To the Graduate Council:

I am submitting herewith a dissertation written by Lyndell Monroe Kerley entitled "(C,β) Summability of Walsh-Fourier Series." I recommend that it be accepted in partial fulfillment of the requirements for the degree of Doctor of Philosophy, with a major in Mathematics.

William R. Wade, Major Professor

We have read this dissertation
and recommend its acceptance:

Accepted for the Council:

Vice Chancellor
Graduate Studies and Research

(C,β) SUMMABILITY OF WALSH-FOURIER SERIES

A Dissertation

Presented for the

Doctor of Philosophy

Degree

The University of Tennessee, Knoxville

Lyndell Monroe Kerley

August 1977

1332580

ACKNOWLEDGMENTS

The author wishes to express his deepest gratitude and appreciation to his adviser, Dr. William R. Wade, whose profound insight and many helpful suggestions have been of immeasurable assistance.

The author dedicates this dissertation to his wife, Linda, and his son, Chris.

ABSTRACT

A sufficiency theorem for both pointwise and uniform convergence of Walsh-Fourier series is obtained. This result is an analogue of the trigonometric Lebesque's Test. Moreover a sufficiency theorem for pointwise (C,β) summability, $\beta > 0$ is developed.

Having defined the class of harmonic bounded variation (HBV), it is proved if $f \in$ HBV and if f is continuous at x, then f satisfies the hypotheses of the aforementioned analogue of the trigonometric Lebesque's Test. Therefore if $f \in$ HBV and if f is continuous at x, then $S_n(f,x)$ converges to $f(x)$ where $S_n(f,x) = \sum_{k=0}^{n-1} a_k(f)\psi_k(x)$, $a_k(f) = \int_0^1 f(t)\psi_k(t)dt$, and $\{\psi_k\}_{k=0}^{\infty}$ are the Walsh functions.

Let $-1/2 < \alpha < 1/2$ and E be a measurable subset of $[0,1)$. Define

$$\mu E = \int_0^1 (\omega(t))^{2\alpha} dt,$$

where $\omega(t) = \sum_{n=0}^{\infty} 2^{-n} \chi_{[2^{-n-1},2^{-n})}(t)$. This class of measures is used to investigate the size of $S_n(f,x)$.

Defining

$$K_n^{\beta *} = \chi_n K_n^{\beta}$$

where $K_n^{\beta} = \frac{1}{A_{n-1}^{\beta}} \sum_{k=0}^{n-1} A_{n-k-1}^{\beta}\psi_k$ and $A_n^{\beta} = \binom{n+\beta}{n}$ for $n = 1, 2, \ldots$, the

growth of $\sigma_{2n}^{\beta *}(f,x)$ is studied where $\sigma_{2n}^{\beta *} = \int_0^1 f(t)K_n^{\beta *}(x + t)dt$. Theorems related to boundedness and convergence of $\sigma_{2n}^{\beta *}$ are developed.

If $f \in L^1[0,1]$, $-1 < \beta < 0$, and $U(x) = \sum_{n=0}^{\infty} A_n^{\beta} a_n(f)\psi_n(x)$, then $U(x)$ is proved to be (C,β) summable a.e. on $[0,1]$.

TABLE OF CONTENTS

CHAPTER		PAGE
INTRODUCTION		1
I.	SUFFICIENT CONDITIONS FOR POINTWISE CONVERGENCE.	19
II.	UNIFORM CONVERGENCE AND FUNCTIONS OF HARMONIC BOUNDED VARIATION.	28
III.	GROWTH OF WALSH-FOURIER SERIES AND THE HIRSCHMAN WEIGHTS	37
IV.	(C,β) SUMMABILITY, $-1 < \beta < 0$.	76
BIBLIOGRAPHY		99
VITA		102

INTRODUCTION

Made relevant around the turn of the century by Hilbert's work on integral equations, legitimatized in 1905 by Schmidt's refinement of the Gramm orthogonalization process, the study of orthonormal systems earnestly began in 1907 with the Riesz-Fischer theorem.

Particular orthonormal systems had been used in mathematics for almost a century. Examples which come to mind include the trigonometric functions, the Bessel functions, and the orthogonal polynomials of Laguerre, Hermite, and Legendre. However, analysis of these systems was very difficult, and far from complete. Moreover, these systems shared a common origin; each consisted of solutions to particular differential equations, which arose in the study of physics.

The Riesz-Fischer theorem made the mathematical community aware of a need for further examples of orthonormal systems. They were needed to test hypotheses, to generate counterexamples, and to ferret out pathology. Haar (1909), Radamacher (1922), Walsh (1924), and Franklin (1928) responded to this challenge, introducing systems which still bear their names. Early work on these systems was primarily concerned with the general question, how are these systems similar or different from the older systems mentioned above?

A function by function comparison of the Walsh system with the trigonometric system (see Figure 1) leads one to believe that these two systems should share many of the same properties. This belief has been corroborated over the years. In fact, over half the results obtained for

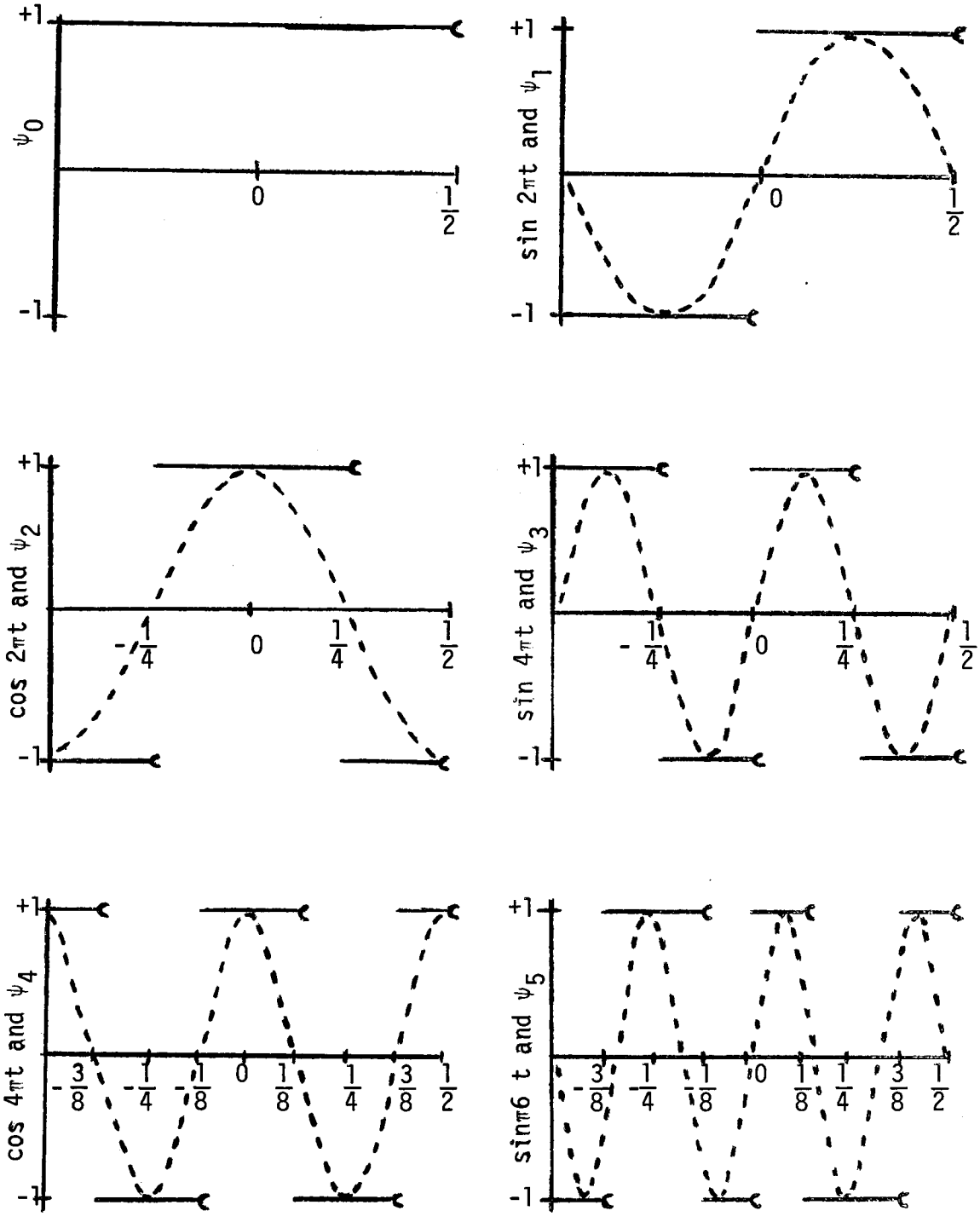


Figure 1. Comparison of Walsh functions as defined in [4], sine functions, and cosine functions.

the Walsh system have older trigonometric analogues. The proofs of these results, however, are not routine extensions of the trigonometric. Instead, the proofs often take on a flavor peculiar to the study of Walsh series. The necessary techniques which have been developed in this study have led to several important generalizations. Three very prominent examples are the Vilenkin groups, the martingale H^p spaces, and multiplicative orthonormal systems.

Certainly, one of the most striking peculiarities of the study of Walsh series is related to this last example, namely, that Walsh functions are generated by products of Radamacher functions. This fact was first observed by Paley (1932), and we repeat his definition verbatim.

Let n be a non-negative integer. The Radamacher function ϕ_n is defined for each real number $x \in [0,1)$ by

$$\phi_n(x) = \begin{cases} 1 \text{ if } x \in [2p2^{-n-1},(2p+1)2^{-n-1}) \text{ for some } p = 0, 1, \ldots, 2^n - 1. \\ -1 \text{ if } x \in [(2p+1)2^{-n-1},2(p+1)2^{-n-1}) \text{ for some } p = 0, 1, \ldots, 2^n-1. \end{cases}$$

It is technically convenient to extend the range of each ϕ_n to $(-\infty,\infty)$ by insisting that it be periodic, of period 1.

The Walsh function ψ_0 is identically 1. For each positive integer n, the Walsh function ψ_n is defined by

$$\psi_n(x) = \phi_{n_1}(x)\phi_{n_2}(x) \ldots \phi_{n_r}(x), \quad x \in (-\infty,\infty)$$

where $n = 2^{n_1} + \ldots + 2^{n_r}$ and $n_1 > n_2 > \ldots > n_r \geq 0$. One observes that $\psi_{2^n} = \phi_n$ for $n = 0, 1, \ldots$, that $\int_0^1 \psi_n(x)dx = 0$ for $n = 1, 2, \ldots$, and $\int_0^1 \psi_0(x)dx = 0$.

We have observed that the Walsh functions are a locally constant approximation to the trigonometric functions. Fine [2] showed that this similarity encompassed structure as well as form. Indeed, it turns out that the Walsh functions are essentially the continuous characters of a compact abelian group, the group 2^ω, in quite the same way that the exponential form of the trigonometric system, $\{e^{inx}\}_{n=-\infty}^{\infty}$ are the continuous characters of the compact group $T = \{z \in \mathbb{C} : |z| = 1\}$.

This group 2^ω is the set of sequences $\bar{x} = (x_1, x_2, \ldots)$, where $x_i = 0$ or 1. The group structure denoted by $\dot{+}$ is provided by adding coordinatewise, modulo 2. The topology is provided by using Tychonov's product topology, viewing the group 2^ω as the cartesian product of countably many copies of the discrete group $\{0,1\}$.

Clearly, the additive identity for the group 2^ω is the element $\bar{0}$, whose entries are all 0, and each element of the group 2^ω is its own inverse: $\bar{x} \dot{+} \bar{x} = \bar{0}$.

In the trigonometric case, the map $e^{it} \to t$ from T onto $[0,2\pi)$ is used to restructure the interval $[0,2\pi)$. In the Walsh case, we use the map $\lambda : 2^\omega \to [0,1)$, introduced by Fine, which is defined as follows. If $\bar{x} = (x_1, x_2, \ldots)$ belongs to the group 2^ω, then

$$\lambda(\bar{x}) = \sum_{i=1}^{\infty} x_i 2^{-i}.$$

By using the binary expansions of elements in [0,1), it is easy to see that the map is onto. It is not one-to-one since each dyadic rational has two distinct preimages. This is the case since each dyadic rational has two binary expansions: one terminates in zeros and the other terminates in ones (e.g., .0111 ... = .1000 ...). If, however, we denote by Z those points in the group 2^{ω} which terminate in ones, then the map $\lambda : 2^{\omega} \sim Z \to [0,1)$ is 1 - 1. We shall denote its inverse by μ. The new topological and algebraic structure for the interval [0,1) is as follows.

If $x, y \in [0,1)$, then the <u>dyadic sum</u> of x and y is

$$x \dot{+} y = \lambda(\mu(x) \dot{+} \mu(y)),$$

and the <u>dyadic distance</u> from x to y is

$$d(x,y) = \sum_{i=1}^{\infty} |x_i - y_i| 2^{-i}$$

where $(x_1, x_2, \ldots) = \mu(x)$ and $(y_1, y_2, \ldots) = \mu(y)$.

Fine went on to verify the following facts.

1. The interval [0,1) is a metric space under the dyadic distance d. A relationship between this metric and the original metric is given by

$$|x - y| \leq d(x,y).$$

Furthermore, the map $\lambda : 2^{\omega} \sim Z \to [0,1)$ is a continuous homomorphism, provided we use the dyadic structure on [0,1).

2. The characters of the group 2^ω are given by $\overline{\psi}_n(\overline{x}) = \psi_n(\lambda(\overline{x}))$. In particular, each Walsh function ψ_n is continuous on $[0,1)$ with respect to the dyadic metric, and $\psi_n(x \dot{+} y) = \psi_n(x)\psi_n(y)$ for $x, y \in [0,1)$ a.e. and each $n = 0, 1, \ldots$.

3. Haar measure for the group 2^ω is given by $\mu E = m(\lambda(E))$, where E is a Borel subset of the group 2^ω and m is Lebesque measure. It follows that if $y_0 \in [0,1)$ and if f is Lebesque integrable on $[0,1)$, then

$$\int_0^1 f(x \dot{+} y_0)dx = \int_0^1 f(x)dx.$$

Using these facts, Fine investigated many Walsh analogues of trigonometric results. They were particularly useful for his study of summability.

Let $S = \sum_{n=0}^{\infty} u_n$ be a series and $\beta > -1$. Recall that S is said to be (C,β) summable to a real number L if σ_n^β converges to L as $n \to \infty$. The sequence σ_n^β is defined by

$$\sigma_n^\beta = \frac{1}{A_{n-1}^\beta} \sum_{k=0}^{n-1} A_{n-k-1}^\beta u_k,$$

where $A_0^\beta = 1$ and $A_n^\beta = \binom{n+\beta}{n}$ for $n = 1, 2, \ldots$. It is well known [21] that $A_n^\beta = n^\beta/\Gamma(\beta+1)$ and that a series is $(C,\beta+\varepsilon)$ summable for $\varepsilon > 0$ when it is is (C,β) summable where Γ is Euler's gamma function.

Notice that if $\beta = 0$, then $A_n^\beta = 1$. Hence a series is $(C,0)$ summable to L if and only if it converges to L in the traditional sense. The special case when $\beta = 1$ is also of interest. Indeed, if $\beta = 1$, then

$$A^{\beta}_{n-k-1} = \binom{n-k}{n-k-1}$$

$$= n - k.$$

Hence the series S is (C,1) summable to L if and only if

$$\sigma_n = \frac{1}{n} \sum_{k=0}^{n-1} (n - k)u_k$$

converges to L. Moreover, since

$$\sigma_n = \sum_{k=0}^{n-1} (1 - k/n)u_k$$

$$= \frac{S_1 + S_2 + \ldots + S_n}{n},$$

where $S_n = \sum_{k=0}^{n-1} u_k$. We see that (C,1) summability is simply Cesaro's method of averaging the sequence of partial sums.

The Walsh-Fourier coefficients of a function f belonging to $L^1[0,1)$ are the numbers

$$a_n(f) = \int_0^1 f(t)\psi_n(t)dt, \quad n = 0, 1, \ldots .$$

A Walsh series $\sum_{n=0}^{\infty} a_n\psi_n$ is said to be a Walsh-Fourier series if there exists an integrable function f such that $a_n = a_n(f)$ for $n = 0, 1, \ldots$. We shall denote the (C,β) partial sum of the Walsh-Fourier series of an integrable function f by $\sigma_n^{\beta}(f,x)$; i.e.,

$$\sigma^{\beta}(f,x) = \frac{1}{A_{n-1}^{\beta}} \sum_{k=0}^{n-1} A_{n-k-1}^{\beta} a_k(f)\psi_k(x).$$

We shall now demonstrate that (C,β) summability questions are actually questions concerning an integral operator on the group 2^{ω} with respect to the kernel

$$K_n^{\beta}(x) = \frac{1}{A_{n-1}^{\beta}} \sum_{k=0}^{n-1} A_{n-k-1}^{\beta} \psi_k(x).$$

Let $D_n = K_n^0$ and observe by definition that

$$\sigma_n^{\beta}(f,x) = \frac{1}{A_{n-1}^{\beta}} \sum_{k=0}^{n-1} A_{n-k-1}^{\beta} \left(\int_0^1 f(t)\psi_k(t)dt\right) \cdot \psi_k(x).$$

Interchanging the summation sign with the integral sign and using the fact that since $\psi_n(t)\psi_n(x) = \psi_n(t \dot{+} x)$ a.e., we conclude that

$$\sigma_n^{\beta}(f,x) = \int_0^1 f(t)K_n^{\beta}(x \dot{+} t)dt.$$

Finally, by changing variables and recalling that all elements of the group 2^{ω} are of order two, we conclude that

$$\sigma_n^{\beta}(f,x) = \int_0^1 f(x \dot{+} t)K_n^{\beta}(t)dt.$$

For technical reasons we shall find it convenient to replace K_n^{β} with

$$K_n^{\beta *}(x) = \psi_n(x)K_n^{\beta}(x).$$

Corresponding to this modified (C,β) kernel, we have a modified (C,β) sum:

$$\sigma_n^{\beta *}(f,x) = \int_0^1 f(x \dot{+} t)\, K_n^{\beta *}(t)dt.$$

Notice that $|K_n^{\beta}| = |K_n^{\beta *}|$, so that $K_n^{\beta *}$ can be viewed as a rotation of K_n^{β}. The relationship between $\sigma_n^{\beta *}$ and σ_n^{β} is easy to derive:

$$\begin{aligned}
\sigma_n^{\beta *}(f,x) &= \int_0^1 f(t)K_n^{\beta *}(x \dot{+} t)dt \\
&= \int_0^1 f(t)\psi_n(x \dot{+} t)K_n^{\beta}(x \dot{+} t)dt \\
&= \psi_n(x) \int_0^1 f(t)\psi_n(t)K_n^{\beta}(x \dot{+} t)dt \\
&= \psi_n(x)\sigma_n^{\beta}(\psi_n f,x).
\end{aligned}$$

It follows that many properties are shared by both the (C,β) sums and the modified (C,β) sums. For example, since $\psi_n f \in L^p$ is equivalent to $f \in L^p$, σ_n^{β} is of strong type or weak type (p,q) if and only if $\sigma_n^{\beta *}$ is. Or, σ_n^{β} is bounded on L^p if and only if $\sigma_n^{\beta *}$ is. For these reasons, we shall examine the modified (C,β) sums for $\beta < 0$ in Chapter IV.

Return to the integral form of the (C,β) sums of a Walsh-Fourier series. Since

$$\int_0^1 K_n(t)dt = 1, \quad n = 0, 1, \ldots$$

but $K_n(x) \to 0$ as $n \to \infty$ for x not a dyadic rational, $\beta \geq 0$, it is clear that K_n^β forms an approximate identity in L^p, $p \geq 1$. In particular, we are led to the following natural question. Does $\sigma_n^\beta(f,x)$ approximate $f(x)$, as $n \to \infty$? This general question, and its variations, has served to motivate a large portion of the study of harmonic analysis on the group 2^ω. And, it is this question which shall be dealt with again in this dissertation.

Let f be a real function defined on [0,1). Let Λ denote a non-decreasing sequence of positive numbers λ_n such that $\sum \lambda_n^{-1}$ diverges, $\{I_n\}$ a sequence of disjoint or abutting intervals $I_n = [a_n, b_n] \subseteq [0,1)$, and let $f(I_n) = f(b_n) - f(a_n)$. The function f is said to be of Λ-bounded variation (ΛBV) if the Λ-variation of f,

$$V_\Lambda(f) = \sup \{\sum |f(I_n)|/\lambda_n : \{I_n\}\} < \infty.$$

Let $\Lambda^m = \{\lambda_{n+m}\}$, $m = 0, 1, 2, \ldots$. A function f of ΛBV is said to be continuous in Λ-variation if $V_{\Lambda^m}(f) \to 0$ as $m \to \infty$. If $\lambda_n = n$ for every positive integer n, Λ-bounded variation is called harmonic bounded variation (HBV). This terminology was introduced by Waterman [17], who used it to study summability problems for the trigonometric system. Notice that if $\lambda_n \geq 1$, then any function of bounded variation is necessarily of Λ-bounded variation.

