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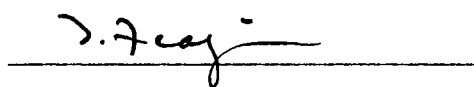
I am submitting herewith a thesis written by Richard Allen Dill entitled "Eigenvalue Computation Concerning the Stability of the Laminar Compressible Boundary Layer." I have examined the final copy of this thesis for form and content and recommend that it be accepted in partial fulfillment of the requirements for the degree of Master of Science, with a major in Engineering Science.



Basil N. Antar, Major Professor

We have read this thesis
and recommend its acceptance:





Accepted for the Council:



Vice Chancellor
Graduate Studies and Research

EIGENVALUE COMPUTATION CONCERNING THE STABILITY OF THE
LAMINAR COMPRESSIBLE BOUNDARY LAYER

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ABSTRACT

This study is a comparative examination of two methods of obtaining the eigenvalues for stability of the laminar compressible boundary layer. The derivation of the equations used in the two methods and explanations of the two methods are given along with the results obtained by using each of the methods.

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INTRODUCTION

Although no unified theory explaining the complete process of transition from laminar to turbulent regimes has been successful, experiments [1]¹ on the boundary layer for incompressible flow over a flat plate have shown that it is manifested in three major stages. A primary stage in which infinitesimal disturbances either amplify or damp exponentially with time as predicted by linear stability theory; a second stage of finite amplitude disturbances where strong three-dimensional nonlinear effects are observed; and finally, a third stage involving the birth of turbulent spots. The belief that this situation also exists at supersonic flow velocities has prompted this investigation pertaining to the linear stability of the laminar compressible boundary layer. Specifically, an attempt is made to identify the type of spectrum belonging to the linear operator describing the linear stability of the laminar compressible boundary layer. The method employed for this investigation is a global eigenvalue computation technique developed by Antar [2] for the linear stability investigation of the incompressible boundary layer. This method will be explained in Chapter III.

A brief explanation of the terms "global" and "local," referring to the eigenvalue computation method and used in the previous paragraph, is now necessary. There are basically two types

¹Numbers in brackets refer to similarly numbered references in the List of References.

of numerical methods used for computing eigenvalues related to problems of this type.

The first type is the collection of local methods which are iterative in nature and comprise most of the accurate techniques for solving numerical eigenvalue problems of the type considered in this work. Figure 1 illustrates a simple one-dimensional Newton-Raphson technique belonging to the class of local methods. In this type of method an initial approximation to the root is guessed (1, in the figure) and by taking the slope at this point another approximation to the root is obtained (2, in the figure). It is hoped that this second approximation is a better approximation to the root than the previous one. This process is continued until an approximation close enough to the true root is obtained, which is to say, the process converges. Chapter V contains information concerning the local technique considered in this work. As can be seen, these methods are disadvantageous in the sense that the convergence of the process and its associated speed depend upon how good the initial guess of the eigenvalue is. Because of the requirement for a good initial guess of the particular eigenvalues sought, when local methods are utilized there must be some independent procedure for constructing a good initial guess, either from known solutions or generated by techniques such as the variational method of Lee and Reynolds [3].

The second class of methods, global techniques, to which the shooting techniques belong, contrast the first class of methods because they utilize fundamental properties of the solution of linear differential operators to solve a particular problem. (Figure 2

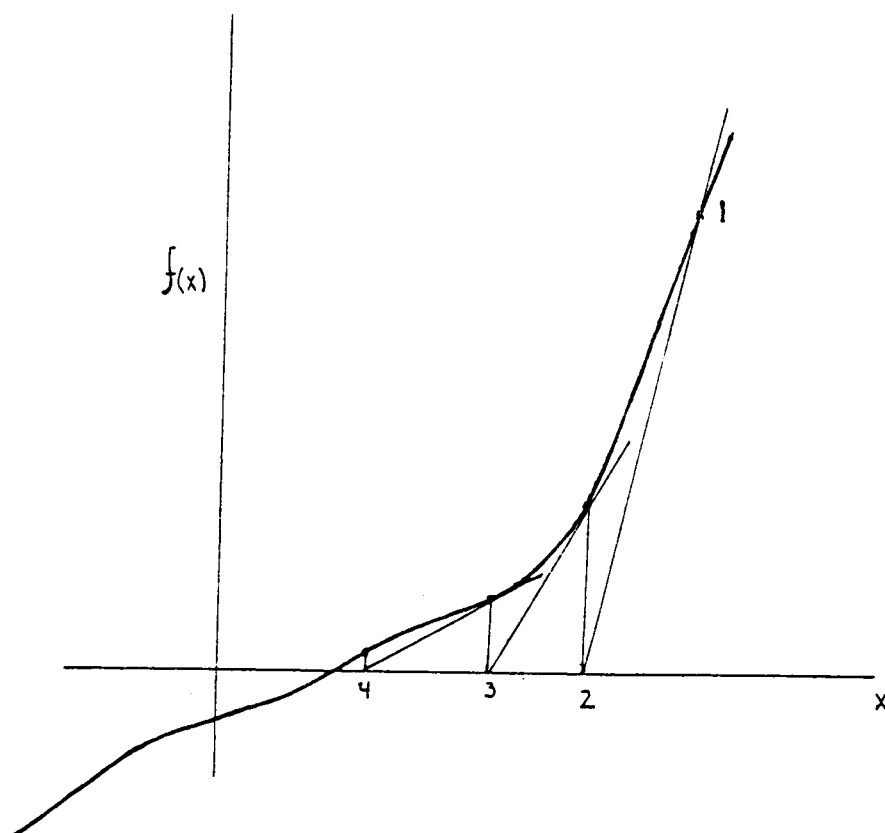


Figure 1. Illustration of one-dimensional Newton-Raphson search procedure.

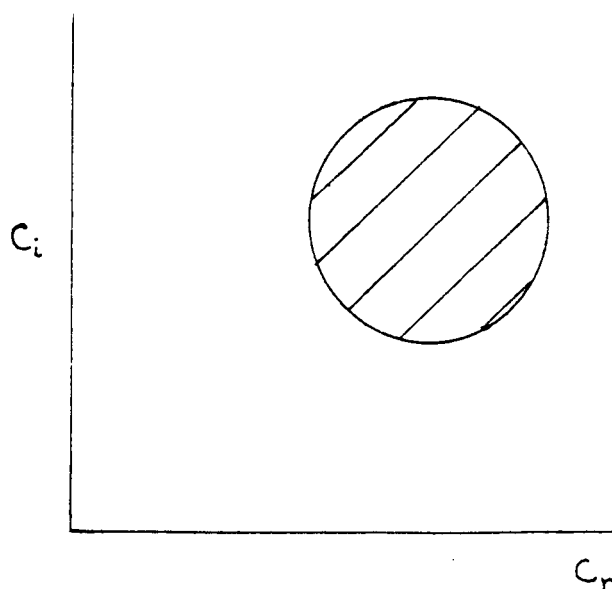


Figure 2. Illustration of the search area for the global method eigenvalue computation method.

illustrates the fact that these methods do not require any a priori knowledge of the quantity or location of an eigenvalue to be determined. This method searches a particular area such as the shaded region in Figure 2 and determines all the eigenvalues in this area.) It is also worth noting that the global method used in this investigation can be used to locate either real or complex eigenvalue spectra belonging to the associated real or complex operators.

The actual problem formulated for investigation in this study is an idealized problem in which air flows over a flat plate and is assumed to obey the perfect gas law. In Figure 3, the general features of the flow and the boundary layer are illustrated. The boundary layer thickness at any point is $\delta(x)$. In general, δ is a function of x ; however, the section of the flow investigated will be taken far enough downstream so that it is a reasonable assumption that δ be regarded as a constant (parallel flow assumption is invoked). The basic flow is taken to be parallel to the solid wall boundary (flat plate) with specified velocity and temperature profiles which are independent of their position in the flow direction and vary only in the direction perpendicular to the flow. The region lying outside the boundary layer will be referred to as the free-stream and all quantities measured in this region will be subscripted with a "1."

The final point necessary for clarification of this flow problem is that the infinitesimal disturbances which are to be considered are linearized and are taken to be in the most general form of traveling sinusoidal waves which amplify or damp with respect to time (i.e., temporal stability analysis).

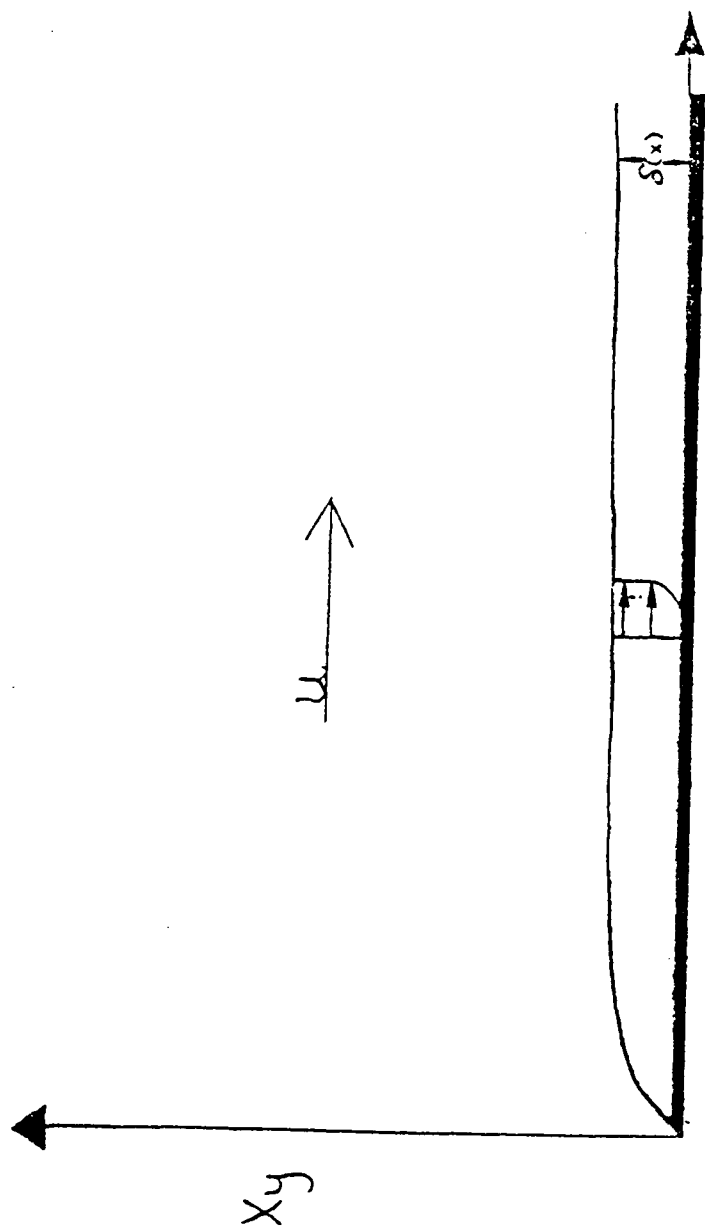


Figure 3. General physical flow features for the problem considered.

CHAPTER I

EQUATIONS GOVERNING THE MEAN FLOW OF THE BOUNDARY LAYER

For the investigation of the stability of the laminar compressible boundary layer undertaken in this work, several tabulated quantities of the mean laminar compressible boundary layer are necessary as a result of considering the actual flow to be composed of a mean and disturbance term. This procedure will be elaborated upon in Chapter II, but it is sufficient for now to know that this consideration necessitates the solution of the system of equations describing a boundary layer flow with zero pressure gradient for the needed mean flow quantities. And as was stated in the Introduction, it will be assumed that the air flowing over a flat plate obeys the perfect gas assumption.

Because of the well-behaved nature of the system of equations to be solved, the equations describing the mean laminar flow can be integrated numerically by simply guessing the unknown initial conditions and correcting these guesses by utilizing an iterative technique based on information obtained from the initial and end conditions of the system of equations. A method employing this type of reasoning, in which the original numerically unsuitable boundary value problem is transformed into an initial value problem, is referred to as a shooting method. The particulars of this method will be elaborated upon after the derivation of the system of equations governing the mean flow.

A. Derivation of the Equations Governing the Mean Flow

The two-dimensional laminar compressible boundary layer equation with zero pressure gradient for the flow of a perfect gas with transport properties and specific heat functions of temperature, but with the temperature below the temperature of dissociation are

$$\frac{\partial}{\partial x^*}(\rho^* u^*) + \frac{\partial}{\partial y^*}(\rho^* v^*) = 0, \quad (1.1)$$

$$\rho^* u^* \frac{\partial u^*}{\partial x^*} + \rho^* v^* \frac{\partial u^*}{\partial y^*} = \frac{\partial}{\partial y^*} \left(\mu^* \frac{\partial u^*}{\partial y^*} \right), \quad (1.2)$$

$$\rho^* u^* \frac{\partial i^*}{\partial x^*} + \rho^* v^* \frac{\partial i^*}{\partial y^*} = \frac{\partial}{\partial y^*} \left(k^* \frac{\partial T^*}{\partial y^*} \right) + \mu^* \left(\frac{\partial u^*}{\partial y^*} \right)^2, \quad (1.3)$$

$$p^* = \rho^* R T^*, \quad (1.4)$$

where the asterisks refer to dimensional quantities and the equations are, respectively, the mass conservation equation, the x-momentum equation, the energy equation, and the equation of state for a perfect gas. The definition of certain quantities to be used are as follows:

x^* Coordinate in the direction of the free-stream flow velocity ,

y^* Coordinate in the direction normal to the free-stream velocity ,

u^* Velocity in the x^* -direction ,

v^* Velocity in the y^* -direction ,

ρ^* Density ,

T^* Temperature ,

- μ^* Viscosity coefficient ,
- k^* Thermal conductivity coefficient ,
- $\bar{R}$ Gas constant for air ,
- p^* Pressure (constant, normalized) ,
- i^* Enthalpy (function of temperature alone) ,
- C_p^* Specific heat at constant pressure ,
- η Nondimensional Blasius variable ,
- R_x x-Reynolds number ,
- U^* Free-stream velocity ,
- ν^* Kinematic viscosity ,
- σ Prandtl number .

The above quantities will be defined with respect to how they are computed as they are utilized in the derivation of the equations governing the mean flow. Quantities which are not directly used in the following derivation will be defined after the derivation is completed.

