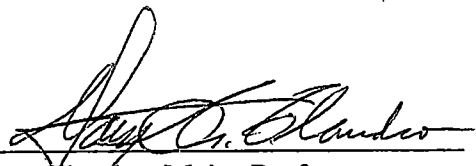
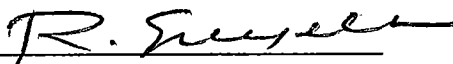


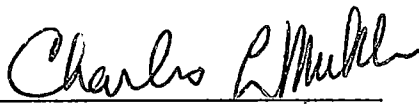
To the Graduate Council:

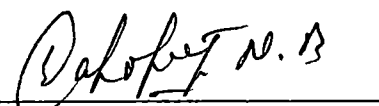
I am submitting herewith a dissertation written by Christel Brigitte van de Kolk entitled "Complex Shaped Solar Sails: A Study of the Coupled Attitude and Trajectory Dynamics". I have examined the final copy of this dissertation for form and content and recommend that it be accepted in partial fulfillment of the requirements for the degree of Doctor of Philosophy, with a major in Aerospace Engineering.


Gary Flandro, Major Professor

We have read this dissertation
and recommend its acceptance:


Remi Engels


Charles Merkle


Narendra Dahotre

Accepted for the Council:


Associate Vice Chancellor and
Dean of the Graduate School

COMPLEX SHAPED SOLAR SAILS:
A STUDY OF THE COUPLED ATTITUDE
AND TRAJECTORY DYNAMICS

A Dissertation
Presented for the
Doctor in Philosophy
Degree
The University of Tennessee, Knoxville

Christel Brigitte van de Kolk
May 2000

Copyright © 2000 by Christel Brigitte van de Kolk

All rights reserved

Acknowledgments

I would like to thank everybody who contributed to the completion of this work in one way or another.

Special thanks goes to my advisor, Dr. Gary Flandro, who introduced me to the topic of solar sailing and to the students who worked on this topic before me, Dan Thomas, Allen McCarley, Brian Funk, Martin Langenbacher. Their theses and computer codes were very helpful to me.

Special thanks also to the librarians Mary Lo, Brenda Brooks and Marge Joseph. They always managed to locate the papers and books I requested.

Last but not least I would like to thank my parents. Without their support I would not have made it this far.

Abstract

The concept of solar sailing is to reflect sunlight of a large surface in space to generate a low thrust, but constantly present, force. By varying the angle between sail normal and incident sunlight, a solar sail can fly inward (to the sun) and outward. The lifetime of a solar sail is not limited by the amount of fuel it can carry, since its fuel is sunlight. Degradation of the reflective surface, due to micro-meteorite impact, etc, is the main limiting factor for the lifetime, but it is safe to say that the lifetime of a solar sail will be decades. This extensive lifetime and the possibility of inward and outward travel within the solar system, make solar sails good candidates for cargo missions within our solar system. They are less useful for manned flight, due to the long flight times of solar sail propelled vehicles. For example, the flight time from Earth to Mars is about a year.

The solar sails studied previously were flat, single surfaces. The required stiffness and rigidity are provided by either a supporting structure or by spinning the sail. What both concepts have in common is that it was assumed that the attitude dynamics and trajectory dynamics were uncoupled. This assumption eliminates an entire family of promising flight modes in which the coupled motion provides automatic passive attitude dynamics and control.

The research presented here will focus on the development of a full 3 dimensional model for the coupled attitude/trajectory dynamics problem of a complex shaped solar sail. This model will then be used to investigate the possible trajectory types and the stability of the attitude dynamics. It will be shown that it is possible to fly either inward to the Sun or out away from the Sun, depending on the dimensions of the individual sail planes and the angle between the two sail planes. However, passive attitude stability for all three axes will not be possible. The roll motion about the sail-sun line is unstable and some form of active control will be needed here.

Table of Contents

1	Introduction	1
1.1	The principle behind solar sailing	1
1.2	Mission types of a solar sail spacecraft	4
1.3	Complex shaped sails	5
1.4	Structure of report	7
2	Literature Review	9
2.1	The first designs	9
2.1.1	Garwin	9
2.1.2	McNeal	11
2.1.3	Wright	21
2.2	Comet Halley rendezvous mission	22
2.2.1	Spar supported sail	22
2.2.2	Spinning disk sail	25
2.2.3	Heliogyro	27
2.2.4	Sail material and coatings	30
2.3	The World Space Foundation and U3P	30
2.4	The past decade	34
3	The Attitude Dynamics and Trajectory Dynamics Equations	38
3.1	The geometry of the sail	38
3.2	Moments of inertia	44
3.2.1	Contribution from plane 1	48
3.2.2	Contribution from plane 2	48
3.2.3	Contribution from central bus	49
3.2.4	Total moments and products:	49
3.3	Solar sail forces and moments	50
3.3.1	Direct radiation terms	51
3.3.1.1	Plane 2 shadows plane 1	52
3.3.1.2	Plane 1 shadows plane 2	57
3.3.1.3	Moments due to direct radiation	59
3.3.2	Indirect radiation terms	59

3.3.2.1 From plane 1 to plane 2	61
3.3.2.2 From plane 2 to plane 1	63
3.4 Gravity gradient moments	64
3.5 Attitude dynamics	67
3.5.1 Angular rotation vector	68
3.5.2 Quaternion formulation	70
3.6 Flight trajectory	75
3.6.1 Generalized coordinate r	75
3.6.2 Generalized coordinate ϕ	76
3.6.3 Generalized coordinate ψ	76
3.6.4 Dimensionless trajectory equations	77
4 The Coplanar Case	79
4.1 Attitude dynamics and trajectory equations	79
4.2 Equilibrium points	81
4.3 Trajectory types	111
5 The 3D Case	142
5.1 Comparison with the coplanar model	142
5.2 Non-coplanar trajectories	147
5.3 Stability	155
6 Conclusions and Recommendations	165
6.1 Conclusions	165
6.2 Recommendations	166
Bibliography	167
Appendices	170
A Quaternion Algebra	171
A.1 Basic definitions	171
A.2 Definition of quaternion product	172
A.3 Transformation matrices	172
A.4 Solving the quaternion problem of chapter 3	174
Vita	176

List of Tables

4.1: Accelerations at 1 AU for $k = 0.0$, $h = 24.23 \mu\text{m}$, $l_3 = 5.0 \text{ m}$	114
4.2: Accelerations at 1 AU for $k = 0.4$, $h = 24.23 \mu\text{m}$, $l_3 = 5.0 \text{ m}$	114
4.3: Accelerations at 1 AU for $k = 0.0$, $h = 24.23 \mu\text{m}$, $l_3 = -5.0 \text{ m}$	122
4.4: Accelerations at 1 AU for $k = 0.4$, $h = 24.23 \mu\text{m}$, $l_3 = -5.0 \text{ m}$	122
4.5: Accelerations at 1 AU for $k = 0.0$, $h = 7.5 \mu\text{m}$, $l_3 = -5.0 \text{ m}$	125
4.6: Accelerations at 1 AU for $k = 0.0$, $h = 7.5 \mu\text{m}$, $l_3 = -15.0 \text{ m}$	125
4.7: Accelerations at 1 AU for $k = 0.0$, $h = 24.23 \mu\text{m}$, $l_2 = 25.0 \text{ m}$, $l_3 = 15.0 \text{ m}$	129
4.8: Accelerations at 1 AU for $k = 0.0$, $h = 24.23 \mu\text{m}$, $l_2 = 40.0 \text{ m}$, $l_3 = -15.0 \text{ m}$	129
4.9: Accelerations at 1 AU for $k = 0.0$, $h = 7.5 \mu\text{m}$, $l_2 = 25.0 \text{ m}$, $l_3 = 15.0 \text{ m}$, $\alpha = 220^\circ$	137
4.10: Accelerations at 1 AU for $k = 0.0$, $h = 7.5 \mu\text{m}$, $l_2 = 25.0 \text{ m}$, $l_3 = 15.0 \text{ m}$, $\alpha = 270^\circ$	137
4.11: Accelerations at 1 AU for $k = 0.0$, $h = 7.5 \mu\text{m}$, $l_2 = 25.0 \text{ m}$, $l_3 = 15.0 \text{ m}$, $\alpha = 320^\circ$	138
4.12: Accelerations at 1 AU for $k = 0.0$, $h = 7.5 \mu\text{m}$, $l_2 = 40.0 \text{ m}$, $l_3 = -15.0 \text{ m}$, $\alpha = 220^\circ$	138

4.13: Accelerations at 1 AU for $k = 0.0$, $h = 7.5 \mu\text{m}$, $l_2 = 40.0 \text{ m}$, $l_3 = -15.0 \text{ m}$, $\alpha = 270^\circ$	
.....	139
4.14: Accelerations at 1 AU for $k = 0.0$, $h = 7.5 \mu\text{m}$, $l_2 = 40.0 \text{ m}$, $l_3 = -15.0 \text{ m}$, $\alpha = 320^\circ$	
.....	139

List of Figures

1.1: The force vector of an ideal sail.	3
1.2: The range of potential solar sailing applications.	6
1.3: Sailing ships can maneuver in toward the Sun and outward from the Sun by setting the sail angle.	6
1.4: Sketch of the two-plane solar sail used by Kirpichnikov <i>et al</i>	8
2.1: Garwin's solar sail design.	10
2.2: Use of the solar sail to increase the altitude of a satellite to escape Earth.	12
2.3: Sketch of the two blade heliogyro.	14
2.4: Deployment sequence for the two blade heliogyro.	15
2.5: Control responses for the two blade heliogyro.	17
2.6: The pinwheel heliogyro.	18
2.7: The parallel blade heliogyro.	20
2.8: The spar supported sail.	24
2.9: The disk sailer configuration.	26
2.10: Disk sailer central region, perspective view.	26
2.11: The Heliogyro.	28
2.12: Heliogyro blade attachment details.	29

2.13: Baseline blade design.	31
2.14: Solar sail vehicle mockup with two control vanes deployed, sail is still stowed. .	33
2.15: The deployed Russian solar sail.	35
2.16: Sail design for the ODISSEE mission.	37
2.17: Spar design for the ODISSEE mission.	37
3.1: Two plane solar sail	40
3.2: The sail orientation in spherical coordinates	40
3.3: Center of mass	43
3.4: Calculation of moments of inertia of sail planes	45
3.5: Calculation of moments of inertia for central bus.	45
3.6: The reference frames	47
3.7: Coordinates of tip in rotated frame	54
3.8: Location of projection of tip plane 2	54
3.9: Projected area plane 2	56
3.10: Location of projection of tip plane 1	58
3.11: Moments due to the direct radiation	60
3.12: Calculation of reflected radiation vector	60
3.13: Flow chart of the calculation of the sail forces.	65
3.14: Definition of the quaternion rotation angles	72
4.1: Equilibrium values of θ as function of α , $m_p = 0.0$ kg, $k = 0.0$	82
4.2: α and θ ranges for which the two sail planes reflect	82
4.3: Equilibrium values for $k = 0.0$ m, payload is pointmass	87

4.4: Equilibrium values for $k=0.4$, payload is pointmass	89
4.5: Equilibrium values for $m_p = 0.0$ kg, payload is pointmass	91
4.6: Equilibrium values for $m_p = 5.0$ kg, payload is pointmass	92
4.7: Equilibrium values for $m_p = 25.0$ kg, payload is pointmass	93
4.8: Equilibrium values for $m_p=50.0$ kg, payload is pointmass	94
4.9: Equilibrium values for $m_p=100.0$ kg, payload is pointmass	95
4.10: Equilibrium values for $k = 0.0$, payload has shape	96
4.11: Equilibrium values for $k = 0.4$, payload has shape	98
4.12: Equilibrium values for $m_p = 0.0$ kg, payload has shape	100
4.13: Equilibrium values for $m_p = 5.0$ kg, payload has shape	101
4.14: Equilibrium values for $m_p = 25.0$ kg, payload has shape	102
4.15: Equilibrium values for $m_p = 50.0$ kg, payload has shape	103
4.16: Equilibrium values for $m_p = 100.0$ kg, payload has shape	104
4.17: Equilibrium values for $m_p = 0.0$ kg.	105
4.18: Equilibrium values for $m_p = 5.0$ kg	106
4.19: Equilibrium values for $m_p = 25.0$ kg.	107
4.20: Equilibrium values for $m_p = 50.0$ kg.	108
4.21: Equilibrium values for $m_p = 100.0$ kg.	109
4.22: Radial position as function of time for $h = 24.23 \mu\text{m}$, $l_3 = 5.0$ m	112
4.23: Radial position as function of time for $h = 7.5 \mu\text{m}$, $l_3 = 5.0$ m	115
4.24: Radial position as function of time for $h = 24.23 \mu\text{m}$, $l_3 = 15.0$ m	117
4.25: Radial position as function of time for $h = 7.5 \mu\text{m}$, $l_3 = 15.0$ m	118

4.26: Radial position as function of time for $h = 24.23 \mu\text{m}$, $l_3 = -5.0 \text{ m}$	119
4.27: Radial position as function of time for $h = 7.5 \mu\text{m}$, $l_3 = -5.0 \text{ m}$	120
4.28: Radial position as function of time for $h = 24.23 \mu\text{m}$, $l_3 = -15.0 \text{ m}$	123
4.29: Radial position as function of time for $h = 7.5 \mu\text{m}$, $l_3 = -15.0 \text{ m}$	124
4.30: Radial position as function of time for $h = 24.23 \mu\text{m}$, $l_2 = 40.0 \text{ m}$, $l_3 = -15.0 \text{ m}$	126
4.31: Radial position as function of time for $h = 7.5 \mu\text{m}$, $l_2 = 40.0 \text{ m}$, $l_3 = -15.0 \text{ m}$...	127
4.32: Radial position as function of time and α for $l_3 = 15.0 \text{ m}$	130
4.33: Radial position as function of time and α for $l_3 = -15.0 \text{ m}$	134
4.34: Radial position as function of time for $\alpha = 320^\circ$	141
5.1: Comparison between coplanar model and 3d model	143
5.2: Influence of changing the initial values of ϕ and ψ	148
5.3: Radial position over time with perturbed initial angular velocities, no control. ...	156
5.4: Two plane solar sail with two small control planes	156
5.5: Influence of control planes on the perturbed initial angular velocities	160
5.6: Influence of control planes and thruster on the perturbed initial angular velocities	163

List of Symbols and Abbreviations

Symbols

a	Magnitude of sail acceleration
a_c	Characteristic acceleration
F	Solar sail force vector
GM	Gravitational constant of the sun
H	Angular momentum vector
I	Inertia tensor
I_{xx}	Moment of inertia about the x-axis
I_{yy}	Moment of inertia about the y-axis
I_{zz}	Moment of inertia about the z-axis
i	Direction vector of incident sunlight
k	Structural mass factor
L	Length of the central bus
l_1	Base length of sail planes
l_2	Height of sail plane 1
l_3	Difference in height between sail plane 1 and 2
m	Mass

M	Moment vector
M_{pl}	Mass of the payload
M_{sail}	Mass of the sail
n	Normal vector of a flat sail
P_{sol}	Solar radiation pressure
q_i	Quaternion element
q_i	Generalized coordinate
Q	Quaternion
Q_i	Generalized force
r	Distance from sail to the Sun
r_{Earth}, r_o	Radial distance from Earth to the Sun
R	Radius of central bus
R	Reflectivity parameter
S_o	Sail area
t'	Coordinate vector of sail tip in local rotated reference frame
T	Transformation matrix
T	Kinetic energy
x_0'	x coordinate of center of mass of the bus in local rotated frame
y_0'	y coordinate of center of mass of the bus in local rotated frame
x_1'	x coordinate of center of mass of sail plane 1 in local rotated frame
y_1'	y coordinate of center of mass of sail plane 1 in local rotated frame
x_2'	x coordinate of center of mass of sail plane 2 in local rotated frame

y_2'	y coordinate of center of mass of sail plane 2 in local rotated frame
x_{cm}'	x coordinate of center of mass of the system in local rotated frame
y_{cm}'	y coordinate of center of mass of the system in local rotated frame
α	angle between the two sail planes
α_x	angle between the two sail planes normal vectors
β	clock angle of the first sail pane
γ	angle between normal vector control plane and the symmetry plane
θ	cone angle of the first sail plane
ρ	density
σ	Sail loading
ω	Angular rotation vector
Ω	rotation angle about n_1

Subscripts

1	Plane 1
2	Plane 2
p	Projected or payload
s	Sail
g	Gravity

Abbreviations

AU Astronomical unit

DLR German Aerospace Research Establishment

ODISSEE Orbital Demonstration of an Innovative Solar Sail driven Expandable structure
Experiment

U3P Union Pour la Promotion de la Propulsion Photonique

WSF World Space Foundation

Chapter 1

Introduction

Solar sails generate a propulsive force by reflecting sun-emitted photons from large, passive reflectors. Therefore, no onboard power generation is needed, leading to considerable mass and cost savings. No fuel needs to be carried along and this, in turn, allows for long duration flights within our solar system, such as shipping cargo to and from Mars. Another advantage is that a solar sail is reusable. The disadvantage, however, is that large sails are needed. Jerome Wright's book (1992), *Space Sailing* and Colin R McInnes' book (1999) *Solar Sailing: Technology, Dynamics and Mission Applications*, give a good introduction to the concept of solar sailing.

1.1 The principle behind solar sailing

A solar sail propels itself through space by reflecting the photons emitted by the sun, and is an example of photonic propulsion. Since photons carry momentum, reflecting them will change this momentum and a resultant force will act against the reflective surface. This force is proportional to the intensity, or power, of the incident sunlight. The intensity of the

sunlight, in turn, is inversely proportional to the square of the distance of the sail to the sun. It is also proportional to the cosine of the angle between the sail and the incident sunlight. Figure 1.1 shows the force vector diagram for an ideal flat sail.

In general, the magnitude a of the reflective force can be written as (Grodzovskii, p. 316):

$$a = a_c \left(\frac{r_{Earth}}{r} \right)^2 (\mathbf{n} \cdot \mathbf{i})^2 \quad (1.1)$$

In this equation, a_c is the characteristic acceleration, r_{Earth} is the distance from the Earth to the Sun, r is the distance from the sail to the Sun, $\mathbf{n}$ is the outward normal unit vector of the sail and $\mathbf{i}$ is the unit vector in the direction of the incident sunlight. The characteristic acceleration a_c is the rate at which the sail would accelerate at the Earth's distance from the Sun if the sail were directly facing the Sun, and can be calculated by using the following formula:

$$a_c = \frac{P_{sol} S_o}{M_{sail} + M_{pl}} \quad (1.2)$$

Here, P_{sol} is the solar radiation pressure, S_o is the sail area, M_{sail} the total mass of the sail and the supporting structure and M_{pl} the mass of the payload. The total mass of the sail can be calculated by multiplying the density of the sail material with the sail area and the thickness. This shows that the density of the sail material is an important factor in the solar sail design.

An other parameter often used is the sail loading, σ . The sail loading is the ratio of the total mass of the sail and payload to the sail area, it is normally given in g/m^2 . The sail loading determines the acceleration a sail can experience.

PRESSURE

$$P = 9.126 \cos^2 \theta / R^2, \mu\text{N/m}^2$$

SAIL LOADING

$$\sigma = \text{Total Mass} / \text{Sail Area}, \text{g/m}^2$$

CHARACTERISTIC ACCELERATION

$$a_c = 8.12 / \sigma, \text{mm/s}^2$$

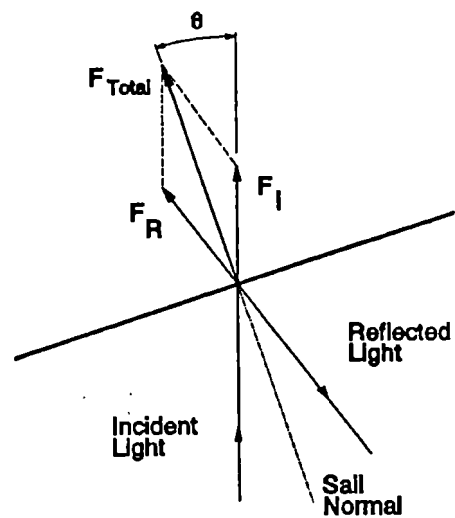


Figure 1.1: The force vector of an ideal sail.

Source: Wright, J.L., *Space Sailing*. Amsterdam, the Netherlands: Gordon and Breach Science Publishers, 1992.

Some photons will be scattered, some absorbed and some transmitted through the sail. Because of this, the sail performance will be about 90% of that of an ideal sail. The absorbed sunlight results in energy being deposited in the sail. This energy needs to be radiated back into space, in order to keep the sail at a steady temperature. This can be achieved by making the back side of the sail a radiator.

For a flat sail an attitude control system is needed to put the sail in the desired position for it to follow the trajectory to the desired destination. Since the sail will billow under influence of the radiation force, the pressure distribution will not be uniform across the sail and the control system is also needed for balancing the entire solar sail spacecraft. The control system also needs to control the pointing direction, which can be achieved by either shifting the center of mass of the entire spacecraft or by shifting the center of pressure on the sail.

Besides the photons, the solar wind also acts on the sail. However, this solar wind is orders of magnitude less than the radiation pressure and can therefore be neglected. A solar sail travels on waves of photons, not on the solar wind.

1.2 Mission types of a solar sail spacecraft

Solar sails can be used to reach every body in our solar system. They can be used to maneuver away and toward the sun. They can change the orientation of their orbital planes around the sun. The sails that travel to the inner planets of our solar system can return to Earth. They have the ability to enter and leave parking orbits around planets and small bodies

in our solar system. This is very well illustrated in figure 1.2. The trajectory the sail will follow depends on the departure and destination points, the characteristic acceleration of the sail and possibly also on the thermal capabilities of the sail and/or the payload.

The force that acts on the sail is radially outward from the sail. The component of the force perpendicular to the solar radius line will cause a lateral acceleration. The lateral component alters the angular momentum of the orbit. The total force affects the energy associated with the orbit. If the lateral component is against the direction of motion, angular momentum and energy will be lost, and the result is an inward spiral, see figure 1.3. Otherwise the sail will spiral out. The inclination of the orbit can be changed by pointing the lateral acceleration out of the plane of motion of the sail. Some amount of this type of control is needed in essentially every interplanetary transfer.

1.3 Complex shaped sails

The solar sails studied previously are flat, single surfaces and it was assumed that the attitude dynamics and trajectory dynamics were uncoupled. This assumption greatly simplified the solution of the flight dynamics problem. However, this eliminates an entire family of promising flight modes in which the coupled motion provides passive attitude stability and control.

Kirpichnikov *et al* (1996) carried out a detailed study of the coupled solar sail trajectory problem restricted to the coplanar case. The spacecraft configuration in this study was a sail

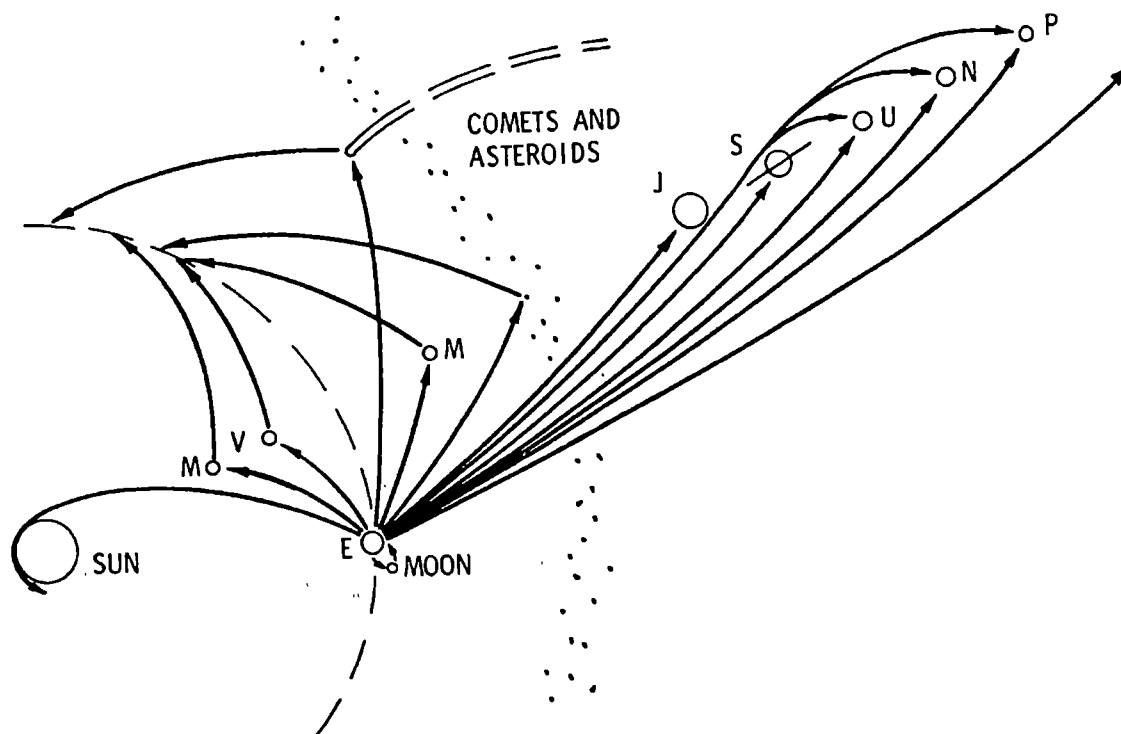


Figure 1.2: The range of potential solar sailing applications.

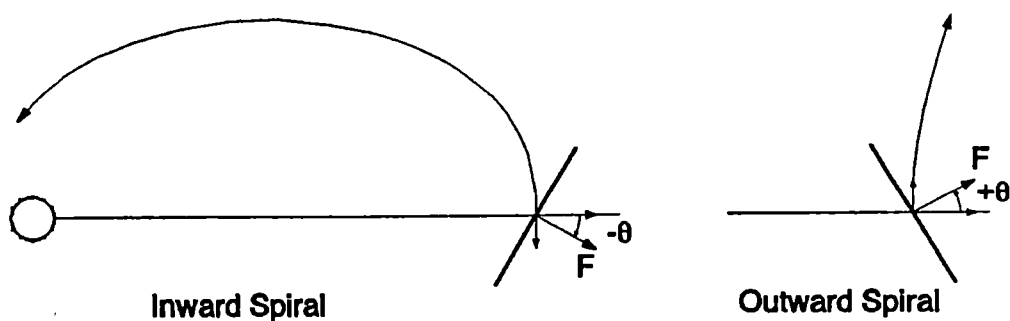


Figure 1.3: Sailing ships can maneuver in toward the Sun and outward from the Sun by setting the sail angle.

Source (figure 1.2 and 1.3): Wright, J.L., *Space Sailing*. Amsterdam, the Netherlands: Gordon and Breach Science Publishers, 1992.

consisting of two reflective planes of unequal size, oriented at each other rigidly at a 90° angular separation. A sketch of this two-plane sail is given in figure 1.4. The feasibility of interplanetary flight by means of an uncontrolled two-plane solar sail was demonstrated in this study.

The research presented here will focus on the development of a full 3 dimensional model for the coupled attitude/trajectory dynamics problem of a complex shaped solar sail. This model will then be used to investigate the possible trajectory types and the stability of the attitude dynamics.

1.4 Structure of report

The structure of this report is as follows. Chapter 2 gives a historical overview of past solar sail designs. In chapter 3 the equations for the attitude and trajectory dynamics will be developed. Since there are now two sail planes, positioned at an angle with each other, it will be possible that one sail plane is shadowing the other sail plane, or that the reflected light of one plane hits the other plane. This will be accounted for. Chapter 4 shows results for a coplanar case. These results show the stability positions of the sail and the corresponding trajectories for different dimensions of the two sail planes. Simulations of the trajectories by using the full 3D model will be done in chapter 5. In this chapter the flight dynamics of the spacecraft will be investigated. The conclusions and recommendations follow in chapter 6.

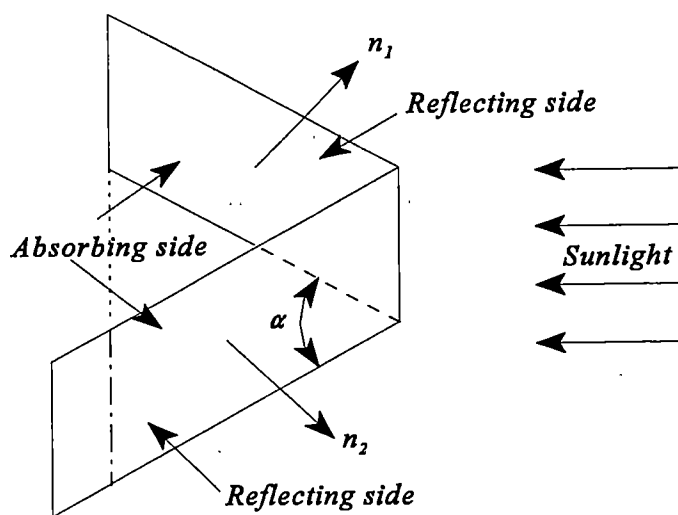


Figure 1.4: Sketch of the two-plane solar sail used by Kirpichnikov *et al*

Chapter 2

Literature Review

2.1 The first designs

The first time the use of a device like a solar sail was mentioned, was in 1924 by F. Tsander, possibly based on suggestions by K. Tsiolkovskii (1921). It was not until the fifties that the solar sailing concept appeared in the American literature (Garwin, 1958).

2.1.1 Garwin

The sail that Garwin (1958) proposed was a parachute-like sail. The sail is circular and is attached to the payload by "shroud lines". The sail and the shroud lines are made out of plastic with a thickness of 0.1 mil (1 mil = 10^{-6} inch). The sail itself may be aluminized to make the sail reflective, thus doubling the solar pressure force. It has no supporting structure, and the payload is trailing behind the sail. Figure 2.1 shows the concept of this sail.

Garwin proposed to vary the thrust of this sail by manipulating the shroud lines. This way, the thrust can be varied very smoothly from the maximum calculated thrust to almost zero.

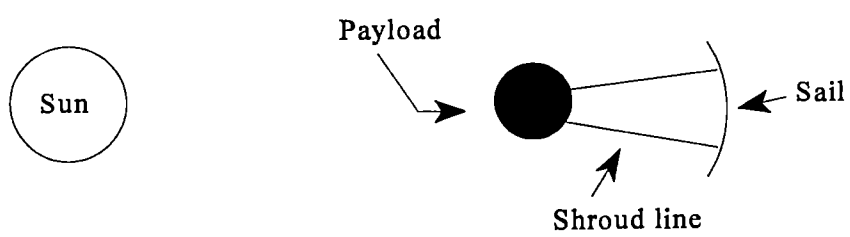


Figure 2.1: Garwin's solar sail design.

Also the thrust can be directed there will hardly be a loss of thrust when the angle between the incident sunlight and the normal vector of the sail is less than 45° . For larger angles, the thrust will change significantly.

According to Garwin, this sail can be used for raising the orbits of satellites in Earth orbit, including escape, and also for interplanetary travel. Orbit raising is achieved by furling the sail by slacking half the shroud lines when approaching the Sun and unfurling the sail when receding from the Sun, see figure 2.2.

Unlike a parachute, where there is a uniform pressure distribution inside, the pressure distribution inside this sail will not be uniform. The pressure will be very low at the edge, since the incidence angle there is large. As a result, there is not enough pressure force present to withstand the tension forces in the shroud lines and the sail will not be very open. Therefore, the actual radiation force will be significantly less than the maximum calculated force. This led to the notion that the sail structure would have to be rigidized. Two main concepts were developed to achieve this. One was the use of spars and wires, the other was to use centrifugal forces generated by spinning the sail. The second method led to the concept of the heliogyro (MacNeal, 1969).

2.1.2 McNeal

MacNeal investigated 3 different types of heliogyro's: the 2 blade heliogyro, the pinwheel heliogyro and the parallel blade heliogyro.

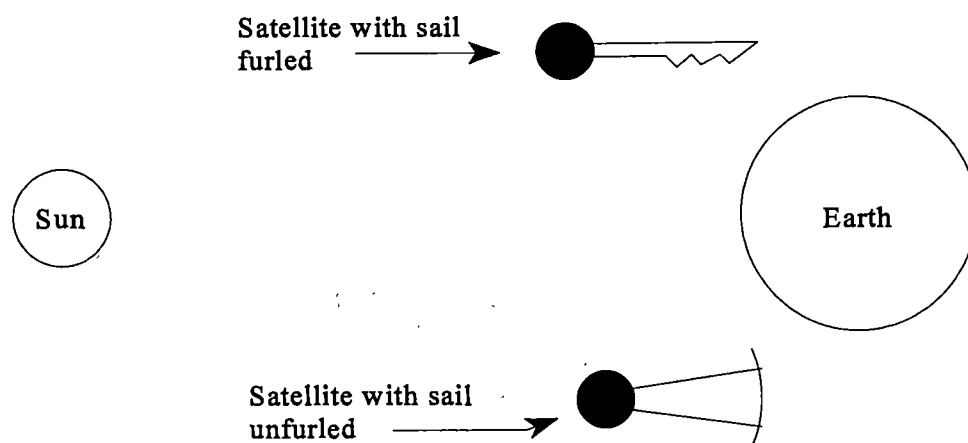


Figure 2.2: Use of the solar sail to increase the altitude of a satellite to escape Earth.

Source: Garwin, R.L. 'Solar Sailing-A Practical Method of Propulsion Within the Solar System'. *Jet Propulsion* March 1958, pp 188-190.

A sketch of the 2 blade heliogyro is given in figure 2.3. The main feature is the two narrow, long blades. The blades get their required stiffness by rotating the entire system, similar to the manner in which helicopter rotor blades get their stiffness. The blades are made from 1/4 mil Mylar with a 1500 Å layer of aluminum coating on each side. The blade tip radius is 18,600 ft and fully deployed it will have a rotational period of 6 minutes.

Deployment, as shown in figure 2.4, begins with spinning up the vehicle to 2 revolutions per second in order to develop initial tension in the blades. The blades are then being unrolled to about 300 ft. At this point the blades are pitched collectively. This generates a spin torque from the solar radiation pressure and supplies the added angular momentum needed as the blades gradually unroll. During flight, cyclic pitch can be used to produce a steady component of the solar radiation pressure force in the plane of the blades, see figure 2.5 a. Collective pitch produces a spin torque and reduces the component of the pressure force normal to the blade plane. A rolling moment is achieved by a combination of cyclic and collective pitch (figure 2.5 b).

