k_n - (q, \varphi_h^n) k_n - (g, \varphi_h^n) k_n = 0.$$

Since, $(\varphi_h^n, \varphi_h^{n-1}) \leq \frac{1}{2} \|\varphi_h^n\|^2 + \frac{1}{2} \|\varphi_h^{n-1}\|^2$ for $n=1,2,3, \dots, N$,

$$E(\psi) \|\varphi_h^n\|^2 - \frac{E(\psi)}{2} \|\varphi_h^n\|^2 - \frac{E(\psi)}{2} \|\varphi_h^{n-1}\|^2 + a(\psi; \varphi_h^n, \varphi_h^n) k_n - (q, \varphi_h^n) k_n - \langle g, \varphi_h^n \rangle k_n \leq 0.$$

By summing over n , we obtain:

$$E(\psi) \|\varphi_h^n\|^2 + 2 \sum_{m=1}^n a(\psi; \varphi_h^m, \varphi_h^m) k_m - 2 \sum_{m=1}^n (q, \varphi_h^m) k_m - 2 \sum_{m=1}^n \langle g, \varphi_h^m \rangle k_m - E(\psi) \|\varphi_h^0\|^2 \leq 0.$$

The above expression can be written as:

$$E(\psi) \|\varphi_h^n\|^2 + 2 \sum_{m=1}^n a(\psi; \varphi_h^m, \varphi_h^m) k_m \leq E(\psi) \|\varphi_h^0\|^2 + 2 \sum_{m=1}^n (q, \varphi_h^m) k_m + 2 \sum_{m=1}^n \langle g, \varphi_h^m \rangle k_m.$$

Since $\sum_{m=1}^n a(\psi; \varphi_h^m, \varphi_h^m) k_m \geq 0$, the above inequality can also be written as:

$$E(\psi) \|\varphi_h^n\|^2 \leq E(\psi) \|\varphi_h^0\|^2 + 2 \sum_{m=1}^n (q, \varphi_h^m) k_m + 2 \sum_{m=1}^n \langle g, \varphi_h^m \rangle k_m.$$

In particular, if $q = 0$ and $g = 0$, then

$$\|\varphi_h^n\|^2 \leq \|\varphi_h^0\|^2 \text{ or, } \|\varphi_h^n\| \leq \|\varphi_h^0\|.$$

Setting $v = \varphi_h(t)$ in equation (3.11) with $g=0, q=0$,

$$\frac{1}{2} \frac{d}{dt} \|E\varphi_h(t)\|^2 + a(\psi; \varphi_h(t), \varphi_h(t)) = 0.$$

Integrating from 0 to t , we get

$$\|E(\psi)\varphi_h(t)\|^2 - \|E(\psi)\varphi_h(0)\|^2 + 2 \int_0^t a(\psi; \varphi_h(s), \varphi_h(s)) ds = 0,$$

Therefore,

$$\|E(\psi)\varphi_h(t)\|^2 - \|E(\psi)\varphi_h(0)\|^2 \leq 0.$$

From which, we get

$$\|\varphi_h(t)\| \leq \|\varphi_h(0)\|.$$

For a linear homogeneous equation (2.10) with constant hydraulic conductivity and soil property so that $\alpha = K/E$, the solution for φ can be written as

$$\varphi(x, z, t) \sim \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} \varphi^0 e^{-\alpha \left\{ \left(\frac{m\pi}{M} \right)^2 + \left(\frac{n\pi}{L} \right)^2 \right\} t} \sin \frac{m\pi z}{M} \sin \frac{n\pi x}{L}$$

Based on the above equation, we can write

$$\|\varphi(t)\| \leq \|\varphi^0\|,$$

Therefore, in particular, we can write

$$\|\varphi_h(t)\| \leq \|\varphi_h(0)\| \leq \|\varphi^0\|.$$

Growth Estimate

With $q=0$ and $g=0$, a growth estimate is done to prove the reliability of the finite element formulation. It can be proved that

$$\max_n \|\varphi_h^n - \varphi_h^{n-1}\| \leq C \|\varphi^0\| \left(1 + \log \frac{T}{k_1}\right)^{1/2}.$$

In equation (3.13) with $f=0$, $g=0$, we can write

$$(\varphi_h^n - \varphi_h^{n-1}, v) + k_n a(\psi; \varphi_h^n, v) = 0.$$

Setting $v = t_n (\varphi_h^n - \varphi_h^{n-1})$, we have

$$(\varphi_h^n - \varphi_h^{n-1}, t_n (\varphi_h^n - \varphi_h^{n-1})) + k_n a(\psi; \varphi_h^n, t_n (\varphi_h^n - \varphi_h^{n-1})) = 0, \quad n = 1, 2, \dots, N$$

or, summing over n ,

$$\sum_{n=1}^N t_n \left\| \frac{\varphi_h^n - \varphi_h^{n-1}}{k_n} \right\|^2 k_n = \sum_{n=1}^N -a(\psi; \varphi_h^n, t_n (\varphi_h^n - \varphi_h^{n-1})).$$

By expanding and summing the right hand side, we have

$$\sum_{n=1}^N t_n \left\| \frac{\varphi_h^n - \varphi_h^{n-1}}{k_n} \right\|^2 k_n = \sum_{n=1}^N a(\psi; \varphi_h^n - \varphi_h^{n-1}, k_n \varphi_h^{n-1}).$$

Since $a(\psi; \dots)$ is continuous, therefore, for a constant $\gamma > 0$

$$a(\psi; \varphi_h^n - \varphi_h^{n-1}, k_n \varphi_h^{n-1}) \leq \gamma \|\varphi_h^n - \varphi_h^{n-1}\| \|k_n \varphi_h^{n-1}\|. \quad (3.27)$$

Since $\|\varphi_h^n - \varphi_h^{n-1}\| \leq \|\varphi^0\|$ and $\|\varphi_h^{n-1}\| \leq \|\varphi^0\|$ therefore,

$$a(\psi; \varphi_h^n - \varphi_h^{n-1}, k_n \varphi_h^{n-1}) \leq \gamma k_n \|\varphi^0\|^2. \quad (3.28)$$

Using (3.28) in (3.27), we get

$$\left(\sum_{n=1}^N t_n \left\| \frac{\varphi_h^n - \varphi_h^{n-1}}{k_n} \right\|^2 k_n \right)^{1/2} \leq C \|\varphi^0\| \text{ where } C = \sqrt{\gamma k_n}. \quad (3.29)$$

Now $\sum_{n=1}^N \left\| \frac{\varphi_h^n - \varphi_h^{n-1}}{k_n} \right\| k_n$ can be written as

$$\sum_{n=1}^N \left\| \frac{\varphi_h^n - \varphi_h^{n-1}}{k_n} \right\| k_n = \sum_{n=1}^N \left\{ \left\| \frac{\varphi_h^n - \varphi_h^{n-1}}{k_n} \right\| k_n^{1/2} t_n^{1/2} \right\} \frac{k_n^{1/2}}{t_n^{1/2}}.$$

Then, by Cauchy's inequality,

$$\sum_{n=1}^N \left\| \frac{\varphi_h^n - \varphi_h^{n-1}}{k_n} \right\| k_n \leq \left(\sum_{n=1}^N t_n \left\| \frac{\varphi_h^n - \varphi_h^{n-1}}{k_n} \right\|^2 k_n \right)^{1/2} \left(\sum_{n=1}^N \frac{k_n}{t_n} \right)^{1/2}. \quad (3.30)$$

From (3.29) and (3.30)

$$\sum_{n=1}^N \left\| \frac{\varphi_h^n - \varphi_h^{n-1}}{k_n} \right\| k_n \leq C \|\varphi^0\| \left(\sum_{n=1}^N \frac{k_n}{t_n} \right)^{1/2}. \quad (3.31)$$

The last summation term in equation (3.31) can be expressed as:

$$\sum_{n=1}^N \frac{k_n}{t_n} \approx \int_0^T \frac{1}{t} dt = \int_0^{t_1} \frac{1}{t} dt + \int_{t_1=k_1}^T \frac{1}{t} dt.$$

By approximating $\int_0^{t_1} \frac{1}{t} dt \approx 1$, we can write

$$\sum_{n=1}^N \frac{k_n}{t_n} \leq 1 + \log \frac{T}{k_1}.$$

Therefore, (3.46) can be written as

$$\sum_{n=1}^N \left\| \frac{\varphi_h^n - \varphi_h^{n-1}}{k_n} \right\| k_n \leq C \|\varphi^0\| \left(1 + \log \frac{T}{k_1} \right)^{1/2}.$$

From which, we get

$$\max_n \|\varphi_h^n - \varphi_h^{n-1}\| \leq C \|\varphi^0\| \left(1 + \log \frac{T}{k_1} \right)^{1/2}.$$

Following the stability estimate, the time step control suggested by Eriksson and Johnson (1985) can be adopted as

Suppose $\delta > 0$ is a given tolerance and suppose that we want the discrete solution $\varphi_h(t)$ to satisfy

$$\max_{t \leq t_N} \|\varphi(t) - \varphi_h(t)\| \leq \delta.$$

The time step k_n can be chosen so that for $n=1,2,\dots,N$

$$k_n \max_{t \in I} \|\dot{\varphi}\| \approx \frac{\delta}{C}.$$

It is assumed that the size of the constant is known approximately (up to, say, a factor 2).

The above condition can be replaced by the condition

$$\|\varphi_h^n - \varphi_h^{n-1}\| \approx \frac{\delta}{C}.$$

Based on the above conditions, an algorithm can be adopted for choosing k_n , assuming that φ_h^{n-1} has been computed.

Algorithm:

- (i) Choose $k_n = k_{n-1}$,
- (ii) Given k_n , compute the corresponding approximate solution φ_h^n ,
- (iii) If $\frac{\delta}{\beta C} \leq \|\varphi_h^n - \varphi_h^{n-1}\| \leq \frac{\delta}{C}$ then stop and accept the time step k_n . Otherwise, decrease or increase k_n by a factor of, say, 2 and return to (i).

3.6 SOLUTION OF THE SYSTEM OF EQUATIONS

The system of equations (3.14) constitute a symmetric positive definite system that is solved using conjugate gradient method. In a concise form, equation (3.14) can be rewritten, replacing $B + k_n A$ by $A \in \mathbb{R}^{n \times n}$ with unknowns as $X \in \mathbb{R}^n$ and right hand side as $b \in \mathbb{R}^n$, as

$$AX = b. \tag{3.32}$$

The Hestenes-Stiefel conjugate gradient method described by Golub and Loan (1996) is used to solve the system of equations (3.32). The algorithm used for solution is outlined briefly as follows

For $X^{(0)} \in \mathbb{R}^n$, let

$$r^{(0)} = AX^{(0)} - b,$$

$$d^{(0)} = r^{(0)}.$$

$$\text{Set } \alpha_k = -\frac{r^{(k)} \cdot d^{(k)}}{Ad^{(k)} \cdot d^{(k)}} = \frac{-r^{(k)} \cdot d^{(k)}}{d^{(k)T} Ad^{(k)}}.$$

The new solution is

$$X^{(k+1)} = X^{(k)} + \alpha_k d^{(k)}.$$

The updated values of $d^{(k+1)}$ and $d^{(k+1)}$ are obtained as

$$r^{(k+1)} = AX^{(k+1)} - b,$$

$$\beta_k = -\frac{Ar^{(k+1)} \cdot d^{(k)}}{Ad^{(k)} \cdot d^{(k)}} = -\frac{d^{(k)T} Ar^{(k+1)}}{d^{(k)T} Ad^{(k)}},$$

$$d^{(k+1)} = r^{(k+1)} + \beta_k d^{(k)}.$$

The coefficient matrix A consists of a symmetric banded system. Only the elements in the bandwidth are stored and used in the operation for conjugate gradient method. The eigenvalues and the condition number of the coefficient matrix is investigated. The

eigenvalues of the coefficient matrix are computed using Jacobi method described by Golub and Loan (1996).

