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To the Graduate Council:

I am submitting herewith a thesis written by Tzu-Pang Li entitled "Optimal Attitude Control of Space Vehicle to Minimize Fuel." I recommend that it be accepted in partial fulfillment of the requirements for the degree of Master of Science, with a major in Electrical Engineering.

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OPTIMAL ATTITUDE CONTROL OF SPACE VEHICLE TO MINIMIZE FUEL

A Thesis

Presented for the

Master of Science

Degree

The University of Tennessee

Tzu-Pang Li

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ABSTRACT

This thesis discusses the fuel optimization of reaction jets when used for the attitude control of a space vehicle. A closed-loop control system with a limiter and on-off relay controller is used for one-axis attitude control. A performance index is chosen which is directly related to the fuel consumed. Since the steepest descent method is not directly applicable to the minimization of this performance index, the calculus of variations is first used to estimate the changes in the performance index with small changes in the system parameters. Then the steepest descent algorithm is used to repeatedly calculate the small changes which reduce the performance index, finally resulting in a minimum for the performance index. A numerical example is presented to illustrate this technique.

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CHAPTER I

INTRODUCTION

Prior to 1940 the design of a feedback control system was carried out by trial and error procedures. The tremendous progress in automatic control is mainly due to World War II. Due to the increased complexity of the controlled system and the interest in achieving optimum performance, analytical methods became increasingly important in control system design (1).[1] With the advent of the space age, the aim of modern control system theory has become the design of an optimal system.

Most space vehicles in earth orbit must maintain fixed angular velocity and position (as a reference attitude) for conducting experiments. When a deviation from this reference occurs, it is necessary to realign the space vehicle. Because of its simplicity, light weight, and low cost, a reaction jet control system is generally used (2). One disadvantage of such a control system is that it is nonlinear; the mathematical treatment of such a nonlinear system is extremely difficult (3). In the past, the parameters of this reaction jet control system have been chosen by empirical methods. Recently, successive approximation

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[1] Numbers in parentheses refer to similarly numbered references in the bibliography.

methods have been developed for determining an optimum attitude control system (4).

For a reaction jet control system, the forces used to correct the attitude error come from the burning of fuel. Since a space vehicle fuel capacity is limited, fuel consumption should be minimized. Therefore, it is desirable to have a closed-loop control system with parameters which minimize fuel consumption; currently this system includes a limiter and on-off relay (5, 6). In this type of attitude control angular velocity and position are described by a second order differential equation constrained by rate limiting and linear switching lines with a dead zone. The rate limiting is used to improve fuel consumption while keeping the attitude error recovery time within an acceptable limit. Linear switching lines with a dead zone prevents large limit cycles from occurring and also minimizes the fuel consumption (6).

The above considerations can be used to define a performance index which gives a direct indication of the fuel consumed. Therefore, the main purpose of this study will be to show in detail how this performance index can be minimized by selecting the parameters of the control system. The steepest descent method is used to set up an iteration from which the optimal parameters can be obtained.

A numerical optimization of this type attitude control has been carried out on a digital computer. The

computer program is given in the last chapter. The results of this computer algorithm indicate that this method can be used to find the optimal parameters efficiently, and that it is more dependable than a trial-and-error method.

CHAPTER II

OPTIMIZATION OF THE ATTITUDE CONTROL OF A SPACE VEHICLE

I. ATTITUDE CONTROL SYSTEM

Because of its simplicity and low cost a closed loop control system as shown in Figure 1, is generally used for space vehicle control (2). The parameters of this system are usually selected by considering the stability and maximum allowable attitude error of the space vehicle. However, much less attention has been given the problem of fuel consumption.

Since fuel capacity is an important factor in space flights, it is necessary to determine the parameters of the control system which minimize its use. The present control system as shown in Figure 1, consists of a forward path and two feedback paths. The forward path contains a limiter and an on-off relay. The angular position (X_1) and velocity (X_2) of the vehicle will be measured and considered as state variables. The signal, here, will be the difference between the measured state variables and the input reference of the angular position or velocity.

For actual use, an adaptive controller (an on-board computer with electronic circuits) is also included in the control system. This controller is used to adjust the

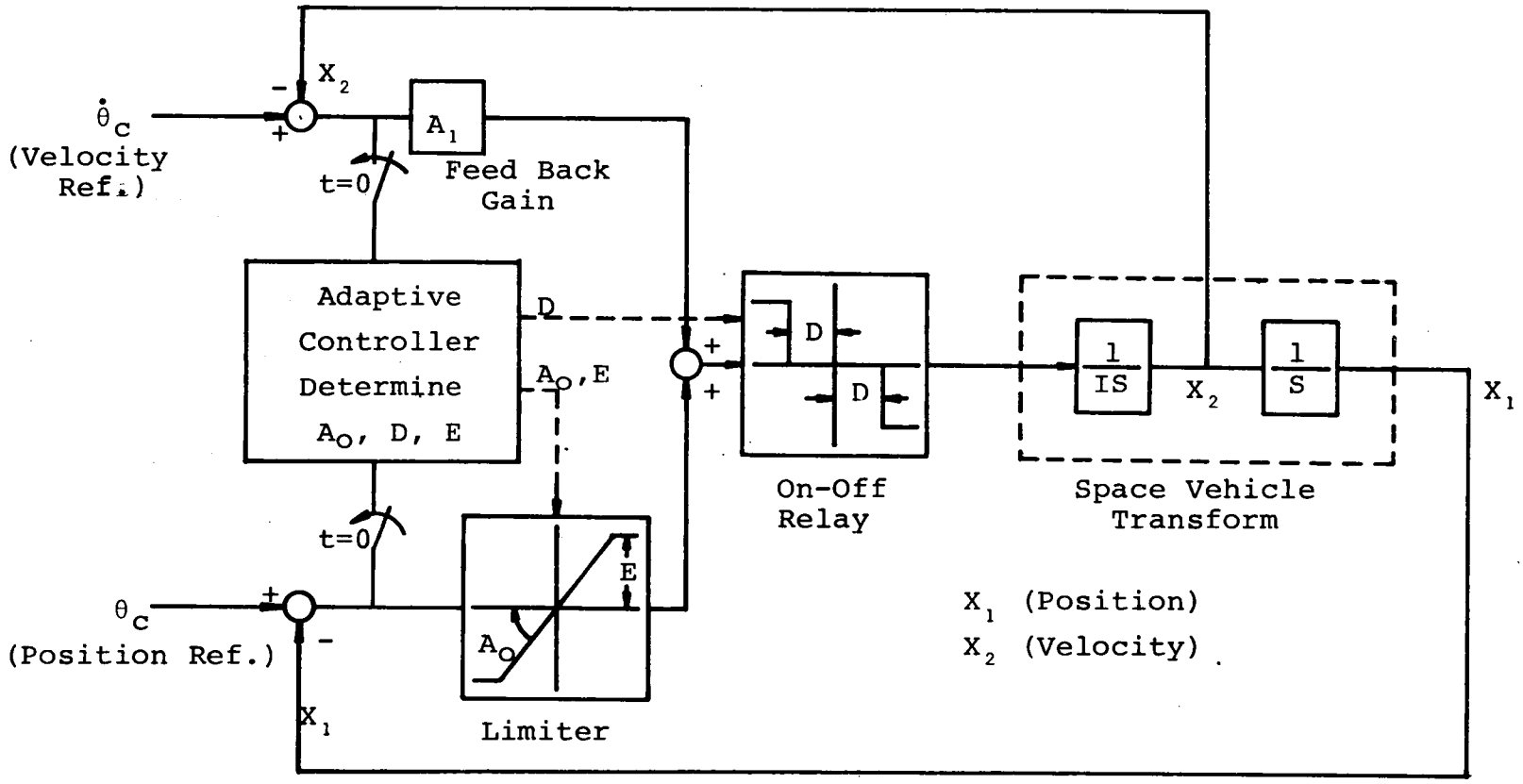


Figure 1. The present control system with an adaptive controller.

parameters for a particular attitude error. The details of this adaptive controller are not included in this study.

