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I am submitting herewith a thesis written by Thomas Edward Hodges entitled "Representations of Fractions: Promoting Students' Mathematical Understanding." I have examined the final electronic copy of this thesis for form and content and recommended that it be accepted in partial fulfillment of the requirements for the degree Master of Science, with a major in Teacher Education.

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(Original signatures are on file with official student records.)

Representations of Fractions: Promoting Students' Mathematical Understanding

A Thesis Presented for
the Master of Science
Degree
The University of Tennessee, Knoxville

Thomas Edward Hodges
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First, I thank God for the gifts He has provided me. The greatest of these gifts is my wife, Jennifer. Her continued support and understanding have been fundamental to my successes thus far. Additionally, I thank my committee, in particular Dr. JoAnn Cady, for their guidance. Work on the representations study has deepened my understanding of both the research process and the ways in which representations support mathematical understanding.

ABSTRACT

Representations of mathematical ideas serve as a foundation for understanding mathematical concepts. Using and translating among a variety of representations promotes deep conceptual understanding and connections between mathematics topics, in addition to providing contexts for more advanced mathematics concepts. The importance placed on mathematical representations led to an examination of sixth grade middle school mathematics textbooks' representation of fraction concepts. The chapters included in this thesis represent a portion of the work completed during the examination and an example of how representations of fractions can impact topics in secondary mathematics.

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CHAPTER I
INTRODUCTION AND GENERAL INFORMATION

Mathematical Representation

One of the most powerful aspects of mathematics is the use of abstraction (National Council of Teachers of Mathematics [NCTM], 2000). Gaining access to this abstraction, however, requires the problem be translated into a mathematical representation. Therefore, one must create and interpret physical representations of mathematical ideas. These representations serve as tools for communicating and thinking about mathematics. They allow one's personal mathematical ideas to be externalized, shared and preserved (Greeno & Hall, 1997). Representations also support reasoning and enhance understanding by clarifying ideas.

"The ways in which mathematical ideas are represented are fundamental to how people can understand and use those ideas" (NCTM, 2000, p.67). Therefore, much of school mathematics is concerned with the interpretation and use of representations of mathematical ideas (Kaput, 1987). Representations are not merely pedagogical tricks but rather they form a significant part of the mathematical content of school mathematics (National Research Council [NRC], 2001). They also serve as a source of mathematical reasoning. Students develop and deepen their understanding of mathematical concepts and relationships as they create, compare, and use mathematical representations. "Understanding a mathematical idea thoroughly requires that several possible

representations be available to allow a choice of those most useful for solving a particular problem" (NRC, 2001).

Exploring Fraction Representation

The importance placed on mathematical representations for the understanding of mathematics led to an examination of the representations used by textbooks. Dr. JoAnn Cady, Randy Collins, and I analyzed three widely-used middle grades textbooks to determine how fraction concepts were represented. We focused our attention on fractions since a significant portion of the middle school curriculum is devoted to this topic. Additionally, learning about fractions is more complicated and difficult than learning about whole numbers because fractions can not only be represented in several ways, but also used in many ways. Chapter Four represents part of the work completed in the examination of the texts. The article, titled "Fraction Representation: The Not-So-Common Denominator Among Textbooks" presents highlights of some of our findings and provides implications for middle grades mathematics teachers.

Connections to Secondary Mathematics

Implications for mathematical representations of fractions extend far beyond elementary and middle grades mathematics. An understanding of fraction concepts serves as the foundation for future study in algebra, proportional reasoning, and similarity at the secondary level (Conference Board for the Mathematical Sciences [CBMS], 2001). In addition, the ways in which fractions

are represented can provide a context for investigating more advanced mathematics. Chapter Five contains an article titled “Redefining a Model” that explores the mathematics of using circular models to represent different fractional parts. This article, written for practicing secondary mathematics teachers, not only provides an activity to engage students in exploring this representation, but includes the pedagogical approaches recommended in the *Standards* (NCTM, 2000).

CHAPTER II
LITERATURE REVIEW

Representations are effective tools used to enhance the understanding of mathematical ideas. Yet developing representations to indicate one's thinking is difficult and it takes work for others to understand representations of our own thinking (NCTM, 2000). Different representations often illuminate different aspects of a complex concept or relationship, suggesting that students need to compare and critique multiple representations (NCTM, 2000). Creating their own representations can also help students organize their thinking. The mathematical concepts that children are in the process of constructing are not the same as the ideas conceived by adults (Van de Walle, 2007). The representations that students create are merely an external model of the ideas they have mentally constructed. They are open to interpretation and often need to be "tested" against an external reality.

Representation Modes

Lesh, Post, and Behr (1987) identify five representations for mathematical ideas: (a) manipulative models, (b) pictures, (c) oral language, (d) written symbols, and (e) real world situations. Their research also suggests that children who have difficulty translating from one representation to another have difficulty solving problems and understanding computations; therefore, providing opportunities that strengthen students' ability to move between and among many representational models enhances their understanding. This suggests that real-

world situations or contexts for mathematical problems could aid students in developing an understanding of the mathematics involved since children have a better chance of connecting new ideas to their existing schema if they have more ways to think about and test their emerging mathematical ideas.

Models

Cramer and Henry (2002) suggest that the use of physical models is important when studying rational numbers. Unfortunately, teachers in the middle grades fail to use models when developing rational number concepts since manipulative materials are not as frequently available as in the elementary school (Van de Walle, 2007). However, using models can help students clarify developing ideas that may be confusing when using a purely symbolic mode. When representing rational numbers using manipulatives or pictures, Van de Walle (2007) suggests distinguishing among and using three different models, area or region, set, and length, as each model imposes a different perspective regarding rational numbers (see Figure 1).