To facilitate computation, the conservation of mass equation can be rearranged in such a way as to yield an expression for v^* in terms of u^* , p^* , and the nondimensional Reynolds number parameter. By also introducing the Blasius variable and utilizing the chain rule for differentiation, the continuity equation can be transformed to

$$\rho^* v^* = \frac{1}{2} \sqrt{R_x} \left(\rho^* v^* \eta - \int_0^\eta \rho^* u^* d\eta \right), \quad (1.5)$$

where

$$\eta = \frac{y^*}{x^*} \sqrt{R_x}, \quad (1.6)$$

$$R_x = \frac{U^* x^*}{\nu^*}; \quad (1.7)$$

the x-momentum equation can similarly be reconstructed to yield

$$\left(\frac{1}{2} \int_0^\eta \rho^* u^* d\eta \right) \frac{\partial u^*}{\partial \eta} + \frac{u^*}{v^*} \left[\frac{\partial}{\partial \eta} \left(\mu^* \frac{\partial u^*}{\partial \eta} \right) \right] = 0. \quad (1.8)$$

Finally, the energy equation can also be transformed to the following form:

$$\left(\frac{1}{2} \int_0^\eta \rho^* u^* d\eta \right) \frac{\partial i^*}{\partial \eta} + \frac{u^*}{v^*} \left[\frac{\partial}{\partial \eta} \left(\frac{k^*}{C_p^*} \frac{\partial i^*}{\partial \eta} \right) + \mu \left(\frac{di^*}{d\eta} \right) \right] = 0. \quad (1.9)$$

Having rearranged the equations in their dimensional form, they can be nondimensionalized and further transformed into a final set of differential equations which will be solved to obtain the quantities necessary for solution of the stability equations. The free-stream variables of the unknowns will be used to form nondimensional forms of the unknown variables in the following way:

$$\rho = \frac{\rho^*}{\rho_1^*} \quad u = \frac{u^*}{U^*} \quad \mu = \frac{\mu^*}{\mu_1^*} \quad T = \frac{T^*}{T_1^*} \quad v^* = \frac{\mu^*}{\rho_1^*}, \quad (1.10)$$

where the subscript "1" refers to the free-stream measurement of the quantities. Inserting these nondimensionalizing forms into the transformed equations yields

$$\left(\frac{1}{2} \int_0^\eta \rho u d\eta \right) \frac{\partial u}{\partial \eta} + \frac{\partial}{\partial \eta} \left(\mu \frac{\partial u}{\partial \eta} \right) = 0 . \quad (1.11)$$

The equation of state in the same way becomes

$$\rho = \rho_1^* \bar{R} T_1^* , \quad (1.12)$$

and the free-stream form,

$$\rho^* = \rho_1^* \bar{R} T_1^* . \quad (1.13)$$

By using the equation of state from above and also

$$g(\eta) = \frac{1}{2} \int_0^\eta \frac{u}{T} d\eta , \quad (1.14)$$

the following differential equation results as a product of the continuity and momentum equations,

$$\frac{g}{\eta} \left(\mu \frac{\partial u}{\partial \eta} \right) + \frac{\partial}{\partial \eta} \left(\mu \frac{\partial u}{\partial \eta} \right) = 0 , \quad (1.15)$$

where the μ has been added in the first term for symmetry.

The equation derived by using the continuity equation and the energy equation may also be nondimensionalized and transformed by using

$$\left. \begin{aligned}
 \theta &= \frac{i_0^* - i_1^*}{i_0^* - i_1^*} = \frac{i_0^* - i_1^*}{U^{*2}/2} \\
 i_0^* &= i_1^* + \left(\frac{U^*}{2} \right)^2 \\
 \theta &= \frac{C_p^* U^*}{k^*} \\
 i^* &= \frac{\theta U^*}{2} + i_1^*
 \end{aligned} \right\} \quad (1.16)$$

The "0" subscript above refers to the measurement of the quantity at the stagnation point. The final form of this equation after various manipulations is

$$\frac{g\sigma}{\mu} \left(\frac{\mu}{\sigma} \frac{\partial \theta}{\partial \eta} \right) + \frac{\partial}{\partial \eta} \left(\frac{\mu}{\sigma} \frac{\partial \theta}{\partial \eta} \right) = -2\mu \left(\frac{\partial u}{\partial \eta} \right)^2 \quad (1.17)$$

The system of governing differential equations to be solved has now been obtained.

These equations are

$$\frac{g}{\mu} \left(\mu \frac{\partial u}{\partial \eta} \right) + \frac{\partial}{\partial \eta} \left(\mu \frac{\partial u}{\partial \eta} \right) = 0 \quad (1.18)$$

$$\frac{g\sigma}{\mu} \left(\frac{\mu}{\sigma} \frac{\partial \theta}{\partial \eta} \right) + \frac{d}{d\eta} \left(\frac{\mu}{\sigma} \frac{\partial u}{\partial \eta} \right) = -2\mu \left(\frac{\partial u}{\partial \eta} \right)^2 \quad (1.19)$$

This pair of equations can be written as a first-order system of four equations. This is accomplished by letting

$$F = \mu \frac{\partial u}{\partial \eta} , \quad (1.20)$$

$$\frac{\partial F}{\partial \eta} = \frac{\partial}{\partial \eta} \left(\mu \frac{\partial u}{\partial \eta} \right) = - \frac{g}{\mu} \left(\mu \frac{\partial \theta}{\partial \eta} \right) = - \frac{g}{\mu} F$$

$$G = \frac{\mu}{\sigma} \frac{\partial \theta}{\partial \eta} , \quad (1.21)$$

$$\frac{\partial G}{\partial \eta} = \frac{\partial}{\partial \eta} \left(\frac{\mu}{\sigma} \frac{\partial \theta}{\partial \eta} \right) = - \frac{g\sigma}{\mu} G - 2\mu \left(\frac{\partial u}{\partial \eta} \right)^2$$

so that the second-order system becomes a first-order system which is given by

$$\frac{\partial u}{\partial \eta} = \frac{F}{\mu} , \quad (1.22)$$

$$\frac{\partial F}{\partial \eta} = - \frac{g}{\mu} F , \quad (1.23)$$

$$\frac{\partial \theta}{\partial \eta} = \frac{\sigma}{\mu} G , \quad (1.24)$$

$$\frac{\partial G}{\partial \eta} = - \frac{g}{\mu} \sigma G - 2\mu \left(\frac{\partial u}{\partial \eta} \right)^2 , \quad (1.25)$$

$$g(\eta) = \frac{1}{2} \int_0^\eta \frac{u}{T} d\eta . \quad (1.26)$$

The boundary conditions to be imposed on these equations, which are obtained from physical considerations, are

$$\begin{aligned} \text{at } \eta = 0 \quad u(0) = 0 \quad \text{and} \quad G(0) = 0 \\ \text{at } \eta \rightarrow \infty \quad u(\infty) = 1 \quad \text{and} \quad \theta(\infty) = 0 \end{aligned} , \quad (1.26a)$$

where the thermal boundary condition is the condition describing an insulated surface [4].

Since the set of equations is to be integrated numerically by an initial value method, the boundary conditions are not suitable for the solution of the equations. For such a numerical integration process the initial conditions must be specified for each variable so that in this work a Runge-Kutta integrator can be used. As previously mentioned, the well-behaved nature of the system of equations simplifies the solution by allowing the unknown initial conditions to be guessed. So the unknown initial conditions, F and θ , can be guessed for this problem. After guessing these unknown initial conditions, the equations are integrated numerically from the wall to η_δ , or the free-stream ($\eta \rightarrow \infty$). Since the initial conditions used for F and θ are in error, the boundary conditions at η_δ will be missed by a finite amount. To find the missing initial conditions, a search procedure is required which utilizes information from the specified initial and final boundary conditions. Thus, the velocity, u , and the enthalpy function, θ , are functions of the missing initial conditions, $F(0)$ and $\theta(0)$, at $u(\eta_\delta)$ and $\theta(\eta_\delta)$,

$$u(\eta_\delta) = u(F(0), \theta(0)), \quad (1.27)$$

$$\theta(\eta_\delta) = \theta(F(0), \theta(0)). \quad (1.28)$$

A Taylor series expansion about the true final values, $u(\eta_\delta)$ and $\theta(\eta_\delta)$, yields

$$u(\eta_\delta + \epsilon) = u(\eta_\delta) + \left(\frac{\partial u}{\partial a} \right)_{\eta_\delta} (F(0) - F'(0)) + \left(\frac{\partial u}{\partial b} \right)_{\eta_\delta} (\theta(0) - \theta'(0)), \quad (1.29)$$

$$\begin{aligned} \theta(\eta_\delta + \varepsilon) = \theta(\eta_\delta) + \left(\frac{\partial \theta}{\partial a} \right)_{\eta_\delta} (F(0) - F'(0)) \\ + \left(\frac{\partial \theta}{\partial b} \right)_{\eta_\delta} (\theta(0) - \theta'(0)), \quad (1.30) \end{aligned}$$

where

$$a = F(0) \quad \text{and} \quad b = \theta(0),$$

and $F'(0)$ and $\theta'(0)$ are the guessed initial values, or the values causing the perturbation, ε .

Since from the physical boundary conditions,

$$u(\eta_\delta) = 1,$$

$$\theta(\eta_\delta) = 0,$$

letting

$$(F'(0) - F(0)) = \Delta a$$

$$(\theta'(0) - \theta(0)) = \Delta b$$

yields

$$1 - u(\eta_\delta) = \left(\frac{\partial u}{\partial a} \right)_{\eta_\delta} \Delta a + \left(\frac{\partial u}{\partial b} \right)_{\eta_\delta} \Delta b + O(\Delta a^2, \Delta b^2), \quad (1.31)$$

$$-\theta(\eta_\delta) = \left(\frac{\partial \theta}{\partial a} \right)_{\eta_\delta} \Delta a + \left(\frac{\partial \theta}{\partial b} \right)_{\eta_\delta} \Delta b + O(\Delta a^2, \Delta b^2). \quad (1.32)$$

In these equations, $u(\eta_\delta)$ and $\theta(\eta_\delta)$ are the free-stream values of the variables after they have been integrated using the erroneous missing initial conditions. The partial derivatives in these equations can be

evaluated by forming another set of equations and integrating them to η_δ , together with the original system of equations. These equations are

$$\begin{aligned}
 Y_1 &= \frac{\partial u}{\partial a} & Y_2 &= \frac{\partial u}{\partial b} & Y_4 &= \frac{\partial \theta}{\partial b} & Y_3 &= \frac{\partial \theta}{\partial a} \\
 Y_5 &= \frac{\partial F}{\partial a} & Y_6 &= \frac{\partial F}{\partial b} & Y_7 &= \frac{\partial G}{\partial a} & Y_8 &= \frac{\partial G}{\partial b} \\
 \frac{\partial Y_1}{\partial \eta} &= \frac{1}{\mu} Y_5 & \frac{\partial Y_2}{\partial \eta} &= \frac{1}{\mu} Y_6 \\
 \frac{\partial Y_3}{\partial \eta} &= \frac{\sigma}{\mu} Y_7 & \frac{\partial Y_4}{\partial \eta} &= \frac{\sigma}{\mu} Y_8 \\
 \frac{\partial Y_5}{\partial \eta} &= -\frac{g}{\mu} Y_5 & \frac{\partial Y_7}{\partial \eta} &= -\frac{1}{\mu} (4FY_5 + \sigma_g Y_7) \\
 \frac{\partial Y_6}{\partial \eta} &= -\frac{g}{\mu} Y_6 & \frac{\partial Y_8}{\partial \eta} &= -\frac{1}{\mu} (4FY_6 + \sigma_g Y_8)
 \end{aligned} \tag{1.33}$$

The initial conditions necessary for the numerical integration of these equations along with the original system equations (1.22), (1.23), (1.24), (1.25), and (1.26) follow directly from the definitions of the quantities involved.

$$\left. \begin{aligned}
 Y_1(0) &= 0 & Y_2(0) &= 0 & Y_3(0) &= 0 & Y_4(0) &= 1 \\
 Y_5(0) &= 1 & Y_6(0) &= 0 & Y_7(0) &= 0 & Y_8(0) &= 0
 \end{aligned} \right\} \tag{1.34}$$

These equations can be integrated by using a Fehlberg seventh-order Runge-Kutta integrator and all computations are performed by using double-precision arithmetic [5].

B. Integration of the Mean Flow Equations

The system of equations which has been derived in Section A, that is, the set of equations (1.22), (1.23), (1.24), (1.25), (1.26), and (1.33), can be integrated together with the conditions (1.26a) and (1.34) by using a shooting method as was discussed in Section A. These equations are integrated numerically by the Fehlberg seventh-order variable step-size Runge-Kutta integrator referenced in Section A. The double precision arithmetic available on the IBM 360/65 is utilized in all calculations involved in the computation of the mean profiles sought in this chapter.

