

Title: The Structure of Matricially Quasinormal Tuples

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Abstract

Quasinormal operators are a class of operators that exists between normal operators, those that commute with their adjoint, and subnormal operators, normal operators restricted to an invariant subspace. Specifically, a quasinormal operator on an infinite-dimensional Hilbert space commutes with its adjoint times itself. In this paper we examine one possible generalization of the concept of quasinormal operators to the multivariable setting: matricially quasinormal tuples. To accomplish this purpose we follow the research of single variable matricially quasinormal tuples performed by John B. Conway and Pei Yuan Wu and expand their research to the more general multivariable setting. We study some basic properties of these commuting tuples and create a related structure theorem.

The Structure of Matricially Quasinormal Tuples

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In this paper we look at one possible generalization of the concept of quasinormal operators to the multivariable setting. We then study some basic properties of these commuting tuples and create a related structure theorem.

1 Introduction

Quasinormal operators are a class of operators that exists between normal operators, those that commute with their adjoint, and subnormal operators, normal operators restricted to an invariant subspace. They include normal operators, and all quasinormal operators are subnormal. Specifically, a quasinormal operator, T , on an infinite-dimensional Hilbert space, $\mathcal{H}$, commutes with its adjoint times itself, or $T(T^*T) = (T^*T)T$. The quasinormal operator's curious position between normal and subnormal operators makes quasinormal operators an interesting example of operators which are close to being normal, with the basic example being the unilateral shift, U_+ .

The most useful structural theorem of a quasinormal operator, T , is that it is unitarily equivalent to an operator

$$N \oplus \begin{pmatrix} 0 & & & \\ A & 0 & & \\ & A & 0 & \\ & & \ddots & \ddots \end{pmatrix}$$

where N is a normal operator and A is a positive self-adjoint operator. If T is pure, which means it has no reducing subspace on which it is normal, then the normal operator is unnecessary in the previous structural theorem [1].

A natural direction in which to continue researching an operator is to examine its properties in the multivariable case. Commuting normal tuples of operators act similarly to their single variable counterparts and provide no important surprises. Subnormal tuples of operators have also been well-studied. However, a multivariable generalization for quasinormal operators has received far less attention. Theorems regarding multivariable quasinormal tuples and

their structures give a useful glimpse at a transition class between subnormal and normal tuples of operators.

Three types of multivariable quasinormal tuples have been introduced: matrixially, jointly, and spherically quasinormal tuples [3]. The class closest to the normal tuples, and therefore most readily studied, is the matrixially quasinormal tuples.

Definition 1.1. A commuting tuple $T = (T_1, \dots, T_d)$ is called *matrixially quasinormal* if T_i commutes with $T_j^* T_k$ for all $i, j, k = 1, \dots, d$.

In this article, we follow the path of [2], and develop representation theorems for matrixially quasinormal tuples.

2 Representation Theorems

The first of the representations that we will examine is related to a matrix structure of the matrixially quasinormal tuple. This representation was given in [3] as one possible generalization of quasinormal operators to the multivariable setting.

Some notational explanation is necessary before we begin. For commuting tuple, $T = (T_1, \dots, T_d)$, and multi-index, $\alpha = (\alpha_1, \dots, \alpha_d) \in \mathbb{Z}_+^d$, we define $|T|^\alpha = |T_1|^{\alpha_1} \dots |T_d|^{\alpha_d}$ where $|T_i| = (T_i^* T_i)^{\frac{1}{2}}$ and $T^\alpha = T_1^{\alpha_1} \dots T_d^{\alpha_d}$.

