July 30, 1973

To the Graduate Council:

I am submitting herewith a thesis written by Kenneth Carl Gilbert entitled "A Method for Computing the Drazin Inverse of K-Circulant Matrices." I recommend that it be accepted for twelve quarter hours of credit in partial fulfillment of the requirements for the degree of Master of Science, with a major in Mathematics.

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A METHOD FOR COMPUTING THE DRAZIN INVERSE OF K-CIRCULANT MATRICES

A Thesis

Presented to

the Graduate Council of

The University of Tennessee

In Partial Fulfillment

of the Requirements for the Degree

Master of Science

by

Kenneth Carl Gilbert

August 1973

ACKNOWLEDGMENT

The author expresses his deepest appreciation to Dr. Randall E. Cline for his patient criticism and aid in the preparation of this paper.

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ABSTRACT

An algorithm proposed by Cline is developed for finding the Drazin inverse of k-circulant matrices. This procedure uses the polynomial representation of any k-circulant matrix to factor it into matrices for which a closed form expression for the Drazin inverse exists. A computer program implementing this method was developed. A description of the program and some numerical results are also given.

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CHAPTER I

INTRODUCTION

Matrices with special structure frequently occur in applied mathematics and techniques for performing numerical operations on the matrices can, in many cases, be developed to take advantage of the structure. An example of a numerical operation for which this is often the case is the construction of the various generalized inverses of matrices of special structures. In this paper a class of matrices is discussed for which the Drazin inverse can be constructed by factoring a polynomial and combining the inverses of matrices associated with each of the factors.

Chapter II describes the notation and terminology used; Chapter III introduces k-circulant matrices and the Drazin inverse. In Chapter IV a method for finding the Drazin inverse of a k-circulant matrix is presented. A description of the computer implementation of this method is given in Chapter V. Chapter VI illustrates some further application; Chapter VII is a summary. The appendices contain a computer listing and some numerical results.

CHAPTER II

NOTATION AND TERMINOLOGY

For the mathematical results discussed in this paper the matrices are assumed to have complex elements. Computational considerations, however, are restricted to matrices with real elements. Matrices are assumed to be square, of arbitrary order, subject to the conditions necessary to make products conformable.

Matrices will be designated by capital letters; underscored lower case Latin letters will denote column vectors. The notation $\underline{e}^t$ will denote the row vector which is the transpose of the column vector $\underline{e}$. Scalars will be represented by lower case Latin letters.

The determinant of a matrix A will be written as det(A). The inverse of a nonsingular matrix A will be denoted by A^{-1}. The n x n identity matrix will be denoted by I and the n x n matrix having ones on the secondary diagonal and zeros elsewhere will be denoted by J.

A matrix A raised to the power i, where i is a positive integer, will be written A^i. The sum and product of the matrices $A_1, A_2, \ldots, A_n$ will be designated as $\sum_{i=1}^{n} A_i$ and $\prod_{i=1}^{n} A_i$, respectively.

Additional notation and terminology will be introduced as required.

CHAPTER III

BACKGROUND

An n x n matrix $L = [\ell_{ij}]$ is called a Toeplitz matrix if there exist 2n-1 complex numbers $a_{-(n-1)},\ldots,a_{(n-1)}$ such that $\ell_{ij} = a_{j-i}$. The general form of a Toeplitz matrix is:

$$L = \begin{bmatrix} a_0 & a_1 & a_2 & \cdots & a_{n-1} \\ a_{-1} & a_0 & a_1 & \ddots & a_{n-2} \\ a_{-2} & a_{-1} & a_0 & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & a_2 \\ a_{-(n-1)} & \cdots & a_{-2} & a_{-1} & a_0 \end{bmatrix}$$

Trench [6] has shown that the inverse of a nonsingular Toeplitz matrix is persymmetric. Huang and Cline [5] subsequently determined necessary and sufficient conditions for a nonsingular persymmetric matrix to have a Toeplitz inverse. Applying this result to Toeplitz matrices, they showed that the inverse of a nonsingular Toeplitz matrix is Toeplitz if and only if L is of the form

$$L = a_0 \begin{bmatrix} 1 & \underline{e}^t \\ g & M \end{bmatrix} \tag{3.1}$$

where $a_0 \neq 0$ and $g = kJ\underline{e}$.

In the terminology of Cline, Plemmons and Worm [3], a Toeplitz matrix of the form (3.1) is called a k-circulant matrix. (A k-circulant matrix with k = 1 is frequently called simply a circulant or cyclic matrix [7].) In terms of elements, a general n x n k-circulant matrix has the form

$$A = \begin{pmatrix} a_0 & a_1 & a_2 & \cdots & a_{n-1} \\ ka_{n-1} & a_0 & a_1 & \cdots & a_{n-2} \\ ka_{n-2} & ka_{n-1} & a_0 & \cdots & a_{n-3} \\ \vdots & & & & \vdots \\ ka_1 & ka_2 & ka_3 & \cdots & a_0 \end{pmatrix}$$

The particular n x n k-circulant matrix with $a_1=1$ and $a_i=0$ for $i\neq 1$ is of special importance. This matrix, which will be noted by Q, is of the form

$$Q = \begin{pmatrix} 0 & 1 & \cdots & 0 & 0 \\ 0 & & & & 0 \\ \vdots & & & & \vdots \\ 0 & & & & 1 \\ k & 0 & \cdots & 0 & 0 \end{pmatrix}$$

We observe that Q^i, $1 \leq i \leq n-1$ is of the form:

$$Q^i = \begin{pmatrix} 0 & 0 & \dots & 1 & \dots & 0 & 0 \\ 0 & & & & & & 0 \\ \vdots & & & & & & \vdots \\ k & & & & & & 1 \\ \vdots & & & & & & \vdots \\ & & & & & & 0 \\ 0 & \dots & k & \dots & 0 & 0 & \end{pmatrix}$$

where the 1's are on super-diagonal i and the k's are on sub-diagonal (n-i). Hence if we let $Q^0 = I$, it follows that a matrix A can be written as $A = \sum_{i=0}^{n-1} a_i Q^i$ if and only if A is k-circulant. It can be verified that an n x n matrix A is k-circulant if and only if AQ = QA. Hence, it follows, that the inverse of a nonsingular k-circulant is also k-circulant. It was shown in [3], moreover, that the Drazin inverse of any k-circulant matrix is also k-circulant. This inverse, which exists for every square complex matrix A, will now be introduced.