Typical mean profiles for velocity and temperature are shown in Figures 4 and 5, respectively. These mean profiles are calculated for the flow parameters shown in the figures. In these figures, the velocity and temperature profiles are observed to approach a specific value as η becomes progressively larger, the values which the profiles approach are referred to as the free-stream values. The boundary conditions for this problem are specified at $\eta = 0$ and $\eta \rightarrow \infty$, and although the profiles are continually changing because of wall effects, the wall effects are continually decreasing. This shows that after a certain η the change observed in the profiles are very insignificant. It is this feature of the flow that makes it possible to very reasonably reduce the boundary conditions for $\eta \rightarrow \infty$ to a finite value, $\eta = C$. The value of C for a particular case depends upon how close the free-stream values of the flow are to be approached. For this work, $\eta = 10$ will be taken to be the position where the free-stream

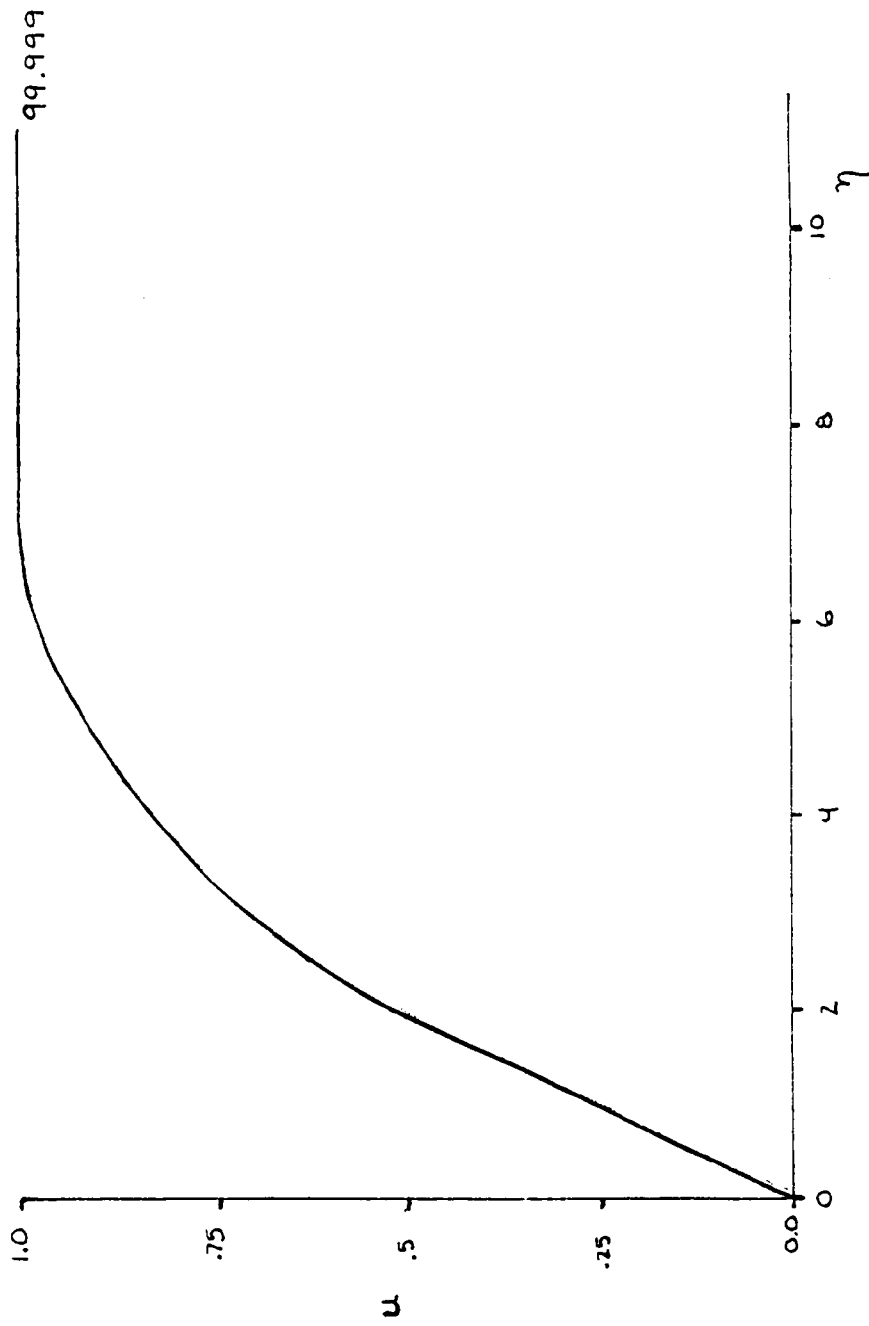


Figure 4. Velocity distribution.

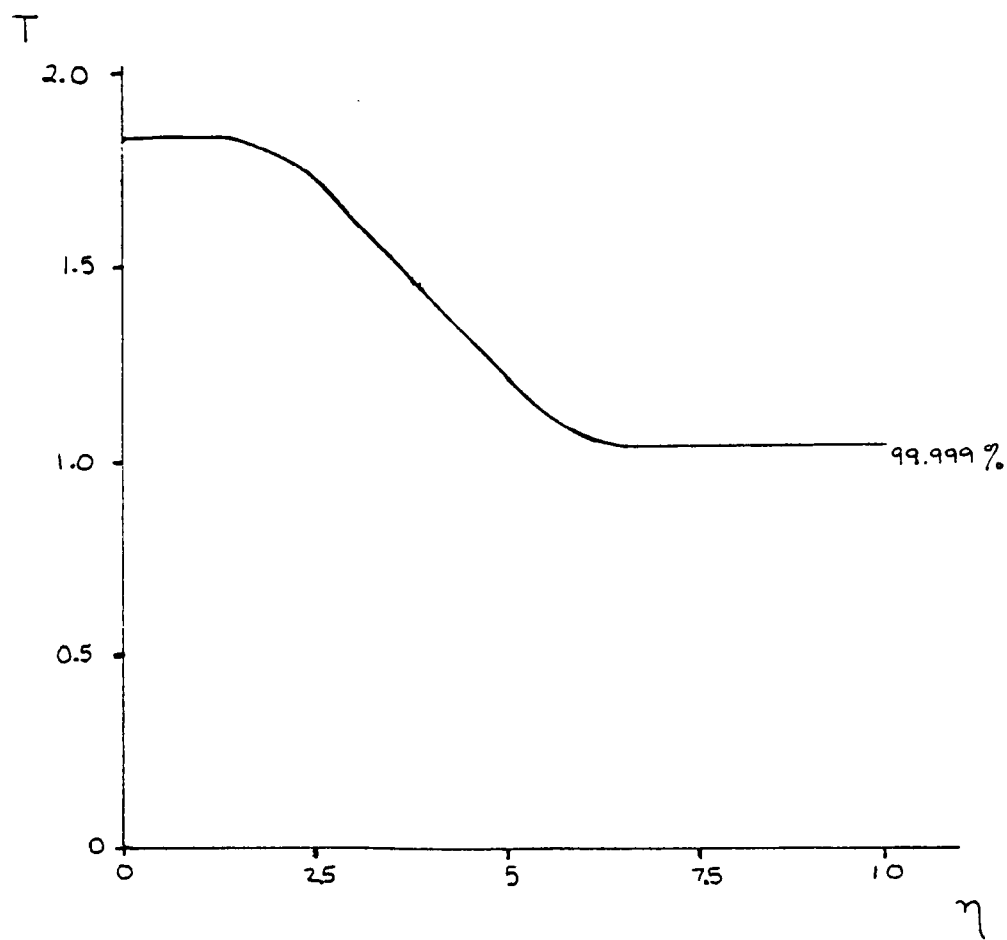


Figure 5. Temperature distribution.

values have been reasonably obtained, here, 99.99999%. This value which is taken as a reasonable approximation of the free-stream is written as η_δ .

The choice of the finite value taken to be η_δ also affects the converged values of the initial conditions. The operation of the search procedure for various choices of η_δ is illustrated in Table 1. The table shows that after a certain value of η_δ there is very little change in the converged initial conditions.

The operation of the Newton-Raphson search procedure is shown in Table 2. The unknown initial conditions which must be initially guessed are $u'(0)$ and $\theta(0)$. The flow parameters for which the integration is performed are given in Table 2. Convergence seems to be relatively independent of the initial guess for $u'(0)$ and $\theta(0)$. This process seems to converge somewhere between linearly and quadratically depending upon conditions such as the step-size and initial guesses.

The numerical integration step-size also affects the convergence of the Newton-Raphson search procedure. However, the effects are not noticed to six digits accuracy until the step-size is over 1.0 and there is still convergence to three or four digits up to 1.8. For step-sizes of greater than 1.8, there is no convergence, but for even step-sizes of 1.8, which is very large, the integration method works very well for this problem. Convergence speed is decreased as the step-size increases, but only noticeably at step-sizes over 1.0.

Table 3 contains information relative to the converged initial conditions for various Mach numbers. These converged initial

Table 1. Effect of the Choice of η_δ on the Initial Conditions

η_δ	$u'(0)$	$\theta(0)$
14	0.1940131	0.8481627
13	0.1940131	0.8481627
12	0.1940131	0.8481624
11	0.1940136	0.8481574
10	0.1940208	0.8480885
9	0.1940965	0.8474741
8	0.1946406	0.8440864

Note: $M_1 = 2.2$; $T_{01}^* = 310.94\text{K}$.

Table 2. Operation of the Search Procedure for the Unknown Initial Values

Iteration Guess	$u'(0)$	$\theta(0)$
1	0.23200000	1.00000000
2	0.21195620	0.82069714
3	0.19951972	0.84752251
4	0.19465417	0.84822222
5	0.19423360	0.84815923
6	0.19403910	0.84817260
7	0.19403935	0.84816621
8	0.19402221	0.84816393
9	0.19401631	0.84816314
10	0.19401427	0.84816287
11	0.19401357	0.84816277
12	0.19401333	0.84816279
13	0.19401325	0.84816273

Note: $\eta_\delta = 14$; $M_1 = 2.2$; $T_{01}^* = 310.94\text{K}$.

Table 3. Operation of the Search Procedure
at Different Mach Numbers

Number of Iterations	Guess $u'(0)$	Guess $\theta(0)$	Final $u'(0)$	Final $\theta(0)$	M
11	0.19400000	0.84800000	0.20164964	0.84763163	2.1
8	0.21000000	0.84600000	0.20947732	0.84707980	2.0
11	0.21000000	0.84650000	0.21747676	0.84650872	1.9
9	0.22700000	0.84600000	0.22563137	0.84591978	1.8
11	0.22000000	0.84600000	0.23389218	0.84531714	1.7
10	0.24000000	0.84500000	0.24221846	0.84470588	1.6
10	0.24600000	0.84420000	0.25056272	0.84409025	1.5

Note: $T_{01}^* = 310.94\text{K}$; $\eta_\delta = 14$; Integration Step-size = 0.5;
Relative Error of Convergence = 10^{-6} .

conditions can be used for later work in which various Mach numbers are considered, but in this work, because of computer time constraints, a Mach number of 2.2 will be the only one considered for the eigenvalue problem solution in later chapters.

Convergence is observed to be very good for this particular problem, even when the integration step-size is large. Because of this well-behaved nature of the problem of obtaining the mean flow profiles, the simple Newton-Raphson iterative technique seems adequate for calculation of the unknown initial conditions and no further convergence acceleration technique is needed. The mean profiles that are discussed in this section will be used as input for the actual solution of the eigenvalue problem considered in the next several chapters. Both the global and local numerical computer codes will utilize these profiles.

Chapter II

DERIVATION OF THE STABILITY EQUATIONS

The equations used to investigate the stability of the laminar compressible boundary layer are the linearized equations of Lees and Lin (1946) [4] which are derived from the Navier-Stokes equations. These equations, as was previously stated, are for air behaving as a perfect gas and flowing parallel or nearly so in the streamline direction. The dimensionless disturbance quantities to be introduced later will be taken as two-dimensional sinusoidal waves (acting in a two-dimensional motion).

The derivation of these equations will be considered starting with conservation of mass in its compressible form,

$$\frac{\partial \rho^*}{\partial t^*} + \frac{\partial}{\partial x^*} (\rho^* u^*) + \frac{\partial}{\partial y^*} (\rho^* v) = 0 \quad (2.1)$$

and dividing the flow quantities into mean and disturbance terms as follows:

$$\rho^* = \bar{\rho}^* + \rho'^* , \quad (2.2)$$

$$u^* = \bar{u}^* + u'^* , \quad (2.3)$$

$$v^* = \bar{v}^* + v'^* . \quad (2.4)$$

Upon substituting these quantities into the continuity equation, neglecting all terms which contain products of the disturbance quantities, subtracting out the mean-flow terms, and using the

following physical assumptions:

$$\bar{u}^* \gg \bar{v}^* , \quad (2.5)$$

$$\frac{\partial}{\partial x^*} \gg \frac{\partial}{\partial y^*} , \quad (2.6)$$

the continuity equation in the perturbation quantities becomes,

$$\frac{\partial \rho'^*}{\partial t^*} + \rho^* \frac{\partial u'^*}{\partial x^*} + \bar{u}^* \frac{\partial \rho'^*}{\partial x^*} + \bar{\rho}^* \frac{\partial v'^*}{\partial y^*} + v' \frac{\partial \rho'^*}{\partial y^*} = 0 . \quad (2.7)$$

The nondimensionalizing quantities to be used in the derivation are

$$\begin{aligned} \rho &= \frac{\bar{\rho}^*}{\rho} & \rho' &= \frac{\rho'^*}{\rho} & u &= \frac{\bar{u}^*}{u_1} & u' &= \frac{u'^*}{u_1} \\ v &= \frac{\bar{v}^*}{v_1} & v' &= \frac{v'^*}{v_1} & x &= \frac{x^*}{\delta} & y &= \frac{y^*}{\delta} & t &= \frac{t^* u_1}{\delta} \end{aligned} , \quad (2.8)$$

where δ is the boundary thickness and u is the free-stream velocity.

The introduction of the appropriate quantities of Eq. (2.8) into Eq. (2.7) yields upon grouping similar terms,

$$\frac{\partial \rho'}{\partial t} + \rho \left(\frac{\partial u'}{\partial x} + \frac{\partial v'}{\partial y} \right) + v \frac{\partial \rho'}{\partial y} + u \frac{\partial \rho'}{\partial x} = 0 . \quad (2.9)$$

The two-dimensional disturbance quantities which have been previously mentioned have phase velocity, c_r , wave number, α , and time rate of amplification, αc_i . These are the forms which the disturbances take:

$$u'(x, \eta, t) = f(\eta) e^{i\alpha(x-ct)} , \quad (2.10)$$

$$v'(x, \eta, t) = \alpha \phi(\eta) e^{i\alpha(x-ct)} , \quad (2.11)$$

$$p'(x, \eta, t) = \pi(\eta) e^{i\alpha(x-ct)} , \quad (2.12)$$

$$T'(x, \eta, t) = \xi(\eta) e^{i\alpha(x-ct)} , \quad (2.13)$$

$$\rho'(x, \eta, t) = r(\eta) e^{i\alpha(x-ct)} , \quad (2.14)$$

where,

$$\eta = \frac{y^*}{x^*} \sqrt{R_x} , \quad (2.15)$$

$$\alpha = \frac{2\pi x^*}{R\lambda} , \quad (2.16)$$

$$R = \sqrt{R_x} , \quad (2.17)$$

λ = wavelength .