3.7 MOVING PHREATIC SURFACE

The variational formulation (V) in equation (3.6) for the problem (D) in equation (3.1) will provide the solutions for ϕ . Using appropriate test functions v , the partial derivatives

$\frac{\partial \phi}{\partial x}$ and $\frac{\partial \phi}{\partial z}$ can be obtained as

$$\frac{\partial \phi}{\partial x} = \sum_{j=1}^{n_e} \frac{\partial L_j}{\partial x} \phi_j,$$

$$\frac{\partial \phi}{\partial z} = \sum_{j=1}^{n_e} \frac{\partial L_j}{\partial z} \phi_j,$$

where

n_e = number of nodes in an element.

Let $0 = t_0 < t_1 < t_2 < \dots < t_N = T$ denote a partitioning of I so that $I_n = (t_{n-1}, t_n)$. Let $k_n = t_n$

– t_{n-1} be the time step size. The derivatives are evaluated as follows

$$\begin{aligned}\frac{\partial \tilde{\varphi}}{\partial x} &= \lambda \left(\frac{\partial \varphi}{\partial x} \right)^{n+1} + (1-\lambda) \left(\frac{\partial \varphi}{\partial x} \right)^n, \\ \frac{\partial \tilde{\varphi}}{\partial z} &= \lambda \left(\frac{\partial \varphi}{\partial z} \right)^{n+1} + (1-\lambda) \left(\frac{\partial \varphi}{\partial z} \right)^n,\end{aligned}$$

where $0 \leq \lambda \leq 1$. Practically, λ is generally taken as 0.5.

Therefore, equation (2.24) can be written in a finite difference form as follows

$$\eta^{n+1} = \eta^n + \frac{k_n}{\theta_e} \left[K_x \left(\frac{\partial \tilde{\varphi}}{\partial x} \right)^2 + K_z \left(\frac{\partial \tilde{\varphi}}{\partial z} \right)^2 - K_z \frac{\partial \tilde{\varphi}}{\partial z} \right]. \quad (3.33)$$

Here η is loosely coupled with φ .

4. BOUNDARY ELEMENT FORMULATION

4.1 INTEGRAL EQUATION FORMULATION

The solution ϕ of the initial boundary value problem given in equation (2.10) can be rewritten with initial and boundary conditions (Figure 3.1) as follows:

$$\begin{aligned} E(x; \psi) \frac{\partial \phi}{\partial t} &= \nabla \cdot [K(x; \psi) \nabla \phi] + q, & \text{in } \Omega \times I, \\ \phi &= \phi_d, & \text{on } \Gamma_d \times I, \\ \frac{\partial \phi}{\partial n} &= g, & \text{on } \Gamma_n \times I, \\ \phi(x, 0) &= \phi^0, & \text{in } \Omega. \end{aligned} \tag{4.1}$$

To develop an integral formulation, it is assumed that the medium is homogeneous and isotropic. Let $\alpha = K/E$ and so q in the following equation is obtained by dividing the original source term by E . Therefore, the linear parabolic partial differential equation is written as follows

$$\begin{aligned} \frac{\partial \phi}{\partial t} &= \alpha \Delta \phi + q, & \text{in } \Omega \times I, \\ \phi &= \phi_d, & \text{on } \Gamma_d \times I, \\ \frac{\partial \phi}{\partial n} &= g, & \text{on } \Gamma_n \times I, \\ \phi(x, 0) &= \phi^0, & \text{in } \Omega. \end{aligned} \tag{4.2}$$

Let u^* be the fundamental solution of the homogeneous form of the parabolic equation.

The fundamental solution u^* of the homogeneous form of the parabolic operator $Hu^* = 0$

(where $H = \frac{\partial}{\partial t} - \alpha \Delta$) is (Morse and Feshbach, 1953):

$$u^*(x, z, t; \xi, \zeta, \tau) = \frac{1}{4\pi\alpha(t-\tau)} e^{-\frac{[(x-\xi)^2 + (z-\zeta)^2]}{4\alpha(t-\tau)}} \quad (4.3)$$

The fundamental solution u^* satisfies the following equations

$$\begin{aligned} \alpha \Delta u^* - u_t^* &= 0, \quad -\infty < x, z < \infty, \quad t > \tau, \\ \alpha \Delta u^* + u_\tau^* &= 0, \quad -\infty < \xi, \zeta < \infty, \quad t > \tau. \end{aligned} \quad (4.4)$$

The normal derivative $\frac{\partial u^*}{\partial n}$ of the fundamental solution can be written as

$$\frac{\partial u^*}{\partial n} = -\frac{\eta}{8\pi\alpha^2(t-\tau)^2} e^{-\frac{[(x-\xi)^2 + (z-\zeta)^2]}{4\alpha(t-\tau)}}, \quad (4.5)$$

where

η = perpendicular distance from the point under consideration to the boundary.

Multiplying equation (4.2) by the fundamental solution u^* and integrating over domain Ω , we obtain:

$$\int_{\Omega} \left[\frac{\partial \phi}{\partial t} - \alpha \Delta \phi - q \right] u^* d\Omega = 0 \text{ for all } \phi \in C^2(\Omega) \cup C^1(\overline{\Omega}).$$

Integrating $\int_{\Omega} \Delta \phi u^* d\Omega$ twice by parts, we obtain

$$\int_{\Omega} \frac{\partial \phi}{\partial t} u^* d\Omega = \alpha \left[\int_{\Gamma} \frac{\partial \phi}{\partial n} u^* d\Gamma - \int_{\Gamma} \phi \frac{\partial u^*}{\partial n} d\Gamma + \int_{\Omega} \phi \Delta u^* d\Omega \right] + \int_{\Omega} q u^* d\Omega. \quad (4.6)$$

The above integral equation can also be obtained by directly using Green's second identity. The Green's second identity for functions ϕ and u^* can be written as

$$\int_{\Omega} [u^* \Delta \phi - \phi \Delta u^*] d\Omega = \int_{\Gamma} \left[u^* \frac{\partial \phi}{\partial n} - \phi \frac{\partial u^*}{\partial n} \right] d\Gamma.$$

Substituting in the expression for $\Delta \phi$ from equation (4.2), an integral equation the same as in equation (4.6) is obtained

$$\int_{\Omega} [u^* (\phi_t - q) - \alpha \phi \Delta u^*] d\Omega = \alpha \int_{\Gamma} \left[u^* \frac{\partial \phi}{\partial n} - \phi \frac{\partial u^*}{\partial n} \right] d\Gamma.$$

Let $0 = t_0 < t_1 < t_2 < \dots < t_N = T$ denote a partitioning of I . Integrating the integral equation with respect to τ from 0 to t_n , we get

$$\int_{\Omega} \int_0^{t_n} [u^*(\varphi_t - q) - \alpha \varphi \Delta u^*] d\tau d\Omega = \alpha \int_0^{t_n} \int_{\Gamma} \left[u^* \frac{\partial \varphi}{\partial n} - \varphi \frac{\partial u^*}{\partial n} \right] d\Gamma d\tau. \quad (4.7)$$

Integrating $u^* \varphi_t$, we have

$$\begin{aligned} \int_0^{t_n} u^* \varphi_t d\tau &= u^*(x, z, t; \xi, \zeta, t_n) \varphi(\xi, \zeta, t_n) - u^*(x, z, t; \xi, \zeta, 0) \varphi(\xi, \zeta, 0) \\ &\quad - \int_0^{t_n} \frac{\partial u^*}{\partial \tau} \varphi d\tau. \end{aligned} \quad (4.8)$$

Substituting (4.8) in (4.7) and rearranging, we get

$$\begin{aligned} \int_{\Omega} \varphi(., t_n) u^*(., t_n) d\Omega &= \int_0^{t_n} \left\{ \alpha \left[\int_{\Gamma} \frac{\partial \varphi}{\partial n} u^* d\Gamma - \int_{\Gamma} \varphi \frac{\partial u^*}{\partial n} d\Gamma + \int_{\Omega} \left(\frac{1}{\alpha} \frac{\partial u^*}{\partial \tau} + \Delta u^* \right) \varphi d\Omega \right] + \int_{\Omega} q u^* d\Omega \right\} d\tau \\ &\quad + \int_{\Omega} \varphi^0 u^* d\Omega, \end{aligned}$$

where φ^0 is the initial condition.

Since u^* satisfies equation (4.4), therefore,

$$\int_{\Omega} \varphi(t_n) u^*(t_n) d\Omega = \int_0^{t_n} \left\{ \alpha \left[\int_{\Gamma} \frac{\partial \varphi}{\partial n} u^* d\Gamma - \int_{\Gamma} \varphi \frac{\partial u^*}{\partial n} d\Gamma \right] + \int_{\Omega} q u^* d\Omega \right\} d\tau + \int_{\Omega} \varphi^0 u^* d\Omega.$$

Now taking $t=t_n$, we get

$$\int_{\Omega} \varphi(\xi, \zeta, t_n) u^*(x, z, t_n; \xi, \zeta, t_n) d\Omega = \lim_{h \rightarrow 0} \int_{\Omega} \varphi(\xi, \zeta, t_n) \frac{1}{4\pi\alpha h} e^{-\frac{(x-\xi)^2 + (z-\zeta)^2}{4\alpha h}} d\Omega.$$

The above integral acts like a delta function for $(x, z) \in \Omega$ then the integral can be written with some constant $c > 0$ as

$$c\varphi(x, z, t_n) = \int_0^{t_n} \left\{ \alpha \left[\int_{\Gamma} \frac{\partial \varphi}{\partial n} u^* d\Gamma - \int_{\Gamma} \varphi \frac{\partial u^*}{\partial n} d\Gamma \right] + \int_{\Omega} q u^* d\Omega \right\} d\tau + \int_{\Omega} \varphi^0 u^* d\Omega. \quad (4.9)$$

Here $c=1$ for $(x, z) \in \Omega$ and $c = \theta/2\pi$ for $(x, z) \in \Gamma$, (θ = interior angle at the boundary).

Equation (4.9) is an integral equation of the second kind for $\varphi|_{\Gamma}$ if $\frac{\partial \varphi}{\partial n}|_{\Gamma}$ is known and an

equation of the first kind for $\frac{\partial \varphi}{\partial n}|_{\Gamma}$ if $\varphi|_{\Gamma}$ is known. The above integral equation is known

as a representation formula or as Green's third identity for the parabolic equation (4.2).

4.2 VARIATIONAL FORMULATION

Let $T_h = \{K_1, K_2, \dots, K_M\}$ be a subdivision of Γ into elements K_i of length at most h .

Let's introduce the finite element space $W_h = \{v \in L_2(\Gamma): v|_{K_i} \text{ is linear, } i=1, 2, \dots, M\}$.