The pinwheel heliogyro (figure 2.6) consists of an axisymmetric array of blades connected to a structural ring. There is a centrally located payload module that is attached to the ring by cables. The ring is divided into 24 segments which are separated by hinges. The hinges are mechanically actuated and kept in synchronism during deployment by a control cable within the ring. An electrical motor is located at the center of each ring segment to provide blade pitch. The blades are made from 1/20 mil Mylar with a 1500 Å aluminum coating on

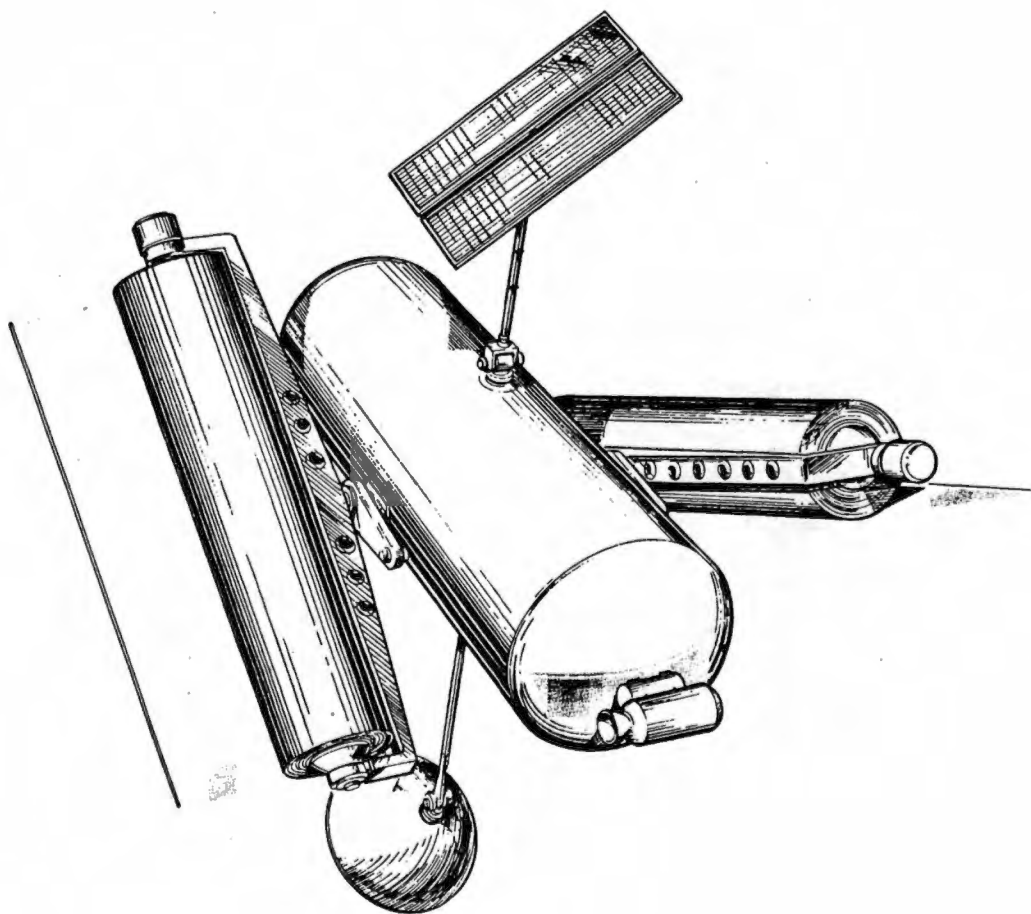


Figure 2.3: Sketch of the two blade heliogyro.

Source: MacNeal, R.H., J.M. Hedgepeth and H.U. Schuerch, *Heliogyro Solar Sailer Summary Report*. NASA CR-1329, June 1969.

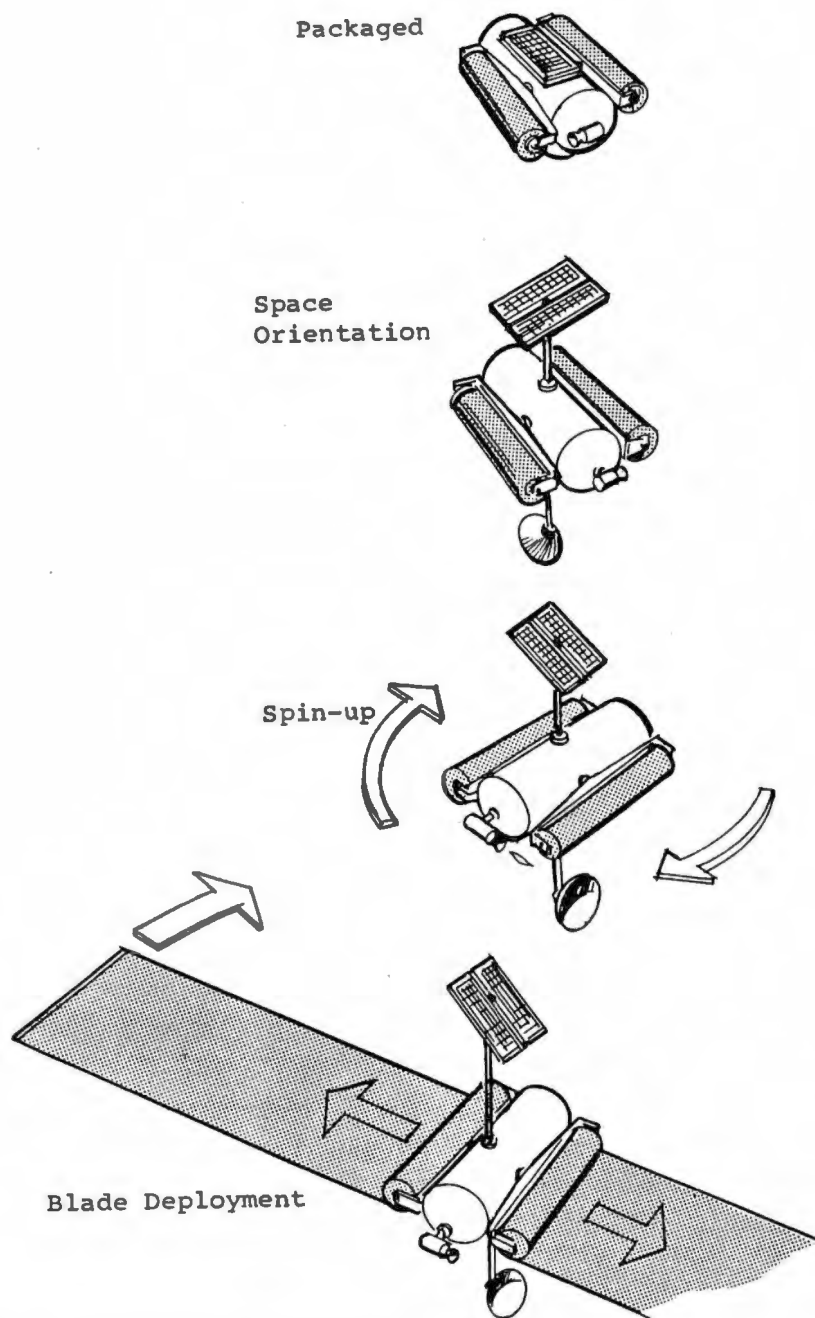


Figure 2.4: Deployment sequence for the two blade heliogyro.

Source: MacNeal, R.H., J.M. Hedgepeth and H.U. Schuerch, *Heliogyro Solar Sailer Summary Report*. NASA CR-1329, June 1969.

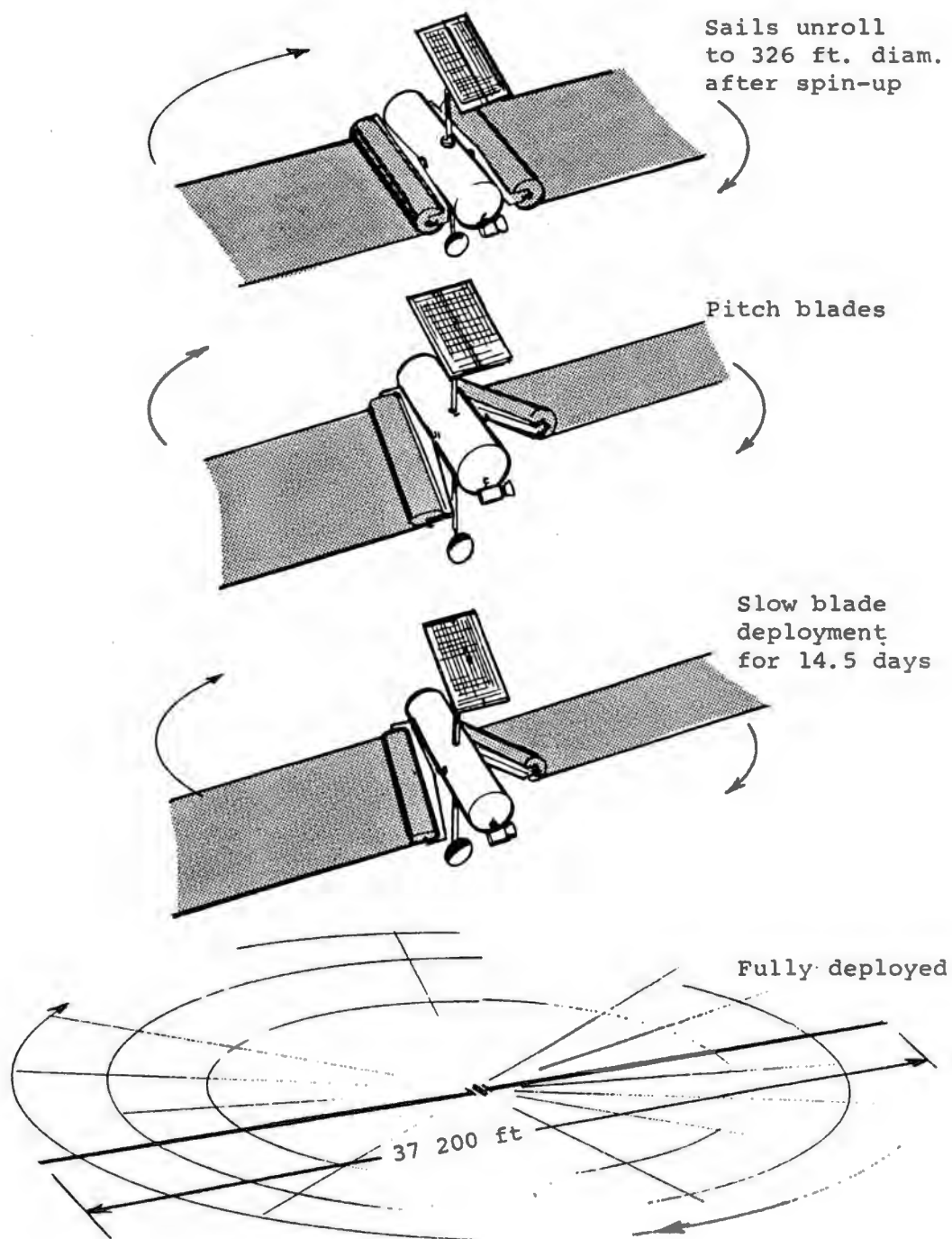
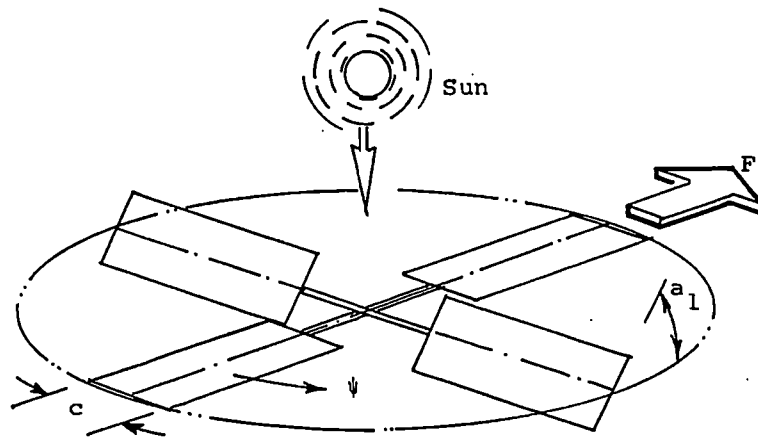
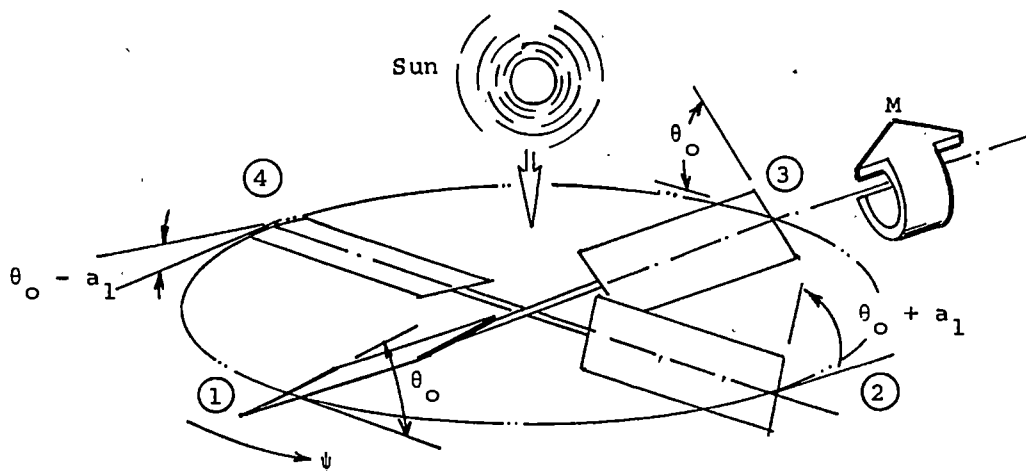


Figure 2.4 (continued)



a) Lateral force produced by cyclic pitch. $\theta = a_1 \sin\psi$



b) Rolling moment produced by combined collective and cyclic pitch. $\theta = \theta_0 + a_1 \sin\psi$

Figure 2.5: Control responses for the two blade heliogyro.

Source: MacNeal, R.H., J.M. Hedgepeth and H.U. Schuerch, *Heliogyro Solar Sailer Summary Report*. NASA CR-1329, June 1969.

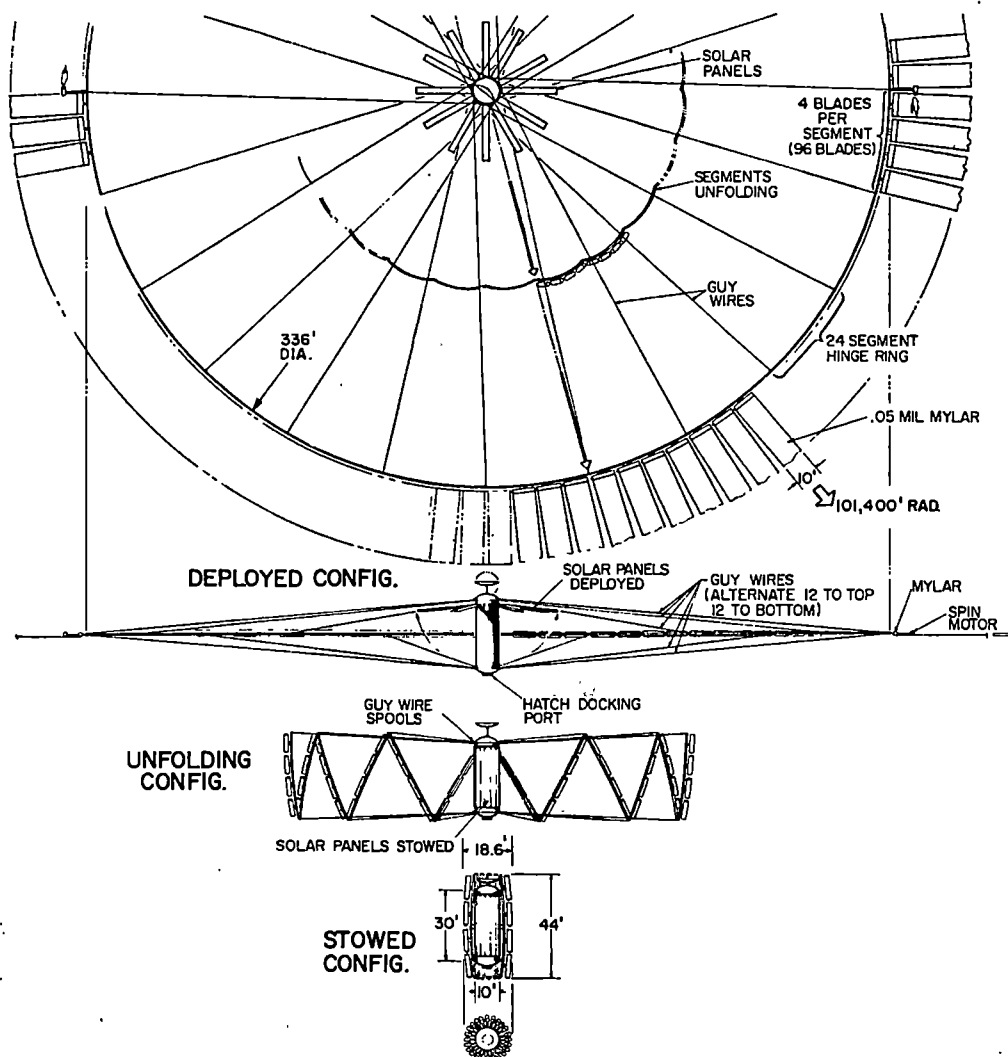


Figure 2.6: The pinwheel heliogyro.

Source: MacNeal, R.H., J.M. Hedgepeth and H.U. Schuerch, *Heliogyro Solar Sailer Summary Report*. NASA CR-1329, June 1969.

each side. The rotational period when fully deployed will be 18.4 min, the blade tip radius is 101,400 ft.

Final deployment from the unfolded ring condition is achieved in a way similar to that of the 2 blade heliogyro. Again rocket motors are used to spin up the system, the blades are unrolled to a predetermined radius, then pitched and gradually unrolled as the radiation pressure increases the angular momentum.

The parallel blade heliogyro is shown in figure 2.7. It consists of a number of parallel blades with two payload modules located half way from the center to blade tips. The pitch mechanism of the blades are slaved together by a control cable which is driven by an electric motor. The blades between the two capsules are not movable. All the blades are attached to booms that extend from the payload capsules and that are hinged to permit folding into the stowed configuration. The blades are made from 1/20 mil Mylar with an aluminum coating on each side of 1500 Å. The blade tip radius is 122,000 ft and the rotational period is 17.5 min.

To deploy this system, first the blade support booms are unfolded and rocket motors at the boom tips will spin up the system. The non-movable blades are then unrolled to a predetermined distance, after which the movable blades are unrolled a small distance and pitched. Finally, both sets of blades are unrolled gradually as the solar radiation pressure increases the angular momentum.

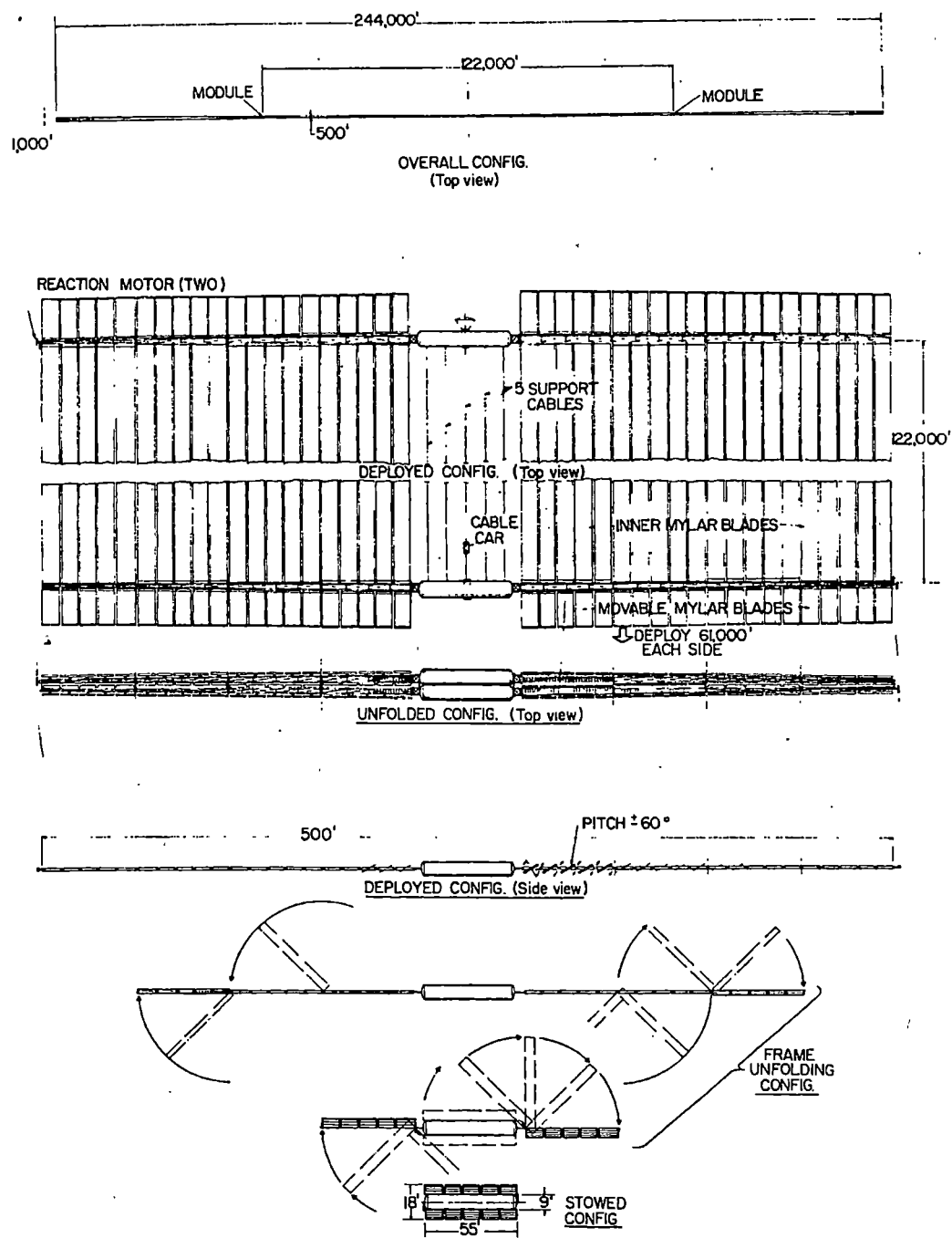


Figure 2.7: The parallel blade heliogyro.

Source: MacNeal, R.H., J.M. Hedgepeth and H.U. Schuerch, *Heliogyro Solar Sailer Summary Report*. NASA CR-1329, June 1969.

2.1.3 Wright

Wright (1974) did a study into the concept and potential of solar sailing. Some of the technical aspects he discussed included:

- Operating problems, like dynamics, stability and structural adequacy cannot be simulated in ground tests.
- Capability to develop the sail material is not an issue. However the sail needs to be rigidized.
- Sail control is a major problem. Conventional thrusters are undesirable due to propellant usage, but the use of pressure vanes might be an option.

The sail material he proposed was Mylar with a thickness of 0.10 mil with an aluminum coating of 1000 Å. Some of the concerns mentioned were erosion and aging resistance, tear resistance following puncture, damage risk during deployment, and fabrication considerations. These items may lead to a thicker Mylar being required, a thicker aluminum coating, reinforced with Mylar strips or a combination of these. The mass estimate was between 4 to 10 g/m².

Wright proposed a flat, fairly rigid sail, either in the form of a disk or a rectangle shape, possibly slightly bowed for stability. The sail would be controllable in both pitch, yaw and roll. Rigidizing would be achieved either by metal structural members or by inflation. The final selection of the shape and rigidizing technique requires a careful study of the structural dynamics of the sail, including sail size, sail deployment (automated or manually assisted), and type of launch vehicle. Previous estimates of the mass of the rigidizing structure was

10% of the total sail mass assembly, but a thorough study of the sail size, maneuvering requirements, and deployment might increase this number. Also thinner sails might not lead to a significant reduction of structural weight, since a minimum amount of support structure will always be required.

2.2 Comet Halley rendezvous mission

In 1976 Jerome Wright discovered a trajectory that showed the possibility for a solar sail spacecraft to rendezvous with Halley's comet during its 1986 passage. The discovery of this trajectory led to an intensive study at JPL. The two main sail configurations studied were the square sail, based on suggestions by Wright (1974), and the heliogyro sail (MacNeal, 1969). Other designs, such as spinning discs, inflatable structures, and parachute designs, were considered and set aside as less promising (Wright, 1976 and 1992, Friedman et al, 1978). The rendezvous trajectory required a "cranking" of the orbit, i.e., to change the inclination from the ecliptic to 162° , which is the plane of Halley's comet. Since the solar sail accelerations have a continuous r^{-2} behavior, it is desirable to approach the Sun as closely as possible for this inclination change.

2.2.1 Spar supported sail

A spar supported sail is a square sail which has spars behind the sail, running along the diagonals, which are used to support the sail and hold it in a relatively flat position even under the influence of solar pressure. To maintain adequate rigidity and at the same time minimize the weight of the spars, stays can be attached to the spars to help carry some of the

loads. This configuration can be seen in figure 2.8. The stays are lines running between pairs of spars and between the spars and the mast. The spars are attached to the mast at less than 90° angles. This approach keeps all of the spars and stays on the lee side of the sail, allowing unrestricted deployment of the sail and avoiding double illumination of the rigging by reflected light.

The square sail studied for the rendezvous mission was a 3-axis stabilized sail, approximately 800 m on each side. The control is provided by center of pressure and/or center of gravity shifts which are implemented by two means. The principal control is provided by moving the apex height which changes the sail shape and provides imbalance torques which are a function of the Sun cone angle. Additional control could be provided by sail translation which is enabled by sets of winches that can move the sail several meters in its plane. Roll control vanes were also designed.

The square sail is deployed by first erecting its structure, i.e. the six booms and associated inter-boom stays. These booms are truss type structures which are coiled up and rotated out of the canisters. After deployment of the structure, the sail is pulled out of canisters located at the center, first in one direction and then in the other, from an accordion fold packing arrangement. Although this deployment scheme was considered to be feasible, it was also regarded as inherently complicated with an unproven reliability. Especially since the sail deployment, the folding and unfolding, could not be tested or simulated on the ground prior to implementation in flight.

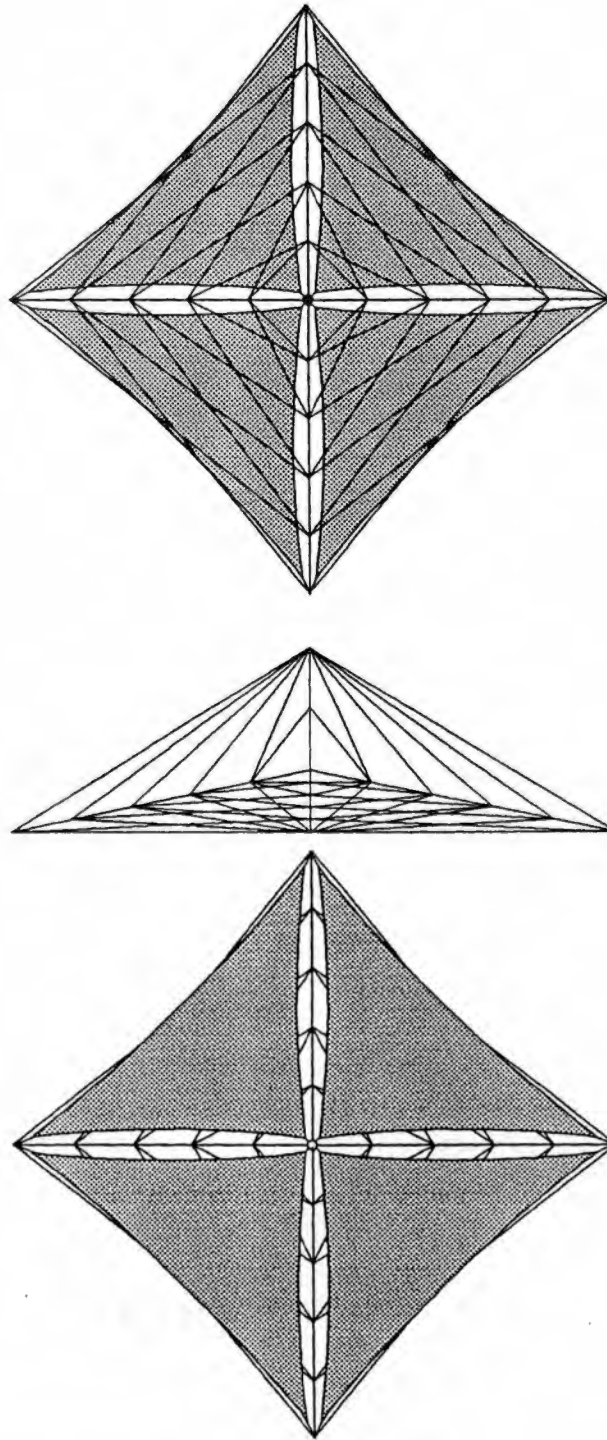


Figure 2.8: The spar supported sail.

Source: Wright, J.L., *Space Sailing*. Amsterdam, the Netherlands: Gordon and Breach Science Publishers, 1992.

2.2.2 Spinning disk sail

A spinning disk sail uses centrifugal forces to deploy and rigidize the sail and has a performance advantage over the square sail because of the lack of a supporting structure. Attitude control will be possible by means of solar pressure attitude control vanes. The sail will have a curved shape based on a balance between solar pressure force and radial inertia force on a thin section of the sail. Thus the solar pressure force will deform the spinning disk sail into an extremely shallow cone.

The basic configuration is that of a circular disk, although the perimeter could have straight segments, or some inward curvature, as shown in figure 2.9. A set of sails is desired over a single sail, for reasons like deployment handling, control, replacement, and sail smoothness during coning. The central portion of the disk is open, with a diameter of about 10% of the overall diameter, as shown in figure 2.10. A short central mast anchors tension lines serving as spokes that stabilize the hub rim and transfer precessional torques between the center body and the sail. The lines from the inner rim to the mast attach to disks at each end of the mast. The lever arms provided by these disks transfer spin torques between the sail and the mast. The segments of the inner rim carry the furled sails. The segments can be long rollers that provide a controlled deployment of the sail, or the sails can be folded and constrained in some manner by mechanisms on the segments. The different sail segments are linked periodically along their edges. The links are massy chords that flex outward under the centrifugal effect to draw the sail segments tight.

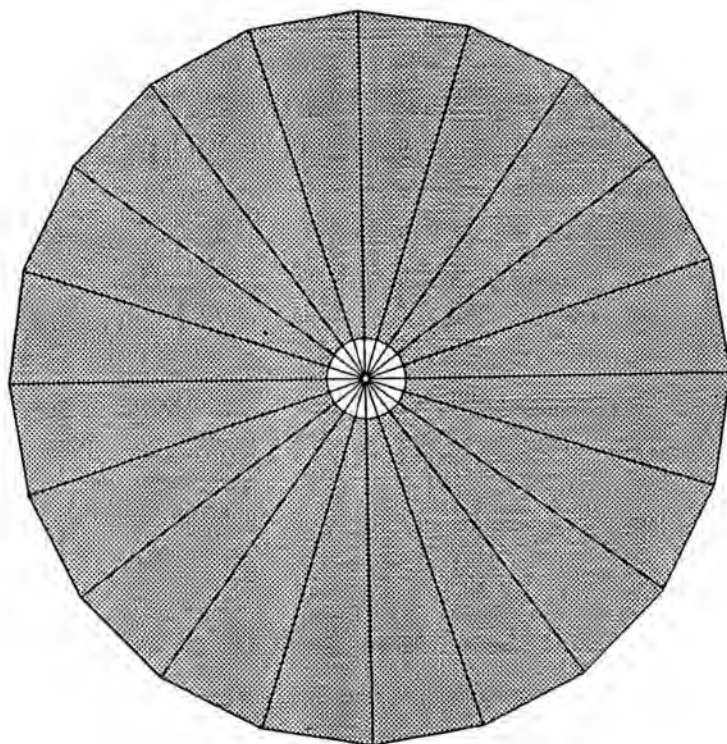


Figure 2.9: The disk sailer configuration.

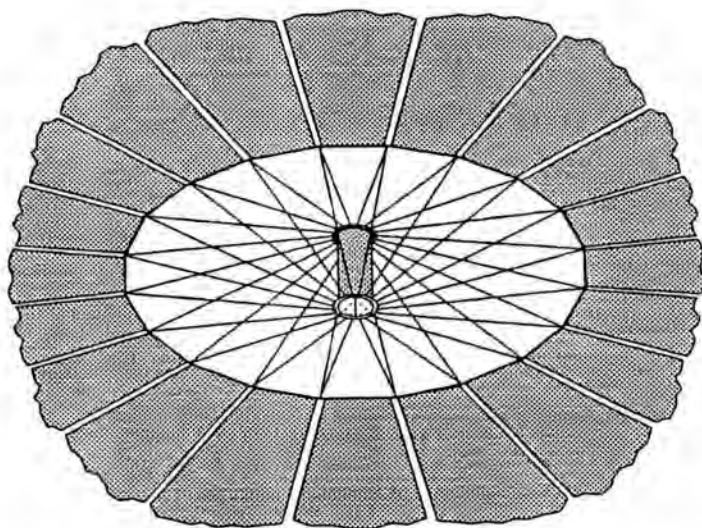


Figure 2.10: Disk sailer central region, perspective view.

Source (figure 2.9 and 2.10): Wright, J.L., *Space Sailing*. Amsterdam, the Netherlands: Gordon and Breach Science Publishers, 1992.

Moment balance for sun incidence angle control will be accomplished either by torquing at the central spacecraft with inertia wheels or by cyclically warping the sail shape with tension lines.

2.2.3 Heliogyro

The heliogyro design considered for the comet Halley rendezvous mission was based on the 2 blade heliogyro design as proposed by MacNeal (1969), see also section 2.1.2. The design was expanded to 12 blades in 2 banks of 6 blades, see figure 2.11, each 7500 m by 8 m. Because of its easy packing and deployment, this design was the final design considered for the mission.

A spin rate of 0.3 rpm provides the necessary longitudinal stiffening to the structure. Battens spaced approximately every 100 m. along the blades provide crosswise stiffening. Each blade is stored on a reel which is attached to the center body structure, or hub, through a yoke-boom assembly, see figure 2.12.

The heliogyro's most serious dynamic problem is the possibility of dynamic instability. Although the environmental forces are small, the available structural damping is also small, so that the susceptibility to instability is comparable to that of a conventional helicopter rotor. The types of instabilities investigated included mechanical instability, thermo-elastic instability and flutter. It was found that only flutter appears to be a serious threat.

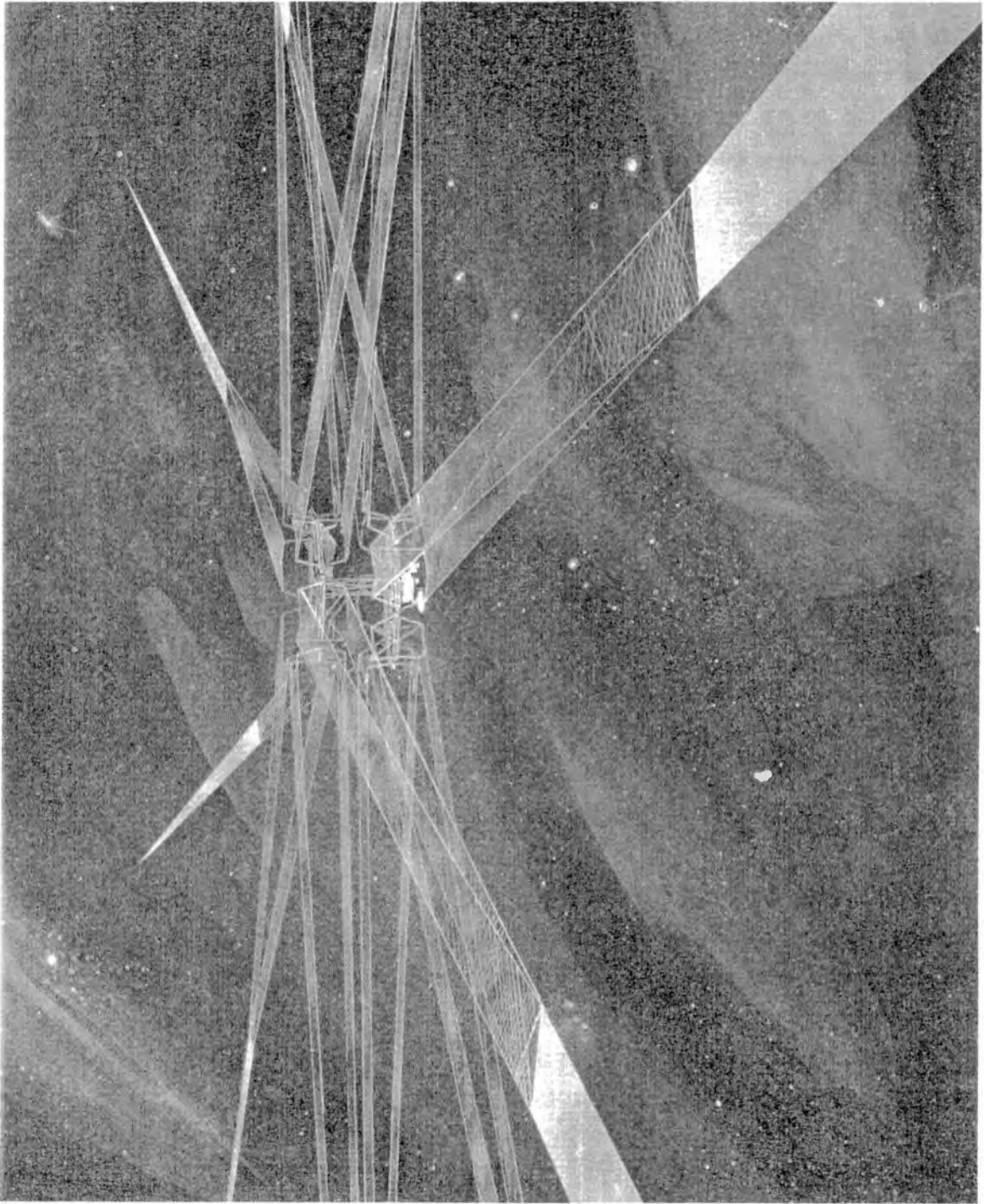


Figure 2.11: The Heliogyro.