If the necessary disturbance forms are substituted into Eq. (2.9) for the corresponding terms, the necessary differentiation is performed, and the Blasius variable is introduced, the final form of the perturbation continuity equation becomes

$$i(u-c)r + \rho \left(\frac{\partial \phi}{\partial \eta} + i f \right) + \phi \frac{\partial \rho}{\partial \eta} = 0 . \quad (2.18)$$

Along similar lines of reasoning, the equation of state can be transformed as follows. Starting with

$$p^* = \rho^* R^* T^* , \quad (2.19)$$

where R^* is the dimensional gas constant. Again dividing the flow quantities into a mean and a disturbance term, and substituting them into Eq. (2.19) yields

$$p'p_1 = R^* (\rho\rho_1 TT_1 + \rho'\rho_1 TT_1 + \rho'\rho_1 T_1T') .$$

As before, if the products of the disturbance terms are discarded and the necessary substitutions are made, the result is

$$p' = \rho T' + \rho' T . \quad (2.20)$$

After the substitution of the disturbance forms into Eq. (2.20), it becomes

$$\pi e^{id} = \rho \xi e^{id} + T \rho e^{id} .$$

Utilizing the fact that there is no pressure gradient in the stream-wise direction and that the flow quantities are all normalized, gives

$$p = \rho T = 1 . \quad (2.21)$$

$$\pi = \frac{\xi}{T} + \frac{\rho}{T}$$

Equation (2.21) is the final form of the equation of state with the necessary disturbance forms contained in it.

For brevity, the x-momentum, y-momentum, and energy equations will not be fully derived here as the continuity and state equations were; however, the equations are obtained in an exactly analogous manner. First, the flow quantities are divided into mean and disturbance terms, the mean flow equations then subtract from the full equations, the equations are linearized by discarding all terms which contain products of disturbances, considering the flow to be parallel, and nondimensionalizing the flow quantities. These steps yield the

x-momentum, y-momentum, and energy equations, respectively, shown below:

$$\rho \left(\frac{\partial \bar{u}'}{\partial t} + u \frac{\partial u'}{\partial x} + v' \frac{\partial u}{\partial y} \right) = - \frac{10}{\gamma M_1^2} \frac{\partial p'}{\partial x} + \frac{1}{R} \left[2\mu \frac{\partial^2 u'}{\partial x^2} + \mu \left(\frac{\partial^2 u'}{\partial y^2} + \frac{\partial^2 v'}{\partial x \partial y} \right) + \frac{2}{3} (\lambda - \mu) \left(\frac{\partial^2 u'}{\partial x^2} + \frac{\partial^2 v'}{\partial x \partial y} \right) + \frac{d\mu}{dT} \frac{dT}{dy} \left(\frac{\partial u'}{\partial y} + \frac{\partial v'}{\partial x} \right) + \frac{d\mu}{dT} \left(\frac{d^2 u}{dy^2} T' + \frac{du}{dy} \frac{dT'}{dy} \right) + \frac{d^2 \mu}{dT^2} \frac{dT}{dy} \frac{du}{dy} T' \right], \quad (2.22)$$

$$\rho \left(\frac{\partial v'}{\partial t} + u \frac{\partial v'}{\partial x} \right) = - \frac{1}{\gamma M_1^2} \frac{\partial p'}{\partial y} + \frac{1}{R} \left[2\mu \frac{\partial^2 v'}{\partial y^2} + \mu \left(\frac{\partial^2 v'}{\partial x^2} + \frac{\partial^2 u}{\partial x \partial y} \right) + \frac{2}{3} (\lambda - \mu) \left(\frac{\partial^2 u'}{\partial x \partial y} + \frac{\partial^2 v'}{\partial y^2} \right) + \frac{d\mu}{dT} \left(2 \frac{dT}{dy} \frac{\partial v'}{\partial y} + \frac{du}{dy} \frac{\partial T'}{\partial x} \right) + \frac{2}{3} \left(\frac{d\lambda}{dT} - \frac{d\mu}{dT} \right) \frac{dT}{dy} \left(\frac{\partial u'}{\partial x} + \frac{\partial v'}{\partial y} \right) \right], \quad (2.23)$$

$$\rho \left(\frac{\partial T'}{\partial t} + u \frac{\partial T'}{\partial x} + v' \frac{\partial T}{\partial y} \right) = - (\gamma - 1) \left(\frac{\partial u'}{\partial x} + \frac{\partial v'}{\partial x} \right) + \frac{\gamma}{\sigma R} \left[\mu \left(\frac{\partial^2 T'}{\partial x^2} + \frac{\partial^2 T'}{\partial y^2} \right) + \frac{d\mu}{dT} \frac{d^2 T}{dy^2} T' + 2 \frac{d\mu}{dT} \frac{dT}{dy} \frac{dT'}{dy} + \frac{d^2 \mu}{dT^2} \left(\frac{dT}{dy} \right)^2 T' \right] + \gamma(\gamma - 1) \frac{M_1^2}{R} \left[2\mu \frac{du}{dy} \left(\frac{\partial u'}{\partial y} + \frac{\partial v'}{\partial y} \right) + \frac{d\mu}{dT} \left(\frac{du}{dy} \right)^2 T' \right], \quad (2.24)$$

where R is the Reynolds number defined as $R = \frac{U\delta}{\nu}$.

In the derivation of the above equations, use is made of the following:

$$\mu_2 = \lambda, \quad (2.25)$$

$$s = \theta \frac{d\mu}{dT}, \quad (2.26)$$

$$\frac{ds}{dy} = \theta \frac{d\mu}{dydT} + \frac{d\theta}{dy} \frac{d\mu}{dT} = \theta \frac{d^2\mu}{dT^2} \frac{dT}{dy} + \frac{d\theta}{dy} \frac{d\mu}{dT}. \quad (2.27)$$

After introducing Eqs. (2.10), (2.11), (2.12), (2.13), and (2.15) into Eqs. (2.25) (2.26), and (2.27) and performing necessary manipulations, the final set of equations for the perturbation functions become

$$\begin{aligned} \rho [i(u-c) f + u' \phi] &= - \frac{i\pi}{\gamma M_1^2} + \frac{\mu}{\alpha R} [f'' + \alpha^2 (i\phi' - 2f)] \\ &+ \frac{2}{3} \frac{(\mu_2 - \mu)}{\alpha R} \alpha^2 (i\phi' - f) + \frac{1}{\alpha R} [su'' + s'u' + \mu' (f' + i\alpha^2 \phi)], \end{aligned} \quad (2.28)$$

$$\begin{aligned} \rho [i(u-c)\phi] &= - \frac{1}{\alpha^2} \frac{\pi'}{\gamma M_1^2} + \frac{\mu}{\alpha R} (2\phi'' + if' - \alpha^2 \phi) \\ &+ \frac{2}{3} \frac{(\mu_2 - \mu)}{\alpha R} (\phi'' - if') + \frac{1}{\alpha R} [isu' + 2\mu'\phi' + \frac{2}{3} (\mu_2' - \mu')(\phi' + if)], \end{aligned} \quad (2.29)$$

$$\begin{aligned} \rho [i(u-c) \xi + T'\phi] &= - (\gamma-1)(\phi' + if) + \frac{\gamma}{\alpha R_0} [\mu(\xi'' - \alpha^2 \xi) \\ &+ (sT')' + \mu'\xi'] + \frac{\gamma(\gamma-1)}{\alpha R} M_1^2 [su'^2 + 2\mu u' (f' + i\alpha^2 \phi)]. \end{aligned} \quad (2.30)$$

In the above equations, the primes denote differentiation with respect to η . The above set of equations is not in a suitable form for direct numerical integration. Therefore, these equations will be transformed

into a more suitable form according to the derivation of Lees and Lin [4].

The following notation is introduced for the dependent variables:

$$\begin{aligned} Z_1 &= f & Z_2 &= f'' & Z_3 &= \phi \\ Z_4 &= \frac{\pi}{\gamma M_1^2} & Z_5 &= \xi & Z_6 &= \xi' \end{aligned} \quad (2.31)$$

This notation is identical to Lees and Lin except for Z_4 , which is the notation used by Mack [6] and is $1/\gamma$ times the Lees-Lin Z_4 . Upon substitution of these new dependent variables into the equations derived in the first part of the chapter, equations analogous to the ones Mack derived are obtained. The equations will be written in matrix form as follows:

$$\frac{dZ_i}{d\eta} = \sum_{j=1}^6 a_{ij} Z_j \quad i = 1, \dots, 6 \quad (2.32)$$

The coefficients, a_{ij} , are

$$\begin{aligned} a_{11} &= 0 \\ a_{12} &= 1 \\ a_{13} &= 0 \\ a_{14} &= 0 \\ a_{15} &= 0 \\ a_{16} &= 0 \end{aligned} \quad (2.33)$$

$$a_{21} = \frac{i\alpha R}{\mu} \frac{(u-c)}{T} p + \alpha^2$$

$$a_{22} = \frac{T'}{\mu} \frac{\partial \mu}{\partial t}$$

$$a_{23} = \left[\frac{\alpha R u'}{\mu T} - i\alpha^2 \frac{T'}{\mu} \frac{\partial u}{\partial T} - i \frac{1}{3} (1 + 2k) \alpha^2 \frac{2T'}{T} \right]$$

(2.34)

$$a_{24} = \frac{1}{3} (1 + 2k) \alpha^2 \frac{(u-c)}{T} - \frac{u''}{\mu} \frac{d\mu}{dT} - \frac{u'T}{\mu} \frac{d^2\mu}{dT^2}$$

$$a_{25} = \frac{1}{3} (1 + 2k) \alpha^2 \frac{(u-c)}{T} - \frac{u''}{\mu} \frac{d\mu}{dT} - \frac{u'T'}{\mu} \frac{d^2\mu}{dT^2}$$

$$a_{26} = - \frac{u'}{\mu} \frac{d\mu}{dT}$$

$$a_{31} = - i$$

$$a_{32} = 0$$

$$a_{33} = - \frac{1}{\rho} \frac{d\rho}{d\eta} \frac{dT}{dT} = \frac{T'}{T}$$

(2.35)

$$a_{34} = (-i(u-c)\gamma M_1^2)$$

$$a_{35} = \frac{i(u-c)}{T}$$

$$a_{36} = 0$$

$$a_{41} = -\frac{i}{L} \left[\frac{2}{3} (2+k) \frac{T'}{T} + 2 \frac{T'}{\mu} \frac{d\mu}{dT} \right]$$

$$a_{42} = -\frac{i}{L}$$

$$a_{43} = \frac{1}{L} \left[-\alpha^2 + \frac{2}{3} (2+k) \frac{T'^2}{T} \frac{1}{\mu} \frac{d\mu}{dT} + \frac{2}{3} (2+k) \frac{T''}{T} - \right.$$

$$\left. \frac{i\alpha R}{\mu T} (u - c_r) - \frac{\alpha R c_i}{\mu T} \right] , \quad (2.36)$$

$$a_{44} = -i \frac{2}{3} (2+k) \gamma M_1^2 \frac{1}{L} \left[(u-c) \frac{T'}{\mu} \frac{d\mu}{dT} + u' + (u-c) \frac{T'}{T} \right]$$

$$a_{45} = \frac{i}{L} \left[\frac{u'}{\mu} \frac{d\mu}{dT} + \frac{2}{3} (2+k) \frac{(u-c)}{T} \frac{T'}{\mu} \frac{d\mu}{dT} + \frac{2}{3} (2+k) \frac{u'}{T} \right]$$

$$a_{46} = \frac{i}{L} \left[\frac{2}{3} (2+k) \frac{(u-c)}{T} \right]$$

$$a_{51} = 0$$

$$a_{52} = 0$$

$$a_{53} = 0$$

$$a_{54} = 0$$

$$a_{55} = 0$$

$$a_{56} = 1$$

(2.37)

$$a_{61} = 0$$

$$a_{62} = -2\sigma (\gamma - 1) M_1^2 u'$$

$$a_{63} = \frac{\alpha T \sigma}{\mu} \frac{T'}{T} - i 2\sigma (\gamma - 1) M_1^2 \alpha^2 u'$$

. (2.38)

$$a_{64} = -\frac{i\alpha R \sigma}{\mu} (\gamma - 1) M_1^2 (u-c)$$

$$a_{65} = \frac{i\alpha R \sigma}{\mu T} (u-c) + \alpha^2 - \sigma (\gamma - 1) M_1^2 \frac{u'^2}{\mu} \frac{d\mu}{dT} - \frac{T''}{\mu} \frac{d\mu}{dT} - \frac{T'^2}{\mu} \frac{d^2\mu}{dT^2}$$

$$a_{66} = -2 \frac{d\mu}{dT} \frac{T'}{\mu}$$

In these equations,

$$c = c_r + ic_i$$

$$L = \frac{R}{\alpha\mu} + \frac{2}{3} i (2 + k) \gamma M_1^2 (u-c) , \quad (2.39)$$

$$k = \frac{\mu^2}{\mu}$$

and again, all primes denote differentiation with respect to the Blasius variable, η .