Drazin [4] has called an element x of an associative ring R pseudoinvertible if there exists an element c of R such that cx = xc, $x^m = x^{m+1}c$, for some positive integer m, and $c = c^2x$. It was shown, moreover, that the element c, if it exists, is unique and that a necessary and sufficient condition for existence is that x be strongly π-regular, that is, there exist elements a and b of R such that $x^p = x^{p+1}a$ and $x^q = bx^{q+1}$ for some positive integers of p and q. The element c is called the pseudoinverse of x and the smallest positive

integer m for which $x^m = x^{m+1}c$ is called the index of x. (Clearly, if x is invertible $c = x^{-1}$.)

Since it can be shown that elements of a finite dimensional algebra are strongly π-regular, it follows, therefore, that every square complex matrix is pseudoinvertible. Thus if A is an n x n matrix, there exists a unique n x n matrix X such that $XA = AX$, $A^m = A^{m+1}X$, for some positive integer m, and $X = X^2A$. Following Cline [1], we will call the matrix X the Drazin inverse of A and denote it as $X = A_d$, where again the smallest positive integer m for which $A^m = A^{m+1}X$ is called the index of A. If A is nonsingular, it immediately follows that $A_d = A^{-1}$. On the other hand, if A is singular, the index is the smallest positive integer k such that rank (A^k) = rank (A^{k+1}).

It was shown in [3] that for a fixed value of k, the set of n x n k-circulant matrices form a multiplicative semigroup and that the index of a k-circulant matrix is 1. Thus it follows that the defining equations for the Drazin inverse of a k-circulant matrix reduce to:

$$A_dA = AA_d \tag{3.2}$$

$$A = A^2A_d \tag{3.3}$$

$$A_d = A_d{}^2A \tag{3.4}$$

From these equations it can be observed that if A is a k-circulant matrix then $(A_d)_d = A$ and that A_d is also the group inverse [3] of A.

A method for constructing the Drazin inverse of a k-circulant matrix was developed in [3] which requires the use of complex arithmetic. However, by noting the special form of one particular type of k-circulant matrix, Cline [2] developed an alternative method which frequently does not require the use of complex arithmetic. The results in [2], to be discussed in the next chapter, serve as the basis for the algorithm described in Chapter V.

CHAPTER IV

A METHOD FOR FINDING THE DRAZIN INVERSE OF A k-CIRCULANT MATRIX

In the development of a method for constructing the Drazin inverse of a k-circulant matrix, it was noted in [2] that a k-circulant matrix A for which $a_0 \neq 0$, $a_1 \neq 0$ and $a_i = 0$ otherwise has determinant $(-1)^{n-1}a_1{}^n k + a_0{}^n$. Since the determinant of matrices of this form is a linear function of k, these matrices were called linear k-circulant matrices. The following theorem, for which the proof is included for completeness, gives an expression for the Drazin inverse of a linear k-circulant matrix.

Theorem 1. For any linear n x n k-circulant matrix A,

$$A_d = \begin{cases} \dfrac{1}{(-1)^{n-1}ka_1{}^n + a_0{}^n} \displaystyle\sum_{j=0}^{n-1} (-1)^j a_0{}^{n-1-j} a_1{}^j Q^j, & \text{if } k \neq (-\frac{a_0}{a_1})^n \\ \dfrac{1}{2na_0{}^n} \displaystyle\sum_{j=0}^{n-1} (-1)^j (n-1-2j) a_0{}^{n-1-j} a_1{}^j Q^j, & \text{if } k = (-\frac{a_0}{a_1})^n \end{cases}$$

Proof: Clearly, any n x n linear k-circulant matrix can be written as $a_0 I + a_1 Q$. If $k \neq (-\frac{a_0}{a_1})^n$, A is nonsingular and $A_d = A^{-1}$. Thus we need to show that if X is the matrix of the above form then AX = I. Forming the product of A and X and using the fact that $Q^n = kI$ yields:

$$AX = (a_0 I + a_1 Q)\left[\frac{1}{(-1)^{n-1} k a_1^{\,n} + a_0^{\,n}}\left(a_0^{\,n-1} I - a_0^{\,n-2} a_1 Q\right.\right.$$

$$\left.\left. + a_0^{\,n-3} a_1^{\,2} Q^2 + \ldots + (-1)^{n-1} a_1^{\,n-1} Q^{n-1}\right)\right]$$

$$= \frac{1}{(-1)^{n-1} k a_1^{\,n} + a_0^{\,n}}\left(a_0^{\,n} I + (-1)^{n-1} a_1 k I\right) = I \ .$$

Hence $X = A^{-1} = A_d$.

If $k = (-\frac{a_0}{a_1})^n$, A is singular and it must be verified that equations (3.2), (3.3), and (3.4) are satisfied when X is of the given form.