The given problem then is to determine: (1) the slope A_o, (2) the saturation limit, E, of the limiter, and (3) the dead zone, D, of the on-off relay, such that a minimum amount of fuel will be used to correct a given attitude error. The determination of the system parameters is an optimization problem.

II. SWITCHING BOUNDARIES AND TRAJECTORY PHASE PLANE

Consider again the closed loop control system shown in Figure 1. This system, with torques produced by gas jets only, can be described by the following equations:

$$-A_o X_1 - A_1 X_2 \pm D = 0 \tag{2-1a}$$

$$-A_1 X_2 \pm E \pm D = 0 \tag{2-1b}$$

$$\dot{X}_2 = \frac{M}{I} \text{ sat. } (X_1, X_2) \tag{2-1c}$$

where

A_o = slope of the limiter

A_1 = feedback constant of angular velocity

D = dead zone of on-off relay

E = rate limiting of limiter

X_1 = Angular position of space vehicle

X_2 = Angular velocity of space vehicle

M = Moment or torque of space vehicle produced by gas jets

I = Moment of inertia of space vehicle

Sat.(X_1,X_2) is the control state sequence

It is evident that Equations 2-1a and 2-1b will determine the switching lines in the X_1 and X_2 trajectory phase plane (shown in Figure 2). The phase plane is divided into three areas--Region I, Region II, and Region III. If the state is in Region I, then Sat.$(X_1,X_2) = -1$ is used; if the state is in Region II, then Sat.$(X_1,X_2) = 0$ is used; if the state is in Region III, then Sat.$(X_1,X_2) = +1$ is used. The three regions can be defined mathematically by the following conditions.

1. Region I (Sat.$(X_1,X_2) = -1$):

$X_2 \geq (E + D)/A_1$

$-(E - D)/A_1 \leq X_2 \leq (E + D)/A_1$ and $A_1X_2 + A_oX_1 - D \geq 0$

2. Region III (Sat.$(X_1,X_2) = +1$):

$X_2 \leq -(E + D)/A_1$

$-(E + D)/A_1 \leq X_2 \leq (E - D)/A_1$ and $A_1X_2 + A_oX_1 + D \leq 0$

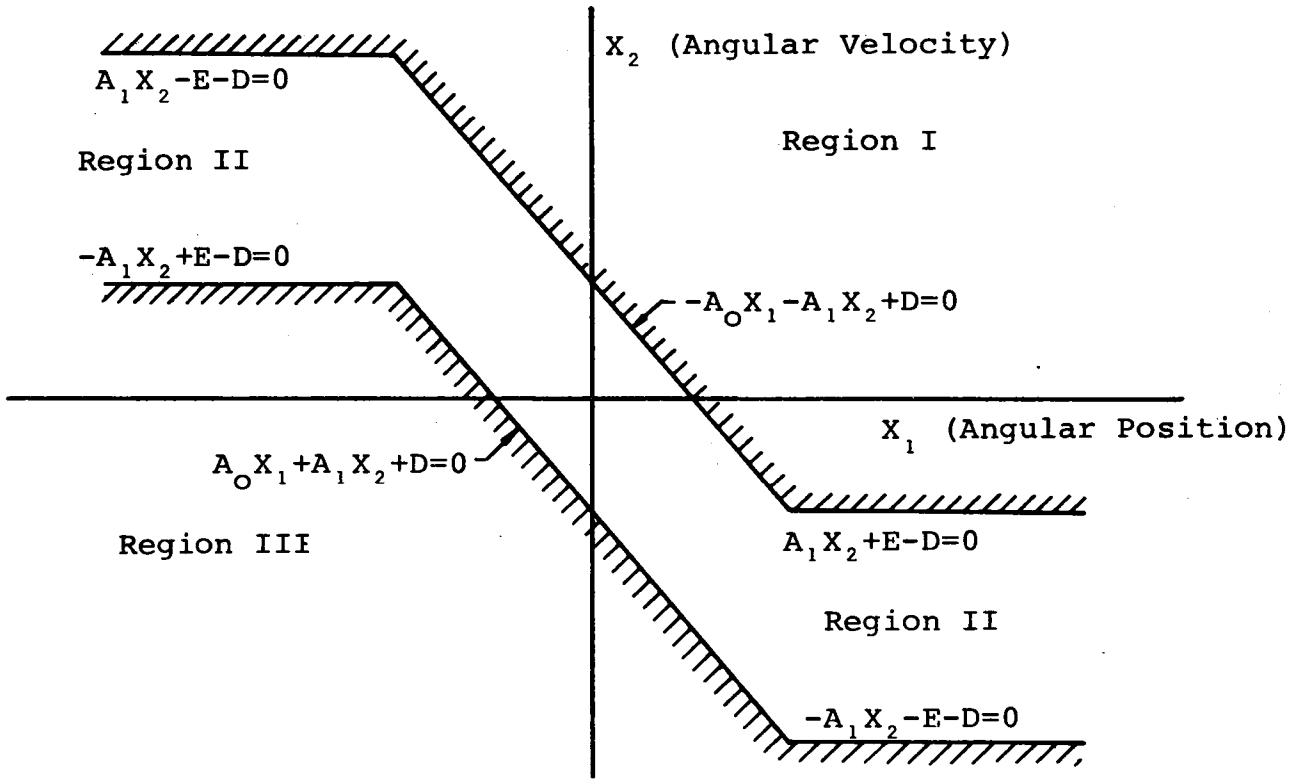
3. Region II (Sat.$(X_1,X_2) = 0$):

otherwise.

If a step function is introduced, the above equations can be rewritten in the compact form:

$$\text{Sat.}(X_1,X_2) = (U_1 - U_2)$$

where



Region I: Negative moment produced by the gas jets.

Region II: Zero moment produced by the gas jets.

Region III: Positive moment produced by the gas jets.

Figure 2. The switching line of the present control system.

$$U_1 = h[A_1(-aX_1 - X_2 - d)]\, h[-X_2 + e - d] + h[A_1(aX_1 + X_2 + d)]\, h[-X_2 - e - d]$$

$$U_2 = h[A_1(+aX_1 + X_2 - d)]\, h[+X_2 + e - d] + h[A_1(-aX_1 - X_2 + d)]\, h[+X_2 - e - d]$$

and

$a = A_o/A_1$

$d = D/A_1$

$e = E/A_1$

$h[\cdot]$ = step function

The state equation of this system becomes

$$\dot{X}_1 = X_2$$

$$\dot{X}_2 = \frac{M}{I}(U_1 - U_2)$$

$$\underline{X}(t_o) = \underline{X}_o \qquad (2\text{-}2)$$

III. PERFORMANCE INDEX

For this system, a performance index must be chosen which relates directly to the fuel consumption of the space vehicle (1, 3). The system will be considered optimal when the parameters are adjusted so that this index reaches a minimum. For the system being considered here the following form is used,

$$\mathcal{J} = \int_o^{t_f} [Q + \frac{M}{I}(U_1 + U_2)]dt + |X_2(t_f)| \qquad (2\text{-}3)$$

where

Q = weighting factor for the error recovery time

t_f = error recovery time

Consider Figure 3 where $a(X_{10}, X_{20})$ is the given initial point. Because of the state Equations 2-2, X_1 versus X_2 will follow a parabolic trajectory (Chapter IV) and intersect the switching lines first at point $b(X_{11}, X_{21})$. The time required to go from $a(X_{10}, X_{20})$ to $b(X_{11}, X_{21})$ will be called T_{ab}, during which the moment produced by the gas jets is negative (or control state sequence is minus one). Similarly, the time required from $b(X_{11}, X_{21})$ to $c(X_{12}, X_{22})$ is T_{bc} (zero moment). The time required from $c(X_{12}, X_{22})$ to $d(X_{13}, X_{23})$ is T_{cd} (positive moment), and so on. The "error recovery time" ($t_f = T_{ab} + T_{bc} + T_{cd} + ---$) then is defined as the total time elapsed until the trajectory is finally restricted to lie entirely within the dead zone.