Source: Wright, J.L., *Space Sailing*. Amsterdam, the Netherlands: Gordon and Breach Science Publishers, 1992.

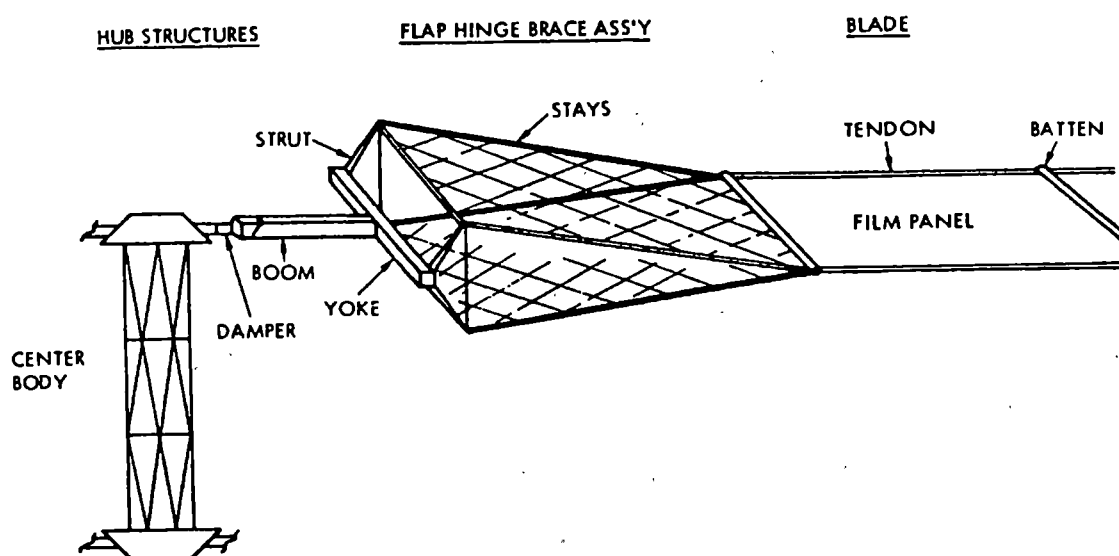


Figure 2.12: Heliogyro blade attachment details.

Source: Wright, J.L., *Space Sailing*. Amsterdam, the Netherlands: Gordon and Breach Science Publishers, 1992.

The heliogyro blade design can be seen in figure 2.13. The film panels have to be kept wrinkle free, and therefore they are "hung" in sections along the blade. Battens provide the crosswise stiffening and edge members provide the structure for film support. Graphite polyamide was selected as the material for the battens and edge members, having both the strength and temperature properties required for the mission. Multi-edge members were used as redundancy to insure high reliability against potential micro-meteoroid damage.

2.2.4 Sail material and coatings

The sail film material chosen was Kapton with a thickness of 2 μm . It has a surface density of 2.9 g/m² and ultimate tensile strength of 1.72 N/m² at room temperature and > 1 N/m² at 250 °C. Kapton was also found to have a good dimensional stability (thermal expansion < 1.5%, creep < 0.1%, shrinkage < 0.5%) and radiation resistance .

An aluminum coating of 1000 Å was found to be satisfactory for the reflector, yielding .88 reflectance. Several thermal coatings were found to give high emittance (> 0.6) and chromium with a thickness of 125 Å was selected as the baseline. The metal coatings also provided for the possibility of shorting the front-to-back in case of unacceptably large voltage build-ups on the sail.

2.3 The World Space Foundation and U3P

After the cancellation of the comet Halley rendezvous mission a group of scientists in the United States, most of them from JPL, decided to set up a private, nonprofit organization to

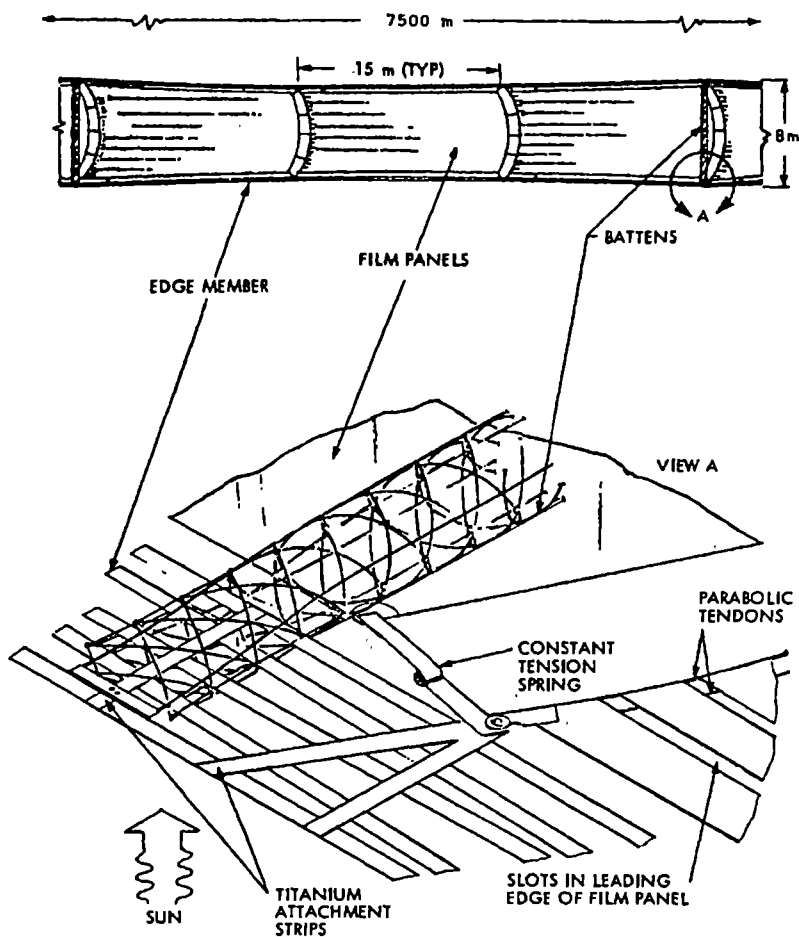


Figure 2.13: Baseline blade design.

Source: Friedman, L. et al (1978), 'Solar Sailing-The Concept Made Realistic'. *Proceedings of the AIAA 16th Aerospace Sciences meeting*, Huntsville, Alabama. AIAA paper 78-82.

continue work on the sail. They called their organization the World Space Foundation (WSF). One of the goals of the WSF was to place a small sail in space. Once it was proven that a sail could be successfully launched and controlled, more sails would follow. They could be used to explore the asteroids or to transport equipment to various planets. On August 25, 1981 a square-shaped sail was successfully deployed in a ground test, see figure 2.14. This sail was made out of Mylar and measured 46 feet on each side. It was about a quarter in size of the sail they hoped to eventually launch in space.

At about the same time, in France a group of scientists from the French National Space Agency were conducting a study of solar sails. They called themselves the Union Pour la Promotion de la Propulsion Photonique (U3P). In 1981, in an effort to stimulate interest in their project, they proposed a solar sailing race to the moon. Initially, in November 1989 eight finalists were selected: three from the United States, one from a joint effort by France and Spain, and one each from Japan, the Soviet Union, Canada and England.

The English sail was disk-shaped with flexible plastic ribbing running through it. The disk would cover about 15 acres (6.07 hectares). The French/Spanish design had four masts, each 148 feet (45 m) long. The sail itself would be either Kapton or a similar material, aluminized on both sides. Japan's entry was initially 330 feet on each side, but was scaled down to 98 feet on a side. The Canadian scientists proposed a hexagonal design of 12,000 square yards (10,000 sq m).



Figure 2.14: Solar sail vehicle mockup with two control vanes deployed, sail is still stowed.

Source: Souza, D.M. (1994), *Space Sailing*. Minneapolis: Lerner Publications Company.

One of the three designs from the United States was from a group of graduate students at the Massachusetts Institute of Technology (MIT). It was a heliogyro with eight blades, each 275 feet (84 m) long and 5 feet (1.5 m) wide. The sail film material would be either Kapton or Lexan, a lightweight film used in model aircraft construction. A group of scientists at the John Hopkins University in Maryland submitted another of the U.S. designs. The sail was sunflower shaped, it had 480 petal-shaped sails held flat by a large hoop. When constructed, the entire craft would measure about 560 feet (170 m) in diameter. The third U.S. design was submitted by the WSF scientists. Their design was a square Kapton sail, 180 feet (55 m) on each side. At each of the sail's corners were smaller triangular sails to help control the sail. This race was originally scheduled to take place in 1995, but it still has to take place.

2.4 The past decade

In early 1993, Russia launched a solar sail aboard a Progress cargo spacecraft. After the Progress docked with the Mir Space Station, two cosmonauts assembled the sail. The spacecraft then backed about 500 feet away from the space station and released the sail. The sail began to unfold, forming a disk that measured 65 feet (20 m) in diameter. Figure 2.15 shows a picture of this deployed sail. Not only did it move, but the sail also reflected sunlight that was visible in various places on Earth.

One of the most recent solar sail studies is a joint undertaking of the German Aerospace Research Establishment DLR and NASA/JPL. The ODISSEE (Orbital Demonstration of an Innovative Solar Sail driven Expandable structure Experiment) study has defined a practical



Figure 2.15: The deployed Russian solar sail.

Source: Souza, D.M. (1994), *Space Sailing*. Minneapolis: Lerner Publications Company.

low-cost solar sail demonstration mission. The goals of the mission are to demonstrate fabrication, packing and deployment of a solar sail along with attitude control through center of mass displacement.

The sail will be a 40 m by 40 m square solar sail with a separate sail module and bus. As shown in figure 2.16. The sail film material is Kapton with a thickness of $7.6\text{ }\mu\text{m}$ coated with an $0.1\text{ }\mu\text{m}$ film of aluminum on the front surface and an $0.015\text{ }\mu\text{m}$ film of chromium on the rear surface for thermal control. The sail film is held in tension using deployable light-weight composite structure booms manufactured from two laminated sheets of carbon fiber reinforced plastic. These two sheets are bonded at the edges to give the spars a tubular form which can be pressed flat and rolled for packing. It results in a low-mass structure of less than 0.1 kg per meter . Figure 2.17 shows this spar design.

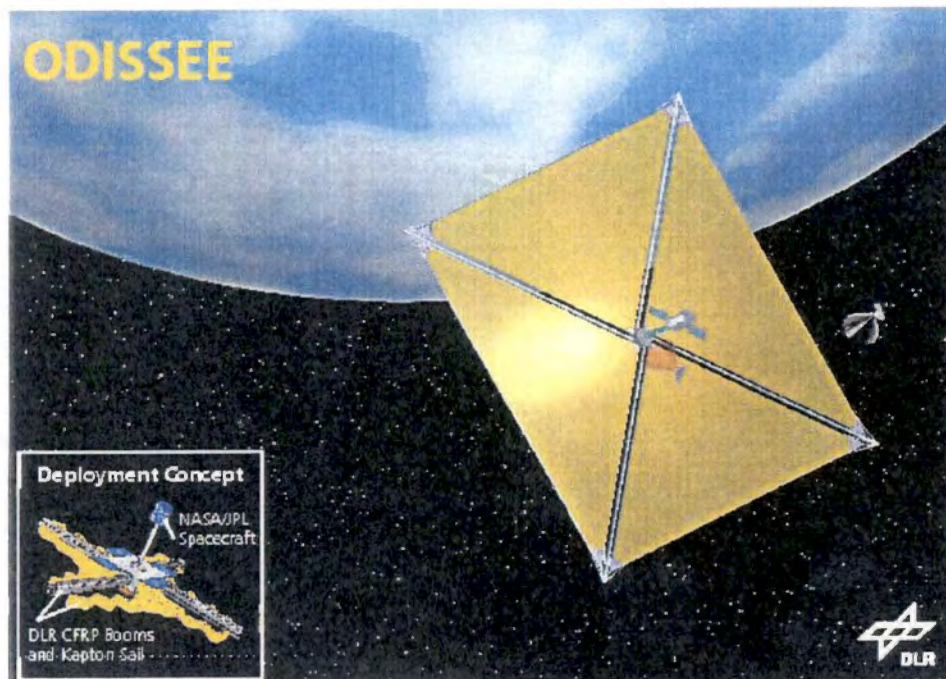


Figure 2.16: Sail design for the ODISSEE mission.

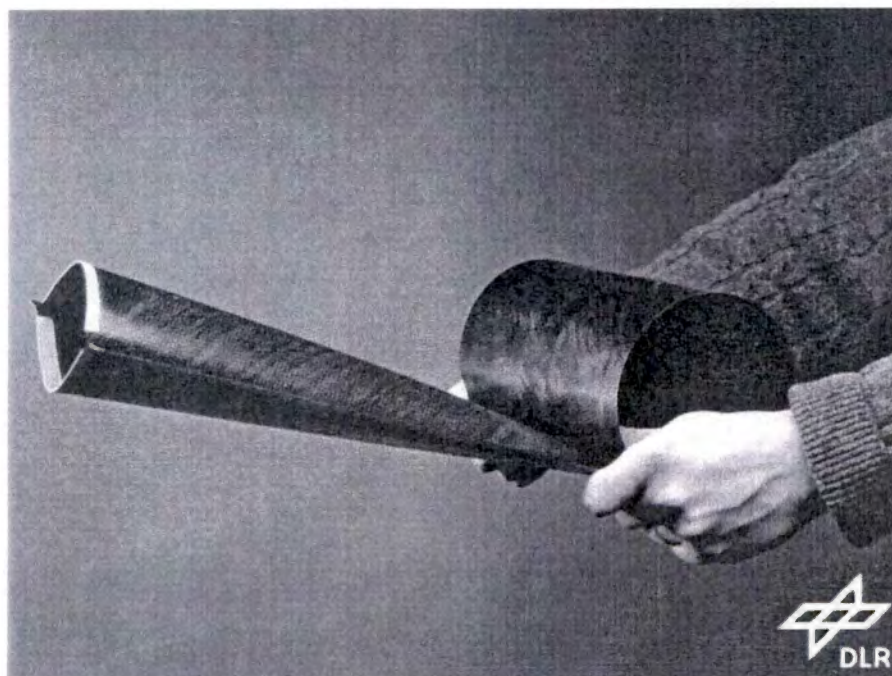


Figure 2.17: Spar design for the ODISSEE mission.

Source (figures 2.16 and 2.17): DLR, <http://www.kp.dlr.de/solarsail>

Chapter 3

The Attitude Dynamics and Trajectory

Dynamics Equations

In this chapter the equations that govern the attitude and trajectory dynamics will be derived. For the attitude dynamics equations, knowledge of the moments and products of inertia of the sail are needed. This in its turn requires knowledge about the geometry of the sail and the materials used to manufacture the sail. Therefore, first a geometry is assumed and the products and moments of inertia that result from this geometry will be calculated. After that, the actual attitude and trajectory dynamics equation will be derived. This all is done for a full 3D model of the system.

3.1 The geometry of the sail

In the previous chapter the various design concepts of solar sails over the years were discussed, and the main designs still in consideration are the heliogyro, the spinning disc sail or a (square) spar supported sail. The first two get their required stiffness from centrifugal

forces generated by spinning the sail, the last one is supported by spars to achieve this. Since in this work the sail will not be flat, centrifugal forces cannot be used to give the sail its required stiffness. This means that the spar supported sail concept is the best way to give the sail its stiffness. In this concept the supporting spars run along the diagonals of the sail and these spars are in their turn deployed from a central bus. By deploying two of those spars under an angle with each other, a sail layout will be achieved as depicted in figure 3.1. There are two triangular sail planes of homogenous density ρ . Plane 1 has a base of length l_1 , height of length l_2 with normal vector $\mathbf{n}_1$. Plane 2 has the same base length, but the height is $l_2 + l_3$ and has normal vector $\mathbf{n}_2$. The angle between the two normal vectors is $\alpha_x = 180^\circ - \alpha$, where α is the angle between the two sail planes. The central bus is modeled as a homogeneous cylinder, with radius R and length L .

Figure 3.2 shows the orientation of the two sail planes in spherical coordinates. There are 4 control angles: the cone angle θ and the clock angle β of the first sail, the angle α_x between the two normal vectors of the sail planes and the angle Ω , a rotation about $\mathbf{n}_1$. The vector $\mathbf{n}_1$ can readily be found in spherical coordinates and is given by:

$$\mathbf{n}_1 = \cos \theta \mathbf{e}_r + \sin \theta \sin \beta \mathbf{e}_\phi + \sin \theta \cos \beta \mathbf{e}_\psi \quad (3.1)$$

Finding $\mathbf{n}_2$ is more complicated. First, a local coordinate system xyz needs to be built up. It will have as x-axis the vector $\mathbf{n}_1$. Having found the x-axis, it is possible to find the z-axis as the cross product between the incident sunlight vector and the vector $\mathbf{n}_1$:

$$\mathbf{z} = \mathbf{i} \times \mathbf{n}_1 \quad (3.2)$$

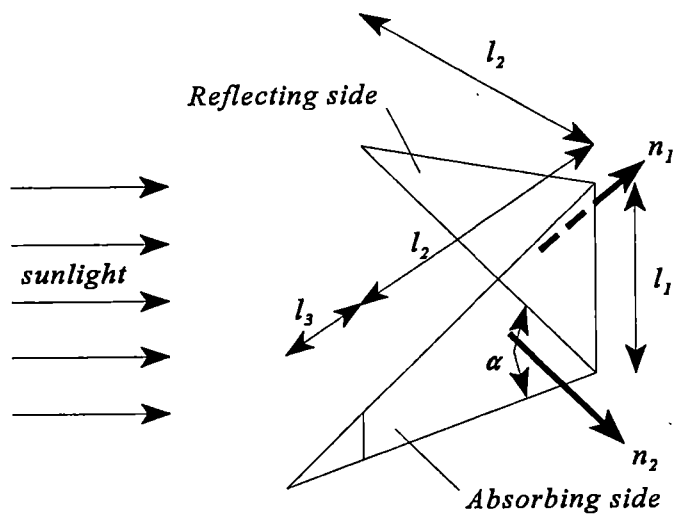


Figure 3.1: Two plane solar sail

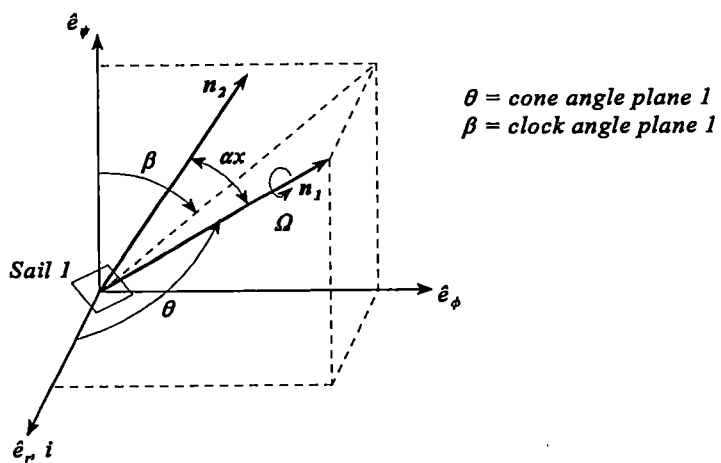


Figure 3.2: The sail orientation in spherical coordinates

Making this a vector of unit length leads to the z-axis to be given by:

$$\mathbf{z} = \begin{pmatrix} 0 \\ -\cos\beta \\ \sin\beta \end{pmatrix} \quad (3.3)$$

The y-axis can be found by making this coordinate system a right handed system:

$$\mathbf{y} = \mathbf{z} \times \mathbf{x} = \mathbf{z} \times \mathbf{n}_1 = \begin{pmatrix} -\sin\theta \\ \cos\theta \sin\beta \\ \cos\theta \cos\beta \end{pmatrix} \quad (3.4)$$

It is now possible to find the rotation matrix T that transforms spherical coordinates into the local coordinates. This matrix is given by:

$$\begin{pmatrix} i \\ j \\ k \end{pmatrix} = T \begin{pmatrix} e_r \\ e_\phi \\ e_\psi \end{pmatrix} = \begin{pmatrix} \cos\theta & \sin\theta \sin\beta & \sin\theta \cos\beta \\ -\sin\theta & \cos\theta \sin\beta & \cos\theta \cos\beta \\ 0 & -\cos\beta & \sin\beta \end{pmatrix} \begin{pmatrix} e_r \\ e_\phi \\ e_\psi \end{pmatrix} \quad (3.5)$$

This local reference frame can be rotated about its x-axis over an angle Ω . The resulting reference frame is the $x'y'z'$ frame, the local rotated frame. The rotation matrix for this rotation is:

$$R = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\Omega & \sin\Omega \\ 0 & -\sin\Omega & \cos\Omega \end{pmatrix} \quad (3.6)$$

The coordinates $\mathbf{n}_2'$ of $\mathbf{n}_2$ in this local rotated frame are:

$$\mathbf{n}_2' = \begin{pmatrix} \cos \alpha x \\ \sin \alpha x \\ 0 \end{pmatrix} \quad (3.7)$$

Then the coordinates of $\mathbf{n}_2$ in the local frame will be given by:

$$\mathbf{n}_{2_{local}} = R^T \mathbf{n}_2' = \begin{pmatrix} \cos \alpha x \\ \cos \Omega \sin \alpha x \\ \sin \Omega \sin \alpha x \end{pmatrix} \quad (3.8)$$

and the coordinates of $\mathbf{n}_2$ in the spherical reference system are given by:

$$\mathbf{n}_2 = T^T \mathbf{n}_{2_{local}} = \begin{pmatrix} \cos \alpha x \cos \theta - \cos \Omega \sin \alpha x \sin \theta \\ \cos \alpha x \sin \beta \sin \theta + \cos \theta \cos \Omega \sin \alpha x \sin \beta - \cos \beta \sin \alpha x \sin \Omega \\ \cos \beta (\cos \alpha x \sin \theta + \cos \theta \cos \Omega \sin \alpha x) + \sin \alpha x \sin \beta \sin \Omega \end{pmatrix} \quad (3.9)$$

Within the local rotated reference system it is easy to find the center of mass of the two sail planes, the central bus and the combined center of mass of the system, refer also to figure 3.3. With x_0', y_0' being the center of mass of the bus, x_1', y_1' the center of mass of plane 1, x_2', y_2' the center of mass of plane 2 and m_p, m_1 and m_2 the masses of the bus, plane 1 and plane 2 respectively, then:

$$x_{cm}' = \frac{x_0' m_p + x_1' m_1 + x_2' m_2}{m_p + m_1 + m_2}, \quad y_{cm}' = \frac{y_0' m_p + y_1' m_1 + y_2' m_2}{m_p + m_1 + m_2} \quad (3.10)$$

The center of mass of the two sail planes are known in this local rotated system, $x_1', y_1' = (0, -\frac{1}{3}l_2)$ and $x_2', y_2' = (-\frac{1}{3}(l_2+l_3)\sin\alpha, -\frac{1}{3}(l_2+l_3)\cos\alpha)$. The bus is assumed to be located at the origin, $x_0', y_0' = (0,0)$.

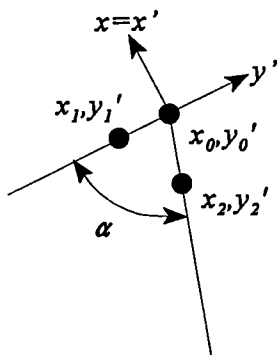


Figure 3.3: Center of mass

3.2 Moments of inertia

First, the moments of inertia for both sail planes are found. The planes can be modeled as homogeneous triangular plates, as can be seen in figure 3.4. Their center of mass is located at 1/3 of the height from the base.

$$I_{xx} = \int_B y^2 dm = 2 \int_0^{\frac{1}{2}l_1} y^2 \frac{m}{\frac{1}{2}l_1 a} a(1-2\frac{y}{l_1}) dy = \frac{4m}{l_1} \int_0^{\frac{1}{2}l_1} (y^2 - 2\frac{y^3}{l_1}) dy = \frac{1}{24}ml_1^2 \quad (3.11)$$

$$I_{yy} = \int_B x^2 dm = \int_{-\frac{1}{3}a}^{\frac{2}{3}a} x^2 \frac{m}{\frac{1}{2}l_1 a} l_1(\frac{2}{3} - \frac{x}{a}) dx = \frac{2m}{a} \int_{-\frac{1}{3}a}^{\frac{2}{3}a} (\frac{2}{3}x^2 - \frac{x^3}{a}) dx = \frac{1}{18}ma^2 \quad (3.12)$$

and

$$I_{zz} = I_{xx} + I_{yy} \quad (3.13)$$

For plane 1, $a = l_2$, $m = \frac{1}{2}\rho l_1 l_2$, for plane 2, $a = l_2 + l_3$, $m = \frac{1}{2}\rho l_1 (l_2 + l_3)$.

The structural mass of the system needs to be accounted for also. For simplicity, it will be assumed that the distribution of the structural over the sail area is homogeneous. Multiplying the mass of the two sail planes by the factor “1+k”, where k is the structural mass factor, will account for the structural mass.

Secondly, the moments of inertia for the central bus have to be found. The bus is modeled as a homogeneous cylinder, with length L and radius R , see also figure 3.5. In the coordinate

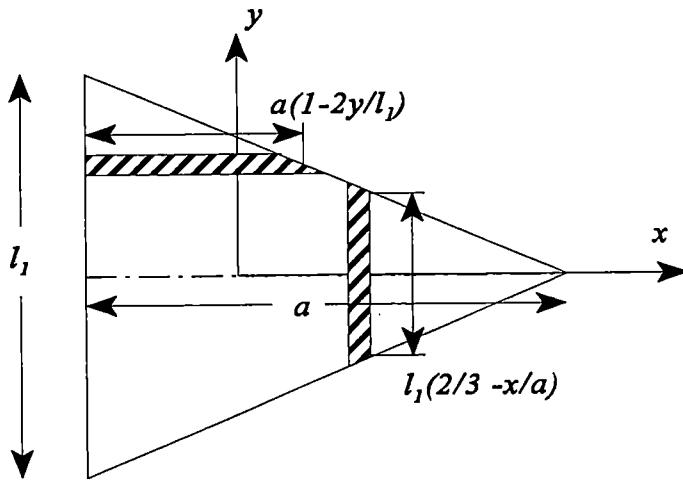


Figure 3.4: Calculation of moments of inertia of sail planes

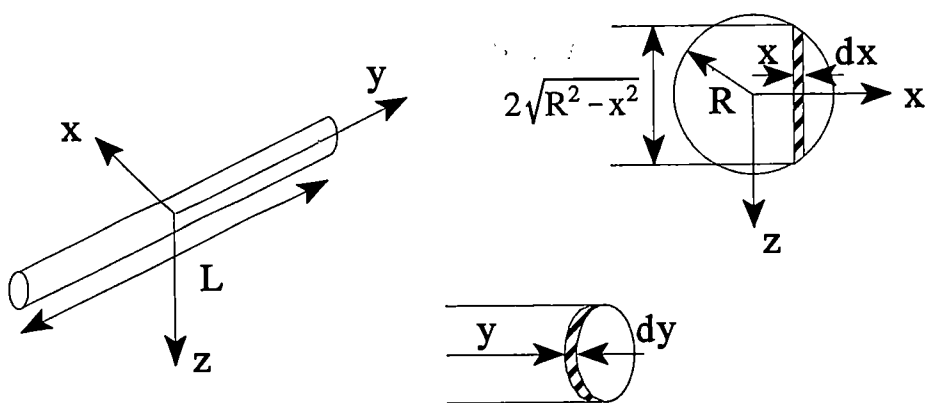


Figure 3.5: Calculation of moments of inertia for central bus.

system depicted in this figure, the following values for the moments of inertia can be found:

$$I_{yy} = \int_B (x^2 + z^2) dm = \int_0^R r^2 \frac{m}{\pi R^2 L} 2\pi r L dr = \frac{2m}{R^2} \int_0^R r^3 dr = \frac{1}{2} m R^2 \quad (3.14)$$

and

$$\begin{aligned} I_{xx} = I_{zz} &= \int_B (y^2 + x^2) dm = \int_{-\frac{1}{2}L}^{\frac{1}{2}L} y^2 \frac{m}{\pi R^2 L} \pi R^2 dy + \int_{-R}^R x^2 \frac{m}{\pi R^2 L} 2\sqrt{R^2 - x^2} L dx \\ &= \frac{1}{12} m L^2 + \frac{2mR^2}{\pi} \int_{-1}^1 u^2 \sqrt{1-u^2} du = \frac{1}{12} m L^2 + \frac{2mR^2}{\pi} \int_{-\frac{1}{2}\pi}^{\frac{1}{2}\pi} \sin^2 \gamma \cos^2 \gamma d\gamma \quad (3.15) \\ &= \frac{1}{12} m L^2 + \frac{mR^2}{4\pi} \int_{-\frac{1}{2}\pi}^{\frac{1}{2}\pi} (1 - \cos 4\gamma) d\gamma = \frac{1}{12} m (L^2 + 3R^2) \end{aligned}$$

Now that all the individual components in local reference frames are known, it will be possible to find the overall system moments and products of inertia by applying the translation and rotation rules to these local values. Figure 3.6 shows all the reference frames needed to calculate the combined moments of inertia. Reference frame x_1, y_1, z_1 : At center of mass of plane 1, directions as in figure 3.4. Reference frame x_2, y_2, z_2 : At center of mass of plane 2, directions as in figure 3.4. Reference frame x_0, y_0, z_0 : At center of mass of bus, directions as in figure 3.5. Reference frame x', y', z' is the local rotated reference frame as defined in section 3.1. Reference frame X, Y, Z : Origin at combined center of mass, parallel to the local rotated frame.

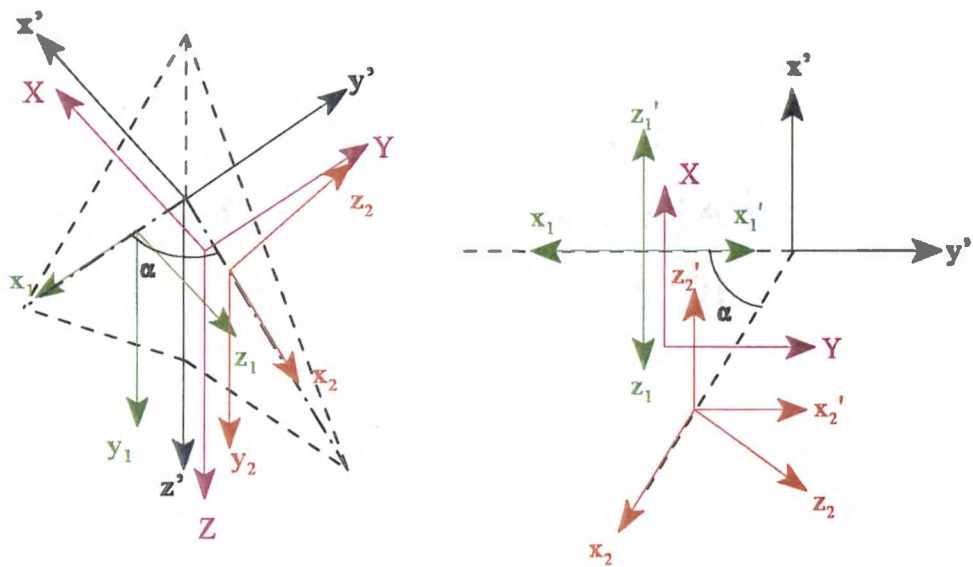


Figure 3.6: The reference frames

3.2.1 Contribution from plane 1

First, the x_1, y_1, z_1 frame needs to be rotated about the y_1 axis over an angle γ of 180° . The resulting frame (z_1', x_1', y_1') will be parallel to the X, Y, Z frame. The moments of inertia in the rotated frame are:

$$\begin{aligned} I_{x_1'/x_1'} &= I_{xx} \cos^2 \gamma + I_{zz} \sin^2 \gamma = I_{xx} \\ I_{z_1'/z_1'} &= I_{xx} \sin^2 \gamma + I_{zz} \cos^2 \gamma = I_{zz} \\ I_{y_1'/y_1'} &= I_{yy} \end{aligned} \quad (3.16)$$

And the contribution of plane 1 to the global moments and products of inertia:

$$\begin{aligned} I_{XX_1} &= I_{z_1'/z_1'} + m_1 (y_1 - y_{cm})^2 = I_{zz} + m_1 (y_1 - y_{cm})^2 \\ I_{YY_1} &= I_{x_1'/x_1'} + m_1 (x_1 - x_{cm})^2 = I_{xx} + m_1 x_{cm}^2 \\ I_{ZZ_1} &= I_{yy} + m_1 \left((x_1 - x_{cm})^2 + (y_1 - y_{cm})^2 \right) \\ I_{XX_1} &= m_1 (x_1 - x_{cm})(y_1 - y_{cm}) \end{aligned} \quad (3.17)$$

3.2.2 Contribution from plane 2

First, the x_2, y_2, z_2 frame needs to be rotated about its y_2 axis over an angle γ of $-\alpha$. The resulting frame (z_2', x_2', y_2') will be parallel to the X, Y, Z frame. The moments of inertia in the rotated frame are:

$$\begin{aligned} I_{x_2'/x_2'} &= I_{xx} \cos^2 \gamma + I_{zz} \sin^2 \gamma = I_{xx} \cos^2 \alpha + I_{zz} \sin^2 \alpha \\ I_{z_2'/z_2'} &= I_{xx} \sin^2 \gamma + I_{zz} \cos^2 \gamma = I_{xx} \sin^2 \alpha + I_{zz} \cos^2 \alpha \\ I_{y_2'/y_2'} &= I_{yy} \\ I_{x_2'/z_2'} &= -\cos \gamma \sin \gamma I_{xx} + \sin \gamma \cos \gamma I_{zz} = \sin \alpha \cos \alpha (I_{zz} - I_{xx}) \end{aligned} \quad (3.18)$$

And the contribution of plane 2 to the global moments and products of inertia:

$$\begin{aligned}
I_{XX_2} &= I_{z'z'} + m_2(y_2 - y_{cm})^2 \\
I_{YY_2} &= I_{x'x'} + m_2(x_2 - x_{cm})^2 \\
I_{ZZ_2} &= I_{yy} + m_2((x_2 - x_{cm})^2 + (y_2 - y_{cm})^2) \\
I_{XY_2} &= I_{x'z'} + m_2(x_2 - x_{cm})(y_2 - y_{cm})
\end{aligned} \tag{3.19}$$

3.2.3 Contribution from central bus

First, the x_0, y_0, z_0 needs to be rotated about the z_0 axis over an angle γ of $\alpha/2$. The resulting frame will coincide with the local rotated frame. In the rotated frame, the moments and products of inertia are:

$$\begin{aligned}
I_{x'x'} &= I_{xx} \cos^2 \gamma + I_{yy} \sin^2 \gamma = I_{xx} \cos^2 \frac{\alpha}{2} + I_{yy} \sin^2 \frac{\alpha}{2} \\
I_{y'y'} &= I_{xx} \sin^2 \gamma + I_{yy} \cos^2 \gamma = I_{xx} \sin^2 \frac{\alpha}{2} + I_{yy} \cos^2 \frac{\alpha}{2} \\
I_{z'z'} &= I_{zz} \\
I_{x'y'} &= \cos \gamma \sin \gamma I_{xx} - \sin \gamma \cos \gamma I_{yy} = \frac{1}{2} \sin \alpha (I_{xx} - I_{yy})
\end{aligned} \tag{3.20}$$

And the contribution of the central bus to the global moments and products of inertia are:

$$\begin{aligned}
I_{XX_{bus}} &= I_{x'x'} + m_p(y_0 - y_{cm})^2 \\
I_{YY_{bus}} &= I_{y'y'} + m_p(x_0 - x_{cm})^2 \\
I_{ZZ_{bus}} &= I_{z'z'} + m_p((x_0 - x_{cm})^2 + (y_0 - y_{cm})^2) \\
I_{XY_{bus}} &= I_{x'y'} + m_p(x_0 - x_{cm})(y_0 - y_{cm})
\end{aligned} \tag{3.21}$$

3.2.4 Total moments and products:

The total moments and products of inertia for the sail are found by adding the individual contributions from the two sail planes and the central bus system together:

$$\begin{aligned}
I_{XX} &= I_{XX_1} + I_{XX_2} + I_{XX_{bus}} \\
I_{YY} &= I_{YY_1} + I_{YY_2} + I_{YY_{bus}} \\
I_{ZZ} &= I_{ZZ_1} + I_{ZZ_2} + I_{ZZ_{bus}} \\
I_{XY} &= I_{XY_1} + I_{XY_2} + I_{XY_{bus}} \\
I_{XZ} &= I_{YZ} = 0
\end{aligned} \tag{3.22}$$

Two products of inertia, I_{XZ} and I_{YZ} , are zero. This is expected since the XY-plane is the symmetry plane of the system. The inertia tensor is thus given by:

$$I = \begin{bmatrix} I_{XX} & -I_{XY} & 0 \\ -I_{XY} & I_{YY} & 0 \\ 0 & 0 & I_{ZZ} \end{bmatrix} \tag{3.23}$$

3.3 Solar sail forces and moments

The forces F acting on the sail planes due to the solar radiation can, in general, be written as:

$$F = P_{sol} S_o \left(\frac{r_o}{r} \right)^2 \left[R(n \cdot i)^2 n - \frac{1}{2}(R-1)(n \cdot i)i \right] \tag{3.24}$$

where n is the normal vector of the sail, i is the direction vector of the incident sunlight, r_o is the radial distance Sun-Earth, r is the radial distance Sun-Sail, P_{sol} is the solar radiation pressure at Earth's distance from the sun and S_o is the sail area. $R = 0$ for complete absorption and 1 for specular reflection. As can be seen from equation 3.24, for specular reflection the resulting force is always in the direction of the sail normal, for complete absorption the force will always be in the direction of the incident sunlight vector. This means that the direction of the force can only be controlled for the specular reflection case, since it will be possible to change the orientation of the sail, but not the direction of the incident sunlight.