From the fact that any k-circulant matrix is a polynomial in Q, it follows that any two k-circulant matrices commute. Since A and X are both k-circulant, AX = XA so that equation (3.2) is immediately satisfied.

To show that equation (3.3) is satisfied we compute A^2X as

$$A^2X = (a_0^{\,2} I + 2a_0 a_1 Q + a_1^{\,2} Q^2)\,\frac{1}{2n a_0^{\,n}} \sum_{j=0}^{n-1} (-1)^j (n-1-2j) a_0^{\,n-1-j} a_1^{\,j} Q^j \ .$$

Whence, combining terms and using the fact that $Q^n = kI$ gives:

$$A^2X = \frac{1}{2n a_0^{\,n}}\left[(n-1) a_0^{\,n+1} I - (n-3) a_0^{\,n} a_1 Q + 2(n-1) a_0^{\,n} a_1 Q\right.$$

$$\left. + (-1)^n 2(n-1) a_0 a_1^{\,n} Q^n + (-1)^{n-1}(n-3) a_0 a_1^{\,n} Q^n + (-1)^n (n-1) a_1^{\,n+1} Q^{n+1}\right]$$

$$= \frac{1}{2n a_0^{\,n}}\left[(n-1) a_0^{\,n+1} I - (n-3) a_0^{\,n} a_1 Q + 2(n-1) a_0^{\,n} a_1 Q\right.$$

$$+ 2(n-1)a_0^{n+1}I - (n-3)a_0^{n+1}I + (n-1)a_0^{n}a_1Q]$$

$$= \frac{1}{2na_0^{n}}[2na_0^{n+1}I + 2na_0^{n}a_1Q] = a_0I + a_1Q = A .$$

Hence $A^2X = A$ as required.

Computationally it would be tedious to show that A and X satisfy $X^2A = X$. However, by first observing that the equation

$$\sum_{j=0}^{n-1} (-1)^j (n - 1 - 2j) a_0^{n-1-j} a_1^{j} x^j = 0$$

has $x = \frac{-a_0}{a_1}$ as a root, it is seen that the given polynomial form for X may be written as $X = (a_0I + a_1Q)\, p_{n-2}(Q) = Ap_{n-2}(Q)$ where $p_{n-2}(Q)$ is a polynomial in Q of degree n - 2. Using the previously verified equation $A^2X = A$ and the commutativity of k-circulant matrices then gives

$$X^2A = XAX = XAAp_{n-2}(Q) = XA^2p_{n-2}(Q) = Ap_{n-2}(Q) = X .$$

It can be verified from the definition of the Drazin inverse that if matrices A and B commute, then $(AB)_d = A_dB_d$. More generally if $A = \prod_{i=1}^{n-1} A_i$ and $A_iA_j = A_jA_i$ for all i and j, then $A_d = \prod_{i=1}^{n} (A_i)_d$. Using this result Theorem 1 can be used to find the inverse of any k-circulant matrix: given any n x n k-circulant matrix $A = \sum_{i=0}^{n-1} a_iQ^i$, the associated polynomial $\sum_{i=0}^{n-1} a_ix_i$ can be factored, over the field of complex numbers, into n - 1 linear factors: $\sum_{i=0}^{n-1} a_ix^i = \prod_{i=1}^{n-1} (b_i + c_ix)$. It now follows that

$$A = \prod_{i=1}^{n-1} (b_i I + c_i Q) \tag{4.1}$$

in which case

$$A_d = \left[\prod_{i=1}^{n-1} (b_i I + c_i Q) \right]_d = \prod_{i=1}^{n-1} (b_i I + c_i Q)_d \ .$$

Since each of the factors $b_i I + c_i Q$ is a linear k-circulant matrix, Theorem 1 can be used to construct $(b_i I + c_i Q)_d$. These inverses can then be multiplied together to form the Drazin inverse of A. Thus we have a method, suitable for computer application, for finding the Drazin inverse of any n x n k-circulant matrix.

CHAPTER V

COMPUTER IMPLEMENTATION

In this chapter the computer implementation of the method described in Chapter III will be discussed. Given an n x n k-circulant matrix A, the computational procedure for finding its Drazin inverse consists of three steps:

1. Find the roots of the polynomial associated with A and factor A into linear k-circulant matrices.

2. Apply Theorem 1 to construct the inverses of the linear k-circulant matrices.

3. Multiply these inverses together to form A_d.

In order to implement step 1 observe that only the elements of the first row of A are needed since these elements are the coefficients of the polynomial which is to be factored. The factorization of this polynomial provides a representation of the linear k-circulant matrices which are the factors of A. Since all of the elements of a k-circulant matrix are determined by the elements of its first row, the Drazin inverse of each linear k-circulant matrix can be represented by the elements of the first row of each inverse. Thus steps 1 and 2 can be completed by storing only the first row of each of the matrices involved, hence a significant amount of storage space could be saved if the matrix multiplication of step 3 could be performed using only the first row of each matrix. The following computations show that it is possible to do so.