The above final criterion can be obtained by considering Figure 3. If the trajectory $[\frac{2M}{I}(X_1^2 - \frac{d}{a}) = X_2^2]$ passes through the point $(\frac{d}{a}, 0)$, it will meet the switching line $(-aX_1 - X_2 + d = 0)$ at the point $(X_2 = 2M/Ia)$; the midpoint $(X_2 = M/Ia)$ will be the least upper bound for X_2 at $t = t_f$. Similarly, $X_2 = -M/Ia$ is the largest lower bound. This means that either

$+aX_1 + X_2 + d = 0$ and $0 \geq X_2 \geq -M/Ia$

or

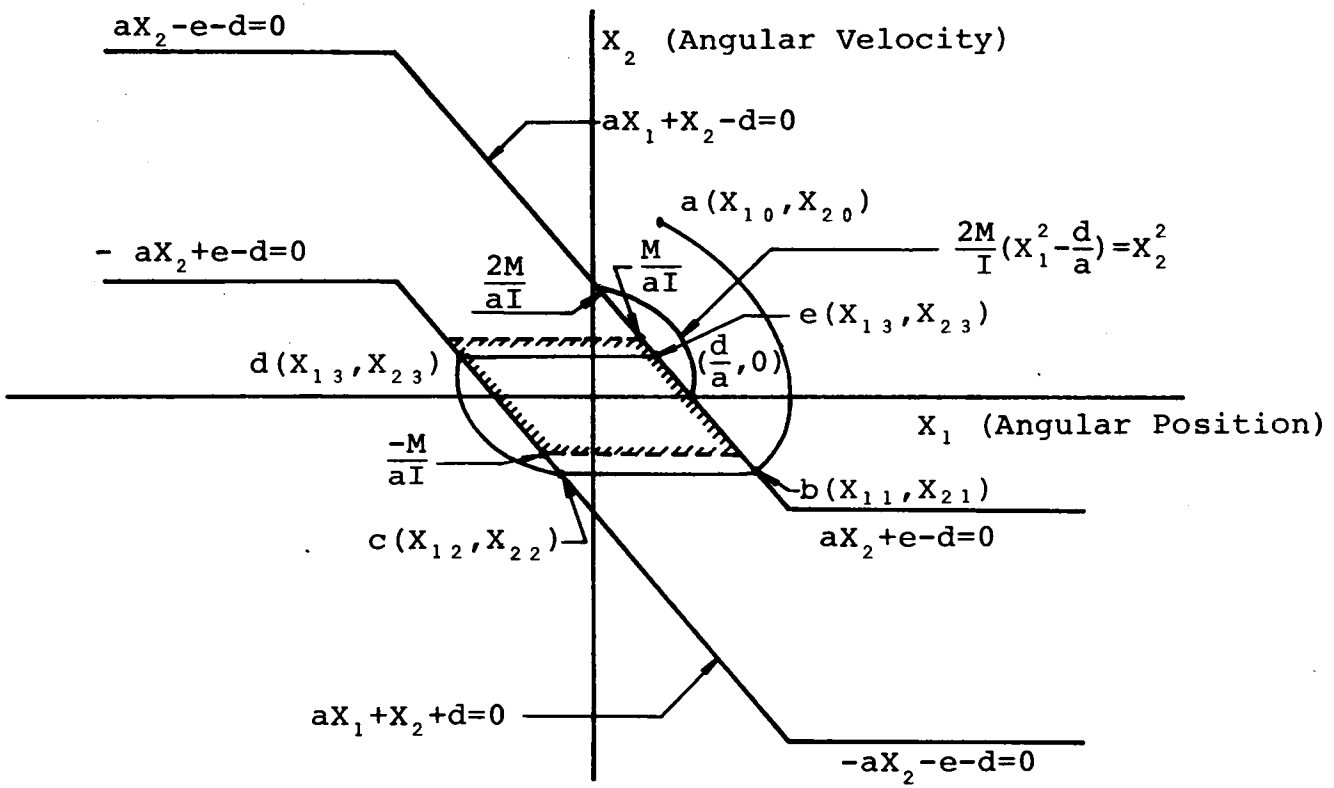


Figure 3. The procedure of error recovery time and final criterion of present control system.

$-aX_1 - X_2 + d = 0$ and $0 \leq X_2 \leq M/Ia$

These conditions define a region (shadow in Figure 3) in the dead zone; if the trajectory intersects the switching line in this region, the trajectory will not go outside the region (because the slope of the trajectory is smaller than the slope of the switching lines). Instead, the trajectory will go toward $X_2 = 0$ and end with a steady-state limit cycle (5, 6).

From these considerations we choose to include three factors in the performance index:

1. Q--since the error correction should occur within a reasonable time for a realistic mission, t_f should be included. However, since time, t_f, is not fixed, it is weighted by Q (7).

2. $\frac{M}{I}(U_1 + U_2)$--this is the total moment expended by the gas jets and is directly related to the fuel consumed during the error recovery time.

3. $X_2(t_f)$--the system must ultimately reach the steady state limit cycle. Thus it is necessary to consider the fuel consumed from the final point to the limit cycle. For simplicity, the absolute value of this angular velocity at t_f is considered to be proportional to this consumption.

Because of hardware limitations, inequality constraints must also be imposed on the physical parameters a, d, and e:

$\alpha \leq a,$

$\Delta_1 \leq d \leq \Delta_2$, and

$\varepsilon_1 \leq e \leq \varepsilon_2$ (2-5)

The complete optimization problem can then be mathematically stated thus:

Determine a set of parameters, a, d, and e, subject to the state equations in Equation 2-2, the final constraints of Equation 2-4, and the inequality constraints of Equation 2-5, such that performance index in Equation 2-3 is minimized.

CHAPTER III

ALGORITHM FOR CONTROL SYSTEM OPTIMIZATION

I. DEVELOPMENT OF PERFORMANCE INDEX

In this section, the measurement of the state variables and their use in developing a control system is considered so that the performance of the control system is optimized.

By introducing slack variable (8, 9), the inequalities of Equation 2-4 become:

$$a - \alpha - X_3^2 = 0$$

$$(d - \Delta_1)(\Delta_2 - d) - X_4^2 = 0$$

$$(e - \varepsilon_1)(\varepsilon_2 - e) - X_5^2 = 0 \qquad (3\text{-}5)$$

where X_3, X_4 and X_5 are slack variables.