Area or region models involve using a region, such as a circle or rectangle, to represent the whole. Fractional parts are represented by partitioning the region into equal portions. Commercial models include fraction squares, circles, or rectangles, and pattern blocks. Students and teachers can fold paper for a model that shows fractional representations of regions. In contrast, the set model uses a collection of objects to represent the whole. Subsets represent the fractional part. For example, four red cars represent one fourth of a set of twelve

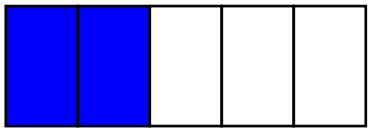
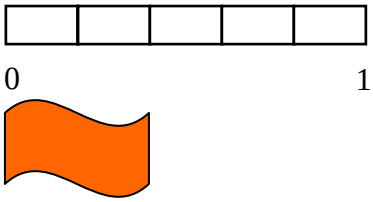
Model	Example	Description
Area/Region		2/5 of the picture is blue
Set		2/5 of the counters are red
Length		The object is 2/5 of a unit long

Figure 1. Examples of Three Types of Models

cars. Referring to a collection of objects as a single entity makes set models difficult for children. They tend to focus on the number of objects in the set or subset, rather than on the number of equal sets in the whole (Van de Walle, 2007). However, the set model encourages connections to real-world situations that use fractions and also connect easily with ratio concepts. Two-color counters and drawings using discrete objects are examples of set models.

Length models compare lengths rather than regions. Lines are drawn and subdivided or materials are compared based on length. Cuisenaire rods and fraction towers are examples of commercially available length models. Teachers and students can make their own length models by folding strips of paper. Manipulative versions of length models provide more opportunities for trial and error and for exploration (Van de Walle, 2007). A significantly more sophisticated

length model is the number line (Bright, Behr, Post, & Wachsmuth, 1988). For children, placing a number on a number line is very different than comparing one length to another, since each number on a line denotes the distance of the labeled point from zero. To enhance students' understanding of fractions, it is important that opportunities to partition regions, lengths, and sets are provided in addition to opportunities requiring them to identify fractional parts with models that are already partitioned (Mack, 2001). Thus, teachers should encourage students to create and partition their own models and use these drawings to solve fraction problems (Charles & Nason, 2000; Lamon, 1996; Pothier & Sawada, 1983, 1989).

Interpretations

In addition to these representations for rational numbers, fractions are used (or interpreted) in a variety ways. Kieren (1988) identifies several different uses or interpretations for rational numbers for which students must become familiar. These interpretations are summarized below:

1. A part-whole interpretation describes a specified number of parts in comparison to the number of parts in the whole (i.e. 5 out of 6 equal-sized shares).

2. A quotient interpretation describes the relationship between the numerator and the denominator using division (i.e. 5 divided by 6).

3. A measure interpretation describes the relationship from the beginning to the end of the unit (i.e. 5/6 of the way from the beginning to the end).
4. A ratio interpretation describes a comparison of two quantities (i.e. 5 blue hats to 6 yellow hats).
5. An operator interpretation uses fractions to enlarge or reduce the size of something (i.e. 5/6 of 30) (Kieren, 1988, 1992).

Students must recognize the nuances between interpretations while constructing relationships among representations in order to create a coherent understanding of rational numbers.

Textbooks use a variety of models and interpretations and suggest certain manipulatives, which may or may not be available in the classroom. Since teachers rely on the textbook as their main source for content and pedagogical strategies – especially in the first few years (Cady, Meier, & Lubinski, 2006) -- textbooks serve as a valuable resource for understanding how students encounter various representations.

CHAPTER III
COMPLETED RESEARCH

A research team consisting of Dr. JoAnn Cady, Randy Collins, and myself analyzed three widely used sixth grade textbooks to determine what, if any, differences there were in fraction representations among the textbooks. In order to provide a context for the included articles, the following sections outline the completed study. This abbreviated form is part of a larger research article under consideration for publication. My use of "we" in this chapter refers to my co-authors and myself. My primary contributions to this paper include a majority of the writing in the discussion section, one-third of the data collection, coding reliability and refinement, completion of statistical analysis, writing of the first draft of the results section, and editing the entire manuscript content for grammatical and stylistic errors.

Coding

Based on the current literature described above, the coding schema consisted of five categories: real-world situational context, visual models (set, area, and length), interpretation or construct, mathematical concept, and one of the five representation modes identified by (Lesh et al., 1987) and mentioned in the *Standards* (NCTM, 2000). We subdivided each of these five categories into smaller components. For example, the mathematical concept category was divided into magnitude/comparing /ordering, fraction-to-fraction equivalence,

decimal/percent/fraction equivalence, improper/mixed numbers equivalence, benchmarks/estimation, unit fraction, and computation.

Data Analysis

To compare the textbooks, we used a chi-square test to determine if the texts were statistically different in their use of context, model, interpretation, concept, and representation. A chi-square test was appropriate since the entire population of fraction problems was categorized by text and also by each of the coding categories and since the data were nominal (Gay, Mills, & Airasian, 2006; Yates, Moore, & McCabe, 1999). Adjusted residual values with an absolute value above two indicated which observed frequencies (category and text) varied significantly from the expected frequencies if each of the texts represented fractions in the same manner.

Results

The results of the chi-square tests indicated that the manner in which fractions were represented in each text varied significantly across all coding categories (problem context, models, interpretation, concepts, and multiple representations). Table 1 includes a summary of the results of the chi-square tests. Appendices B – F contain the frequency counts, results of the Chi-square tests, adjusted residual values, and within text percentages for each coding category.

Table 1. Results of Chi-square Tests

Coding category	Chi-square results
Problem Context	$\chi^2(10) = 163.983,\ p < .001$
Model	$\chi^2(8) = 250.783,\ p < .001$
Interpretation	$\chi^2(8) = 506.088,\ p < .001$
Concept	$\chi^2(12) = 167.093,\ p < .001$
Multiple Representations	$\chi^2(4) = 89.343,\ p < .001$

Since some of the problems in each text had multiple representations, a chi-square test was inappropriate. In lieu of the chi-square test, we reported with-in text percentages of problems utilizing each of the representation modes (see Table 2).

Conclusions

Many teachers rely on textbooks for their curriculum, especially in their first few years of teaching (Cady, Meier, & Lubinski, 2006; Gonzales et. al., 2004). The results of our examination of textbooks reveals that if textbooks are relied upon by teachers and, if teachers implement the intended curriculum, then students using these textbooks would receive very different mathematics instruction. Real-world contexts and applications advocated by NCTM would be missing from the majority of the problems in some texts and from most problems in other texts.