Suppose $A = \sum_{i=0}^{n-1} a_i Q^i$ and $B = \sum_{i=0}^{n-1} b_i Q^i$ are n x n k-circulant matrices. Then

$$AB = a_0b_0I + (a_0b_1 + a_1b_0)\,Q + \ldots \sum_{i=0}^{\ell} a_i b_{\ell-i} Q^{\ell} + \ldots$$

$$+ \sum_{i=0}^{n-1} a_i b_{n-1-i} Q^{n-1} + \sum_{i=1}^{n-1} a_i b_{n-i} Q^{n} + \sum_{i=2}^{n-1} a_i b_{n+1-i} Q^{n+1} + \ldots$$

$$\ldots \sum_{i=\ell+1}^{n-1} a_i b_{n+\ell-i} Q^{n+\ell} - \ldots + a_{n-1}b_{n-1}Q^{2n-2} .$$

Using the fact that $Q^{n+i} = Q^nQ^i = kQ^i$ gives

$$AB = \sum_{\ell=0}^{n-1} \left(\sum_{i=0}^{\ell} a_i b_{\ell-i} + k \sum_{i=\ell+1}^{n-1} a_i b_{n+\ell-i}\right) Q^{\ell} . \qquad (5.1)$$

From this equation it can be seen that only the elements of the first row of both A and B are needed to form AB. Using (5.1) a computer program for finding the Drazin inverse of a k-circulant matrix based upon the method of Chapter IV was developed which uses only one-dimensional storage arrays. In the following paragraphs a description of this program, together with some observations concerning its effectiveness, is given.

Inputs to the program consist of the dimension of the k-circulant matrix A which is to be inverted, the first row $(a_0, a_1, \ldots, a_{n-1})$ of A, and the value of k. The program first determines the n - 1 roots of the polynimial equation $a_0 + a_1x + a_2x^2 + \ldots a_{n-1}x^{n-1} = 0$. If these roots are designated $r_1, r_2 \ldots r_{n-1}$, then $(x-r_1)$, $(x-r_2)$, ..., $(x-r_n)$ are the linear factors of the monic polynomial $\frac{a_0}{a_{n-1}} + \frac{a_1}{a_{n-1}} x + \ldots + \frac{a_{n-2}}{a_{n-1}} x^{n-2}$

$+ x^{n-1}$. It follows from equation (4.1) that A can be written as a scalar multiple of linear k-circulant matrices in the form:

$$A = a_{n-1} \sum_{i=1}^{n-1} (-r_i I + Q) .$$

Hence the factorization of A into linear k-circulant matrices is implicitly represented by $r_1, r_2, \ldots, r_{n-1}$; the linear k-circulant matrices need not be generated.

To form the Drazin inverse of A the determinant of the first linear k-circulant is calculated. If this determinant is less than .001 an indicator is set to show that A is singular and the form for A_d with $k = (-\frac{a_0}{a_1})^n$ from Theorem 1 is used to construct the first row of the inverse of the linear factor. If the determinant is greater than or equal to .001, the form for A_d with $k \neq (-\frac{a_0}{a_1})^n$ from Theorem 1 is used. The first row of the second linear k-circulant factor is formed in a similar fashion after which the first row of the product of the first and second Drazin inverses is formed using equation (4.1). Having formed the first row of the product of the first two inverses, the first row of the inverses of each of the other linear factors are formed sequentially and combined using (5.1). This yields the first row of the product of the inverses. Multiplication of this row by $\frac{1}{a_{n-1}}$ gives the first row of A_d after which the program prints the entire matrix.

One of the desirable characteristics of this program is that it permits efficient storage of large order k-circulant matrices by working

with only the elements of the first row. To invert an n x n k-circulant matrix the program requires only four one-dimensional arrays of size n.

The program was used to invert some k-circulant matrices having all nonsingular linear factors so that these matrices could also be inverted for comparison purposes using the subroutine INVERT, which uses Gauss-Jordon elimination. In general, the program used more computing time than INVERT due to the amount of time required to solve the associated polynomial. However, if the roots of the polynomial are known in advance and are used as inputs to the program, it uses less time than INVERT.

An important factor in determining the accuracy of the program is the degree of accuracy with which the associated polynomial is solved. When greater accuracy is desired, a double precision polynomial solving routine with a smaller convergence criteria results in greater accuracy at the expense of longer execution times. In situations where the roots of the polynomial are known exactly, an increase in accuracy can be achieved by using these roots as inputs to the program.

A listing of the program and some numerical results are included in Appendices A and B.

CHAPTER VI

APPLICATIONS IN CONSTRUCTING TEST MATRICES

In addition to serving as a basis for a computer algorithm for finding the Drazin inverse of a k-circulant matrix, the theory of Chapter IV can also be used to construct matrices for use in testing computer programs. Some of these applications are discussed in this chapter.

Theorem 1 provides an efficient method for constructing an n x n matrix for which both the determinant and Drazin inverse can be easily computed. Such matrices can be used, for example, in testing computer programs either for finding the Drazin inverse of a matrix or for finding the inverse of a nonsingular matrix. For example, consider the 5 x 5 linear k-circulant matrix

$$A = \begin{pmatrix} 2 & 3 & 0 & 0 & 0 \\ 0 & 2 & 3 & 0 & 0 \\ 0 & 0 & 2 & 3 & 0 \\ 0 & 0 & 0 & 2 & 3 \\ 3k & 0 & 0 & 0 & 2 \end{pmatrix}$$

which has determinant $(2)^5 + k(3)^5$ and is singular if and only if $k = -\frac{32}{243}$, in which case

$$A_d = \frac{1}{2(5)(2)^4}\left[(4)(2)^4 I - 2(2)^3(3)A + 0(2)^2(3)^2Q^2 - (2)(2)(3)^3Q^3 + 4(3)^4Q^4\right]$$

$$= \begin{pmatrix} .400 & -.300 & .000 & -.675 & 2.025 \\ -.267 & .400 & -.300 & .000 & -.675 \\ .089 & -.267 & .400 & -.300 & .000 \\ .000 & .089 & -.267 & .400 & -.300 \\ .040 & .000 & .089 & -.267 & .400 \end{pmatrix}$$

If $k \neq -\frac{32}{243}$, A is nonsingular, in which case

$$A_d = A^{-1} = \frac{1}{2^5 + k(3)^5}\left[2^4 I - 2^3(3)Q + 2^2(3)^2Q^2 - 2(3)^3Q^3 + (3)^4Q^4\right]$$

$$= \frac{1}{2^5 + k(3)^5}\begin{pmatrix} 16 & -24 & 36 & -18 & 18 \\ 81k & 16 & -24 & 36 & -18 \\ -18k & 81k & 16 & -24 & 36 \\ 36k & -18k & 81k & 16 & -24 \\ -24k & 36k & -18k & 81k & 16 \end{pmatrix}$$

and $\det A^{-1} = \frac{1}{2^5 + k(3)^5}$. In both of the above cases either A or A_d could be used as input to the program to be tested.