Multiplying the boundary integral equation (4.9) by a test function v in the test space W_h and integrating over Γ , we are led to a variational formulation of the boundary integral equation (4.9) as follows:

To find $\varphi \in W_h$ such that

$$(\varphi, v) + (H\varphi, v) - (G\varphi, v) = (\ell, v) \quad \forall v \in W_h, \quad (4.10)$$

where

$$(\varphi, v) = \int_{\Gamma} \varphi v \, d\Gamma,$$

$$H\varphi = \frac{\alpha}{c} \int_0^{t_n} \int_{\Gamma} \frac{\partial u^*}{\partial n} \varphi \, d\Gamma \, d\tau,$$

$$(H\varphi, v) = \int_{\Gamma} H\varphi \, v \, d\Gamma = \frac{\alpha}{c} \int_{\Gamma} \left\{ \int_0^{t_n} \int_{\Gamma} \frac{\partial u^*}{\partial n} \varphi \, d\Gamma \, d\tau \right\} v \, d\Gamma,$$

$$G\varphi = \frac{\alpha}{c} \int_0^{t_n} \int_{\Gamma} u^* \frac{\partial \varphi}{\partial n} \, d\Gamma \, d\tau,$$

$$(G\varphi, v) = \int_{\Gamma} G\varphi \, v \, d\Gamma = \frac{\alpha}{c} \int_{\Gamma} \left\{ \int_0^{t_n} \int_{\Gamma} u^* \frac{\partial \varphi}{\partial n} \, d\Gamma \, d\tau \right\} v \, d\Gamma,$$

$$\ell = \frac{1}{c} \left(\left\{ \int_{\Omega} u^* \varphi^0 \, d\Omega \right\} + \int_0^{t_n} \int_{\Omega} u^* q \, d\Omega \, d\tau \right),$$

$$(\ell, v) = \int_{\Gamma} \ell v \, d\Gamma = \frac{1}{c} \int_{\Gamma} \left\{ \int_{\Omega} u^* \varphi^0 \, d\Omega \right\} v \, d\Gamma + \frac{1}{c} \int_{\Gamma} \left\{ \int_0^{t_n} \int_{\Omega} u^* q \, d\Omega \, d\tau \right\} v \, d\Gamma.$$

4.3 EVALUATION OF INTEGRALS

The elements K_i ($i=1,2,\dots,M$) are subdivisions of Γ . In each element there are two linear basis functions ψ_1 and ψ_2 . The two nodal linear basis functions ψ_1 and ψ_2 used for an element K_i ($i=1,2, \dots, M$) can be expressed as

$$\psi_1 = \frac{1}{2} \left(1 - \frac{2\xi}{L_{ei}} \right) \text{ for } -\frac{L_{ei}}{2} \leq \xi \leq \frac{L_{ei}}{2},$$
$$\psi_2 = \frac{1}{2} \left(1 + \frac{2\xi}{L_{ei}} \right) \text{ for } -\frac{L_{ei}}{2} \leq \xi \leq \frac{L_{ei}}{2},$$

where

L_{ei} = length of an element K_i ($i=1,2, \dots, M$).

In any element K_i ($i=1,2, \dots, M$), ϕ and $\frac{\partial \phi}{\partial n}$ vary as follows:

$$\phi = \phi_1 \psi_1 + \phi_2 \psi_2,$$
$$\frac{\partial \phi}{\partial n} = \frac{\partial \phi_1}{\partial n} \psi_1 + \frac{\partial \phi_2}{\partial n} \psi_2.$$

Here ϕ_1 , ϕ_2 , $\frac{\partial \phi_1}{\partial n}$, and $\frac{\partial \phi_2}{\partial n}$ are the nodal values at points 1 and 2 of any element K_i .

Using test function v as the basis function ψ_i ($i = 1,2,3, \dots, M$), the integrals in equation

(4.10) can be evaluated for any K_i of length L_{ei} ($i=1,2, \dots, M$) as:

$$(\phi, v) = \frac{1}{4} \int_{-L_{ei}/2}^{L_{ei}/2} \left\{ \left(1 - \frac{2\xi^2}{L_{ei}} \right)^2 \phi_1 + \left(1 - \frac{4\xi^2}{L_{ei}^2} \right) \phi_2 \right\} d\xi = \frac{L_{ei}}{3} \phi_1 + \frac{L_{ei}}{6} \phi_2.$$

Similarly,

$$(H\phi, v) = \frac{L_{ei}}{3} H\phi_1 + \frac{L_{ei}}{6} H\phi_2,$$

$$(G\phi, v) = \frac{L_{ei}}{3} G\phi_1 + \frac{L_{ei}}{6} G\phi_2,$$

$$(\ell, v) = \frac{L_{ei}}{3} F_1 + \frac{L_{ei}}{6} F_2.$$

The integrals $H\phi_1$, $H\phi_2$, $G\phi_1$, $G\phi_2$, F_1 , and F_2 for nodal points 1 and 2 of any element K_i ($i=1,2, \dots, M$) under consideration and associating field element K_j ($j=1,2,3, \dots, M$) are evaluated as follows:

$$\begin{aligned} H\phi_1 = \frac{\alpha}{c} \int_0^{t_n} \int_{\Gamma} \left(\frac{\partial u_1^*}{\partial n} \psi_1 \phi_1 + \frac{\partial u_1^*}{\partial n} \psi_2 \phi_2 \right) d\Gamma \, d\tau = \int_0^{t_n} \beta_1 \left\{ \gamma e_{ij} + \lambda_2 e^{-\frac{\eta_1^2}{4\alpha(t-\tau)}} g_{ij} \right\} \phi_1 \, d\tau \\ + \int_0^{t_n} \beta_1 \left\{ -\gamma e_{ij} - \lambda_1 e^{-\frac{\eta_1^2}{4\alpha(t-\tau)}} g_{ij} \right\} \phi_2 \, d\tau, \end{aligned}$$

$$\begin{aligned} H\phi_2 = \frac{\alpha}{c} \int_0^{t_n} \int_{\Gamma} \left(\frac{\partial u_2^*}{\partial n} \psi_1 \phi_1 + \frac{\partial u_2^*}{\partial n} \psi_2 \phi_2 \right) d\Gamma \, d\tau = \int_0^{t_n} \beta_2 \left\{ \gamma e_{ij} + \lambda_2 e^{-\frac{\eta_2^2}{4\alpha(t-\tau)}} g_{ij} \right\} \phi_1 \, d\tau \\ + \int_0^{t_n} \beta_2 \left\{ -\gamma e_{ij} - \lambda_1 e^{-\frac{\eta_2^2}{4\alpha(t-\tau)}} g_{ij} \right\} \phi_2 \, d\tau, \end{aligned}$$

$$G\varphi_1 = \frac{\alpha}{c} \int_0^{t_n} \int_{\Gamma} \left(\frac{\partial u_1^*}{\partial n} \psi_1 \varphi_1 + \frac{\partial u_1^*}{\partial n} \psi_2 \varphi_2 \right) d\Gamma d\tau = \int_{t_{n-1}}^{t_n} \beta_3 \left\{ \gamma e_{ij} + \lambda_2 e^{-\frac{\eta_1^2}{4\alpha(t-\tau)}} g_{ij} \right\} \frac{\partial \varphi_1}{\partial n} d\tau$$

$$+ \int_0^{t_n} \beta_3 \left\{ -\gamma e_{ij} - \lambda_1 e^{-\frac{\eta_1^2}{4\alpha(t-\tau)}} g_{ij} \right\} \frac{\partial \varphi_2}{\partial n} d\tau,$$

$$G\varphi_2 = \frac{\alpha}{c} \int_0^{t_n} \int_{\Gamma} \left(\frac{\partial u_2^*}{\partial n} \psi_1 \varphi_1 + \frac{\partial u_2^*}{\partial n} \psi_2 \varphi_2 \right) d\Gamma d\tau = \int_0^{t_n} \beta_3 \left\{ \gamma e_{ij} + \lambda_2 e^{-\frac{\eta_2^2}{4\alpha(t-\tau)}} g_{ij} \right\} \frac{\partial \varphi_1}{\partial n} d\tau$$

$$+ \int_0^{t_n} \beta_3 \left\{ -\gamma e_{ij} - \lambda_1 e^{-\frac{\eta_2^2}{4\alpha(t-\tau)}} g_{ij} \right\} \frac{\partial \varphi_2}{\partial n} d\tau,$$

$$F_1 = \int_{\Omega} \frac{1}{4\pi\alpha(t-\tau)} e^{-\frac{(x_1-\xi)^2+(z_1-\zeta)^2}{4\alpha(t-\tau)}} d\Omega + \int_0^{t_n} \int_{\Omega} \frac{q}{4\pi\alpha(t-\tau)} e^{-\frac{(x_1-\xi)^2+(z_1-\zeta)^2}{4\alpha(t-\tau)}} d\Omega d\tau,$$

$$F_2 = \int_{\Omega} \frac{1}{4\pi\alpha(t-\tau)} e^{-\frac{(x_2-\xi)^2+(z_2-\zeta)^2}{4\alpha(t-\tau)}} d\Omega + \int_0^{t_n} \int_{\Omega} \frac{q}{4\pi\alpha(t-\tau)} e^{-\frac{(x_2-\xi)^2+(z_2-\zeta)^2}{4\alpha(t-\tau)}} d\Omega d\tau,$$

where

$$\beta_1 = \frac{\eta_1}{8\pi\alpha(t-\tau)^2},$$

$$\beta_2 = \frac{\eta_2}{8\pi\alpha(t-\tau)^2},$$

η_1 = perpendicular distance from nodal point 1 of a source element i to a field element j ,

η_2 = perpendicular distance from nodal point 2 of a source element i to a field element j ,

$$\gamma = \frac{2\alpha(t-\tau)}{\xi_{j+1} - \xi_j},$$

$$\lambda_1 = -\frac{1}{2}\sqrt{\pi\alpha(t-\tau)},$$

$$\lambda_2 = \frac{1}{2}\sqrt{\pi\alpha(t-\tau)},$$

$$e_{ij} = e^{\frac{\eta_i^2 + \xi_{j+1}^2}{4\alpha(t-\tau)}} - e^{\frac{\eta_i^2 + \xi_j^2}{4\alpha(t-\tau)}}, \quad g_{ij} = \operatorname{erf}\left(\frac{\xi_{j+1}}{\sqrt{4\alpha(t-\tau)}}\right) - \operatorname{erf}\left(\frac{\xi_j}{\sqrt{4\alpha(t-\tau)}}\right),$$

ξ_j = coordinate measured by arc length of node 1 of a field element j ,

ξ_{j+1} = coordinate measured by arc length of node 2 of a field element j ,

$$\beta_3 = \frac{1}{4\pi(t-\tau)}.$$

The integrals associated with the coefficients of ϕ and $\frac{\partial\phi}{\partial n}$ are evaluated analytically as shown above. The integrals involving coefficient matrix, initial condition, and forcing function (source/sink) are evaluated numerically using a Gaussian quadrature formula.