Proposition 2.1 ([3], Theorem 2.2). *Let $T = (T_1, \dots, T_d)$ be a pure commuting tuple. Then T_i commutes with $T_j^* T_k$ for all i, j, k if and only if $T_i^* [T_j^*, T_k] = 0$ for all i, j, k . In this case, there is a subspace $\mathcal{L}$ and positive commuting operators A_i on $\mathcal{L}$ such that*

$$T_i = \begin{pmatrix} 0 & & & \\ A_i & 0 & & \\ & A_i & 0 & \\ & & \ddots & \ddots \end{pmatrix}$$

on $\mathcal{H} = \oplus_{n=1}^{\infty} \mathcal{L}$.

Using this representation we can see that matrixially quasinormal tuples are subnormal, and if they are pure, we can find the structure of its minimal normal extension.

Proposition 2.2 ([3]). *If $T = (T_1, \dots, T_d)$ is a pure matrixially quasinormal tuple, then T is subnormal with normal extension, $N = (N_1, \dots, N_d)$ of the form*

$$N_i = \begin{pmatrix} T_i^* & 0 \\ C_i & T_i \end{pmatrix}$$

on $\mathcal{H} \oplus \mathcal{H}$ where $C_i = ([T_i^*, T_i])^{\frac{1}{2}}$. Furthermore, N is the minimal normal extension of T if and only if T is pure.

Proof. Let $T = (T_1, \dots, T_d)$ be a matricially quasinormal tuple. Let

$$N_i = \begin{pmatrix} T_i^* & 0 \\ C_i & T_i \end{pmatrix}$$

where $C_i = ([T_i^*, T_i])^{\frac{1}{2}}$. To check that each N_i is normal, examine the self-commutator.

$$\begin{aligned} [N_i^*, N_i] &= N_i^* N_i - N_i N_i^* \\ &= \begin{pmatrix} T_i & C_i \\ 0 & T_i^* \end{pmatrix} \begin{pmatrix} T_i^* & 0 \\ C_i & T_i \end{pmatrix} - \begin{pmatrix} T_i^* & 0 \\ C_i & T_i \end{pmatrix} \begin{pmatrix} T_i & C_i \\ 0 & T_i^* \end{pmatrix} \\ &= \begin{pmatrix} C_i^2 - [T_i^*, T_i] & C_i T_i - T_i^* C_i \\ T_i^* C_i - C_i T_i & [T_i^*, T_i] - C_i^2 \end{pmatrix} = 0 \end{aligned}$$

since $C_i = ([T_i^*, T_i])^{\frac{1}{2}}$ and $C_i T_i = ([T_i^*, T_i])^{\frac{1}{2}} T_i = T_i^* ([T_i^*, T_i])^{\frac{1}{2}} = T_i^* C_i$.
 N is a commuting tuple since

$$\begin{aligned} N_i N_j - N_j N_i &= \begin{pmatrix} T_i^* & 0 \\ C_i & T_i \end{pmatrix} \begin{pmatrix} T_j^* & 0 \\ C_j & T_j \end{pmatrix} - \begin{pmatrix} T_j^* & 0 \\ C_j & T_j \end{pmatrix} \begin{pmatrix} T_i^* & 0 \\ C_i & T_i \end{pmatrix} \\ &= \begin{pmatrix} T_i^* T_j^* & 0 \\ C_i T_j^* + T_i C_j & T_i T_j \end{pmatrix} - \begin{pmatrix} T_j^* T_i^* & 0 \\ C_j T_i^* + T_j C_i & T_j T_i \end{pmatrix} = 0. \end{aligned}$$

Thus, since T is pure $N = (N_1, \dots, N_d)$ is the minimum normal extension. $\square$

Since A_i commutes with A_j for all j , we see that if T is a matricially quasinormal tuple, then $T = U_+ \otimes A$ where U_+ is the unilateral shift of multiplicity 1 and $A = (A_1, \dots, A_d)$ is a commuting tuple of positive operators.

Definition 2.3. A tuple $A = (A_1, \dots, A_d)$ on a Hilbert space $\mathcal{H}$ is called *cyclic* if there is a vector e in $\mathcal{H}$ such that the linear span of $\{A^\alpha e : \alpha \in \mathbb{Z}_+^d\}$ is dense in $\mathcal{H}$.