Since the system trajectory's final point will be at the switching line, the terminal constraints of Equation 2-3 can be specified as:

$$\psi = \pm aX_1 \pm X_2 + d = 0 \qquad (3\text{-}6)$$

After substituting Equations 3-5 and 3-6 in Equation 2-2, a new performance index can be formed:

$$J = \int_0^{t_f} (H - \underline{\lambda}^T \dot{\underline{X}})dt + \phi + v\psi \qquad (3\text{-}7)$$

where

$$H = Q + \frac{M}{I}(U_1 + U_2) + \underline{\lambda}^T \underline{f}$$

$$\psi = \pm aX_1 \pm X_2 + d$$

$$\phi = |X_2(t_f)|$$

V, $\underline{\lambda}$ = multipliers

Now, the optimization problem is to minimize Equation 3-7 subject to:

$$\dot{\underline{X}} = \underline{f}(\underline{X}, \underline{u}, t)$$

$$\underline{X}(t_o) = \underline{X}_o$$

where

$$\underline{X} = [X_1, X_2, X_2, X_3, X_4, X_5]^T$$

$$\underline{u} = [a, d, e]^T$$

$$\underline{f} = \begin{Bmatrix} X_2 \\ \frac{M}{I}(U_1 - U_2) \\ a - \alpha - X_3^2 \\ (d - \Delta_1)(\Delta_2 - d) - X_4^2 \\ (e - \varepsilon_1)(\varepsilon_2 - e) - X_5^2 \end{Bmatrix}$$

II. NECESSARY CONDITIONS FOR OPTIMIZATION

In order to minimize the performance index, consider the first variation of the performance index Equation 3-7 can be written:

$$dJ = \int_o^{t_f} \left(\frac{\partial H}{\partial \underline{X}} \delta \underline{X} + \frac{\partial H}{\partial \underline{u}} \delta \underline{u} - \underline{\lambda}^T \delta \dot{\underline{X}} \right) dt + (H - \underline{\lambda}^T \dot{\underline{X}}) \Big|_{t_f} dt_f$$

$$+ V \left(\frac{\partial \psi}{\partial \underline{X}} d\underline{X} + \frac{\partial \phi}{\partial \underline{u}} d\underline{u} + \frac{\partial \psi}{\partial t} dt \right) + d\phi$$

Rearranging the above equation, gives:

$$dJ = \left(\int_o^{t_f} \frac{\partial H}{\partial \underline{u}} dt + V \frac{\partial \psi}{\partial \underline{u}} + \frac{\partial \phi}{\partial \underline{u}} \right) d\underline{u} + \int_{t_o}^{t_f} \left(\dot{\underline{\lambda}}^T + \frac{\partial H}{\partial \underline{X}} \right) \delta \underline{X} \, dt$$

$$+ \left(V \frac{\partial \psi}{\partial \underline{X}} + \frac{\partial \phi}{\partial \underline{X}} - \underline{\lambda}^T \right) d\underline{X}$$

$$+ \left(V \frac{\partial \psi}{\partial t} + \frac{\partial \phi}{\partial t} + \underline{\lambda}^T \dot{\underline{X}} + Q + \frac{M}{I}(U_1 + U_2) \right) dt$$

A necessary condition for an extremum of the performance index is given by the following conditions (10, 11, 12):

$$\dot{\underline{\lambda}}^T + \frac{\partial H}{\partial \underline{X}} = 0 \qquad (3\text{-}8)$$

$$\int_o^{t_f} \frac{\partial H}{\partial \underline{u}} \, dt + \left(V \frac{\partial \psi}{\partial \underline{u}} + \frac{\partial \phi}{\partial \underline{u}} \right) = 0 \qquad (3\text{-}9)$$

with the final condition:

$$V \frac{\partial \psi}{\partial \underline{X}} + \frac{\partial \phi}{\partial \underline{X}} - \underline{\lambda}^T = 0$$

$$V\frac{\partial\psi}{\partial t} + \frac{\partial\phi}{\partial t} + \underline{\lambda}^T \dot{\underline{X}} + Q + \frac{M}{I}(U_1 + U_2) = 0$$

The final condition can be expressed explicitly as:

$$\lambda_1 = -a \left. \frac{Q + \frac{M}{I}[(U_1 + U_2) + (U_1 - U_2)\ \mathrm{sgn}(X_2)]}{aX_2 + \frac{M}{I}(U_1 - U_2)} \right|_{t_f}$$

$$\lambda_2 = \left. \frac{\lambda_1 + a\ \mathrm{sgn}(X_2)}{a} \right|_{t_f} \qquad (3\text{-}10)$$

Since Equation 3-9 is nonlinear, no analytical solution can be obtained for this problem. Successive approximations will be used in the next section to obtain a numerical solution of the problem.

III. ALGORITHM FOR SUCCESSIVE APPROXIMATION

The change of performance index, dJ, due to a small variation of $d\underline{u}$, can be estimated (as shown in Appendix A) by finding the $\underline{\lambda}(t)$'s from Equation 3-8 (as shown in Appendix B). The change of performance index is given by:

$$dJ = \Delta\underline{H}^T\, d\underline{u}$$

where

$$\Delta H^T = \frac{\partial\phi}{\partial\underline{u}} + V\frac{\partial\psi}{\partial\underline{u}} + \int_{t_o}^{t_f} \frac{\partial H}{\partial\underline{u}}\, dt$$

As shown in Appendix A, $\Delta\underline{H}$ can be expressed as:

$$\Delta\underline{H} = \sum_{i,j} \begin{bmatrix} \mp\left(\frac{X_1(1+\lambda_2^+)}{aX_2+\dot{X}_2^-}\right)_i \pm \left(\frac{X_1(1-\lambda_2^+)}{aX_2+\dot{X}_2^-}\right)_j \\ \mp\left(\frac{1+\lambda_2^+}{aX_2+\dot{X}_2^-}\right)_i \mp \left(\frac{1-\lambda_2^+}{aX_2+\dot{X}_2^-}\right)_j - \left(\frac{1+\lambda_2^+}{\dot{X}_2^-}\right)_{\substack{m' \\ m}} + \left(\frac{1-\lambda_2^+}{\dot{X}_2^-}\right)_{\substack{n \\ n}} \\ \mp\left(\frac{1+\lambda_2^+}{\dot{X}_2^-}\right)_{m'} + \left(\frac{1+\lambda_2^+}{\dot{X}_2^-}\right)_m - \left(\frac{1-\lambda_2^+}{\dot{X}_2^-}\right)_n + \left(\frac{1-\lambda_2^+}{\dot{X}_2^-}\right)_{n'} \end{bmatrix} + \begin{bmatrix} \mp X|_{t_f} \\ 1 \\ 0 \end{bmatrix} \tag{3-11}$$

where the variables i, j, m', n, n' represent points of intersection of the following switching lines:

$aX_1 + X_2 + d = 0$

$-aX_1 - X_2 + d = 0$

$X_2 = e - d$

$X_2 = e + d$

$X_2 = -e + d$

$X_2 = -e - d$

respectively.

It should be noticed that:

1. The upper sign in the equation for $\Delta\underline{H}$ is for the intersection of $X_2 < 0$ and the lower sign for that of $X_2 > 0$.

2. Because it is possible to intersect only one of the switching lines $X_2 = +e \pm d$ and $X_2 = -e \pm d$, there is only one value, if any, for the terms with the subscripts m, m', n, and n'.

Since $\Delta\underline{H}$ corresponds to the gradient of J with respect to $\underline{u}$ for the linearized case, it follows that the

method of steepest descent (10, 13, 14) can be used to find the $d\underline{u}$ which reduces J. (The gradient method can also be used to approach a similar result (10, 15, 16).) The minimum of J can be reached by repeatedly changing the control parameters $\underline{u}$ by $d\underline{u}$ until dJ approaches zero.

In this study, the method of steepest descent is used to obtain $d\underline{u}$. By adding the positive definite quadratic constraint,

$$(dp)^2 = d\underline{u}^T W \, d\underline{u}$$

where W is a weighting matrix and using the technique of Lagrange multipliers to determine an extremum (14), one has the final form for the variation of the control $d\underline{u}$,

$$d\underline{u} = \frac{dp}{(I_{HH})^{1/2}} W^{-1} \Delta\underline{H} \qquad (3\text{-}12)$$

where

$$I_{HH} = \Delta\underline{H}^T W^{-1} \Delta\underline{H}$$

The value of dp should be selected in such a way that the linearization is valid in the derivation of Equations 3-11 and 3-12.

A new control variable is written as $\underline{u}_{new} = \underline{u}_{old} + d\underline{u}$, where $d\underline{u}$ is obtained from Equation 3-12. Then, $\underline{u}_{new}$ is used in the original nonlinear Equation 3-7 resulting in new values of $\Delta\underline{H}$ and $d\underline{u}$. This process is repeated until the performance index is minimized. The block diagram

for this iteration is shown in Figure 4. The final value of $\underline{u}$ gives the optimal control for this system.