Let $-1/2 < \alpha < 1/2$ and E be a measurable subset of [0,1). Throughout the remainder of this dissertation, μ will represent a measure of the form

$$\mu E = \int_E (\omega(t))^{2\alpha} dt,$$

where $\omega(t) = \sum_{n=0}^{\infty} 2^{-n} \chi_{[2-n-1,2-n)}(t)$. This class of measures was introduced by Hirschman [5], who used it to study the growth of the partial sums

$$(S_n f)(x) = S_n(f,x) = \sum_{k=0}^{n-1} a_k(f)\psi_k(x).$$

The reader should observe that $u >> m$ when $-1/2 < \alpha < 0$, that $\mu << m$ when $0 < \alpha < 1/2$, and that $\mu = m$ when $\alpha = 0$.

In Chapter I we shall obtain the following analogue to Lebesque's Test. (See [21].)

THEOREM 1. Suppose

(a) $\int_0^{2^{-n}} |\phi_x(t)|dt = o(2^{-n})$ as $n \to \infty$ where $\phi_x(t) = f(x \dot{+} t) - f(x)$

and

(b) $\int_{2^{-n}}^{1} |\phi_x(t) - \phi_x(t + 2^{-n-1})|t^{-1}dt = o(1)$ as $n \to \infty$.

Then $S_n(f,x)$ converges to $f(x)$ at each $x \in [0,1)$. Also if f is continuous on $[c,d] \subseteq [0,1)$ and (b) holds uniformly on $[c,d]$, then $S_n(f,x)$ converges uniformly on $[c,d]$. It follows that if f is a continuous function which satisfies the hypotheses of Theorem 1, and if $\beta \geq 0$, then $\sigma_n^{\beta}(f,x)$ converges to f uniformly as $n \to \infty$.

As mentioned earlier (C,β) summability implies (C,α) summability when $\alpha > \beta$. Theorem 2 will establish a partial converse, namely, that

(C,β) summability implies (C,α) summability when $0 < \alpha < \beta$ provided condition (a) of Theorem 1 holds.

THEOREM 2. Let $x \in [0,1)$ be fixed. If

(a) the Walsh-Fourier series of f is $(C,1)$ summable at x

and

(b) $\int_0^{2^{-n}} |\phi_x(t)|dt = o(2^{-n})$, as $n \to \infty$,

then the Walsh-Fourier series of f is (C,β) summable to $f(x)$ for $\beta > 0$.

This result is best possible in the sense that the conclusions cannot be strengthened to include $\beta = 0$. Indeed, using a theorem of Haar and the fact that the Lebesque constants $L_n = \int_0^1 D_n(x \dot{+} t)dt$ are unbounded, Walsh has proved that for any given x_0, there exists a continuous function whose Walsh-Fourier series diverges at x_0. Clearly such a function satisfies condition (b) of Theorem 2 and by Fine [2] such a function also satisfies condition (a) of Theorem 2.

It is only fair to point out that Fine [3] has shown that the Walsh-Fourier series of any $L^1[0,1)$ function is (C,β) summable almost everywhere, $\beta > 0$. Hence, by our theorem, (b) need only hold on a certain set of measure zero for each function f considered.

In Chapter II, we combine Theorem 1 with some delicate estimates to prove the following theorem.

THEOREM 3. If $f \in HBV$ and if f is continuous at x, then the following conditions hold:

$$(a) \quad \int_0^{2^{-n}} |\phi_x(t)|dt = o(2^{-n}) \text{ as } n \to \infty$$

and

$$(b) \quad \int_{2^{-n}}^{1} |\phi_x(t) - \phi_x(t + 2^{-n-1})|t^{-1}dt = o(1) \text{ as } n \to \infty.$$

Moreover if $f \in C[a,b]$, $[a,b] \subseteq [0,1)$, then $S_n(f,x)$ converges to f uniformly on $[a,b]$.

It is interesting to note that this theorem generalizes the first result ever obtained concerning convergence of Walsh-Fourier series. If f is continuous and of bounded variation, then $S_n(f,x)$ converges to $f(x)$ uniformly, as $n \to \infty$ (see Walsh [15]).

In Chapter III, we investigate the size of $S_n(f,x)$ in the L_μ^p norms. We prove

THEOREM 4. Suppose $f_n \in L^2[0,1)$ for every $n = 1, 2, \ldots$ and p_n is a sequence of integers. If $S_{n,\nu}$ denotes the ν^{th} partial sum of the Walsh series of f_n, then there exists a constant A independent of f_n such that

$$\int_0^1 \sum_{n=1}^{\infty} |S_{n,p_n}(t)|^2 d\mu \leq A \int_0^1 \sum_{n=1}^{\infty} |f_n(t)|^2 d\mu.$$

Theorem 4 represents an extension of an early result by Watari [16] for the case $\mu = m$ and $p = 2$.

After proving Theorem 4, we digress to present proofs of the following two results.

THEOREM 5. Given $f \in L^2(\omega_0)$, ω_0 a dyadic interval, that is, $\omega_0 = (k2^{-n}, (k+1)2^{-n})$, for some $n \geq 0$ and some $k = 0, \ldots, 2^n - 1$, $y > 0$ and positive integer $N > N_0$, $|\omega_0| = 2^{-N_0}$, there is a set $E = E(f,y,N)$ such that

$$mE \leq Cy^{-2} \int_a^b |f(x)|^2 dx$$

and $x \in \omega_0 \sim E$, $0 < n < 2^{-N}$ implies

$|S_n f(x)| \leq Cy$ where $\omega_0 = (a,b)$ and C is a constant independent of f.

THEOREM 6. If $1 < p < \infty$, then there is a constant C (depending only on p) such that

$$\int_{\omega_0} |Mf|^p \, dt \leq C \int_{\omega_0} |f|^p \, d\mu$$

for every $f \in L^p(\omega_0)$ where $Mf = \sup_n |S_n f|$ and $\omega_0 \subseteq [0,1)$.

The proofs of the two previous theorems are due to Hunt [6] and [7] and are included here for completeness and to motivate the use of the modified (C,β) sums in Chapter IV.

It should be remarked as Hunt [6] does that Theorem 5 implies $Mf(x) = \sup_n |S_n(f,x)|$ is of weak type (2,2) on ω which implies $S_n(f,x)$ converges to $f(x)$ a.e. on ω if $f \in L^2(\omega)$.

Theorems 4 and 6 are then applied to obtain the following result.

THEOREM 7. If $f \in L^p_\mu[0,1) \cap L^2_\mu[0,1)$, $1 < p < \infty$, and $\beta > 0$, then

$$-\infty < B \int_0^1 \sum_{n=1}^{\infty} \frac{|S_n(t) - \sigma_n(t)|^p}{n^{1+\beta}} d\mu \leq \int_0^1 |f(t)|^2 d\mu$$

$$\leq A \int_0^1 \sum_{n=1}^{\infty} \frac{|S_n(t) - \sigma_n(t)|^2}{n} d\mu < \infty$$

where we assume f is redefined such that f has a_0 subtracted from it.

Theorem 6 also allows us to establish that

$$\lim_{n\to\infty} \frac{1}{n+1} \sum_{\nu=1}^{n} |S_\nu(f,t) - f(t)| = 0 \text{ a.e. } [\mu] \text{ for } f \in L^p_\mu[0,1),\ 1 < p < \infty.$$

In order to place this result in the context of earlier research, recall that Fine [2] has shown: if $f \in L^1[0,1)$, then $\sigma_n(f,x)$ converges a.e. to $f(x)$ as $n \to \infty$. Since

$$|\sigma_{n+1}(f,t) - f(t)| = (n+1)^{-1} |(n+1)\sigma_{n+1}(f,t) - (n+1)f(t)|$$

$$= (n+1)^{-1} |S_1(f,t) + \ldots + S_n(f,t) - (n+1)f(t)|$$

$$\leq \frac{1}{n+1} \sum_{k=1}^{n} |S_k(f,t) - f(t)|,$$

our result represents a considerable improvement of Fine's classical result in the case $p > 1$.

Using an argument similar to the proof of Theorem 7 and the fact that the Walsh polynomials are dense in $L^p[0,1]$ for $1 < p < \infty$, we obtain the following theorem.

THEOREM 8. Let $f \in L^p[0,1]$, $1 < p < \infty$. Then

$$(a) \quad \int_0^1 \sup_n \frac{1}{n+1} \sum_{\nu=0}^{n} |S_\nu(f,t) - f(t)|^p d\mu \leq C_p \int_0^1 |f(t)|^p d\mu$$

and

$$(b) \quad \lim_{n\to\infty} \frac{1}{n+1} \sum_{\nu=0}^{n} |S_\nu(f,t) - f(t)| = 0 \quad \text{a.e. } [\mu].$$

Finally, in Chapter IV we expand the techniques introduced in Chapter I to obtain results concerning the growth of $\sigma^{\beta^*}_{2^n}(f,x)$ for $-1 < \beta < 0$. We begin by establishing the existence of a continuous function g such that $\sigma^{\beta^*}_n(g,0)$ is unbounded for $-1 < \beta < 0$. Boundedness is established in the following theorem.

THEOREM 9. If $f \in \Lambda BV$, where $\Lambda = \{n^{1+\beta}\}$, $-1 < \beta < 0$, then $|\sigma^{\beta^*}_{2^n}(f,x)| \leq M$ for every $x \in [0,1)$ and for every n.

A sufficiency theorem, similar to one by Waterman [18], for convergence of 2^n partial (C,β^*) sums is proved.

THEOREM 10. $\sigma^{\beta*}_{2^n}(f,x) \to f(x)$ and $\sigma^{\beta}_{2^n}(f,x) \to f(x)$, $-1 < \beta < 0$, for every $x \in Z_f \equiv \{x : f(x) = 0\}$ at which

(a) $\Phi(h) = o(h)$ as $h \to 0$ where $\Phi(h) = \int_0^h |\phi_x(t)|dt$

and

(b) $\int_\eta^1 |\phi_x(t) - \phi_x(t + 2^{-n-1})|t^{-\beta-1}dt = o(2^{n\beta})$ as

$$\eta = 1/(2^n + (\beta + 1)/2) \to 0$$

and summability is uniform on $Z_f \cap \{x : f \text{ is continuous at } x\}$ where (a) is satisfied uniformly. If f is continuous at x, (a) may be omitted.

If (a') and (b') denote (a) and (b) with "o" replaced by "0", $\sigma^{\beta*}_{2^n}(f,x)$ will be bounded at every $x \in Z_f$ at which (a') and (b') hold, and uniformly bounded on every set on which the conditions are satisfied uniformly. If f is bounded, condition (a') may be omitted.

Theorem 10 is then applied to obtain the following theorem.

THEOREM 11. Suppose $f \in \{n^{\beta+1}\}$ BV, $-1 < \beta < 0$, and f is continuous in $\{n^{\beta+1}\}$-variation. Then $\alpha^{\beta*}_{2^n}(f,x) \to f(x)$ and $\alpha^{\beta}_{2^n}(\psi_n f,x) \to f(x)$ for each $x \in Z_f \cap \{x : f \text{ is continuous at } x\}$.

Yano [19] has proved that $\{n^\beta\}$, $-1 < \beta < 0$, is a (C,β) summability factor for the Walsh-Fourier series of integrable functions. Specifically, he has proved for $f \in L^1[0,1]$ that

$\sum_{n=0}^{\infty} a_n(f)\,(n+1)^{\beta}\psi_n(x)$ is (C,β) summable a.e., $-1 < \beta < 0$.

We prove that if $f \in L^1[0,1]$, $-1 < \beta < 0$, and $U(x) = \sum_{n=0}^{\infty} A_n^{\beta} a_n(f)\psi_n(x)$, then $U(x)$ is (C,β) summable a.e. on $[0,1]$.

For the sake of clarity, each of the statements of the theorems which have been mentioned in the Introduction will be repeated in the chapters as needed.

CHAPTER I

SUFFICIENT CONDITIONS FOR POINTWISE CONVERGENCE

The purpose of this chapter is to prove the following theorem.

THEOREM 1. Suppose

(a) $\int_0^{2^{-n}} |\phi_x(t)|dt = o(2^{-n})$ as $n \to \infty$ where $\phi_x(t) = f(x \dot{+} t) - f(x)$

and

(b) $\int_{2^{-n}}^{1} |\phi_x(t) - \phi_x(t + 2^{-n-1})|t^{-1}dt = o(1)$ as $n \to \infty$.

Then $S_n(f,x)$ converges to $f(x)$ at each $x \in [0,1)$. Also, if f is continuous on $[c,d] \subseteq [0,1)$ and (b) holds uniformly on $[c,d]$, then $S_n(f,x)$ converges uniformly on $[c,d]$.

The trigonometric analogue of this result is known as Lebesque's Test. The following version of this classical test for convergence is found in [21]. Suppose

(a) $\int_0^{h} |\phi_x(t)|dt = o(h)$ as $h \to 0$ where $\phi_x(t) = (f(x+t)-f(x-t) - 2f(x))/2$

and

(b) $\int_{\pi n^{-1}}^{\pi} |\phi_x(t) - \phi_x(t + \pi n^{-1})|t^{-1}dt = o(1)$ as $n \to \infty$.

Then $S_n(f,x)$ converges to $f(x)$ at each $x \in [0,\pi]$ and if (b) holds uniformly on $[c,d] \subseteq [0,\pi]$, then $S_n(f,x)$ converges uniformly on $[c,d]$. The reader should observe that since each element of the dyadic group

2^{ω} is its own inverse, the following identity holds:

$$f(x\dot{+}t) - f(x) = (f(x\dot{+}t) + f(x\dot{+}(-t)) - 2f(x))/2.$$

This is the reason why ϕ_x in Theorem 1 takes on the simpler form.

PROOF OF THEOREM 1. For every positive integer n, there is a k which satisfies $2^k \leq n < 2^{k+1}$. Put $\nu = n - 2^k$. Then $\nu \geq 0$ and $\nu < 2^k$. Fine [2] has shown that $D_n = D_{2^k} + \psi_k D_\nu$. In particular,

$$\begin{aligned} S_n(f,x) - f(x) &= \int_0^1 \phi_x(t) D_n(t)\,dt \\ &= \int_0^1 \phi_x(t) D_{2^k}(t)\,dt + \int_0^1 \phi_x(t)\psi_{2^k}(t) D_\nu(t)\,dt \\ &= A_k + B_k. \end{aligned}$$

Moreover,

$$\begin{aligned} B_k &= \int_0^1 \phi_x(t)\psi_{2^k}(t) D_\nu(t)\,dt \\ &= \int_0^{2^{-k}} \phi_x(t)\phi_k(t) D_\nu(t)\,dt + \int_{2^{-k}}^1 \phi_x(t)\phi_k(t) D_\nu(t)\,dt \\ &= B_k^{(1)} + B_k^{(2)}. \end{aligned}$$

Now, an easy inductive proof using the Radamacher decomposition for Walsh functions shows us that $D_{2^k}(t) = 2^k \chi_{[0,2^{-k})}(t)$. In particular,

$$A_k = 2^k \int_0^{2^{-k}} \phi_x(t)dt;$$

consequently,

$$|A_k| \leq 2^k \int_0^{2^{-k}} |\phi_x(t)|dt$$

$$= 2^k o(2^{-k})$$

$$= o(1) \text{ as } k \to \infty.$$

Before estimating B_k, we examine D_ν in more detail. Now, $D_\nu(t) = \sum_{j=0}^{\nu-1} \psi_j(t)$, and for each integer j, subject to the restriction $0 \leq j < \nu < 2^k$, the corresponding Walsh function, ψ_j, is constant on $[p2^{-k}, (p+1)2^{-k})$, for $p = 0, \ldots, 2^\nu - 1$. Hence $t\varepsilon[2p2^{-k-1},(2p+1)2^{-k-1})$ implies $D_\nu(t) = D_\nu(t+2^{-k-1}) = D_\nu(2p2^{-k-1})$.

Using this fact and a change of variables, we obtain

$$B_k^{(2)} = \int_{2^{-k}}^{1} \phi_x(t)\phi_k(t)D_\nu(t)dt$$

$$= \sum_{p=1}^{2^k-1} \left(\int_{2p2^{-k-1}}^{(2p+1)2^{-k-1}} \phi_x(t)D_\nu(t)dt - \int_{(2p+1)2^{-k-1}}^{(2p+2)2^{-k-1}} \phi_x(t)D_\nu(t)dt\right)$$

$$= \sum_{p=1}^{2^k-1} \left(\int_{2p2^{-k-1}}^{(2p+1)2^{-k-1}} \phi_x(t)D_\nu(t)dt - \int_{2p2^{-k-1}}^{(2p+1)2^{-k-1}} \phi_x(t+2^{-k-1})D_\nu(t)dt\right)$$

$$= \int_{2^{-k}}^{1} (\phi_x(t) - \phi_x(t+2^{-k-1}))D_\nu(t)dt.$$

Furthermore, we know that $|D_\nu(t)| \leq t^{-1}$ for $t \in (0,1)$ (Fine [2]). Hence by hypothesis, we conclude that

$$|B_k^{(2)}| \leq \int_{2^{-k}}^{1} |\phi_x(t) - \phi_x(t + 2^{-k-1})|t^{-1}dt$$

$$= o(1) \text{ as } k \to \infty.$$

Estimating $B_2^{(1)}$ is easier. Indeed, since $|D_\nu(t)| \leq \nu$ and $\nu < 2^k$, we conclude that

$$|B_k^{(1)}| \leq \int_0^{2^{-k}} |\phi_x(t)||D_\nu(t)|dt$$

$$\leq \int_0^{2^{-k}} \nu|\phi_x(t)|dt$$

$$< 2^k \int_0^{2^{-k}} |\phi_x(t)|dt$$

$$= 2^k o(2^{-k})$$

$$= o(1) \text{ as } k \to \infty.$$

Therefore $|B_k| \leq o(1)$ as $k \to \infty$ and $S_n(f,x) - f(x) = o(1)$ as $k \to \infty$.

We shall presently obtain a Tauberian type theorem for (C,β) summability. Before doing this, we find it convenient to establish the following lemma.

LEMMA 1. Let $n = \sum_{i=1}^{r} 2^{n_i}$, $n_1 > n_2 > \ldots > n_r \geq 0$ and suppose that $\varepsilon_i \to 0$ as $i \to \infty$ where $\varepsilon_i > 0$ for every i. Then $\lim_{n\to\infty} \sum_{i=1}^{r} \frac{2^{n_i}}{n} \varepsilon_{n_i} = 0$.

PROOF. $\varepsilon_i \to 0$ as $i \to \infty$ implies there exists an N_1 such that

$$\varepsilon_i < \frac{\varepsilon}{2} \quad \text{when } i > N_1.$$

There exists an N_2 such that

$$\left(\sum_{i=1}^{N_1} 2^{n_i} \varepsilon_{n_i} \right) \Big/ n < \frac{\varepsilon}{2} \quad \text{when} \quad n > N_2.$$

Then

$$\sum_{i=1}^{r} \frac{2^{n_i}}{n} \varepsilon_{n_i} = \sum_{i=1}^{N_1} \frac{2^{n_i}}{n} \varepsilon_{n_i} + \sum_{i=N_1+1}^{r} \frac{2^{n_i}}{n} \varepsilon_{n_i}$$

$$< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} \left(\sum_{i=N_1+1}^{r} 2^{n_i} \right) \Big/ n$$

$$< \varepsilon$$

when $n > N = \max \{N_1, N_2\}$.

THEOREM 2. For a fixed $x \in [0,1)$, the hypotheses

(a) Walsh-Fourier series of f is $(C,1)$ summable at x

and

(b) $\int_0^{2^{-n}} |\phi_x(t)| dt = o(2^{-n})$ as $n \to \infty$

imply that the Walsh-Fourier series of f is (C,β) summable to $f(x)$ for $\beta > 0$.