The method which is employed for the solution of the stability equations also requires the derivatives of the above set of equations with respect to c , the wave speed. The reason for this will be discussed in a later section. The set of equations can be written in an analogous matrix form as

$$\beta_i' = \sum_{j=1}^6 [(h_{ij} Z_j) + (a_{ij} \beta_j)] \quad \text{for } i = 1, \dots, 6, \quad (2.40)$$

here,

$$h_{ij} = \frac{da_{ij}}{dc},$$

$$\beta_i' = \frac{dZ_i'}{dc},$$

$$\beta_j = \frac{dZ_{ij}}{dc},$$

For brevity, the equations will just be given in the form of a system of equations.

$$\beta_1' = \beta_2, \quad (2.41)$$

$$\begin{aligned} \beta_2' = & \left[\frac{i\alpha R}{\mu T} (u-c) + \alpha^2 \right] \beta_1 - \frac{i\alpha R}{\mu T} Z_1 - \frac{T'}{\mu} \frac{d\mu}{dT} \beta_2 + \left[\frac{\alpha R}{\mu T} u' - i\alpha^2 \frac{T'}{\mu} \frac{d\mu}{dT} \right. \\ & \left. - i \frac{1}{3} (1 + 2k) \alpha^2 \frac{T'}{T} \right] \beta_3 + \left[\frac{i\alpha R}{\mu} - \frac{1}{3} (1 + 2k) \gamma M_1^2 \alpha^2 (u-c) \right] \beta_4 \\ & + \frac{1}{3} (1 + 2k) \gamma M_1^2 \alpha^2 Z_4 + \left[\frac{1}{3} (1 + 2k) \alpha^2 \frac{(u-c)}{T} \right. \\ & \left. - \frac{u''}{dT} \frac{d\mu}{\mu} - \frac{u'T'}{\mu} \frac{d^2\mu}{dT^2} \right] \beta_5 - \frac{1}{3} (1 + 2k) \frac{\alpha^2}{T} Z_5 - \frac{u'}{\mu} \frac{d\mu}{dT} \beta_6, \quad (2.42) \end{aligned}$$

$$\begin{aligned} \beta_3' = & -i\beta_1 + \frac{T'}{T} \beta_3 - i(u-c) \gamma M_1^2 \beta_4 + i\gamma M_1^2 Z_4 \\ & + i \frac{(u-c)}{T} \beta_5 - \frac{i}{T} Z_5, \quad (2.43) \end{aligned}$$

$$\begin{aligned}
\beta_4' = & -\frac{1}{L} \left[2 \frac{T'}{\mu} \frac{d\mu}{dT} + \frac{2}{3} (2+k) \frac{1}{T} \right] \beta_1 - \frac{1}{L'} \left[2 \frac{T'}{\mu} \frac{d\mu}{dT} + \frac{2}{3} (2+k) \frac{T'}{T} \right] Z_1 \\
& - \frac{i}{L} \beta_2 - \frac{1}{L'} Z_2 + \frac{1}{L} \left[-\alpha^2 + \frac{2}{3} (2+k) \frac{T'^2}{T} \frac{1}{\mu} \frac{d\mu}{dT} \right. \\
& + \left. \frac{2}{3} (2+k) \frac{T''}{T} - \frac{i\alpha R(u-c)}{\mu T} \right] \beta_3 + \frac{1}{L'} \left[-\alpha^2 + \frac{2}{3} (2+k) \frac{T'^2}{T} \frac{1}{\mu} \frac{d\mu}{dT} \right. \\
& + \left. \frac{2}{3} (2+k) \frac{T''}{T} - \frac{i\alpha R}{\mu T} (u-c) \right] Z_3 + \frac{1}{L} \frac{i\alpha R}{\mu T} Z_3 - i \frac{2}{3} (2+k) \gamma M_1^2 \frac{1}{L} \\
& \left[(u-c) \frac{T'}{\mu} \frac{d\mu}{dT} + u' + (u-c) \frac{T'}{T} \right] \beta_4 - i \frac{2}{3} (2+k) \gamma M_1^2 \frac{1}{L'} \\
& \left[(u-c) \frac{T'}{\mu} \frac{d\mu}{dT} + u' + (u-c) \frac{T'}{T} \right] Z_4 - i \frac{2}{3} (2+k) \gamma M_1^2 \frac{1}{L} \\
& \left[-\frac{T'}{\mu} \frac{d\mu}{dT} - \frac{T'}{T} \right] Z_4 + \frac{i}{L} \left[\frac{u'}{\mu} \frac{d\mu}{dT} + \frac{2}{3} (2+k) \frac{(u-c)}{T} \frac{T'}{\mu} \frac{d\mu}{dT} \right. \\
& + \left. \frac{2}{3} (2+k) \frac{u'}{T} \right] \beta_5 + \frac{i}{L'} \left[\frac{u'}{\mu} \frac{d\mu}{dT} + \frac{2}{3} (2+k) \frac{(u-c)}{T} \frac{T'}{\mu} \frac{d\mu}{dT} \right. \\
& + \left. \frac{2}{3} (2+k) \frac{u'}{T} \right] Z_5 + \frac{i}{L} \left[-\frac{2}{3} (2+k) \frac{T'}{\mu T} \frac{d\mu}{dT} \right] Z_5 \\
& + \frac{i}{L} \frac{2}{3} (2+k) \frac{(u-c)}{T} \beta_6 + \frac{i}{L'} \frac{2}{3} (2+k) \frac{(u-c)}{T} Z_6 \\
& - \frac{i}{L} \frac{2}{3} (2+k) \frac{Z_6}{T} \quad , \quad (2.44)
\end{aligned}$$

$$\beta_5' = \beta_6 \quad , \quad (2.45)$$

$$\begin{aligned}
\beta_6' = & -2\sigma (\gamma-1) M_1^2 u' \beta_2 + \left[\frac{\alpha R \sigma}{\mu T} T' - i 2\sigma (\gamma-1) M_1^2 \alpha^2 u' \right] \beta_3 \\
& - \frac{i \alpha R \sigma}{\mu} (\gamma-1) M_1^2 (u-c) \beta_4 + \frac{i \alpha R \sigma}{\mu} (\gamma-1) M_1^2 Z_4 \\
& + \left[\frac{i \alpha R \sigma}{\mu T} (u-c) + \alpha^2 - \sigma (\gamma-1) M_1^2 \frac{u'^2}{\mu} \frac{d\mu}{dT} - \frac{T''}{\mu} \frac{d\mu}{dT} - \frac{T'^2}{\mu} \frac{d^2\mu}{dT^2} \right] \beta_5 \\
& - \frac{i \alpha R \sigma}{\mu T} Z_5 - 2 \frac{T'}{\mu} \frac{d\mu}{dT} \beta_6 \quad , \quad (2.46)
\end{aligned}$$

and where,

$$L' = -i \frac{2}{3} (2+k) \gamma M_1^2 .$$

The next step which must be taken is the specification of the boundary conditions which the physical flow and thus, the governing stability equations, must satisfy. The boundary conditions on this set of equations are: The two disturbance velocity components u' are v' zero at $\eta = 0$; the heat transfer is continuous at $\eta = 0$; and all disturbances go to zero as $\eta \rightarrow \infty$:

$$Z_1(0) = 0 \quad Z_3(0) = 0 \quad Z_5(0) = 0$$

$$Z_1(\eta_\delta) = 0 \quad Z_3(\eta_\delta) = 0 \quad Z_5(\eta_\delta) = 0$$

The frequency of the oscillations considered is also sufficiently high so that the solid wall remains at a fixed temperature,

The six boundary conditions can be written

$$\begin{aligned}
Z_1(0) = 0 \quad Z_3(0) = 0 \quad Z_5(0) = 0 \\
Z_1, Z_3, Z_5 \rightarrow 0 \quad \text{as } \eta \rightarrow \infty
\end{aligned} \quad , \quad (2.47)$$

and since these six boundary conditions are homogeneous, the boundary conditions themselves are insufficient to establish any solution of the six equations other than the trivial solution of zero. Therefore, the formulated problem is an eigenvalue problem in which only certain combinations of the parameters α , R , and c are sufficient to satisfy the boundary conditions for nonzero solutions.

A factor which must be considered while solving this problem is that the presence of the term αR in the equations, which can be numerically large, can present a difficulty in the numerical integration of the equations. The asymptotic theory shows that for large αR , these equations have exponentially growing solutions with the exponent proportional to $\pm\sqrt{\alpha R}$, and the error growth which follows this solution presents problems in a step-by-step integration. The solution which is only slightly incorrect will quickly diverge in the step-by-step integration process. Consequently, the approach which was used to solve the mean flow equations, which was to guess the unknown initial conditions and correct them, cannot be used for this system of less well-behaved equations. There must be some more systematic method of obtaining initial conditions for solution of these equations.

The method which will be utilized will be to build up the solution to the problem from three linearly independent solutions, which are obtained from the solutions of the equations with constant coefficients to which the complete equations reduce for $\eta \geq \eta_\delta$. These three linearly independent solutions go to zero as $\eta \rightarrow \infty$, so

that the boundary conditions at infinity are automatically satisfied. The analytic solution of the equations for $\eta \geq \eta_\delta$ will be given in Chapter IV.

From this point, the method of solution begins by using the three linearly independent solutions obtained at η_δ , so initial conditions for solving the stability equations from η_δ to the wall, $\eta = 0$, for certain values of α , R , μ , and c . This process will yield three linearly independent solutions at $\eta = c$. Along with these equations, also numerically integrated are the derivatives of the system of equations with respect to

$$\frac{\partial Z_i}{\partial c} = \frac{\partial}{\partial c} \left[\sum_{j=1}^6 a_{ij} Z_j \right], \quad i = 1, \dots, 6. \quad (2.48)$$

The initial conditions for these equations will also be given in Chapter IV, and they are just the derivatives of the set of initial conditions for the Z_i system differentiated with respect to c . This means that there will be three linearly independent solutions at $\eta = 0$ for the Z -system and three for the $\frac{\partial Z}{\partial c}$ -system. There is also an orthonormalization process utilized in the integration of these equations because of the tendency of the originally linearly independent vectors to become more and more independent. The use of this orthonormalizing technique assures that the solutions at $\eta = 0$ will be linearly independent. This technique is further explained in Chapter VI.

CHAPTER III

SOLUTION OF EQUATIONS FOR $\eta \geq \eta_\delta$

The boundary conditions imposed at infinity may be recast in a more quantitative form such that it may be used in the numerical integration scheme. Namely, the perturbation equations under the condition $\eta \rightarrow \infty$ reduce to a linear system with constant coefficients, which may be solved explicitly.

In the free-stream, $\eta \geq \eta_\delta$, the normalized velocity and temperature are asymptotically equal to one; that is,

$$\begin{aligned} u &\rightarrow 1 \\ T &\rightarrow 1 \end{aligned} ,$$

and all derivatives of the flow quantities are zero:

$$\frac{\partial}{\partial u} = 0 \quad \frac{\partial}{\partial T} = 0 .$$

The system of equations governing the perturbation functions under the above conditions reduces to

$$Z_1' = Z_2 , \tag{3.1}$$

$$\begin{aligned} Z_2' = & (i\alpha R (1-c) + \alpha^2) Z_1 + i\alpha R Z_4 - \frac{1}{3} (1 + 2k) \gamma M_1^2 \alpha^2 (1-c) Z_4 \\ & + \frac{1}{3} (1 + 2k) \alpha^2 (1-c) Z_5 , \end{aligned} \tag{3.2}$$

$$Z_3' = -iZ_1 = i(1-c) \gamma M_1^2 Z_4 + i(1-c) Z_5, \quad (3.3)$$

$$Z_4' = -\frac{i}{L} Z_2 - \frac{\alpha^2}{L} Z_3 - \frac{i\alpha R}{L} (1-c) Z_3 + \frac{i}{L} \frac{2}{3} (2+k)(1-c) Z_6, \quad (3.4)$$

$$Z_5' = Z_6, \quad (3.5)$$

$$Z_6' = -i\alpha R\sigma (\gamma-1) M_1^2 (1-c) Z_4 + [i\alpha R\sigma (1-c) + \alpha^2] Z_5, \quad (3.6)$$

and the general solution to these equations is of the form

$$Z_i = \sum_{j=1}^6 c_j A_i^{(j)} e^{\lambda_j(\eta - \eta_\delta)} \quad i = 1, \dots, 6, \quad (3.7)$$

where the λ_j are the six characteristic values of the coefficient matrix; a_{ij} are the 36 components of the six characteristic vectors, $A^{(j)}$; and the six c_j are arbitrary constants to be determined by the six boundary conditions for the problem.

A further simplification of the algebra required for the problem can be obtained by transforming the first-order system into a second-order system as follows: Let

$$V_1 = Z_1 \quad V_2 = Z_2 \quad V_3 = Z_3, \quad (3.8)$$

and the resulting set of equations will have the form

$$V_1'' = b_{11} V_1 + b_{12} V_2 + b_{13} V_3, \quad (3.9)$$

$$V_2'' = b_{22} V_2 + b_{23} V_3, \quad (3.10)$$

$$V_3'' = b_{32} V_2 + b_{33} V_3. \quad (3.11)$$

If

$$V_1' = Z_1' = Z_2$$

$$V_1'' = Z_2'$$

the coefficients for the V_1'' -equation will be

$$b_{11} = i\alpha R (1-c) + \alpha^2, \quad (3.12)$$

$$b_{12} = i\alpha R - \frac{1}{3} (1 + 2k) \gamma M_1^2 \alpha^2 (1-c), \quad (3.13)$$

$$b_{13} = \frac{1}{3} (1 + 2k) \alpha^2 (1-c). \quad (3.14)$$

For the V_2'' -equation,

$$V_2' = Z_4'$$

$$V_2'' = Z_4'' = -\frac{i}{L} Z_2' - \frac{\alpha^2}{L} Z_3' - \frac{i}{L} \alpha R (1-c) Z_3' + \frac{i}{L} \frac{2}{3} (2 + k)(1-c) Z_6',$$

and upon substituting for the Z quantities the necessary quantities of Eq. (3.8) are obtained in the coefficients for V_2'' ,

$$b_{21} = 0,$$

$$b_{22} = \frac{1}{L} \left[\alpha R + i \frac{1}{3} (1 + 2k) \gamma M_1^2 \alpha^2 (1-c) - \alpha R (1-c)^2 \gamma M_1^2 + i \alpha^2 \gamma M_1^2 (1-c) + \frac{2}{3} (2 + k) \alpha R \sigma (\gamma - 1)(1-c)^2 M_1^2 \right], \quad (3.15)$$

$$b_{23} = \frac{1}{L} \left[-i \frac{1}{3} (1 + 2k)(1-c) \alpha^2 - i \alpha^2 (1-c) + \alpha R (1-c)^2 - \left[\frac{2}{3} (2 + k) \alpha R \sigma (1-c)^2 \right] + i \frac{2}{3} (2 + k)(1-c) \alpha^2 \right]. \quad (3.16)$$

In the same way, by differentiation and substitution, the coefficients for the V_3'' -equation are obtained:

$$b_{31} = 0 ,$$

$$b_{32} = - \frac{i\alpha R\sigma}{\mu} (\gamma-1) M_1^2 (1-c) , \quad (3.17)$$

$$b_{33} = \frac{i\alpha R\sigma}{\mu} (1-c) + \alpha^2 . \quad (3.18)$$

The characteristic values of this second-order system of equations are the same as for the original system of equations in first-order form (Eqs. (3.1) through (3.6)), and the form of the solutions to the equations for $\eta \geq \eta_\delta$ are

$$V_i = \sum_{j=1}^6 D_j B_i^{(j)} e^{\lambda_j(\eta - \eta_\delta)} . \quad (3.19)$$

In this equation the $B_i^{(j)}$ are the 18 components of the six characteristic vectors, $B_i^{(j)}$, and the six D_j are arbitrary constants of integration. The substitution of an individual term of Eq. (3.19) into (3.9), (3.10), and (3.11) yields

$$\begin{aligned} (b_{11} - \lambda_j^2) B_1^{(j)} + b_{12} B_2^{(j)} + b_{13} B_3^{(j)} &= 0 \\ (b_{22} - \lambda_j^2) B_2^{(j)} + b_{23} B_3^{(j)} &= 0 , \\ b_{33} B_3^{(j)} + (b_{33} - \lambda_j^2) B_3^{(j)} &= 0 \end{aligned} \quad (3.20)$$

and the determinant of the coefficient matrix is set equal to zero to yield the three characteristic values given in (3.19) above.

$$\begin{bmatrix} (b_{11} - \lambda_j^2) & b_{12} & b_{13} \\ 0 & (b_{22} - \lambda_j^2) & b_{23} \\ 0 & b_{32} & (b_{33} - \lambda_j^2) \end{bmatrix} = 0$$

$$(b_{11} - \lambda_j^2) \left[(b_{22} - \lambda_j^2)(b_{33} - \lambda_j^2) - b_{23} b_{32} \right] = 0, \quad (3.21)$$

$$(b_{11} - \lambda_j^2) = 0, \quad (3.22)$$

$$\lambda_1^2 = b_{11}, \quad (3.23)$$

$$\lambda_1 = -b_{11}^{1/2}$$

where the positive square root is eliminated because it does not satisfy the boundary conditions of the perturbation function going to zero as $\eta \rightarrow \infty$.