Proposition 2.4. A pure matricially quasinormal tuple $T = (T_1, \dots, T_d)$ on $\mathcal{H}$ is unitarily equivalent to $U_+ \otimes A$, where $A = (A_1, \dots, A_d)$ is a positive cyclic tuple on a Hilbert space $\mathcal{L}$, if and only if there is a vector e in $\mathcal{H}$ such that $\mathcal{H}$ is the closed linear span of $\{|T|^\alpha T^\beta e : \alpha, \beta \in \mathbb{Z}_+^d\}$.

Proof. Assume there exists an $e \in \mathcal{H}$ such that $\mathcal{H} = \vee \{|T|^\alpha T^\beta e : \alpha, \beta \in \mathbb{Z}_+^d\}$. It is sufficient to say that $\mathcal{H} = \mathcal{L}^{(\infty)}$ and $T_i = U_+ \otimes A_i$. If such a vector e exists, let $e = (e_0, e_1, \dots)$, with e_i in $\mathcal{L}$. Note that

$$|T| = |(T_1, \dots, T_d)| = (|T_1|, \dots, |T_d|)$$

and each

$$|T_i| = \begin{pmatrix} A_i & 0 & & \\ 0 & A_i & & \\ & & \ddots & \\ & & & \ddots \end{pmatrix}.$$

For $|\beta| \geq 1$, the zeroth coordinate of $|T|^\alpha T^\beta e = |T_1|^{\alpha_1} \dots |T_d|^{\alpha_d} T_1^{\beta_1} \dots T_d^{\beta_d} e$ is 0. However, the zeroth coordinate of $|T|^\alpha e$ is $A^\alpha e_0$. Thus, e_0 is a cyclic vector for A .

Now assume e_0 is a cyclic vector for A and let $e = (e_0, 0, 0, \dots)$ in $\mathcal{L}^{(\infty)}$. The for every $|\alpha|, |\beta| \geq 0$, every coordinate of $|T|^\alpha T^\beta e$ is zero except for the $|\beta|$ -th coordinate, which is $A_1^{\alpha_1} \dots A_d^{\alpha_d} A_1^{\beta_1} \dots A_d^{\beta_d} e_0 = A_1^{\alpha_1 + \beta_1} \dots A_d^{\alpha_d + \beta_d} e_0 = A^{\alpha + \beta} e_0$.

To show that A has dense range, fix $j \geq 1$. Let $f \in \mathcal{L}$ such that for all $\alpha \in \mathbb{Z}_+^d$

$$\begin{aligned} 0 &= \langle f, A^\alpha (\Sigma_{i=1}^d A_i^j e_0) \rangle = \langle f, \Sigma_{i=1}^d A^\alpha A_i^j e_0 \rangle \\ &= \Sigma_{i=1}^d \langle f, A^\alpha A_i^j e_0 \rangle = \Sigma_{i=1}^d \langle f, A_i^j A^\alpha e_0 \rangle \\ &= \Sigma_{i=1}^d \langle A_i^j f, A^\alpha e_0 \rangle = \langle \Sigma_{i=1}^d A_i^j f, A^\alpha e_0 \rangle \end{aligned}$$

As a result, $\Sigma_{i=1}^d A_i^j f = 0$ and $\langle \Sigma_{i=1}^d A_i^j f, f \rangle = 0$. Thus, $A_i^j f = 0$ for all $i = 1, \dots, d$ and $f \in \bigcap_{i=1}^d \ker A_i = \{0\}$.