The above method of constructing test matrices may be unsatisfactory in some instances since either the input matrix or its inverse will have

only 2n nonzero entries. Thus it is of interest to note how arbitrary k-circulant matrices can be used as test matrices. Any n x n k-circulant matrix with $a_0, a_1, \ldots, a_{n-1}$ fixed has at most n - 1 values of k for which A is singular. As will be illustrated by the following example, it is not difficult to construct an n x n k-circulant matrix for which the values of k that make det A = 0 are known. If, for example, we wish to construct such a matrix of order 8, we first would construct a seventh degree polynomial for which we know the linear factors. This can be accomplished by expanding an expression of the form $(x + a)^7$. However, to be more general, we form the product of seven distinct linear factors:

$$(x-1)(x+1)(x-2)(x+2)(x-3)(x-\sqrt{3})(x+\sqrt{3}) =$$

$$x^7 - 5x^6 + 28x^4 - 17x^3 - 51x^2 + 24x + 36 .$$

Next an 8 x 8 k-circulant matrix A is formed by replacing x with Q in the above polynomial:

$$A = \begin{pmatrix} 36 & 24 & -51 & -17 & 28 & 0 & 5 & 1 \\ k & 36 & 24 & -51 & -17 & 28 & 0 & 5 \\ 5k & k & 36 & 24 & -51 & -17 & 28 & 0 \\ 0 & 5k & k & 36 & 24 & -51 & -17 & 28 \\ 28k & 0 & 5k & k & 36 & 24 & -51 & -17 \\ -17k & 28k & 0 & 5k & k & 36 & 24 & -51 \\ -51k & -17k & 28k & 0 & 5k & k & 36 & 24 \\ 24k & -51k & -17k & 28k & 0 & 5k & k & 36 \end{pmatrix}$$

It follows that A can be factored into linear k-circulant matrices as follows:

$$A = (-I+Q)(I+Q)(-2I+Q)(2I+Q)(-3I+Q)(-\sqrt{3}I+Q)(\sqrt{3}I+Q) \ .$$

Thus A is singular for $k = 1, 2^8, 3^8, 3^4$.

This example shows how it is possible to construct a nontrivial test matrix which is known to be singular (or nonsingular). Furthermore, since the determinant of A is a continuous function of k, it can be made arbitrarily small in absolute value by making k nearly equal to one of the values of k for which det A = 0. In this manner nonsingular matrices with small determinants can be constructed. Since matrices with small determinants often are ill conditioned, such matrices are useful for testing the effects of ill conditioning on computer routines.

CHAPTER VII

SUMMARY

The method described in this paper is illustrative of the techniques that can be developed for finding the generalized inverses of structured matrices. The utility of this method for computer application and construction of test matrices indicates that further exploration of the construction of the generalized inverses of structured matrices would be warranted.

It would appear that other techniques for finding the Drazin inverse of k-circulant matrices could be developed. One possibility is suggested by the fact that the inverse of a nonsingular k-circulant matrix can be constructed by solving the system of equations $A\underline{x} = (1, 0, \ldots, 0)^t$. The solution, $\underline{x}$, to this system of equations is the first column of A^{-1}, from which the entire matrix can be constructed. Thus iterative techniques such as Point Jacobi and Gauss-Siedel methods [7] for solving systems of equations can be used to invert nonsingular k-circulant matrices provided the conditions necessary for convergence are satisfied. Furthermore, computer algorithms based on iterative techniques can be programmed which use only the first row of the k-circulant matrix in computations in order to reduce storage requirements. It seems possible that iterative techniques might be used to construct the Drazin inverse of singular k-circulant matrices—perhaps by combining the defining equations into a single equation.

A characterization of the structure of the Moore-Penrose inverse of k-circulant matrices is another area for further research. It was shown in [3] that the Moore-Penrose inverse of a singular k-circulant matrix is k-circulant if and only if k lies on the unit circle. However, the structure of the Moore-Penrose inverse of arbitrary k-circulant matrices has not yet been characterized.

The results derived from the polynomial representation of k-circulant matrices gives rise to the question of the existence of matrices with some special representation. Construction of such classes of matrices, examination of the structure of the various generalized inverses of these matrices and development of special techniques for finding these inverses are possible areas for research.