Following the same line of reasoning as above yields λ_3 as

$$\lambda_3 = - \left\{ \frac{1}{2} (b_{22} + b_{23}) + \left[\frac{1}{4} (b_{22} - b_{23})^2 + b_{23} b_{32} \right]^{1/2} \right\}^{1/2}, \quad (3.24)$$

so that for $j = 1$ the set (3.20) becomes

$$(b_{11} - b_{11}) \beta_1^{(1)} + b_{12} \beta_2^{(1)} + b_{13} \beta_3^{(1)} = 0$$

$$(b_{22} - b_{11}) \beta_2^{(1)} + b_{23} \beta_3^{(1)} = 0, \quad (3.25)$$

$$b_{32} \beta_2^{(1)} + (b_{23} - b_{11}) \beta_3^{(1)} = 0$$

and upon solving these,

$$B_3^{(1)} = 0 \quad B_2^{(1)} = 0 \quad B_1^{(1)} = 1 ,$$

where the arbitrary magnitude of $B_1^{(1)}$ has been set equal to (1,0). For $j = 2,3$ the equations specify only the ratios $B_2^{(j)}/B_3^{(j)}$. Upon normalizing the individual magnitudes are

$$\begin{aligned} j &= 2,3 \\ B_1^{(j)} &= b_{12} (b_{33} - \lambda_j^2) - b_{13} b_{32} \\ B_2^{(j)} &= b_{33} - \lambda_j^2 \\ B_3^{(j)} &= -b_{32} \end{aligned} \quad (3.26)$$

The components of the $A^{(j)}$ for the original first-order system in terms of the second-order system are taken from the definitions given in Eq. (3.8):

$$\begin{aligned} A_1^{(j)} &= B_1^{(j)} \\ A_2^{(j)} &= \lambda_j B_1^{(j)} \\ A_3^{(j)} &= \frac{i}{\lambda_j} \left[-B_1^{(j)} - \lambda M_1^2 (1-c) B_2^{(j)} + (1-c) B_3^{(j)} \right] \\ A_4^{(j)} &= B_2^{(j)} \\ A_5^{(j)} &= B_3^{(j)} \\ A_6^{(j)} &= \lambda_j B_3^{(j)} \end{aligned} \quad (3.27)$$

Before the actual vectors are given which will be used as the three linearly independent sets of initial values, the limit of large αR will be taken to the same order of accuracy as used in the Dunn-Linn theory. The eigenvalues then become

$$\lambda_1 = - \left[i\alpha R (1-c) \right]^{1/2}, \quad (3.28)$$

$$\lambda_2 = - \left[i\alpha R\sigma (1-c) \right]^{1/2}, \quad (3.29)$$

$$\lambda_3 = - \alpha \left[1 - M_1^2 (1-c)^2 \right]^{1/2}. \quad (3.30)$$

After using the information obtained from Eqs. (3.25), (3.26), (3.27), (3.28), (3.29), and (3.30), the three linearly independent vectors to be used as initial conditions for the numerical integration of the stability equation for $\eta \geq \eta_\delta$ are given as

$$\begin{aligned} A_1^{(1)} &= 1 \\ A_2^{(1)} &= - (i\alpha R (1-c))^{1/2} \\ A_3^{(1)} &= - \frac{i}{(i R(1-c))^{1/2}}, \\ A_4^{(1)} &= 0 \\ A_5^{(1)} &= 0 \\ A_6^{(1)} &= 0 \end{aligned} \quad (3.31)$$

$$\begin{aligned}
A_1^{(2)} &= 0 \\
A_2^{(2)} &= 0 \\
A_3^{(2)} &= - (i\alpha R\sigma)^{1/2} (1-c)^{1/2} \left[i(\gamma-1) M_1^2 (1-c) \right] \\
A_4^{(2)} &= 0 \\
A_5^{(2)} &= i\alpha R\sigma (\gamma-1) M_1^2 (1-c_r) + \alpha R\sigma (\gamma-1) M_1^2 c_i \\
A_6^{(2)} &= - \left[i\alpha R\sigma (1-c) \right]^{1/2} i\alpha R\sigma (\gamma-1) M_1^2 (1-c)
\end{aligned} \tag{3.32}$$

$$\begin{aligned}
A_1^{(3)} &= - i\alpha R\sigma \\
A_2^{(3)} &= - i\alpha^2 \left[1 - M_1^2 (1-c)^2 \right]^{1/2} R\sigma \\
A_3^{(3)} &= - i\alpha R\sigma \frac{i}{\alpha} \left[1 - M_1^2 (1-c)^2 \right]^{1/2} \\
A_4^{(3)} &= i\alpha R\sigma (1-c) \\
A_5^{(3)} &= i\alpha R\sigma (\gamma-1) M_1^2 (1-c) \\
A_6^{(3)} &= - i\alpha^2 \left[1 - M_1^2 (1-c)^2 \right]^{1/2} R\sigma (\gamma-1) M_1^2 (1-c)
\end{aligned} \tag{3.33}$$

Similarly, the three linearly independent vectors that are derivatives with respect to c are

$$\frac{dA_1^{(1)}}{dc} = 0$$

$$\frac{dA_2^{(1)}}{dc} = \frac{1}{2} \left(\frac{i+1}{\sqrt{2}} \right) (\alpha R)^{1/2} \left[\frac{1}{(1-c)^{1/2}} \right]$$

$$\frac{dA_3^{(1)}}{dc} = - \frac{1}{2} \left(\frac{1+1}{\sqrt{2}} \right) \frac{1}{(\alpha R)^{1/2}} \left[\frac{1}{(1-c)^{3/2}} \right], \quad (3.34)$$

$$\frac{dA_4^{(1)}}{dc} = 0$$

$$\frac{dA_5^{(1)}}{dc} = 0$$

$$\frac{dA_6^{(1)}}{dc} = 0$$

$$\frac{dA_1^{(2)}}{dc} = 0$$

$$\frac{dA_2^{(2)}}{dc} = 0$$

$$\frac{dA_3^{(2)}}{dc} = \frac{3}{2} i \left(\frac{i+1}{\sqrt{2}} \right) (\alpha R \sigma)^{1/2} (\gamma-1) M_1^2 (1-c)^{1/2}, \quad (3.35)$$

$$\frac{dA_4^{(2)}}{dc} = 0$$

$$\frac{dA_5^{(2)}}{dc} = - i \alpha R \sigma (\gamma-1) M_1^2$$

$$\frac{dA_6^{(2)}}{dc} = \frac{3}{2} \left(\frac{i-1}{\sqrt{2}} \right) (\alpha R \sigma)^{3/2} (1-c)^{1/2} (\gamma-1) M_1^2$$

$$\frac{dA_1^{(3)}}{dc} = 0$$

$$\frac{dA_2^{(3)}}{dc} = -i\alpha^2 \sigma R \left[\frac{M_1^2(1-c)}{(1-M_1^2(1-c))^{1/2}} \right]$$

$$\frac{dA_3^{(3)}}{dc} = \sigma R M_1^2 (1-c) \left[\frac{1}{[1-M_1^2(1-c)]^{1/2}} \right]$$

$$\frac{dA_4^{(3)}}{dc} = -i\alpha R \sigma \quad (3.36)$$

$$\frac{dA_5^{(3)}}{dc} = -i\alpha R \sigma (\gamma-1) M_1^2$$

$$\begin{aligned} \frac{dA_6^{(3)}}{dc} = & i\alpha^2 \left[1 - M_1^2 (1-c)^2 \right]^{1/2} R \sigma M_1^2 (\gamma-1) \\ & - \left[i\alpha^2 \sigma R (\gamma-1) M_1^2 (1-c)^2 M_1^2 \right] \\ & \left[\frac{1}{(1-M_1^2(1-c)^2)^{1/2}} \right] \end{aligned}$$

These are the initial values to be used in the computer program for the numerical integration of the stability equations across the boundary layer. Since these are the analytic solution for the stability equations for $\eta \geq \eta_\delta$, they are not guessed initial values and there will be no need for an iteration process to correct guess values.

CHAPTER IV

MATHEMATICAL BASIS OF THE GLOBAL SOLUTION OF THE STABILITY EQUATIONS

The first-order system of differential equations, that is, Eq. (2.32), describing the stability of the laminar compressible boundary along with the associated physical boundary conditions, Eq. (2.47), constitute a boundary-value problem. Since the system of equations and its boundary conditions are homogeneous, the problem posed will yield nontrivial solutions only for certain values of the parameters upon which the differential equations depend. The values of these parameters for which nontrivial solutions exist are referred to as eigenvalues, and the corresponding nontrivial solutions are referred to as eigenfunctions or eigensolutions.

Shooting-type methods of numerical integration schemes are not suited to solving such boundary-value problems because initial conditions are necessary for all the dependent variables in order to begin the numerical integration procedure. Therefore, the boundary-value problem must be transformed into an initial value problem in which all of the boundary conditions are specified at the same value of the independent variable. This transformation can be made by various techniques; however, if certain physical aspects of the flow are considered, an analytic transformation is available for the problem considered in this investigation. The full stability equations can be reduced to a form which can be solved analytically

for $\eta \geq \eta_\delta$. This is the free-stream solution of the stability equations and the solution method used was the one originated by Mack [7]. Using this method of solution, three linearly independent solution vectors are obtained for the free-stream and are applicable for $\eta \geq \eta_\delta$. So the original boundary-value problem, where the boundary conditions were specified at two points, has been transformed to an initial value problem in which the three linearly independent vectors are used as initial conditions for a numerical integration from η_δ , the edge of the boundary layer, across the boundary layer to the wall, $\eta = 0$.

This numerical integration is performed by a Runge-Kutta-Fehlberg sixth-seventh-order scheme [5] which provides the option of using variable step-size control. Even with the control of the step-size, there is an error problem associated with the finite word length of the computer and also because of the ill-conditioned matrix of coefficients. These factors cause the original three linearly independent vectors to become more dependent as the step-wise process continues. This type of situation can lead to poor or incorrect results. This problem can, however, be overcome with the aid of a method proposed by Godunov [8], which is found to belong to the multiple shooting class. He proposed that the solution vectors be kept mutually orthogonal at each step of the integration process. This method was somewhat modified by Conte [9] to make it computationally more efficient and more accurate. It is a method designed in the spirit of Conte's method, and derived by using projection operators, which will be utilized in this work. The method is

generally applicable in either a complex or real space and will be detailed in Chapter VII.

Assuming that the three linearly independent vectors can be integrated across the boundary layer to yield accurate solutions, the next step is to make use of several mathematical theorems. By using a theorem from Naimark [10], an analytic function can be constructed and the zeros of this function are the zeros of the original differential system cast into an operator system. This analytic function is constructed from the fundamental system of the three Cauchy solutions of the differential system collectively satisfying linearly independent sets of initial conditions at $\eta = 0$. Or, as in the problem at hand, the analytic function is constructed from the results obtained at $\eta = 0$ from the integration of the three linearly independent vectors forming three linearly independent Cauchy problems; that is,

$$f(c) = \det \begin{vmatrix} z_1^1(0) & z_1^2(0) & z_1^3(0) \\ z_3^1(0) & z_3^2(0) & z_3^3(0) \\ z_5^1(0) & z_5^2(0) & z_5^3(0) \end{vmatrix}. \quad (4.1)$$

Now, since the differential system is nonsingular in the region of interest, $f(c)$ is an integral analytic function. Now that the integral analytic function has been obtained whose zeros are the zeros of the stability differential system of Eq. (2.32), the problem has been reduced to one of finding the zeros of the analytic function,

$f(c)$. This can be accomplished by either a local iterative technique, as discussed in Chapter V, or a global technique, as will be presented now. In this work, the global technique of locating eigenvalues developed by Antar [2] is employed which first uses an appropriate theorem from complex variables. The theorem states that in a closed region, which is taken to be a circle in this work, the contour integral [11] is,

$$\frac{1}{2\pi i} \int_{\gamma} \frac{f'(c)}{f(c)} dc = M, \quad (4.2)$$

where M is the number of eigenvalues, if there are no poles or other singularities in the region of interest. Since the characteristic determinant is an integral analytic function having no singularities in the simply closed region, this integral definitely yields the number of eigenvalues. A generalization of this contour integral admits a simple relation that will be utilized later; that is,

$$\frac{1}{2\pi i} \int_{\gamma} w(c) \frac{f'(c)}{f(c)} dc = \sum_{s=1}^N q_s w(c_s), \quad (4.3)$$

where the sum is taken over the zeros of $f(c)$, and q_s is the order of the zeros. The proof of this generalization is found in Goodstein [11].