In case of the two-plane sail, four different contributions to the solar radiation force vector can be identified:

1. Forces due to the direct radiation (either absorption or reflection) of plane 1.
2. Forces due to the direct radiation (either absorption or reflection) of plane 2.
3. Forces due to the indirect radiation (reflection only) of plane 2 by reflected radiation coming of plane 1.
4. Forces due to the indirect radiation (reflection only) of plane 1 by reflected radiation coming of plane 2.

For components 1 and 2, partial shadowing by the other plane is possible. The indirect radiation terms are only present when $\alpha < 180^\circ$

3.3.1 Direct radiation terms

It is assumed that for both sail planes one side is perfectly reflecting and the other side absorbs the sunlight completely, see also figure 3.1. Then, when the reflecting side ($R=1$) faces the sun, the forces are:

$$\begin{aligned}
 \mathbf{F}_{1,2} &= P_{sol} S_{1,2} \left(\frac{r_o}{r} \right)^2 (\mathbf{n}_{1,2} \cdot \mathbf{i})^2 \mathbf{n}_{1,2} \Rightarrow \\
 \mathbf{F}_1 &= P_{sol} S_1 \left(\frac{r_o}{r} \right)^2 \cos^2 \theta \mathbf{n}_1 \\
 \mathbf{F}_2 &= P_{sol} S_2 \left(\frac{r_o}{r} \right)^2 (\cos \alpha \cos \theta - \cos \Omega \sin \alpha \sin \theta)^2 \mathbf{n}_2
 \end{aligned} \tag{3.25}$$

and when the absorbing side ($R=0$) faces the sun the forces can be written as:

$$F_{1,2} = \frac{1}{2} P_{sol} S_{1,2} \left(\frac{r_o}{r} \right)^2 (-n_{1,2} \cdot i) i \Rightarrow$$

$$F_1 = -\frac{1}{2} P_{sol} S_1 \left(\frac{r_o}{r} \right)^2 \cos \theta i \quad (3.26)$$

$$F_2 = -\frac{1}{2} P_{sol} S_2 \left(\frac{r_o}{r} \right)^2 (\cos \alpha x \cos \theta - \cos \Omega \sin \alpha x \sin \theta) i$$

When shadowing occurs, the sail area exposed to the sunlight will reduce in size. In order to calculate the actual sail area, the projection of the sail tip of the plane that shadows onto the plane that is being shadowed needs to be calculated. All these calculations will be done in the local rotated reference frame. The approach used to calculate the shadowed area:

- 1) Find the coordinates t' of the sail plane tip in a local rotated reference frame
- 2) Add an unknown multiple of the incident radiation vector to this:

$$t' + \lambda i' \quad (3.27)$$

- 3) Calculate where this line intersects with the receiving plane. Now the projection of the tip onto the receiving plane is found and thus the projection of the entire plane can be easily found.

3.3.1.1 Plane 2 shadows plane 1

In the local rotated reference frame the coordinates of the tip of the second sail plane t_2' are given by:

$$t_2' = \begin{pmatrix} -(l_2 + l_3) \sin \alpha x \\ (l_2 + l_3) \cos \alpha x \\ 0 \end{pmatrix} \quad (3.28)$$

see also figure 3.7. The incident sunlight vector is given by:

$$\mathbf{i}' = \begin{pmatrix} \cos \theta \\ -\sin \theta \cos \Omega \\ \sin \theta \sin \Omega \end{pmatrix} \quad (3.29)$$

By calculating the intersection of the line $\mathbf{t}_2' + \lambda \mathbf{i}'$ with the plane $x=0$, the position of the projection of the sail tip of plane 2 onto plane 1 can be calculated, and is found to be:

$$\mathbf{t}_{2p}' = \frac{l_2 + l_3}{\cos \theta} \begin{pmatrix} 0 \\ \cos \alpha x \cos \theta - \sin \theta \cos \Omega \sin \alpha x \\ \sin \alpha x \sin \theta \sin \Omega \end{pmatrix} \quad (3.30)$$

In order to check whether the projection lies in plane 1, the position of the tip of plane 1 has to be found. The coordinates $\mathbf{t}_1'$ of the tip of plane 1 in the local rotated reference frame are:

$$\mathbf{t}_1' = \begin{pmatrix} 0 \\ -l_2 \\ 0 \end{pmatrix} \quad (3.31)$$

It is now possible to define 4 different lines, see also figure 3.8:

$$\text{line 1-}\mathbf{t}_{2p}': (0 \ 0 \ 1/2 l_1)^T + \lambda (\mathbf{t}_{2p}' - (0 \ 0 \ 1/2 l_1)^T)$$

$$\text{line 2-}\mathbf{t}_{2p}': (0 \ 0 \ -1/2 l_1)^T + \mu (\mathbf{t}_{2p}' - (0 \ 0 \ -1/2 l_1)^T)$$

$$\text{line 1-}\mathbf{t}_1': (0 \ 0 \ 1/2 l_1)^T + \nu (\mathbf{t}_1' - (0 \ 0 \ 1/2 l_1)^T)$$

$$\text{line 2-}\mathbf{t}_1': (0 \ 0 \ -1/2 l_1)^T + \eta (\mathbf{t}_1' - (0 \ 0 \ -1/2 l_1)^T)$$

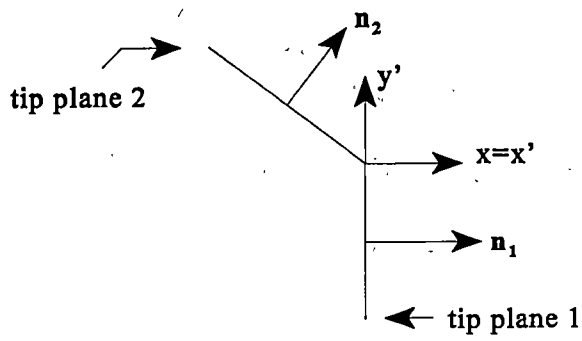


Figure 3.7: Coordinates of tip in rotated frame

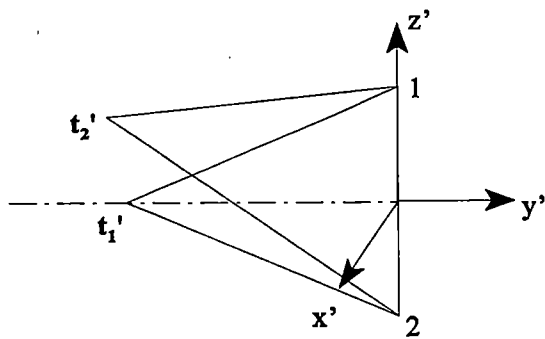


Figure 3.8: Location of projection of tip plane 2

In case the z' -coordinate of t_{2_p}' is greater than 0, it will be possible to calculate the intersection of the lines $1-t_1'$ and $2-t_{2_p}'$. The projection lies in plane 1 when $\mu > 1$, and the coordinates of t_{2_p}' are used in the area calculation. When $\mu < 1$ and $\nu > 1$, plane 1 is completely shadowed by plane 2 and the coordinates of t_1' are used in the area calculation. Otherwise the coordinates of the intersection point are used in the area calculation.

In case the z' -coordinate of t_{2_p}' is smaller than 0, it will be possible to calculate the intersection of the lines $1-t_{2_p}'$ and $2-t_1'$. The projection lies in plane 1 when $\lambda > 1$, and the coordinates of t_{2_p}' are used in the area calculation. When $\lambda < 1$ and $\eta > 1$, plane 1 is completely shadowed by plane 2 and the coordinates of t_1' are used in the area calculation. Otherwise the coordinates of the intersection point are used in the area calculation.

Now that the exact projection is known, the area of plane 1 that is shadowed by plane 2 can be calculated, see also figure 3.9. Point 3 is either the projection of the tip of plane 2, the tip of plane 1 or the intersection point, as calculated above. The value of h is found by:

$$h = \sqrt{x_3'^2 + y_3'^2} \quad (3.32)$$

the area is found by:

$$S = 1/2 h l_1 \quad (3.33)$$

and the center of mass of this area in the local frame is found by:

$$\begin{pmatrix} x_4 \\ y_4 \\ z_4 \end{pmatrix} = \frac{1}{3} \begin{pmatrix} x_1 + x_2 + x_3 \\ y_1 + y_2 + y_3 \\ z_1 + z_2 + z_3 \end{pmatrix} \quad (3.34)$$

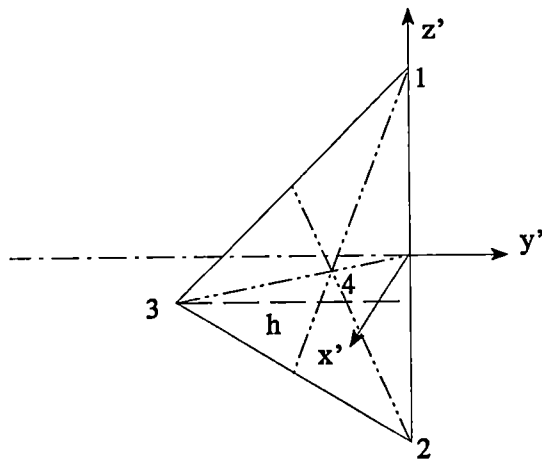


Figure 3.9: Projected area plane 2

By subtracting the area calculated in 3.33 from the area of plane 1, and calculating the center of mass of the remaining area, the exact force and locating where the force is acting on the sail plane can be calculated.

3.3.1.2 Plane 1 shadows plane 2

In the local rotated reference frame the coordinates of the tip of the second sail plane t_1' are given by equation 3.31. By calculating the intersection of the line $t_1' + \lambda i'$ with the plane $(\cos \alpha x) + (\sin \alpha x)y = 0$, the position of the projection of the sail tip of plane 1 onto plane 2 can be calculated, and is found to be:

$$t_{1p}' = \frac{l_2}{\cos \theta \cos \alpha x - \sin \theta \cos \Omega \sin \alpha x} \begin{pmatrix} \sin \alpha x \cos \theta \\ -\cos \alpha x \cos \theta \\ \sin \alpha x \sin \theta \sin \Omega \end{pmatrix} \quad (3.35)$$

The coordinates t_2' of the tip of plane 2 in the local rotated reference frame is given by equation 3.28. Again, it is now possible to define 4 different lines, see also figure 3.10:

$$\text{line 1-}t_{1p}': (0 \ 0 \ 1/2 l_1)^T + \lambda (t_{1p}' - (0 \ 0 \ 1/2 l_1)^T)$$

$$\text{line 2-}t_{1p}': (0 \ 0 \ -1/2 l_1)^T + \mu (t_{1p}' - (0 \ 0 \ -1/2 l_1)^T)$$

$$\text{line 1-}t_2': (0 \ 0 \ 1/2 l_1)^T + v (t_2' - (0 \ 0 \ 1/2 l_1)^T)$$

$$\text{line 2-}t_2': (0 \ 0 \ -1/2 l_1)^T + \eta (t_2' - (0 \ 0 \ -1/2 l_1)^T)$$

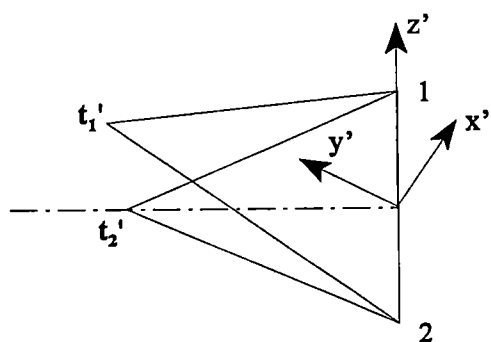


Figure 3.10: Location of projection of tip plane 1

The same procedure as before can now be used for finding the coordinates of point 3, the area that is shadowed by plane 1, and the center of mass of the shadowed area.

3.3.1.3 Moments due to direct radiation

The direct radiation forces act on the center of mass of the illuminated parts of the individual planes, thus resulting in moments around the combined center of mass of the system, see figure 3.11. These moments can be written as:

$$\mathbf{M}_{1,2} = \begin{pmatrix} x_{1,2} - x_{cm} \\ y_{1,2} - y_{cm} \\ z_{1,2} - z_{cm} \end{pmatrix} \times \mathbf{F}_{1,2} \quad (3.36)$$

3.3.2 Indirect radiation terms

For the indirect radiation force components, the direction of the reflected radiation coming of the other plane has to be found first. Figure 3.12 shows the geometry. The projection of the incident radiation vector $\mathbf{i}$ onto the normal vector $\mathbf{n}_i$ of either plane is given by $(\mathbf{i} \cdot \mathbf{n}_i) \mathbf{n}_i$ and the component of $\mathbf{i}$ perpendicular to $\mathbf{n}_i$ is the difference between the incident radiation vector and its projection onto the normal vector: $\mathbf{i} - (\mathbf{i} \cdot \mathbf{n}_i) \mathbf{n}_i$. Then $\mathbf{i}_i$, the direction of the reflected radiation of either plane, is the sum of the perpendicular component and the negative of the projection:

$$\begin{aligned} \mathbf{i}_i &= [\mathbf{i} - (\mathbf{i} \cdot \mathbf{n}_i) \mathbf{n}_i] - (\mathbf{i} \cdot \mathbf{n}_i) \mathbf{n}_i = \\ &= \mathbf{i} - 2(\mathbf{i} \cdot \mathbf{n}_i) \mathbf{n}_i \end{aligned} \quad (3.37)$$

Calculation of the forces and moments resulting from the indirect radiation requires knowledge of the exact area of the sail that receives the indirect radiation. Calculating this

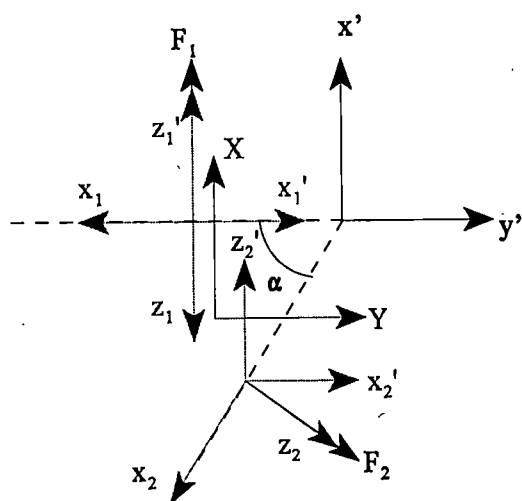


Figure 3.11: Moments due to the direct radiation

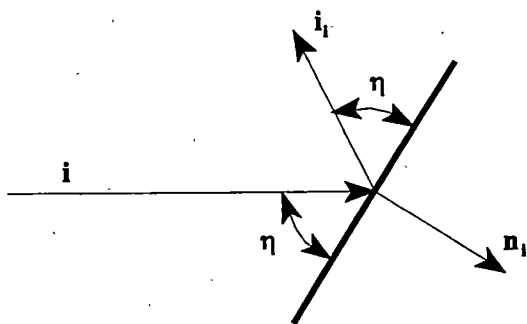


Figure 3.12: Calculation of reflected radiation vector

area is done by projecting the reflecting surface onto the receiving surface, the same way as for calculating the part of the sail planes that were shadowed by the other sail plane. This projecting needs to be done along the reflected radiation vector.

3.3.2.1 From plane 1 to plane 2

The indirect radiation vector in the local rotated frame, i_1' is:

$$i_1' = i' - 2(i' \cdot n_1')n_1'$$

$$= \begin{pmatrix} \cos \theta \\ -\sin \theta \cos \Omega \\ \sin \theta \sin \Omega \end{pmatrix} - 2 \cos \theta \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} -\cos \theta \\ -\sin \theta \cos \Omega \\ \sin \theta \sin \Omega \end{pmatrix} \quad (3.38)$$

The components of n_2' in this local rotated frame are given by equation 3.7. There will be no indirect radiation force when the angle between the i_1' and n_2' vectors is greater than 90° .

The coordinates of the tip t_1' is given by equation 3.31. The projection of the tip onto the second plane along the indirect radiation vector lies on the intersection of the line defined by:

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{pmatrix} 0 \\ -l_2 \\ 0 \end{pmatrix} + \lambda \begin{pmatrix} -\cos \theta \\ -\sin \theta \cos \Omega \\ \sin \theta \sin \Omega \end{pmatrix} \quad (3.39)$$

and the plane defined by:

$$(\cos \alpha x)x + (\sin \alpha x)y = 0 \quad (3.40)$$

The coordinates of the projected tip are then found to be:

$$t_{1_p}' = \frac{l_2}{\cos \alpha x \cos \theta + \cos \Omega \sin \alpha x \sin \theta} \begin{pmatrix} \sin \alpha x \cos \theta \\ -\cos \alpha x \cos \theta \\ -\sin \alpha x \sin \theta \sin \Omega \end{pmatrix} \quad (3.41)$$

The coordinates t_2' of the tip of plane 2 in the rotated frame are given by equation 3.28. It is now possible to define 4 different lines, see also figure 3.10:

$$\text{line 1-}t_{1_p}': (0 \ 0 \ 1/2 l_1)^T + \lambda (t_{1_p}' - (0 \ 0 \ 1/2 l_1)^T)$$

$$\text{line 2-}t_{1_p}': (0 \ 0 \ -1/2 l_1)^T + \mu (t_{1_p}' - (0 \ 0 \ -1/2 l_1)^T)$$

$$\text{line 1-}t_2': (0 \ 0 \ 1/2 l_1)^T + \nu (t_2' - (0 \ 0 \ 1/2 l_1)^T)$$

$$\text{line 2-}t_2': (0 \ 0 \ -1/2 l_1)^T + \eta (t_2' - (0 \ 0 \ -1/2 l_1)^T)$$

The same procedure as for the calculation of the shadowed area can be used for finding the coordinates of point 3, the area that is illuminated and the center of mass of the illuminated area. The force is:

$$F_3 = P_{sol} \left(\frac{r_o}{r} \right)^2 S(n_2 \cdot i_1)^2 n_2 = F_3 n_2 \quad (3.42)$$

acting on the center of mass of the projected area. The resulting moment in the X, Y, Z frame is:

$$M_3 = \begin{pmatrix} -(z_4 - z_{cm}) F_3 \sin \alpha \\ -(z_4 - z_{cm}) F_3 \cos \alpha \\ (x_4 - x_{cm}) F_3 \sin \alpha + (y_4 - y_{cm}) F_3 \cos \alpha \end{pmatrix} \quad (3.43)$$

3.3.2.2 From plane 2 to plane 1

The indirect radiation vector in the local rotated frame, i_2' is:

$$i_2' = i' - 2(i' \cdot n_2')n_2' = \begin{pmatrix} \cos \theta (1 - 2 \cos^2 \alpha x) + \sin \theta \cos \Omega \sin 2 \alpha x \\ \sin \theta \cos \Omega (2 \sin^2 \alpha x - 1) - \cos \theta \sin 2 \alpha x \\ \sin \Omega \sin \theta \end{pmatrix} \quad (3.44)$$

The direction of n_1' in the local rotated frame is along the x-axis. There will be no indirect radiation force when the angle between the i_2' and n_1' vectors is greater than 90° . The coordinates of the tip of plane 2 in the local rotated frame are given in equation 3.28. Now the intersection of the line defined by:

$$\begin{pmatrix} -(l_2 + l_3) \sin \alpha x \\ (l_2 + l_3) \cos \alpha x \\ 0 \end{pmatrix} + \lambda \begin{pmatrix} \cos \theta (1 - 2 \cos^2 \alpha x) + \sin \theta \cos \Omega \sin 2 \alpha x \\ \sin \theta \cos \Omega (2 \sin^2 \alpha x - 1) - \cos \theta \sin 2 \alpha x \\ \sin \Omega \sin \theta \end{pmatrix} \quad (3.45)$$

with the plane $x = 0$ has to be found. The resulting coordinates of the projected tip are:

$$t_{2p}' = (l_2 + l_3) \begin{pmatrix} 0 \\ \frac{\sin \alpha x \sin \theta \cos \Omega - \cos \alpha x \cos \theta}{\cos \theta (1 - 2 \cos^2 \alpha x) + \sin \theta \cos \Omega \sin 2 \alpha x} \\ \frac{\sin \alpha x \sin \theta \sin \Omega}{\cos \theta (1 - 2 \cos^2 \alpha x) + \sin \theta \cos \Omega \sin 2 \alpha x} \end{pmatrix} \quad (3.46)$$

In order to check whether the projection lies in plane 1, the position of the tip of plane 1 has to be found. The coordinates t_1' of the tip of plane 1 in the local rotated frame are given in equation 3.31. Now it is possible again to define 4 different lines, see also figure 3.8:

$$\text{line 1-} t_{2p}': (0 \ 0 \ 1/2 l_1)^T + \lambda (t_{2p}' - (0 \ 0 \ 1/2 l_1)^T)$$

$$\text{line } 2-t_2': (0 \ 0 \ -1/2 l_1)^T + \mu (t_2' - (0 \ 0 \ -1/2 l_1)^T)$$

$$\text{line } 1-t_1': (0 \ 0 \ 1/2 l_1)^T + \nu (t_1' - (0 \ 0 \ 1/2 l_1)^T)$$

$$\text{line } 2-t_1': (0 \ 0 \ -1/2 l_1)^T + \eta (t_1' - (0 \ 0 \ -1/2 l_1)^T)$$

The same procedure as before can now be used in finding the coordinates of point 3, the area that is illuminated and its center of mass in the local rotated reference frame. Thus, the force is:

$$F_4 = P_{sol} \left(\frac{r_o}{r} \right)^2 S (n_1 \cdot i_2)^2 n_1 = F_4 n_1 \quad (3.47)$$

acting on the center of mass of the projected plane. The resulting moment in the X, Y, Z frame is:

$$M_4 = \begin{pmatrix} 0 \\ (z_4 - z_{cm}) F_4 \\ -(y_4 - y_{cm}) F_4 \end{pmatrix} \quad (3.48)$$

Figure 3.13 shows the flow chart that indicates how the forces are calculated in the computer program.

3.4 Gravity gradient moments

Besides moments caused by the solar sail forces not acting on the center of mass of the system, gravity gradient moments will also be acting on the sail. These moments are given by the relation:

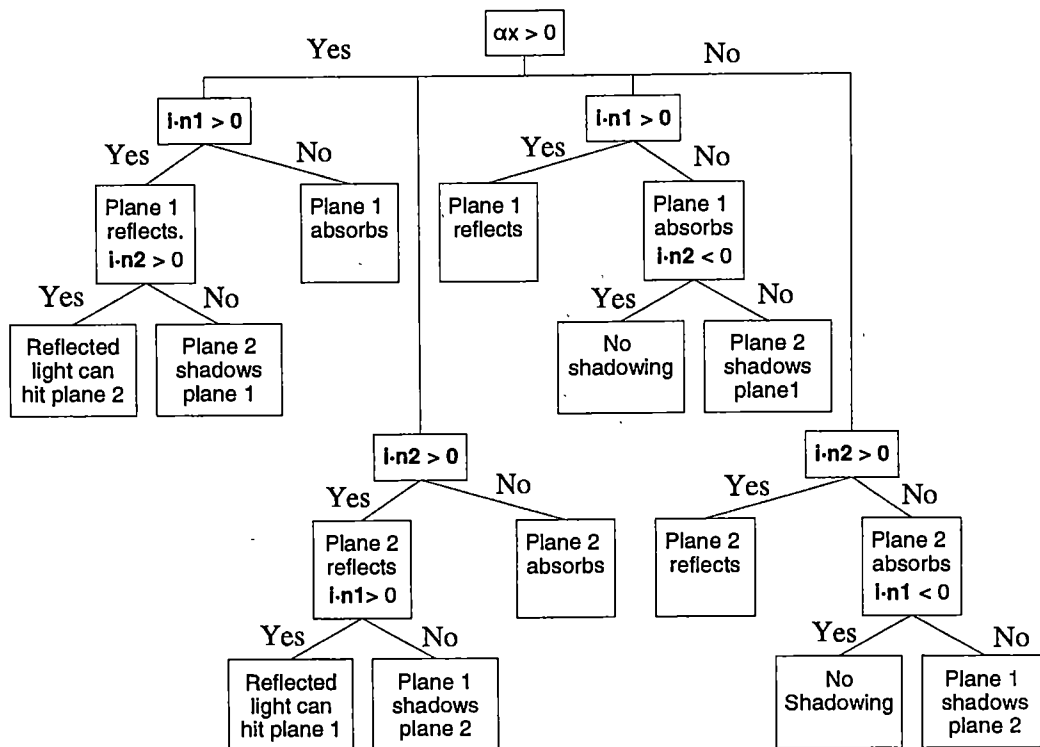


Figure 3.13: Flow chart of the calculation of the sail forces.

$$M_g = \frac{3GM}{r_o^3} \int_B (\mathbf{r} \cdot \mathbf{r}_o)(\mathbf{r} \times \mathbf{r}_o) dm \quad (3.49)$$

In this equation, $\mathbf{r}$ is the vector from the center of mass of the system to an element of mass dm , $\mathbf{r}_o$ is the vector from the origin of the coordinate system to the center of mass of the system, r_o is the length of this vector, which is the distance Sun-Sail, GM is the gravitational constant of the sun. In this case, these two vectors are given by:

$$\mathbf{r} = \begin{pmatrix} x \\ y \\ z \end{pmatrix}, \quad \mathbf{r}_o = r_o \begin{pmatrix} \cos \theta \\ -\sin \theta \cos \Omega \\ \sin \theta \sin \Omega \end{pmatrix} \quad (3.50)$$

Substituting equation (3.56) into (3.55) and calculating the dot and cross product, results in:

$$M_g = \frac{3GM}{r_o^3} \int_B (\cos \theta x - \sin \theta \cos \Omega y + \sin \theta \sin \Omega z) \begin{pmatrix} \sin \theta \sin \Omega y + \sin \theta \cos \Omega z \\ \cos \theta z - \sin \theta \sin \Omega x \\ -\sin \theta \cos \Omega x - \cos \theta y \end{pmatrix} dm \quad (3.51)$$

The definitions of the moments and products of inertia are:

$$\begin{aligned} I_{XX} &= \int_B (y^2 + z^2) dm, & I_{YY} &= \int_B (x^2 + z^2) dm, & I_{ZZ} &= \int_B (x^2 + y^2) dm \\ I_{XY} &= \int_B xy dm, & I_{XZ} &= \int_B xz dm, & I_{YZ} &= \int_B yz dm \end{aligned} \quad (3.52)$$

In equation 3.23 it was found that both I_{XZ} and I_{YZ} are zero, due to the symmetry of the system. By using the definitions as posted in equation 3.52, equation 3.51 can be written in terms of these moments and products of inertia:

$$\mathbf{M}_g = \frac{3GM}{r_o^3} \begin{pmatrix} \sin^2 \theta \sin \Omega \cos \Omega (I_{YY} - I_{ZZ}) + \sin \theta \cos \theta \sin \Omega I_{XY} \\ \sin \theta \cos \theta \sin \Omega (I_{XX} - I_{ZZ}) + \sin^2 \theta \sin \Omega \cos \Omega I_{XY} \\ \sin \theta \cos \theta \cos \Omega (I_{XX} - I_{YY}) + (\sin^2 \theta \cos^2 \Omega - \cos^2 \theta) I_{XY} \end{pmatrix} \quad (3.53)$$

3.5 Attitude dynamics

The general equation that describes dynamic motion of a system in a body fixed, non-inertial reference system is given by the following relation:

$$\frac{d\mathbf{H}}{dt} = \frac{\partial \mathbf{H}}{\partial t} + \boldsymbol{\omega} \times \mathbf{H} = \mathbf{M} \quad (3.54)$$

In this equation, $\mathbf{H}$ is the angular momentum vector of the system, $\boldsymbol{\omega}$ is the angular rotation vector of the system and $\mathbf{M}$ are the moments acting on the system. In the case of the two plane solar sail the inertia tensor I is given by equation 3.23, which leads to the following relation for the angular momentum vector:

$$\mathbf{H} = I\boldsymbol{\omega} = \begin{bmatrix} I_{XX} & -I_{XY} & 0 \\ -I_{XY} & I_{YY} & 0 \\ 0 & 0 & I_{ZZ} \end{bmatrix} \begin{pmatrix} \omega_X \\ \omega_Y \\ \omega_Z \end{pmatrix} = \begin{bmatrix} I_{XX}\omega_X - I_{XY}\omega_Y \\ -I_{XY}\omega_X + I_{YY}\omega_Y \\ I_{ZZ}\omega_Z \end{bmatrix} \quad (3.55)$$

Substituting 3.55 into 3.54 and expanding out, leads to:

$$\begin{aligned} I_{XX}\dot{\omega}_X - I_{XY}\dot{\omega}_Y + (I_{ZZ} - I_{YY})\omega_Y\omega_Z + I_{XY}\omega_X\omega_Z &= M_X \\ -I_{XY}\dot{\omega}_X + I_{YY}\dot{\omega}_Y + (I_{XX} - I_{ZZ})\omega_X\omega_Z - I_{XY}\omega_Y\omega_Z &= M_Y \\ I_{ZZ}\dot{\omega}_Z + (I_{YY} - I_{XX})\omega_X\omega_Y + I_{XY}(\omega_Y^2 - \omega_X^2) &= M_Z \end{aligned} \quad (3.56)$$

These equations 3.56 can be solved for $\dot{\omega}_X$, $\dot{\omega}_Y$, and $\dot{\omega}_Z$:

$$\begin{aligned}
(I_{XX}I_{YY} - I_{XY}^2)\dot{\omega}_X &= -(I_{YY}I_{ZZ} - I_{YZ}^2 - I_{XY}^2)\omega_Y\omega_Z - I_{XY}(I_{XX} + I_{YY} - I_{ZZ})\omega_X\omega_Z + I_{YY}M_X + I_{XY}M_Y \\
(I_{XX}I_{YY} - I_{XY}^2)\dot{\omega}_Y &= -(I_{XX}^2 - I_{XX}I_{ZZ} + I_{XY}^2)\omega_X\omega_Z + I_{XY}(I_{XX} + I_{YY} - I_{ZZ})\omega_Y\omega_Z + I_{XY}M_X + I_{XX}M_Y \\
I_{ZZ}\dot{\omega}_Z &= -(I_{YY} - I_{XX})\omega_X\omega_Y - I_{XY}(\omega_Y^2 - \omega_X^2) + M_Z
\end{aligned} \tag{3.57}$$

3.5.1 Angular rotation vector

Now, relations between $\omega_x, \omega_y, \omega_z$ on one side and the varying control angles (θ, β, Ω) and the varying spherical angles (ϕ, ψ) on the other side has to be found. The spherical angles need to be included since the spherical reference system is not an inertial system. The spherical system is found by applying 2 successive rotations to an inertial system X, Y, Z. First a rotation about the Z-axis over an angle ϕ . The rotation matrix R_1 and angular velocity vector ω_1 are:

$$R_1 = \begin{pmatrix} \cos \phi & \sin \phi & 0 \\ -\sin \phi & \cos \phi & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad \omega_1 = \begin{pmatrix} 0 \\ 0 \\ \dot{\phi} \end{pmatrix} \tag{3.58}$$

The second rotation is about the new Y-axis over an angle $-\psi$. For this rotation, the rotation matrix R_2 and angular velocity vector ω_2 are:

$$R_2 = \begin{pmatrix} \cos \psi & 0 & \sin \psi \\ 0 & 1 & 0 \\ -\sin \psi & 0 & \cos \psi \end{pmatrix} \quad \omega_2 = \begin{pmatrix} 0 \\ -\dot{\psi} \\ 0 \end{pmatrix} + R_2 \omega_1 = \begin{pmatrix} \dot{\phi} \sin \psi \\ -\dot{\psi} \\ \dot{\phi} \cos \psi \end{pmatrix} \tag{3.59}$$

The relation with the control angles can be found by using the rotation matrices as defined in equations 3.6 and 3.5. The rotation matrix T from equation 3.5 is the product of two other rotation matrices R_3 and R_4 . The rotation R_3 is a rotation about the $\hat{e}_r$ -axis over an angle $\beta_x = 90^\circ - \beta$. The rotation matrix is:

$$R_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \sin \beta & \cos \beta \\ 0 & -\cos \beta & \sin \beta \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos \beta x & \sin \beta x \\ 0 & -\sin \beta x & \cos \beta x \end{pmatrix} \quad (3.60)$$

and the angular velocity vector is:

$$\omega_3 = \begin{pmatrix} -\dot{\beta} \\ 0 \\ 0 \end{pmatrix} + R_3 \begin{pmatrix} \dot{\phi} \sin \psi \\ -\dot{\psi} \\ \dot{\phi} \cos \psi \end{pmatrix} = \begin{pmatrix} -\dot{\beta} + \dot{\phi} \sin \psi \\ -\dot{\psi} \sin \beta + \dot{\phi} \cos \psi \cos \beta \\ \dot{\psi} \cos \beta + \dot{\phi} \cos \psi \sin \beta \end{pmatrix} \quad (3.61)$$

The second rotation is a rotation about the current z-axis over an angle θ . The rotation matrix

R_4 is therefore:

$$R_4 = \begin{pmatrix} \cos \theta & \sin \theta & 0 \\ -\sin \theta & \cos \theta & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad (3.62)$$

and the combined angular velocity is:

$$\omega_4 = \begin{pmatrix} 0 \\ 0 \\ \dot{\theta} \end{pmatrix} + R_4 \omega_3 = \begin{pmatrix} (-\dot{\beta} + \dot{\phi} \sin \psi) \cos \theta + (-\dot{\psi} \sin \beta + \dot{\phi} \cos \psi \cos \beta) \sin \theta \\ (\dot{\beta} - \dot{\phi} \sin \psi) \sin \theta + (-\dot{\psi} \sin \beta + \dot{\phi} \cos \psi \cos \beta) \cos \theta \\ \dot{\theta} + \dot{\psi} \cos \beta + \dot{\phi} \cos \psi \sin \beta \end{pmatrix} \quad (3.63)$$

The final rotation is a rotation about the current x-axis over an angle Ω . This rotation matrix is the matrix R from equation 3.6. The combined angular velocities of these five rotations are the $\omega_x, \omega_y, \omega_z$ of the sail. These combined angular velocities are:

$$\omega = \begin{pmatrix} \omega_x \\ \omega_y \\ \omega_z \end{pmatrix} = \begin{pmatrix} \dot{\Omega} \\ 0 \\ 0 \end{pmatrix} + R \omega_4$$

$$= \begin{pmatrix} \dot{\Omega} + (-\dot{\beta} + \dot{\phi} \sin \psi) \cos \theta + (-\psi \sin \beta + \dot{\phi} \cos \psi \cos \beta) \sin \theta \\ \left[\{(\dot{\beta} - \dot{\phi} \sin \psi) \sin \theta + (-\psi \sin \beta + \dot{\phi} \cos \psi \cos \beta) \cos \theta\} \cos \Omega + \right. \\ \left. + (\dot{\theta} + \psi \cos \beta + \dot{\phi} \cos \psi \sin \beta) \sin \Omega \right] \\ \left[-\{(\dot{\beta} - \dot{\phi} \sin \psi) \sin \theta + (-\psi \sin \beta + \dot{\phi} \cos \psi \cos \beta) \cos \theta\} \sin \Omega + \right. \\ \left. + (\dot{\theta} + \psi \cos \beta + \dot{\phi} \cos \psi \sin \beta) \cos \Omega \right] \end{pmatrix} \quad (3.64)$$

Equations 3.64 can now be solved for $\dot{\Omega}$, $\dot{\beta}$, and $\dot{\theta}$:

$$\begin{aligned} \dot{\Omega} &= \omega_x + \frac{\cos \Omega}{\tan \theta} \omega_y - \frac{\sin \Omega}{\tan \theta} \omega_z + \frac{\sin \beta}{\sin \theta} \psi - \frac{\cos \psi \cos \beta}{\sin \theta} \dot{\phi} \\ \dot{\beta} &= \frac{\cos \Omega}{\sin \theta} \omega_y - \frac{\sin \Omega}{\sin \theta} \omega_z + \frac{\sin \beta}{\tan \theta} \psi + \left(\sin \psi - \frac{\cos \psi \cos \beta}{\tan \theta} \right) \dot{\phi} \\ \dot{\theta} &= \sin \Omega \omega_y + \cos \Omega \omega_z - \cos \beta \psi - \cos \psi \sin \beta \dot{\phi} \end{aligned} \quad (3.65)$$

Equations 3.65 together with equations 3.57 form the attitude dynamics equations. This is a set of 6 coupled, non-linear, first order differential equations.