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APPENDICES

APPENDIX A

COMPUTER PROGRAM

This appendix contains a listing of the computer program for finding the Drazin inverse of a k-circulant matrix.

```
C
C
C     PROGRAM K-CRC
C
C     PURPOSE
C        COMPUTES THE DRAZIN INVERSE OF A K-CIRCULANT
C     MATRIX
C
C     DESCRIPTION OF PARAMETERS
C        Z          -VECTOR CONTAINING THE FIRST ROW
C                   OF THE MATRIX TO BE INVERTED
C        N          -ORDER OF INPUT MATRIX
C        RK         -REAL VALUE OF K
C        B,C,D,CDF  -WORKING VECTORS OF LENGTH N
C        M          -DEGREE OF ASSOCIATED POLYNOMIAL
C        ROOTR      -RESULTANT VECTOR CONTAINING THE
                    REAL ROOTS OF THE ASSOCIATED POLYNOMIAL
C        ROOTI      -VECTOR CONTAINING THE CORRESPONDING
C                   IMAGINARY ROOTS
C        ISING      -INDICATOR,
C                   ISING = 0 MATRIX IS SINGULAR
C                   ISING = 1 MATRIX IS NONSINGULAR
C        ICODE      -PRINTING CODE
C                   ICODE = 0 PRINT A,AD
C                   ICODE = 1 PRINT A,AD, (A**2)*AD,(AD**2)*A
C
C     REMARKS
C        PROGRAM CAN BE USED ONLY TO INVERT K-CIRCULANT
C        MATRICES WHOSE ASSOCIATED POLYNOMIAL HAS ALL
C        REAL ROOTS
C
C     SUBROUTINES REQUIRED
C        RITE,POLRT,MULT
C
C     METHOD
C        FACTORS INPUT MATRIX INTO LINEAR K-CIRCULANT
C        MATRICES, INVERTS EACH FACTOR AND COMBINES
C        THE RESULTING INVERSES TO FORM THE DRAZIN
C        INVERSE OF THE INPUT MATRIX
C
      DIMENSION A(10),B(10),C(10),D(10),ROOTR(10),ROOTI(10),CDF(10)
      EQUIVALENCE(C(1),ROOTI(1)),(D(1),CDF(1))
      ISING = 1
C
C     READ INPUT DATA
C
      READ(5,10) N,ICODE,RK
      READ(5,20) (A(I),I=1,N)
   10 FORMAT(215,F10.3)
```

```
   20 FORMAT(8F10.3)
C
C     GENERATE AND PRINT THE ROWS OF THE ROWS OF MATRIX
C
      WRITE(6,21)
   21 FORMAT(30H THE INPUT MATRIX)
      WRITE(6,22)
      CALL RITE(A,B,N,RK)
      WRITE(6,23)
   22 FORMAT(1HO)
   23 FORMAT(1H1)
C
C     DETERMINE DEGREE OF ASSOCIATED POLYNOMIAL
C
      M=N
      DO 25 I=1,N
      IF(A(M).NE.0.) GO TO 26
   25 M=M-1
   26 M=M-1
      L = N-1
C
C     FIND THE ROOTS OF THE POLYNOMIAL
C
      CALL POLRT(A,COF,M,ROOTR,ROOTI,IER)
C
C     CHECK FOR ERRORS
C
      IF(IER.EQ.)) GO TO 40
      WRITE(5,30)IER
   30 FORMAT(5H IER=,12)
      STOP
C
C     CHECK FOR COMPLEX ROOTS
C
   40 DO 50 I=1,M
      IF(ABS(ROOTI(I)).LE..001) GO TO 50
      GO TO 60
   50 CONTINUE
      GO TO 70
   60 WRITE(6,65)
   65 FORMAT(50H POLYNOMIAL HAS COMPLEX ROOTS                          )
      STOP
C
C  70 DO 130 I=1,M
      AO= -ROOTR(I)
C
C     CALCULATE THE DETERMINANT OF THE LINEAR FACTOR
C
      DET=(-1)**L*RK + AO**N
C
C     IF SINGULAR GO TO 85
```

```
C
      IF(ABS(DET).LE..001) GO TO 85
C
C     FACTOR MATRIX IS NONSINGULAR CONSTRUCT FIRST ROW
C        OF INVERSE
C
      B(N)=(-1)**(L)/DET
      DO 80 JJ=1,L
      J=N-JJ
      J1=J+1
   80 B(J) = B(J1)*AO*(-1.)
      GO TO 100
C
C     FACTOR MATRIX IS SINGULAR CONSTRUCT FIRST ROW
C     OF DRAZIN INVERSE
C
   85 ISING=0
      DO 90 J=1,N
      JJ=J-1
   90 B(J)=(-1.)**JJ*(L-2*JJ)*AO**(1-JJ)/(2*N)
  100 CONTINUE
      IF(I.NE.1) GO TO 120
C
C     STORE FIRST ROW OF FIRST INVERSE
C
      DO 110 J=1,N
  110 C(J)=B(J)
      GO TO 130
C
C     COMBINE FIRST ROW OF INVERSE JUST OBTAINED WITH
C     PREVIOUS INVERSES
C
  120 CALL MULT(B,C,D,N,RK)
C
C     STORE PRODUCT
      DO 125 J=1,N
  125 C(J) = D(J)
  130 CONTINUE
C
C     DIVIDE ROW BY A(N) TO FORM FIRST ROW OF AD
C
      DO 135 I=1,N
  135 C(I)=C(I)/A(N)
C
C     PRINT DRAZIN INVERSE OF INPUT MATRIX
C
      IF(ISING.EQ.0) GO TO 140
      WRITE(6,141)
      GO TO 145
  140 WRITE(6,142)
```

```
  145 CONTINUE
  141 FORMAT(50H INPUT MATRIX IS NONSINGULAR THIS IS THE INVERSE      )
  142 FORMAT(50H INPUT MATRIX IS SINGULAR DRAZIN INVERSE              )
      WRITE(6,22)
      CALL RITE(C,B,N,RK)
      IF(ICODE.EQ.1) GO TO 150
      STOP
C
C     CALCULATE AND PRINT (A**2)*AD,(AD**2)A
C
  150 WRITE(6,23)
      IF(ISING.EQ.1) GO TO 160
      CALL MULT(A,C,B,N,RK)
      CALL MULT(A,B,D,N,RK)
      WRITE(6,151)
  151 FORMAT(50H(A**2)*AD                                             )
      WRITE(6,22)
      CALL RITE(D,A,N,RK)
      WRITE(6,23)
      CALL MULT(C,B,D,N,RK)
      WRITE(6,153)
  153 FORMAT(20H (AD**2)*A                                            )
      WRITE(6,22)
      CALL RITE(D,A,N,RK)
      STOP
  160 CONTINUE
      WRITE(6,161)
      WRITE(6,22)
  161 FORMAT(30H A*A-INVERSE                                          )
      CALL MULT(A,C,D,N,RK)
      CALL RITE(B,D,N,RK)
      STOP
      END
```

```
      SUBROUTINE RITE(A,B,N,RK)
C
C     THIS SUBROUTINE PRINTS A K-CIRCULANT MATRIX
C     GIVEN THE FIRST ROW
C
C     DESCRIPTION OF PARAMETERS
C        A          -VECTOR CONTAINING FIRST ROW OF MATRIX
C                    TO BE PRINTED
C        B          -WORKING VECTOR
C        N          -ORDER OF THE MATRIX
C        RK         -REAL VALUE OF K
C
      DIMENSION A(1),B(1)
      DO 1 I=1,N
    1 B(I) = A(I)
      DO 5 I = 1,N
      WRITE(6,2) (B(J),J=1,N)
      X=B(N)
      L=N
      DO 4 J=2,N
      B(L) = B(L-1)
    4 L=L-1
    5 B(1) = RK*X
    2 FORMAT(1X,10F13.5)
      RETURN
      END
```

```
      SUBROUTINE MULT(A,B,R,N,RK)
C
C     THIS SUBROUTINE COMBINES THE FIRST ROW OF TWO
C     K-CIRCULANT MATRICES TO FORM THE FIRST ROW OF
C     THEIR PRODUCT
C     DESCRIPTION OF PARAMETERS
C        A,B        -VECTORS CONTAINING THE FIRST ROW
C                   OF THE MATRICES TO BE MULTIPLIED
C                   TOGETHER
C        R          -RESULTANT VECTOR CONTAINING THE
C                   FIRST ROW OF THE PRODUCT
C        N          -ORDER OF THE MATRICES
C        RK         -REAL VALUE OF K
C
      DIMENSION A(1),B(1),R(1)
      DO 5 I-1,N
      SUM = 0.
      DO 4 J=1,N
      IF (J.GT.I) GO TO 2
      SUM = SUM + A(J)*B(I-J+1)
      GO TO 4
    2 SUM = SUM + RK*A(J)*B(N+I-J+1)
    4 CONTINUE
    5 R(I) = SUM
      RETURN
      END
```