4.4 NUMERICAL ALGORITHM

After evaluating the integrals, the coefficients associated with ϕ and $\frac{\partial\phi}{\partial n}$ are obtained. In matrix form, they can be written as

$$(D + W)\phi - P \frac{\partial\phi}{\partial n} = S, \quad (4.11)$$

where

$D = (d_i)$, a diagonal matrix,

$$d_i = \int_{\Gamma_i} \psi_i^2 d\Gamma_i,$$

$W = (w_{ij})$,

$$w_{ij} = \int_{\Gamma_i} H_j \psi_i d\Gamma_i \text{ for } j \neq i+1,$$

$$w_{ij} = \int_{\Gamma_i} \psi_i \psi_j d\Gamma_i + \int_{\Gamma_i} H_j \psi_i d\Gamma_i \text{ for } j = i+1,$$

$P = (p_{ij})$,

$$p_{ij} = \int_{\Gamma_i} G_j \psi_i d\Gamma_i,$$

$S = (s_i)$,

$$s_i = \int_{\Gamma_i} (\ell_i \psi_i + \ell_{i+1} \psi_{i+1}) \psi_i d\Gamma_i.$$

Here $\Gamma = \Gamma_d \cup \Gamma_n$. Along Γ , the system of equations (4.11) reveals that M_d values of φ

on Γ_d and M_n values of $\frac{\partial \varphi}{\partial n}$ on Γ_n are known and therefore, there are M_d values of

$\frac{\partial \varphi}{\partial n}$ on Γ_d and M_n values of φ on Γ_n which are unknowns. So, there are $M = M_d + M_n$

unknowns in the expression. Rearranging the coefficients in such a way that all the

unknowns are in the left hand side, one can write the following matrix equation:

$$AX = B,$$

where

$$a_{ij} = d_i + w_{ij} \text{ if } (x,z) \in \Gamma_n,$$

$$a_{ij} = p_{ij} \text{ if } (x,z) \in \Gamma_d,$$

$$b_i = s_i - (d_i + w_{ij})\phi_j \text{ if } (x,z) \in \Gamma_d,$$

$$b_i = s_i - c_{ij} \frac{\partial \phi_j}{\partial n} \text{ if } (x,z) \in \Gamma_n,$$

$$X_i = \phi_i \text{ if } (x,z) \in \Gamma_n,$$

$$X_i = \frac{\partial \phi_i}{\partial n} \text{ if } (x,z) \in \Gamma_d.$$

Alternatively, equation (4.11) can be arranged to obtain a Fredholm equation of the second kind for $\phi|_\Gamma$ if $\frac{\partial \phi}{\partial n}|_\Gamma$ is known and a Fredholm equation of the first kind for $\frac{\partial \phi}{\partial n}|_\Gamma$

if $\phi|_\Gamma$ is known. For unknown boundary conditions ϕ or $\frac{\partial \phi}{\partial n}$, Liggett and Liu (1979)

suggest a different procedure to solve for the unknown values. The system of equations for M_n nodes with $(x,z) \in \Gamma_n$ can be written as

$$(D + W)\phi = P \frac{\partial \phi}{\partial n} + S \text{ for } i = 1, 2, \dots, M_n.$$

In the right hand side, the $\frac{\partial \phi}{\partial n}$ values for $(x,z) \in \Gamma_n$ are known and the $\frac{\partial \phi}{\partial n}$ values for

$(x,z) \in \Gamma_d$ are unknown but are used from the values obtained at the previous time steps.

After obtaining the φ values for $(x,z) \in \Gamma_n$, the unknown values of $\frac{\partial \varphi}{\partial n}$ values for $(x,z) \in \Gamma_d$ can be obtained as follows

$$-P \frac{\partial \varphi}{\partial n} = S - (D + W)\varphi \text{ for } i = 1, 2, \dots, M_d$$

Once all the values of φ and $\frac{\partial \varphi}{\partial n}$ for $(x,z) \in \Gamma = \Gamma_d \cup \Gamma_n$ are known, the values of φ for $(x,z) \in \Omega$ can be obtained readily from equation (4.9):

4.5 MATRIX CHARACTERISTICS

The matrix W can be written in a simpler form as

$$w_{ij} = \frac{\alpha}{c} \int_{t_{n-1}}^{t_n} \int_{\Gamma_i} \int_{\Gamma_j} \frac{\partial u^*}{\partial n} \psi_i \psi_j d\Gamma_i d\Gamma_j d\tau,$$

$$w_{ji} = \frac{\alpha}{c} \int_{t_{n-1}}^{t_n} \int_{\Gamma_i} \int_{\Gamma_j} \frac{\partial u^*}{\partial n} \psi_j \psi_i d\Gamma_j d\Gamma_i d\tau,$$

$$\Rightarrow w_{ij} = w_{ji}.$$

Therefore, W is a symmetric matrix.

The matrix P is not symmetric because

$$p_{ij} = \frac{\alpha}{c} \int_0^{t_n} \int_{\Gamma_i} \int_{\Gamma_j} u^* \frac{\partial \psi_j}{\partial n} \psi_i d\Gamma_i d\Gamma_j d\tau$$

$$p_{ji} = \frac{\alpha}{c} \int_0^{t_n} \int_{\Gamma_j} \int_{\Gamma_i} u^* \frac{\partial \psi_i}{\partial n} \psi_j d\Gamma_j d\Gamma_i d\tau$$

$$\Rightarrow p_{ij} \neq p_{ji}$$

Therefore, P is not symmetric matrix.

5. VERIFICATION OF NUMERICAL SOLUTIONS

5.1 ANALYTICAL SOLUTION

An analytical solution is obtained for a two-dimensional domain using some simplifying assumptions. The assumptions used are as follows:

- the domain is assumed to be rectangular in shape
- the medium is assumed to be homogeneous and isotropic
- the hydraulic conductivity is assumed to be constant with an effective value for the medium

Based on the above assumptions, the governing equation can be written as

$$\frac{\partial \phi}{\partial t} = \alpha \Delta \phi + q \text{ for } \phi \in C^2(\Omega) \cup C^1(\Gamma). \quad (5.1)$$

Two analytical solutions are obtained for different boundary conditions. In the first problem, Dirichlet boundary conditions are used on all sides of a rectangular domain. The second problem consists of both Dirichlet and Neumann boundary conditions. The method of separation of variables are used to derive analytical solution for both the problems. The derivation is given below.

Dirichlet Problem

The Dirichlet boundary condition is applied on all four sides of a two-dimensional rectangular domain to obtain a solution of the simplified equation (5.1). The boundary conditions are:

$$\varphi(0,z,t) = f_1(z), \quad 0 < z < M, t > 0,$$

$$\varphi(x,0,t) = f_2(x), \quad 0 < x < L, t > 0,$$

$$\varphi(L,z,t) = f_3(z), \quad 0 < z < M, t > 0,$$

$$\varphi(x,M,t) = f_4(x), \quad 0 < x < L, t > 0.$$

Here, $f_1(z)$, $f_2(x)$, $f_3(z)$, and $f_4(x)$ can be expressed as follows:

$$f_1(z) = a + b \frac{z}{M},$$

$$f_2(x) = a + \frac{bx}{L} \left(1 - \frac{x}{L} \right),$$

$$f_3(z) = a + c \frac{z}{M},$$

$$f_4(x) = a - \frac{bx}{L} \left(1 - \frac{x}{L} \right),$$

where $a, b, c, d \in \mathbb{R}$.

The initial condition is expressed as follows:

$$\varphi(x,z,0) = \varphi^0, \quad 0 < x < L, \quad 0 < z < M$$

Based on the principle of superposition, let the solution for φ can be written as:

$$\varphi(x,z,t) = \eta(x,z,t) + \zeta(x,z), \quad (5.2)$$

where

$$\zeta(x,z) = a + b \frac{z}{M} \left(1 - \frac{x}{L} \right) + c \frac{xz}{LM}.$$

Substituting (5.2) in the simplified equation (5.1) for φ -problem, the partial differential equation for η -problem can be obtained as:

$$\frac{\partial \eta}{\partial t} = \alpha \Delta \eta + q, \quad (5.3)$$

$$\eta(0,z,t) = 0, \quad 0 < z < M,$$

$$\eta(x,0,t) = f_2(x) - a, \quad 0 < x < L,$$

$$\eta(L,z,t) = 0, \quad 0 < z < M,$$

$$\eta(x,M,t) = f_4(x) - a - b \left(1 - \frac{x}{L} \right) - c \frac{x}{L}, \quad 0 < x < L,$$

$$\eta(x,z,0) = \varphi^0 - \zeta(x,z).$$

The linear nonhomogeneous parabolic equation (5.3) for η describing an Initial-Boundary Value Problem (I-BVP) can be solved using separation of variables and the method of variation of parameters. To use the method of the variation of parameters, we

must know the eigenfunctions of the eigen value problem which can be obtained by considering the I-BVP for the associated homogeneous partial differential equation.

Based on the method of separation of variables, $\eta(x,z,t)$ for the homogeneous equation (5.3) can be written as

$$\eta(x, z, t) = v(z, t)X(x). \quad (5.4)$$

Substituting equation (5.4) in equation (5.3) with $q=0$ results in the eigenvalues for the X- problem as

$$\lambda_n = \left(\frac{n\pi}{L} \right)^2, \quad n = 1, 2, 3, \dots$$

The eigen functions of the X- problem are:

$$X_n \approx \sin \frac{n\pi x}{L}.$$

Therefore, η can be written as

$$\eta = \sum_{n=1}^{\infty} v(z, t) \sin \frac{n\pi x}{L},$$

where

$$v(z, t) = \frac{2}{L} \int_0^L \eta(x, z, t) \sin \frac{n\pi x}{L} dx. \quad (5.5)$$

Differentiating equation (5.5) with respect to t and substituting $\frac{\partial \eta}{\partial t}$ from equation (5.3),

we can come up with the v -problem as

$$\frac{\partial v}{\partial t} = \alpha \frac{\partial^2 v}{\partial z^2} - \alpha \lambda_n v + \tilde{q},$$

where

$$v(0, t) = \frac{2}{L} \int_0^L \eta(x, 0, t) \sin \frac{n\pi x}{L} dx = \frac{2}{L} \int_0^L (f_2(x) - a) \sin \frac{n\pi x}{L} dx,$$

$$v(M, t) = \frac{2}{L} \int_0^L \eta(x, M, t) \sin \frac{n\pi x}{L} dx = \frac{2}{L} \int_0^L \left(f_4(x) - a - b \left(1 - \frac{x}{L} \right) - c \frac{x}{L} \right) \sin \frac{n\pi x}{L} dx,$$

$$v(z, 0) = v^0 = \frac{2}{L} \int_0^L \eta(x, z, 0) \sin \frac{n\pi x}{L} dx = \frac{2}{L} \int_0^L (\varphi^0 - \zeta(x, z)) \sin \frac{n\pi x}{L} dx,$$

$$\tilde{q} = \frac{2}{L} \int_0^L q \sin \frac{n\pi x}{L} dx.$$

Based on the principle of superposition, let the solution for $v(z, t)$ be written as

$$v(z, t) = u(z, t) + w(z),$$

where

$$w(z) = \left(1 - \frac{z}{M} \right) v(0, t) + \frac{z}{M} v(M, t).$$

Using the above relationship, we can come up with the u -problem as

$$\frac{\partial u}{\partial t} = \alpha \frac{\partial^2 u}{\partial z^2} - \alpha \lambda_n u + \hat{q}, \quad (5.6)$$