PROOF. Without loss of generality, we may assume that $0 < \beta < 1$. Using Abel's transformation twice, Yano [20] has shown that

$$K_n^\beta(t) = \frac{1}{A_n^\beta} \sum_{i=1}^{r} \psi_{n-n_{i-1}}(t) \left\{ \sum_{k=0}^{2^{n_i}-3} kA^{\beta-2}_{n_i+k+1} K_{k+1}(t) + (2^{n_i}-2)A^{\beta-1}_{n_{i-1}-2} K_{2^{n_i}-1}(t) + A^{\beta}_{n_{i-1}-1} D_{2^{n_i}}(t) \right\}.$$

Therefore

$$|\sigma_n^\beta(x,f) - f(x)| \le \frac{1}{A^\beta_{n-1}} \sum_{i=1}^{r} \left\{ \sum_{k=1}^{2^{n_i}-3} k|A^{\beta-2}_{n_i+k+1}| \int_0^1 |f(x\dot{+}t) - f(x)||K_{k+1}(t)|dt + 2^{n_i}A^{\beta-1}_{n_{i-1}-2} \int_0^1 |f(x\dot{+}t) - f(x)||K_{2^{n_i}-1}(t)|dt + A^{\beta}_{n_{i-1}-1} \int_0^1 |f(x\dot{+}t) - f(x)|D_{2^{n_i}}(t)dt \right\}$$

$$= A_n + B_n + C_n.$$

Before estimating A_n, we observe that

$$0 < \sum_{k=1}^{2^n} k^{\alpha-1}/2^{n\alpha} \le 1 + \int_1^{2^n} t^{\alpha-1}dt/2^{n\alpha} \text{ where } 0 < \alpha < 1.$$

Furthermore, a straightforward application of L'Hôpital's rule proves that

$$\lim_{n\to\infty} \int_1^{2^n} t^{\alpha-1}dt/2^{n\alpha} = 1/\alpha.$$

It follows that there is a constant $C_\alpha > 0$ such that

$$\sum_{k=1}^{2^n} k^{\alpha-1}/2^{n\alpha} \leq C_\alpha \text{ for every } n.$$

In particular, we can conclude that

$$\sum_{k=1}^{2^{n_i}-3} k|A^{\beta-2}_{n^i+k+1}| \leq C \sum_{k=1}^{2^{n_i}} k(n^i + k + 1)^{\beta-2}$$

$$\leq C \sum_{k=1}^{2^{n_i}} kk^{\beta-2}$$

$$= C \sum_{k=1}^{2^{n_i}} k^{\beta-1}.$$

This inequality, in turn, implies that

$$\sum_{k=1}^{2^{n_i}-3} k|A^{\beta-2}_{n^i+k+1}|2^{-n\beta} \leq C \sum_{k=1}^{2^{n_i}} k^{\beta-1}/2^{n\beta}$$

$$\leq CC_\beta.$$

Thus

$$|A_n| \leq CC_\beta/n^\beta \sum_{i=1}^{r} 2^{n_i\beta}\varepsilon_{n_i}$$

where $\varepsilon_{n_i} = \int_0^1 |f(x + t) - f(x)||K_{n_i}(t)|dt$. By Lemma 1 and the fact

that the Walsh-Fourier series of f, S(f,x), is (C,1) summable to f(x), we are led to conclude that

$$A_n = o(1) \text{ as } n \to \infty.$$

We now estimate B_n.

$$|B_n| \leq \frac{2^{n_i}}{A^{\beta}_{n-1}} A^{\beta-1}_{n^{i-1}-2} \int_0^1 |f(x \dot{+} t) - f(x)||K_{2^{n_i-1}}(t)|dt.$$

Now $n^{i-1} = \sum_{k=i}^{r} 2^{n_k}$ implies $n^{i-1} \geq 2 \cdot 2^{n_i}$ which in turn implies $n^{i-1} - 2 \geq 2^{n_i}$. It follows that

$$\frac{2^{n_i}}{A^{\beta}_{n-1}} A^{\beta-1}_{n^{i-1}-2} \leq C 2^{n_i} (2^{n_i})^{\beta-1}/n^{\beta}$$

$$\leq C\, 2^{n_i \beta}/n^{\beta}$$

$$\leq C.$$

Using this fact and hypothesis (a), we obtain

$$B_n = o(1) \text{ as } n \to \infty.$$

Finally, since $D_{2^{n_i}}(t) = 2^{n_i} \chi_{[0,2^{-n_i})}(t)$,

$$|C_n| \leq \frac{A^{\beta}_{n_{i-1}-1}}{A^{\beta}_{n-1}} \int_0^1 |f(x \dot{+} t) - f(x)| D_{2^{n_i}}(t)dt$$

$$\leq 1 \cdot 2^{n_i} \int_0^{2^{-n_i}} f(x \dot{+} t) - f(x)|dt$$

$$= 2^{n_i} \int_0^{2^{-n_i}} |\phi_x(t)|dt.$$

However, this last term tends to zero by hypothesis. Hence

$$C_n = o(1) \text{ as } n \to \infty$$

and the proof is complete.

CHAPTER II

UNIFORM CONVERGENCE AND FUNCTIONS OF HARMONIC BOUNDED VARIATION

In this chapter, we apply Theorem 1 to resolve the convergence question for functions of harmonic bounded variation. Indeed, we shall show that all continuous functions belonging to this class have uniformly convergent Walsh-Fourier series. Also, we show that if f is of harmonic bounded variation and if f is continuous at x, then $S_n(f,x)$ converges to $f(x)$.

Waterman has extended the concept of Λ-bounded variation to a group G which is compact, 0-dimensional, metrizable, and abelian. Indeed, a function f defined on such a group is of Λ-bounded fluctuation if there is a finite M such that for every collection $\{I_n\}$ of disjoint cosets in G, we have $\sum_{n=1}^{\infty} \frac{1}{\lambda_n} \operatorname{osc}(f,I_n) < M$ where $\operatorname{osc}(f,H) = \sup \{|f(x_1) - f(x_2)|: x_1, x_2 \in H \subseteq G\}$. For $\Lambda = \{n\}$, we say f is of harmonic bounded fluctuation.

We now comment on the relationship between harmonic bounded variation of [0,1) and the concept of harmonic bounded fluctuation on a group. Identifying [0,1) with the dyadic group 2^{ω}, we observe for any positive integer n, $[0,2^{-n})$ is a subgroup of 2^{ω} since $x \dot{+} t \in [0,2^{-n})$ if and only if x and t belong to the same dyadic interval of length 2^{-n} and inverse of $x \in [0,2^{-n})$ is x itself. In particular, for any $x \in 2^{\omega}$, $x \dot{+} [0,2^{-n}) = [k2^{-n}, (k+1)2^{-n})$ for some $k = 0, 1, \ldots, 2^n - 1$. That is, a coset of $[0,2^{-n})$ is simply an interval of the form $[k2^{-n},(k+1)2^{-n})$ for some $k = 0, 1, 2^n - 1$.

It is then immediate that the set of functions of harmonic bounded fluctuation is a subset of the functions of harmonic bounded variation. Indeed, if $\{[a_n,b_n]\}_{n=1}^{\infty}$ is a sequence of non-overlapping dyadic intervals, then

$$\sum_{n=1}^{\infty} |f(b_n) - f(a_n)|/n \leq \sum_{n=1}^{\infty} \operatorname{osc}(f,I_n)/n < \infty$$

where $I_n = [a_n,b_n]$ for every positive integer n.

We now define Φ-bounded variation and establish a corollary relating it to harmonic bounded variation.

Let us suppose Φ and Ψ are two functions complementary in the sense of W. H. Young, i.e., $\Phi(x) = \int_0^x \phi(t)dt$, where ϕ is continuous, strictly increasing, and $\phi(0) = 0$, and $\Psi(x) = \int_0^x \phi^{-1}(t)dt$. A function f on [0,1] is said to be of Φ-bounded variation (Φ - BF) if there is an $M < \infty$ such that, for every partition $0 = x_0 < x_1 < \ldots < x_n = 1$, we have

$$\sum_{i=1}^{n-1} \Phi(|f(x_{i+1}) - f(x_i)|) < M.$$

Proceeding as Waterman [17] did in the trigonometric case, we obtain the following result.

If $f \in \Phi$ - BF and $\sum_{n=1}^{\infty} \Psi(n^{-1}) < \infty$, then $f \in$ HBV. Indeed, under these assumptions and application of Young's inequality from Zygmund [21] yields the following.

$$\sum_{n=1}^{N} f(I_n)/n = \sum_{n=1}^{N} f(I_n)/n$$

$$\leq \sum_{n=1}^{N} \Phi(f(I_n)) + \sum_{n=1}^{N} \Psi(1/n)$$

$$< M + \sum_{n=1}^{\infty} \Psi(n^{-1}) < \infty$$

for every N. Therefore $f \in HBV$.

THEOREM 3. If $f \in HBV$ and if f is continuous at x, then

$$(a) \quad \int_0^{2^{-n}} |\phi_x(t)|dt = o(2^{-n}) \text{ as } n \to \infty$$

and

$$(b) \quad \int_{2^{-n}}^{1} |\phi_x(t) - \phi_x(t \dot{+} 2^{-n-1})|t^{-1}dt = o(1) \text{ as } n \to \infty.$$

Moreover if $f \in C[a,b]$, $[a,b] \subseteq [0,1)$, then $S_n(f,x) \to f(x)$ uniformly on $[a,b]$.

PROOF. Assume to the contrary, that (b) does not hold uniformly on $[a,b]$. Without loss of generality, assume $a = 0$. Let $L(n,x) = \int_{2^{-n}}^{1} |\phi_x(t) - \phi_x(t + 2^{-n-1})|t^{-1}dt$. Then by assumption, there exists $\alpha > 0$, $x_k \downarrow 0$, $x_k < 2^{-n_k-1}$, and $\{n_k\}$ such that $L(n_k,x_k) > \alpha$ for every k where $n_k \uparrow \infty$. Break L(n,x) into two pieces, as follows:

$$L(n,x) = \int_{2^{-n}}^{2^{-n_0}} |\phi_x(t) - \phi_x(t \dot{+} 2^{-n-1})|t^{-1}dt + \int_{2^{-n_0}}^{1} |\phi_x(t) - \phi_x(t+2^{-n-1})|t^{-1}dt$$

$$= L_1(n,x,n_0) + L_2(n,x,n_0).$$

To estimate the second piece, observe that

$$L_2(n,x,n_0) \le 2^{n_0} \int_{2^{-n_0}}^{1} |f(x \dot{+} t) - f(x \dot{+} t \dot{+} 2^{-n-1})|dt$$

$$= 2^{n_0} \int_{2^{-n_0} \dot{+} x}^{1} |f(t) - f(t \dot{+} 2^{-n-1})|dt.$$

This last term converges to zero as $n \to \infty$ since $f \in L[0,1)$. In fact, if we set $F(n) = \int_0^1 |f(t) - f(t \dot{+} 2^{-n-1})|dt$ for $n = 0, 1, \ldots$, then $F(n) \to 0$ as $n \to \infty$ whenever f is integrable.

Let ε_i be any sequence of real numbers which decreases monotonically to zero, as $i \to \infty$. Also, let n'_k be any sequence of integers increasing monotonically to $+\infty$, as $k \to \infty$. Then for every integer i, there exists n_{k_i} of $\{n_k\}_{k=1}^{\infty}$ such that $n_{k_i} - n'_i > i$ and $L_2(n_{k_i},x_{k_i},n'_i) < \varepsilon_1$ since $L_2(n, \cdot, \delta) \to 0$ as $n \to \infty$. Therefore $n_{k_i} - n'_i \uparrow \infty$, $\varepsilon_i \downarrow 0$, and $L_2(n_{k_i},x_{k_i},n'_1) \to 0$ as $i \to \infty$.

Rewriting we have $n_k - n'_k \uparrow \infty$, $\varepsilon_k \downarrow 0$ and $L_2(n_k,x_k,n'_k) \to 0$ as $k \to \infty$. Define $m_k = 2^{n_k - n'_k + 1} - 1$. Since $L(n_k,x_k) > \alpha$ for every k and $L_2(n_k,x_k,n'_k) \to 0$ as $k \to \infty$, there exists a K such that $k > K$ implies

$$\frac{3}{4}\alpha < L_1(n_k,x_k,n'_k)$$

$$\le \int_{2^{-nk-1}}^{2^{-n'k}} |\phi_{x_k}(t) - \phi_{x_k}(t + 2^{-n_k-1})|t^{-1}dt$$

$$= \sum_{j=1}^{m_k} \int_{j2^{-n_k-1}}^{(j+1)2^{-n_k-1}} |\phi_{X_k}(t) - \phi_{X_k}(t + 2^{-n_k-1})| t^{-1} dt$$

$$= \sum_{j=1}^{m_k} \int_{0}^{2^{-n_k-1}} |\phi_{X_k}(t + j2^{-n_k-1})$$

$$- \phi_{X_k}(t + (j+1))2^{-n_k-1} \frac{1}{t + j2^{-n_k-1}} dt$$

where a change of variables is used to obtain the last equality.

An application of the mean value theorem for integrals on each interval $[0,2^{-n_k-1}]$ guarantees the existence of $\theta_k \in (0, 2^{-n_k-1})$ such that

$$\int_{0}^{2^{-n_k-1}} |\phi_{X_k}(t + j2^{-n_k-1}) - \phi_{X_k}(t + (j+1)2^{-n_k-1})| \frac{1}{t + j2^{-n_k-1}} dt$$

$$= 2^{-n_k-1} |\phi_{X_k}(\theta_k + j2^{-n_k-1}) - \phi_{X_k}(\theta_k + (j+1)2^{-n_k-1})| \frac{1}{\theta_k + j2^{-n_k-1}}$$

$$\leq 2^{-n_k-1} |\phi_{X_k}(\theta_k + j2^{-n_k-1}) - \phi_{X_k}(\theta_k + (j+1)2^{-n_k-1})| j^{-1} 2^{n_k+1}$$

$$= |\phi_{X_k}(\theta_k + j2^{-n_k-1}) - \phi_{X_k}(\theta_k + (j+1)2^{-n_k-1})| j^{-1}.$$

Therefore we obtain for $k > K$,

$$\frac{3}{4}\alpha < \sum_{j=1}^{m_k} |\phi_{x_k}(\theta_k + j2^{-n_k-1}) - \theta_{x_k}(\theta_k + (j+1)2^{-n_k-1})|j^{-1}$$

Define $\Delta k = \sup \{|f(x \dot{+} h) - f(x)| : |h| \leq 2^{-n_k}, x \in [0,b] \dot{+} [0,2^{-n'_k}]\}$.

Clearly $\Delta k \downarrow 0$ as $k \to \infty$. Let $k_1 > K_1$ such that

$$\Delta(k_1) < \frac{\alpha}{8}.$$

For each $j = 2, 3, \ldots$, choose $k_j > k_{j-1}$ such that (i) $n'_{k_j} > n_{k_{j-1}}$

and (ii) $\sum_{i=1}^{m_{k_{j-1}}} \frac{1}{i} \Delta k_j < \frac{\alpha}{8}$.

For each $j = 1, 2, 3, \ldots$, there is a $\theta_j \in (0, 2^{-n_{k_j}-1})$ such

that if $\phi(i,j) \equiv |\phi_{x_{k_j}}(\theta_j + i2^{-n_{k_j}-1}) - \theta_{x_{k_j}}(\theta_j + (i+1)2^{-n_{k_j}-1})|$,

$$\sum_{i=1}^{m_{k_j}} \frac{\phi(i,j)}{i} > \frac{3}{4}\alpha.$$

By (ii), $\sum_{i=1}^{m_{k_{j-1}}} \frac{\phi(i,j)}{i} \leq \left(\sum_{i=1}^{m_{k_{j-1}}} \frac{1}{i}\right) \Delta k_j < \frac{\alpha}{8}$.

Therefore

$$\sum_{i=m_{k_{j-1}}+1}^{m_{k_j}} \frac{\phi(i,j)}{i} = \sum_{i=1}^{m_{k_j}} \frac{\phi(i,j)}{i} - \sum_{i=1}^{m_{k_{j-1}}} \frac{\phi(i,j)}{i} \geq \frac{3}{4}\alpha - \frac{1}{8}\alpha = \frac{5}{8}\alpha \text{ for } j > 1$$

We now construct a collection of non-overlapping intervals which will contradict the fact that $f \in HBV$.

We observe that

$$[0,2^{-n_{k_j}}] = \{x \in [0,1] : x = \sum_{i=n_{k_j}}^{\infty} b_i 2^{-i-1},\ b_i = 0 \text{ or } 1\}$$

implies

$$[2^{-n_{k_j}}, 2^{-n'_{k_j}}] = \bigcup_{i=1}^{m_{k_j}} [i2^{-n_{k_j}}, (i+1)2^{-n_{k_j}}] = \bigcup_{i=1}^{m_{k_j}} \{z_i^{(n_{k_j})} \dot{+} [0,2^{-n_{k_j}}]\}$$

where

$$z_i^{(n_{k_j})} \in \{x \in [0,1] : x = \sum_{i=0}^{n_{k_j}-1} b'_i 2^{-i-1},\ b'_i \text{ not all zero, } b'_i = 0 \text{ or } 1\}.$$

Then for $i \in \{1, 2, \ldots, m_{k_j}\}$, we obtain

$$x_{k_j} \dot{+} \theta_j + i2^{-n_{k_j}-1},\ x_{k_j} \dot{+} \theta_j + (i+1)2^{-n_{k_j}-1} \in x_{k_j} \dot{+} \theta_j + [i2^{-n_{k_j}-1},\ (i+1)2^{-n_{k_j}-1}]$$

$$= x_{k_j} \dot{+} \theta_j + z_i^{(n_{k_j})} \dot{+} [0,\ 2^{-n_{j_k}-1}]$$

$$= z_i^{n_{k_j}} \dot{+} [0,2^{-n_{k_j}-1}]$$

$$= [i2^{-n_{k_j}-1}, (i+1)2^{-n_{k_j}-1}] \subseteq \bigcup_{i=1}^{m_k} [i2^{-n_{k_j}-1}, (i+1)2^{-n_{k_j}-1}]$$

$$= [2^{-n_{k_j}},\ 2^{-n'_{k_j}}].$$

Set $I_{ij} = [x_{k_j} \dot{+} \theta_j + i2^{n_{k_j}-1}, x_{k_j} \dot{+} \theta_j + (i+1)2^{-n_{k_j}-1}]$. Condition (i) implies for $j = 2, 3, \ldots$ $\{I_{ij}\}_{i=1}^{m_{k_j}}$ overlaps at most one interval corresponding to $\phi(i,j)$, $1 \leq i \leq m_{k_{j-1}}$.

Then for $N = 1, 2, \ldots$, we have

$$\sum_{j=1}^{N} \sum_{i=m_{k_{j-1}}}^{m_{k_j}} |f(x_{k_j} \dot{+} \theta_j + i2^{-n_{k_j}-1}) - f(x_{k_j} \dot{+} \theta_j + (i+1)2^{-n_{k_j}-1})| i^{-1}$$

$$= \sum_{j=1}^{N} \sum_{i=m_{k_{j-1}}}^{m_{k_j}} \frac{|f(I_{ij})|}{i}$$

$$= \sum_{j=1}^{N} \sum_{i=m_{k_{j-1}}}^{m_{k_j}} \frac{\phi(i,j)}{i}$$

$$> \frac{5}{8} N\alpha.$$

If for each $j = 1, \ldots, N-1$, we eliminate from the inner sum the at most one term whose interval contains the intervals of the $(j+1)$st stage, we make at most $N-1$ deletions, and the sum after these deletions will exceed

$$\sum_{j=1}^{N} \sum_{i=m_{k_{j-1}}}^{m_{k_j}} \frac{|f(I_{ij})|}{i} - N\frac{\alpha}{8} \text{ since } \Delta(k_1) < \frac{\alpha}{8}.$$

In the above, denote I_{ij} by I_i. If $\{i_n\}$, $n = 1, 2, \ldots, m(N)$, denotes

the indices of the terms remaining after the deletions, then the resulting intervals are non-overlapping.

Since $i_n \geq n$, we have

$$\sum_{n=1}^{m(N)} \frac{1}{n} f(I_{i_n}) \geq \sum_{i=1}^{m_{k_N}} \frac{f(I_{i_n})}{i_n} > \frac{5}{8}N\alpha - \frac{1}{8}N\alpha = \frac{1}{2}N\alpha. \quad \text{Thus } f \notin \text{HBV}.$$

This argument is easily modified to include the pointwise case. The condition (a) is immediate for HBV functions. If we assume (b) does not hold at $x = 0$ and that $f(0) = 0$, then we proceed as before, setting $x_k = 0$ for all k thereby yielding a contradiction, namely, $f \notin \text{HBV}$.

CHAPTER III

GROWTH OF WALSH-FOURIER SERIES AND THE HIRSCHMAN WEIGHTS

For the remainder of this dissertation,

$$\omega(t) = \sum_{n=0}^{\infty} 2^{-n} \chi_{[2^{-n-1},2^{-n})}(t), \qquad t \in [0,1).$$

For each parameter α, with $-1/2 < \alpha < 1/2$, and each exponent p, with $1 < p < \infty$, we shall define a measure $\mu = \mu_{\alpha,p}$ by setting

$$\mu E = \int_E (\omega(t))^{p\alpha} dt, \tag{3.1}$$

for each measurable subset E of $[0,1)$. For f a characteristic function, we have

$$\int_0^1 f(t)d\mu = \int_0^1 f(t)(\omega(t))^{p\alpha} dt. \tag{3.2}$$

Using the standard approach of approximating a function by simple functions and the monotone convergence theorem, (3.2) can be shown to hold for f a measurable function. Define

$L_\mu^p[0,1) = \{f : [0,1) \to \text{Reals} \mid \int_0^1 |f(t)|d\mu < \infty\}$, and observe that $L^p[0,1) \subseteq L_\mu^p[0,1)$. Indeed, for each $f \in L^p[0,1)$ we have

$$\int_0^1 |f(t)|^p d\mu = \int_0^1 |f(t)|^p (\omega(t))^{p\alpha} dt \leq \int_0^1 |f(t)|^p dt < \infty.$$

We now make certain definitions and state some results due to Paley [11] and Hirschman [5].