APPENDIX B

NUMERICAL EXAMPLES

The following two examples show Drazin inverses obtained by the program in Appendix A. In the first example the input matrix is the following 8 x 8 nonsingular k-circulant matrix.

A

Row 1					
36.00000	24.00000	-51.00000	-17.00000	28.00000	0.0
-5.00000	1.00000				
Row 2					
2.00000	36.00000	24.00000	-51.00000	-17.00000	28.00000
0.0	-5.00000				
Row 3					
-10.00000	2.00000	36.00000	24.00000	-51.00000	-17.00000
28.00000	0.0				
Row 4					
0.0	-10.00000	2.00000	36.00000	24.00000	-51.00000
-17.00000	28.00000				
Row 5					
56.00000	0.0	-10.00000	2.00000	36.00000	24.00000
-51.00000	-17.00000				
Row 6					
-34.00000	56.00000	0.0	-10.00000	2.00000	36.00000
24.00000	-51.00000				
Row 7					
-102.00000	-34.00000	56.00000	0.0	-10.00000	2.00000
36.00000	24.00000				
Row 8					
48.00000	-102.00000	-34.00000	56.00000	0.0	-10.00000
2.00000	36.00000				

The following matrix is the computed form for the inverse

A^{-1}

Row 1					
0.22728	-0.18589	0.19400	-0.16037	0.16068	-0.13596
0.12935	-0.11036				
Row 2					
-0.22073	0.22728	-0.18589	0.19400	-0.16037	0.16068
-0.13596	0.12935				
Row 3					
0.25870	-0.22073	0.22728	-0.18589	0.19400	-0.16037
0.16068	-0.13596				
Row 4					
-0.27192	0.25870	-0.22073	0.22728	-0.18589	0.19400
-0.16037	0.16068				
Row 5					
0.32136	-0.27192	0.25870	-0.22073	0.22728	-0.18589
0.19400	-0.16037				
Row 6					
-0.32073	0.32136	-0.27192	0.25870	-0.22073	0.22728
-0.18589	0.19400				
Row 7					
0.38801	-0.32073	0.32136	-0.27192	0.25870	-0.22073
0.22728	-0.18589				
Row 8					
-0.37177	0.38801	-0.32073	0.32136	-0.27192	0.25870
-0.22073	0.22728				

To verify this result, AA^{-1} has been computed.