Since the problem is solved numerically, the above-mentioned contour integral (Eq. (4.2)) must be approximated numerically. This is accomplished by using a suitable quadrature formula for approximating the integral. Once the number of zeros inside the simply

closed region have been determined, the next step is to determine their location. This can be done by letting the weighting function, $w(c)$ in Eq. (4.3), take the successive values, $c, c^2, \dots, c^N$. Thus, all of the power sums of the zeros inside γ may be determined by performing the N contour integrations.

$$\begin{aligned} s_1 &= \int_{\gamma} c \frac{f'(c)}{f(c)} dc = \sum_{j=1}^N c_j \\ s_2 &= \int_{\gamma} c^2 \frac{f'(c)}{f(c)} dc = \sum_{j=1}^N c_j^2, \\ s_3 &= \int_{\gamma} c^N \frac{f'(c)}{f(c)} dc = \sum_{j=1}^N c_j^N \end{aligned} \quad (4.4)$$

where c_j denotes all the zeros of $f(c)$ inside the simply closed region. These contour integrals are evaluated numerically by the same type of quadrature method used to calculate Eq. (4.2). After the complex numbers have been found, a polynomial $P(c)$ of degree N can be constructed whose zeros coincide with the zeros of $f(c)$ inside γ and are not an approximation of $f(c)$. This polynomial is constructed by using Newton's formula in which the coefficients of the polynomial are evaluated from the power sums of the roots of the polynomial. After this polynomial is constructed, any root finder may be employed to extract the zeros of the polynomial $P(c)$.

The roots of the polynomial can easily be extracted and, therefore, in principle, the problem of finding all the eigenvalues for the stability of the laminar compressible boundary layer has been

reduced to the evaluation of the necessary contour integrals. The evaluation of a contour integral, however, requires the evaluation of the function at several points around the simply closed region in order to approximate the contour integral, Figure 6. As in Figure 6, the approximation of a contour integral is performed by beginning with eight function evaluations at an equal interval around the circle, as represented by the long slashes. If this is not a good enough approximation to the contour integral, the intervals are halved and the number of function evaluations are therefore doubled. This process continues until a good enough approximation of the contour integral has been obtained.

Each of these function evaluations requires an integration of the differential system, Eqs. (2.33) through (2.38) and (2.41) through (2.46), along with the associated initial conditions, Eqs. (3.31) through (3.36). So, as the number of function evaluations around the circle increases, the computer time consumed grows very rapidly. It may be possible to reduce the required number of function evaluations by several methods. Future considerations should include an examination of some functional approximation techniques, such as discussed by Lyness and Delves [12] or Gaster [13]. More will be discussed concerning these aspects in Chapter VI, which discusses the numerical results obtained from the global method.

The techniques contained in this chapter are incorporated in a numerical program that searches for the eigenvalues associated with the governing equations (Eqs. (2.28) through (2.30)). The operation of this global method code is discussed in Chapter VI.

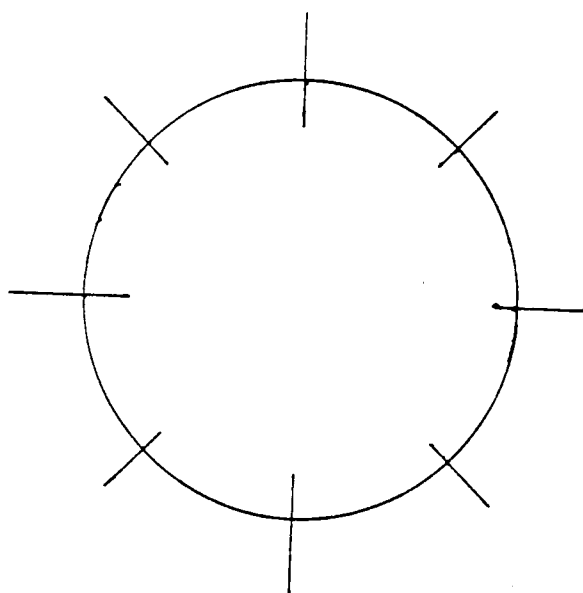


Figure 6. Illustration of the approximation scheme for a contour integral.

CHAPTER V

SOLUTION OF THE STABILITY EQUATIONS BY LOCAL ITERATIVE METHODS

The local method which is considered in this work is to be used as a companion numerical program for inspecting the operation and performance of the global technique presented previously (Chapter III). The first part of the present chapter contains the information necessary for obtaining the local iterative method considered, while the latter part contains the results obtained from the solution of the system of stability equations derived in Chapter II ((2.33) through (2.38)) under various physical conditions.

The system of stability equations ((2.33) through (2.38)), along with the associated initial conditions, Eq. (2.47), is integrated numerically using the variable step-size Runge-Kutta-Fehlberg method as in the global technique. Completion of this integration yields $Z^{(1)}(0)$, $Z^{(2)}(0)$, and $Z^{(3)}(0)$ for a specified α , R , and c . These are the three linearly independent vectors at the wall, $\eta = 0$. Combining these solutions linearly will satisfy two of the three boundary conditions at $\eta = 0$. The final boundary condition can only be satisfied by this combined solution when a set of eigenvalues has been found. In addition to these two equations obtained from the consideration of the boundary conditions, a third equation specifying a magnitude is needed to avoid producing the trivial combined solution of zero. The selected condition for this purpose is that $Z_1(\eta_0) = (1,0)$,

where $Z_1(\eta_\delta)$ is complex and the values in parentheses are its real and imaginary parts, respectively. The term N is a normalization constant which is also a complex number equal to $(1,0)$. The three equations therefore become

$$c_1 Z_1^{(1)}(0) + c_2 Z_1^{(2)}(0) + c_3 Z_1^{(3)}(0) = 0, \quad (5.1)$$

$$c_1 Z_3^{(1)}(0) + c_2 Z_3^{(2)}(0) + c_3 Z_3^{(3)}(0) = 0, \quad (5.2)$$

$$c_1 Z_1^{(1)}(\eta_\delta) + c_2 Z_1^{(2)}(\eta_\delta) + c_3 Z_1^{(3)}(\eta_\delta) = N, \quad (5.3)$$

This is then in the form

$$\begin{Bmatrix} Z_1^{(1)}(0) & Z_1^{(2)}(0) & Z_1^{(3)}(0) \\ Z_3^{(1)}(0) & Z_3^{(2)}(0) & Z_3^{(3)}(0) \\ Z_1^{(1)}(\eta_\delta) & Z_1^{(2)}(\eta_\delta) & Z_1^{(3)}(\eta_\delta) \end{Bmatrix} \begin{Bmatrix} c_1 \\ c_2 \\ c_3 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ N \end{Bmatrix}, \quad (5.4)$$

and the solution of this equation can then be written as

$$\begin{Bmatrix} c_1 \\ c_2 \\ c_3 \end{Bmatrix} = \begin{Bmatrix} Z_1^{(1)}(0) & Z_1^{(2)}(0) & Z_1^{(3)}(0) \\ Z_3^{(1)}(0) & Z_3^{(2)}(0) & Z_3^{(3)}(0) \\ Z_1^{(1)}(\eta_\delta) & Z_1^{(2)}(\eta_\delta) & Z_1^{(3)}(\eta_\delta) \end{Bmatrix}^{-1} \begin{Bmatrix} 0 \\ 0 \\ N \end{Bmatrix}. \quad (5.5)$$

Because of the two zeros in the vector to the far right in the above equation, the inverse here is easily solved analytically so that the solution of the set of equations (5.1), (5.2), and (5.3) becomes

$$\left. \begin{aligned} c_1 &= z_1^{(2)}(0) z_3^{(3)}(0) - z_3^{(2)}(0) z_1^{(3)}(0) \quad N/D \\ c_2 &= z_1^{(1)}(0) z_3^{(3)}(0) - z_1^{(3)}(0) z_3^{(1)}(0) \quad N/D \\ c_3 &= z_1^{(1)}(0) z_3^{(3)}(0) - z_1^{(2)}(0) z_3^{(1)}(0) \quad N/D \end{aligned} \right\}, \quad (5.6)$$

where

$$D = \det \begin{vmatrix} z_1^{(1)}(0) & z_1^{(2)}(0) & z_1^{(3)}(0) \\ z_3^{(1)}(0) & z_3^{(2)}(0) & z_3^{(3)}(0) \\ z_1^{(1)}(\eta_\delta) & z_1^{(2)}(\eta_\delta) & z_1^{(3)}(\eta_\delta) \end{vmatrix}. \quad (5.7)$$

Once these coefficients have been obtained, a total $Z_5(0)$ vector can be formed as follows:

$$Z_5(0) = c_1 Z_5^{(1)}(0) + c_2 Z_5^{(2)}(0) + c_3 Z_5^{(3)}(0), \quad (5.8)$$

so that a search procedure to find a set of eigenvalues remains the only step to be done. Two of the four parameters, α , R , $c (= c_r + ic_i)$ are held constant while the other two vary. This is because the real and imaginary parts of the boundary condition, $Z_5(0)$, establish two conditions.

A Newton-Raphson search procedure is the final step in the eigenvalue search procedure. This can be done by remembering that $Z_5(0)$ is now a function of the two parameters which are not being held constant. So, if c_r and c_i are held constant, then,

$$Z_{5_r}(0) = Z_{5_r}(\alpha, R), \quad (5.9)$$

$$Z_{5_i}(0) = Z_{5_i}(\alpha, R), \quad (5.10)$$

If a Taylor series expansion about the eigensolution, $\tilde{Z}_5(0)$, is taken, the following equations are obtained:

$$Z_{5_r}(\tilde{Z}_{5_r}(0) + \epsilon) = \tilde{Z}_{5_r}(0) + \left(\frac{\partial Z_{5_r}(0)}{\partial \alpha} \right) (\tilde{\alpha} - \alpha) + \left(\frac{\partial Z_{5_r}(0)}{\partial R} \right) (\tilde{R} - R) + O(\Delta\alpha^2, \Delta R^2), \quad (5.11)$$

$$Z_{5_i}(\tilde{Z}_{5_i}(0) + \epsilon) = \tilde{Z}_{5_i}(0) + \left(\frac{\partial Z_{5_i}(0)}{\partial \alpha} \right) (\tilde{\alpha} - \alpha) + \left(\frac{\partial Z_{5_i}(0)}{\partial R} \right) (\tilde{R} - R) + O(\Delta\alpha^2, \Delta R^2), \quad (5.12)$$

where $Z_5(\tilde{Z}_5(0) + \epsilon)$ represents a perturbation to the true eigensolution; $\tilde{\alpha}, \tilde{R}$ are the eigenvalues associated with the eigensolution; and α, R are the initial guesses of the eigenvalues. If the equation is rewritten using

$$\begin{aligned} (\tilde{\alpha} - \alpha) &= \Delta\alpha, \\ (\tilde{R} - R) &= \Delta R, \end{aligned}$$

and from the physical boundary condition, $Z_5(Z_5(0) + \epsilon) = 0$, an equation which can be used as a search procedure is obtained,

$$\frac{\partial Z_{5_r}(0)}{\partial \alpha} \Delta\alpha + \frac{\partial Z_{5_r}(0)}{\partial R} \Delta R = -Z_{5_r}(0), \quad (5.13)$$

$$\frac{\partial Z_{5_i}(0)}{\partial \alpha} \Delta\alpha + \frac{\partial Z_{5_i}(0)}{\partial R} \Delta R = -Z_{5_i}(0). \quad (5.14)$$

The derivatives for this search procedure are obtained by integrating the system of equations ((2.33) through (2.38)) for different parameters, (α, R, c_r, c_i) , $(\alpha + \Delta\alpha, R, c_r, c_i)$, and $(\alpha, R + \Delta R, c_r, c_i)$ and forming differences from these to approximate the derivatives.

In an analogous manner, if the parameters α and R are held constant and c_r and c_i varied, a similar formula is obtained:

$$\frac{\partial Z_{5_i}(0)}{\partial c_r} \Delta c_r + \frac{\partial Z_{5_i}(0)}{\partial c_i} \Delta c_i = - Z_{5_i}(0) , \quad (5.15)$$

$$\frac{\partial Z_{5_r}(0)}{\partial c_r} \Delta c_r + \frac{\partial Z_{5_r}(0)}{\partial c_i} \Delta c_i = - Z_{5_r}(0) . \quad (5.16)$$

So now that a method has been derived to correct an initial approximation of the set of eigenvalues, by holding either α , R , or c_r , c_i constant and varying the other two, a set of eigenvalues can be found. The results for this method are found in Chapter VI.

CHAPTER VI

COMPARISON OF THE OPERATION AND RESULTS FOR THE LOCAL AND GLOBAL SOLUTIONS OF THE STABILITY EQUATIONS

This chapter examines and discusses the operation of the global and local method described in Chapters IV and V, respectively. The results obtained from these computer codes are discussed along with details about the operation of the program. Possibilities are also considered relative to further work which may be necessary in this area. The discussion begins with a consideration of the results obtained from the local method computer code.

The operation of the computer code for the local method of locating the eigenvalues of the different equations ((2.28) through (2.30)) requires initial guesses of the eigenvalues which are to be iterated upon to hopefully converge to the true eigenvalues. If these initial guesses of the eigenvalues are close enough to the true eigenvalues the iterations will converge to the true eigenvalues. The initial guesses will be mentioned again when the results are considered relative to how good the initial guesses must be to obtain convergence. It is enough now to mention that there can be convergence problems if the initial guesses of the eigenvalues are not close to the true eigenvalues. The computer code for the local method, as was explained in Chapter V, holds two of the four eigenvalues constant and varies the other two. The results considered in this work are for fixing c_r and c_i , guessing α and R , and iterating

on the initial guesses of α and R . Each of these iterations requires three integrations of the differential equations ((2.28) through (2.30)) across the boundary layer, requiring approximately a minute and a half of computer time per integration. This means that each iteration requires about four minutes when the simple Newton-Raphson method presented in this work is utilized. This method converges somewhere between linearly and quadratically, so that in future work a method of convergence acceleration might be considered relative to decreasing the computer time requirements for the program.

An initial guess of two sets of eigenvalues is obtained from the work of Mack [6]. The guesses considered for the initial testing of the local eigenvalue search program are

$$\alpha = 0.061 \quad R = 560 \quad c_r = 0.61 \quad c_i = 0.0$$

$$\alpha = 0.050 \quad R = 300 \quad c_r = 0.61 \quad c_i = 0.0$$

The actual operation of the local method Newton-Raphson iteration procedure is given in Table 4.