Suppose

$$f(t) \sim \sum_{k=0}^{\infty} c_k \psi_k(t).$$

Define

$$\Delta_n(f(t)) = \begin{cases} c(0) & \text{if } n = 0 \\ \sum_{k=2^{n-1}}^{2^n-1} c_k \psi_k & \text{if } n = 1, 2, \ldots . \end{cases}$$

Paley [11] has shown the following result. If $f \in L^p$, $1 < p < \infty$, then

$$0 < A_1 \int_0^1 |f(t)|^p dt \leq \int_0^1 \{ \sum_{n=0}^{\infty} |\Delta_n(f(t))|^2 \}^{p/2} dt \leq A_2 \int_0^1 |f(t)|^p dt < \infty. \quad (3.3)$$

Hirschman [5] extended this result as follows. If $-1/2 < \alpha < 1/2$, if $1 < p < \infty$, and if $f(t) \in L^1(0,1)$, then

$$0 < A_1 \int_0^1 (\omega(t))^{p\alpha} |f(t)|^p dt \leq \int_0^1 (\omega(t))^{p\alpha} \{ \sum_{n=0}^{\infty} |\Delta_n(f(t))|^2 \}^{p/2} dt$$

$$\leq A_2 \int_0^1 (\omega(t))^{p\alpha} |f(t)|^p dt < \infty. \quad (3.4)$$

Using (3.2), we observe this result may be written in terms of the measure spaces L^1_μ. If $-1/2 < \alpha < 1/2$, if $1 < p < \infty$, and if $f(t) \in L^1(0,1)$, then

$$0 < A_1 \int_0^1 |f(t)|^p d\mu \le \int_0^1 \sum_{n=0}^{\infty} \{|\Delta_n(f(t))|^2\}^{p/2} d\mu \le A_2 \int_0^1 |f(t)|^p d\mu. \quad (3.5)$$

LEMMA 2. Suppose that $f_k \in L^2(0,1)$ for $k = 1, 2, \ldots, -1/2 < \alpha < 1/2$. Then

$$A_1 \int_0^1 \sum_{k=0}^{\infty} |f_k(t)|^2 d\mu \le \int_0^1 \sum_{n=1}^{\infty} \sum_{k=1}^{\infty} |\Delta_n(f_k(t))|^2 d\mu \le A_2 \int_0^1 \sum_{k=1}^{\infty} |f_k(t)|^2 d\mu.$$

PROOF. From (3.5), for each k,

$$A_1 \int_0^1 |f_k(t)|^2 d\mu \le \int_0^1 \sum_{n=0}^{\infty} |\Delta_n(f_k(t))|^2 d\mu \le A_2 \int_0^1 |f_k(t)|^2 d\mu.$$

Thus

$$A_1 \sum_{k=1}^{\infty} \int_0^1 |f_k(t)|^2 d\mu \le \sum_{k=1}^{\infty} \int_0^1 \sum_{n=0}^{\infty} |\Delta_n(f_k(t))|^2 d\mu \le A_2 \sum_{k=1}^{\infty} \int_0^1 |f_k(t)|^2 d\mu.$$

By the monotone convergence theorem,

$$A_1 \int_0^1 \sum_{k=1}^{\infty} |f_k(t)|^2 d\mu \le \int_0^1 \sum_{n=1}^{\infty} \sum_{k=1}^{\infty} |\Delta_n(f_k(t))|^2 d\mu \le A_2 \int_0^1 \sum_{k=1}^{\infty} |f_k(t)|^2 d\mu.$$

THEOREM 4. Suppose that $f_n \in L^2(0,1)$ for every n and that p_n is a sequence of integers. If $S_{n,\nu}$ denotes the $\nu^{\underline{th}}$ partial sum of the Walsh-Fourier series of $f_n(t)$, then there exists a constant A independent of f_n such that

$$\int_0^1 \sum_{n=0}^{\infty} |S_{n,p_n}(t)|^2 d\mu \le A \int_0^1 \sum_{n=0}^{\infty} |f_n(t)|^2 d\mu.$$

PROOF. For every integer k,

$$k = \sum_{j=1}^{\lambda(k)} 2^{k_j - 1}, \text{ where } k_{\lambda(k)} > \ldots > k_1 \geq 1.$$

Fix n for the moment and define

$$f_n(t)\psi_{p_n}(t) \sim \sum_{m=0}^{\infty} c'_m \psi_m(t).$$

Observe that $\Delta_k(f_n\psi_{p_n}(t)) = \sum_{m=2^{k-1}}^{2^k-1} c'_m\psi_m(t)$, and set

$$g_{n,k} = \sum_{m=2^{k-1}}^{2^k-1} c'_m\psi_m(t).$$

Define $m \dot{+} k = \sum_{m_i \neq k_j} 2^{m_i} + \sum_{k_j \neq m_i} 2^{k_j}$ where

$$m = \sum_{i=1}^{\lambda(m)} 2^{m_i}, \; m_{\lambda(m)} > \ldots > m_1 \geq 1$$

and

$$k = \sum_{j=1}^{\lambda(k)} 2^{k_j}, \; k_{\lambda(k)} > \ldots > k_1 \geq 1.$$

Since $\psi_m\psi_k = \psi_{m+k}$ and $m = n \dot{+} k$ if and only if $n = m \dot{+} k$,

$$\psi_k f \sim \sum_{n=0}^{\infty} c_n\psi_n\psi_k = \sum_{n=0}^{\infty} c_n\psi_{n\dot{+}k}$$

$$= \sum_{m=0}^{\infty} c_{m\dot{+}k}\psi_m.$$

It follows that $c'_m = c_{m\dot{+}k}$, for m = 0, 1, ... and k = 1, 2, ...

Hence

$$\sum_{r=1}^{\lambda(k)} \Delta_{k_r} (f_n \psi_{p_n}) = \sum_{r=1}^{\lambda(k)} \sum_{m=2^{k_r-1}}^{2^{k_r}-1} c'_m \psi_m$$

$$= \sum_{r=1}^{\lambda(k)} \sum_{m=2^{k_r-1}}^{2^{k_r}-1} \psi_k c_{m \dot{+} k} \psi_{m \dot{+} k}$$

$$= \psi_k \sum_{n \in A} c_n \psi_n$$

where $A = \{m \dot{+} k \mid 2^{k_r-1} \leq m < 2^{k_r}, r = 1, 2, \ldots, \lambda(k)\}$. We now establish that

$$A = \{0, 1, 2, \ldots, k - 1\}.$$

Indeed, if $m = 2^{k_r} + \mu$, where $0 \leq \mu < 2^{k_r}$, and $k = 2^{k_r} + (k - 2)^{k_r}$, then

$$m \dot{+} k = \mu \dot{+} (k - 2^{k_r})$$

$$\leq \mu + (k - 2^{k_r})$$

$$< k.$$

Hence $0 \leq m \dot{+} k < k$. However, it is clear that A contains k indices, and that no two indices of A are equal since the non-negative integers form a group under $\dot{+}$.

This reformulation for A means that

$$\sum_{n\in A} c_n\psi_n = \sum_{n=0}^{k-1} c_n\psi_n = S_{n,k}.$$

In particular,

$$\sum_{r=1}^{\lambda(k)} g_{n,k_r} = \sum_{r=1}^{\lambda(k)} \Delta_{k_r}(f_n\psi_{p_n}) = \psi_k S_{n,k}. \tag{3.6}$$

Also by definition, we observe for $1 \leq j \leq \lambda$, that

$$\Delta_{k_j}\left(\sum_{r=1}^{\lambda} g_{n,k_r}\right) = g_{n,k_j}. \tag{3.7}$$

Consequently, if we use (3.6) together with Lemma 2 and (3.7), we obtain

$$\int_0^1 \sum_{n=1}^{\infty} |S_{n,p_n}(t)|^2 d\mu \leq \int_0^1 \sum_{n=1}^{\infty} \left| \sum_{r=1}^{\lambda(p_n)} \Delta_{k_r}(f_n\psi_{p_n})\right|^2 d\mu$$

$$= \int_0^1 \sum_{n=1}^{\infty} \left| \sum_{r=1}^{\lambda(p_n)} g_{n,k_r}\right|^2 d\mu$$

$$\leq A \int_0^1 \sum_{n=1}^{\infty} \sum_{\nu=1}^{\lambda(p_n)} \left|\Delta_{k_\nu}\left(\sum_{r=1}^{\lambda(p_n)} g_{n,k_r}\right)\right|^2 d\mu$$

$$= A \int_0^1 \sum_{n=1}^{\infty} \sum_{\nu=1}^{\lambda(p_n)} |g_{n,k_\nu}|^2 d\mu.$$

Another application of Lemma 2 yields

$$A\int_0^1 \sum_{n=1}^{\infty}\sum_{\nu=1}^{\lambda(p_n)} |g_{n,k_\nu}|^2 d\mu \le A\int_0^1 \sum_{n=1}^{\infty}\sum_{\nu=0}^{\infty} |g_{n,\nu}|^2 d\mu$$

$$= A\int_0^1 \sum_{n=1}^{\infty}\sum_{\nu=0}^{\infty} |\Delta_\nu(f_n\psi_{p_n})|^2 d\mu$$

$$\le AB\int_0^1 \sum_{n=1}^{\infty} |f_n\psi_{p_n}|^2 d\mu$$

$$= AB\int_0^1 \sum_{n=1}^{\infty} |f_n|^2 d\mu.$$

Therefore the conclusion follows.

THEOREM 5. Given $f \in L^2(\omega_0)$, ω_0 a dyadic interval, $y > 0$, and positive integer $N > N_0$, $|\omega_0| = 2^{-N_0}$, there is a set $E = E(f,y,N)$ such that

$$mE \le Cy^{-2}\int_a^b |f(x)|^2 dx \text{ and } x \in \omega_0 \sim E, \tag{3.8}$$

$$0 < n < 2^N \text{ implies } |S_n f(x)| \le Cy \text{ where } \omega_0 = (a,b).$$

PROOF. Let ω_0^* be the unique dyadic subinterval of ω_0 and $|\omega^*| = 2|\omega|$ where ω is a dyadic interval, $\omega \subseteq \omega_0$. If $|\omega| = 2^{-\mu}$ and $n = \sum_{j=0}^{\infty} \varepsilon_j 2^j$, $\varepsilon_j = 0$ or 1, we write $n(\omega) = \sum_{j=\mu}^{\infty} \varepsilon_j 2^j$. Observe that $\psi_n = \psi_{n(\omega)}\psi_s$, where $0 \le s = n - n(\omega) < 2^\mu = |\omega|^{-1}$. $0 \le s < |\omega|^{-1}$ implies ω_s is identically ± 1 on ω. Define $D_n^* = \psi_n D_n$. If $n = \sum_{j=0}^{\infty} \varepsilon_j 2^j$, $\varepsilon_j = 0$ or 1,

then $D_n^*(t) = \sum_{j=0}^{\infty} \varepsilon_j \delta_j^*(t)$, [8], where

$$\delta_j^* = \sum_{\nu=2^j}^{2^{j+1}-1} \psi_\nu = \psi_{2^j} D_{2^j}.$$

For fixed x, $\delta_j^*(x \dot{+} t)$ is constant for $t \in \omega$ if $x \notin \omega$ [8]. Define

$$T_n f(x) = \int_a^b f(t) D_n^*(x \dot{+} t)dt.$$

Since $T_n f(x) = \psi_n(x) S_n(\psi_n f)(x)$, Paley's theorem [12] implies that

$$\int_a^b |T_n f|^p \leq C_p \int_a^b |f|^p, \quad 1 < p < \infty. \tag{3.9}$$

Define

$$Hf(x) = \sup \{|S_{2^n} f(x)| : n \geq 0\}.$$

Since

$$Hf(x) = \sup \left\{ \left| |\omega|^{-1} \int_\omega f(t)dt \right| : x \in \omega \right\},$$

Hf is majorized by the Hardy-Littlewood maximal function of f, so

$$\int_a^b |Hf|^p \leq C_p \int_a^b |f|^p, \; 1 < p < \infty. \tag{3.10}$$

Define

$$a_n(\omega) = a_n(\omega;f) = |\omega|^{-1} \int_{\omega} f(t)\psi_n(t)dt, \qquad \omega \subset \omega_0.$$

We then observe $a_{n(\omega)}(\omega) = \pm a_n(\omega)$. Define

$$A_n(\omega) = \max \{|a_n(\overline{\omega})| : \overline{\omega}^* = \omega\}.$$

$\left\{\psi_{n|\omega|^{-1}}\right\}_{n=0}^{\infty}$ is a complete orthonormal set on ω with respect to the measure $|\omega|^{-1}dx$ since for $|\omega| = 2^{-\nu}$,

$$\int_{\omega} f(t)\psi_{n|\omega|^{-1}}(t)|\omega|^{-1}dt = 2^{\nu} \int_{\omega} f(t)\psi_{n2^{\nu}}(t)dt$$

$$= 2^{\nu} \int_{\omega} f(t)\psi_n(2^{\nu}t)dt$$

$$= \int_0^1 f(t2^{-\nu})\psi_n(t)dt$$

and $\{\psi_n\}_{n=0}^{\infty}$ is known to be a complete orthonormal set on (0,1). Plancherel's formula on ω is

$$\sum_{n=0}^{\infty} |a_{n|\omega|^{-1}}(\omega;f)|^2 = \frac{1}{|\omega|} \int_{\omega} |f(t)|^2 dt.$$

The main idea in the proof is to estimate $S_n f(x)$ by using predetermined estimates of certain selected sums of the form $S_{n(\omega)}f(x) - S_{n(\overline{\omega})}f(x)$. We will have "enough" selected sums to write $S_n f(x) = \sum_{j=0}^{\mu} (S_{n(\omega_j)}f(x) - S_{n(\omega_{j+1})}f(x))$, where each term

$S_{n(\omega_j)}f(x) - S_{n(\omega_{j+1})}f(x)$ is the difference of two selected sums. In order that we have appropriate estimates of our selected sums, we must avoid certain x for each selected sum.

To determine our selected sums we define a collection $G^* = \bigcup_{k=1}^{\infty} G_k^*$ of selected pairs $(n(\omega),\omega)$. For each pair $(n(\omega),\omega) \in G_k^*$ we define a partition

$$\Omega = \Omega(n(\omega),\omega,k) \text{ of } \omega.$$

The partition Ω determines intervals $\bar{\omega}$ for which $S_{n(\omega)}f(x) - S_{n(\bar{\omega})}f(x)$ is a selected sum. Also, Ω is used to obtain appropriate estimates of selected sums. The selection of pairs G^*, the partitions Ω and the corresponding estimates are coordinated by use of the numbers $A_n(\omega)$.

The numbers $A_n(\omega)$ are controlled by the set

$$S^* = \bigcup \{\omega^* \subseteq \omega_0 : \int_{\omega} |f(t)|^2 dt \geq y^2|\omega|\}.$$

For any n and an application of Holder's inequality,

$$A_n(\omega) = \max \{|a_n(\bar{\omega})| : \bar{\omega}^* = \omega\}$$

$$\leq \max \{|\bar{\omega}|^{-1} \int_{\bar{\omega}} |f(t)\psi_n(t)|dt : \bar{\omega}^* = \omega\}$$

$$= \max \{|\bar{\omega}|^{-1} \int_{\bar{\omega}} |f(t)|dt : \bar{\omega}^* = \omega\}$$

$$\leq \max \{|\bar{\omega}|^{-1} \left(\int_{\bar{\omega}} |f(t)|^2\right)^{1/2} |\bar{\omega}|^{1/2} dt : \bar{\omega}^* = \omega\}$$

$$\leq \max \{(|\bar{\omega}|^{-1} \int_{\bar{\omega}} |f(t)|^2 dt)^{1/2} : \bar{\omega}^* = \omega\}$$

$$< \max \left\{ \left(\frac{|\bar{\omega}| y^2}{|\bar{\omega}|} \right)^{1/2} : \bar{\omega}^* = \omega \right\} \text{ for } \omega \not\subset S^*$$

$$= y.$$

We therefore have

$$\omega \not\subset S^* \text{ implies } A_n(\omega) < y \text{ for all } n. \tag{3.11}$$

Since we can write S^* as a union of disjoint intervals ω_j^* with $\int_{\omega_j} |f(t)|^2 dt \geq y^2 |\omega_j|$ for all j, we have

$$mS^* = \sum_j |\omega_j^*| = 2 \sum_j |\omega_j| \leq 2y^{-2} \int_a^b |f(x)|^2 dx. \tag{3.12}$$

For each positive integer k, let G_k be the collection of all pairs $(n(\omega),\omega)$ such that

$$|a_{n(\omega)}(\omega)| \geq 2^{-k} y \tag{3.13}$$

and

$$|\omega| = |\omega_0|/2 \text{ and } |a_{n(\omega)}(\omega)| < 2^{-k+1} y$$

or

$2^{-N} \leq |\omega| < |\omega_0|/2$ and $|a_{\bar{n}}(\bar{\omega})| < 2^{-k}y$ for all $\bar{n}$, $\bar{\omega}$ with $\bar{n}(\omega) = n(\omega)$ and $\bar{\omega} \supsetneq \omega$, $|\bar{\omega}| \leq |\omega_0|/2$. (3.14)

Let

$$P_k(x;\omega) = \sum_{\substack{(n(\bar{\omega}),\bar{\omega})\in G_k \\ \bar{\omega} \supset \omega}} a_{n(\bar{\omega})}(\bar{\omega})\psi_{n(\bar{\omega})}(x).$$

Note that if $(n(\omega),\omega) \in G_k$ then $P_k(x;\omega^*)$ contains no term $a_m\psi_m$ with $m(\omega) = n(\omega)$, so $\int_\omega P_k(x;\omega^*)\psi_{n(\omega)}(x)dx = 0$. Hence, $(n(\omega),\omega) \in G_k$ implies $a_{n(\omega)}(\omega) = a_{n(\omega)}(\omega;f) = a_{n(\omega)}(\omega;f - P_k(\cdot,\omega^*))$. Plancherel's formula implies

$$\int_\omega |f(x) - P_k(x,\omega)|^2dx$$

$$= \int_\omega |f(x) - P_k(x,\omega^*) - \sum_{(n(\omega),\omega)\in G_k} a_{n(\omega)}(\omega)\psi_{n(\omega)}(x)|^2dx$$

$$= \int_\omega |f(x) - P_k(x,\omega^*)|^2dx - \sum_{(n(\omega),\omega)\in G_k} |a_{n(\omega)}(\omega)|^2|\omega|. \quad (3.15)$$

In particular,

$$\sum_{|\omega|=2^{-n}} \int_\omega |f(x) - P_k(x,\omega)|^2dx$$

$$= \sum_{|\omega|=2^{-(n-1)}} \int_\omega |f(x) - P_k(x,\omega)|^2dx - \sum_{\substack{(n(\omega),\omega)\in G_k \\ |\omega| = 2^{-n}}} |a_{n(\omega)}(\omega)|^2|\omega| \quad (3.16)$$

since

$$\sum_{\substack{\omega \\ |\omega|=2^{-n}}} \int_{\omega} |f(x) - P_k(x;\omega^*)|^2 dx = \sum_{\substack{\omega \\ |\omega|=2^{-n+1}}} \int_{\omega} |f(x) - P_k(x;\omega)|^2 dx.$$

Repeating the same argument on the first term on the right of (3.16), we obtain after a finite number of steps

$$0 \leq \sum_{\substack{\omega \\ |\omega|=2^{-N}}} \int_{\omega} |f(x) - P_k(x,\omega)|^2 dx = \int_a^b |f(x)|^2 dx - \sum_{\substack{(n(\omega),\omega)\in G_k \\ |\omega|\geq 2^{-N}}} |a_{n(\omega)}(\omega)|^2 |\omega|$$

Equation (3.13) then implies

$$\sum_{(n(\omega),\omega)\in G_k} |\omega| \leq 2^{2k_y-2} \int_a^b |f(x)|^2 dx. \tag{3.17}$$

Let G_k^* be the collection of all pairs (n,ω^*) so that

$$n = n(\omega^*) \text{ and } (n(\omega),\omega) \in G_k.$$

Note that for each pair $(n(\omega),\omega) \in G_k$ there correspond two pairs $(n(\omega),\omega^*)$ and $(n(\omega) + |\omega^*|^{-1},\omega^*)$, in G_k^* since if $|\omega| = 2^{-\mu}$,

$$n(\omega^*) = \sum_{j=\mu-1}^{\infty} \varepsilon_j 2^j = \varepsilon_{\mu-1} 2^{\mu-1} + \sum_{j=\mu}^{\infty} \varepsilon_j 2^j = \varepsilon_{\mu-1} 2^{\mu-1} + n(\omega)$$

$$= \varepsilon_{\mu-1} |\omega^*|^{-1} + n(\omega).$$

Since $|\omega^*| = 2|\omega|$, (3.17) yields

$$\sum_{(n,\omega)\in G_k^*} |\omega| \leq 4 \cdot 2^{2k_y-2} \int_a^b |f(x)|^2 \, dx. \tag{3.18}$$

Equation (3.18) tells us G_k^* does not contain "too many" pairs. The next three observations tell us that collection $G^* = \bigcup_{k=1}^{\infty} G_k^*$ contains "enough" pairs.

$$2^{-\tilde{k}}y \leq A_{n(\omega)}(\omega) < 2^{-\tilde{k}+1}y, \ \omega = \omega_0 \text{ implies } (n(\omega),\omega) \in G_{\tilde{k}}^*. \tag{3.19}$$

If $2^{-k}y \leq A_{n(\omega)}(\omega)$ and $\omega \not\subset S^*$ then there exist $(\tilde{n},\tilde{\omega},\tilde{k})$

with $\tilde{n}(\hat{\omega}) = n(\hat{\omega})$, $2|\hat{\omega}| = |\omega|$, $\tilde{\omega} \supset \omega$, $1 \leq \tilde{k} \leq k$, and

$$(\tilde{n}(\tilde{\omega}),\tilde{\omega}) \in G_{\tilde{k}}^*. \tag{3.20}$$

Equation (3.19) is obvious. We now show (3.20). If $(n(\omega),\omega) \in G_k^*$, we are done. If $(n(\omega),\omega) \notin G_k^*$, $\omega \subsetneq \omega_0$, then $(n(\hat{\omega}),\hat{\omega}) \notin G_k$ where $\hat{\omega}^* = \omega$. Also $\omega \subsetneq \omega_0$ implies $|\hat{\omega}| < |\omega_0|/2$. By (3.14), there exists $\bar{n}$ and ω' so that $\bar{n}(\hat{\omega}) = n(\hat{\omega})$, $\omega' \supsetneq \hat{\omega}$, $|\omega'| \leq |\omega_0|/2$, $a_{\bar{n}}(\omega') \geq 2^{-k}y$. Then $A_{\bar{n}}(\omega'^*) \geq 2^{-k}y$. Also $\omega'^* \supsetneq \omega' \supsetneq \hat{\omega}$ implies $\omega'^* \supsetneq \omega' \supseteq \omega \supsetneq \hat{\omega}$. Put $\bar{\omega} = \omega'^*$. Then $A_{\bar{n}}(\bar{\omega}) \geq 2^{-k}y$, $\bar{\omega} \supsetneq \omega$, and $\bar{n}(\hat{\omega}) = n(\hat{\omega})$. If $(\bar{n}(\bar{\omega}),\bar{\omega}) \notin G_k^*$, we repeat the argument until we find $(\bar{n}(\bar{\omega}),\bar{\omega}) \in G_k^*$ or reach $\bar{\omega} = \omega_0$. When $\bar{\omega} = \omega_0$ we can apply (3.19). We also note $\omega \not\subset S^*$ implies $k \geq 1$ for $k = 0$ and $\omega \not\subset S^*$ implies $y \leq A_{n(\omega)}(\omega) < y$ by (3.11) if $\omega = \omega_0$. For $\omega \subsetneq \omega_0$, clearly $\tilde{k} \geq 1$.