AA^{-1}

Row 1					
1.00000	-0.00000	0.00002	0.00001	-0.00002	-0.00002
-0.00001	0.00001				
Row 2					
0.00002	1.00000	-0.00000	0.00002	0.00001	-0.00002
-0.00002	-0.00001				
Row 3					
-0.00001	0.00002	1.00000	-0.00000	0.00002	0.00001
-0.00002	-0.00002				
Row 4					
-0.00003	-0.00001	0.00002	1.00000	-0.00000	0.00002
0.00001	-0.00002				
Row 5					
-0.00003	-0.00003	-0.00001	0.00002	1.00000	-0.00000
0.00002	0.00001				

Row 6					
0.00002	-0.00003	-0.00003	-0.00001	0.00002	1.00000
-0.00000	0.00002				
Row 7					
0.00005	0.00002	-0.00003	-0.00003	-0.00001	0.00002
1.00000	-0.00000				
Row 8					
-0.00001	0.00005	0.00002	-0.00003	-0.00003	-0.00001
0.00002	1.00000				

In the second example the input matrix is the following 8 x 8 singular k-circulant matrix.

A

Row 1					
36.00000	24.00000	-51.00000	-17.00000	28.00000	0.0
-5.00000	1.00000				
Row 2					
1.00000	36.00000	24.00000	-51.00000	-17.00000	28.00000
0.0	-5.00000				
Row 3					
-5.00000	1.00000	36.00000	24.00000	-51.00000	-17.00000
28.00000	0.0				
Row 4					
0.0	-5.00000	1.00000	36.00000	24.00000	-51.00000
-17.00000	28.00000				
Row 5					
28.00000	0.0	-5.00000	1.00000	36.00000	24.00000
-51.00000	-17.00000				
Row 6					
-17.00000	28.00000	0.0	-5.00000	1.00000	36.00000
24.00000	-51.00000				
Row 7					
-51.00000	-17.00000	28.00000	0.0	-5.00000	1.00000
36.00000	24.00000				
Row 8					
24.00000	-51.00000	-17.00000	28.00000	0.0	-5.00000
1.00000	36.00000				

The following is the computed form for A_j.

A_d

Row 1					
0.01085	0.00994	0.01343	0.00442	0.00852	0.00443
-0.00155	0.01246				
Row 2					
0.01246	0.01085	0.00994	0.01343	0.00442	0.00852
0.00443	-0.00155				
Row 3					
-0.00155	0.01246	0.01085	0.00994	0.01343	0.00442
0.00852	0.00443				
Row 4					
0.00443	-0.00155	0.01246	0.01085	0.00994	0.01343
0.00442	0.00852				
Row 5					
0.00852	0.00443	-0.00155	0.01246	0.01085	0.00994
0.01343	0.00442				
Row 6					
0.00442	0.00852	0.00443	-0.00155	0.01246	0.01085
0.00994	0.01343				
Row 7					
0.01343	0.00442	0.00852	0.00443	-0.00155	0.01246
0.01085	0.00994				
Row 8					
0.00994	0.01343	0.00442	0.00852	0.00443	-0.00155
0.01246	0.01085				

To verify this result A^2A_d and $A_d{}^2A$ have been computed.

A^2A_d

Row 1					
35.99945	23.99963	-50.99930	-16.99973	27.99959	-0.00005
-4.99987	1.00005				
Row 2					
1.00005	35.99945	23.99963	-50.99930	-16.99973	27.99959
-0.00005	-4.99987				
Row 3					
-4.99987	1.00005	35.99945	23.99963	-50.99930	-16.99973
27.99959	-0.00005				
Row 4					
-0.00005	-4.99987	1.00005	35.99945	23.99963	-50.99930
-16.99973	27.99959				
Row 5					
27.99959	-0.00005	-4.99987	1.00005	35.99945	23.99963
-50.99930	-16.99973				

Row 6

-16.99973	27.99959	-0.00005	-4.99987	1.00005	35.99945
23.99963	-50.99930				

Row 7

-50.99930	-16.99973	27.99959	-0.00005	-4.99987	1.00005
35.99945	23.99963				

Row 8

23.99963	-50.99930	-16.99973	27.99959	-0.00005	-4.99987
1.00005	35.99945				

$A_d{}^2A$

Row 1

0.01085	0.00994	0.01343	0.00442	0.00852	0.00443
-0.00155	0.01246				

Row 2

0.01246	0.01085	0.00994	0.01343	0.00442	0.00852
0.00443	-0.00155				

Row 3

-0.00155	0.01246	0.01085	0.00994	0.01343	0.00442
0.00852	0.00443				

Row 4

0.00443	-0.00155	0.01246	0.01085	0.00995	0.01343
0.00442	0.00852				

Row 5

0.00852	0.00443	-0.00155	0.01246	0.01085	0.00994
0.01343	0.00442				

Row 6

0.00442	0.00852	0.00443	-0.00155	0.01246	0.01085
0.00994	0.01343				

Row 7

0.01343	0.00442	0.00852	0.00443	-0.00155	0.01246
0.01085	0.00994				

Row 8

0.00994	0.01343	0.00442	0.00852	0.00443	-0.00155
0.01246	0.01085				

VITA

Kenneth Carl Gilbert was born in Jonesville, Virginia, on October 11, 1947. He was graduated from Pennington High School in June, 1965. In June, 1969, he received a Bachelor of Arts degree in Mathematics from Berea College, Berea, Kentucky. After graduation he was employed as a Statistical Clerk by Holston Defense Corporation, Kingsport, Tennessee. He entered the U.S. Army in October, 1969, and served as Mathematics-Statistics Assistant in the Safeguard Communications Agency, Fort Huachuca, Arizona, until August, 1971. From September, 1971, to August, 1973, he served as a graduate assistant in the Department of Mathematics at The University of Tennessee and received the Master of Science degree with a major in Mathematics. He is presently working toward a Doctor of Philosophy degree with a major in Management Science.

He is married to the former Peggy Elizabeth Sisson of Oak Ridge, Tennessee, and has a son, Mark Preston.