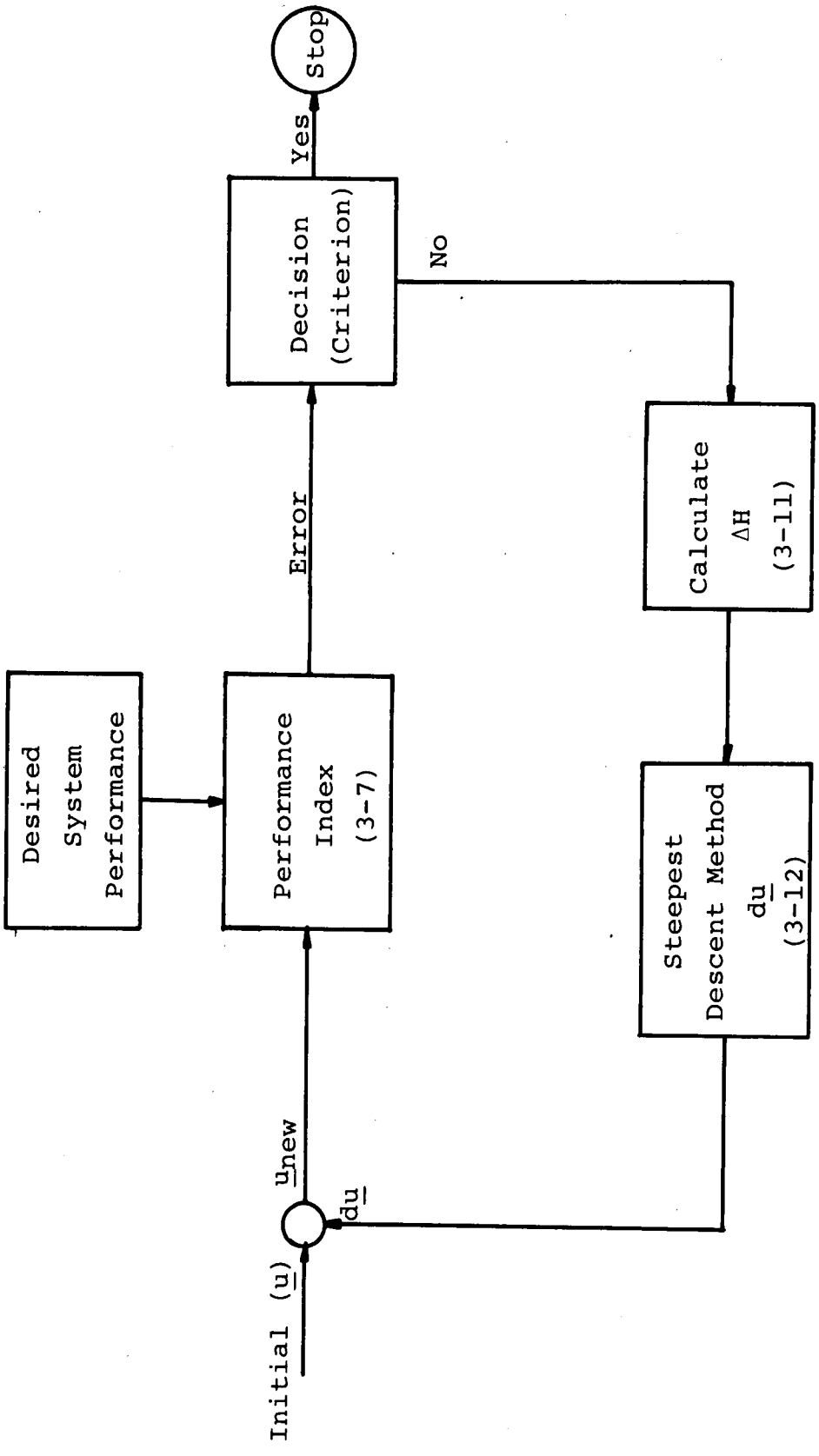


Figure 4. Iteration on system parameters u.

CHAPTER IV

NUMERICAL PROCEDURE FOR OPTIMIZATION

I. COMPUTATIONAL PROCEDURE

The actual procedures required to carry out one iteration in the optimization are outlined in the following section.

Determination of the Region of Given Initial Condition

The state variables equations (Equations 2-2) for this control system are:

$$\dot{X}_1 = X_2$$

$$\dot{X}_2 = \frac{M}{I} \text{ Sat.}(U_1 - U_2)$$

The above two equations give a parabola in the X_1 and X_2 phase plane, which is:

$$2 \frac{M}{I}(X_1 - X_{10}) = X_2^2 - X_{20}^2$$

where X_{10} and X_{20} is any given initial point in the trajectory plane. As shown in Figure 5, the region is divided into eight parts by several parabolas. From this figure, the region of the given initial condition can be determined.

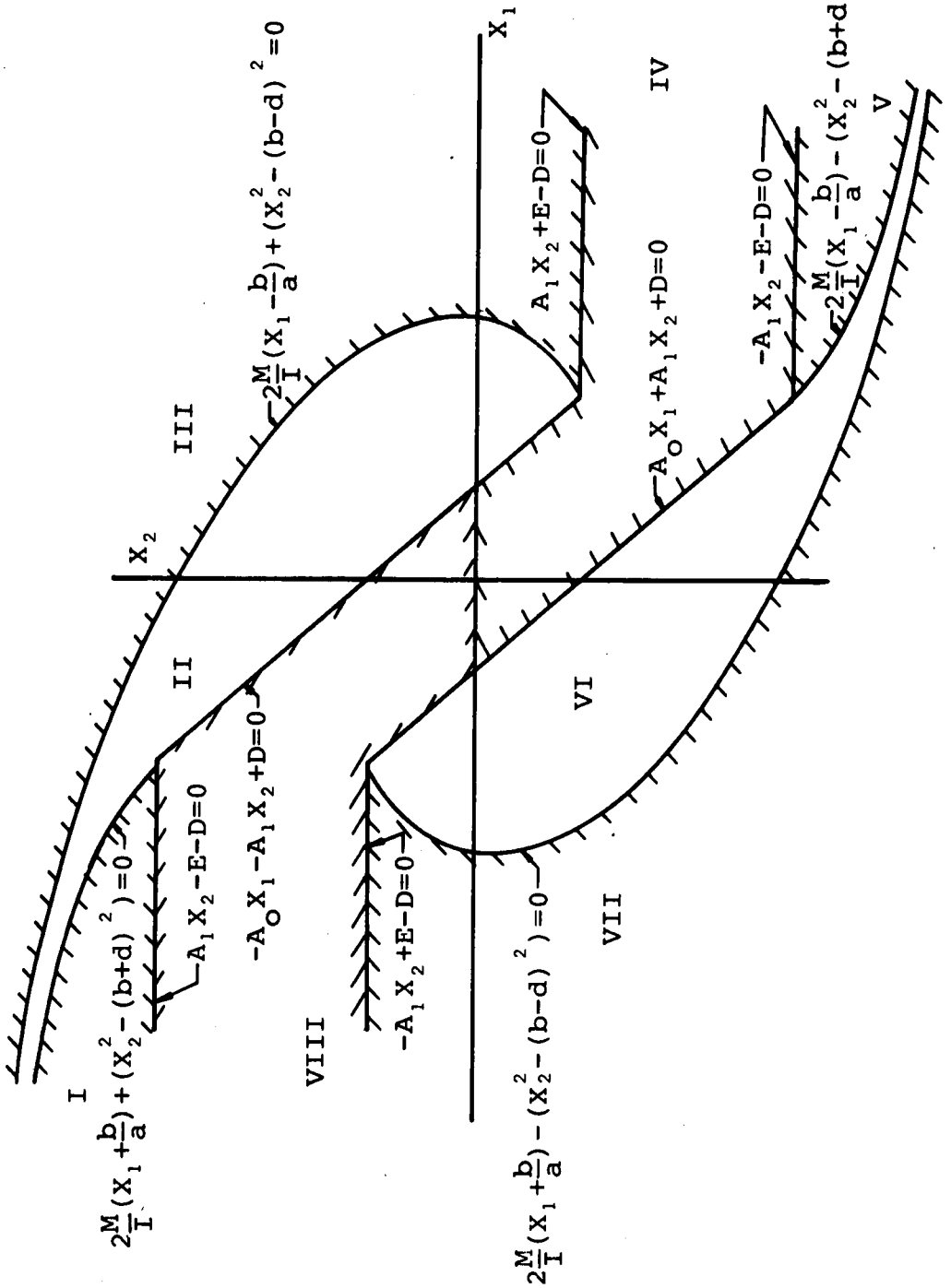


Figure 5. The trajectory plane of the present control system.