If we choose $(\tilde{n},\tilde{\omega},\tilde{k})$ in (3.20) such that $\tilde{k}$ is minimal, we have

$$A_{\tilde{n}(\bar{\omega})}(\bar{\omega}) < 2^{-\tilde{k}+1} \text{ for all } \bar{\omega},\ \omega \subset \bar{\omega} \subset \tilde{\omega}. \tag{3.21}$$

If (3.21) were not true, we could apply (3.20) to $(\tilde{n}(\bar{\omega}),\bar{\omega},\tilde{k}-1)$ in place of $(n(\omega),\omega,k)$ and contradict the fact that $\tilde{k}$ is minimal.

We now establish the identity

$$a_{n(\omega)}(\omega) = a_{n(\omega)}(\omega^*) \pm a_{n(\omega)+|\omega^*|-1}(\omega^*). \tag{3.22}$$

Now

$$a_{n(\omega)}(\omega^*) \pm a_{n(\omega)+|\omega^*|-1}(\omega^*) = \frac{1}{2|\omega|} \int_{\omega^*} f(t)\psi_{n(\omega)}(t)[1 \pm \psi_{|\omega^*|-1}(t)]dt.$$

If ω is the left half of ω^*, then

$$1 + \psi_{|\omega^*|-1}(t) = \begin{cases} 2 \text{ on } \omega \\ 0 \text{ on } \omega^* \sim \omega. \end{cases}$$

If ω is the right half of ω^*, then

$$1 - \psi_{|\omega^*|-1}(t) = \begin{cases} 2 \text{ on } \omega \\ 0 \text{ on } \omega^* \sim \omega. \end{cases}$$

Therefore

$$a_{n(\omega)}(\omega^*) \pm a_{n(\omega)+|\omega^*|-1}(\omega^*) = \frac{1}{|\omega|} \int_{\omega} f(t)\psi_{n(\omega)}(t)dt = a_{n(\omega)}(\omega).$$

Using (3.22), we establish

$$(n(\omega),\omega) \in G_k^* \text{ implies } A_{n(\omega)}(\omega) < 2^{-k+1}y. \tag{3.23}$$

Let $(n(\omega),\omega) \in G_k^*$ where $\overline{\omega}^* = \omega$ and $|\omega| < |\omega_0|$

$$A_{n(\omega)}(\omega) \equiv \max \{|a_{n(\omega)}(\overline{\omega})| : \overline{\omega}^* = \omega\}$$

$$= \max \{|a_{n(\overline{\omega})}(\overline{\omega})| : \overline{\omega}^* = \omega\}$$

$$= \max \{|a_{n(\overline{\omega})}(\omega) \pm a_{n(\overline{\omega})+|\overline{\omega}|-1}(\omega)| : \overline{\omega}^* = \omega\}$$

implies $A_{n(\omega)}(\omega) \leq |a_{n(\overline{\omega})}(\omega)| + |a_{n(\overline{\omega})+|\omega|-1}(\omega)|$. Since $(n(\omega),\omega) \in G_k^*$ and $|\omega|<|\omega_0|$, it follows from (3.14) that $|a_{n(\overline{\omega})}(\omega)| < 2^{-k}y$ and $|a_{n(\overline{\omega})+|\omega|-1}(\omega)| < 2^{-k}y$. Therefore

$$A_{n(\omega)}(\omega) < 2^{-k+1}y \text{ if } |\omega| < |\omega_0|$$

Note that (n, ω_0) might be in two sets $G_{k_1}^*$ and $G_{k_2}^*, k_1<k_2$. In that case delete (n, ω_0) from $G_{k_2}^*$. Then (3.23) holds and (3.23) insures (3.24), (3.25), and (3.26) below yield a collection Ω of disjoint intervals ω' with $\omega = \bigcup \omega'$, $\omega' \in \Omega$.

If $(n(\omega),\omega) \in G_k^*$, choose $\omega' \in \Omega = \Omega(n(\omega),\omega,k)$ if

$$\omega' \subset \omega, \ |\omega'| \geq 2^{-N}, \tag{3.24}$$

$$|a_{n(\omega)}(\bar{\omega})| < 2^{-k+1}y \text{ for all } \omega \supset \bar{\omega} \supset \omega', \tag{3.25}$$

$$|\omega'| > 2^{-N} \text{ and } A_{n(\omega)}(\omega') \geq 2^{-k+1}y \text{ or } |\omega'| = 2^{-N}. \tag{3.26}$$

(Constructively, starting with $\bar{\omega} = \omega$ we split $\bar{\omega}$ in half if $A_{n(\omega)}(\bar{\omega}) < 2^{-k+1}y$ and $|\bar{\omega}| > 2^{-N}$. Otherwise, $\bar{\omega} \in \Omega$.)

For each pair $(n(\omega),\omega) \in G_k^*$ the corresponding selected sums are those sums $S_{n(\omega)}f(x) - S_{n(\bar{\omega})}f(x)$, where $x \in \bar{\omega}$, $\omega' \subset \bar{\omega} \subset \omega$, $\omega' \in \Omega$. (Hence, $\omega'' \in \Omega$ implies $\omega'' \cap \bar{\omega} = \omega''$ or ϕ.)

Suppose $S_{n(\omega)}f(x) - S_{n(\bar{\omega})}f(x)$ is a selected sum which corresponds to $(n(\omega),\omega) \in G_k^*$. If $n(\omega) = \sum_{j=\mu}^{\infty} \varepsilon_j 2^j$, $\varepsilon_j = 0$ or 1, and $n(\bar{\omega}) = \sum_{j=\bar{\mu}}^{\infty} \varepsilon_j 2^j$, $\bar{\mu} > \mu$, then

$$\begin{aligned} S_{n(\omega)}f(x) - S_{n(\bar{\omega})}f(x) &= \int_a^b f(t)(D_{n(\omega)}(x \dot{+} t) - D_{n(\bar{\omega})}(x \dot{+} t))dt \\ &= \int_a^b f(t)\psi_{n(\omega)}(x \dot{+} t) \sum_{j=\mu}^{\bar{\mu}-1} \varepsilon_j \delta_j^*(x \dot{+} t)dt \\ &+ \int_a^b f(t)\psi_{n(\omega)}(x \dot{+} t) \sum_{j=\bar{\mu}}^{\infty} \varepsilon_j \delta_j^*(x \dot{+} t)dt \\ &- \int_a^b f(t)\psi_{n(\omega)}(x \dot{+} t) \sum_{j=\mu}^{\infty} \varepsilon_j \delta_j^*(x \dot{+} t)dt \end{aligned}$$

$$= \sum_{j=\mu}^{\bar{\mu}-1} \varepsilon_j \int_a^b f(t)\psi_{n(\omega)}(x \dot{+} t)\delta_j^*(x \dot{+} t)dt$$

since $\psi_{n(\bar{\omega})}(t)\delta_j^*(t) = \psi_{n(\omega)}(t)\delta_j^*(t)$ for $j \geq \bar{\mu}$. Define

$$g(t) = \begin{cases} a_{n(\omega)}(\omega') & \text{for } t \in \omega' \in \Omega \\ 0 & \text{for } t \notin \omega. \end{cases}$$

We then observe that (3.25) implies

$$|g(t)| < 2^{-k+1}y, \; t \in \omega. \tag{3.27}$$

For fixed $x \in \bar{\omega}$, $\delta_j^*(x \dot{+} t)$ is constant for $t \in \bar{\omega}$, $j < \bar{\mu}$. Hence, $\delta_j^*(x \dot{+} t)$ is constant for $t \in \omega'$, where $x \in \omega'$, $\omega' \in \Omega$, $\omega' \subset \bar{\omega}$, $j < \bar{\mu}$. From Hunt [8] $\delta_j^*(x \dot{+} t)$ is also constant for t in any interval $\omega'' \in \Omega$, $x \notin \omega''$. Also $\delta_j^*(x \dot{+} t) = 0$, $t \notin \omega$ for $j \geq \mu$ and $x \in \omega$. Define

$$c_j = \delta_j^*(x \dot{+} t) \text{ for } t \in \omega_j$$

and

$$a_j = a_{n(\omega)}(\omega_j) \text{ where } \omega = \dot{\bigcup_{j=1}^{N(\omega)}} \omega_j, \; \omega_j \in \Omega(\omega).$$

Then

$$S_{n(\omega)}f(x)-S_{n(\bar{\omega})}f(x) = \sum_{j=\mu}^{\bar{\mu}-1} \varepsilon_j \int_{\omega} f(t)\psi_{n(\omega)}(x \dot{+} t)\delta_j^*(x \dot{+} t)dt$$

$$= \sum_{j=\mu}^{\bar{\mu}-1} \varepsilon_j \psi_{n(\omega)}(x) \sum_{k=1}^{N(\omega)} [\int_{\omega_k} f(t)\psi_{n(\omega)}(t)dt]\delta_j^*(x \dot{+} t)dt$$

$$= \sum_{j=\mu}^{\bar{\mu}-1} \varepsilon_j \psi_{n(\omega)}(x) \sum_{k=1}^{N(\omega)} a_{n(\omega)}(\omega_k)|\omega_k|c_j$$

$$= \sum_{j=\mu}^{\bar{\mu}-1} \varepsilon_j \psi_{n(\omega)}(x) \sum_{k=1}^{N(\omega)} a_k|\omega_k|c_j$$

$$= \sum_{j=\mu}^{\bar{\mu}-1} \varepsilon_j \psi_{n(\omega)}(x) \sum_{k=1}^{N(\omega)} \int_{\omega_k} a_{n(\omega)}(\omega_k)c_j dt$$

$$= \sum_{j=\mu}^{\bar{\mu}-1} \varepsilon_j \psi_{n(\omega)}(x) \sum_{k=1}^{N(\omega)} \int_{\omega_k} g(t)\delta_j^*(x \dot{+} t)dt$$

$$= \sum_{j=\mu}^{\bar{\mu}-1} \varepsilon_j \psi_{n(\omega)}(x) \int_a^b g(t)\delta_j^*(x \dot{+} t)dt.$$

Therefore

$$S_{n(\omega)}f(x)-S_{n(\bar{\omega})}f(x) = \sum_{j=\mu}^{\bar{\mu}-1} \varepsilon_j \psi_{n(\omega)}(x) \int_a^b g(t)\delta_j^*(x + t)dt.$$

Observe that

$$\sum_{j=\mu}^{\bar{\mu}-1} \varepsilon_j \int_a^b g(t)\delta_j^*(x \dot{+} t)dt = S_{2^{\bar{\mu}}}(T_{n(\omega)}g)(x)$$

by the following argument. Since

$$(T_{n(\omega)}g)(x) = \int_a^b g(t)D^*_{n(\omega)}(x \dot{+} t)dt$$

$$= \sum_{j=\mu}^{\infty} \varepsilon_j \int_a^b g(t) \sum_{k=2^j}^{2^{j+1}-1} \psi_k(x \dot{+} t)dt$$

$$= \sum_{j=\mu}^{\infty} \varepsilon_j (S_{2^{j+1}} g - S_{2^j} g)(x)$$

$$= \Big(\sum_{j=\mu}^{\bar{\mu}-1} \varepsilon_j + \sum_{j=\bar{\mu}}^{\infty} \varepsilon_j\Big)(S_{2^{j+1}} g - S_{2^j} g)(x),$$

$$S_{2^{\bar{\mu}}}(T_{n(\omega)} g)(x) = \sum_{j=\mu}^{\bar{\mu}-1} \varepsilon_j \, S_{2^{\bar{\mu}}}(S_{2^{j+1}} g - S_{2^j} g)(x) + 0$$

$$= \sum_{j=\mu}^{\bar{\mu}-1} \varepsilon_j (S_{2^{j+1}} g - S_{2^j} g)(x)$$

$$= \sum_{j=\mu}^{\bar{\mu}-1} \varepsilon_j \int_a^b g(t) \delta_j^*(x \dot{+} t)\, dt.$$

By the definition of H,

$$|S_{n(\omega)} f(x) - S_{n(\bar{\omega})} f(x)| \leq H(T_{n(\omega)} g)(x).$$

Let $U^* = U^*(n(\omega),\omega,k) = \{x \in \omega_0 : H(T_{n(\omega)} g)(x) > 2^{-k/2} y\}$.

Noting (3.27) and applying (3.9) and (3.10) with, say, p = 6, we obtain

$$mU^* \leq (2^{-k/2} y)^{-6} \Big|\Big| H(T_{n(\omega)} g) \Big|\Big|_6^6 \leq C 2^{-3k} |\omega|. \tag{3.28}$$

If $x \in \bar{\omega}$, $\omega' \subset \bar{\omega} \subset \omega$, $\omega' \in \Omega$, and $x \notin U^*$, then

$$|S_{n(\omega)} f(x) - S_{n(\bar{\omega})} f(x)| \leq 2^{-k/2} y. \tag{3.29}$$

We can now assemble the various parts of the proof. Let $E = S^* \cup \bigcup_{k=1}^{\infty} \bigcup_{(n(\omega),\omega) \in G_k^*} U^*(n(\omega),\omega,k)$. From (3.12), (3.18), and (3.28)

we obtain $mE \leq (2 + C \sum_{k=1}^{\infty} 4 \cdot 2^{-k})y^{-2} \int_a^b |f(x)|^2 dx$.

Consider $S_n f(x)$, $0 < n < 2^N$, $x \notin E$. We may assume $A_n(\omega_0) > 0$. Then

$$2^{-\tilde{k}_0} y \leq A_n(\omega_0) < 2^{-\tilde{k}_0+1} y \text{ for some } \tilde{k}_0 \geq 1,$$

so $(n,\omega_0) \in G^*_{\tilde{k}_0}$ by (3.19) and the partition $\Omega_0 = \Omega(n,\omega_0,\tilde{k}_0)$ is defined. Let ω_1 be the interval of Ω_0 which contains x. Equation (3.29) implies

$$|S_n f(x) - S_{n(\omega_1)} f(x)| \leq 2^{-\tilde{k}_0/2} y.$$

From our comment on construction of Ω, we see $\omega_1 \subsetneq \omega_0$. If $n(\omega_1) = 0$, we stop. If $n(\omega_1) \neq 0$, we continue with a typical step.

$$n(\omega_1) \neq 0 \text{ and } 2^N > n = \sum_{j=0}^{p} \varepsilon_j 2^j \text{ implies } p < N.$$

If $|\omega_1| = 2^{-\mu}$, $\sum_{j=\mu}^{p} \varepsilon_j 2^j = n(\omega_1) \neq 0$ implies $\mu < N$. Then $|\omega_1| = 2^{-\mu} > 2^{-N}$ and so (3.26) implies $2^{-k_1} y \leq A_{n(\omega_1)}(\omega_1) < 2^{-k_1+1} y$ for some $1 \leq k_1 < \tilde{k}_0$. Choose $(\tilde{n}_1,\tilde{\omega}_1,\tilde{k}_1)$ as in (3.20) and (3.21). Then $\Omega_1 = \Omega(\tilde{n}_1(\tilde{\omega}_1),\tilde{\omega}_1,\tilde{k}_1)$ is defined. Let ω_2 be the interval in Ω_1 which contains x noting $\omega_1 \subset \tilde{\omega}_1$. Now $x \in \omega_1 \cap \omega_2$. Therefore $\omega_1 \subset \omega_2$ or $\omega_2 \subset \omega_1$. Suppose $\omega_1 \subseteq \omega_2$. Now $x \in \omega_1 \subsetneq \omega_0$ and $\tilde{\omega}_1 \supset \omega_1$ implies by definition of Ω_1, $A_{\tilde{n}_1(\tilde{\omega}_1)}(\omega_2) \geq 2^{-\tilde{k}_1+1} y$ since $\omega_2 \in \Omega$. However,

$\omega_1 \subseteq \omega_2$ implies by (3.21) that $A_{\tilde{n}_1(\omega_2)}(\omega_2) < 2^{-\tilde{k}_1+1}$. Since this is a contradiction, $\omega_2 \not\subseteq \omega_1$.

From (3.20), $\tilde{n}_1(\hat{\omega}_1) = n(\hat{\omega}_1)$ and $\omega_2 \not\subseteq \omega_1$ implies $\tilde{n}_1(\omega_2) = n(\omega_2)$ which we now argue. Put $|\hat{\omega}_1| = 2^{-\mu}$. Then $\sum_{k=\mu}^{\infty} \varepsilon_k 2^k = \tilde{n}_1(\hat{\omega}_1) = n(\hat{\omega}_1)$ $= \sum_{k=\mu}^{\infty} \varepsilon'_k 2^k$ implies $\varepsilon_k = \varepsilon'_k$ for $k \geq \mu$. Moreover $\omega_2 \not\subseteq \omega_1$ implies $\omega_2 \subseteq \hat{\omega}_1$. So if $|\omega_2| = 2^{-m}$, $m \geq \mu$. Hence $\tilde{n}_1(\omega_2) = \sum_{k=m}^{\infty} \varepsilon_k 2^k = \sum_{k=m}^{\infty} \varepsilon'_k 2^k$ $= n_1(\omega_2)$ since $\varepsilon_k = \varepsilon'_k$ for $k \geq \mu$ and $m \geq \mu$.

Before proceeding further, we establish

$$|S_{n(\omega_1)}f(x) - S_{n(\hat{\omega}_1)}f(x)| \leq A_n(\omega_1). \tag{3.30}$$

Let $|\omega_1| = 2^{-\mu}$. Then $n(\omega_1) = \sum_{j=\mu}^{\infty} \varepsilon_j 2^j$ and $n(\hat{\omega}_1) = \sum_{j=\mu+1}^{\infty} \varepsilon_j 2^j$, $\varepsilon_j = 0$ or 1.

$$|S_{n(\omega_1)}f(x) - S_{n(\hat{\omega}_1)}f(x)| = \left|\int_a^b f(t) \sum_{j=n(\hat{\omega}_1)}^{n(\omega_1)-1} \psi_j(x \dot{+} t)dt\right|$$

$$= \left|\int_a^b f(t) \sum_{j=0}^{2^\mu-1} \psi_j(x \dot{+} t)\psi_{n(\hat{\omega}_1)}(x \dot{+} t)dt\right|$$

$$= \left|\int_a^b f(t)\psi_{n(\hat{\omega}_1)}(t)D_{2^\mu}(x \dot{+} t)dt\right|$$

$$= \left|\frac{1}{|\omega_1|}\int_{\omega_1} f(t)\psi_{n(\hat{\omega}_1)}(t)dt\right|$$

$$\leq A_{n(\hat{\omega}_1)}(\omega_1) = A_n(\omega_1).$$

Applying (3.30) and (3.29) twice, we obtain

$$|S_{n(\omega_1)}f(x) - S_{n(\omega_2)}f(x)|$$

$$\leq |S_{n(\omega_1)}f(x) - S_{n(\hat{\omega}_1)}f(x)| + |S_{\tilde{n}_1(\tilde{\omega}_1)}f(x) - S_{n(\hat{\omega}_1)}f(x)|$$

$$+ |S_{\tilde{n}_1(\tilde{\omega}_1)}f(x) - S_{n(\omega_2)}f(x)|$$

$$= |S_{n(\omega_1)}f(x) - S_{n(\hat{\omega}_1)}f(x)| + |S_{\tilde{n}_1(\tilde{\omega}_1)}f(x) - S_{\tilde{n}_1(\hat{\omega}_1)}f(x)|$$

$$+ |S_{\tilde{n}_1(\tilde{\omega}_1)}f(x) - S_{\tilde{n}_1(\omega_2)}f(x)|$$

$$\leq A_n(\omega_1) + 2 \cdot 2^{-\tilde{k}_1/2}y.$$

If $n(\omega_2) \neq 0$, we repeat the above typical step until we obtain

$$\omega_0 \supsetneq \omega_1 \supsetneq \cdots \supsetneq \omega_{\mu+1}, \quad \tilde{k}_0 > \tilde{k}_1 > \cdots > \tilde{k}_\mu \geq 1$$

with $n(\omega_j) \neq 0$, $j = 1, \ldots, \mu$, $n(\omega_{\mu+1}) = 0$ and

$$|S_{n(\omega_j)}f(x) - S_{n(\omega_{j+1})}f(x)| \leq 4 \cdot 2^{-\tilde{k}_j/2}y, \; j = 0, \ldots, \mu.$$

Then

$$|S_n f(x)| = |\sum_{j=0}^{\mu} (S_{n(\omega_j)}f(x) - S_{n(\omega_{j+1})}f(x))| \leq 4(\sum_{k=1}^{\infty} 2^{-k/2})y$$

and the theorem is proved.