Table 2. Within-text Percentage of Problems Utilizing Each Representation Mode

Representation Mode	CMP	Thematics	Glencoe
Words	31.94%	7.28%	6.38%
Symbols	58.15%	88.95%	95.71%
Pictures	27.31%	7.28%	7.36%
Real World	27.53%	2.70%	14.36%
Manipulatives	2.86%	5.12%	0.25%

CHAPTER IV
FRACTION REPRESENTATION IN THE MIDDLE GRADES

This chapter is a version of a paper named "Fraction Representation: The Not-So-Common Denominator Among Textbooks" under consideration for publication in *Mathematics Teaching in the Middle School* by Thomas E. Hodges, JoAnn Cady, and Randy Collins.

My use of "we" in this chapter refers to my co-authors and myself. This paper uses the same data collected for the research paper outlined in Chapter Three. As primary author, my contribution to this paper included writing the paper in a format that could be easily read by middle school mathematics teachers. It was our intent to outline the differences in the textbooks regarding fraction representation and highlight the need for teachers to utilize a variety of curricular resources.

Introduction

"The ways in which mathematical ideas are represented is fundamental to how people can understand and use those ideas… When students gain access to mathematical representations and the ideas they represent, they have a set of tools that significantly expand their capacity to think mathematically" (NCTM, 2000, p. 67). School mathematics has included various forms of representations such as symbols, drawings, and graphs. These forms of representations help students reason about and understand mathematics. They support students' learning and help students communicate mathematically, organize their thinking, make connections among mathematical concepts, and apply mathematics to the real world through modeling (NCTM, 2000). Students in the middle grades use representations to help them make sense of more abstract concepts, such as rational numbers.

One challenge for teachers who wish to develop students' flexibility and fluency with a variety of representations is choosing the resources or curriculum materials for classroom instruction. With a variety of resources available, the teacher's role becomes one of determining which problems or tasks provide opportunities for students to not only demonstrate their mathematical thinking using the representation of their choice, but also encourage students to become flexible translating from one representation to another. This challenge for teachers and the fact that teachers often rely on textbooks as their main source for content and pedagogical strategies, especially in the first few years, (Cady, Meier, & Lubinski, 2006), led us to explore the representations used in textbooks. We chose fractions as they are a significant portion of the middle school curriculum. In addition, learning about fractions is more complicated and difficult than learning about whole numbers. The textbooks we selected were: Co*nnected Mathematics* (CMP); *Middle Grades Math Thematics* (Thematics)*;* and Glencoe's *Mathematics: Applications & Concepts Course 1* (Glencoe)*.* Glencoe represents a traditional text, while CMP represents a reform text. The authors felt that while Thematics was classified as a reform text, it provided more of blend between traditional and reform.

Representation Mode

To begin our comparison, we considered the representations identified by Lesh, Post, and Behr (1987): (a) real world situations, (b) pictures, (c) written

language, (d) manipulatives, and (e) symbols. We reviewed and labeled each fraction problem for its representation mode. During this review, we discovered that some problems used more than one representation mode; therefore, a problem could have more than one representation label.

When comparing the representation modes used in the texts, the most common representation mode used was symbolic (see Table 3). Examining the percentages of the other four representation modes indicates that CMP more evenly addresses the representation modes used in the study, with the exception of manipulatives as the percentages of problems in each of the categories are more evenly distributed. The lower frequency of problems that were represented using written language, pictures, and utilizing real-world contexts in Glencoe and Thematics suggests that students have fewer opportunities to explore alternative ways of representing fractions in those texts. The high percentage of problems using the symbolic mode, might suggest the textbook authors purpose for using the symbolic representation mode was to move students to a higher level of abstraction. One aspect of the middle school mathematics curriculum is to serve as a bridge between the concrete experiences of the elementary school and the more abstract curriculum of the high school mathematics classroom. However, few problems in each of the texts used manipulatives or suggested that students use manipulatives. Therefore, we question the use of 89% and 95% symbolic

Table 3. Percentage of Problems Using Each Representation

Representation Mode	CMP	Thematics	Glencoe
Words	31.94%	7.28%	6.38%
Symbols	58.15%	88.95%	95.71%
Pictures	27.31%	7.28%	7.36%
Real World	27.53%	2.70%	14.36%
Manipulatives	2.86%	5.12%	0.25%

problems in Glencoe and Thematics, respectively, combined with the low frequency of problems in the picture, word, and manipulative representation modes as bridging from concrete to abstract.

Students' understanding is enhanced when using a variety of representations and/or when students are encouraged to translate from one representation to another. Therefore, we felt it also necessary to compare the number of problems in each text using more than one representation. We categorized problems as using multiple representations when a single problem suggested or encouraged a variety of representations either when students solved the problem or when the problem was presented using more than one representation. For example, consider the sample problem provided from Thematics (See Appendix). A diagram is used to present the problem. Students are then expected to translate their thinking about the two samples of trout into written language. Thus, the problem was represented one way, but the students' solutions are represented another. The juice consumption problem from CMP serves as another example. The problem presents the fractions as lengths on a bar chart. This requires students to translate the picture representation to

symbolic form, determine a solution, and then translate the solution into written language based upon the context.

Our comparison indicates the number of problems using more than one representation differed significantly among the three texts. Table 4 shows the percentage of problems using multiple representations within each of the three texts. CMP had the lowest percentage of problems that utilized a single representation and was the only text to include fraction problems in four and five representations. For the most part, problems using only a single representation used the symbolic mode in all three texts and the majority of these problems were also void of context. CMP had a much higher percentage of problems that utilized three or more representation modes, suggesting that CMP offers students more opportunities to use a variety of representations. Even though CMP does offer more opportunities, over 70% of the problems in each of the texts used a single representation mode. Thus, fluency in translating between representation modes may be difficult to achieve by using any of the texts alone.

Model

The term *model* can be used in different ways in mathematics education. For some, it is used interchangeably with *representation*. In this article, we use the term model when referring to manipulative models or diagrams that represent fractions. We felt that the way in which fractions are represented by manipulatives or drawings was also important to consider when comparing

Table 4. Percentage of Problems Using Multiple Representations

Number of Representations	CMP	Thematics	Glencoe
Single	70.0%	88.9%	79.1%
Two	17.2%	9.2%	18.2%
Three or more	12.8%	1.9%	2.7%

textbooks as each has its own strengths and weaknesses.