Time to Trace the Trajectory

Case A. Trajectory in Region I or Region II

(1) Intersection with $\mp aX_1 \mp X_2 + d = 0$

$$t = -\left[\frac{1}{a} + \frac{X_{20}}{\frac{M}{I}}\right] \pm \left[\left[\frac{1}{a} + \frac{X_{20}}{\frac{M}{I}}\right]^2 \mp 2\left[\left(\frac{M}{I}\right)a\right]^{-} \left[aX_{10} + X_{20} + (\mp d)\right]\right]^{1/2}$$

$$X_1 = \frac{\frac{M}{I}}{2} t^2 + X_{20}t + X_{10}$$

$$X_2 = \left(\frac{M}{I}\right)t + X_{20}$$

Where

-d is used for intersection with $-aX_1 - X_2 + d = 0$ and

+d is used for $aX_1 + X_2 + d = 0$

$\left[\left(\frac{M}{I}\right)a\right]^{-1}(aX_{10} \pm X_{20} \pm d) > 0$ using '+' sign

$\left[\left(\frac{M}{I}\right)a\right]^{-1}(aX_{10} \pm X_{20} \pm d) < 0$ using '-' sign

(2) Intersection with $X_2 = \mp(b - d)$

$$t = \frac{\mp(b - d) - X_{20}}{\left(\frac{M}{I}\right)}$$

$$X_1 = \frac{\left(\frac{M}{I}\right)}{2} t^2 + X_{20}t + X_{10}$$

$$X_2 = (\frac{M}{I})t + X_{20}$$

(3) Intersection with $X_2 = \pm(b + d)$

$$t = \frac{\pm(b + d) - X_{20}}{(\frac{M}{I})}$$

$$X_1 = \frac{(\frac{M}{I})}{2} t^2 + X_{20}t + X_{10}$$

$$X_2 = (\frac{M}{I})t + X_{20}$$

Note: for all of Case A replace $\frac{M}{I}$ by $-\frac{M}{I}$ when trajectory is in Region I.

Case B. Trajectory in Dead Zone (Region II)

Intersection with $\mp aX \mp X + d = 0$

$$t = \frac{1}{X_{20}}[\frac{1}{a}(\mp d - X_{20}) - X_{10}]$$

$$X_1 = X_{20}t^2 + X_{10}$$

$$X_2 = X_{20}$$

where - d is used for intersection with $-aX_1 - X_2 + d = 0$ and +d is used for $aX_1 + X_2 + d = 0$.

Checking Final Condition

If $X_2(t_f) \leq -\frac{(-\frac{M}{I})}{a}$, stop at $-aX_1 - X_2 + d = 0$

If $-X_2(t_f) \leq \frac{\frac{M}{I}}{a}$, stop at $aX_1 + X_2 + d = 0$

Calculation of Performance Index

$$\mathcal{J} = \int_0^{t_f} (Q + \frac{M}{I}(U_1 + U_2))dt + |X_2(t_f)|$$

$$= Qt_f + \int_0^{t_f} \frac{M}{I}(U_1 + U_2)dt + |X_2(t_f)|$$

Calculating λ_1, and λ_2 at Switching Lines

Case A. Calculating λ_1, and λ_2 at Final Point

(a) Stopping at $aX_1 + X_2 + d = 0$

(1) Final trajectory in dead zone

$$\lambda_1(t_f) = -\frac{Q}{X_2(t_f)}$$

$$\lambda_2(t_f) = -\frac{Q}{aX_2(t_f)} - 1$$

(2) Final trajectory in Region III

$$\lambda_1(t_f) = -\frac{Qa}{aX_2(t_f) + \frac{M}{I}}$$

$$\lambda_2(t_f) = -\frac{Q}{aX_2(t_f) + \frac{M}{I}} - 1$$

(b) Stopping at $-aX_1 - X_2 + d = 0$

(1) Final trajectory in dead zone

$$\lambda_1(t_f) = -\frac{Q}{X_2(t_f)}$$

$$\lambda_2(t_f) = -\frac{Q}{aX_2(t_f)} + 1$$

(2) Final trajectory in Region I

$$\lambda_1(t_f) = \frac{-aQ}{aX_2(t_f) - \frac{M}{I}}$$

$$\lambda_2(t_f) = \frac{-Q}{aX_2(t_f) - \frac{M}{I}} + 1$$

Case B. Calculating Jump in λ's

$$\lambda_1^+(t_{f-n}) = \lambda_1(t_{f-(n-1)})$$

$$\lambda_2^+(t_{f-n}) = \lambda_1(t_{f-(n-1)})(t_{f-n} - t_{f-(n-1)}) + \lambda_2(t_{f-(n-1)})$$

$$\lambda_2(t_{f-n}) = \lambda_2^+(t_{f-n})\frac{aX_2 + \frac{M}{I}(U_1 - U_2)^+}{aX_2 + \frac{M}{I}(U_1 - U_2)^-} + \frac{\frac{M}{I}(U_1 + U_2)^+ - \frac{M}{I}(U_1+U_2)^-}{aX_2 + \frac{M}{I}(U_1 - U_2)^-}$$

$$\lambda_1(t_{f-n}) = \lambda_1^+(t_{f-n}) + \frac{M}{I}[\lambda_2^+(t_{f-n})(U_1 - U_2)^+ - \lambda_2(t_{f-n})(U_1 - U_2)^- + (U_1 + U_2)^+ - (U_1 + U_2)^-]/X_2$$

where

$n = 1$ to $f-1$

Calculating Vector $\Delta\underline{H}$ and $d\underline{u}$

The vector $\Delta\underline{H}$ can be obtained by using Equation 3-11. It should be noticed that only $\lambda_2^+(t_{f-n})$ is used in this equation. The variation of the control $d\underline{u}$ is evaluated from Equation 3-12 and the value of dp is selected between 0 and 1.

II. COMPUTATIONAL RESULTS

The space vehicle data (moment inertia of space vehicle = 76,865 slug-ft^2, M = 25 lb-ft, A_o = 1.0, A_1 = 10.0, $0.03 \leq D \leq 3.0$, $0.1 + D \leq E \leq 3.0 + D$) was used and one axis control was assumed. The initial conditions of the angular position and velocity were assumed to be 1.0 and 0.2 °/sec. respectively. Using Q = M/I = 0.0186352 and the trial values A_o = 1.0, D = 0.1, E = 3.0 + D, the algorithm given in Equation 3-12 gave the results shown in Table I.