LEMMA 3. Let $S^{**} = \bigcup \{\omega^* : \int_{\omega} |f(t)|^p dt \geq y^p |\omega|\}$ where $y > 0$, $1 < p < \infty$, and $f = \chi_F$ where F is a measurable subset of ω_0. Then $\omega \not\subset S^*$ implies $A_n(\omega) < y$ for every n.

PROOF. $\omega \not\subset S^*$ implies $|\hat{\omega}|^{-1/p} \left\{\int_{\hat{\omega}} |f(t)|^p dt\right\}^{1/p} < y$. By Holder's inequality,

$$a_n(\hat{\omega}) \leq |\hat{\omega}|^{-1} \int_{\hat{\omega}} |f(t)| dt$$

$$\leq |\hat{\omega}|^{-1} \left\{\int_{\hat{\omega}} |f(t)|^p dt\right\}^{1/p} |\hat{\omega}|^{1/q}$$

$$= |\hat{\omega}|^{-p^{-1}} \left\{\int_{\hat{\omega}} |f(t)|^p dt\right\}^{1/p}$$

$$< y.$$

Therefore $A_n(\omega) = \max \{|a_n(\hat{\omega})| : |\omega^*| = |\omega|\} < y$.

LEMMA 4. There exists an integer $L = L(p) \geq 5$ such that $L(p) = O(p^2/(p-1))$ such that $\omega^* \not\subset S$ and $2^{-k} y \leq A_n(\omega^*, \chi_F)$ imply $y^{p/2} \leq b_{kL}^{-1/4} y$ where $b_k = 2^{-k}$.

PROOF. The proof is similar to a result by Hunt [7].

LEMMA 5. Let $N > 0$, $1 < p < \infty$, $y > 0$ and $F \subseteq \omega_0$ be a measurable set. Then there exists an exceptional set $\sum = \sum(y,p,\chi_F)$ in ω_0, and constants C and C_p with $C_p \leq Cp/(p-1)$, such that

$$m\sum \leq C_p^p y^{-p} \int_{\omega_0} |f|^p dx$$

and $x \in \omega_0 \sim \sum$, $0 < n < 2^N$ implies

$$|S_n f(x)| \leq Cy.$$

PROOF. This result is obtained by using the three preceding results with minor changes. S^* as defined in the proof of Theorem 5 is replaced by S^{**} as defined in Lemma 3. Then as in Theorem 5,

$$mS^{**} \leq 2y^{-p} \int_{\omega_0} |\chi_F(x)|^p dx.$$

Define

$$U^{**} = U^{**}(n(\omega),\omega,k) = \{x \in \omega_0 : H(T_{n(\omega)}g)(x) > C_0 Lk 2^{-k+1} y\}.$$

By Sjolin [13], Lemma 1.9,

$$m\{x \in \omega_0 : H(T_{n(\omega)}g)(x) > C_0 Lk 2^{-k+1} y\} \leq C|\omega| \exp(-CC_0 Lk 2^{-k+1} y/||g||_\infty)$$

$$\leq C|\omega| \exp(-CC_0 Lk)$$

since $||g||_\infty < 2^{-k+1} y$ by (3.27) in the proof of Theorem 5.

Choose C_0 so that

$$mU^{**} \leq 2^{-5kL}|\omega|.$$

From an argument similar to the proof of Lemma 4,

$$y^{p-2} \leq 2^{kL}, \quad k = 1, 2, \ldots .$$

From (3.18) of the proof of Theorem 5 as well as the two previous inequalities, we obtain

$$m\left[\bigcup_{(n(\omega),\omega)\in G_k^*} U^{**}(n(\omega),\omega,k)\right] \leq 4 \cdot 2^{-k}y^{-p} \int_{\omega_0} |\chi_F|^2 dx.$$

Therefore

$$m\left[\bigcup_{k=1}^{\infty} \bigcup_{(n(\omega),\omega)\in G_k^*} U^{**}(n(\omega)\omega,k)\right] \leq C^p y^{-p} \int_{\omega_0} |\psi_F|^2 dx.$$

The proof is complete by letting

$$\sum = S^{**} \cup \bigcup_{k=1}^{\infty} \bigcup_{(n(\omega),\omega)\in G_k^*} U^{**}(n(\omega),\omega,k).$$

LEMMA 6. Let $M_N f(x) = \sup_{n<2^N} |(S_n f)(x)|$ and $Mf(x) = \sup_{n\geq 1} |(S_n f)(x)|$ where $f \in L^p(\omega_0)$, $1 < p < \infty$. Then there is a constant C so that

$$m\{x \in \omega_0 : M\chi_F(x) > y\} \leq Cy^{-p}mF, \quad 1 < p < \infty,$$

for all $y > 0$ and all measurable $F \subseteq \omega_0$.

PROOF. Set $E_N = \{x \in \omega_0 : M_N f(x) > y\}$. Clearly $E_N \subseteq E_{N+1}$ and

$$m\{x \in \omega_0 : Mf(x) > y\} = \lim_{N\to\infty} mE_N.$$

An application of Lemma 5 yields

$$m\{x \in \omega_0 : M_N f(x) > Cy\} = m\{x \in \omega_0 : \sup_{n<2^N} |S_n f(x)| > Cy\}$$

$$< Cy^{-p} \int_{\omega_0} |f(x)|^p dx.$$

Therefore for $f = \chi_F$,

$$m\{x \in \omega_0 : M\chi_F(x) > y\} = \lim_{N\to\infty} mE_N$$

$$\leq Cy^{-p} mF.$$

LEMMA 7. There is a constant C so that for each p, $1 < p < \infty$

$$\int_{\omega_0} |M\chi_F(x)|^p dx \leq C_p \int_{\omega_0} |\chi_F(x)|^p dx$$

for all measurable $F \subseteq \omega_0$.

PROOF. By Lemma 6, M is of restricted weak type $((p+1)/2, (p+1)/2)$ and $(p + 1, p + 1)$. By Mozzochi, Lemma 1.9 [9], M is of restricted type (p_t, q_t) for any $0 < t < 1$ where $p_t = q_t = (2(1 - t)/(p + 1) + t/(p + 1))^{-1}$. If $t = 1 - 1/p$, then $p_t = q_t = p$. In particular, M is of restricted type (p,p).

LEMMA 8. Suppose $\alpha : \omega_0 \to \{0, 1, \ldots, N\}$ is a measurable function and that a linear function T_α is defined on $L^1(\omega_0)$ by

$$T_\alpha f(x) = S_{\alpha(x)}(f,x) \qquad x \in \omega_0.$$

Then there is a constant F_p for each p, $1 < p < \infty$ such that

$$\int_{\omega_0} |T_\alpha f(x)|^p dx \leq F_p \int_{\omega_0} |f(x)|^p dx$$

for any simple function $f \in L^p(\omega_0)$.

PROOF. Since T_α is linear, this result will follow from the fact that T_α is of restricted type (p,p) for all $p > 1$. Indeed, apply a result by Mozzochi, Lemma 1.18 [9] to $((p + 1)/2, (p + 1)/2)$ and $(p + 1, p + 1)$ plus the fact that an operator of restricted type is also of weak restricted type.

To show that T_α is of restricted type (p,p) for all p, $1 < p < \infty$, let $F \subset \omega_0$ be measurable. Then by Lemma 7

$$\int_{\omega_0} |T_\alpha \chi_F(x)|^p \leq \int_{\omega_0} |M\chi_F(x)|^p dx$$

$$\leq F_p \int_{\omega_0} |\chi_F(x)|^p dx.$$

LEMMA 9. There is a constant C_p depending only on p, $1 < p < \infty$ such that

$$\int_{\omega_0} |T_\alpha f|^p dx \leq C_p \int_{\omega_0} |f|^p dx$$

for every $f \in L^p(\omega_0)$ where $\alpha : \omega_0 \to \{0, 1, \ldots, N\}$ is measurable.

PROOF. Since $|f|$ is non-negative and measurable, we can find a sequence $f_1, f_2, \ldots$ of simple functions such that $|f_n| \leq |f|$, $f_n \to f$ everywhere and $\int_{\omega_0} |f_n|^p \to \int_{\omega_0} |f|^p$ as $n \to \infty$. Hence by the Lebesque dominated convergence theorem for each fixed M,

$$\lim_{n\to\infty} S_M(f - f_n, x) = \lim_{n\to\infty} \int_{\omega_0} (f(t) - f_n(t)) D_M(x \dot{+} t)dt$$

$$= 0.$$

In particular, $T_\alpha(f - f_n)(x) \to 0$ as $n \to \infty$. By Lemma 8 and the choice of the f_n's

$$C_p \int_{\omega_0} |f|^p = C_p \lim_{n\to\infty} \int_{\omega_0} |f_n|^p$$

$$\geq \liminf \int_{\omega_0} |T_\alpha f_n|^p$$

$$\geq \int_{\omega_0} |T_\alpha f|^p$$

where the last inequality is obtained using Fatou's lemma.

THEOREM 6. If $1 < p < \infty$, then there exists a constant depending only on p such that

$$\int_{\omega_0} |Mf|^p \leq C \int_{\omega_0} |f|^p$$

for every $f \in L^p(\omega_0)$ where $Mf = \sup_n |S_n f|$ and $\omega_0 \subseteq [0,1)$.

PROOF. Let $\alpha_0(x) = k_0 \in \{0, 1, \ldots, 2^N\}$ if and only if

$$S_{k_0}(f,x) = \sup_{0 \le k < 2^N} |S_k(f,x)| \text{ for each } x \in \omega_0.$$

By induction on N, α_0 is measurable. Hence by Lemma 9,

$$\int_{\omega_0} |M_N f|^p = \int_{\omega_0} |T_{\alpha_0} f|^p$$

$$\le C \int_{\omega_0} |f|^p$$

where $M_N f(x) = \sup_{0 \le k < 2^N} |S_k(f,x)|$. However, $M_1 f \le M_2 f \le \ldots$. So by the monotone convergence theorem,

$$\int_{\omega_0} |Mf|^p = \lim_{N \to \infty} \int_{\omega_0} |M_N f|^p$$

$$\le C \int_{\omega_0} |f|^p.$$

THEOREM 7. If $f \in L^p_\mu(0,1) \cap L^2_\mu(0,1)$, $1 < p < \infty$, and $\beta > 0$, then

$$-\infty < B \int_0^1 \sum_{n=1}^{\infty} \frac{|S_n(t) - \sigma_n(t)|^p}{n^{1+\beta}} d\mu \le \int_0^1 |f(t)|^2 d\mu$$

$$\le A \int_0^1 \sum_{n=1}^{\infty} \frac{|S_n(t) - \sigma_n(t)|^2}{n} d\mu < \infty.$$

PROOF. See page 15 for the restriction on f. Since $S_n(t) = \sum_{k=0}^{n-1} c_k \psi_k(t)$, and $\sigma_n(t) = \sum_{k=1}^{n} S_k(t)/n = \sum_{k=0}^{n-1} (1 - k/n) c_k \psi_k(t)$, we have

$$(n + 1)\sigma_{n+1}(t) - n\sigma_n(t) = S_{n+1}(t).$$

Therefore, $S_{n+1}(t) - \sigma_{n+1}(t) = n(\sigma_{n+1}(t) - \sigma_n(t))$. Moreover,

$$S_n(t) - \sigma_n(t) = \sum_{k=0}^{n-1} c_k\psi_k(t) - \sum_{k=0}^{n-1} (1 - k/n)c_k\psi_k(t)$$

$$= \sum_{k=1}^{n-1} \frac{k}{n} c_k\psi_k(t).$$

Using Schwartz,

$$|\sigma_{2^{n+1}}(t) - \sigma_{2^n}(t)| \leq \sum_{i=2^n}^{2^{n+1}-1} |\sigma_{i+1}(t) - \sigma_i(t)|$$

$$\leq \sum_{i=2^n}^{2^{n+1}-1} i(\sigma_{i+1}(t) - \sigma_i(t))^2 \sum_{i=2^n}^{2^{n+1}-1} \frac{1}{i}$$

$$\leq 2 \sum_{i=2^n}^{2^{n+1}-1} i(\sigma_{i+1}(t) - \sigma_i(t))^2$$

$$\leq 4 \sum_{i=2^n}^{2^{n+1}-1} \frac{|S_{i+1}(t) - \sigma_{i+1}(t)|^2}{i + 1}$$

$$= 4 \sum_{i=2^n+1}^{2^{n+1}} \frac{|S_i(t) - \sigma_i(t)|^2}{i}.$$

Consequently,

$$|\sigma_{2^{n+1}}(t) - \sigma_{2^n}(t)| \leq 4 \sum_{i=2^n+1}^{2^{n+1}} \frac{|S_i(t) - \sigma_i(t)|^2}{i}. \qquad (3.31)$$

Let $f_\nu(t)=\frac{1}{\nu^3}\sum_{i=1}^{\nu-1} ic_i\psi_i(t)$ and $p_\nu = 2^k$ if $2^k \le \nu < 2^{k+1}$. Then

$S_{\nu,p_\nu}(t) = \frac{1}{\nu^3}\sum_{i=0}^{2^k-1} ic_i\psi_i(t)$. By Theorem 4, we conclude that

$$\int_0^1 \sum_{k=0}^{\infty} |S_{2^k}(t) - \sigma_{2^k}(t)|^2 d\mu \le \int_0^1 \sum_{k=0}^{\infty} \frac{1}{2^{2k}}\left(\sum_{i=0}^{2^k-1} ic_i\psi_i(t)\right)^2 d\mu$$

$$\le 2\int_0^1 \sum_{k=0}^{\infty}\left(\sum_{i=0}^{2^k-1} ic_i\psi_i(t)\right)^2 \sum_{\nu=2^k}^{2^{k+1}-1}\frac{1}{\nu^3} d\mu$$

$$\le 2A\int_0^1 \sum_{k=0}^{\infty} \sum_{\nu=2^k}^{2^{k+1}-1}\frac{1}{\nu^3}\left(\sum_{i=1}^{\nu-1} ic_i\psi_i(t)\right)^2 d\mu$$

$$= 2A\int_0^1 \sum_{k=0}^{\infty} \sum_{\nu=2^k}^{2^{k+1}-1}\frac{1}{\nu}\left(\sum_{i=1}^{\nu-1} ic_i\psi_i(t)/\nu\right)^2 d\mu$$

$$= 2A\int_0^1 \sum_{k=0}^{\infty} \sum_{\nu=2^k}^{2^{k+1}-1}\frac{(S_\nu(t) - \sigma_\nu(t))^2}{\nu} d\mu$$

$$= 2A\int_0^1 \sum_{k=1}^{\infty} \frac{(S_k(t) - \sigma_k(t))^2}{k} d\mu.$$

In other words,

$$\int_0^1 \sum_{k=0}^{\infty} |S_{2^k}(t) - \sigma_{2^k}(t)|^2 d\mu \le A\int_0^1 \sum_{k=1}^{\infty} \frac{|S_k(t) - \sigma_k(t)|^2}{k} d\mu. \quad (3.32)$$

Observing

$$(\omega(t))^{2\alpha}|S_{2^{n+1}} - S_{2^n}| = (\omega(t))^{2\alpha}|S_{2^{n+1}} - \sigma_{2^{n+1}} - (S_{2^n} - \sigma_{2^n}) + \sigma_{2^{n+1}} - \sigma_{2^n}|$$

and using (3.31) and (3.32), we obtain

$$\int_0^1 \sum_{n=1}^{\infty} \Delta_n^2(f(t))d\mu = \int_0^1 \sum_{n=1}^{\infty} |S_{2^{n+1}}(t) - S_{2^n}(t)|^2 d\mu$$

$$\leq A \int_0^1 \sum_{n=1}^{\infty} \frac{|S_n(t) - \sigma_n(t)|^2}{n} d\mu.$$

An application of Hirschman's result (3.4) yields

$$\int_0^1 |f(t)|^2 d\mu \leq A \int_0^1 \sum_{n=1}^{\infty} \Delta_n^2(f(t))dt$$

$$\leq A \int_0^1 \sum_{n=1}^{\infty} \frac{|S_n(t) - \sigma_n(t)|^2}{n} d\mu.$$

Therefore the second inequality in the conclusion has been established.

We now establish the first inequality. Let $1 < p < \infty$.

$$\int_0^1 \sum_{n=1}^{m} \frac{|S_n(t) - \sigma_n(t)|^p}{n^{1+\beta}} d\mu = \int_0^1 \sum_{n=1}^{m} \frac{|S_n(t) - \sigma_n(t)|^p}{n^{1+\beta}} (\omega(t))^{p\alpha} dt$$

$$= 2^{p-1} \sum_{j=1}^{\infty} 2^{-pj\alpha} \int_{2^{-j}}^{2^{-j+1}} \sum_{n=1}^{m} \frac{|S_n^p(t)| + |\sigma_n^p(t)|}{n^{1+\beta}} dt$$

$$\leq 2^p \sum_{j=1}^{\infty} 2^{-pj\alpha} \int_{2^{-j}}^{2^{-j+1}} \sum_{n=1}^{m} \frac{(\sup_n |S_n(t)|)^p}{n^{1+\beta}} dt$$

$$= 2^p \sum_{j=1}^{\infty} 2^{-pj\alpha} \sum_{n=1}^{m} \frac{1}{n^{1+\beta}} \int_{2^{-j}}^{2^{-j+1}} (\sup_n |S_n(t)|)^p dt$$

$$\leq 2^p \sum_{n=1}^{m} \frac{1}{n^{1+\beta}} \sum_{j=1}^{\infty} 2^{-pj\alpha} \int_{2^{-j}}^{2^{-j+1}} |f(t)|^p dt$$

$$= 2^p \sum_{n=1}^{m} \frac{1}{n^{1+\beta}} \int_0^1 \sum_{j=1}^{\infty} 2^{-pj\alpha}$$

$$\cdot \chi_{[2^{-j}, 2^{-j+1})}(t) |f(t)|^p dt$$

$$= 2^p \sum_{n=1}^{m} \frac{1}{n^{1+\beta}} \int_0^1 |f(t)|^p (\omega(t))^{p\alpha} dt$$

$$\leq 2^p \sum_{n=1}^{\infty} \frac{1}{n^{1+\beta}} \int_0^1 |f(t)|^p d\mu.$$

Consequently,

$$\int_0^1 \sum_{n=1}^{\infty} \frac{|S_n(t) - \sigma_n(t)|^p}{n^{1+\beta}} d\mu \leq \frac{1}{B} \int_0^1 |f(t)|^p d\mu < \infty.$$

THEOREM 8. Let $f \in L_\mu^p [0,1]$, $1 < p < \infty$. Then

$$(a) \quad \int_0^1 \sup_n \frac{1}{n+1} \sum_{\nu=0}^{n} |S_\nu(f,t) - f(t)|^p d\mu \leq C_p \int_0^1 |f(t)|^p d\mu$$

and

$$(b) \quad \lim_{n\to\infty} \frac{1}{n+1} \sum_{\nu=0}^{n} |S_\nu(f,t) - f(t)| = 0 \quad \text{a.e. } [\mu].$$

PROOF OF (a)

$$\frac{1}{n+1} \sum_{\nu=0}^{n} |S_\nu(t) - f(t)|^p \leq \frac{2^{p-1}}{n+1} \left(\sum_{\nu=0}^{n} |S_\nu(t)|^p + \sum_{\nu=0}^{n} |f(t)|^p \right)$$

$$\leq \frac{2^{p-1}}{n+1} \left(\sum_{\nu=0}^{n} \sup_{1\leq\nu<\infty} |S_\nu(t)| \right)^p + |f(t)|^p)$$

$$\leq 2^{p-1} \left(\sup_{1\leq\nu<\infty} |S_\nu(t)| \right)^p + |f(t)|^p).$$

Therefore

$$\int_0^1 \sup_{1\le n<\infty} \frac{1}{n+1} \sum_{\nu=0}^{n} |S_\nu(t) - f(t)|^p d\mu$$

$$\le \int_0^1 \sup_{1\le n<\infty} \frac{1}{n+1} \sum_{\nu=0}^{n} |S_\nu(t) - f(t)|^p (\omega(t))^{p\alpha} dt$$

$$\le 2^{p-1} \int_0^1 \left(\sup_{1\le \nu<\infty} |S_\nu(t)| \right)^p (\omega(t))^{p\alpha} dt$$

$$+ 2^{p-1} \int_0^1 |f(t)|^p (\omega(t))^{p\alpha} dt$$

$$= A + B.$$

By Theorem 6,

$$|A| \le 2^{p-1} \sum_{n=0}^{\infty} 2^{-pn\alpha} \int_{2^{-n-1}}^{2^{-n}} \left(\sup_{1\le \nu<\infty} |S_\nu(t)| \right)^p dt$$

$$\le 2^{p-1} \sum_{n=0}^{\infty} 2^{-np\alpha} \int_{2^{-n-1}}^{2^{-n}} |f(t)|^p dt$$

$$= 2^{p-1} \int_0^1 |f(t)|^p (\omega(t))^{p\alpha} dt$$

$$= 2^{p-1} \int_0^1 |f(t)|^p d\mu.$$

Therefore

$$\int_0^1 \sup_{1\le n<\infty} \frac{1}{n+1} \sum_{\nu=0}^{n} |S_\nu(t) - f(t)|^p d\mu \le C \int_0^1 |f(t)|^p d\mu.$$

PROOF OF (b) For every positive integer n, define

$$T_n f(t) \equiv \frac{1}{n+1} \sum_{\nu=0}^{n} |S_\nu(f,t) - f(t)|^p$$

and

$$Mf(t) \equiv \sup_n T_n f(t).$$

Since the Walsh polynomials are dense in L^p, $1 < p < \infty$, there exists a polynomial P such that

$$||f - P||_p < \varepsilon.$$

Now $P(x) = \sum_{k=0}^{N-1} a_k \psi_k(x)$ for some integer N, and certain real numbers a_k, $k = 0, 1, \ldots, N-1$. Furthermore, there is a constant M such that

$$\sum_{\nu=0}^{N-1} |S_\nu(P,t) - P(t)|^p < M < \infty \text{ for every } t \in [0,1].$$

Hence, for $n > N$,

$$T_n P(t) = \frac{1}{n+1} \sum_{\nu=0}^{N-1} |S_\nu(P,t) - P(t)|^p + \frac{1}{n+1} \sum_{\nu=N}^{n} |S_\nu(P,t) - P(t)|^p$$

$$\leq M/(n+1) + \frac{1}{n+1} \sum_{\nu=N}^{n} |P(t) - P(t)|$$

$$= M/(n+1),$$

which in turn implies that

$$0 \leq \lim_{n\to\infty} T_n P(t) \leq 0.$$

Now define

$$Tf = \overline{\lim}\ T_n f - \underline{\lim}\ T_n f.$$

Then by (a)

$$||Tf||_p \leq ||T(f-P)||_p + ||T(P)||_p$$

$$\leq 2||M(f - P)||_p + ||T(P)||_p$$

$$\leq 2C||f - P||_p + 0$$

$$< 2C\varepsilon.$$

Hence, $Tf = 0$ a.e. $[\mu]$, so $\lim_{n\to\infty} T_n f$ exists a.e. $[\mu]$ and $\lim_{n\to\infty} T_n f = 0$ a.e. $[\mu]$.