Van de Walle (2007) suggests distinguishing among and using three different models: area or region, set, and length (see Figure 1). Using models can help students clarify developing ideas that may be confusing when using a purely symbolic mode as each model imposes a different perspective regarding fractions. For example, when using the set model, students tend to focus on the number of objects in the set or subset, rather than on the number of equal sets in the whole (Van de Walle, 2007). Additionally, referring to a collection of objects as a single entity makes set models difficult for children. Yet, the set model encourages connections to real-world situations that use discrete objects and also connects easily with ratio concepts. The length model provides another perspective. In this model, lines are drawn and subdivided or materials are compared based on length. A number line is a more sophisticated length model (Bright, Behr, Post, & Wachsmuth, 1988). For children, placing a number on a number line is very different than comparing one length to another. When students place numbers on a number line, they are denoting the distance to the labeled point from zero. No matter which model is used, to enhance students' understanding of fractions, it is important that opportunities to partition regions,

lengths, and sets are provided in addition to opportunities requiring them to identify fractional parts with models that are already partitioned (Mack, 2001).

Even though Cramer and Henry (2002) recommend the use of models when studying rational numbers, all three texts had a high percentage of problems where a model was not provided or where students were not encouraged to make use of or create a model of their own (see Table 5). CMP used thermometers and number lines as the length model in almost one-fifth of their problems. This corresponded with CMP's emphasis on magnitude or ordering and comparing fractions, in addition to preparing students to use length models for more advanced studies in mathematics. As students labeled the points on a number line or thermometer, they focused on magnitude, with the purpose of comparing and ordering the numbers.

Problem Context

NCTM (2000) advocates for problems set in real world contexts. Others suggest that the context in which the problem is set may enhance or hinder understanding (Brown, Collins, & Duguid, 1989). When students translate these real-world situations to drawings and symbols and explain the thinking involved in the translation through oral or written language they have a better chance of connecting new ideas to their existing schema. However, if students are not familiar with the context, they may have difficulty translating from one

Table 5. Percentage of Problems Using Each Model

Model	CMP	Thematics	Glencoe
None	63.9%	84.8%	93.1%
Area/region	10.1%	3.3%	3.9%
Set	2.4%	5.2%	0.1%
Length	18.9%	6.8%	1.2%
Choice	4.6%	0.0%	1.6%

representation and thus have difficulty solving problems and understanding computations (Lesh, Post, & Behr, 1987). Therefore, we decided to compare the contexts used in each of the three textbooks to see if the context were familiar to middle school students.

As we looked at the contexts used with each of the three texts, we noticed over 75% of the fraction problems in all three texts were naked problems, or problems that lacked a context (See Table 6). Using our coding schema, the absence of a context for a problem often suggested that the problem was represented in symbolic form.

As an example, a problem might ask students to order the following fractions from least to greatest: 2/5, 2/3, 4/9, 1/2. Thus, the problem is naked and coded symbolic since the fractions do not represent any real-world situation. The absence of a picture, or suggestion for use of a manipulative, meant that the problem also lacked a model (area, set, or length). Our first reaction was to conclude the lack of a context inhibits the use of representations. Upon further reflection, we conjectured that a context may suggest a particular model or representation. In contrast, the naked problem allows for the student to choose their own representation. For instance in the example of ordering fractions from

Table 6. Percentage of Problems in Context

Context	CMP	Thematics	Glencoe
Contained a context	21.8%	3.3%	4.4%
Lacked a context	78.2%	96.7%	95.6%

least to greatest, students may see the fractions as representing some quantity in a real-world context and then choose the appropriate area, set, or length model in which to represent these fractions. However, if students have not had previous experience with using each of the models to solve problems set in real-world contexts, they may then limit themselves to using a single mathematical model or a solely symbolic representation mode to solve the problem. We did find that the "Investigations" in CMP and the introductory problems in Thematics were set in context. These tasks were designed to aid students as they begin to make sense of a new mathematical concept. Problems void of context appeared in the practice or homework sections of these two texts. In Glencoe, only 34.2% of the introductory problems were set in real-world contexts. The contexts that occurred most frequently -- money, cooking, shopping, and sharing – are contexts familiar to most middle school students.

Conclusion

Our results indicate that if teachers utilize the intended textbook curriculum as their only source for both content and pedagogical approaches, then students using each of these texts would receive dramatically different classroom instruction, especially in the area of representations. The use of multiple representations to support students' learning advocated by NCTM would

be missing from the majority of the problems in some texts and from most problems in other texts. In addition, activities utilizing manipulatives were almost absent from all three texts. Therefore, if teachers want students to become flexible with a variety of representations, they must utilize an assortment of resources. Teachers must also encourage students to reflect upon their use of representations. Evaluating the strengths and weaknesses of various representations for a particular problem enhances students' mathematical understanding. The challenges for teachers when encouraging students' use representations can be met through careful evaluation of textbooks and the problems they provide. Problems that support connections and relationships through the use of multiple modes of representation do exist in these textbooks, but other resources should also be considered.

We hope this information assists teachers and district curriculum specialists in making educational decisions regarding the use of the textbook they have chosen. Teachers are the ultimate decision makers when determining how a curriculum is implemented and whether or not a variety of curriculum resources are used. These curricular decisions are significant when developing students' ability to use a variety of representations. Each of us tends to utilize a predominate representation. However, to support students in their mathematical thinking and understanding, teachers can help students move beyond this predominant representation by encouraging students to solve problems using multiple representations. Fostering an environment where students feel comfortable reflecting upon and critiquing others' representations also enhances

students' mathematical understanding. Listening and questioning skills play a key role in understanding the representations created by students; teachers must use what they know about their students and their professional judgment when deciding when and how to help students move toward conventional representations. It can be counterproductive to introduce students to conventional representations before they are able to use them meaningfully (NCTM, 2000). Thus, the teacher "has an important role in helping middle-grades students develop confidence and competence both in creating their own representations when they are needed to solve a challenging problem and in selecting flexibly and appropriately from an extensive repertoire of conventional representations" (NCTM, 2000, p. 284).

CHAPTER V
FRACTION REPRESENTATION AS A CONTEXT FOR SECONDARY MATHEMATICS

This chapter is a version of a paper named “Redefining a Model” by Thomas E. Hodges accepted for publication in 2007 by the editorial panel of Mathematics Teacher.

Hodges, T. E. (in press). Redefining a Model. Mathematics Teacher.