where

$$u(0,t) = 0,$$

$$u(M,t) = 0,$$

$$u(z,0) = u^0 = v^0 - w(z),$$

$$\hat{q} = \tilde{q} - \alpha \lambda_n w(z).$$

As before, $u(z,t)$ for the homogeneous u -problem can be written as

$$u(z,t) = Z(z)T(t). \quad (5.7)$$

Substituting equation (5.7) in equation (5.6) with $\hat{q} = 0$ results in the eigenvalues for the Z -problem as

$$\lambda_m = \left(\frac{m\pi}{M} \right)^2, \quad m = 1, 2, 3, \dots$$

The eigen functions of the Z -problem are:

$$Z_m \approx \sin \frac{m\pi z}{M}.$$

The solution for $u(z,t)$ can therefore be written as

$$u(z, t) = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} K_{mn}(t) \sin \frac{m\pi z}{M},$$

where

$$K_{mn}(t) = \frac{2}{M} \int_0^M u(z, t) \sin \frac{m\pi z}{M} dz.$$

Differentiating $K_{mn}(t)$ we obtain an ordinary differential equation for $K_{mn}(t)$ as

$$\frac{dK_{mn}(t)}{dt} + \alpha \lambda_{mn} K_{mn}(t) = q_{mn},$$

where

$$\lambda_{mn} = \lambda_m + \lambda_n,$$

$$q_{mn} = \frac{2}{M} \int_0^M \hat{q} \sin \frac{m\pi z}{M} dz.$$

The solution for $K_{mn}(t)$ is written as

$$K_{mn}(t) = \int_0^t e^{-\alpha \lambda_{mn}(t-\tau)} q_{mn}(z, \tau) d\tau + g e^{-\alpha \lambda_{mn} t},$$

where

$$g = K_{mn}(0) = \frac{2}{M} \int_0^M u^0 \sin \frac{m\pi z}{M} dz.$$

Therefore, the solution of the original ϕ -problem is

$$\varphi(x, z, t) = \sum_{n=1}^{\infty} \left[\sum_{m=1}^{\infty} K_{mn}(t) \sin \frac{m\pi z}{M} + w(z) \right] \sin \frac{n\pi x}{L} + \zeta(x, z), \quad (5.8)$$

where $K_{mn}(t)$, $w(z)$, and $\zeta(x, z)$ are described earlier.

Mixed Problem

In the mixed problem, both Dirichlet and Neumann boundary conditions were applied on the sides of a two-dimensional rectangular domain. The boundary conditions used in the mixed problem are as follows

$$\varphi(0, z, t) = f_1(z), \quad 0 < z < M, t > 0,$$

$$\varphi(x, 0, t) = f_2(x), \quad 0 < x < L, t > 0,$$

$$\varphi_x(L, z, t) = f_3(z), \quad 0 < z < M, t > 0,$$

$$\varphi_z(x, M, t) = f_4(x), \quad 0 < x < L, t > 0.$$

Here, $f_1(z)$, $f_2(x)$, $f_3(z)$, and $f_4(x)$ can be expressed as follows:

$$f_1(z) = a + b \frac{z}{M},$$

$$f_2(x) = c - \frac{dx}{L},$$

$$f_3(z) = -\frac{bz}{LM},$$

$$f_4(x) = b - \frac{dx}{L}.$$

The initial condition is expressed as follows:

$$\varphi(x,z,0) = \varphi^0, \quad 0 < x < L, 0 < z < M.$$

Based on the principle of superposition, let the solution for φ can be written as:

$$\varphi(x,z,t) = \eta(x,z,t) + \zeta(x,z), \quad (5.9)$$

where

$$\zeta(x,z) = a + b \frac{z}{M} \left(1 - \frac{x^2}{2L^2} \right)$$

Substituting equation (5.9) in the simplified equation (5.1) for φ -problem, the partial differential equation for η -problem can be written as:

$$\frac{\partial \eta}{\partial t} = \alpha \Delta \eta + q \quad (5.10)$$

$$\eta(0, z, t) = 0 \quad 0 < z < M,$$

$$\eta(x, 0, t) = f_3(x) - a \quad 0 < x < L,$$

$$\eta_x(L, z, t) = 0 \quad 0 < z < M,$$

$$\eta_z(x, M, t) = f_4(x) - \frac{b}{M} \left(1 - \frac{x^2}{2L^2} \right) \quad 0 < x < L,$$

$$\eta(x, z, 0) = \varphi^0 - \zeta(x, z).$$

Based on the method of separation of variables, $\eta(x, z, t)$ for the homogeneous equation (5.10) can be written as

$$\eta(x, z, t) = v(z, t)X(x). \quad (5.11)$$

Substituting equation (5.11) in equation (5.10) with $q=0$ results in the eigen values for the X-problem as

$$\lambda_n = \left(\frac{(2n+1)\pi}{2L} \right)^2, \quad n = 0, 1, 2, 3, \dots$$

The eigen functions of the X- problem are:

$$X_n \approx \sin \frac{(2n+1)\pi x}{2L}.$$

Therefore, η can be written as

$$\eta = \sum_{n=1}^{\infty} v(z, t) \sin \frac{(2n+1)\pi x}{2L}, \quad (5.12)$$

where

$$v(z, t) = \frac{2}{L} \int_0^L \eta(x, z, t) \sin \frac{(2n+1)\pi x}{2L} dx. \quad (5.13)$$

Differentiating equation (5.13) with respect to t and substituting $\frac{\partial \eta}{\partial t}$ from equation (5.10),

we can come up with the v -problem as

$$\frac{\partial v}{\partial t} = \alpha \frac{\partial^2 v}{\partial z^2} - \alpha \lambda_n v + \tilde{q},$$

where

$$v(0, t) = \frac{2}{L} \int_0^L \eta(x, 0, t) \sin \frac{(2n+1)\pi x}{2L} dx = \frac{2}{L} \int_0^L (f_2(x) - a) \sin \frac{(2n+1)\pi x}{L} dx,$$

$$v_z(M, t) = \frac{2}{L} \int_0^L \eta_z(x, M, t) \sin \frac{(2n+1)\pi x}{2L} dx = \frac{2}{L} \int_0^L \left[f_4(x) - \frac{b}{M} \left(1 - \frac{x^2}{2L^2} \right) \right] \sin \frac{(2n+1)\pi x}{2L} dx,$$

$$v(z, 0) = v^0 = \frac{2}{L} \int_0^L \eta(x, z, 0) \sin \frac{(2n+1)\pi x}{2L} dx = \frac{2}{L} \int_0^L (\varphi^0 - \zeta(x, z)) \sin \frac{(2n+1)\pi x}{2L} dx,$$

$$\tilde{q} = \frac{2}{L} \int_0^L \left[q - \frac{\alpha b z}{L^2 M} \right] \sin \frac{(2n+1)\pi x}{2L} dx.$$

Based on the principle of superposition, let the solution for $v(z, t)$ be written as:

$$v(z, t) = u(z, t) + w(z),$$

where

$$w(z) = v(0, t) + \frac{z^2}{2M} v_z(M, t).$$

Using the above relationship, we can come up with the u-problem as

$$\frac{\partial u}{\partial t} = \alpha \frac{\partial^2 u}{\partial z^2} - \alpha \lambda_n u + \hat{q}, \quad (5.14)$$

where

$$u(0, t) = 0,$$

$$u_z(M, t) = 0,$$

$$u(z, 0) = v^0 - w(z),$$

$$\hat{q} = \tilde{q} - \alpha \lambda_n w(z).$$

As before, $u(z, t)$ for the homogeneous u-problem can be written as

$$u(z, t) = Z(z)T(t). \quad (5.15)$$

Substituting equation (5.15) in equation (5.14) with $\hat{q} = 0$ results in the eigen values for the Z-problem as

$$\lambda_m = \left(\frac{(2m+1)\pi}{2M} \right)^2, \quad m = 0, 1, 2, 3, \dots$$

The eigen functions of the Z- problem are:

$$Z_m \approx \sin \frac{(2m+1)\pi z}{2M}.$$

The solution for $u(z,t)$ therefore can be written as

$$u(z,t) = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} K_{mn}(t) \sin \frac{(2m+1)\pi z}{2M},$$

where

$$K_{mn}(t) = \frac{2}{M} \int_0^M u(z,t) \sin \frac{(2m+1)\pi z}{2M} dz.$$

Differentiating $K_{mn}(t)$ we obtain an ordinary differential equation for $K_{mn}(t)$ as

$$\frac{dK_{mn}(t)}{dt} + \alpha \lambda_{mn} K_{mn}(t) = q_{mn},$$

where

$$\lambda_{mn} = \lambda_m + \lambda_n,$$

$$q_{mn} = \frac{2}{M} \int_0^M \hat{q} \sin \frac{(2m+1)\pi z}{2M} dz,$$

The solution for $K_{mn}(t)$ is written as

$$K_{mn}(t) = \int_0^t e^{-\alpha \lambda_{mn}(t-\tau)} q_{mn}(z, \tau) d\tau + g e^{-\alpha \lambda_{mn} t},$$

where

$$g = K_{mn}(0) = \frac{2}{M} \int_0^M (v^0 - w(z)) \sin \frac{(2m+1)\pi z}{2M} dz.$$

Therefore, the solution of the original ϕ -problem is

$$\phi(x, z, t) = \sum_{n=0}^{\infty} \left[\sum_{m=0}^{\infty} K_{mn}(t) \sin \frac{(2m+1)\pi z}{2M} + w(z) \right] \sin \frac{(2n+1)\pi x}{2L} + \zeta(x, z), \quad (5.16)$$

where, $K_{mn}(t)$, $w(z)$, and $\zeta(x, z)$ are described earlier.

5.2 COMPARISON OF NUMERICAL AND ANALYTICAL SOLUTIONS

The finite element solutions are evaluated by comparing with the analytical solutions for a two-dimensional rectangular homogeneous domain with constant characteristics. The characteristics used for the homogeneous medium are given in Table 5.1. A uniform source term (value of q in equation 2.10) of 0.01 is applied in the domain. For finite element solution, the computational domain (15 ft in x -direction and 20-ft in z -direction) is discretized using 120 elements as shown in Figure 5.1.

The finite element and analytical solutions for both Dirichlet and mixed problems are given in Table 5.2 for selected nodes. The absolute values of the error and the relative errors are reasonably small. The finite element and analytical solutions are also compared at two nodal points as shown in Figures 5.2 through 5.5. The eigen values of the coefficient matrix for the Dirichlet problem computed using Jacobi method are

Table 5.1. Values of the parameters of the homogeneous domain

Parameter	Value
Hydraulic conductivity in x-direction (ft/day)	1
Hydraulic conductivity in z-direction (ft/day)	0.5
Saturated moisture content	0.4
Residual moisture content	0.1
Parameter n	2
Parameter m	0.5
Specific storage	0.025

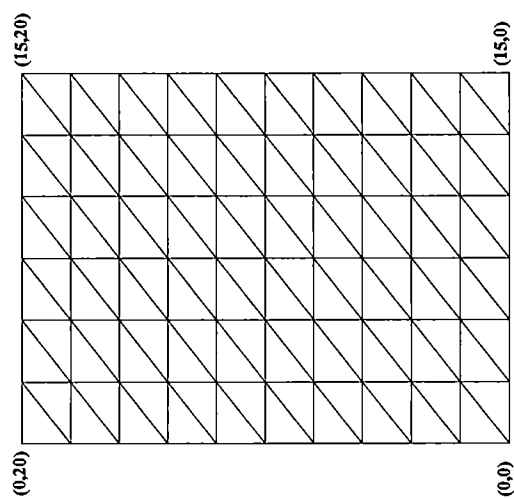


Figure S.1. Finite Elements in the Computational Domain.