Thus, $\mathcal{H}$ is the closed linear span of $\{|T|^\alpha T^\beta e : \alpha, \beta \in \mathbb{Z}_+^d\}$. $\square$

For a matricially quasinormal tuple $T = (T_1, \dots, T_d)$, let $\mathfrak{R}(T)$ be the norm closed algebra generated by $\{T_1, \dots, T_d\}$ and $\{|T_1|, \dots, |T_d|\}$. T is $\mathfrak{R}(T)$ -cyclic if there is a vector e such that $\{Re : R \in \mathfrak{R}(T)\}$ is dense in $\mathcal{H}$. By the previous proposition, T is $\mathfrak{R}(T)$ -cyclic if and only if A is cyclic.

If for $i = 1, \dots, d$, ν is a positive Borel measure on $[0, \infty)^d$, $\tilde{\nu}$ will denote the measure on $\mathbb{C}^d$ defined by

$$d\tilde{\nu}(r_1 e^{i\theta_1}, \dots, r_d e^{i\theta_d}) = \frac{1}{(2\pi)^d} d\theta_1 \dots d\theta_d d\nu(r).$$

This brings us to our main result.

Theorem 2.5. *If $T = (T_1, \dots, T_d)$ is a pure matricially quasinormal tuple that is $\mathfrak{R}(T)$ -cyclic, then there is a measure ν on $K = [0, \|T_1\|] \times [0, \|T_2\|] \times \dots \times [0, \|T_d\|]$ such that, if $\mathfrak{R}$ is the closed linear span of $\{|z|^\alpha z^\beta : \alpha, \beta \in \mathbb{Z}_+^d\}$ in $L^2(\tilde{\nu})$, then, for each $i = 1, \dots, d$, T_i is unitarily equivalent to multiplication by z_i on $\mathfrak{R}$.*

Proof. Assume that $\|T\| = 1$, $\mathcal{H} = \mathcal{L}^{(\infty)}$, and $T_i = U_+ \otimes A_i$. T is $\mathfrak{R}(T)$ -cyclic if there is a vector e such that $\{Re : R \in \mathfrak{R}(T)\}$ is dense in $\mathcal{H}$. Since T is $\mathfrak{R}(T)$ -cyclic, then A is a positive cyclic tuple. Since T is a matricially quasinormal tuple, each A_i is self-adjoint. Thus, $\cap \ker A_i = 0$, and there exists a measure ν on K such that there exists unitary U with $A_i = U^* M r_i U$ for all i . Also, note that $\nu(\{0\}) = 0$.

For all $\beta \in \mathbb{Z}_+^d$, let $\mathfrak{R}_\beta$ be the closed linear span of $\{|z|^\alpha z^\beta : \alpha \in \mathbb{Z}_+^d\}$ in $\mathcal{L}^2(\tilde{\nu})$. Each $\mathfrak{R}_\beta$ is orthogonal since

$$\langle |z|^\alpha z^\beta, |z|^\gamma z^\tau \rangle_{L^2(\tilde{\nu})} = \int_{|z_1| \leq a_1, \dots, |z_d| \leq a_d} |z|^\alpha z^\beta |z|^\gamma z^{-\tau} d\tilde{\nu}(z)$$

$$= \frac{1}{(2\pi)^d} \int r^{\alpha+\beta+\gamma+\tau} (e^{i\beta_1\theta_1} \dots e^{i\beta_d\theta_d}) (e^{-i\gamma_1\theta_1} \dots e^{-i\gamma_d\theta_d}) d\nu d(\theta)^{n \geq 1} = 0$$

if $\beta = \tau$.

For $f \in \mathcal{L}^2(\nu)$ and $\beta \in \mathbb{Z}_+^d$ define g on $\mathbb{C}^d \setminus 0$ by

$$g(z) = |z|^{-\beta} z^\beta f(|z|).$$

Since $\tilde{\nu}(\{0\}) = 0$, this defines g a.e. Also,

$$\|g\|^2 = \int |g|^2 d\tilde{\nu} = \int |f|^2 d\nu = \|f\|^2$$

Define $V_\beta : \mathcal{L}^2(\nu) \rightarrow \mathcal{L}^2(\tilde{\nu})$ by

$$(V_\beta f)(z) = |z|^{-\beta} z^\beta f(|z|).$$

Since $\|V_\beta f\|^2 = \|f\|^2$, V_β is an isometry.