Introduction

Traditionally representing thirds using a circular model requires students to make radial cuts (see Figure 2). Counter to how children are known to best learn with models (Ball, 1992), students are often left with directions as to how to appropriately divide the circle to get the correct representation. This is due in part to students’ intuitive nature not to use radial cuts when partitioning a circle, leaving them with a picture similar to Figure 3.

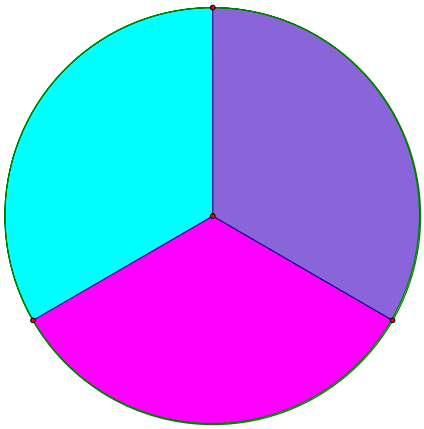


Figure 2. Traditional Circular Model of Thirds

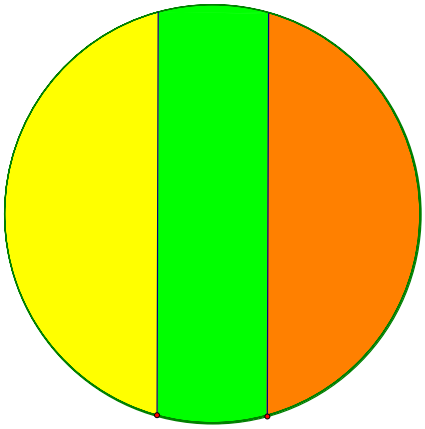


Figure 3. Alternative Circular Model of Thirds

So, in turn, many elementary and middle school teachers use rectangular models to avoid this common mistake. However, is there value (beyond a wrong answer and a talking point) in the students' intuitive model? Clearly there is a correct way to model 1/3 using a circle without making radial cuts, but how might this be accomplished?

A scenario such as this provides teachers an opportunity to present a problem with rich mathematics across multiple grade bands. While younger students should be encouraged to appropriately model 1/3 using radial cuts, more mature learners can investigate an alternative method. The alternative method described here is appropriate for students studying geometry and trigonometry at the high school level. Prerequisite knowledge includes measurement of central and inscribed angles in a circle, tangents, areas of circles and circular segments, SAS formula for the area of a triangle, and solving linear equations using graphical methods.

An Alternative Method

Students are placed into groups to investigate the following task: *Divide a circle into thirds using two non-intersecting chords, or chords intersecting on the circle, and without making any radial cuts*. First, students are not given any form of technology or tools for measurement, only several large circles cut from construction paper. The teacher should encourage students to make rough estimates, visualizing the partitions needed to create three regions of equal area. The process of dividing a circle in such a way is quite foreign to many students who are familiar with the traditional circular model. Once student groups have the opportunity to discuss and create a rough estimate, groups should present their drawings and justify the location of the two chords. This discussion provides a backdrop for further investigation. Student drawings may include parallel chords (Figure 4), non-parallel chords (Figure 5), or chords that create an inscribed angle (Figure 6). This discussion should end with the understanding that there is infinite number of ways to divide a circle into thirds using this method.

Next, the investigation transitions from approximate drawings to more precise calculations. Student groups should be encouraged to use their drawings as an estimate for these precise calculations. Different drawings will allow different groups to incorporate different mathematics when solving and justifying the problem. However, each group will have common elements. For

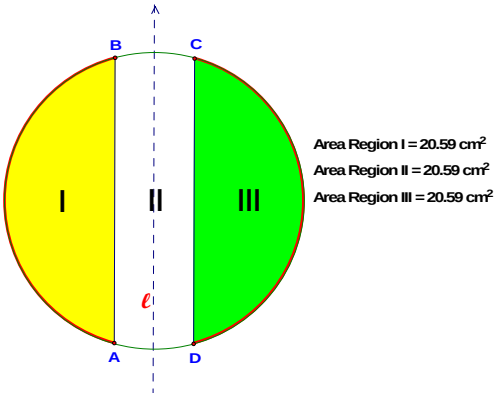


Figure 4. Construction with Parallel Chords

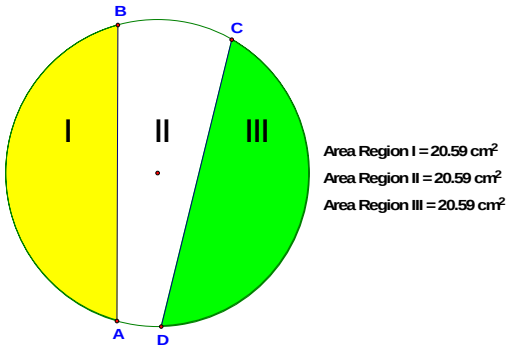


Figure 5. Construction without Parallel Chords

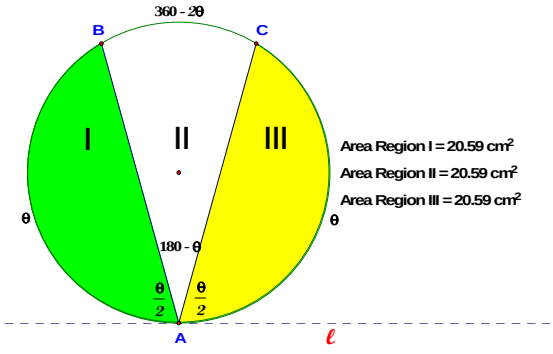


Figure 6. Construction Using Inscribed Angles

instance, each group's drawing has two circular segments (see regions I and III in figures 4, 5, and 6). These circular segments are a nice place to start and are central to solving the problem using the method described in this article. A prompt to assist student groups as they struggle might be as follows: *I see that you have three regions within your drawing. How would you describe the shape of each of those regions and how might you find the area of each?* Once students determine that the outermost regions are circular segments, they will most likely use the formula for the area of a segment commonly found in Geometry textbooks, $A_{segment} = \frac{m\overset{\frown}{AB}}{360} * \pi r^2 - A_{triangle}$, accompanied by a picture of a circular segment with chord $\overline{AB}$. This somewhat generic form must be translated into the variables associated with the task. During their work in groups, students should come to the following conclusions:

1. $A_{segment} = \frac{1}{3}\pi r^2$ since each region represent one-third of the area of the circle.
2. $m\overset{\frown}{AB}$ is unknown (I will call it θ)

3. To find the area of the triangle, the SAS formula ($A = \frac{1}{2}ab\sin\theta$, where *a* and *b* are the distances from the center of the circle to the endpoints of the chord) can be used (see Figure 7).