TABLE I

ITERATION HISTORY

No. of Iter.	Performance Index	System Parameters		
		A_o	D	E
1	7.466659	1.000000	0.100000	3.100000
2	3.370883	0.991962	0.180000	0.190000
3	2.665557	0.998083	0.036000	0.190000
4	2.493607	0.993697	0.129600	0.182346
5	2.483644	0.994396	0.008122	0.188775

CHAPTER V

CONCLUSION

In this study, an optimization technique is used to determine the parameters of the attitude control system for a space vehicle so that minimum fuel is consumed. This optimization technique employs the method of calculus of variations to evaluate the change of performance index due to small variations of the control parameters. Then, the steepest descent method is used to set up an algorithm for finding the control variables which minimize this performance index.

The algorithm illustrated in Figure 4, page 21, provides the optimal parameters to correct a given attitude error using minimum fuel. Thus, an adaptive type of controller (Figure 1, page 5) can be used to adjust the system parameters to their optimal values for a particular attitude error. The controller can be an on-board computer that either calculates the parameters or selects them from values that were calculated previously and stored in the memory. Electronic circuits are then used to change system parameters to the optimal values selected by the on-board computer.

The algorithm which was developed for the successive approximations of the control parameters depends upon the vector, $\Delta\underline{H}$, which is estimated by a linearization procedure.

When the performance index is near the optimum, the calculation of $\Delta \underline{H}$ may involve a large error. A computer procedure is, therefore, needed to determine a proper step size, insuring the decrease of performance index for each iteration.

This study discusses one-axis attitude control, but it is not difficult to extend this to three-axes. In a three-axes control system, if the vehicle is a nonsymmetrical rigid body, the gravitation torque must be considered; hence, the optimization problem will be somewhat different.

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APPENDIXES

APPENDIX A

EVALUATION OF $\Delta\underline{H}$

The vector $\Delta\underline{H}$ is defined as:

$$\Delta\underline{H} = \int_0^{t_f} \frac{\partial H}{\partial \underline{u}}\, dt + V\left(\frac{\partial \psi}{\partial \underline{u}}\right) + \frac{\partial \phi}{\partial \underline{u}} \qquad \text{(A-1)}$$

where

$$\frac{\partial \psi}{\partial \underline{u}} = (\mp X_1,\ 1,\ 0) \text{ and } \frac{\partial \phi}{\partial \underline{u}} = \underline{0}$$

From Equation 3-7a, the above equation becomes:

$$\int_0^{t_f} \frac{\partial H}{\partial \underline{u}} dt = \int_0^{t_f} \frac{M}{I}[(1 + \lambda_2)\frac{\partial}{\partial \underline{u}}U_1 + (1 - \lambda_2)\frac{\partial}{\partial \underline{u}}U_2]dt$$

where

$$\frac{\partial}{\partial \underline{u}}U_1 = [h(-X_2 + e - d) - h(-X_2 - e - d)]\ \frac{\partial}{\partial \underline{u}}h(-A_1Y)$$

$$+ h(-A_1Y)\frac{\partial}{\partial \underline{u}}h(-X_2 + e - d) + h(A_1Y)\frac{\partial}{\partial \underline{u}}h(-X_2 - e - d)$$

$$\frac{\partial}{\partial \underline{u}}U_2 = [h(X_2 + e - d) - h(X_2 - e - d)\ \frac{\partial}{\partial \underline{u}}\ h(-A_1Z)$$

$$+ h(-A_1Z)\frac{\partial}{\partial \underline{u}}h(X_2 + e - d) + h(A_1Z)\frac{\partial}{\partial \underline{u}}h(X_2 - e - d)$$

where $Y = aX_1 + X_2 + d$, $Z = -aX_1 - X_2 + d$

It is known that the trajectory in the phase plane has only one chance to intersect with horizontal switching lines. It has many chances to intersect with the other switching lines, Y = 0 and Z = 0. The intersection with Y = 0 and Z = 0 are given by the first term of $\frac{\partial}{\partial \underline{u}} U_1$ and $\frac{\partial}{\partial \underline{u}} U_2$, respectively. The horizontal switching lines are given by the other terms. Consider the first term of $\frac{\partial}{\partial \underline{u}} U_1$:

$$\int_0^{t_f} \frac{M}{I}(1 + \lambda_2)[h(-X_2 + e - d) - h(-X_2 - e - d)]\frac{\partial}{\partial \underline{u}} h(-A_1 Y)dt$$

$$= \sum_i \frac{M}{I} \int_{t_i-\varepsilon}^{t_i+\varepsilon} (1 + \lambda_2)\left(\frac{\partial}{\partial \underline{u}} h(-A_1 Y)\right)dt \qquad \text{(A-2)}$$

$$= \sum_i \frac{M}{I} \int_{t_i-\varepsilon}^{t_i+\varepsilon} (1 + \lambda_2)\left(\frac{\partial}{\partial a} h(-A_1 Y), \frac{\partial}{\partial d} h(-A_1 Y),\right.$$

$$\left.\frac{\partial}{\partial e} h(-A_1 Y)\right)dt$$

$$= \sum_i \frac{M}{I}\left(\mp \frac{X_1(1 + \lambda_2^+)}{aX_2 + \dot{X}_2^-}, \mp \frac{1 + \lambda_2^+}{aX_2 + \dot{X}_2^-}, 0\right)_i$$

Where

t_i = the ith time at which the trajectory of space vehicle in phase plane intersects the portion of switching line Y = 0 between $X_2 = e - d$ and $X_2 = -e - d$

λ_2^+ = the λ_2 evaluated at $t_i + \varepsilon$

$\dot{X}_2^-$ = the $\dot{X}_2$ evaluated at $t_i - \varepsilon$

The upper sign is for the intersection point when $X_2 > 0$ and the lower sign for that of $X_2 < 0$. The above result is derived by the following procedure:

Because the intersection is on the line Y = 0 and since this point also satisfies the state Equation 2-2, it follows that

$$d(d) = -(adX_1 + dX_2) = -(aX_2 + \dot{X}_2)dt$$

If the intersecting point has a positive value for X_2,

$t_i \pm \varepsilon$ implies $d_i \mp \varepsilon_d$

where

$$\varepsilon_d = [aX_2 + \dot{X}_2]\big|_{t_i} \varepsilon$$

By using this relation, integral A-2 can be evaluated as follows:

$$\int_{t_i-\varepsilon}^{t_i+\varepsilon} (1 + \lambda_2) \frac{\partial}{\partial u} h(-A_1 Y)dt = \int_{d_i+\varepsilon_d}^{d_i-\varepsilon_d} - \frac{1 + \lambda_2}{aX_2 + \dot{X}_2} \frac{\partial}{\partial u} h(-A_1 Y)d(d)$$

$$= - \frac{1 + \lambda_2}{aX_2 + \dot{X}_2} h(-A_1 Y) \Big|_{d_i+\varepsilon_d}^{d_i-\varepsilon_d} + \int_{d_i+\varepsilon_d}^{d_i-\varepsilon_d} h(-A_1 Y) \frac{\partial}{\partial d}\left[\frac{1 + \lambda_2}{aX_2 + \dot{X}_2}\right] d(d)$$

Since the point d_i is on the line Y = 0,

$$d(d) = -(aX_2 + \dot{X}_2)\big|_{t_i} dt$$

should be kept unchanged in the neighborhood of d_i, that is,

$$(aX_2 + \dot{X}_2)\,|_{t_i} = (aX_2 + \dot{X}_2)\,|_{t_i-\varepsilon}$$

Let $\varepsilon_d \to 0$ and use the relation,

$$h(-A_1Y)\,|_{d_i-\varepsilon_d} = 1 \text{ and } h(-A_1Y)\,|_{d_i+\varepsilon_d} = 0$$

Then

$$\int_{t_i-\varepsilon}^{t_i+\varepsilon} (1 + \lambda_2)\,\frac{\partial}{\partial d}\,h(-A_1Y)dt = -\,\frac{1 + \lambda_2^+}{aX_2 + \dot{X}_2^-}$$

If the intersection has a negative value for X_2, $t_i \pm \varepsilon \to d_i \pm \varepsilon_d$, and similarly

$$\int_{t_i-\varepsilon}^{t_i+\varepsilon} (1 + \lambda_2)\,\frac{\partial}{\partial d}\,h(-A_1Y)dt = +\,\frac{1 + \lambda_2^+}{aX_2 + \dot{X}_2^-}$$

The same method can be applied to $Z = 0$ and the horizontal switching lines. After considering all the switching lines for all the terms given in Equation A-1, the expression of $\Delta\underline{H}$ given in Equation 3-11 is obtained.