Table 5.2. Comparison of finite element and analytical solutions

Time	Dirichlet Problem						Mixed Problem					
	Solution at $(x,z)=(5,6)$			Solution at $(x,z)=(7,5,6)$			Solution at $(x,z)=(5,6)$			Solution at $(x,z)=(7,5,6)$		
	Numerical	Analytical	Absolute Error	Relative Error (%)	Numerical	Analytical	Numerical	Analytical	Absolute Error	Relative Error (%)	Numerical	Analytical
0	15.67	15.67	0	0	16	15.67	15.67	15.67	0	0	16	15.67
1	15.39	15.3	0.09	0.59	15.43	15.36	15.3	15.22	0.08	0.53	15.35	15.36
2	15.37	15.37	0	0	15.39	15.4	15.25	15.24	0.01	0.06	15.27	15.31
3	15.43	15.47	0.04	0.26	15.45	15.49	15.29	15.32	0.03	0.20	15.3	15.35
4	15.53	15.57	0.04	0.26	15.54	15.59	15.37	15.41	0.04	0.26	15.36	15.35
5	15.62	15.67	0.05	0.32	15.64	15.69	15.45	15.5	0.05	0.32	15.44	15.51
6	15.72	15.77	0.05	0.32	15.74	15.79	15.55	15.6	0.05	0.32	15.53	15.6
7	15.82	15.87	0.05	0.32	15.84	15.89	15.74	15.8	0.06	0.38	15.63	15.7
8	15.92	15.96	0.04	0.25	15.94	15.99	15.84	15.8	0.06	0.38	15.72	15.8
9	16.02	16.06	0.04	0.25	16.03	16.08	15.93	15.89	0.06	0.38	15.81	15.89
10	16.11	16.15	0.04	0.25	16.12	16.18	15.92	15.98	0.06	0.38	15.91	15.98
11	16.2	16.25	0.05	0.31	16.22	16.27	16.02	16.08	0.06	0.37	16	16.08
12	16.27	16.3	0.03	0.18	16.28	16.32	16.09	16.13	0.04	0.25	16.07	16.13
13	16.32	16.34	0.02	0.12	16.33	16.36	16.14	16.17	0.03	0.19	16.13	16.17
14	16.36	16.38	0.02	0.12	16.37	16.4	16.18	16.21	0.03	0.19	16.18	16.21
15	16.4	16.42	0.02	0.12	16.41	16.44	16.23	16.25	0.02	0.12	16.22	16.25
16	16.41	16.42	0.01	0.06	16.43	16.44	16.24	16.25	0.01	0.06	16.24	16.25
17	16.41	16.42	0.01	0.06	16.43	16.44	16.25	16.25	0	0	16.25	16.25
18	16.41	16.42	0.01	0.06	16.43	16.44	16.26	16.25	0.01	0.06	16.26	16.25
19	16.41	16.41	0	0	16.43	16.44	16.25	16.24	0.01	0.06	16.26	16.24
20	16.38	16.38	0	0	16.4	16.4	16.23	16.21	0.02	0.12	16.24	16.21
21	16.36	16.35	0.01	0.06	16.37	16.37	16.21	16.18	0.03	0.19	16.21	16.18
22	16.32	16.31	0.01	0.06	16.34	16.34	16.18	16.15	0.03	0.19	16.19	16.15
23	16.3	16.3	0	0	16.32	16.32	16.16	16.13	0.03	0.19	16.17	16.13
24	16.3	16.3	0	0	16.31	16.32	16.15	16.13	0.02	0.12	16.16	16.13
25	16.29	16.3	0.01	0.06	16.31	16.32	16.14	16.13	0.01	0.06	16.15	16.13
26	16.29	16.29	0	0	16.31	16.32	16.14	16.13	0.01	0.06	16.15	16.13
27	16.29	16.29	0	0	16.31	16.32	16.14	16.13	0.01	0.06	16.14	16.13
28	16.29	16.29	0	0	16.31	16.32	16.14	16.12	0.02	0.12	16.14	16.12
29	16.29	16.29	0	0	16.31	16.32	16.13	16.12	0.01	0.06	16.14	16.12
30	16.29	16.29	0	0	16.3	16.32	16.13	16.12	0.01	0.06	16.14	16.12

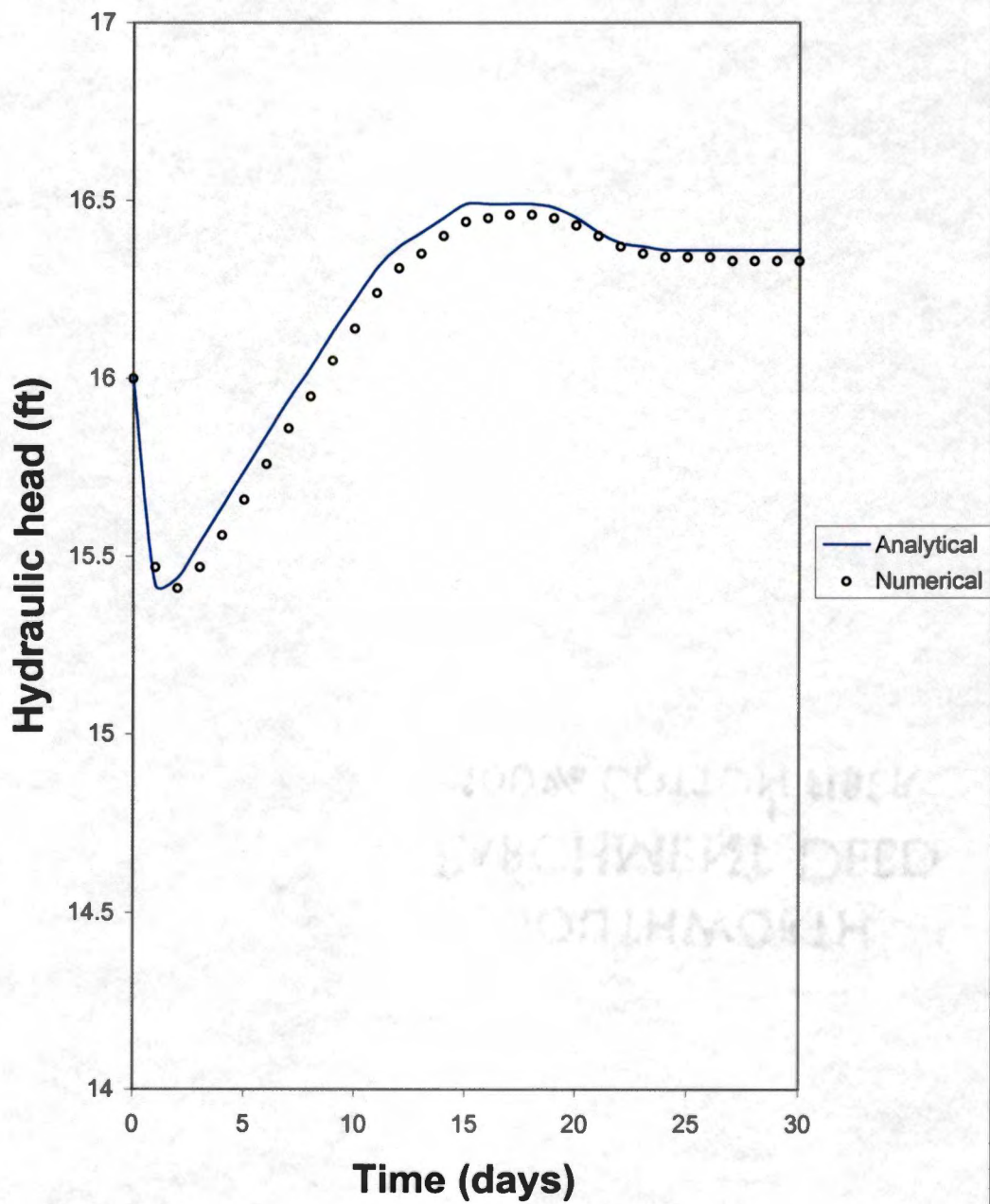


Figure 5.2. Finite element and analytical solutions at $(x,z) = (7.5,10)$ for Dirichlet problem

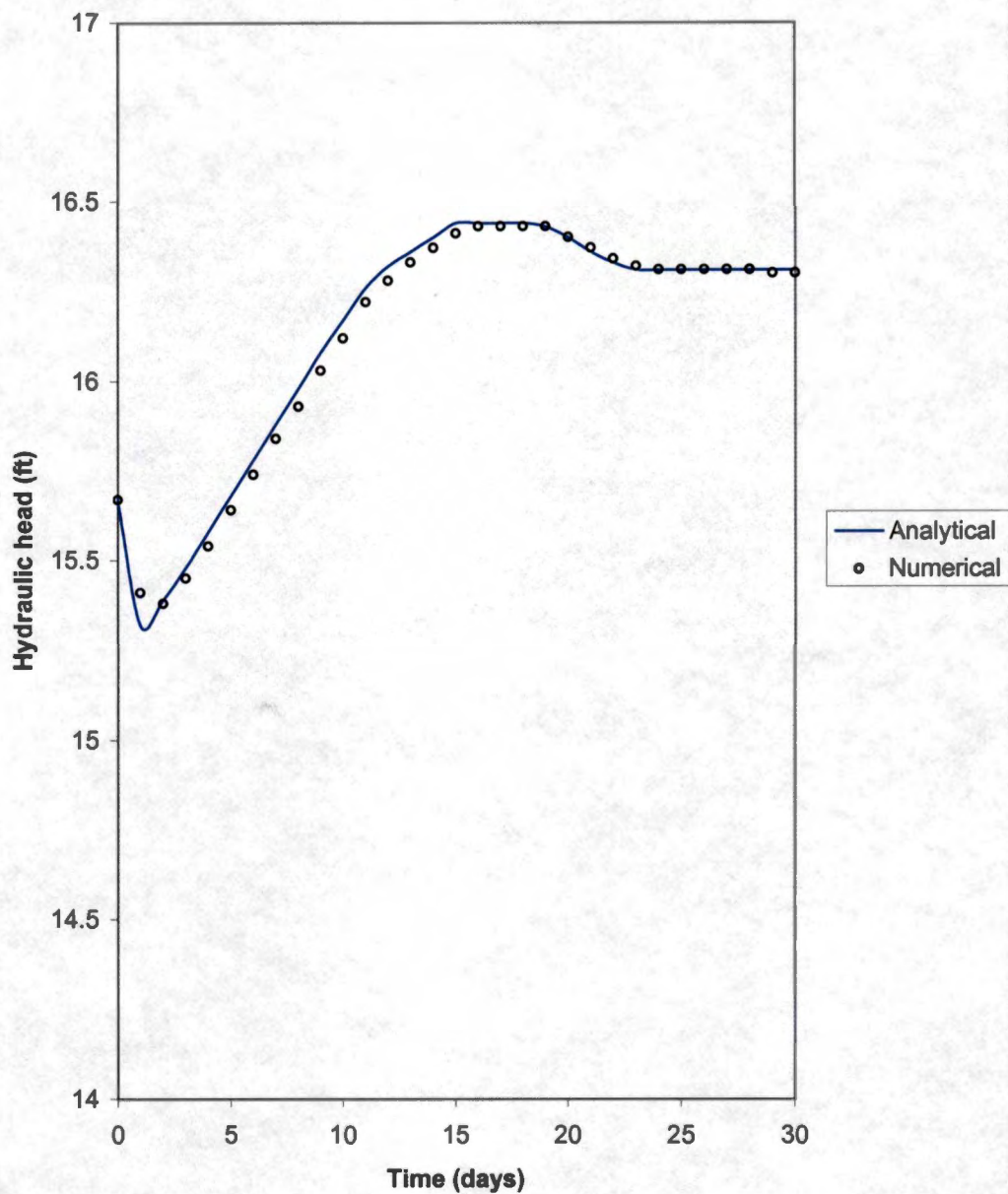


Figure 5.3. Finite element and analytical solutions at $(x,z) = (5,12)$ for Dirichlet Problem