The third conclusion is possibly the most difficult. A prompt to assist students in their investigation might be as follows: *List the different ways to find the area of a*

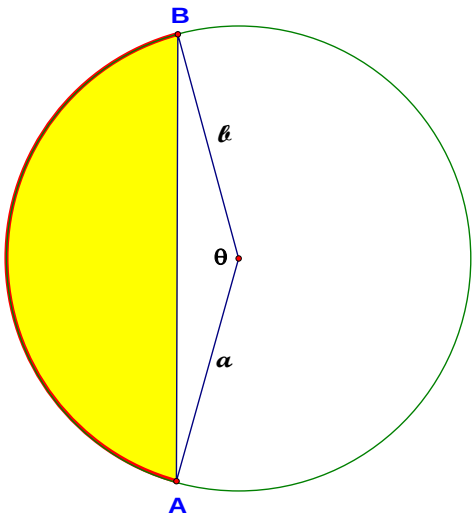


Figure 7. Variables for Using the SAS Formula of a Triangle.

triangle, then list the things you know about this triangle. Given what you know about this triangle, and the other variables involved in the equation, which formula is the best fit? Both *a* and *b* are radii of the circle, so by substituting those variables with *r* and utilizing the conclusions above, students generate the following equation: $\frac{\theta}{360} \times \pi r^2 - \frac{1}{2} r^2 \sin\theta = \frac{1}{3} \pi r^2$. Students should soon realize that they can divide both sides of the equation by r^2, leaving $\frac{\theta}{360} \times \pi - \frac{1}{2} \sin\theta = \frac{1}{3} \pi$. The elimination of r^2 is significant for students to understand. A prompt to reveal whether students are thinking about the need for a specific area might be:

Why is it okay to eliminate r^2*?* And if necessary ask: *How does the value of* r *affect the circle? How is the size of the circle related to what we are trying to find?*

Students will realize that they cannot solve the resulting equation, and they might see that it is impossible to solve by algebraic methods. Ask them what other methods they can use to solve the equation and encourage them to consider graphing so they can use a graphing calculator or computer graphing program. Some may set the equation equal to zero and graph $y = \frac{\theta}{360} \times \pi - \frac{1}{2}\sin\theta - \frac{1}{3}\pi$ to see where the graph crosses the x-axis (see Figure 8).

Others might write two equations such as: $y = \frac{1}{2}\sin x$ and $y = \frac{\pi\theta}{360} - \frac{\pi}{3}$ and look for intersections of their graphs. Either way the graphs will yield a value for the central angle of approximately 149.27417.

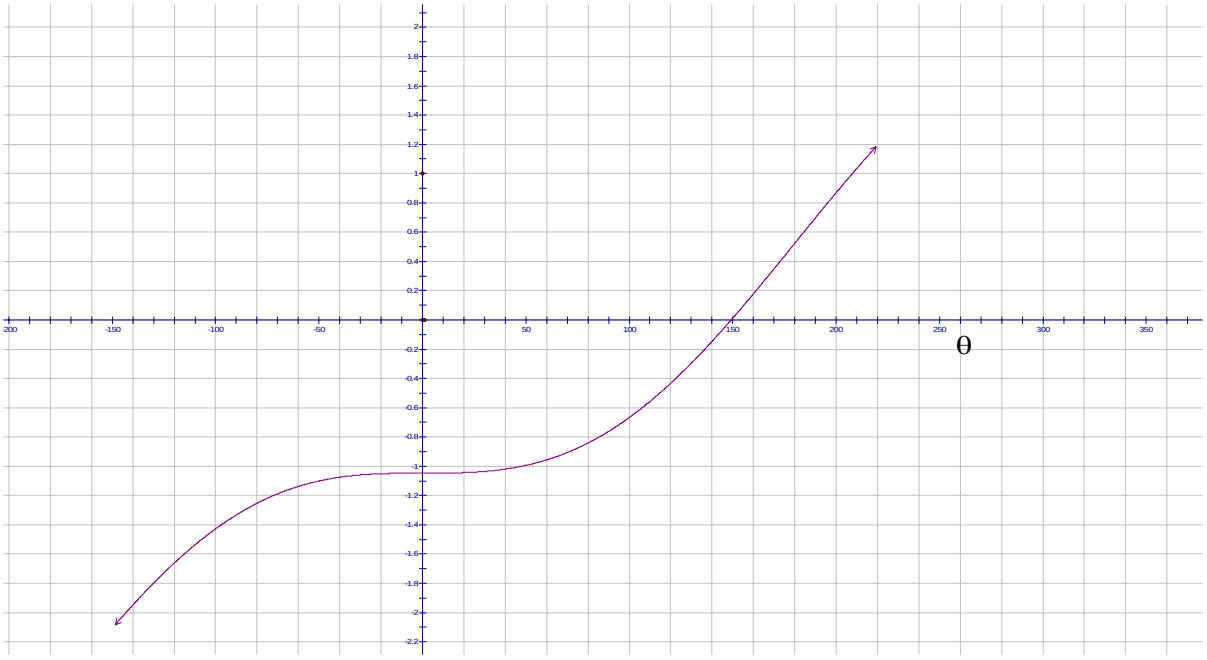


Figure 8. Graph of Equation for Thirds

Students should be encouraged to discuss the reasonableness of their solution. A fairly easy way to go about this is to discuss the area of the sector. In order to find 1/3 of the circle using the traditional method, students generate three circular sectors. The measure of each central angle is 120°. Since the area of the sector is larger than the area of the segment for a given central angle, it makes sense to increase the measure of the central angle to increase the area of the circular segment.

Verifying the Solution Using Technology

Computer access with dynamic geometry software can act as a powerful tool for verification and further exploration of the concept. Student groups should have access to this software in order to verify their value for θ and construct the thirds model. By using dynamic geometry software, other areas of mathematics are incorporated into the task. Student groups should be encouraged to use both their estimate drawings from the beginning of the task as well as their value for θ in order to make the construction. A method for creating parallel chords follows:

1. Construct a single chord by creating a point and rotating the point around the center 149.27417°.
2. Construct a line l parallel to the chord through the center of the circle.
3. Reflect the chord over line l (see Figure 4).