APPENDIX B

EVALUATION OF THE JUMPS FOR $\underline{\lambda}$'s

From the previous section, we know that the relation of $\underline{\lambda}$'s before and after the intersection with the horizontal switching lines are not needed. Therefore, the discontinuities occurring on the lines, $\pm aX_1 \pm X_2 + d = 0$ will be considered here. At those discontinuous points, Equation 3-8 gives

$$\dot{\underline{\lambda}}^T + \frac{\partial H}{\partial \underline{x}} = 0$$

where

$$\underline{\lambda} = [\lambda_1, \lambda_2, \lambda_3, \lambda_4, \lambda_5]^T$$

and

$$\dot{\underline{\lambda}} = [\dot{\lambda}_1, \dot{\lambda}_2, 0, 0, 0]^T$$

By using the function of H, it can be shown that:

$$\dot{\lambda}_2 + \lambda_1 + \frac{M}{I}\lambda_2 \frac{\partial}{\partial X_2}(U_1 - U_2) + \frac{M}{I}\frac{\partial}{\partial X_2}(U_1 + U_2) = 0$$

$$\dot{\lambda}_1 + \frac{M}{I}\lambda_2 \frac{\partial}{\partial X_1}(U_1 - U_2) + \frac{M}{I}\frac{\partial}{\partial X_1}(U_1 + U_2) = 0 \qquad \text{(A-3a)}$$

$$-2\lambda_3 X_3^2 = 0 \rightarrow \lambda_3 = 0, \text{ or } X_3 = 0$$

$$-2\lambda_4 X_4^2 = 0 \rightarrow \lambda_4 = 0, \text{ or } X_4 = 0$$

$$-2\lambda_5 X_5^2 = 0 \rightarrow \lambda_5 = 0, \text{ or } X_5 = 0 \qquad \text{(A-3b)}$$

Integrating the differential Equation A-3a from $t_i-\varepsilon$ to $t_i+\varepsilon$ about these discontinuous points, and using the equation of motion (2-1c) and equations of switching lines (2-1a and 2-1b) to transfer dt to dX_1 and dX_2, respectively, the following results are obtained:

$$\lambda_2\Big|_{t_i-\varepsilon}^{t_i+\varepsilon} + \int_{t_i-\varepsilon}^{t_i+\varepsilon} \lambda_1 dt + \frac{M}{I}\int_{X_2+\varepsilon_X}^{X_2-\varepsilon_X} -\frac{\lambda_2}{aX_2}\frac{\partial}{\partial X_2}(U_1 - U_2)dX_2$$

$$+ \frac{M}{I}\int_{X_2+\varepsilon_X}^{X_2-\varepsilon_X} -\frac{1}{aX_2}\frac{\partial}{\partial X_2}(U_1 + U_2)dX_2 = 0 \qquad \text{(A-3)}$$

$$\lambda_1\Big|_{t_i-\varepsilon}^{t_i+\varepsilon} + \frac{M}{I}\int_{X_1-\varepsilon_X}^{X_1+\varepsilon_X} \frac{\lambda_2}{X_2}\frac{\partial}{\partial X_1}(U_1 - U_2)dX_1$$

$$+ \frac{M}{I}\int_{X_1-\varepsilon_X}^{X_1+\varepsilon_X} \frac{1}{X_2}\frac{\partial}{\partial X_1}(U_1 + U_2)dX_1 = 0 \qquad \text{(A-4)}$$

where

$$X_1 = X_1\Big|_{t_i}$$

$$X_2 = X_2\Big|_{t_i}$$

Integrating Equation A-2 by parts, gives

$$\lambda_2 \Bigg|_{t_i-\varepsilon}^{t_i+\varepsilon} + \int_{t_i-\varepsilon}^{t_i+\varepsilon} \lambda_1 dt - \frac{M}{I} \frac{\lambda_2}{aX_2} (U_1 - U_2) \Bigg|_{X_2+\varepsilon_X}^{X_2-\varepsilon_X}$$

$$+ \frac{M}{I} \int_{X_2+\varepsilon_X}^{X_2-\varepsilon_X} (U_1 - U_2) \frac{\partial}{\partial X_2} \left(\frac{\lambda_2}{aX_2}\right) dX_2 - \frac{\frac{M}{I}(U_1 + U_2)}{aX_2} \Bigg|_{X_2+\varepsilon_X}^{X_2-\varepsilon_X}$$

$$+ \frac{M}{I} \int_{X_2+\varepsilon_X}^{X_2-\varepsilon_X} \frac{\partial}{\partial X_2} \left(\frac{1}{aX_2}\right) dX_2 = 0$$

Using the definition of $(U_1 - U_2)$ and $(U_1 + U_2)$, and letting $\varepsilon \to o$ the above integrals can be evaluated as:

$$\lim_{\varepsilon_X \to o} \int_{X_2+\varepsilon_X}^{X_2-\varepsilon_X} (U_1 - U_2) \frac{\partial}{\partial X_2} \left(\frac{\lambda_2}{aX_2}\right) dX_2$$

$$= \lim_{\varepsilon_X \to o} (U_1 - U_2) \int_{X_2+\varepsilon_X}^{X_2-\varepsilon_X} \frac{\partial}{\partial X_2} \left(\frac{\lambda_2}{aX_2}\right) dX_2 = 0$$

$$\lim_{\varepsilon \to o} \int_{t_i-\varepsilon}^{t_i+\varepsilon} \lambda_1 dt = 0$$

$$\lim_{\varepsilon_X \to o} \int_{X_2+\varepsilon_X}^{X_2-\varepsilon_X} (U_1 + U_2) \frac{\partial}{\partial X_2} \left(\frac{1}{aX_2}\right) dX_2 = 0$$

Consequently,

$$\lambda_2^- = \lambda_2^+ \frac{aX_2 + \frac{M}{I}(U_1 - U_2)^+}{aX_2 + \frac{M}{I}(U_1 - U_2)^-} + \frac{\frac{M}{I}(U_1 + U_2)^+ - \frac{M}{I}(U_1 + U_2)^-}{aX_2 + \frac{M}{I}(U_1 - U_2)^-}$$

where

$$\lambda_2^- = \lim_{\varepsilon \to o} \lambda_2 \Big|_{t_i - \varepsilon}$$

$$\lambda_2^+ = \lim_{\varepsilon \to o} \lambda_2 \Big|_{t_i + \varepsilon}$$

$$\frac{M}{I}(U_1 \pm U_2)^{\pm} = \lim_{\varepsilon \to o} \frac{M}{I}(U_1 \pm U_2) \Big|_{t_i \pm \varepsilon}$$

Similarly, it follows that

$$\lambda_1^- = \lambda_1^+ + \frac{M}{I} \frac{\lambda_2^+(U_1 - U_2)^+ - \lambda_2^-(U_1 - U_2)^- + (U_1 + U_2)^+ - (U_1+U_2)^-}{X_2}$$

where

$$\lambda_1^- = \lim_{\varepsilon \to o} \lambda_1 \Big|_{t_i - \varepsilon}$$

$$\lambda_1^+ = \lim_{\varepsilon \to o} \lambda_1 \Big|_{t_i + \varepsilon}$$

These results for λ_2^- and λ^+ are used for evaluating the jump in the $\underline{\lambda}$'s.

VITA

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