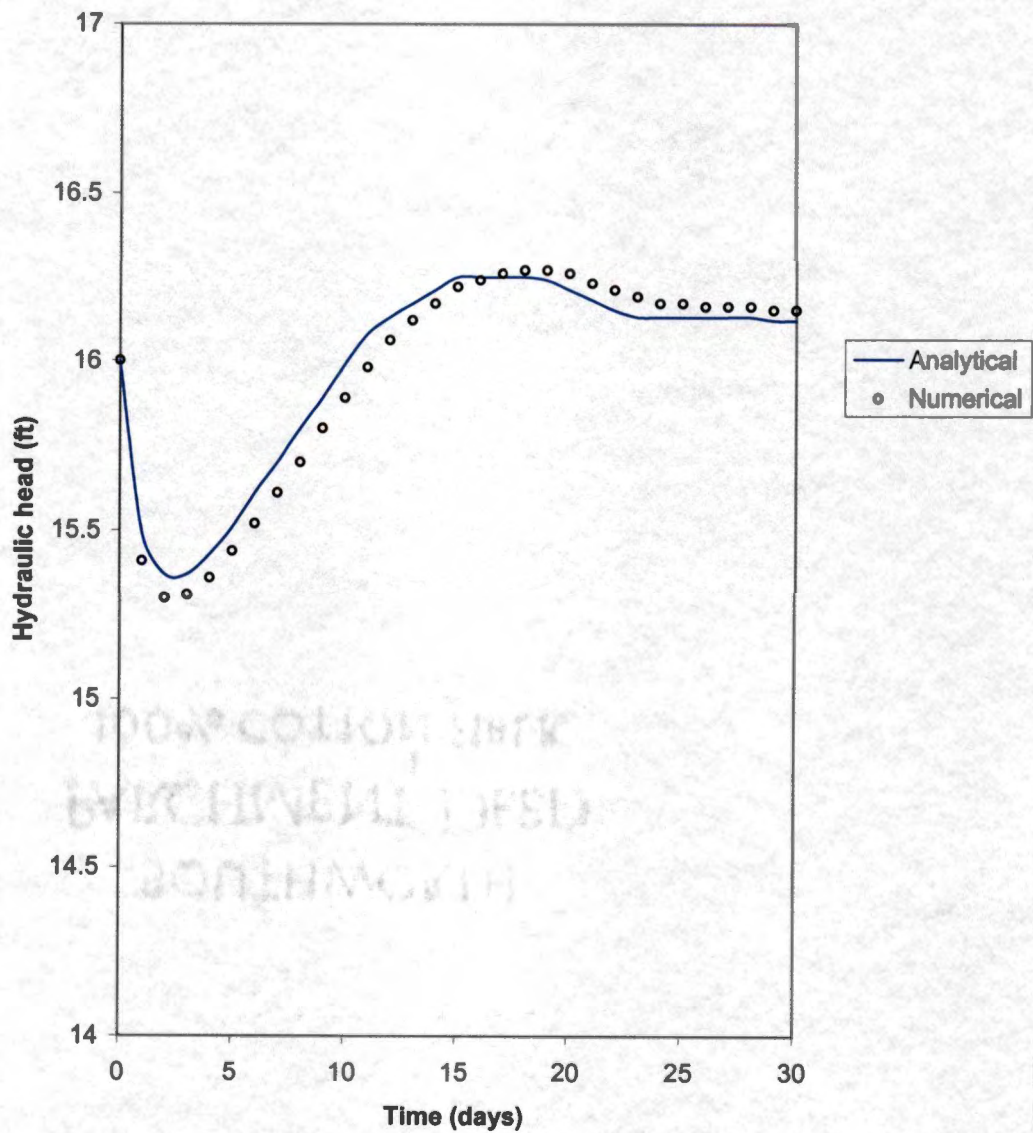


Figure 5.4. Finite element and analytical solutions at $(x,z) = (7.5,10)$ for Mixed problem

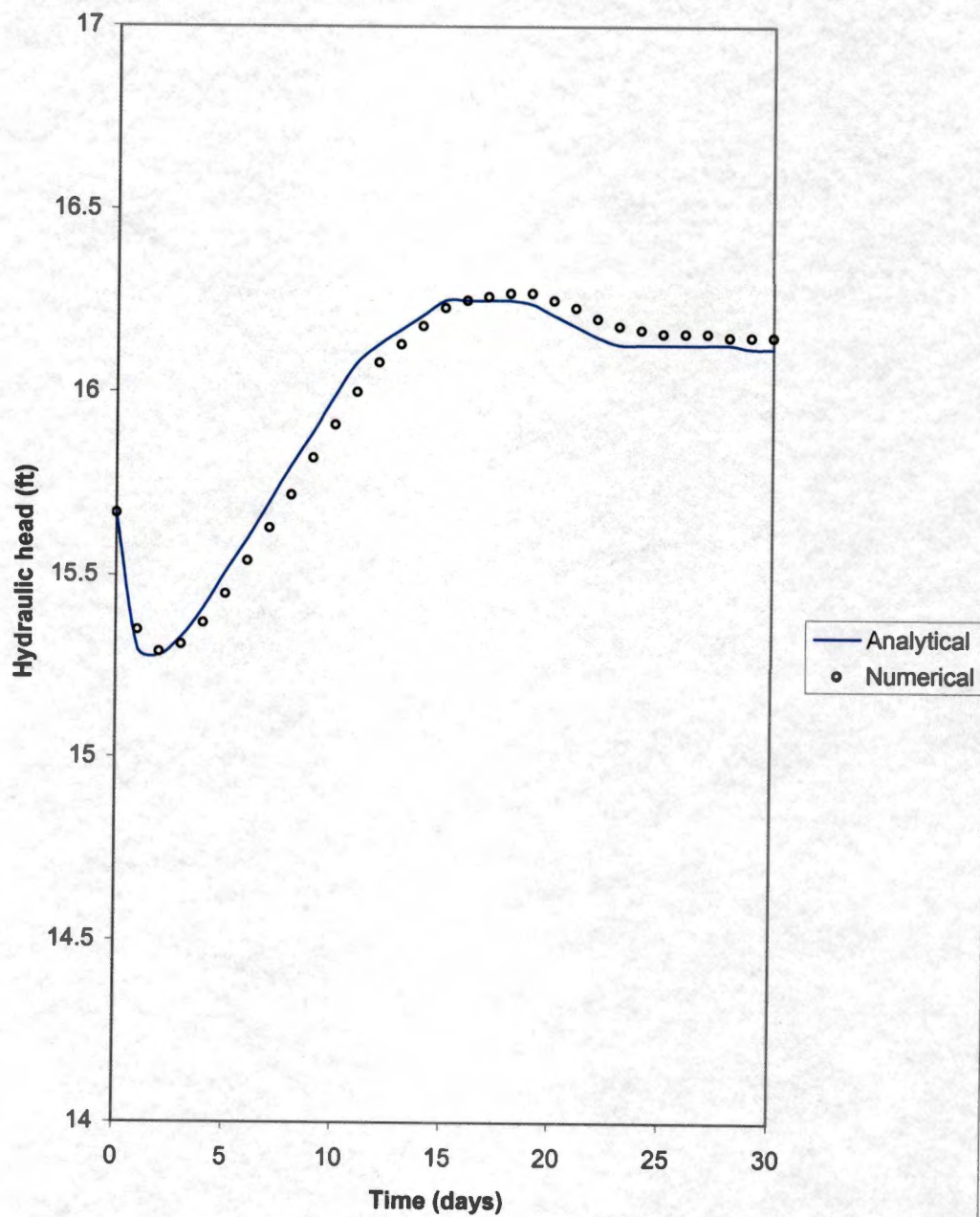


Figure 5.5. Finite element and analytical solutions at $(x,z) = (5,12)$ for Mixed problem

positive and are given in Table 5.3. The positive values of the eigenvalues indicate the symmetric positive definite system that is solved. The maximum eigenvalue is 7.91 and the minimum eigenvalue is 0.454. The condition number is 17.4.

5.3 PHREATIC SURFACE PROBLEM

The phreatic surface is located using equation (3.25) based on the solution of the parabolic equation. Two phreatic surface problems were investigated. The first problem consists of a homogeneous domain with initial location of the surface as shown in Figure 5.6. The properties used in the homogeneous domain are the same as those shown in Table 5.1. The second problem consists of a heterogeneous domain with initial location of the surface as shown in Figure 5.7. The properties used in the heterogeneous domain are given in Table 5.4. The source terms used in the computation are 0.05 for the homogeneous domain and 0.025 for the heterogeneous domain. The time-varying locations of the phreatic surfaces for both the problems are shown in Figures 5.8 and 5.9.

Table 5.3. Eigen values of the coefficient matrix

Node	Eigen Value	Node	Eigen Value
1	1	40	3.49
2	1	41	1.92
3	1	42	1
4	1	43	1
5	1	44	4.93
6	1	45	6.3
7	1	46	3.56
8	1	47	5.72
9	4.16	48	4.13
10	5.53	49	1
11	2.8	50	1
12	4.95	51	3.4
13	3.37	52	4.76
14	1	53	2.03
15	1	54	4.19
16	6.52	55	2.61
17	7.91	56	1
18	5.14	57	1
19	7.32	58	6.17
20	5.72	59	7.55
21	1	60	4.79
22	1	61	6.97
23	1.8	62	5.37
24	3.16	63	1
25	0.454	64	1
26	2.58	65	2.16
27	1.03	66	3.51
28	1	67	0.804
29	1	68	2.94
30	5.62	69	1.38
31	7	70	1
32	4.24	71	1
33	6.42	72	1
34	4.82	73	1
35	1	74	1
36	1	75	1
37	2.71	76	1
38	4.06	77	1
39	1.35		

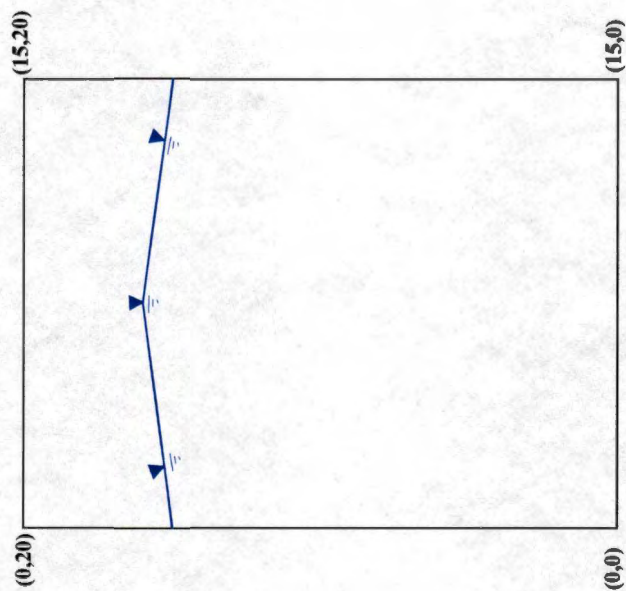


Figure 5.6. Phreatic Surface in a Homogeneous Domain.