Students then find the area of each of the regions within the circle and of the entire circle in order to verify the solution.

Student groups with alternative estimates will generate sketches different from the parallel chords construction, thus they may use different mathematics for their constructions (see Figures 5 and 6). One interesting construction uses inscribed angles.

1. Construct chord $\overline{AB}$ similar to the above method.
2. Construct a tangent through point A.
3. Since the measure of $\overset{\frown}{AB} = \theta$, the measure of the inscribed angle of $\overset{\frown}{AB}$ is $\frac{\theta}{2}$. In order to create another inscribed angle that measures $\frac{\theta}{2}$, the angle of rotation about the center should be 180 - $\frac{\theta}{2}$ - $\frac{\theta}{2}$ or $180 - \theta$ from point A (see Figure 6).

Extending the Problem

Once student groups have verified their construction and value for θ, each group should share their calculations for θ and their design of the model, including their methods used for the construction. To extend the task, student groups can be asked to investigate different numbers other than thirds. For example, *how could you divide a circle into fourths? What about fifths?* One possibility is to have each group chose a number for themselves. Each group could create a poster presentation of their work to share with the class. Some interesting final questions might include:

1. What were some common attributes among all groups? What was different?
2. Must constructions of $\frac{1}{n}$, where *n* is even, have a chord that goes through the center? Why or why not?
3. How might you generalize the process so that others would be able to complete constructions of $\frac{1}{n}$, where *n* is any positive integer?
4. How might we divide a circle into $\frac{1}{n}$'s where the chords intersected but not at the center?

Conclusions

Utilizing technology both to solve and justify the solutions brings deeper meaning to the task and integrates several areas of mathematics that would be nearly impossible to do by hand over such a short period of time. Students' study of fractions brings them to solving trigonometric equations using graphical methods, area measurement, geometric transformations, and a different model for representing 1/3.

Tasks such as this provide valuable learning experiences that require students to actively do mathematics. The students become involved in a problem with no explicit strategy or approach, but use existing knowledge as a tool for solving a unique problem. These demands embody many of the components outlined by Smith and Stein (1998) when designing tasks that call

for high levels of cognitive demand. This task provides another example of how a seemingly simple exercise in representing fractions can be the basis for in-depth study connecting mathematics concepts as called for in the Standards (2000).

CHAPTER VI
CONCLUSIONS

Many students' experiences with mathematics consist primarily of manipulating symbolic forms of mathematics concepts. Students in these settings rarely have opportunities to represent mathematical ideas through various forms of representations. Research indicates that the ways in which fraction concepts are represented help form the lasting ideas students have about a particular mathematics concept. In order to develop a deep understanding of and connections among topics, students should be introduced to a variety of representations, encouraged to use the representation of their choice, and supported in attempts to translate between a variety representations. In addition, teachers should provide an environment where observation and critique of others' representations is promoted.

The article presented in Chapter IV of this thesis shows students' opportunities to learn mathematics consistent with research varies significantly among the studied textbooks. The variety of representations known to enhance students understanding and encourage mathematical reasoning is lacking from the majority of fraction problems in each of the texts. Since a robust understanding of mathematical ideas requires students to have available a variety of representations with which to solve a problem, the textbooks included in the study provide students different access to representations. Therefore, the lasting ideas that students hold about fraction concepts may lack depth in understanding and connections between topics unless teachers supplement the

textbooks with activities that encourage students to explore each of the areas outlined in the study.

To serve as an example of using representations to explore and connect ideas, Chapter V of this thesis includes an article outlining an activity where secondary students use an area model to represent one-third in a non-traditional way. Representing different fractional parts in this manner serves to enhance students' understanding of a variety of concepts in geometry and trigonometry. Physical models of their thinking represent the abstract ideas that students hold about these concepts. As indicated by research, students' presentation of their physical models allows them to communicate their thinking. In addition, sharing their ideas regarding the representations of various fractional parts allows others to critiquing their thinking.

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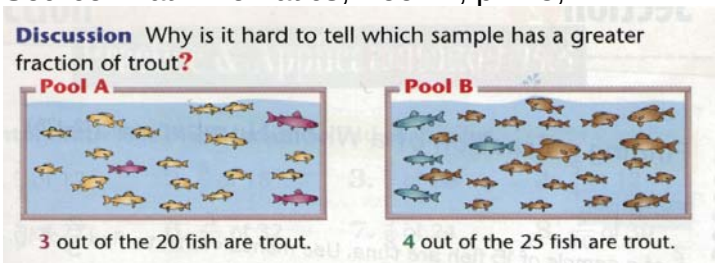
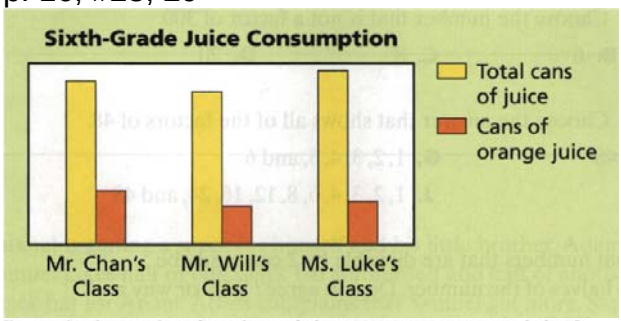
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APPENDICES

APPENDIX A

	Context	Model	Representation(s)
Source: *MathThematics*, Book 1, p.175, #14 	Science	Set	Words, Pictures
Source: *Connected Mathematics*, Bits & Pieces I, p. 16, #28, 29  In each class, what fraction of the cans were orange juice? In which class would you say orange juice was most popular?	Data	Length	Words, Pictures, Symbols, Real-world
Source: *Glencoe Mathematics: Applications and Concepts*, Course 1, p. 205, #32 **BUTTERFLIES** The average wingspan of a western tailed-blue butterfly shown at the right is between 0.875 and 1.125 inches. Find two lengths that are within the given span. Write them as fractions in simplest form.	Science	Length	Symbols, Real-world