Let $\beta \in \mathbb{Z}_+^d$ be fixed. Let $a_\beta = \{p \in \mathbb{C}[r] : p(r) = r^\beta q(r) \text{ for some } q \in \mathbb{C}[r]\}$ where $r = (r_1, \dots, r_d)$.

Then by the Stone Weierstrass Theorem, any continuous function on K can be uniformly approximated by polynomials in a_β which vanish at the origin. Thus, we have that a_β is dense in $\mathcal{L}^2(\nu)$.

Let $p \in a_\beta$, then $V_\beta p \in \mathfrak{R}_\beta$. So $V_\beta : \mathcal{L}^2(\nu) \rightarrow \mathfrak{R}_\beta$.

On the other hand, if $g \in \mathfrak{R}_\beta$, then there is a sequence of polynomials $\{p_n\}$ such that $\int p_n(|z|) z^\beta - g(z)^2 d\tilde{\nu}(z) \rightarrow 0$ as $n \rightarrow \infty$. But

$$\int_K |r^\beta p_n(r) - r^\beta p_m(r)|^2 d\nu(r) = \int |z^\beta p_n(z) - z^\beta p_m(z)|^2 d\tilde{\nu}(z) \rightarrow 0.$$

So $\{r^\beta p_n\}$ is Cauchy in $\mathcal{L}^2(\nu)$; let f be the limit of the sequence $\{r^\beta p_n\}$ in $\mathcal{L}^2(\nu)$. But $V_\beta(r^\beta p_n) = z^\beta p_n$; since V_β is continuous, $V_\beta(f) = g$. So V_β is an isometry of $\mathcal{L}^2(\nu)$ onto $\mathfrak{R}_\beta$.

For each β , $V_\beta : \mathcal{L}^2(\nu) \rightarrow \mathfrak{R}_\beta$ is unitary. Since $\mathcal{H} = \oplus_{\beta \in \mathbb{Z}_+^d} \mathcal{L} = \mathcal{L}^{(\infty)}$ and $V : \mathcal{H} \rightarrow \mathfrak{R}$ by $V = \oplus_{\beta \in \mathbb{Z}_+^d} V_\beta$. Let $M = (M_{z_1}, M_{z_2}, \dots, M_{z_d})$ on $\mathfrak{R}$. If $\alpha, \beta \in \mathbb{Z}_+^d$, then $|z|^\alpha z^\beta \in \mathfrak{R}_\beta$. $V^{-1}(|z|^\alpha z^\beta) = V_\beta^{-1}(|z|^\alpha z^\beta) = r^{\alpha+\beta} \in L^2(\nu)_\beta$. Thus, $(\oplus_{\beta \in \mathbb{Z}_+^d} U) T_i (\oplus_{\beta \in \mathbb{Z}_+^d} U^*) V^{-1}(|z|^\alpha z^\beta) = (\oplus_{\beta \in \mathbb{Z}_+^d} U) T_i (\oplus_{\beta \in \mathbb{Z}_+^d} U^*) (r^{\alpha+\beta}) = M_{r_i}(r^{\alpha+\beta}) = r^{\alpha+\beta} \cdot r_i \in L^2(\nu)_{\beta+e_i}$. Hence,

$$\begin{aligned} V(\oplus U) T_i (\oplus U^*) V^* (|z|^\alpha z^\beta) &= V(\oplus U) T_i (\oplus U^*) (r^{\alpha+\beta}) = V(r^{\alpha+\beta} \cdot r_i) \\ &= |z|^\alpha z^{\beta+e_i} = M_{z_i}(|z|^\alpha z^\beta). \end{aligned}$$

Thus, $M_{z_i} = V(\oplus U) T_i (\oplus U^*) V^*$. □

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