As can be seen from equations 3.65, singularities will occur when $\sin \theta = 0$. Therefore it will be better to use the quaternion formulation for the calculation of Ω , β and θ over time.

3.5.2 Quaternion formulation

A quaternion Q is a four parameter representation of a transformation matrix, consisting of a scalar and a vector:

$$Q = q_1 + q_2 \hat{i} + q_3 \hat{j} + q_4 \hat{k} \quad (3.66)$$

It makes use of Euler's theorem, which states that any rotation of one coordinate system with respect to another may be described by a rotation through some angle about a single fixed axis. The three elements of the vector define this axis of rotation, and the scalar defines the magnitude of the rotation angle, see also figure 3.14. In this figure, μ is the amount of rotation, and α , β and γ are the angles between the rotation axis and the X, Y and Z axis respectively. The elements for the quaternion describing this rotation are, by definition:

$$\begin{aligned} q_1 &= \cos \frac{\mu}{2}, & q_3 &= \sin \frac{\mu}{2} \cos \beta \\ q_2 &= \sin \frac{\mu}{2} \cos \alpha, & q_4 &= \sin \frac{\mu}{2} \cos \gamma \end{aligned} \quad (3.67)$$

And the rotation matrix of the system becomes:

$$T_q = \begin{pmatrix} q_1^2 + q_2^2 - q_3^2 - q_4^2 & 2(q_2 q_3 - q_1 q_4) & 2(q_2 q_4 + q_1 q_3) \\ 2(q_2 q_3 + q_1 q_4) & q_1^2 - q_2^2 + q_3^2 - q_4^2 & 2(q_3 q_4 - q_1 q_2) \\ 2(q_2 q_4 - q_1 q_3) & 2(q_3 q_4 + q_1 q_2) & q_1^2 - q_2^2 - q_3^2 + q_4^2 \end{pmatrix} \quad (3.68)$$

When using the control angles, and by adapting the notation $\beta_x = 90^\circ - \beta$, the rotation matrix for the system is (from equations 3.6, 3.62, 3.60, 3.59 and 3.58):

$$T_a = \begin{pmatrix} \cos \theta \cos \psi \cos \phi & \cos \theta \cos \psi \sin \phi & -\cos \theta \sin \psi \\ -\sin \theta \cos \beta_x \sin \phi & +\sin \theta \cos \beta_x \cos \phi & +\sin \theta \sin \beta_x \cos \psi \\ +\sin \theta \sin \beta_x \sin \psi \cos \phi & +\sin \theta \sin \beta_x \sin \psi \sin \phi & \end{pmatrix} \quad (3.69)$$

$$\begin{pmatrix} -\sin \theta \cos \Omega \cos \psi \cos \phi & -\sin \theta \cos \Omega \cos \psi \sin \phi & \sin \theta \cos \Omega \sin \psi \\ -(\cos \theta \cos \beta_x \cos \Omega - \sin \beta_x \sin \Omega) \sin \phi & +(\cos \theta \cos \beta_x \cos \Omega - \sin \beta_x \sin \Omega) \cos \phi & +(\cos \theta \sin \beta_x \cos \Omega + \cos \beta_x \sin \Omega) \cos \psi \\ +(\cos \theta \sin \beta_x \cos \Omega + \sin \Omega \cos \beta_x) \sin \psi \cos \phi & +(\cos \theta \cos \Omega \sin \beta_x + \sin \Omega \cos \beta_x) \sin \psi \sin \phi & \end{pmatrix}$$

$$\begin{pmatrix} \sin \theta \sin \Omega \cos \psi \cos \phi & \sin \theta \sin \Omega \cos \psi \sin \phi & -\sin \theta \sin \Omega \sin \psi \\ +(\cos \theta \cos \beta_x \sin \Omega + \cos \Omega \sin \beta_x) \sin \phi & -(\cos \theta \cos \beta_x \sin \Omega + \cos \Omega \sin \beta_x) \cos \phi & +(-\cos \theta \sin \beta_x \sin \Omega + \cos \beta_x \cos \Omega) \cos \psi \\ +(-\cos \theta \sin \Omega \sin \beta_x + \cos \Omega \cos \beta_x) \sin \psi \cos \phi & +(-\cos \theta \sin \Omega \sin \beta_x + \cos \Omega \cos \beta_x) \sin \psi \sin \phi & \end{pmatrix}$$

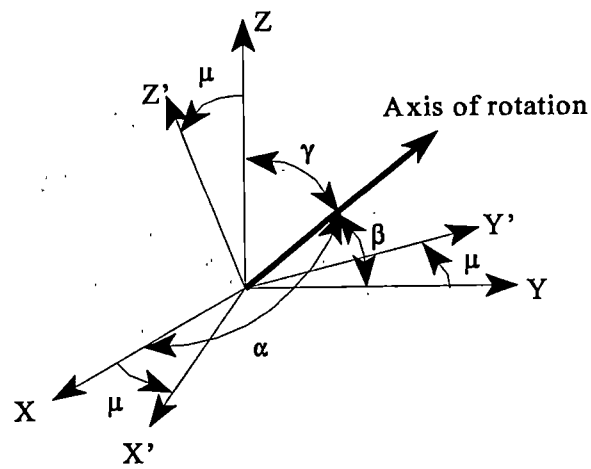


Figure 3.14: Definition of the quaternion rotation angles

Since equations 3.68 and 3.69 have to be equal, relations between the quaternion elements and the angles can be found, see also Appendix A. These relations are:

$$\begin{aligned}
 q_1 &= \cos \frac{\phi}{2} \left(\cos \frac{\theta}{2} \cos \frac{\psi}{2} \cos \frac{\beta x + \Omega}{2} + \sin \frac{\theta}{2} \sin \frac{\psi}{2} \sin \frac{\beta x - \Omega}{2} \right) \\
 &\quad - \sin \frac{\phi}{2} \left(\sin \frac{\theta}{2} \cos \frac{\psi}{2} \cos \frac{\beta x - \Omega}{2} - \cos \frac{\theta}{2} \sin \frac{\psi}{2} \sin \frac{\beta x + \Omega}{2} \right) \\
 q_2 &= -\cos \frac{\phi}{2} \left(\cos \frac{\theta}{2} \cos \frac{\psi}{2} \sin \frac{\beta x + \Omega}{2} + \sin \frac{\theta}{2} \sin \frac{\psi}{2} \cos \frac{\beta x - \Omega}{2} \right) \\
 &\quad - \sin \frac{\phi}{2} \left(\sin \frac{\theta}{2} \cos \frac{\psi}{2} \sin \frac{\beta x - \Omega}{2} - \cos \frac{\theta}{2} \sin \frac{\psi}{2} \cos \frac{\beta x + \Omega}{2} \right) \\
 q_3 &= \cos \frac{\phi}{2} \left(\sin \frac{\theta}{2} \cos \frac{\psi}{2} \sin \frac{\beta x - \Omega}{2} - \cos \frac{\theta}{2} \sin \frac{\psi}{2} \cos \frac{\beta x + \Omega}{2} \right) \\
 &\quad - \sin \frac{\phi}{2} \left(\cos \frac{\theta}{2} \cos \frac{\psi}{2} \sin \frac{\beta x + \Omega}{2} + \sin \frac{\theta}{2} \sin \frac{\psi}{2} \cos \frac{\beta x - \Omega}{2} \right) \\
 q_4 &= -\cos \frac{\phi}{2} \left(\sin \frac{\theta}{2} \cos \frac{\psi}{2} \cos \frac{\beta x - \Omega}{2} - \cos \frac{\theta}{2} \sin \frac{\psi}{2} \sin \frac{\beta x + \Omega}{2} \right) \\
 &\quad - \sin \frac{\phi}{2} \left(\cos \frac{\theta}{2} \cos \frac{\psi}{2} \cos \frac{\beta x + \Omega}{2} + \sin \frac{\theta}{2} \sin \frac{\psi}{2} \sin \frac{\beta x - \Omega}{2} \right)
 \end{aligned} \tag{3.70}$$

The time derivatives for the quaternion elements can be found by taking the time derivative of equations 3.70 and substituting $\dot{\Omega}$, $\dot{\beta}$ and $\dot{\theta}$ by the results found in equation 3.65. The resulting equations then only depend on the three angular velocities (ω_x , ω_y , ω_z) and the quatern elements themselves. The resulting equations are:

$$\begin{aligned}
 \dot{q}_1 &= \frac{1}{2} (\omega_x q_2 + \omega_y q_3 + \omega_z q_4) \\
 \dot{q}_2 &= \frac{1}{2} (-\omega_x q_1 + \omega_z q_3 - \omega_y q_4) \\
 \dot{q}_3 &= \frac{1}{2} (-\omega_y q_1 - \omega_z q_2 + \omega_x q_4) \\
 \dot{q}_4 &= \frac{1}{2} (-\omega_z q_1 + \omega_y q_2 - \omega_x q_3)
 \end{aligned} \tag{3.71}$$

Equations 3.71 and 3.57 together now form the attitude dynamics equations, a set of 7 coupled, non-linear, first order differential equations.

In order to be able to calculate the normal vectors and incident radiation vector, and thus the solar sail forces and moments, it is necessary to find the 9 elements of the rotation matrix that define the rotation from the spherical reference system into the local rotated reference system. It is possible to find this matrix from the quaternion elements and the rotation matrix from an inertial system into the spherical system. Since T_q equals T_a , the following relation holds:

$$T_q = R R_4 R_3 R_2 R_1 \quad (3.72)$$

Thus:

$$T_q R_1^T R_2^T = R R_4 R_3 = \begin{pmatrix} \cos \theta & \sin \theta \sin \beta & \sin \theta \cos \beta \\ -\sin \theta \cos \Omega & \cos \theta \cos \Omega \sin \beta - \cos \beta \sin \Omega & \cos \theta \cos \Omega \cos \beta + \sin \Omega \sin \beta \\ \sin \theta \sin \Omega & -\cos \theta \sin \Omega \sin \beta - \cos \Omega \cos \beta & -\cos \theta \sin \Omega \cos \beta + \cos \Omega \sin \beta \end{pmatrix} \quad (3.73)$$

with:

$$R_1^T R_2^T = \begin{pmatrix} \cos \psi \cos \phi & -\sin \phi & -\sin \psi \cos \phi \\ \cos \psi \sin \phi & \cos \phi & -\sin \psi \sin \phi \\ \sin \psi & 0 & \cos \psi \end{pmatrix} \quad (3.74)$$

being the transpose of the rotation matrix from an inertial system into a spherical system. All elements of this matrix are known, since ψ and ϕ will be calculated directly in the trajectory equations. Thus all elements of the matrix in equation 3.73 can be calculated. Since only

these elements appear in the relations derived before, there is no need to solve for each angle individually.

3.6 Flight trajectory

The equations of motion along the flight trajectory in spherical coordinates will be derived by using Lagrange's equation:

$$\frac{d}{dt} \left(\frac{\partial T}{\partial \dot{q}_i} \right) - \frac{\partial T}{\partial q_i} = Q_i \quad (3.75)$$

Here T is the total kinetic energy of the system, the q_i 's are the generalized coordinates and the Q_i 's are the generalized forces. The kinetic energy of the motion in the β , θ , and Ω direction is small compared to that in the r , ϕ , and ψ direction and will therefore be neglected. The resulting kinetic energy can thus be written as:

$$T = \frac{1}{2} m \left(V_r^2 + V_\phi^2 + V_\psi^2 \right) \quad (3.76)$$

In spherical coordinates, the velocities can be written as:

$$\begin{aligned} V_r &= \dot{r} \\ V_\phi &= r \dot{\phi} \cos \psi \\ V_\psi &= r \dot{\psi} \end{aligned} \quad (3.77)$$

substituting equations 3.77 into 3.76 results in:

$$T = \frac{1}{2} m \left(\dot{r}^2 + r^2 \dot{\phi}^2 \cos^2 \psi + r^2 \dot{\psi}^2 \right) \quad (3.78)$$

3.6.1 Generalized coordinate r

The generalized force in the r -direction is given by:

$$Q_r = -\frac{GMm}{r^2} + F_r \quad (3.79)$$

GM is the gravitational constant of the sun and F_r is the solar sail force in the r -direction. Substituting equations 3.79 and 3.78 into 3.75 gives the following equation for the motion in the radial direction:

$$\ddot{r} - r\dot{\phi}^2 \cos^2 \psi - r\psi^2 = -\frac{GM}{r^2} + \frac{F_r}{m} \quad (3.80)$$

3.6.2 Generalized coordinate ϕ

Since this generalized coordinate is an angle, the generalized force in this case is actually a moment. The only force acting in the ϕ -direction is a solar sail force, and therefore Q_ϕ can be written as:

$$Q_\phi = F_\phi r \cos \psi \quad (3.81)$$

This leads to the following equation for the motion in the ϕ -direction:

$$\ddot{\phi} r \cos \psi + 2\dot{r}\dot{\phi} \cos \psi - 2r\dot{\phi}\dot{\psi} \sin \psi = \frac{F_\phi}{m} \quad (3.82)$$

3.6.3 Generalized coordinate ψ

This is also an angle, so again the generalized force is a moment, caused by the solar sail force in the ψ -direction:

$$Q_\psi = F_\psi r \quad (3.83)$$

and the following equation describes the motion in the ψ -direction:

$$r\ddot{\psi} + 2\dot{r}\dot{\psi} + r\dot{\phi}^2 \cos \psi \sin \psi = \frac{F_\psi}{m} \quad (3.84)$$

The three second order differential equations 3.80, 3.82, and 3.84 can be transformed into the following set of 6 first order differential equations:

$$\begin{aligned}
 \frac{dr}{dt} &= V_r \\
 \frac{d\phi}{dt} &= \frac{V_\phi}{r \cos \psi} \\
 \frac{d\psi}{dt} &= \frac{V_\psi}{r} \\
 \frac{dV_r}{dt} &= \frac{V_\phi^2 + V_\psi^2}{r} - \frac{GM}{r^2} + \frac{F_r}{m} \\
 \frac{dV_\phi}{dt} &= -\frac{V_r V_\phi}{r} + \frac{V_\phi V_\psi \tan \psi}{r} + \frac{F_\phi}{m} \\
 \frac{dV_\psi}{dt} &= -\frac{V_r V_\psi}{r} - \frac{V_\phi^2 \tan \psi}{r} + \frac{F_\psi}{m}
 \end{aligned} \tag{3.85}$$

3.6.4 Dimensionless trajectory equations

Making the equations 3.85 dimensionless will lead to relative small values for the 6 state variables, which will improve the accuracy of the integration process. The following equations will be used for making the state variables dimensionless:

$$\begin{aligned}
 r &= \frac{r^*}{AU} & V &= \frac{V^*}{\sqrt{\frac{GM}{AU}}} \\
 a &= \frac{a^*}{\left(\frac{GM}{AU^2}\right)} & t &= \frac{t^*}{\sqrt{\frac{AU^3}{GM}}}
 \end{aligned} \tag{3.86}$$

In equations 3.86, the asterisk denotes the dimensional variable. GM is the gravitational constant of the Sun and AU , the astronomical unit, is the distance from the Earth to the Sun.

Substituting these equations into 3.85 leads to the following set of dimensionless differential equations:

$$\begin{aligned}
 \frac{dr}{dt} &= V_r \\
 \frac{d\phi}{dt} &= \frac{V_\phi}{r \cos \psi} \\
 \frac{d\psi}{dt} &= \frac{V_\psi}{r} \\
 \frac{dV_r}{dt} &= \frac{V_\phi^2 + V_\psi^2}{r} - \frac{1}{r^2} + \frac{r_o^2}{GM} \frac{F_r}{m} \\
 \frac{dV_\phi}{dt} &= -\frac{V_r V_\phi}{r} + \frac{V_\phi V_\psi \tan \psi}{r} + \frac{r_o^2}{GM} \frac{F_\phi}{m} \\
 \frac{dV_\psi}{dt} &= -\frac{V_r V_\psi}{r} - \frac{V_\phi^2 \tan \psi}{r} + \frac{r_o^2}{GM} \frac{F_\psi}{m}
 \end{aligned} \tag{3.87}$$

Chapter 4

The Coplanar Case

In this chapter it is shown that for a two plane solar sail with a variable angle between the two sail planes, it is indeed possible to achieve passive attitude stability and control for a simplified 2D- model. The resulting trajectory types depend on the angle between the two sail planes and the size difference between the two sail planes, but both inward and outward trajectories are possible.

4.1 Attitude dynamics and trajectory equations

In the coplanar case, $\psi=0^\circ$, $\beta=90^\circ$ and $\Omega=0^\circ$. Also, $\psi=\beta=\Omega=0$. This means that in this case the local frame and the local rotated frame are identical and the spherical reference frame becomes a polar reference system. In this polar system, the coordinates of the two sail plane normals and the incident sunlight vector are:

$$n_1 = \begin{pmatrix} \cos \theta \\ \sin \theta \\ 0 \end{pmatrix}, \quad n_2 = \begin{pmatrix} \cos(\alpha x + \theta) \\ \sin(\alpha x + \theta) \\ 0 \end{pmatrix}, \quad i = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \quad (4.1)$$

and the angular velocity vector ω , equation (3.64), becomes:

$$\omega = \begin{pmatrix} 0 \\ 0 \\ \dot{\phi} + \dot{\theta} \end{pmatrix} \quad (4.2)$$

This equation can be solved for $\dot{\theta}$ without introducing singularities. This means that there is no need for the use the quaternion formulation in this case and the attitude dynamics equations (3.57) and (3.65) reduce to:

$$\begin{aligned} \dot{\theta} &= \omega_z - \dot{\phi} \\ \dot{\omega}_z &= \frac{M_{Gz} + M_{Sz}}{I_{zz}} \end{aligned} \quad (4.3)$$

The gravity gradient moments from equation (3.53) will become:

$$M_G = \frac{3GM}{r_o^3} \begin{pmatrix} 0 \\ 0 \\ (I_{xx} - I_{yy}) \sin \theta \cos \theta - I_{xy} \cos 2\theta \end{pmatrix} \quad (4.4)$$

The flight trajectory equations can be found by substituting the values for ψ , β , Ω and their time derivatives as specified into equations (3.87):

$$\begin{aligned} \frac{dr}{dt} &= V_r, & \frac{d\phi}{dt} &= \frac{V_\phi}{r} \\ \frac{dV_r}{dt} &= \frac{V_\phi^2}{r} - \frac{1}{r^2} + \frac{r_o^2}{GM} \frac{F_r}{m} \\ \frac{dV_\phi}{dt} &= -\frac{V_r V_\phi}{r} + \frac{r_o^2}{GM} \frac{F_\phi}{m} \end{aligned} \quad (4.5)$$

4.2 Equilibrium points

Since it is assumed that the complex shaped sail design leads to passive dynamic stability of the sail, it should be possible to find for each α -value, a θ -value that remains constant over time, i.e. the system is in equilibrium. These equilibrium positions were found by investigation of the rotational motion around the sail center of mass for small parts of the trajectory for various values of α , the angle between the two sail planes. In this case, it can be assumed that $r = \text{constant}$, $\dot{\phi} = \text{constant}$, and equations (4.3) reduce to:

$$\ddot{\theta} = \dot{\omega}_z = \frac{M_{G_z} + M_{S_z}}{I_{zz}} \quad (4.6)$$

Figure 4.1 shows the equilibrium values of θ (deg) as a function of α , for the case $l_1 = 50$ m, $l_2 = 25$ m, $l_3 = 5$ m, 0 m and -5 m, $\rho = 1.3 \cdot 10^3$ kg/m³, $h = 24.23$ μ m, $k=0.0$ and $m_p=0.0$ kg. ρ is the sail film density, h the sail film thickness, k the structural mass factor and m_p the mass of the central bus (=payload). For the definitions of l_1, l_2, l_3 see figure 3.1.

For $0^\circ < \alpha < 180^\circ$, the results are the same for the three different configurations, and is given by the following relation:

$$\theta = 90^\circ + \frac{1}{2}\alpha \quad (4.7)$$

When substituting these values for α and θ into the equations for the two sail normal vectors, equation 4.1, and calculating the dot product with the incident sunlight vector, it is found that both sail planes are absorbing the sunlight for $0^\circ < \alpha < 180^\circ$, see also figure 4.2. This means that the magnitude of the solar sail forces at 1 AU are given by:

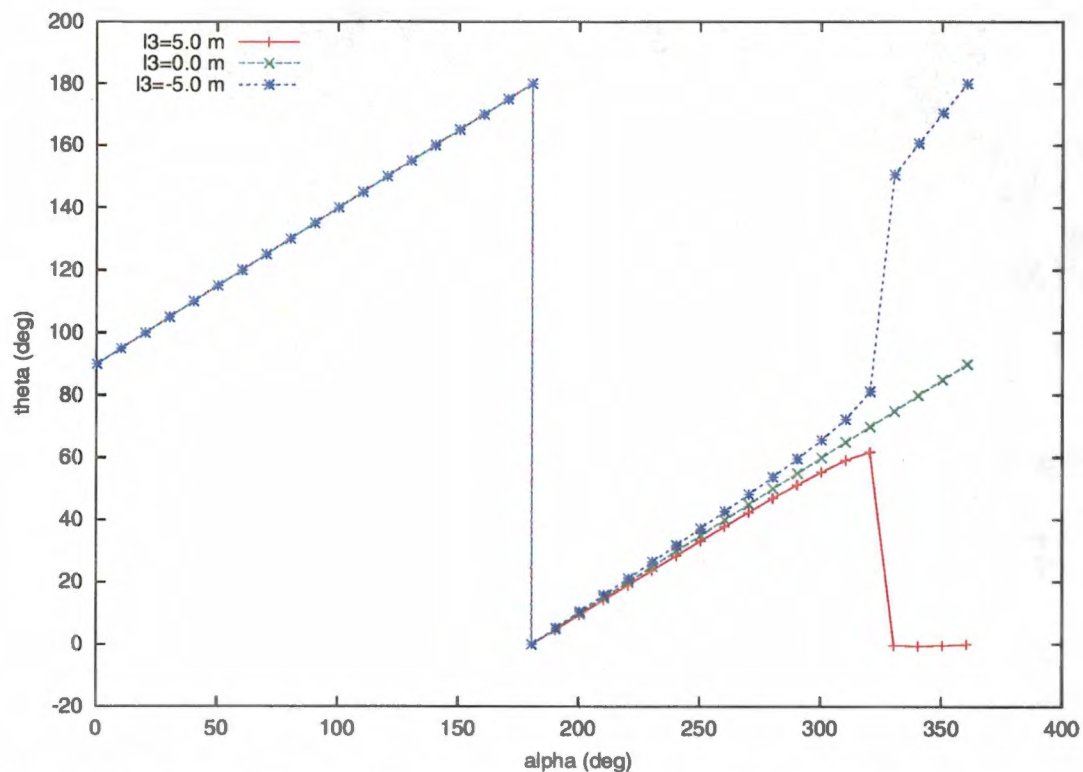


Figure 4.1: Equilibrium values of θ as function of α , $m_p = 0.0$ kg, $k = 0.0$

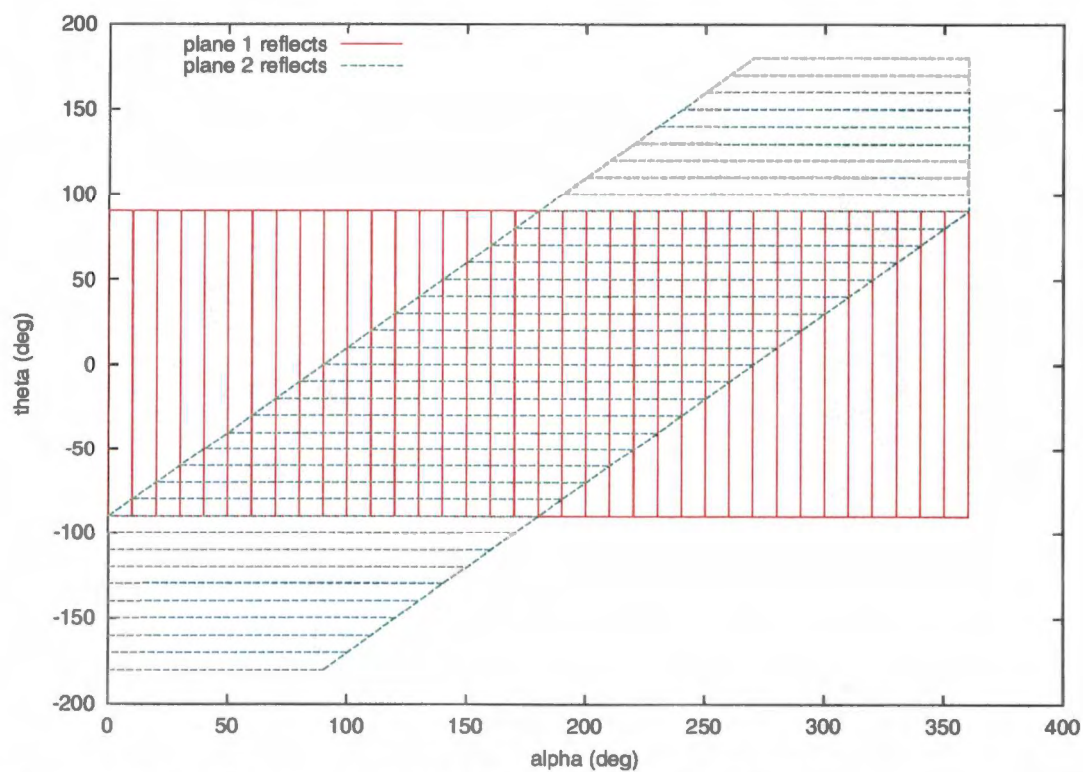


Figure 4.2: α and θ ranges for which the two sail planes reflect

$$F = -\frac{1}{2}P_{sol}(S_1(n_1 \cdot i) + S_2(n_2 \cdot i))i = -1.125 \cdot 10^{-4}(25 \cos \theta + (25 + l_3) \cos(\alpha x + \theta)) \text{ N} \quad (4.8)$$

and the moment by:

$$M_{Z_s} = \sum_{i=1}^2 -(y_i - y_{cm}) = \left(-\frac{25}{3} - y_{cm}\right) 2.8125 \cdot 10^{-3} \cos \theta + \left(\frac{-(25 + l_3) \cos \alpha}{3} - y_{cm}\right) 1.125 \cdot 10^{-4} (25 + l_3) \cos(\alpha x + \theta) \text{ Nm} \quad (4.9)$$

The moments and product of inertia I_{xx} , I_{yy} , I_{zz} and I_{xy} are given by:

$$I_{xx} = 2.734 \cdot 10^3 + 19.69 \left(-\frac{25}{3} - y_{cm}\right)^2 + 0.787(25 + l_3) \left[1.0417 \cdot 10^2 + \frac{(25 + l_3)^2}{18} \cos^2 \alpha + \left(-\frac{25 + l_3}{3} \cos \alpha - y_{cm}\right)^2 \right] \quad (4.10)$$

$$I_{yy} = 2.051 \cdot 10^3 + 19.69 x_{cm}^2 + 0.787(25 + l_3) \left[1.0417 \cdot 10^2 + \frac{(25 + l_3)^2}{18} \sin^2 \alpha + \left(-\frac{25 + l_3}{3} \sin \alpha - x_{cm}\right)^2 \right] \quad (4.11)$$

$$I_{zz} = 6.836 \cdot 10^2 + 19.69 \left(x_{cm}^2 + \left(-\frac{25}{3} - y_{cm}\right)^2 \right) + 0.787(25 + l_3) \left[\frac{(25 + l_3)^2}{18} + \left(-\frac{25 + l_3}{3} \sin \alpha - x_{cm}\right)^2 + \left(-\frac{25 + l_3}{3} \cos \alpha - y_{cm}\right)^2 \right] \quad (4.12)$$

$$I_{xy} = -19.69 x_{cm} \left(-\frac{25}{3} - y_{cm}\right) + 0.787(25 + l_3) \left[\frac{(25 + l_3)^2}{36} \sin 2\alpha + \left(-\frac{25 + l_3}{3} \sin \alpha - x_{cm}\right) \left(-\frac{25 + l_3}{3} \cos \alpha - y_{cm}\right) \right] \quad (4.13)$$

With $GM = 3.96396415708 \cdot 10^{-14} \text{ AU}^3/\text{s}^2$, the gravity gradient moment at 1 AU is:

$$M_{Z_G} = 3.9674 \cdot 10^{-14} \left(0.683 \cdot 10^3 + 19.69 \left[\left(-\frac{25}{3} - y_{xm}\right)^2 - x_{cm}^2 \right] + 0.787(25 + l_3) \left[\frac{(25 + l_3)^2}{18} \cos 2\alpha + \left(-\frac{25 + l_3}{3} \cos \alpha - y_{cm}\right)^2 - \left(-\frac{25 + l_3}{3} \sin \alpha - x_{cm}\right)^2 \right] \right) \quad (4.14)$$

Comparing equations 4.14 and 4.9, it is clear that the gravity gradient moment is much smaller compared to the solar radiation moment, and can be neglected. This means that the only explanation for why the results are the same for the different configurations has to be that there exist a solution to the problem $M_{Z_s} = 0$ which is independent of the dimensions.

When substituting the relations for the solar sail normals (equation 4.1), the incident sunlight vector (equation 4.1), the center of mass (equation 3.10) into the solar radiation force and moment equation (equations 4.8 and 4.9), the resulting moment equation is:

$$M_{Z_s} = \frac{1}{12} \frac{l_1 P_{sol} l_2}{2l_2 + l_3} \left[l_2^2 (\sin \alpha \cos \theta \sin \theta - \cos^2 \theta + \cos^2 \theta \cos \alpha + \sin \alpha \cos(\alpha - \theta) \sin \theta - \cos(\alpha - \theta) \cos \theta + \cos(\alpha - \theta) \cos \theta \cos \alpha) + l_2 l_3 (2 \sin \alpha \cos \theta \sin \theta - \cos^2 \theta + 2 \cos^2 \theta \cos \alpha + 2 \sin \alpha \cos(\alpha - \theta) \sin \theta - \cos(\alpha - \theta) \cos \theta + 2 \cos(\alpha - \theta) \cos \theta \cos \alpha) + l_3^2 (\sin \alpha \cos \theta \sin \theta + \cos^2 \theta \cos \alpha + \sin \alpha \cos(\alpha - \theta) \sin \theta + \cos(\alpha - \theta) \cos \theta \cos \alpha) \right] = 0 \quad (4.15)$$

In order for equation (4.15) to be independent of the geometry of the sail, each of the terms multiplying l_2^2 , $l_2 l_3$, and l_3^2 have to be zero. A solution can be found, and this solution is:

$$\tan \theta = \frac{-\cos \alpha - 1}{\sin \alpha} = \frac{-2 \cos^2 \frac{1}{2} \alpha}{2 \sin \frac{1}{2} \alpha \cos \frac{1}{2} \alpha} = \frac{\cos(-\frac{1}{2} \alpha)}{\sin(-\frac{1}{2} \alpha)} = \frac{\sin(90^\circ + \frac{1}{2} \alpha)}{\cos(90^\circ + \frac{1}{2} \alpha)} \quad (4.16)$$

This equation is identical to the one given in equation 4.7. This all means, that when there is no payload present, there exists a solution to the equilibrium problem for $0^\circ < \alpha < 180^\circ$, which is independent of the dimensions of the sail planes.

For $180^\circ < \alpha < 360^\circ$, $l_3 = 0$ m, and according to figure 4.2 both sail planes are always reflecting. Since in this case both sail planes are of equal dimensions (i.e symmetric), it is expected that the resulting equilibrium values would also be symmetric: the angles between

the two sail planes normals and the incident sunlight vector will be of equal magnitude, but opposite signs. According to figure 4.1, the relation between α and θ is given by:

$$\theta = \frac{1}{2}\alpha - 90^\circ \quad (4.17)$$

When substituting this equation into equation 4.1, the following result can be found:

$$n_1 = \begin{pmatrix} \cos(\frac{1}{2}\alpha - 90^\circ) \\ \sin(\frac{1}{2}\alpha - 90^\circ) \\ 0 \end{pmatrix}, \quad n_2 = \begin{pmatrix} \cos(90^\circ - \frac{1}{2}\alpha) \\ \sin(90^\circ - \frac{1}{2}\alpha) \\ 0 \end{pmatrix} \quad (4.18)$$

Which agrees with the assumption. In the α , θ range in which both sail planes reflect, it would be expected that for $l_3 > 0.0$ meter, the θ equilibrium value would be smaller than that for $l_3 = 0.0$ meter. When $l_3 > 0.0$ meter, the second sail plane is larger and will produce a larger force. This force will reduce the angle between the second sail plane and the incident sunlight angle, thus increasing the angle between the first sail plane and the incident sunlight angle, which means a smaller angle θ between the normal vector of the first sail plane and the incident sunlight vector. The opposite will hold when $l_3 < 0.0$ meter. As can be seen from figure 4.1, indeed for $l_3 < 0.0$ meter, the equilibrium values are larger than for $l_3 = 0.0$ meter, and are smaller when $l_3 > 0.0$ meter. However, there are two consequences to this.