APPENDIX B

*Context * Textbook Crosstabulation with Chi-Square Test*

			Textbook			Total
			CMP	Thematics	Glencoe	
Context	Cooking	Count	21	3	11	35
		% within textbook	5.6%	.9%	1.5%	2.5%
		Adjusted Residual	4.5	-2.1	-2.2	
	Money	Count	25	0	3	28
		% within textbook	6.6%	.0%	.4%	2.0%
		Adjusted Residual	7.6	-2.9	-4.2	
	Naked	Count	295	320	681	1296
		% within textbook	78.2%	96.7%	95.6%	91.3%
		Adjusted Residual	-10.4	4.0	5.9	
	Science	Count	3	6	14	23
		% within textbook	.8%	1.8%	2.0%	1.6%
		Adjusted Residual	-1.5	.3	1.0	
	Sharing	Count	21	0	0	21
		% within textbook	5.6%	.0%	.0%	1.5%
		Adjusted Residual	7.7	-2.5	-4.6	
	Shopping	Count	12	2	3	17
		% within textbook	3.2%	.6%	.4%	1.2%
		Adjusted Residual	4.1	-1.1	-2.7	
Total		Count	377	331	712	1420
		% within textbook	100.0%	100.0%	100.0%	100.0%

$\chi^2(10) = 163.983, p < .001$

APPENDIX C

*Model * Textbook Crosstabulation with Chi-Square Test*

			Textbook			Total
			CMP	Thematics	Glencoe	
Model	Area/region	Count	46	12	32	90
		% within textbook	10.1%	3.3%	3.9%	5.5%
		Adjusted Residual	5.1	-2.1	-2.8	
	Choice	Count	21	0	13	34
		% within textbook	4.6%	.0%	1.6%	2.1%
		Adjusted Residual	4.5	-3.2	-1.4	
	Length	Count	86	25	10	121
		% within textbook	18.9%	6.8%	1.2%	7.4%
		Adjusted Residual	11.1	-.5	-9.5	
	None	Count	290	312	761	1363
		% within textbook	63.9%	84.8%	93.1%	83.2%
		Adjusted Residual	-12.9	.9	10.8	
	Set	Count	11	19	1	31
		% within textbook	2.4%	5.2%	0.1%	1.9%
		Adjusted Residual	1.0	5.2	-5.2	
Total		Count	454	368	817	1639
		% within textbook	100.0%	100%	100%	100.0%

$\chi^2(8) = 250.783, p < .001$

APPENDIX D

*Interpretation * Textbook Crosstabulation with Chi-Square Test*

			Textbook			Total
			CMP	Thematics	Glencoe	
Interpretation	Measurement/quantity	Count	207	313	765	1285
		% within textbook	45.2%	84.4%	93.8%	78.1%
		Adjusted Residual	-20.1	3.3	15.2	
	Operator	Count	139	19	0	158
		% within textbook	30.3%	5.1%	.0%	9.6%
		Adjusted Residual	17.7	-3.3	-13.1	
	Part/whole	Count	87	39	34	160
		% within textbook	19.0%	10.5%	4.2%	9.7%
		Adjusted Residual	7.9	.6	-7.6	
	Quotient	Count	19	0	0	19
		% within textbook	4.1%	0%	.0%	1.2%
		Adjusted Residual	7.1	-2.4	-4.3	
	Ratio	Count	6	0	17	23
		% within textbook	1.3%	.0%	2.1%	1.4%
		Adjusted Residual	-.2	-2.6	2.3	
Total		Count	458	371	816	1645
		% within textbook	100.0%	100%	100%	100.0%

$\chi^2(8) = 506.088, p < .001$

APPENDIX E

*Concept * Textbook Crosstabulation with Chi-Square Test*

			Textbook			Total
			CMP	Thematics	Glencoe	
Concept	Computation	Count	194	158	386	738
		% within textbook	41.3%	41.0%	49.9%	45.3%
		Adjusted Residual	-2.1	-1.9	3.5	
	Equivalence(decimal/percent/fraction)	Count	49	47	128	224
		% within textbook	10.4%	12.2%	16.6%	13.8%
		Adjusted Residual	-2.5	-1.0	3.1	
	Equivalence(fraction/fraction)	Count	54	42	52	148
		% within textbook	11.5%	10.9%	6.7%	9.1%
		Adjusted Residual	2.1	1.4	-3.2	
	Equivalence(improper/mixed number)	Count	6	27	75	108
		% within textbook	1.3%	7.0%	9.7%	6.6%
		Adjusted Residual	-5.5	.3	4.7	
	Estimation	Count	65	10	76	151
		% within textbook	13.8%	2.6%	9.8%	9.3%
		Adjusted Residual	4.0	-5.2	.7	
	Magnitude(comparing/ordering)	Count	91	84	56	231
		% within textbook	19.4%	21.8%	7.2%	14.2%
		Adjusted Residual	3.8	4.9	-7.6	
	Unit fraction	Count	11	17	0	28
		% within textbook	2.3%	4.4%	.0%	1.7%
		Adjusted Residual	1.2	4.7	-5.1	
Total		Count	470	385	773	1628
		% within textbook	100.0%	100%	100%	100.0%

$\chi^2(12) = 167.093, p < .001$

APPENDIX F

*Multiple Representations * Textbook Crosstabulation with Chi-Square Test*

			Textbook			Total
			CMP	Thematics	Glencoe	
Number of Representations	Single	Count	318	330	645	1293
		% within textbook	70.0%	88.9%	79.1%	78.8%
		Adjusted Residual	-5.4	5.4	.3	
	Two	Count	78	34	148	260
		% within textbook	17.2%	9.2%	18.2%	15.9%
		Adjusted Residual	.9	-4.0	2.5	
	Three or more	Count	58	7	22	87
		% within textbook	12.8%	1.9%	2.7%	5.3%
		Adjusted Residual	8.4	-3.3	-4.7	
Total		Count	454	371	815	1640
		% within textbook	100.0%	100%	100%	100.0%

$\chi^2(4) = 89.343, p < .001$

VITA

Thomas Edward Hodges, III (Tommy) was born in Nashville, TN. Tommy graduated from Belmont University and worked as a mathematics teacher at Hillsboro High School and Franklin Road Academy in Nashville. Tommy taught Algebra I, Geometry, Pre-Calculus, and AP Statistics. Currently, Tommy is working towards a Ph.D. in Education, specializing in mathematics education at the University of Tennessee, Knoxville. In addition, he is the proud father of Tyler Andrew Hodges, Jennifer and Tommy's first child.