Consider any p and q where $\frac{1}{p} + \frac{1}{q} = 1$, $1 < p < \infty$ and $1 < q < \infty$. An application of Holder's inequality yields

$$\frac{1}{n+1} \sum_{\nu=0}^{n} |S_\nu(f,t) - f(t)| \cdot 1 \leq \frac{1}{n+1} \{ \sum_{\nu=0}^{n} |S_\nu(f,t) - f(t)|^p \}^{1/p} \{ \sum_{\nu=0}^{n} 1^q \}^{1/q}$$

$$= \left\{ \frac{\sum_{\nu=0}^{n} |S_\nu(f,t) - f(t)|^p}{n+1} \right\}^{1/p} \left\{ \frac{n+1}{n+1} \right\}^{1/q}$$

$$\leq \left\{ \frac{\sum_{\nu=0}^{n} |S_\nu(f,t) - f(t)|^p}{n+1} \right\}^{1/p}$$

$$= o(1) \text{ as } n \to \infty.$$

Therefore conclusion (b) follows.

COROLLARY 1. Let $f \in L^p_\mu[0,1)$, $1 < p < \infty$. Then

$$\text{(a)} \quad \int_0^1 \sup_n \frac{1}{n+1} \sum_{\nu=0}^{n} |S_\nu(f,t) - f(t)|^p d\mu \leq C_p \int_0^1 |f(t)|^p d\mu$$

and

$$\text{(b)} \quad \lim_{n\to\infty} \frac{1}{n+1} \sum_{\nu=0}^{n} |\sigma_\nu(f,t) - f(t)| = 0 \quad \text{a.e. } [\mu].$$

PROOF. This follows using the techniques from Theorem 8 and Theorem 6, observing that

$$|\sigma_n(f,t)| \leq \sum_{\nu=1}^{n} |S_\nu(f,t)|/n$$

$$\leq \frac{n}{n} \sup_{\nu \geq 1} |S_\nu(f,t)|$$

$$= \sup_n |S_n(f,t)|.$$

COROLLARY 2. Let $f \in L^p[0,1)$, $1 < p < \infty$. Then

$$\lim_{n\to\infty} \frac{1}{n+1} \sum_{\nu=0}^{n} |S_\nu(f,t) - \sigma_\nu(f,t)| = 0 \qquad \text{a.e. } [\mu].$$

PROOF. $S_\nu(f,t) - \sigma_\nu(f,t) = S_\nu(f,t) - f(t) + f(t) - \sigma_\nu(f,t)$. An application of Theorem 8(b) and Corollary 1(b) yields the conclusion.

CHAPTER IV

(C,β) SUMMABILITY, $-1 < \beta < 0$

In the preceding chapters, we have dealt exclusively with (C,β) summability for $\beta \geq 0$. In this chapter, we turn our attention toward the more difficult case when $-1 < \beta < 0$.

Recall that for any function f belonging to $L^1(0,1)$,

$$\sigma_n^{\beta *}(f,x) = f * K_n^{\beta *}(x)$$

where * is convolution on the group and $K_n^{\beta *}(x) = \psi_n(x)K_n^{\beta}(x)$. We shall begin this chapter by establishing the existence of a continuous function g such that $\sigma_n^{\beta *}(g,0)$ is unbounded for $-1 < \beta < 0$.

Toward this, recall that $D_n^* = \psi_n D_n$, $S_n^* = f * D_n^*$, and set $F(k) = \sum_{m=1}^{k} 2^m$. Fix k, and decompose the unit interval as follows.

$$[0,1] = [0,2^{-2k}) \cup \bigcup_{j=1}^{2k-1} [2^{-2k+j-1}, 2^{-2k+j}) \cup [\tfrac{1}{2},1]$$

$$= [0,2^{-2k}) \cup \bigcup_{j=1}^{2k-1} F_j \cup [\tfrac{1}{2},1]$$

$$= [0,2^{-2k}) \cup \bigcup_{j=1}^{2k-1} \bigcup_{i=0}^{2^{j+1}-1} [2^{-2k+j-1} + i2^{-2k-1}, 2^{-2k+j-1} + (i+1)2^{-2k-1}) \cup [\tfrac{1}{2},1]$$

$$= [0,2^{-2k}) \cup \bigcup_{j=1}^{2k-1} \bigcup_{i=0}^{2^{j+1}-1} E_{j,i} \cup [\tfrac{1}{2},1].$$

Define

$$f_k = \begin{cases} \dfrac{|D^*_{F(k)}|}{D^*_{F(k)}} \text{ on } E_{j,i},\ i \text{ even} \\ 0 \quad \text{otherwise.} \end{cases}$$

Observe that

$$\bigcup_{j=1}^{2k-1} \bigcup_{i=0}^{2^j-1} E_{j,i} = \bigcup_{j=1}^{2k-1} [2^{-2k+j-1}, 2^{-2k+j})$$

$$= \bigcup_{p=1}^{2k-1} [2^{-p-1}, 2^{-p})$$

$$= \bigcup_{p=1}^{2k-1} A_p.$$

We claim that for $1 \leq p \leq 2k$, if $x \in A_p$, then $|D^*_{F(k)}(x)| > 2^{p-1}$. To see this, recall that Billard [1] has shown that

$$D_{F(k)}(x) = \psi_{F(k)}(x) \sum_{s=0}^{k} \delta^*_{2s}, \text{ where}$$

$$\delta^*_j(x) = \begin{cases} 2^j & \text{if } 0 < x < 2^{-j-1} \\ -2^j & \text{if } 2^{-j-1} < x < 2^{-j} \\ 0 & \text{otherwise.} \end{cases}$$

CASE 1. p is odd. In this case, since $1 \leq p \leq 2k$, we know that $p = 2\ell + 1$ for some $\ell < k$. Let $x \in [2^{-2\ell-2}, 2^{-2\ell-1})$ for $\ell < k$. If s satisfies $0 \leq s \leq \ell < k$, then

$$[2^{-2\ell-2},2^{-2\ell-1}) \subseteq [0,2^{-2s-1}).$$

Therefore $0 \leq s \leq \ell < k$ and $x \in [2^{-2\ell-2},2^{-2\ell-1})$ imply that $\delta^*_{2s}(x) = 2^{2s}$. If s satisfies $0 \leq \ell < s < k$, then

$$\text{support } \delta^*_{2s} \cap [2^{-2\ell-2},2^{-2\ell-1}) = [0,2^{-2s}) \cap [2^{-2(\ell+1)},2^{-2\ell-1})$$

$$= \emptyset$$

since $2s \geq 2(\ell+1)$. Therefore $x \in [2^{-2\ell-2},2^{-2\ell-1})$ implies

$$D^*_{F(k)}(x) = \sum_{s=0}^{\ell} \delta^*_{2s}(x) + \sum_{s=\ell+1}^{k} \delta^*_{2s}(x)$$

$$= \sum_{s=0}^{\ell} 2^{2s}$$

$$> 2^{2\ell}$$

$$= 2^{p-1}.$$

CASE 2. p is even. Let $x \in [2^{-2\ell-1}, 2^{-2\ell})$ and set $p = 2\ell$ for some ℓ, $\ell \leq k$. If s satisfies $0 \leq s < \ell \leq k$, then we can proceed as before, establishing $[2^{-2\ell-1},2^{-2\ell}) \subseteq [0,2^{-2s-1})$. Therefore $\delta^*_{2s}(x) = 2^{2s}$ for $x \in [2^{-2\ell-1}, 2^{-2\ell})$, $0 \leq s < \ell \leq k$. If $s = \ell \leq k$, then

$$x \in [2^{-2\ell-1},2^{-2\ell}) = [2^{-2s-1},2^{-2s})$$

implies

$$\delta^*_{2\ell}(x) = \delta^*_{2s}(x)$$

$$= -2^{2s}$$

$$= -2^{2\ell}.$$

Finally, if s satisfies $0 \leq \ell < s \leq k$, then

$$\text{support } \delta^*_{2s} \cap [2^{-2\ell-1}, 2^{-2\ell}) = [0, 2^{-2s}) \cap [2^{-2\ell-1}, 2^{-2\ell})$$

$$= \emptyset$$

since $2s \geq 2\ell + 1$. Therefore $x \in [2^{-2\ell-1}, 2^{-2\ell})$ implies

$$|D^*_{F(k)}(x)| = \Big|\sum_{s=0}^{\ell-1} \delta^*_{2s} + \delta^*_{2\ell} + \sum_{s=\ell+1}^{k} \delta^*_{2s}\Big|$$

$$= \Big|\sum_{s=0}^{\ell-1} 2^{2s} - 2^{2\ell}\Big|$$

$$= 2^{2\ell} - \sum_{s=0}^{\ell-1} 2^{2s}.$$

Since

$$\sum_{s=0}^{\ell-1} 2^{2s} = \frac{1 - 2^{2\ell}}{1 - 2^2} = (2^{2\ell} - 1)/3 < 2^{2\ell-1},$$

this difference is greater than $2^{2\ell-1} = 2^{p-1}$. Hence, $|D^*_{F(k)}(x)| > 2^{p-1}$, as required.

Using these estimates on $D^*_{F(k)}$, we can show that $f_k * D^*_{F(k)}(0)$ diverges to $+\infty$, as $k \to \infty$. Indeed, since

$$\int_{A_p} f_k(t) D^*_{F(k)}(t)dt = \int_{A_p} |D^*_{F(k)}(t)|dt$$

$$> 2^{p-1},$$

we have

$$f_k * D^*_{F(k)}(0) = \int_0^1 f_k(t) D^*_{F(k)}(t)dt$$

$$> \sum_{p=1}^{2k-1} 2^{p-1} \to +\infty \quad \text{as } k \to \infty.$$

The set of bounded functions on [0,1] forms a Banach space. By the Banach-Steinhaus theorem, there is a bounded function, say g, on [0,1] such that $S^*_{F(k)}(g,0) \to \infty$ as $k \to \infty$.

It then follows that $\sigma^{\beta}_n{}^*(g,0)$ is not bounded for $-1 < \beta < 0$. For if $\sigma^{\beta}_n{}^*(g,0)$ were bounded. $-1 < \beta < 0$, $S^*_{F(k)}(g,0)$ would be bounded noting

$$S^*_{F(k)}(g,0) = \sigma^{\beta}_{F(k)}{}^*(g,0) \text{ where } \beta = 0.$$

It should be remarked that since the set of continuous functions on [0,1] forms a Banach space and f_k could be modified slightly to

obtain a continuous function, one can in fact obtain a function h continuous such that $S^{*}_{F(k)}(h,0) \to +\infty$ as $k \to \infty$.

We now show that when 2^n partial (C^{*},β) sums are used, the function h cannot be of ΛBV.

THEOREM 9. Let $f \in \Lambda BV$, $\Lambda = \{n^{1+\beta}\}$, $-1 < \beta < 0$. Then $|\sigma^{\beta *}_{2^n}(f,x)| \leq M$ for all n and for all $x \in [0,1]$.

PROOF. $$\int_0^1 \phi_x(t)K^{\beta *}_{2^n}(t)dt = \int_0^1 \phi_x(t)\psi_{2^n}(t)K^{\beta}_{2^n}(t)dt.$$

Using an argument similar to the proof of Theorem 1, we obtain

$$\int_0^1 \phi_x(t)\psi_{2^n}(t)K^{\beta}_{2^n}(t)dt = \int_0^1 (\phi_x(t) - \phi_x(t + 2^{-n-1}))K^{\beta}_{2^n}(t)dt.$$

Yano [19] has shown an estimate on $K^{\beta}_{2^n}(t)$ which yields

$$\int_0^1 |\phi_x(t) - \phi_x(t + 2^{-n-1})|\ |K^{\beta}_{2^n}(t)|dt \leq C2^{-n\beta} \int_0^1 |\phi_x(t)-\phi_x(t+2^{-n-1})|t^{-\beta-1}dt.$$

If we integrate by parts, with

$$\Phi(t) = \int_0^t |\phi_x(u)|du$$

and

$$\eta = 1/(2^n + (\beta+1)2^{-1}),$$

we obtain

$$C2^{-n\beta} \int_0^{\eta} |\phi_x(t) - \phi_x(t + 2^{-n-1})| t^{-\beta-1} dt$$

$$\leq 2C2^{-n\beta} \int_0^{\eta} |\phi_x(t)| t^{-\beta-1} dt$$

$$= 2C2^{-n\beta}(t^{-\beta-1}\Phi(t)]_0^{\eta} - (-\beta - 1) \int_0^{\eta} \Phi(t) t^{-\beta-2} dt)$$

$$= 2C2^{-n\beta}(\Phi(\eta)\eta^{-\beta-1} + (\beta+1) \int_0^{\eta} O(t^{-\beta-1}) dt)$$

$$= 2C[(\eta 2^n)^{-\beta} O(1) + \frac{\beta+1}{-\beta} (\eta 2^n)^{-\beta}]$$

$$= O(1) \text{ as } n \to \infty.$$

(Note, $2^n < \eta$ and f is bounded since $f \in \Lambda BV$.)

Also for $m = 2^{n+1} - 2$,

$$C2^{n(-\beta)} \int_{\eta}^{1-\eta} |\phi_x(t) - \phi_x(t + 2^{-n-1})| t^{-\beta-1} dt$$

$$\leq C2^{-n\beta} \sum_{i=1}^{m} \int_{i2^{-n-1}}^{(i+1)2^{-n-1}} |f(x \dot{+} t) - f(x \dot{+} t + 2^{-n-1})| t^{-\beta-1} dt$$

$$= C2^{-n\beta} 2^{-n-1} 2^{(n+1)(\beta+1)} \sum_{i=1}^{m} \frac{\operatorname{osc}(f; [x + i2^{-n-1}, x + (i+2)2^{-n-1}])}{i^{\beta+1}}$$

$$\leq C2^{\beta} V$$

$$\leq C2^{\beta} M$$

$$= O(1) \text{ as } n \to \infty.$$

Moreover, $C2^{-n\beta} \int_{1-\eta}^{1} |\phi_x(t) - \phi_x(t + 2^{-n-1})| t^{-\beta-1} = O(1)$ as $n \to \infty$.

Therefore

$$|\sigma^{\beta*}_{2^n}(f,x)| = |f * K^{\beta*}_{2^n}(x)|$$

$$= |\int_0^1 f(x \dot{+} t)K^{\beta*}_{2^n}(t)dt|$$

$$= |\int_0^1 \phi_x(t)K^{\beta*}_{2^n}(t)dt|$$

$$\leq O(1) \text{ as } n \to \infty.$$

COROLLARY 3. Let $f \in \Lambda BV$, $\Lambda = \{n^{1+\beta}\}$, $-1 < \beta < 0$. Then $|\sigma^{\beta}_{2^n}(f\psi_{2^n},x)| \leq M$ for all n and for all $x \in [0,1]$.

PROOF.

$$|\sigma^{\beta}_{2^n}(f\psi_{2^n},x)| = |\psi_{2^n}(x)\sigma^{\beta}_{2^n}(f\psi_{2^n},x)|$$

$$= |\psi_{2^n}(x) \int_0^1 f(x \dot{+} t)\psi_{2^n}(x \dot{+} t)K^{\beta}_{2^n}(t)dt|$$

$$= |\psi_{2^n}(x)\psi_{2^n}(x) \int_0^1 f(x + t)\psi_{2^n}(t)K^{\beta}_{2^n}(t)dt|$$

$$= \int_0^1 |f(x + t)K^{\beta*}_{2^n}(t)dt|$$

$$= |\sigma^{\beta*}_{2^n}(f,x)|$$

$$\leq O(1) \text{ as } n \to \infty \text{ by Theorem 9.}$$

In view of Theorem 9 and the counterexample which opened this chapter, it is natural to ask whether there exist any continuous functions of $\{n^{\beta+1}\}$-variation whose 2^nth partial (C^*,β) sums diverge? In order to obtain an answer to this question, we first develop sufficient conditions for convergence of the 2^nth partial (C^*,β) sums.

THEOREM 10. $\sigma^{\beta}_{2^n}(f,x) \to f(x)$ and $\sigma^{\beta}_{2^n}(\psi_n f,x) \to f(x)$, $-1 < \beta < 0$ for every $x \in Z_f \equiv \{x : f(x) = 0\}$ at which

(a) $\Phi(h) = o(h)$ as $h \to 0$ where $\Phi(h) = \int_0^h \phi_x(t)dt$

and

(b) $\int_{\eta}^{1} |\phi_x(t) - \phi_x(t + 2^{-n-1})|t^{-\beta-1}dt = o(2^{n\beta})$

as $\eta = 1/(2^n + (\beta + 1)/2) \to 0$ and summability is uniform on $Z_f \cap \{x : f \text{ is continuous at } x\}$ where (a) is satisfied uniformly. If f is continuous at x, (a) may be omitted.

If (a') and (b') denote (a) and (b) with "o" replaced by "0," $\sigma^{\beta}_{2^n}(f,x)$ will be bounded at every $x \in Z_f$ at which (a') and (b') hold, and uniformly bounded on every set on which the conditions are satisfied uniformly. If f is bounded, condition (a') may be omitted.

A somewhat similar result is the following trigonometric test for (C,β) summability which is found in [18]. S[f] is (C,β) summable to $f(x)$, $-1 < \beta < 0$, at every point x at which

(i) $\Phi(h) = o(h)$,

(ii) $\Phi(\pi) - \Phi(\pi - h) = o(h^{-\beta})$,

(iii) $\int_{\eta}^{\pi-\eta} |\phi_x(t) - \phi_x(t + \eta)|t^{-\beta-1}dt = o(\eta^{\beta})$

as $\eta = \pi/(n + (\beta + 1)/2) \to 0$, and summability is uniform over any closed interval of continuity where conditions (ii) and (iii) are satisfied uniformly. If f is bounded, condition (ii) may be omitted; if f is regulated, that is, $f(x) = (f(x+) + f(x-))/2$ for each x, conditions (i) and (ii) may be omitted.

If (i'), (ii'), (iii') denote (i), (ii), (iii) with "o" replaced by "0," S[f] will be (C,β) bounded at every x at which (i'), (ii'), (iii') hold, and uniformly (C,β) bounded on every set on which the conditions are satisfied uniformly. If f is bounded, conditions (i') and (ii') may be omitted.