First of all, for $l_3 = -5.0$ m, it means that the equilibrium θ value will become larger than 90° before α reaches 360° . According to figure 4.2, this means that plane 1 will start to absorb the sunlight, while plane 2 is still reflecting the sunlight. In fact, plane 2 is (partially) shadowing plane 1 and as can be seen in figure 4.1 this leads to a completely different equilibrium position.

For $l_3 = 5.0$ m, the opposite will happen. The θ, α equilibrium pair will enter the region in which plane 2 will be absorbing the sunlight, while plane 1 is still reflecting it. This will also lead to completely different equilibrium position, which figure 4.1 illustrates very well.

Figures 4.3 through 4.9 show the resulting equilibrium values for payload masses m_p of 0 kg, 5 kg, 25 kg, 50 kg, 100 kg for two different structural mass factors k , $k = 0.0$ and $k = 0.4$. The payload is assumed to be a pointmass.

Figures 4.10 through 4.16 show the equilibrium values for the same payload masses and structural factors as above, but now the payload is assumed to be a homogeneous cylinder with a radius of 0.5 m and length of 1.0 m. Finally, figures 4.17 through 4.21 compare the equilibrium results for the payloads with and without shape and the two structural factors for each of the payload masses.

These results show that for $l_3 = 5.0$ m, the equilibrium values for $0^\circ < \alpha < 180^\circ$ lie above those for $m_p = 0.0$ kg, and the difference increases with increasing payload mass, regardless of the value of the structural factor or whether the payload is assumed to be a pointmass or has some shape to it (figures 4.3-a, 4.4-a, 4.10-a and 4.11-a). For $l_3 = -5.0$ m and $0^\circ < \alpha < 180^\circ$, the results lie below those for $m_p = 0.0$ kg, and the difference increases with increasing payload mass, also regardless of the value of the structural factor or the shape of the payload (figures 4.3-b, 4.4-b, 4.10-b and 4.11-b). In all cases, both sailplanes are still absorbing the sunlight.

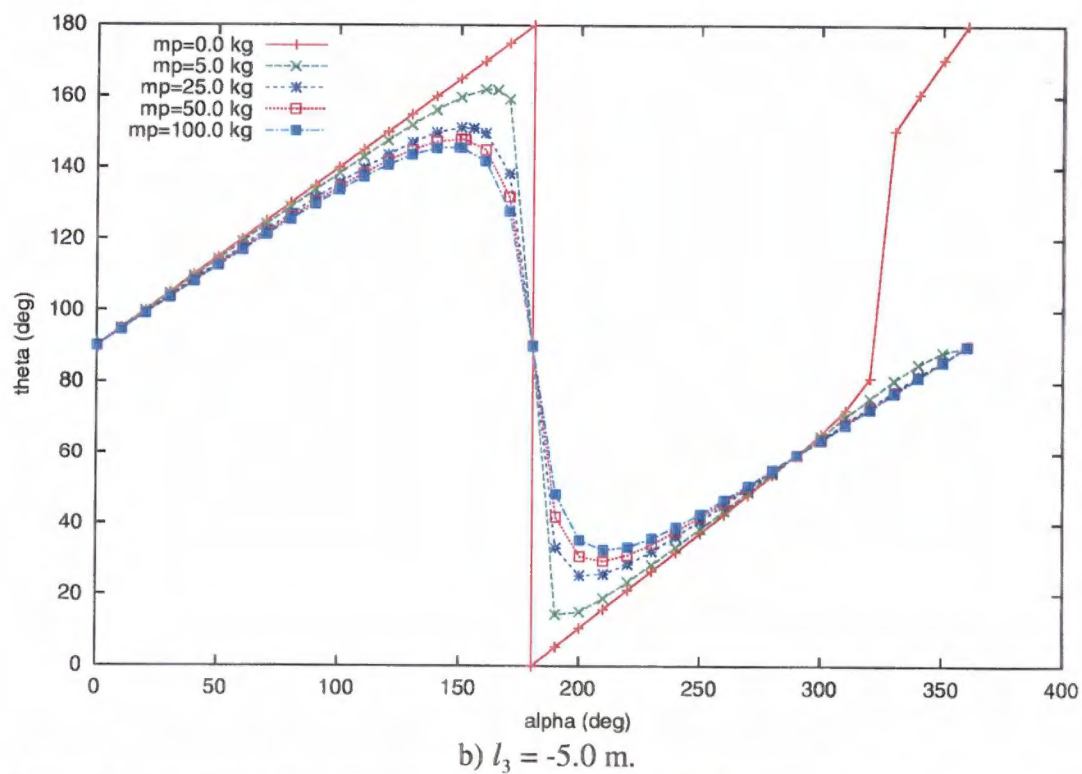
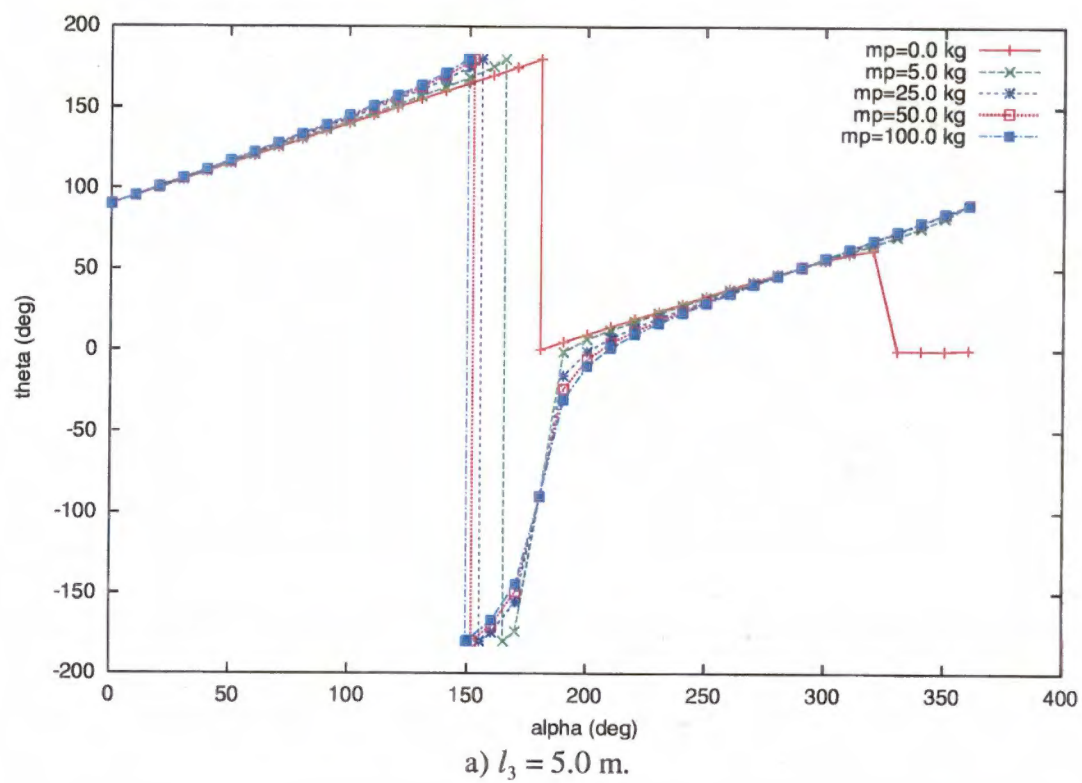
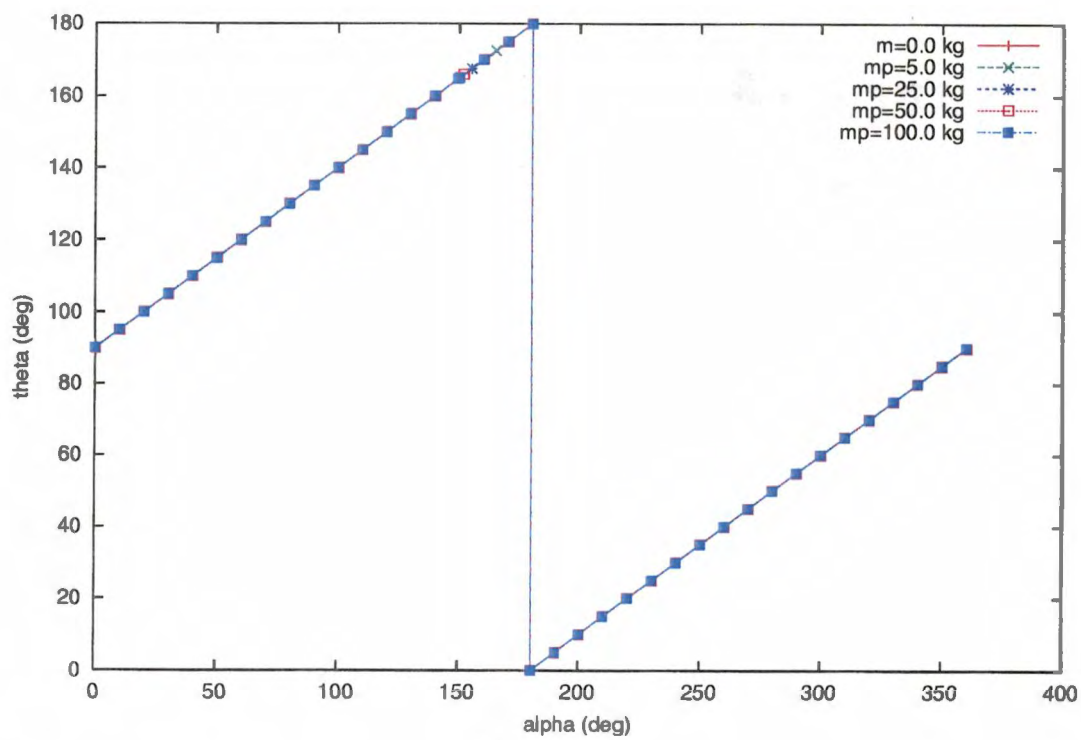
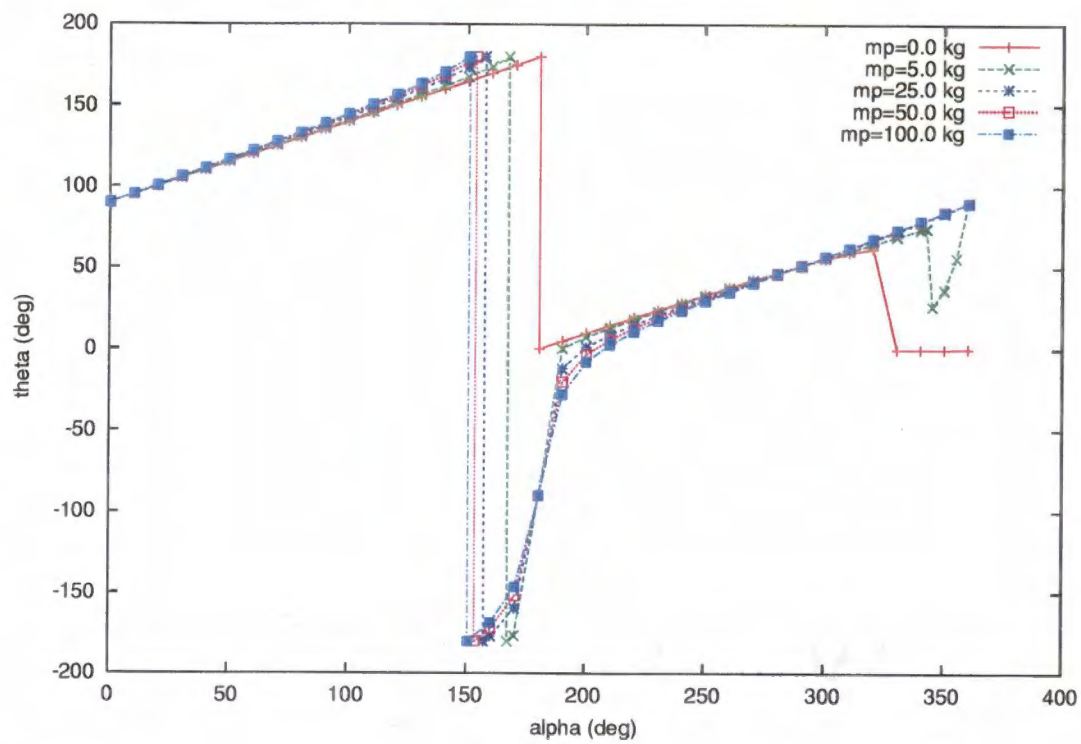


Figure 4.3: Equilibrium values for $k = 0.0$ m, payload is pointmass

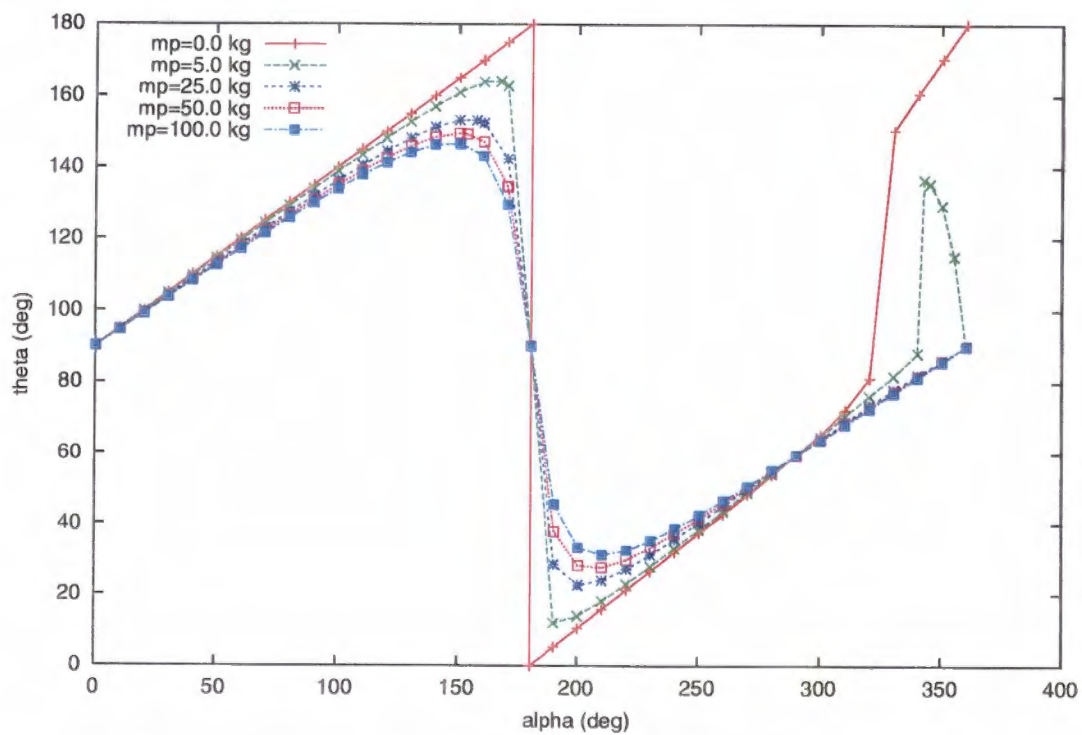


c) $l_3 = 0.0$ m.

Figure 4.3 (continued)

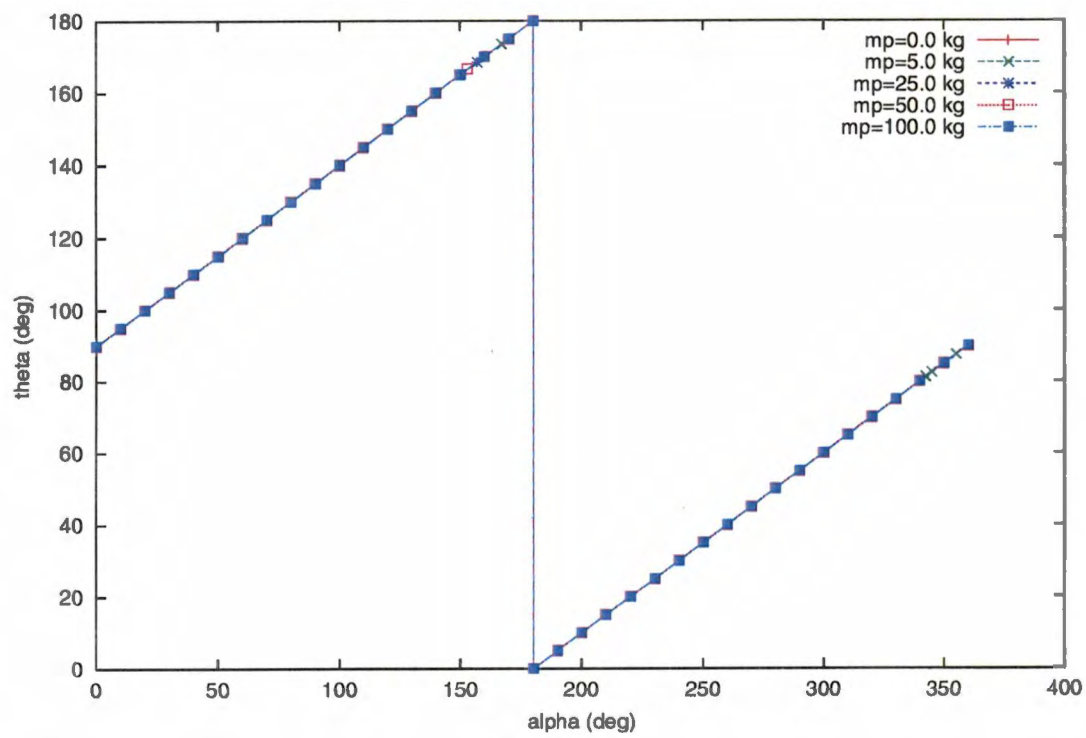


a) $l_3 = 5.0$ m.



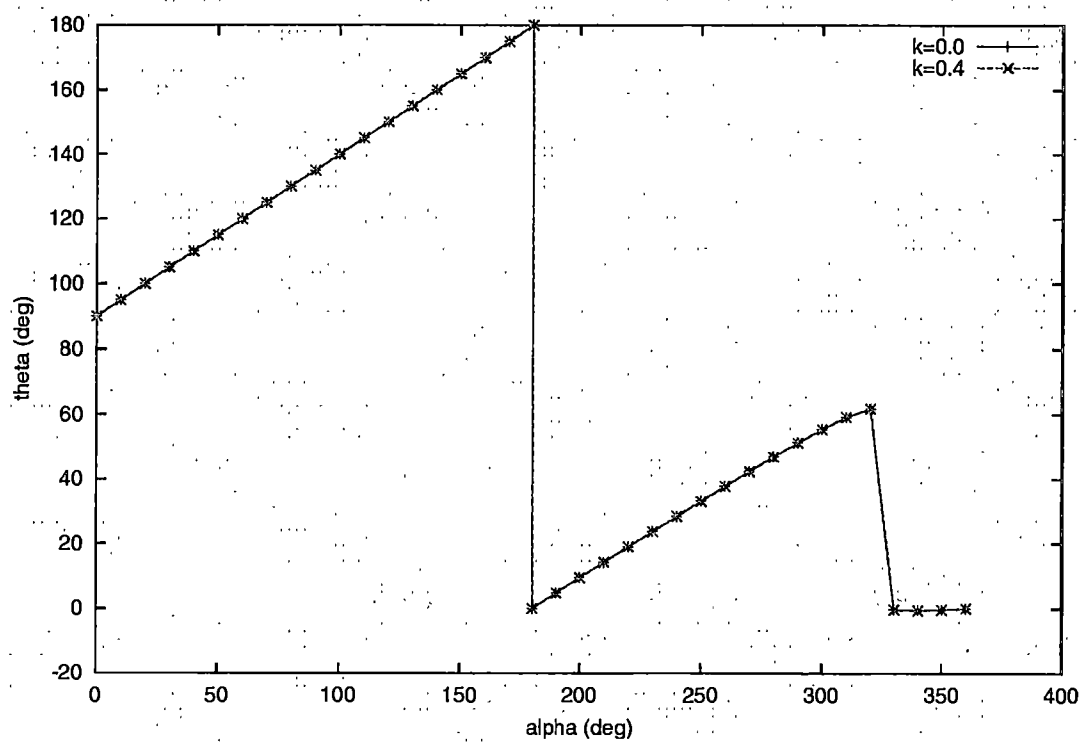
b) $l_3 = -5.0$ m.

Figure 4.4: Equilibrium values for $k=0.4$, payload is pointmass

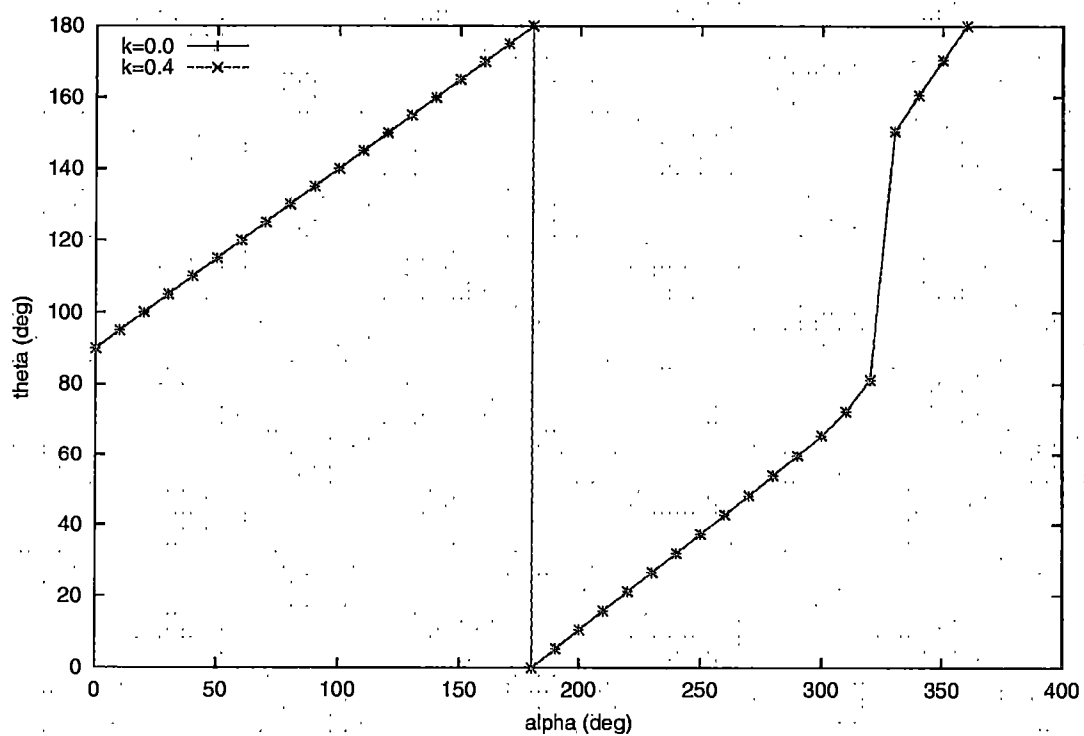


c) $l_3 = 0.0$ m.

Figure 4.4 (continued)

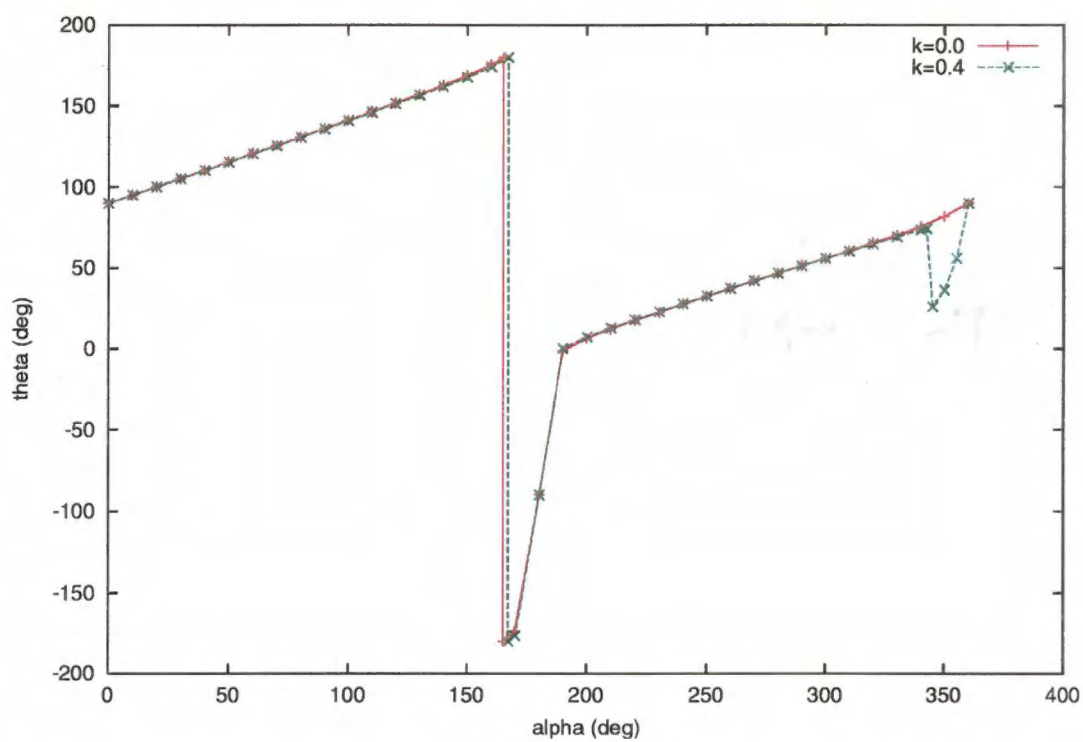


a) $l_3 = 5.0$ m.

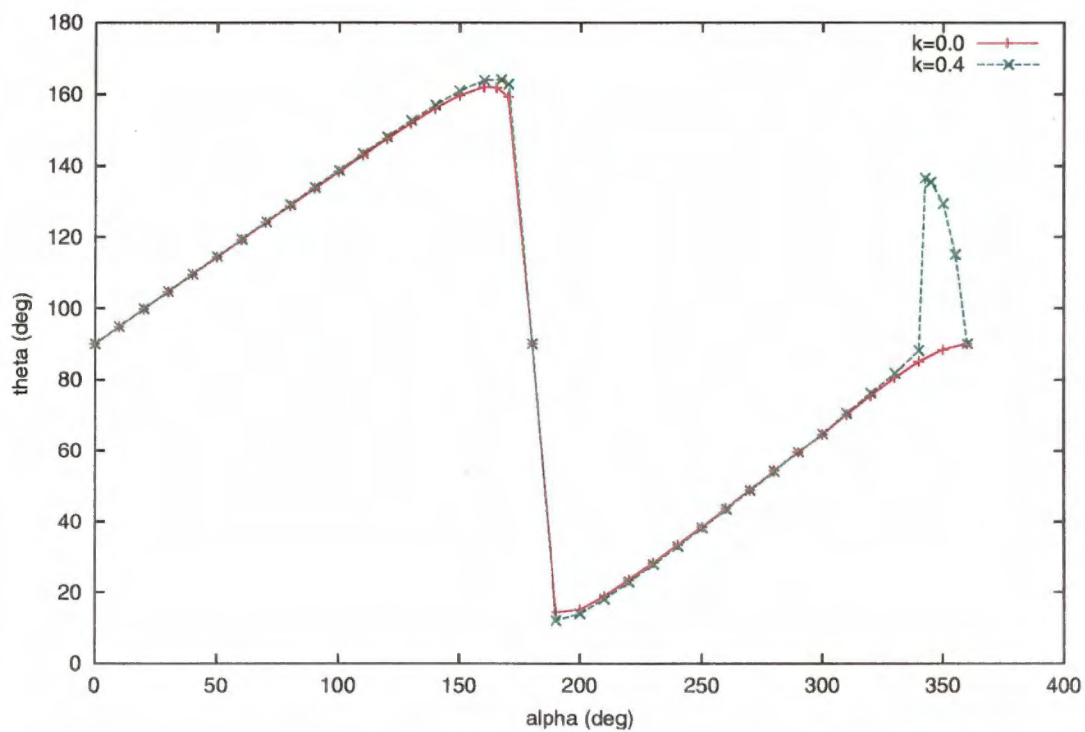


b) $l_3 = -5.0$ m.

Figure 4.5: Equilibrium values for $m_p = 0.0$ kg, payload is pointmass

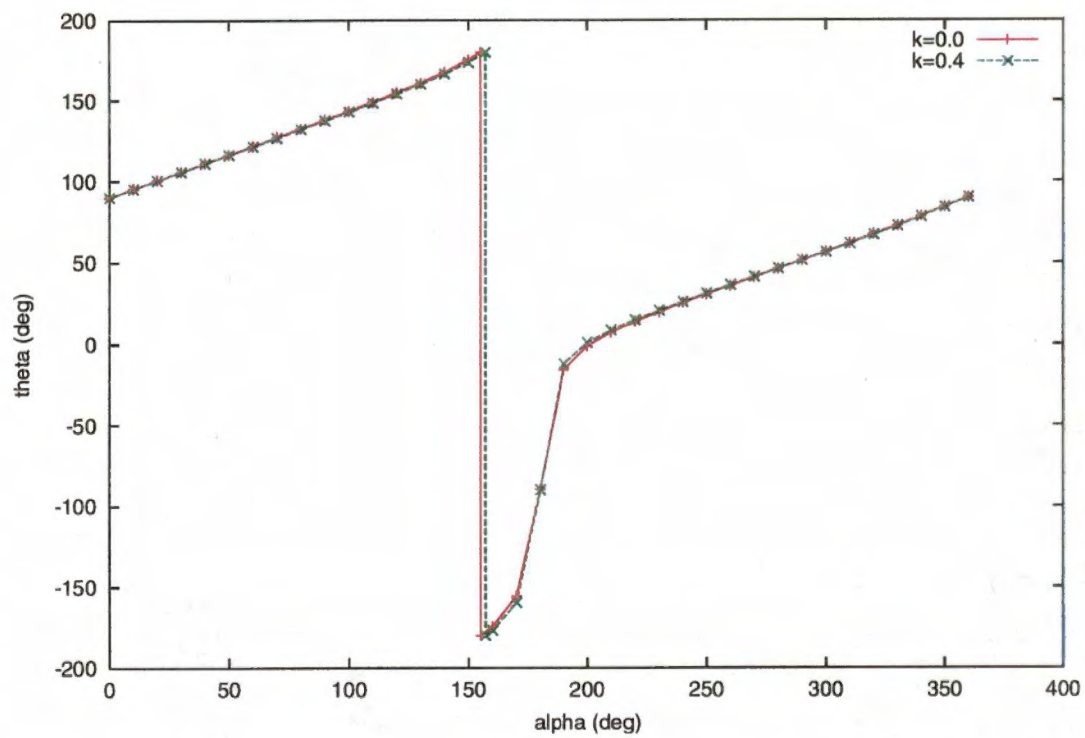


a) $l_3 = 5.0$ m.

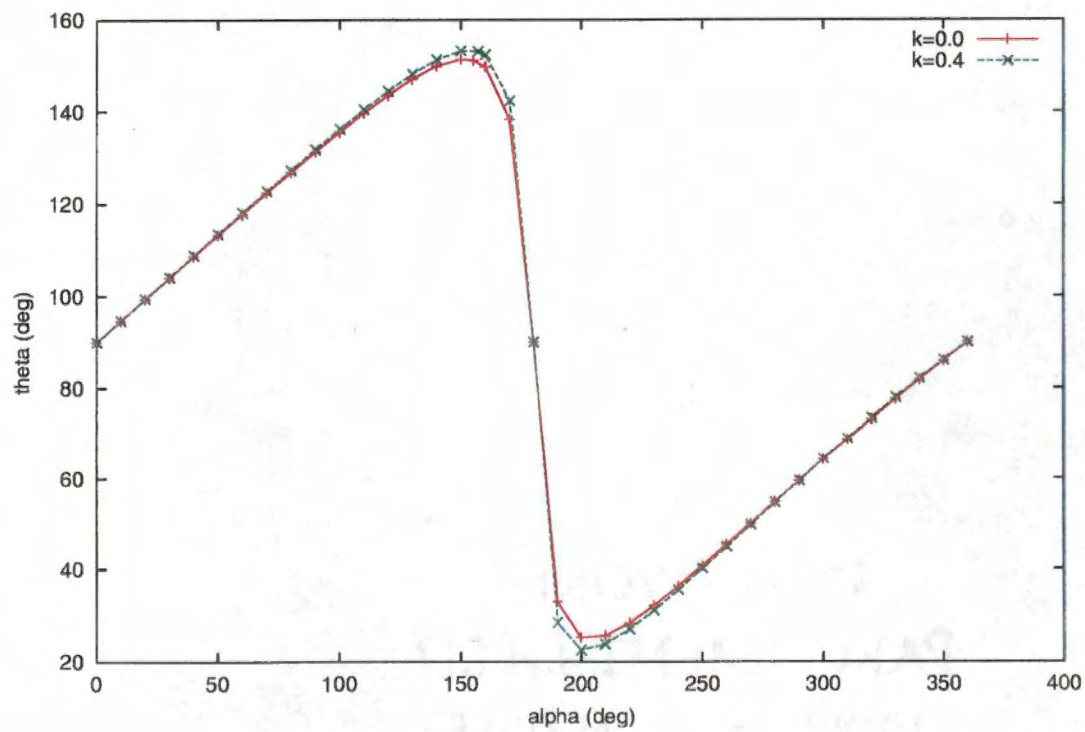


b) $l_3 = -5.0$ m.

Figure 4.6: Equilibrium values for $m_p = 5.0$ kg, payload is pointmass

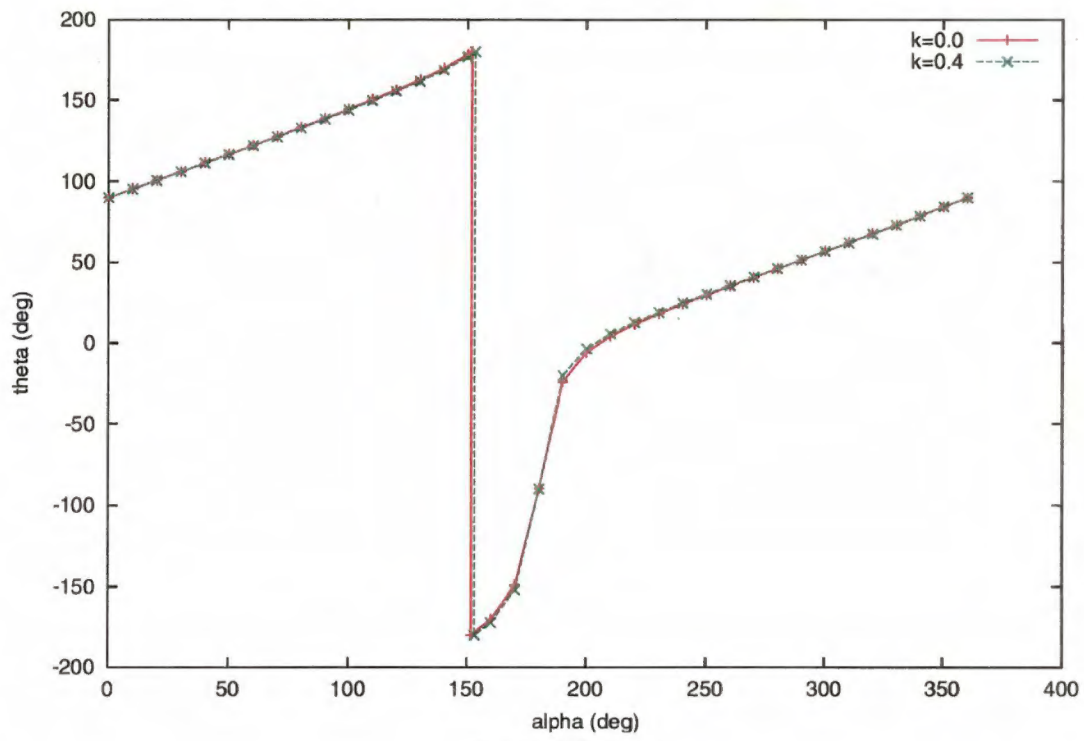


a) $l_3 = 5.0$ m.