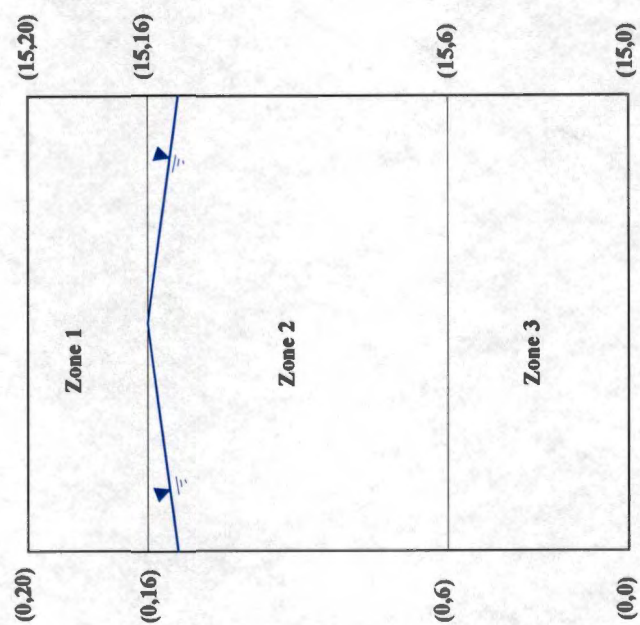


Figure 5.7. Phreatic Surface in a Heterogeneous Domain.

Table 5.4. Values of the parameters of the heterogeneous domain

Parameter	Value in zone 1	Value in zone 2	Value in zone 3
Hydraulic conductivity in x-direction (ft/day)	2	1	0.5
Hydraulic conductivity in z-direction (ft/day)	1	0.5	0.25
Saturated moisture content	0.45	0.4	0.43
Residual moisture content	0.025	0.1	0.15
Parameter n	10	2	1.5
Parameter m	0.9	0.5	0.34
Specific storage	0.025	0.05	0.1

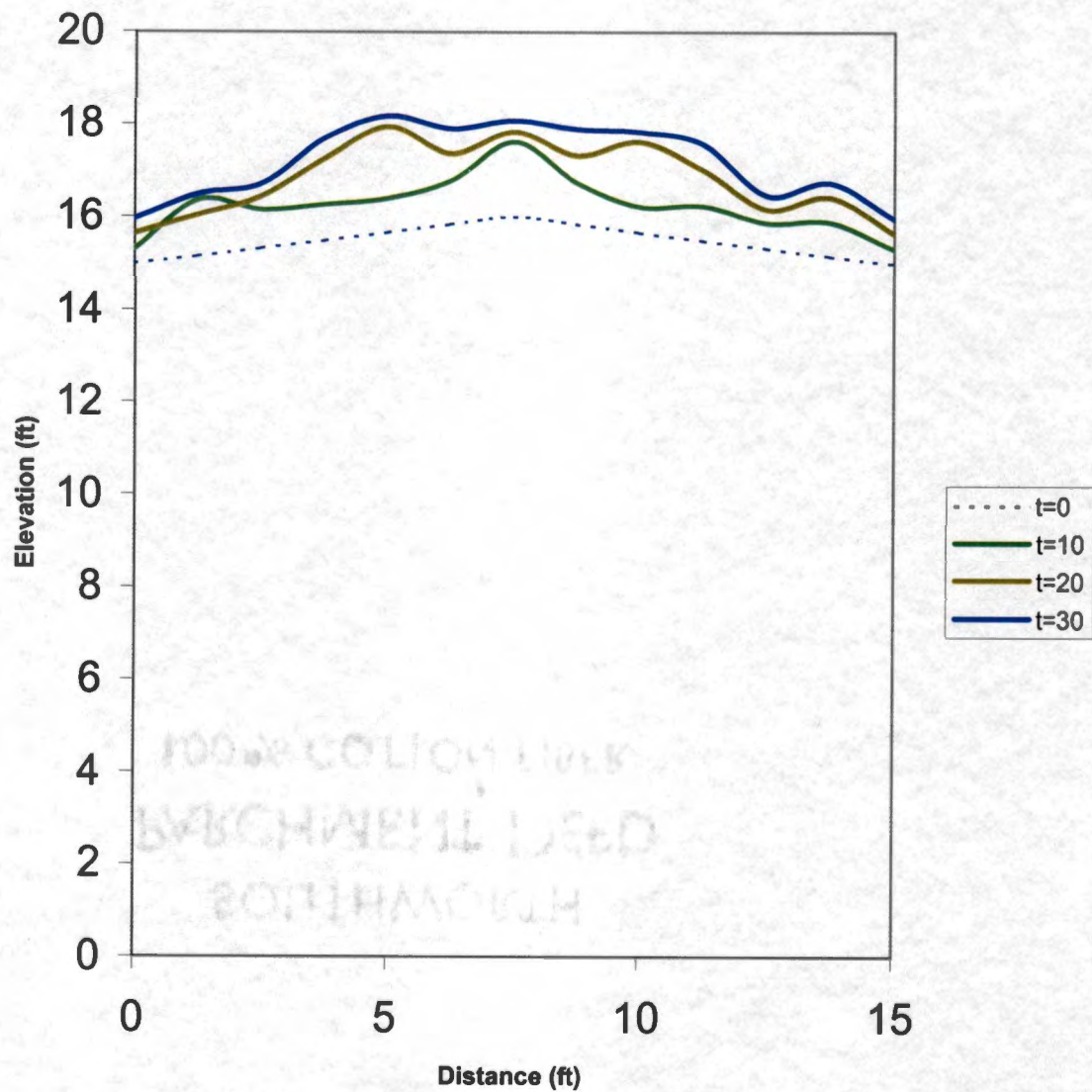


Figure 5.8. Variation of phreatic surface in the homogeneous domain

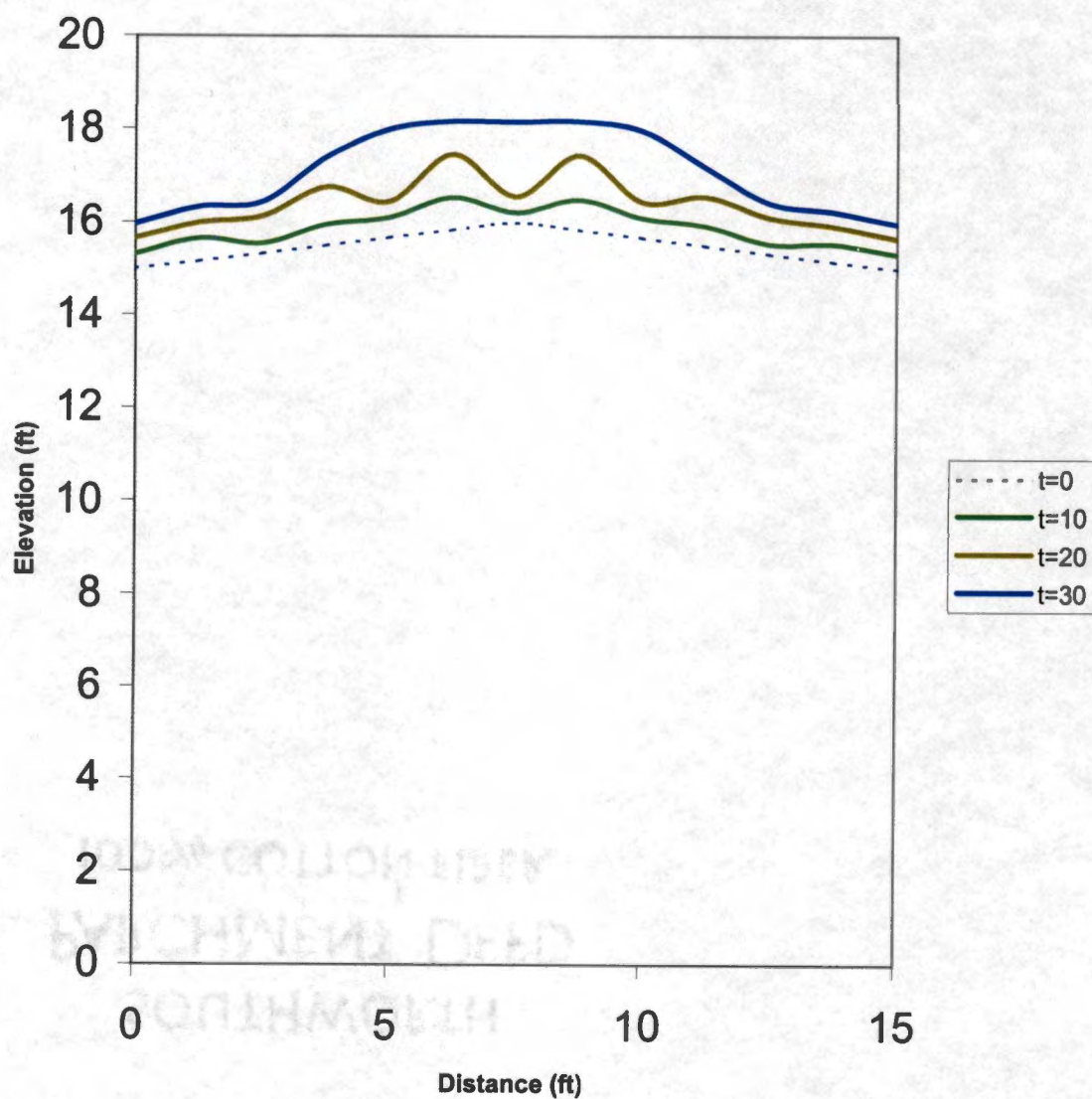


Figure 5.9. Variation of phreatic surface in the heterogeneous domain

6. SUMMARY

In this study, variational formulations were developed for finite element solutions of a parabolic equation describing variably saturated flow. The classical backward Euler method was used to obtain a semi-discrete variational problem which was transformed into a computer program for generating time-varying solutions. A symmetric positive definite system of equations was obtained and solved for the unknowns. The finite element solutions were compared with the analytical solutions. The numerical solutions agreed reasonably with the analytical solutions. The phreatic surface separating unsaturated and saturated zones was simulated for domains with homogeneity and heterogeneity. The movement of the phreatic surface was consistent based on the solutions of the parabolic equation.

The boundary element formulation of the parabolic equation was investigated. The integral representation was obtained. The finite element formulation was developed for the integral equations. The integrals were evaluated based on the test functions. The system of equations was obtained and analyzed for solving the unknown boundary conditions, which can be used to generate solutions of the parabolic partial differential equation in the domain.

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VITA

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