PROOF OF THEOREM . Let $x \in Z_f$. Then

$$\sigma^{\beta}_{2^n}{}^{*}(f,x) - f(x) = \int_0^1 \phi_x(t) K^{\beta}_{2^n}{}^{*}(t)\,dt$$

$$= \int_0^1 \phi_x(t)\psi_{2^n}(t)K^{\beta}_{2^n}(t)\,dt$$

$$= \int_0^{\eta} \phi_x(t)\psi_{2^n}(t)K^{\beta}_{2^n}(t)\,dt$$

$$+ \int_{\eta}^1 \phi_x(t)\psi_{2^n}(t)K^{\beta}_{2^n}(t)\,dt$$

$$= A_n + B_n.$$

Using Lemma 2 of Yano [19] and integration by parts, we obtain

$$|A_n| = \left|\int_0^{\eta} \phi_x(t)\psi_{2^n}(t)K^{\beta}_{2^n}(t)dt\right|$$

$$\leq \int_0^{\eta} |\phi_x(t)|\ |K^{\beta}_{2^n}(t)|dt$$

$$\leq C_{\beta}2^{-n\beta} \int_0^{\eta} \frac{|\phi_x(t)|}{t^{\beta+1}}\,dt$$

$$= C_{\beta}2^{-n\beta}\ \{\Phi(t)t^{-\beta-1}]_0^{\eta} + (\beta+1)\int_0^{\eta}\frac{\Phi(t)}{t^{\beta+2}}\,dt\}$$

$$= C_{\beta}2^{-n\beta}\Phi(\eta)\eta^{-\beta-1} + C_{\beta}(\beta+1)2^{-n\beta}\int_0^{\eta}\Phi(t)t^{-\beta-2}dt$$

$$= C_{\beta}2^{-n\beta}o(2^{n\beta}) + C_{\beta}(\beta+1)2^{-n\beta}\int_0^{\eta} o(t^{-\beta-1})dt$$

$$= o(1) \text{ as } n \to \infty.$$

Using an argument similar to the proof of Theorem 1, we obtain

$$B_n = \int_{\eta}^{1} \phi_x(t)\psi_{2^n}(t)K^{\beta}_{2^n}(t)dt$$

$$= \int_{\eta}^{1} (\phi_x(t) - \phi_x(t + 2^{-n-1}))K^{\beta}_{2^n}(t)dt.$$

By Lemma 2 of Yano [19] and the hypothesis (b),

$$|B_n| \leq C_{\beta}2^{-n\beta}\int_{\eta}^{1}\frac{|\phi_x(t) - \phi_x(t + 2^{-n-1})|}{t^{\beta+1}}dt$$

$$\leq C_{\beta}2^{-n\beta}\ o(2^{n\beta})$$

$= o(1)$ as $n \to \infty$.

Since $x \in Z_f$, the rest of the proof is obvious.

We now apply Theorem 10 to obtain the following answer to our question.

THEOREM 11. Suppose $f \in \{n^{\beta+1}\}BV$, $-1 < \beta < 0$ and f is continuous in $\{n^{\beta+1}\}$-variation. Then $\sigma^{\beta}_{2^n}(f,x) \to f(x)$ and $\sigma^{\beta}_{2^n}(\psi_n f,x) \to f(x)$ for each $x \in Z_f \cap \{x : f \text{ is continuous at } x\}$.

A somewhat similar result to Theorem 9 and Theorem 11 is the following trigonometric theorem which is found in [18]. The trigonometric series, $S[f]$, of a function f of class $\{n^{\beta+1}\}BV$, $-1 \leq \beta < 0$, is everywhere (C,β) bounded and is uniformly (C,β) bounded on each closed interval of continuity. For $\alpha > \beta$, $S[f]$ is everywhere (C,α) summable to sum $f(x)$ and summability is uniform on each closed interval of continuity.

If f is continuous in $\{n^{\beta+1}\}$-variation, $-1 < \beta < 0$, then $S[f]$ is everywhere (C,β) summable to sum $f(x)$ and summability is uniform on each closed interval of continuity.

PROOF OF THEOREM 11. We use Theorem 10 in order to establish the proof. (a) of Theorem 10 is immediate since f is continuous at x. In order to establish (b) of Theorem 10, we write

$$2^{-n\beta}\int_{\eta}^{1} \frac{|\phi_x(t) - \phi_x(t + 2^{-n-1})|}{t^{\beta+1}}\,dt$$

$$= 2^{-n\beta}\left(\int_{\eta}^{\delta} + \int_{\delta}^{1-\eta} + \int_{1-\eta}^{1}\right) \frac{|\phi_x(t) - \phi_x(t + 2^{-n-1})|}{t^{\beta+1}}dt$$

$$= A_n + B_n + C_n$$

where $\eta < \delta < 1 - \eta$ recalling $\eta = 1/(2^n + \frac{\beta+1}{2})$. Let $m = [\delta/n]$ and let $V([a,b])$ denote the $\{n^{\beta+1}\}$-variation of f on $[a,b]$. We now study A_n.

$$A_n = 2^{-n\beta} \int_{\eta}^{\delta} \frac{|\phi_x(t) - \phi_x(t + 2^{-n-1})|}{t^{\beta+1}}\,dt$$

$$\leq 2^{-n\beta} \int_{2^{-n-1}}^{\delta} \frac{|\phi_x(t) - \phi_x(t + 2^{-n-1})|}{t^{\beta+1}}\,dt$$

$$\leq 2^{-n\beta} \sum_{i=1}^{m} \int_{i2^{-n-1}}^{(i+1)^{-n-1}} \frac{|f(x+t) - f(x+t+2^{-n-1})|}{t^{\beta+1}}\,dt$$

$$\leq 2^{-n\beta}2^{(n+1)(\beta+1)}2^{-n-1} \sum_{i=1}^{m} \operatorname{osc}(f;[x+i2^{-n-1},x+(i+2)2^{-n-1}])/i^{\beta+1}$$

$$\leq 2^{\beta+1}V([x + 2^{-n-1}, x + \delta + 2^{-n}]).$$

Applying Lemma 3 of Waterman [18], we see $A_n = o(1)$ as $\delta \to 0$, $n \to \infty$. Clearly $C_n \to 0$ as $n \to \infty$.

Let us consider B_n where $m = [\delta\eta^{-1}]$.

$$B_n = 2^{-n\beta}\int_{\delta}^{1-\eta} \frac{|\phi_x(t) - \phi_x(t + 2^{-n-1})|}{t^{\beta+1}}\,dt$$

$$\leq 2^{\beta} \sum_{i=m}^{2^{n+1}-2} \operatorname{osc}(f;[x + i2^{-n-1},\ x + (i+2)2^{-n-1}])/i^{\beta+1}$$

$$\leq 2^{\beta+1} V_m$$

where V_m denotes the $\{n^{\beta+1}\}^m$-variation of f.

Thus if f is continuous in $\{n^{\beta+1}\}$-variation,

$$B_n = o(1) \text{ as } 2^n\delta \to \infty \text{ uniformly in } x.$$

If our hypothesis is that f is of $\{n^{\beta+1}\}$BV, then V_m is dominated by the $\{n^{\beta+1}\}$-variation of f and so

$$B_n = O(1) \text{ as } 2^n\delta \to \infty$$

uniformly in x.

Now choose $\delta_n \downarrow 0$ so that $2^n\delta_n \to \infty$. If f is of $\{n^{\beta+1}\}$BV, then for each x,

$$A_n + B_n = o(1) + O(1) = O(1) \text{ as } n \to \infty.$$

Thus $\sigma^{\beta*}_{2^n}(f,x)$ is everywhere bounded. If f is continuous in $\{n^{\beta+1}\}$-variation and continuous at $x \in Z_f$, then $A_n + B_n = o(1)$ as $n \to \infty$ implying $\sigma^{\beta*}_{2^n}(f,x) \to f(x)$ which implies $\sigma^{\beta}_{2^n}(\psi_n f,x) \to f(x)$ since $f(x) = 0$.

Our last major result is to establish that if $f \in L^1[0,1]$, then $\sum_{n=0}^{\infty} \frac{a_n(f)}{A_n^{-\beta}} \psi_n(x)$ is (C,β) summable a.e. for $-1 < \beta < 0$.

We first establish several lemmas.

LEMMA 10. Let $L_n^{\beta}(x) = \sum_{k=0}^{n-1} \frac{\psi_k(x)}{A_k^{-\beta}}$, $-1 < \beta < 0$. Then

$$|L_n^{\beta}(x)| \leq Cx^{-\beta-1} \text{ for } x \in (0,1).$$

PROOF. Recall that $A_k^{-\beta} \cong k^{-\beta}$. Hence in the special case that $x \leq 1/n$, we have

$$\begin{aligned} |L_n^{\beta}(x)| &\leq \sum_{k=0}^{n-1} \frac{1}{A_k^{-\beta}} \\ &\leq C \sum_{k=1}^{n-1} \frac{1}{k^{-\beta}} + C \\ &\leq C[2 + \int_1^n t^{\beta} dt] \\ &\leq Cn^{1+\beta} \\ &\leq Cx^{-1-\beta}. \end{aligned}$$

On the other hand, if $x > 1/n$, then break $L_n^{\beta}(x)$ into two pieces.

$$L_n^{\beta}(x) = \sum_{k=0}^{[\frac{1}{x}]} \psi_k(x) \frac{1}{A_k^{-\beta}} + \sum_{k=[\frac{1}{x}]+1}^{n-1} \psi_k(x) \frac{1}{A_k^{-\beta}} \equiv A_1 + A_2.$$

From the previous argument, $|A_1| \leq Cx^{-1-\beta}$.

Applying Abel's transformation, we have

$$A_2 = \sum_{k=[\frac{1}{x}]+1}^{n-2} T_{k+1}(x)\left[\frac{1}{A_k^{-\beta}} - \frac{1}{A_{k+1}^{-\beta}}\right] + \frac{1}{A_{n-1}^{-\beta}} T_n(x)$$

where

$$T_k(x) = \sum_{j=[\frac{1}{x}]+1}^{k-1} \psi_j(x).$$

Since $T_k(x) \leq C\,\frac{1}{x}$, $C > 0$, it follows that

$$|A_2| \leq \frac{C}{x}\left[\frac{1}{A^{-\beta}_{[\frac{1}{x}]+1}} - \frac{1}{A_{n-1}^{-\beta}}\right] + \frac{1}{A_{n-1}^{-\beta}}\frac{C}{x}$$

$$\leq \frac{C}{x}\,\frac{1}{(\frac{1}{x})^{-\beta}} + \frac{1}{(n-2)^{-\beta}}\,\frac{C}{x}$$

$$\leq Cx^{-\beta-1} + Cn^{\beta}x^{-1}$$

$$< Cx^{-\beta-1} + Cx^{-\beta}x^{-1}$$

$$\leq Cx^{-\beta-1}.$$

LEMMA 11. For $-1 < \beta < 0$, set $M_n^{\beta}(x) \equiv \frac{1}{A_{n-1}^{\beta}} \sum_{k=0}^{n-1} A_{n-k-1}^{\beta} \frac{1}{A_k^{-\beta}} \psi_k(x)$.

Then $|M_n^{\beta}(x)| \leq Cx^{-\beta-1}$ for $x \in (0,1)$.

PROOF.

$$M_n(x) = \frac{1}{A_{n-1}^{\beta}} \left(\sum_{k=0}^{[\frac{n}{2}]-1} + \sum_{k=[\frac{n}{2}]}^{n-1} \right) A_{n-k-1}^{\beta} \frac{1}{A_k^{-\beta}} \psi_k(x)$$

$$= U_n(x) + V_n(x).$$

By Abel's transformation,

$$U_n(x) = \frac{1}{A_{n-1}^{\beta}} \left\{ \sum_{k=0}^{[\frac{n}{2}]-2} L_{k+1}^{-\beta}(x)(A_{n-k-1}^{\beta} - A_{n-k}^{\beta}) + L_{[\frac{n}{2}]}^{-\beta}(x) A_{n-[\frac{n}{2}]}^{\beta} \right\}.$$

Since

$$A_{n-k-1}^{\beta} - A_{n-k}^{\beta} = A_{n-k-1}^{\beta} \frac{(-\beta)}{n-k}$$

$$\leq 2 A_{n-k-1}^{\beta-1}$$

$$\leq |A_{n-k-1}^{\beta-1}| C$$

and by the preceding lemma,

$$|U_n(x)| \leq \frac{1}{A_{n-1}^{\beta}} \sum_{k=0}^{[\frac{n}{2}]-2} Cx^{-\beta-1} |A_{n-k-1}^{\beta-1}| + Cx^{-\beta-1} A_{n-[\frac{n}{2}]}^{\beta}$$

$$\leq \frac{1}{(n-1)^{\beta}} Cx^{-\beta-1} \left\{ \sum_{k=0}^{[\frac{n}{2}]-2} |A_{n-k-1}^{\beta-1}| + A_{n-[\frac{n}{2}]}^{\beta} \right\}$$

$$\leq Cn^{-\beta} x^{-\beta-1} \left\{ \sum_{k=[\frac{n}{2}]}^{n} \frac{1}{k^{-\beta+1}} + n^{\beta} \right\}$$

$$\leq Cn^{-\beta}x^{-\beta-1}\left\{\int_{\frac{n}{2}-1}^{n} t^{\beta-1}dt + n^{\beta}\right\}$$

$$\leq Cn^{-\beta}x^{-\beta-1}\left\{\frac{n^{\beta}}{\beta} - \frac{(\frac{n}{2}-1)^{\beta}}{\beta} + n^{\beta}\right\}$$

$$\leq Cx^{-\beta-1}.$$

We now study V_n. By Abel's transformation,

$$V_n(x) = \frac{1}{A_{n-1}^{\beta}} \sum_{k=[\frac{n}{2}]}^{n-2}\left\{\frac{1}{A_k^{-\beta}} - \frac{1}{A_{k+1}^{-\beta}}\right\} \sum_{j=[\frac{n}{2}]}^{k} A_{n-j-1}^{\beta}\,\psi_j(x)$$

$$+ \frac{1}{A_{n-1}^{\beta}} A_{n-1}^{\beta} \sum_{k=[\frac{n}{2}]}^{n-1} A_{n-k-1}\psi_k(x).$$

Hence

$$|V_n(x)| \leq \frac{C}{A_{n-1}^{\beta}}\left[\sum_{k=[\frac{n}{2}]}^{n-2}\left(\frac{1}{k^{-\beta}} - \frac{1}{(k+1)^{-\beta}}\right)\left|\sum_{j=[\frac{n}{2}]} A_{n-j-1}^{\beta}\,\psi_j(x)\right|\right.$$

$$\left. + \left|\sum_{k=[\frac{n}{2}]}^{n-1} A_{n-k-1}^{\beta}\,\psi_k(x)\right|\right].$$

Since

$$\sum_{k=1}^{n-1}\left[\frac{1}{(k+1)^{-\beta}} - \frac{1}{(k+2)^{-\beta}}\right] = \frac{1}{2^{-\beta}} - \frac{1}{(n+1)^{-\beta}}$$

$$= -\beta\int_{2}^{n+1} t^{\beta-1}dt$$

$$\leq -\beta \sum_{k=1}^{n} \frac{1}{k^{-\beta+1}},$$

in a manner similar to the first part of the proof we obtain

$$|V_n| \leq \frac{C}{A_{n-1}^{-\beta}} \left\{ \sum_{k=[\frac{n}{2}]}^{n-1} \frac{1}{k^{-\beta+1}} + n^{\beta} \right\} \max_{[\frac{n}{2}] \leq k < n} \left| \sum_{j=[\frac{n}{2}]}^{k} A_{n-j-1}^{\beta} \psi_j(x) \right|$$

$$\leq C \max_{[\frac{n}{2}] \leq k < n} \left| \sum_{j=[\frac{n}{2}]}^{k} A_{n-j-1}^{\beta} \psi_j(t) \right|.$$

By Lemma 10,

$$|V_n| \leq Cx^{-\beta-1}.$$

Therefore

$$|M_n^{\beta}(x)| \leq Cx^{-\beta-1} \quad \text{for } x \in (0,1).$$

LEMMA 12. For $-1 < \beta < 0$ and $f \in L^1[0,1]$, set

$$M_n^{\beta}(x,f) \equiv \frac{1}{A_{n-1}^{\beta}} \sum_{k=0}^{n-1} A_{n-k-1}^{\beta} \frac{1}{A_k^{-\beta}} c_k \psi_k(x).$$

Then

$$\int_0^1 \overline{\lim_n} |M_n^{\beta}(t,f)| dt \leq C \int_0^1 |f(t)| dt.$$

PROOF. $$M_n^{\beta}(x,f) = \frac{1}{A_{n-1}^{\beta}} \sum_{k=0}^{n-1} A_{n-k-1}^{\beta} \frac{1}{A_k^{-\beta}} \int_0^1 f(t)\psi_k(t)dt\psi_k(x)$$

$$= \int_0^1 f(t)\psi_k(x \dot{+} t)dt \frac{1}{A_{n-1}^{\beta}} \sum_{k=0}^{n-1} A_{n-k-1}^{\beta} \frac{1}{A_k^{-\beta}}$$

$$= \int_0^1 f(x \dot{+} t) \frac{1}{A_{n-1}^{\beta}} \sum_{k=0}^{n-1} A_{n-k-1}^{\beta} \frac{1}{A_k^{-\beta}} \psi_k(t)dt$$

$$= \int_0^1 f(x \dot{+} t)M_n^{\beta}(t)dt.$$

By Lemma 11, for every n,

$$|M_n^{\beta}(x,f)| \leq \int_0^1 |f(x \dot{+} t)| \, |M_n^{\beta}(t)|dt$$

$$\leq C \int_0^1 |f(x \dot{+} t)|t^{-\beta-1}dt.$$

Therefore

$$\overline{\lim_n} \, |M_n^{\beta}(x,f)| \leq C \int_0^1 |f(x \dot{+} t)t^{-\beta-1}dt.$$

Integrating with respect with x,

$$\int_0^1 \overline{\lim_n} \, |M_n^{\beta}(x,f)|dx \leq C \int_0^1 \{ \int_0^1 |f(x \dot{+} t)|t^{-\beta-1}dt\}dx$$

$$= C \int_0^1 \{\int_0^1 |f(x \dot{+} t)|t^{-\beta-1}dx\}dt$$

$$= C \int_0^1 t^{-\beta-1} dt \int_0^1 |f(x \dot{+} t)| dx$$

$$\leq C \int_0^1 |f(x)| dx.$$

THEOREM 12. Let $f \in L^1[0,1]$. Then $\sum_{n=0}^{\infty} \frac{a_n(f)}{A_n^{-\beta}} \psi_n(x)$ is (C,β) summable a.e. for $-1 < \beta < 0$.

PROOF. Since the Walsh polynomials are dense in $L^1[0,1]$, there exists $P(x)$ a Walsh polynomial such that

$$||f(x) - P(x)||_1 < \frac{\varepsilon^2}{2C}.$$

Put

$$g(x) \equiv f(x) - P(x).$$

Then

$$||g(x)||_1 < \frac{\varepsilon^2}{2C}.$$

Define

$$f_1(x) \equiv \overline{\lim_n} \frac{1}{A_{n-1}^{\beta}} \sum_{k=0}^{n-1} A_{n-k-1} \frac{1}{A_n^{-\beta}} c_k \psi_k(x).$$

Similarly define g_1 and P_1.

Since $c_n(f) = c_n(g + P) = c_n(g) + c_n(P)$, we obtain

$$f_1 = g_1 + P_1.$$

Lemma 12 and $g \in L^1[0,1]$ imply

$$\int_0^1 \overline{\lim}\, |M_n^\beta(x,g)|dx \leq C \int_0^1 |g(x)|dx$$

$$= C||g||_1$$

$$< \frac{\varepsilon^2}{2}.$$

Hence

$$\int_0^1 |g_1(x)|dx < \frac{\varepsilon^2}{2}.$$

Let $E_\varepsilon \equiv \{x \in [0,1] : \overline{\lim}\, |M_n^\beta(x,g)| > \varepsilon\} \bigcup \{x \in [0,1] : |g_1(x)| > \varepsilon\}$.
By Chebychev,

$$|E_\varepsilon| \leq \frac{1}{\varepsilon} \int_0^1 \overline{\lim}\, |M_n(x,g)|dx + \frac{1}{\varepsilon} \int_0^1 |g_1(x)|dx$$

$$\leq \frac{1}{\varepsilon} \frac{\varepsilon^2}{2} + \frac{1}{\varepsilon} \frac{\varepsilon^2}{2} = \varepsilon.$$

Since $M_n^\beta(x,f) - f_1(x) = M_n^\beta(x,P_1) - P_1(x) + M_n^\beta(x,g) - g_1(x)$, for $x \notin E_\varepsilon$
the following inequality holds:

$$\overline{\lim_n}\, |M_n^\beta(x,f) - f_1(x)| \leq \overline{\lim}\, |M_n^\beta(x,g)| + |g_1(x)|.$$

$$< \varepsilon.$$

(Observe that $M_n(x,P_1) = P_1(x)$ for n sufficiently large.)

Since $|E_\varepsilon| \to 0$ as $\varepsilon \to 0$, the theorem is established.

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