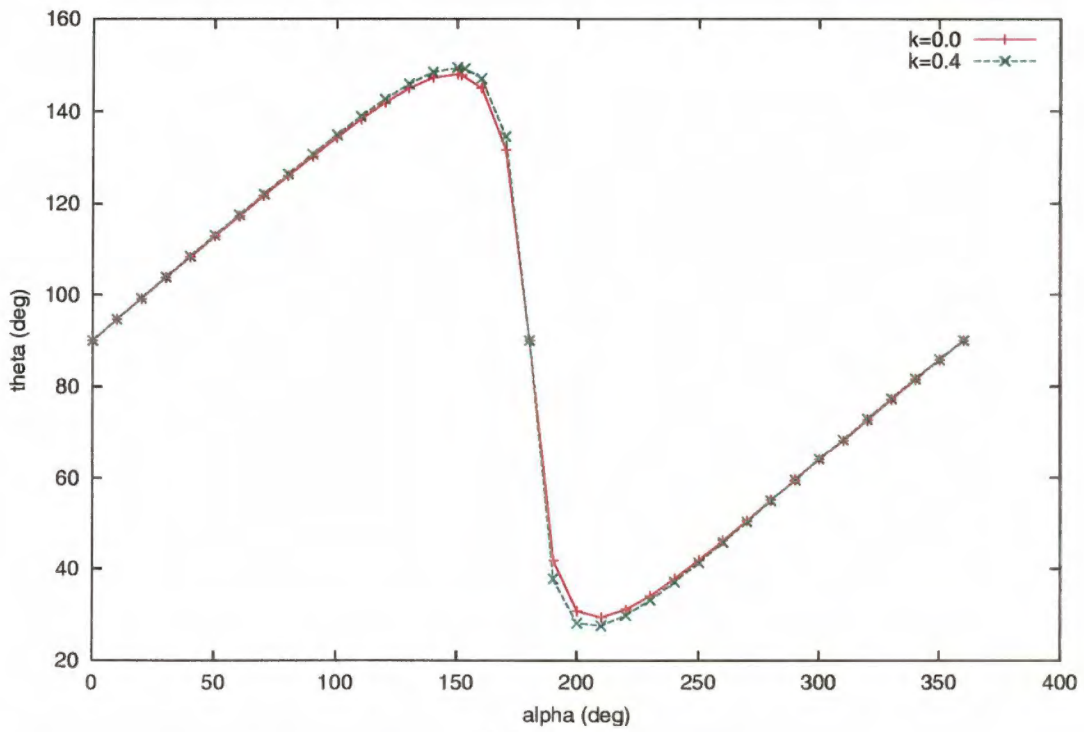


b) $l_3 = -5.0$ m.

Figure 4.7: Equilibrium values for $m_p = 25.0$ kg, payload is pointmass

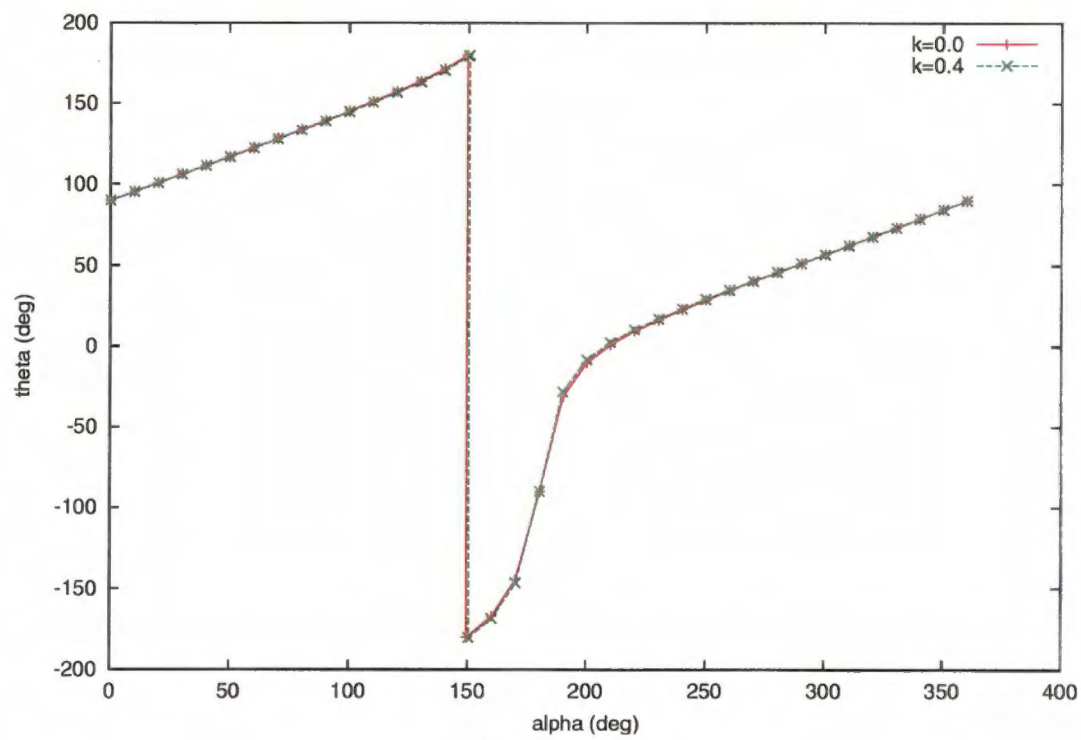


a) $l_3 = 5.0$ m.

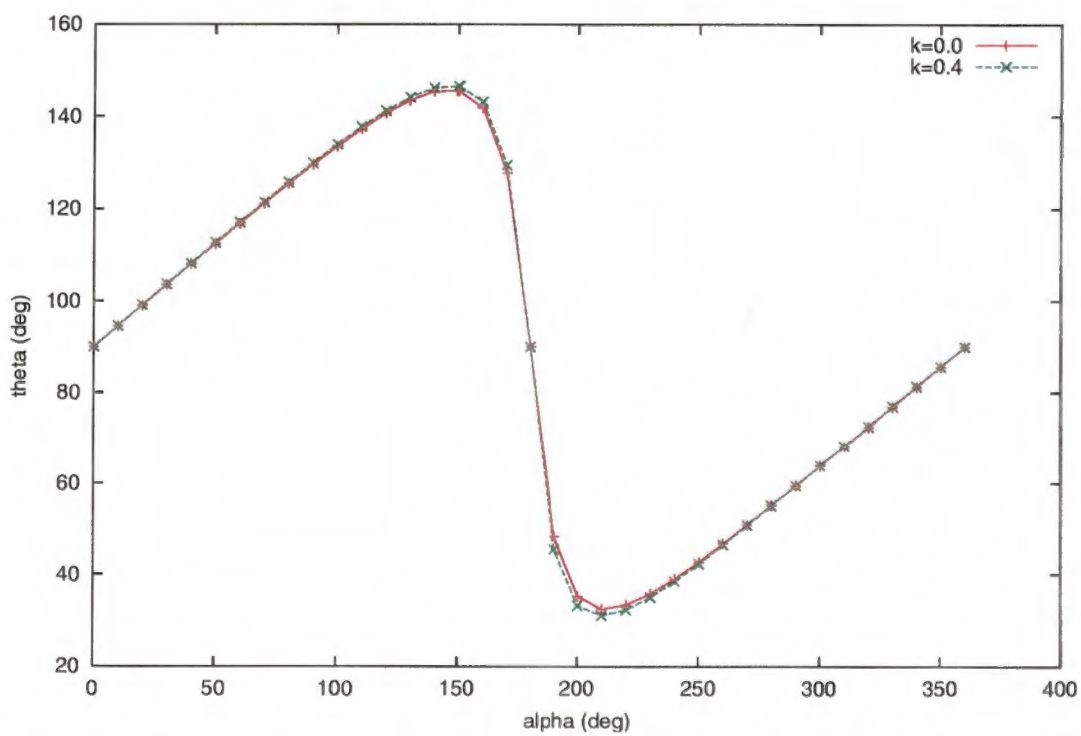


b) $l_3 = -5.0$.

Figure 4.8: Equilibrium values for $m_p=50.0$ kg, payload is pointmass

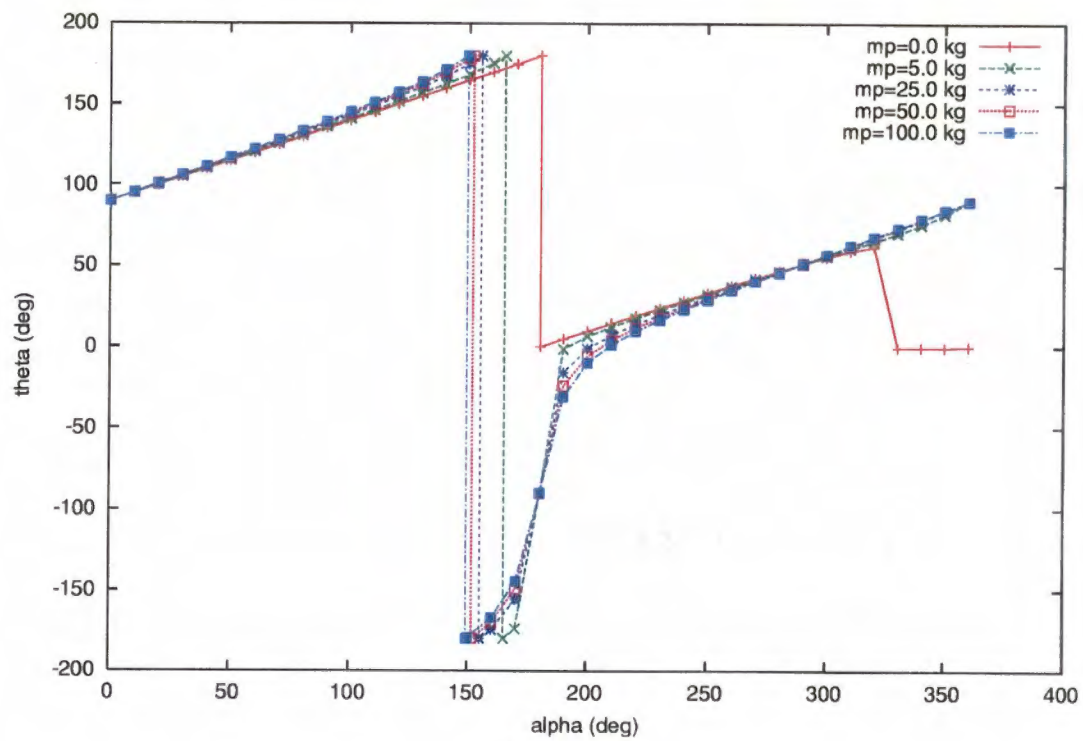


a) $l_3 = 5.0$ m.

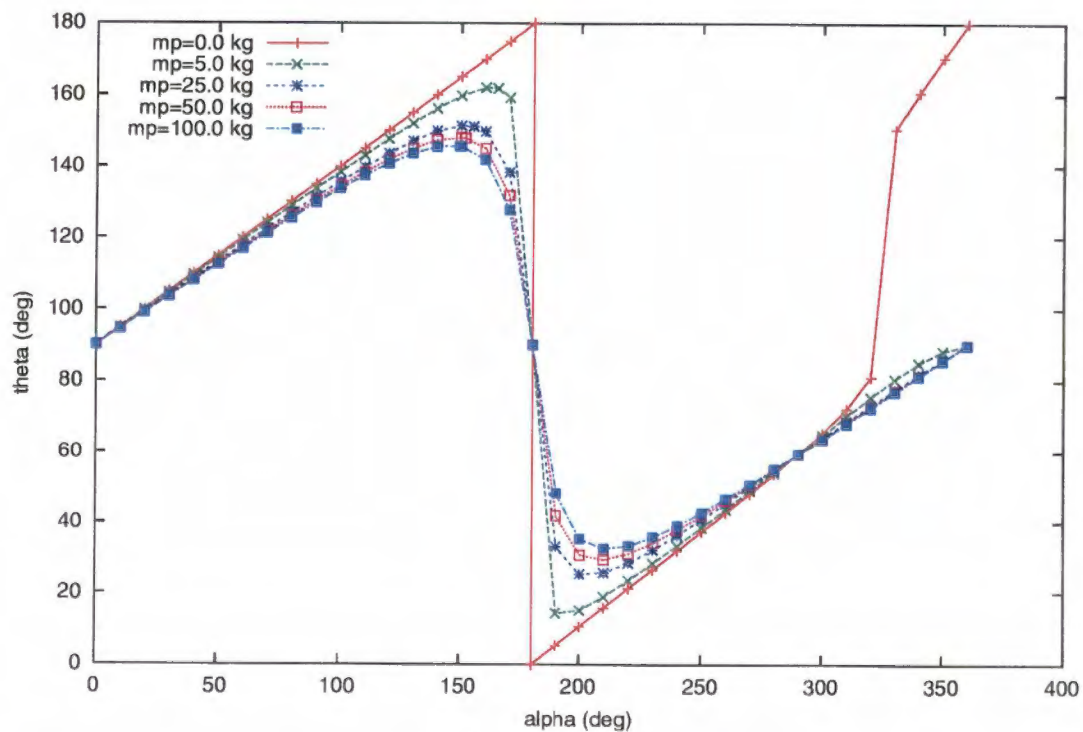


b) $l_3 = -5.0$ m.

Figure 4.9: Equilibrium values for $m_p=100.0$ kg, payload is pointmass

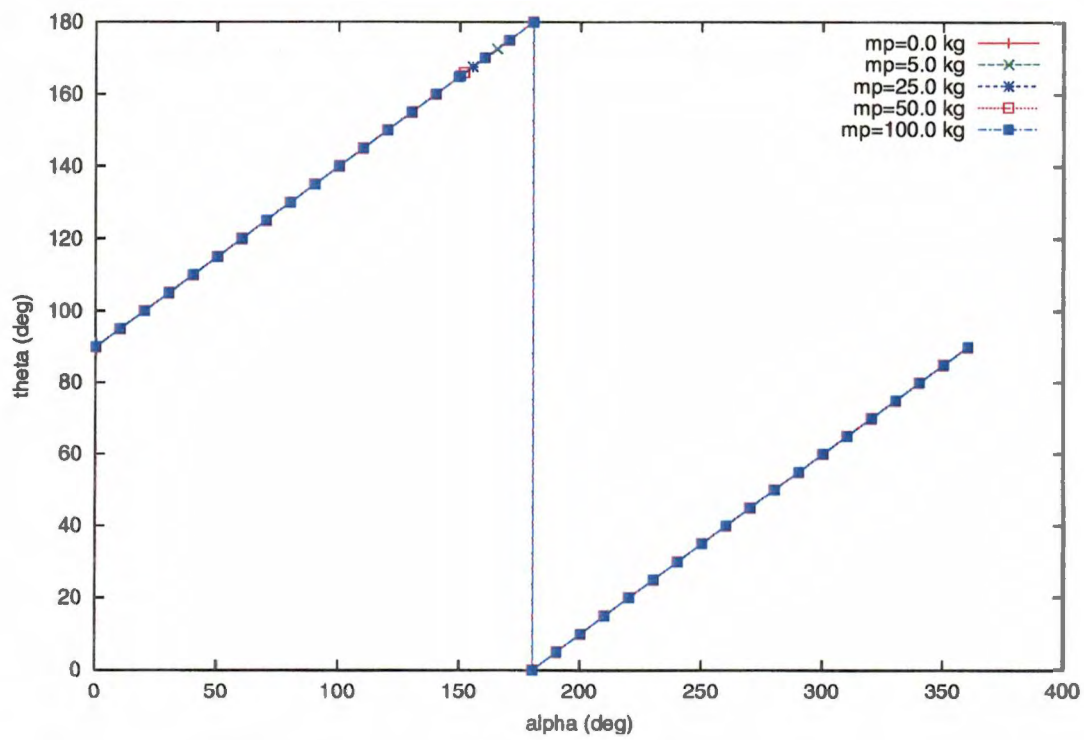


a) $l_3 = 5.0$ m.



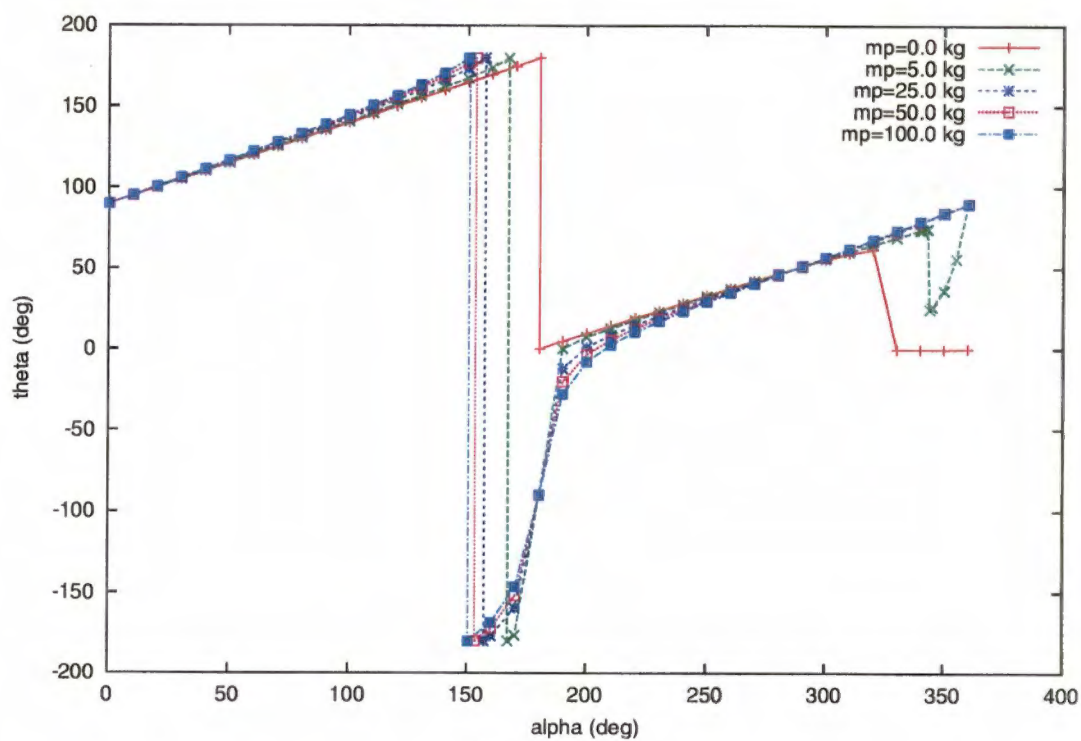
b) $l_3 = -5.0$ m.

Figure 4.10: Equilibrium values for $k = 0.0$, payload has shape

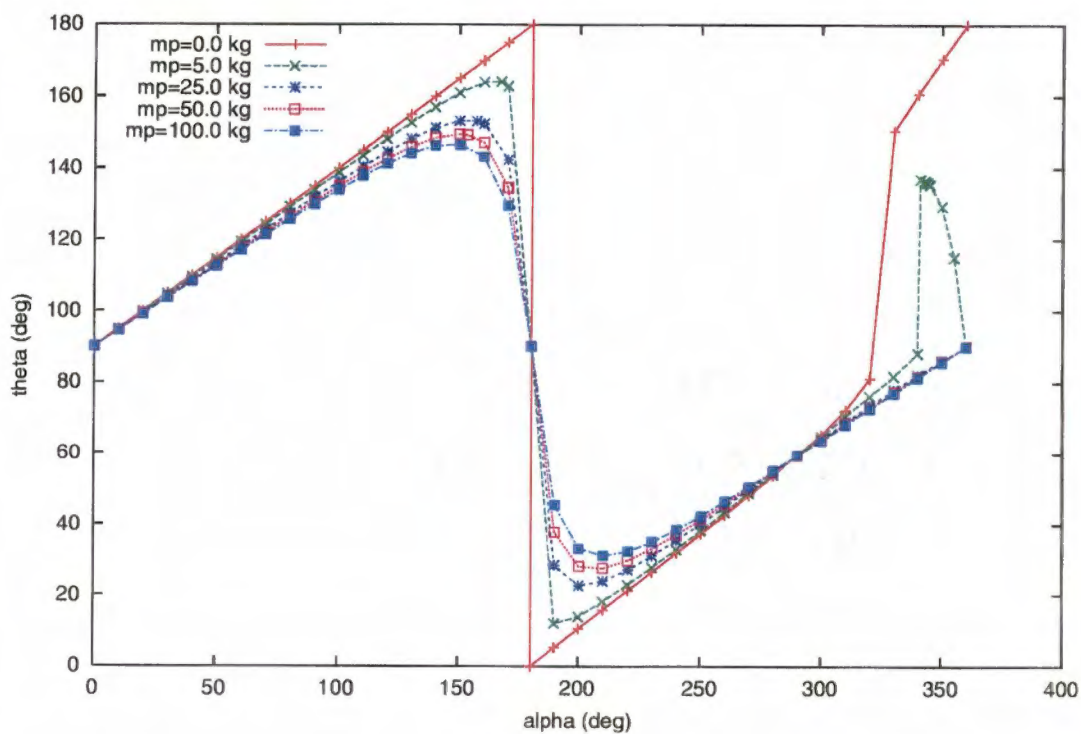


c) $l_3 = 0.0$ m.

Figure 4.10 (continued)

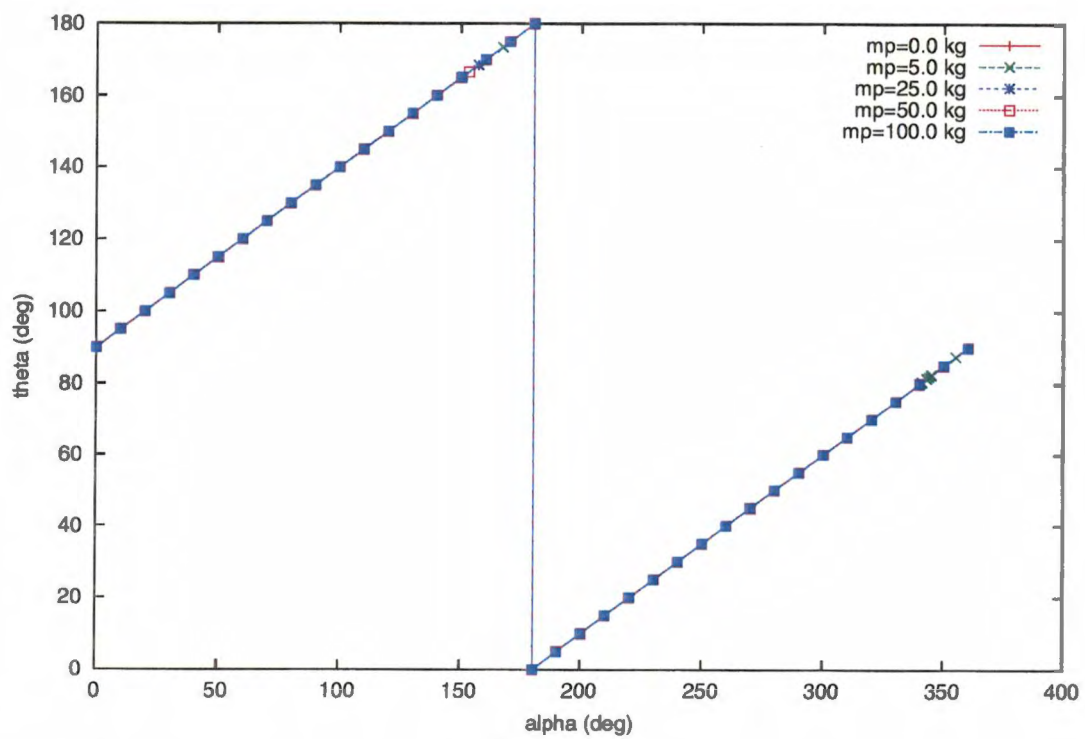


a) $l_3 = 5.0$ m.



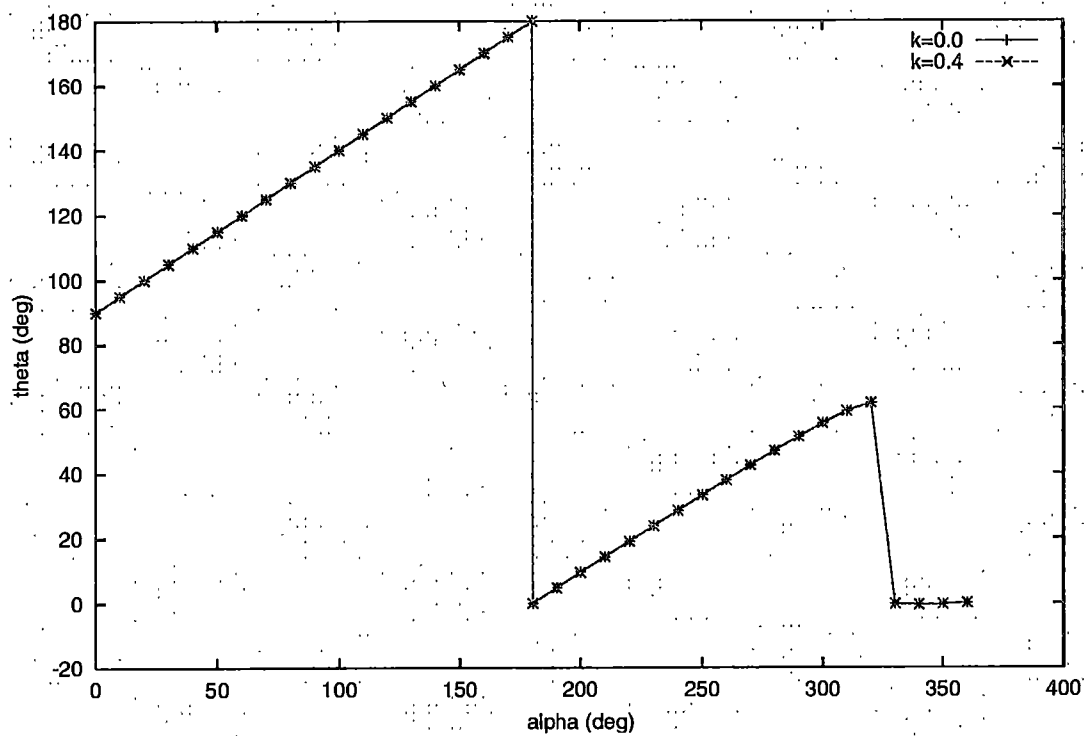
b) $l_3 = -5.0$ m.

Figure 4.11: Equilibrium values for $k = 0.4$, payload has shape

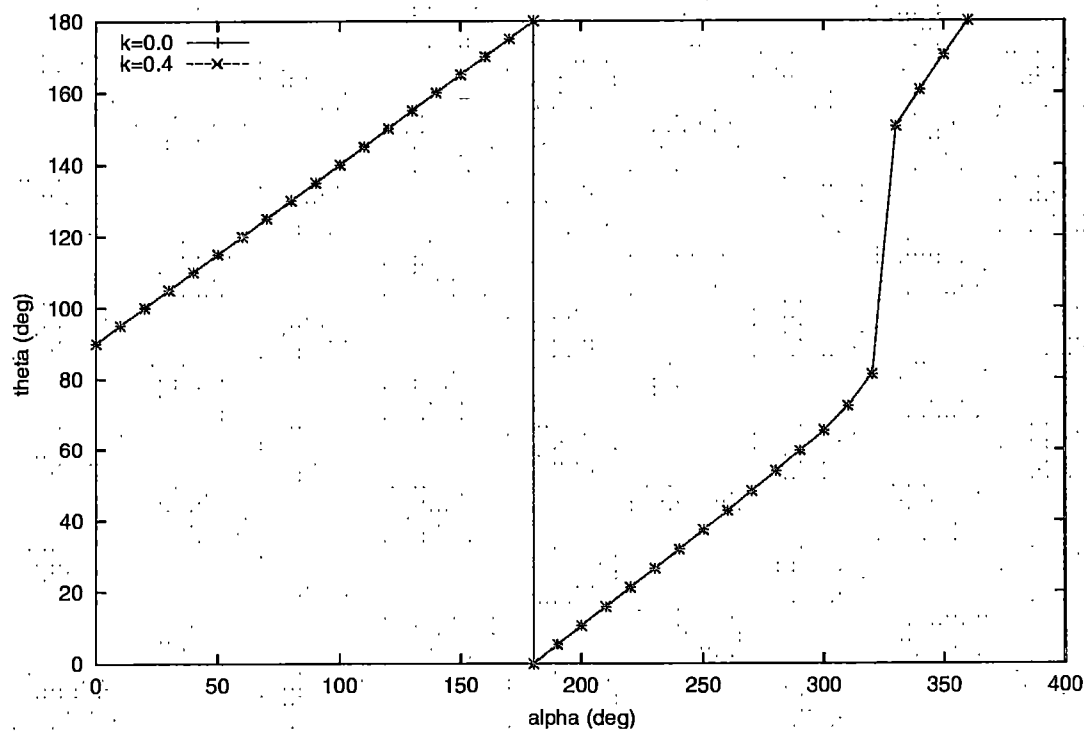


c) $l_3 = 0.0$ m.

Figure 4.11 (continued)

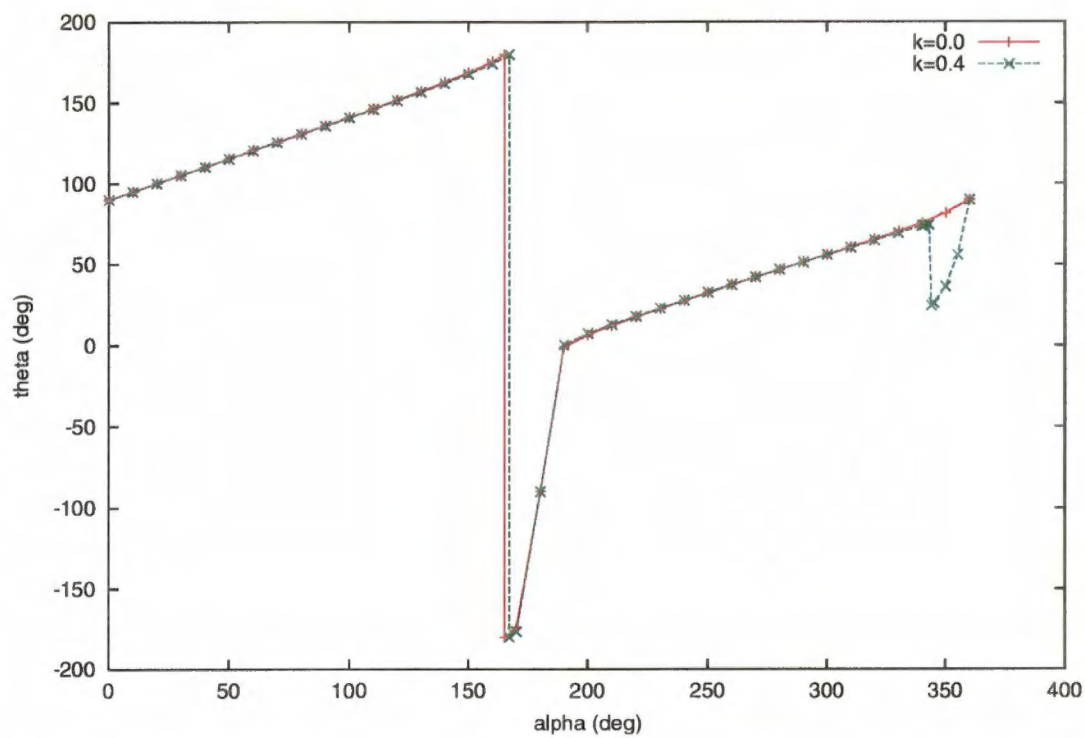


a) $l_3 = 5.0$ m.

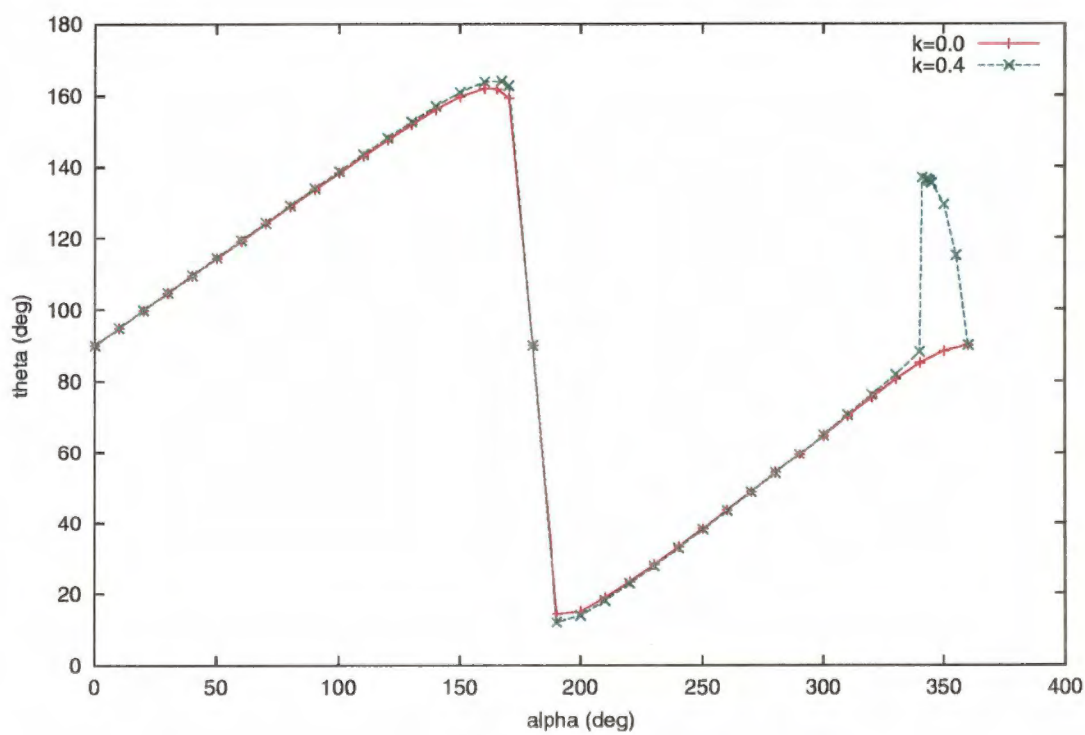


b) $l_3 = -5.0$ m.