Convergence having been obtained for two eigenvalues, the next step is the construction of a neutral stability curve. The neutral stability curve is a plot of α in one direction and R in the other direction for varying values of the real part of the wave speed, c_r , and no imaginary part of the wave speed, c_i . The curve represents the loci of points separating a region of stability and instability. The neutral stability curve is constructed for a specific Mach number in the following way. After one set of eigenvalues has been obtained for which the imaginary part of the wave

Table 4. Operation of Local Method Newton-Raphson Search Procedure

α	R
0.063	565.0
0.06315917719	565.4112629
0.06316427420	566.3911385
0.06316485670	566.4403920
0.06316489944	566.4455230
0.06316490525	566.4458324
0.06316490479	566.4458810
0.050	300.0
0.05274427812	302.5210163
0.05395944681	293.7815989
0.05317168144	285.3800003
0.05315685751	285.7464916
0.05315932585	285.7621721
0.05315933586	285.7639972
0.05315934645	285.7641050
0.05315935008	285.7641612

Note: $M = 2.2$; $c_r = 0.61$; $c_i = 0$.

speed is zero, then the α and R of this set can be used as initial guesses for another point on the neutral stability curve. With this α and R as initial guesses, the values of c_r are incremented and the Newton-Raphson procedure is begun in which the value of α and R are iterated upon to obtain a new set of eigenvalues for the new value of c_r . The result of this type of procedure is shown in Figure 7.

Convergence in this process depends upon the amount the real part of the wave speed, Δc_r , is incremented. If c_r is changed by too great an amount the values of α and R used as initial guesses will not be good enough guesses to converge to a new set of neutrally stable eigenvalues. The amount by which c_r can be incremented is also dependent upon the values of α and R so that in certain regions of α and R the increments of c_r for which convergence is obtained varies.

A neutral stability curve obtained for a Mach number of 2.2 is shown in Figure 8. The different values by which it was necessary to increment c_r to obtain convergence varied from 0.001 to 0.005.

The computer program utilizing the global method of solution has been tested in the section of the eigenvalue plane from which the local method results were obtained and excellent agreement has been observed. As an example, for the set of values,

$$\alpha = 0.053159$$

$$R = 285.7642$$

the eigenvalues were located by the global method as

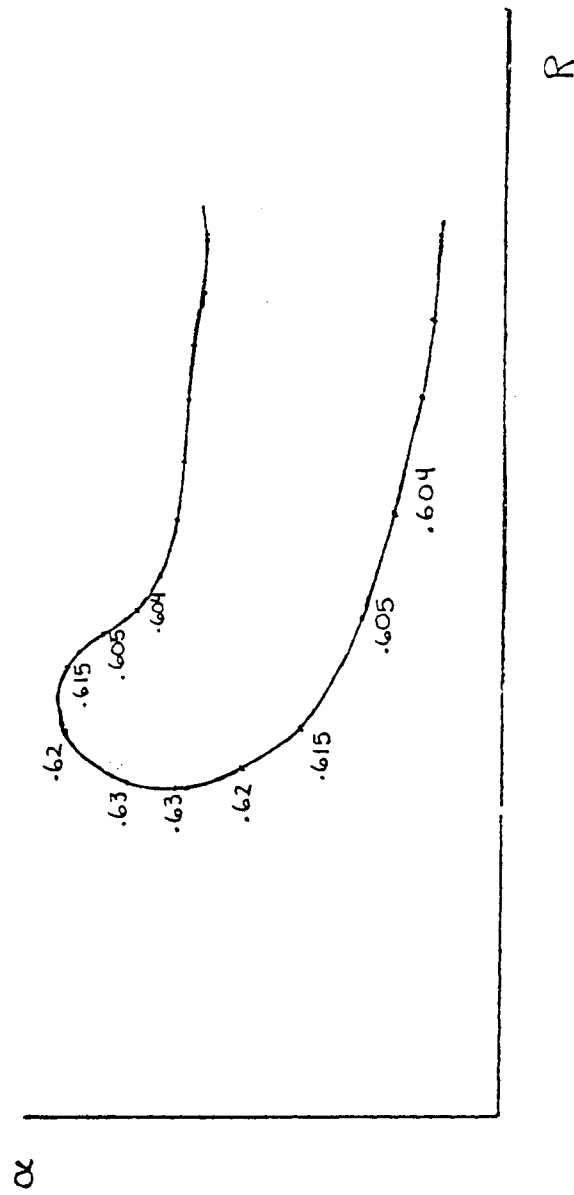


Figure 7. Illustration of the method of determination of a neutral stability curve.

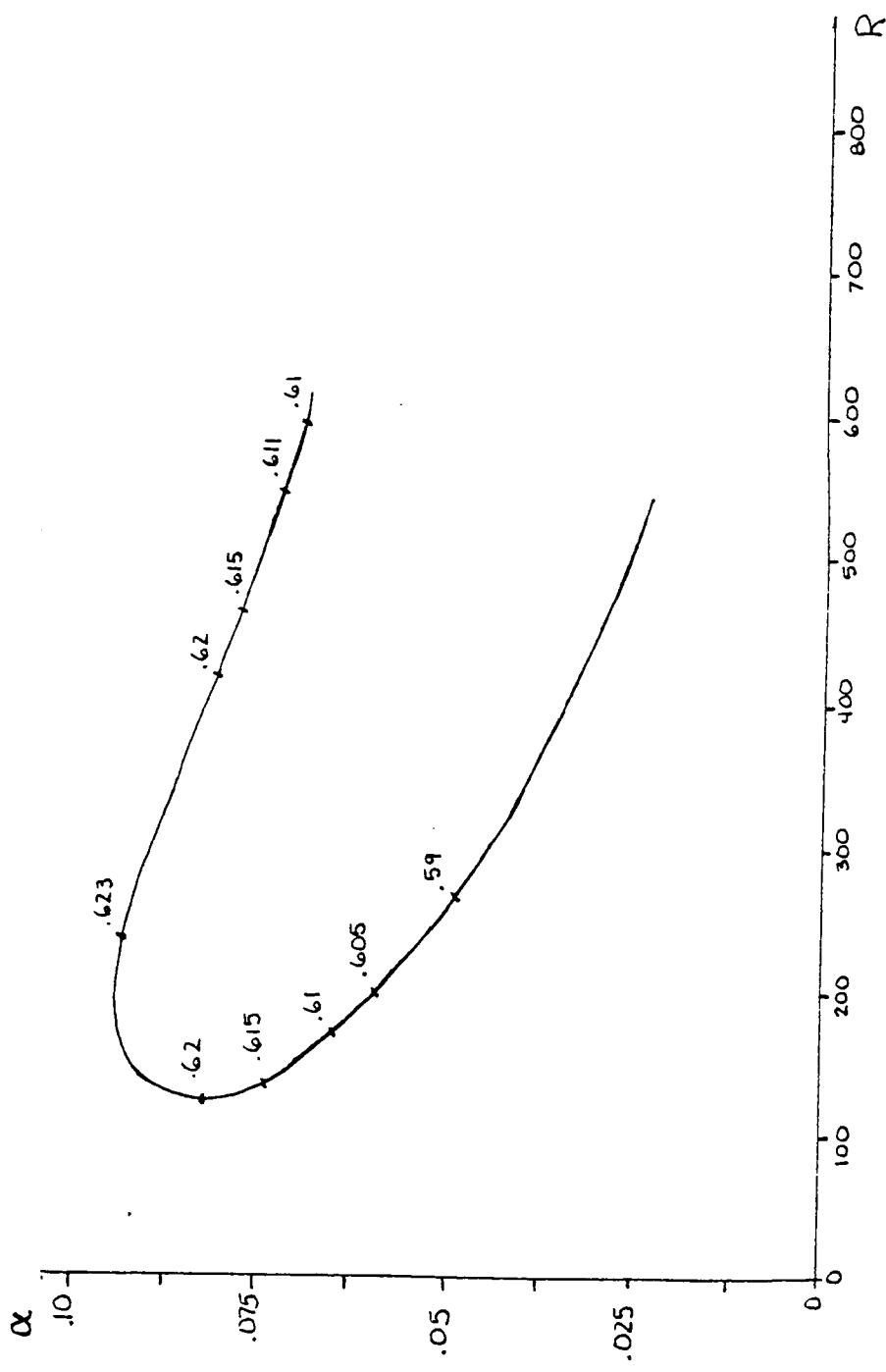


Figure 8. Neutral stability curve.

$$\begin{aligned}c_r &= 0.6101 \\c_i &= 0.0002\end{aligned}$$

as opposed to those obtained from the local method,

$$\begin{aligned}c_r &= 0.6100 \\c_i &= 0.0000\end{aligned}$$

Because of the exorbitant CPU and core memory requirement, a search of the entire eigenvalue plane is not yet possible on the IBM 360/65. Work is now being continued, however, in the area of decreasing the CPU and core requirements for the global method program. There are many areas in which the code can be improved and the possibility of finding a faster computer such as the CDC 7700 is also being pursued. When these tasks are completed, a complete search of the eigenvalue plane for different α and R values will be performed. Eigenvectors will be obtained and this information will be compiled and submitted to journals for an expected technical note publication.

CHAPTER VII

ORTHONORMALIZATION

As explained in Chapter III, the numerical difficulties associated with the integration of three linearly independent initial condition vectors across the boundary layer can be alleviated by an orthonormalization procedure. Each time the vectors begin to lose their numerical linear independence, the solution vectors are reorthonormalized.

In this process, orthonormalizations are performed at necessary points in the integration from η_δ to the wall and a transformation matrix relating the original vectors to the reorthonormalized vectors is stored for each of the orthonormalizations. After the integration is completed, these transformation matrices are reapplied in a reverse manner from the wall to η_δ at the original points where the orthonormalization transformation was previously applied. In this manner the solution at each orthonormalization point is reconstructed taking to account the transformations which have been made. This last step of reconstruction will be unnecessary in the case of this work because of certain mathematical properties of the problem, which will be detailed later.

In the derivation of the orthonormalization procedure applicable to an arbitrary complex vector belonging to a complex space C_n , Dirac notation will be employed. In this notation a vector is denoted by $|l\rangle$, and capital letters, A , denote operators. Let

$$P_n = |n\rangle \langle n|, \quad (7.1)$$

where the right side of the equation is an outer product of two vectors belonging to the orthonormal basis of C_n . Because of the effect it has upon application to a vector, P_n is called a projection operator. Also,

$$P^{(M)} = \sum_{n=1}^M P_n \quad M \leq N \quad (7.2)$$

is a projection operator called a projector onto a subspace of M dimensions. Now, if $\{|n\rangle\}_{n=1}^N$ is a linearly independent set of vectors in the space C_n , the vector product of this set is such that, where δ_{nm} is the Dirac delta,

$$\begin{aligned} \delta_{nm} &= 1, n = m \\ \langle n|m \rangle &= d_{nm} \neq \delta_{nm} \\ \delta_{nm} &= 0, n \neq m \end{aligned} \quad (7.3)$$

What is sought is a set of vectors $\{|n\rangle\}_{n=1}^N$ satisfying the relation

$$\langle n|m \rangle = \delta_{nm}. \quad (7.4)$$

This is formal since $\{|n\rangle\}_{n=1}^N$ is not known in terms of $\{|n'\rangle\}_{n=1}^3$. For the orthonormalization process, first let

$$|1\rangle = \frac{|1'\rangle}{L_1}, \quad (7.5)$$

where

$$L_1 = \sqrt{\langle 1|1 \rangle} , \quad (7.6)$$

so that the new $|1\rangle$ vector is normalized. Now, let

$$|2\rangle = \frac{(1 - P^{(1)}) |2\rangle'}{L_2} , \quad (7.7)$$

where 1 is the unit operator, and

$$P^{(1)} = |1\rangle \langle 1|$$

$$L_2 = \rho(|2\rangle' , P^{(1)} |2\rangle') = \sqrt{\langle 2|2\rangle' - \langle 2|P^{(1)}|2\rangle'} . \quad (7.8)$$

The $\rho(a,b)$ here denotes the distance between a and b . Again, as before, $|2\rangle$ is normalized. In the same manner,

$$|3\rangle = \frac{(1 - P^{(2)}) |3\rangle'}{L_3} , \quad (7.9)$$

where

$$P^{(2)} = \sum_{n=1}^2 |n\rangle \langle n| ,$$

and

$$L_3 = \sqrt{\langle 3|3\rangle' - \langle 3|P^{(2)}|3\rangle'} . \quad (7.10)$$

So now if the multiplication is performed it can be shown that,

$$\langle n|m \rangle = \delta_{nm} , \quad (7.11)$$

which is the result that was sought. An orthonormal set has now been obtained from the original linearly independent set of vectors.

In the present work, there is a set of three linearly independent vectors and these vectors are eigenvectors for the proper parameters. Recalling the property that the superposition of eigen-solutions is also a solution of the problem allows one to express all solutions of the problem as

$$|f\rangle = \sum_{n=1}^3 p_n |p_n\rangle . \quad (7.12)$$

In the above expression, $|f\rangle$ is the solution vector of the problem, and $\{|p_n\rangle\}_{n=1}^3$ is a basis for the space, C_3 . This is because the three eigenvectors are a linearly independent spanning set. Also, the coefficients, p_n , are involved because of the generalized expression. For the problem presented in this work,

$$|p_m\rangle = \left\{ \begin{array}{c} z_1^m(n) \\ z_2^m(n) \\ z_3^m(n) \\ z_4^m(n) \\ z_5^m(n) \\ z_6^m(n) \end{array} \right\} \quad \text{for } m = 1, 2, 3 . \quad (7.13)$$

As was previously mentioned, for the problem considered there is no need to reconstruct the solution by a reversed application of the transformation matrices. This is because it can easily be shown that,

$$|f\rangle = \sum_{n=1}^3 p_n |p_n\rangle = \sum_{n=1}^3 \bar{p}_n |\bar{p}_n\rangle, \quad (7.14)$$

where $\{|p_n\rangle\}_{n=1}^3$ represents the original linearly independent set of vectors, and $\{|\bar{p}_n\rangle\}_{n=1}^3$ represents the orthonormalized form of it. In other words, Eq. (7.14) says that the solution, $|f\rangle$ itself, is not affected by the transformation of the basis, but the effect is absorbed in the coefficients.

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