Figure 4.12: Equilibrium values for $m_p = 0.0$ kg, payload has shape

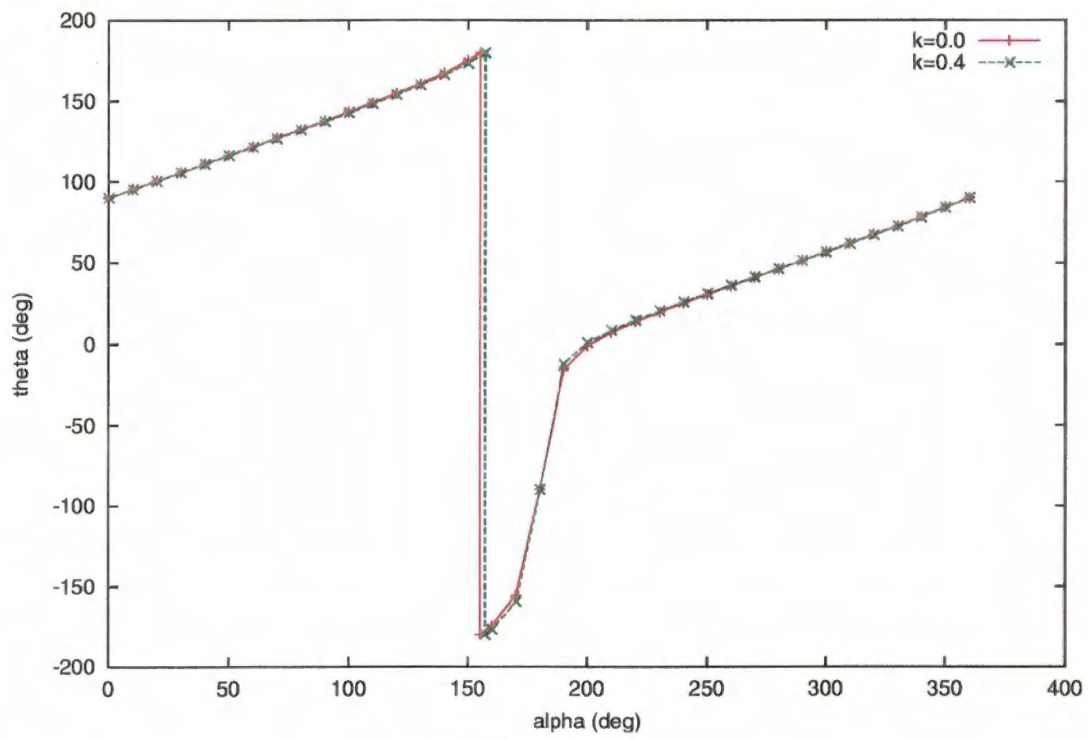


a) $l_3 = 5.0$ m.

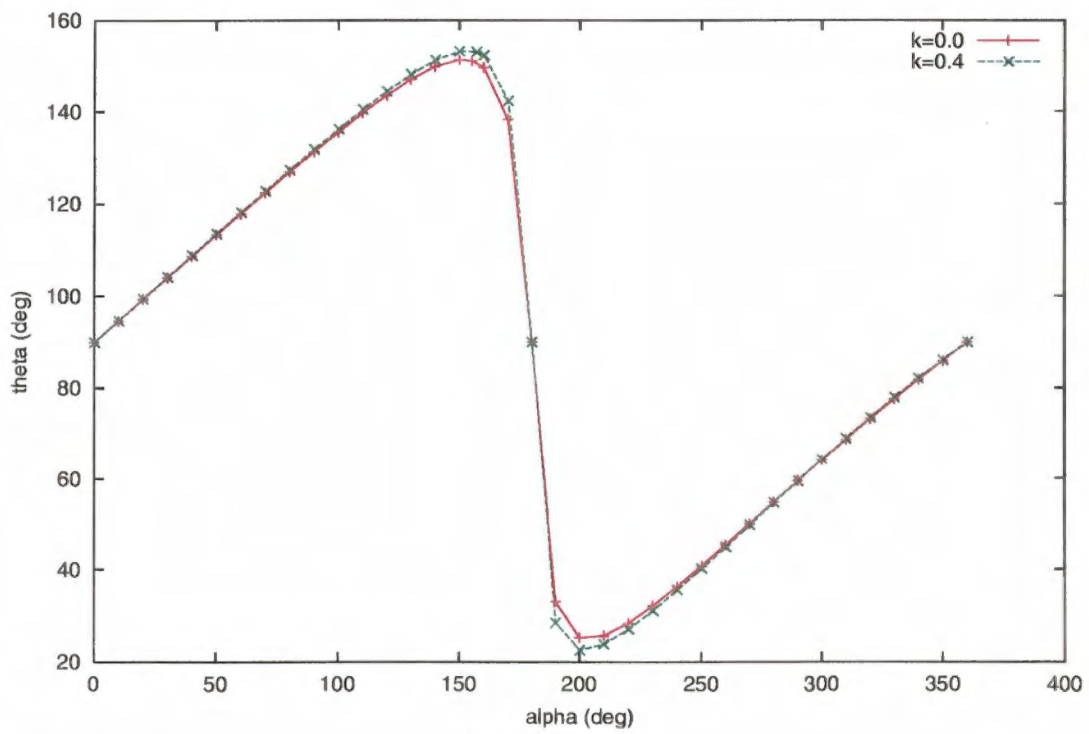


b) $l_3 = -5.0$ m.

Figure 4.13: Equilibrium values for $m_p = 5.0$ kg, payload has shape



a) $l_3 = 5.0$ m.



b) $l_3 = -5.0$ m.

Figure 4.14: Equilibrium values for $m_p = 25.0$ kg, payload has shape

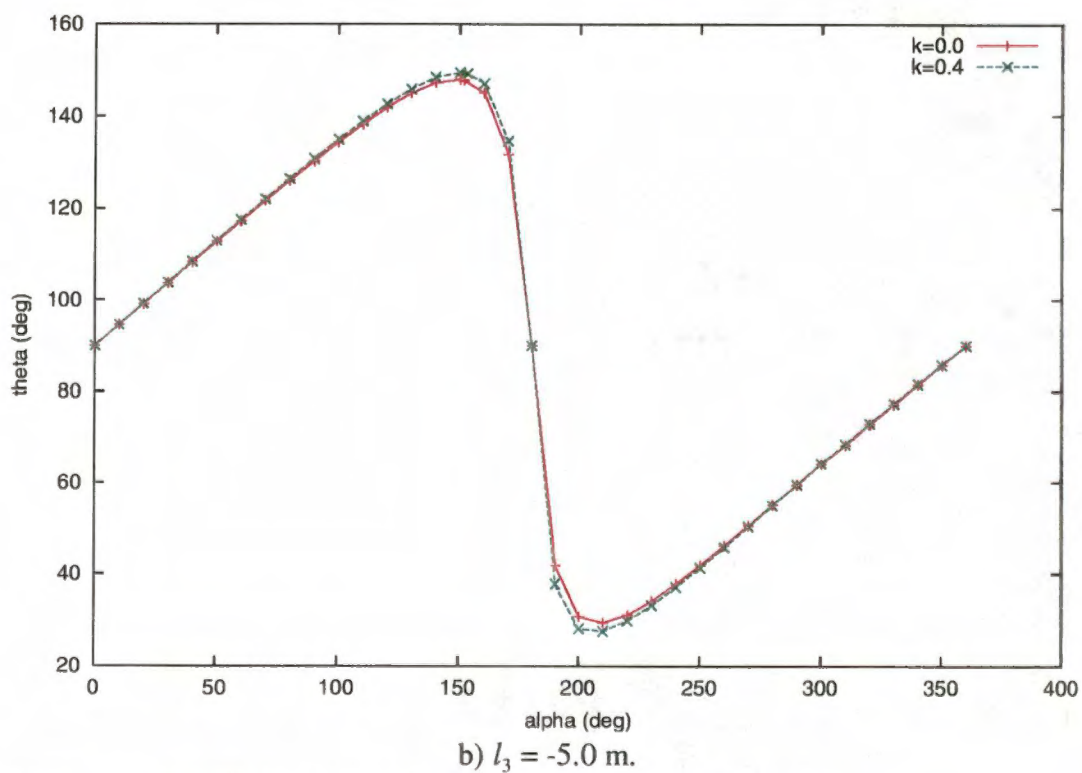
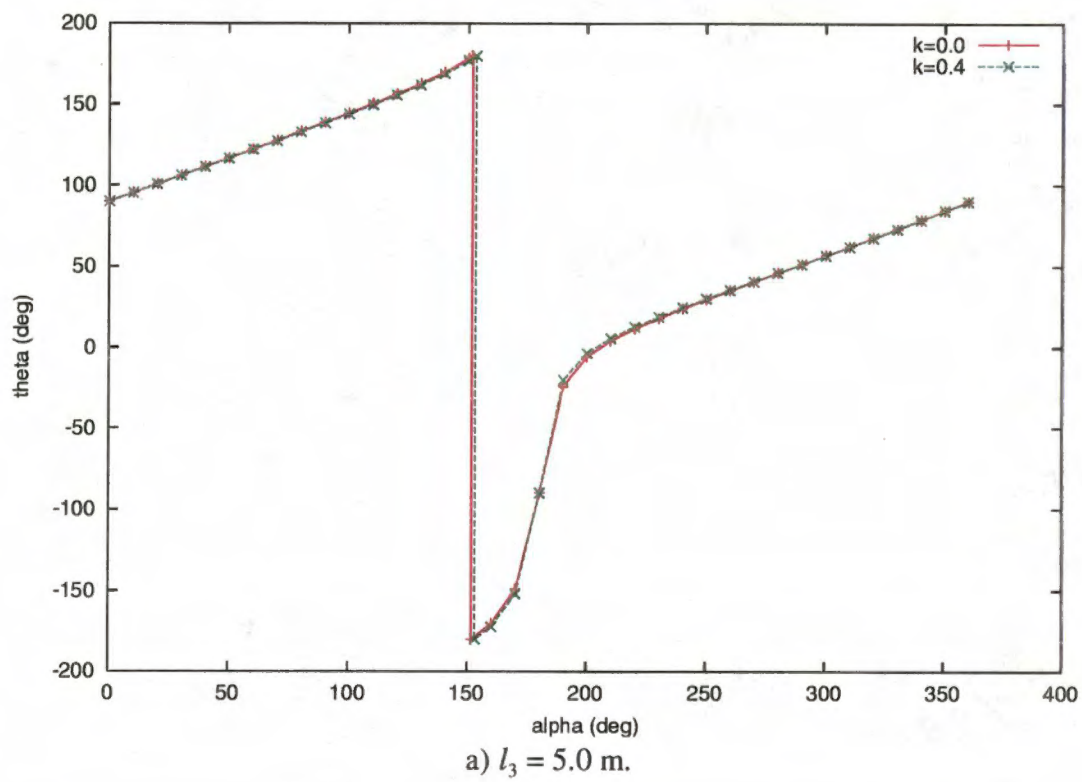


Figure 4.15: Equilibrium values for $m_p = 50.0$ kg, payload has shape

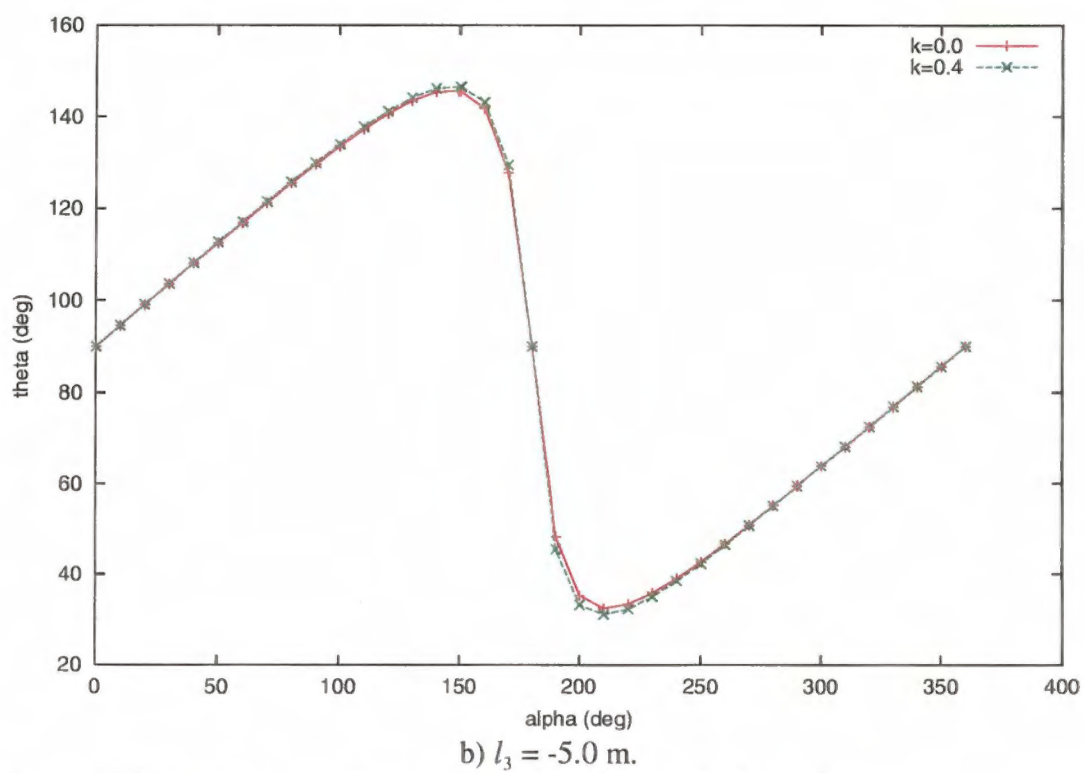
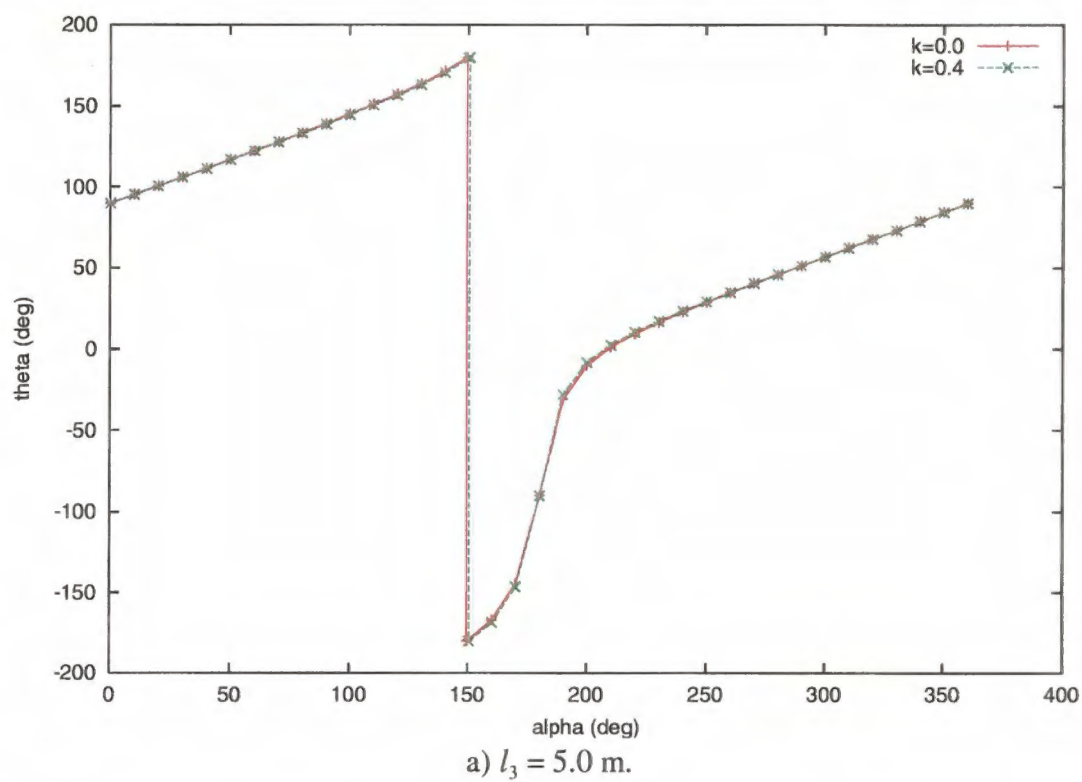
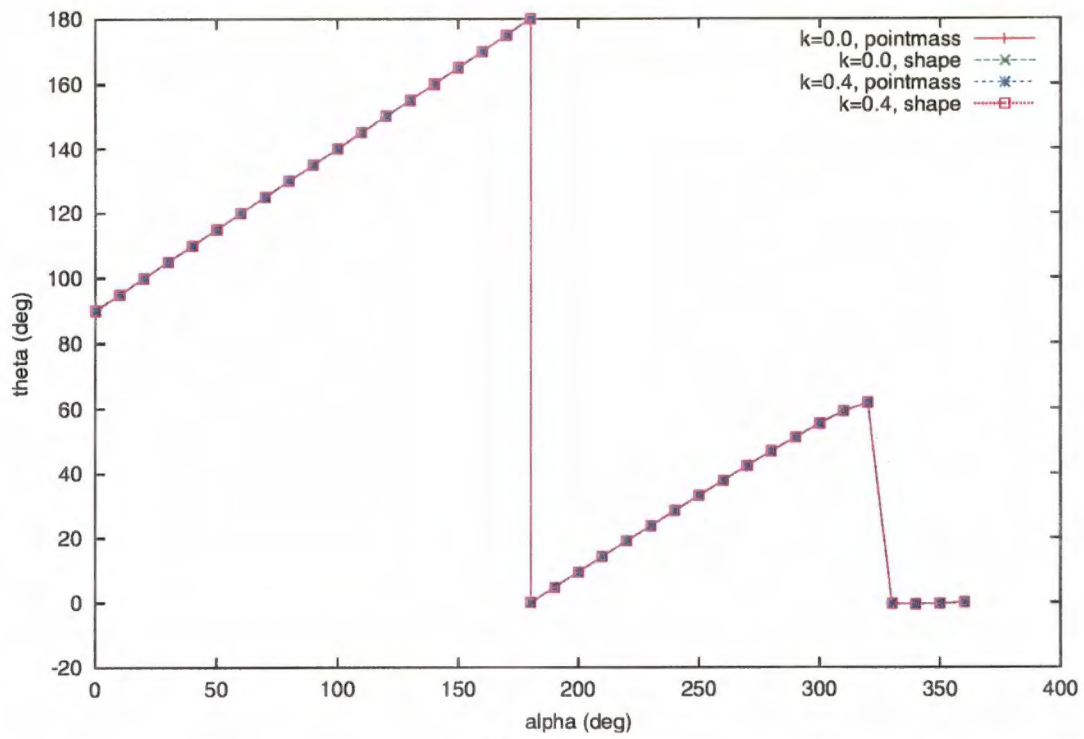
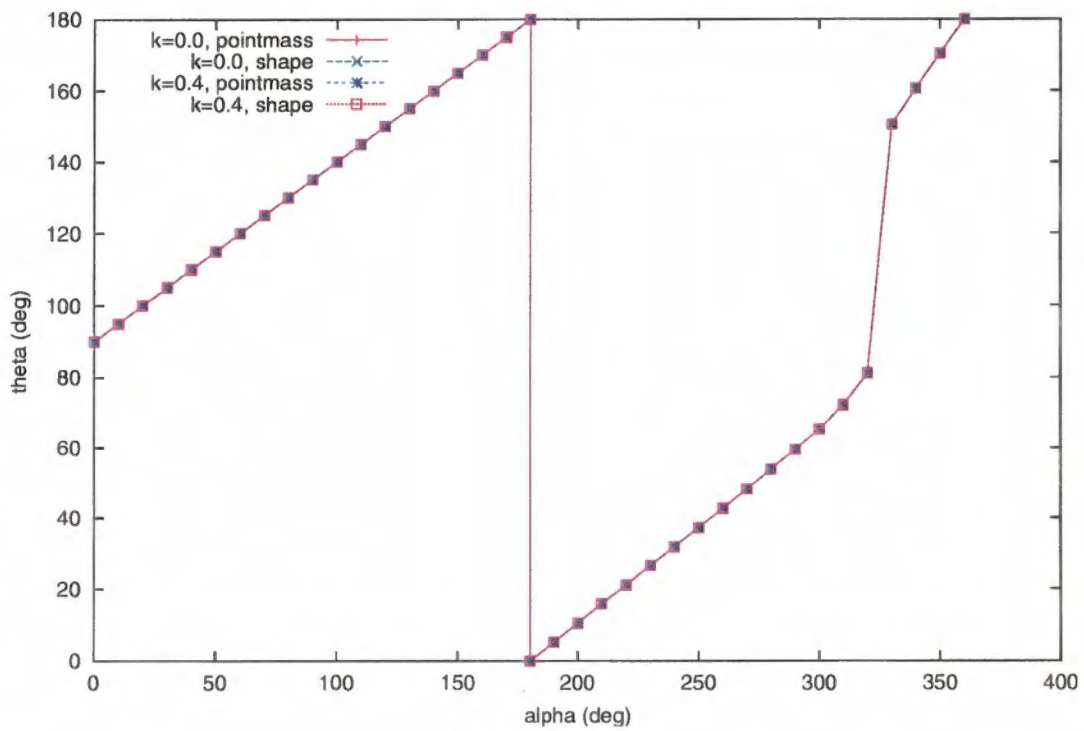


Figure 4.16: Equilibrium values for $m_p = 100.0$ kg, payload has shape



a) $l_3 = 5.0$ m.



b) $l_3 = -5.0$ m.

Figure 4.17: Equilibrium values for $m_p = 0.0$ kg.

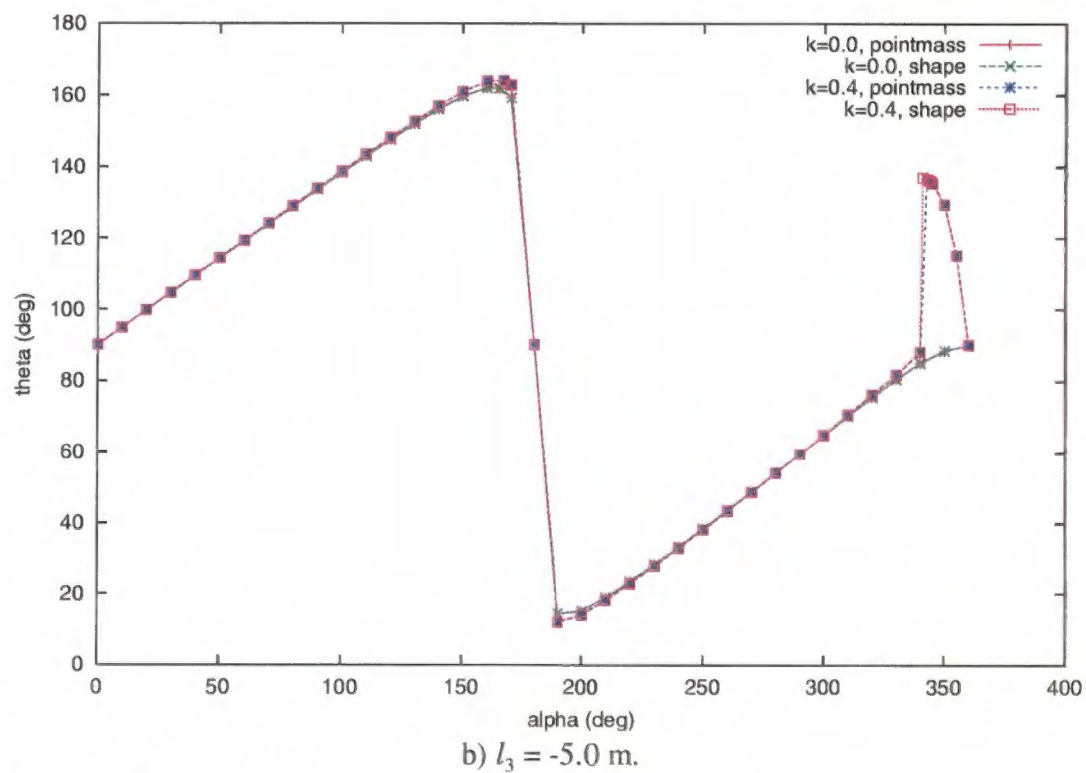
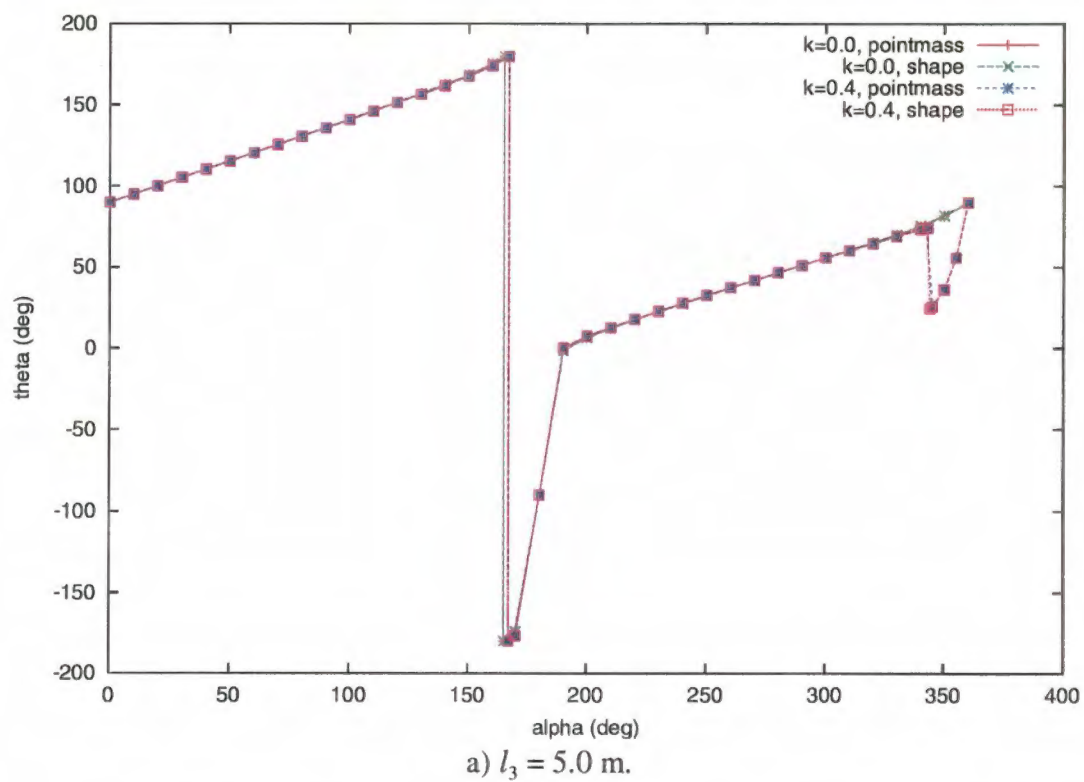
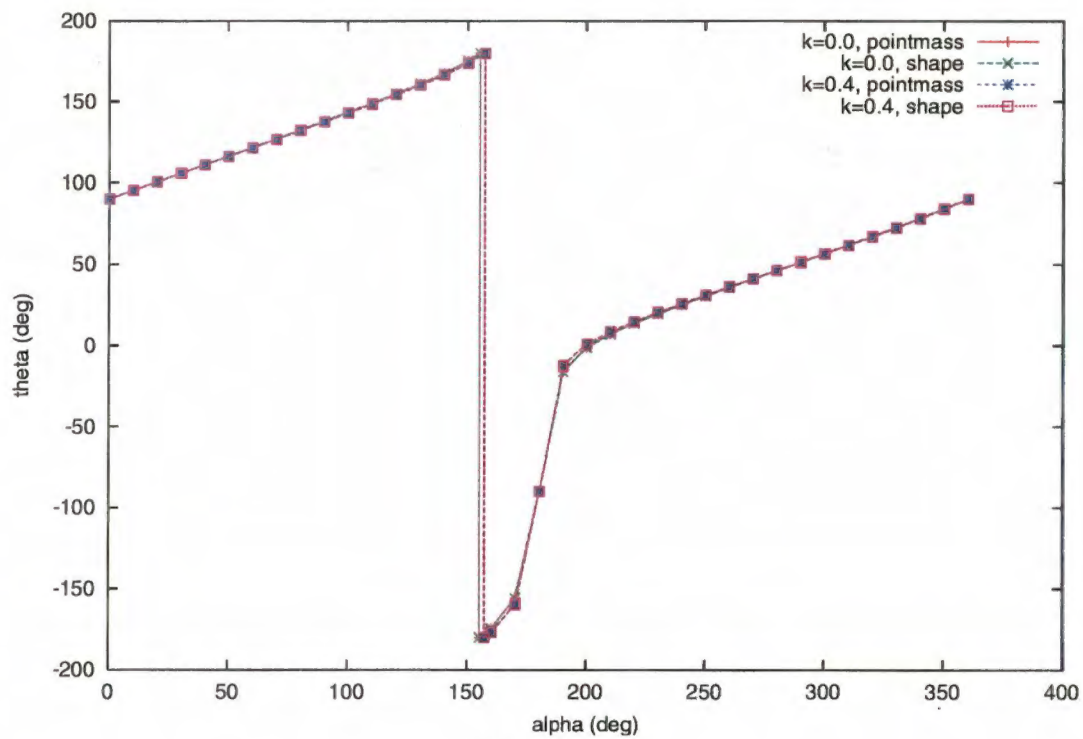
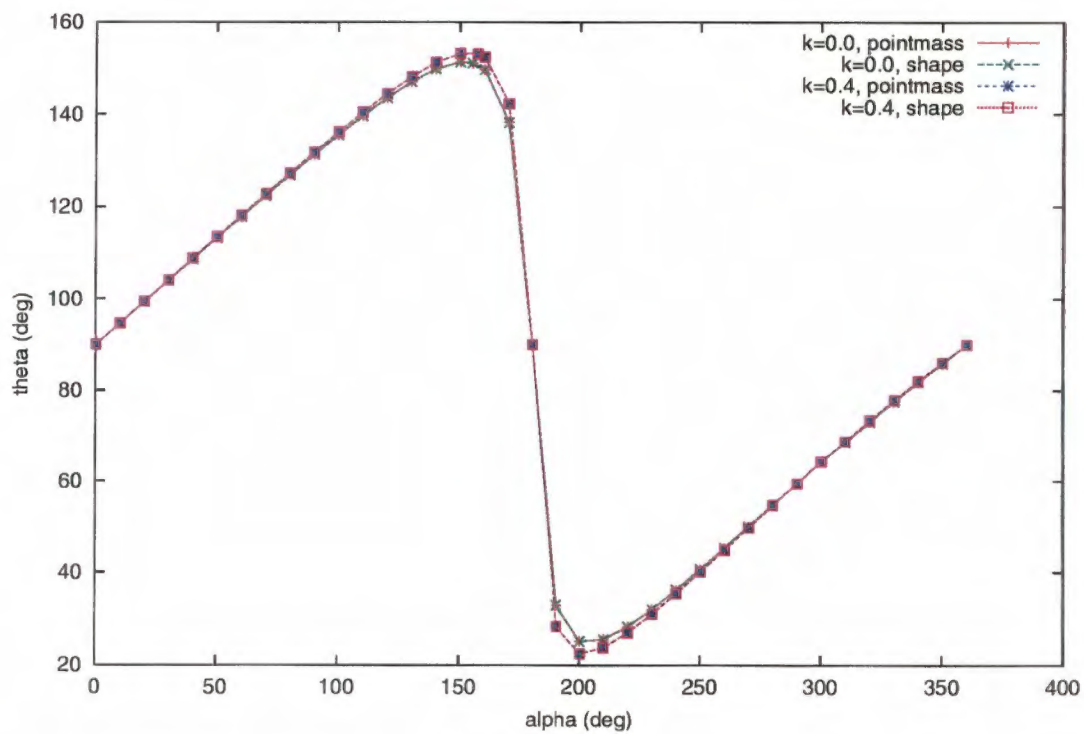


Figure 4.18: Equilibrium values for $m_p = 5.0$ kg

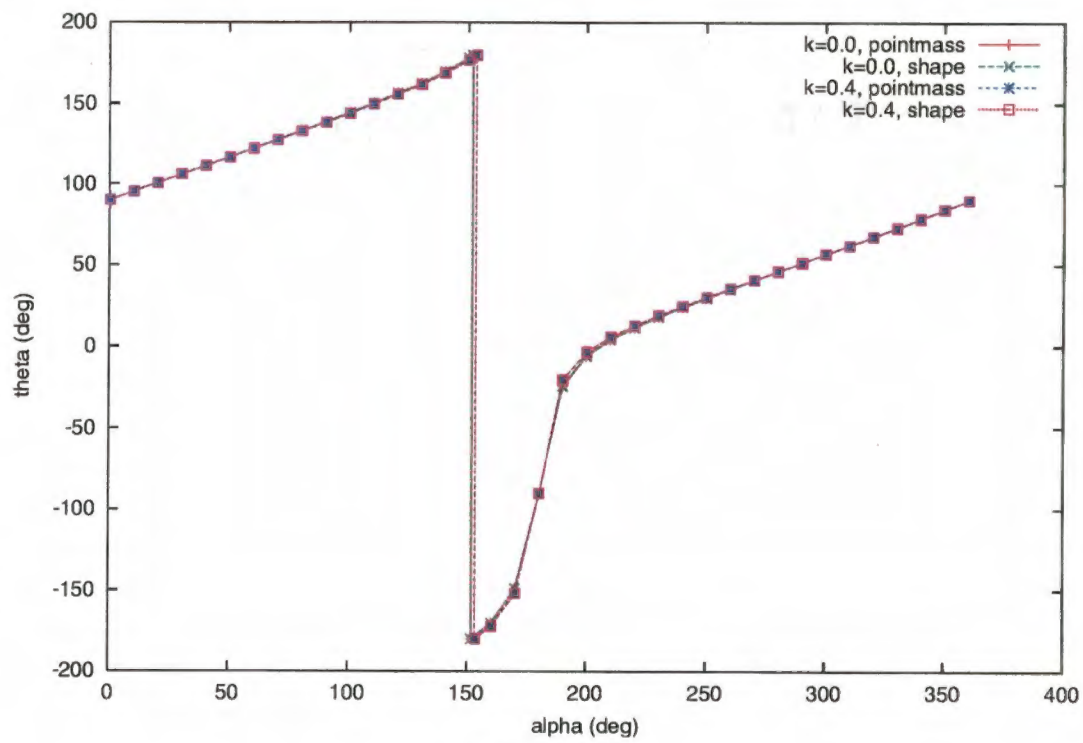


a) $l_3 = 5.0$ m.

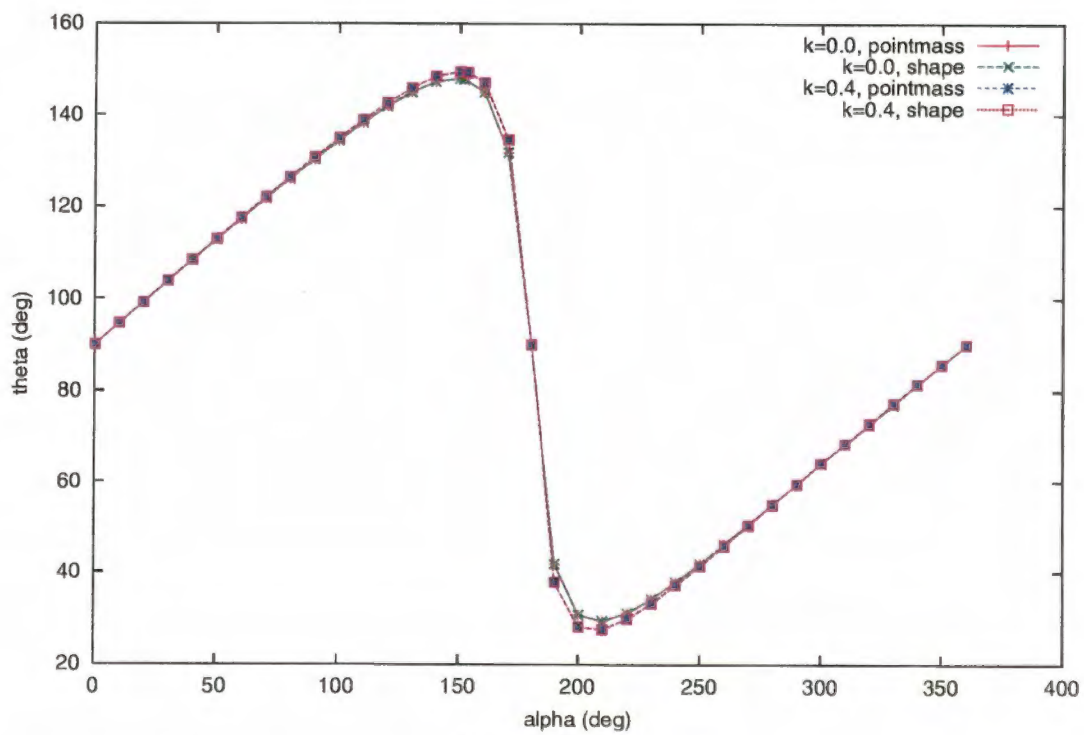


b) $l_3 = -5.0$ m.

Figure 4.19: Equilibrium values for $m_p = 25.0$ kg.

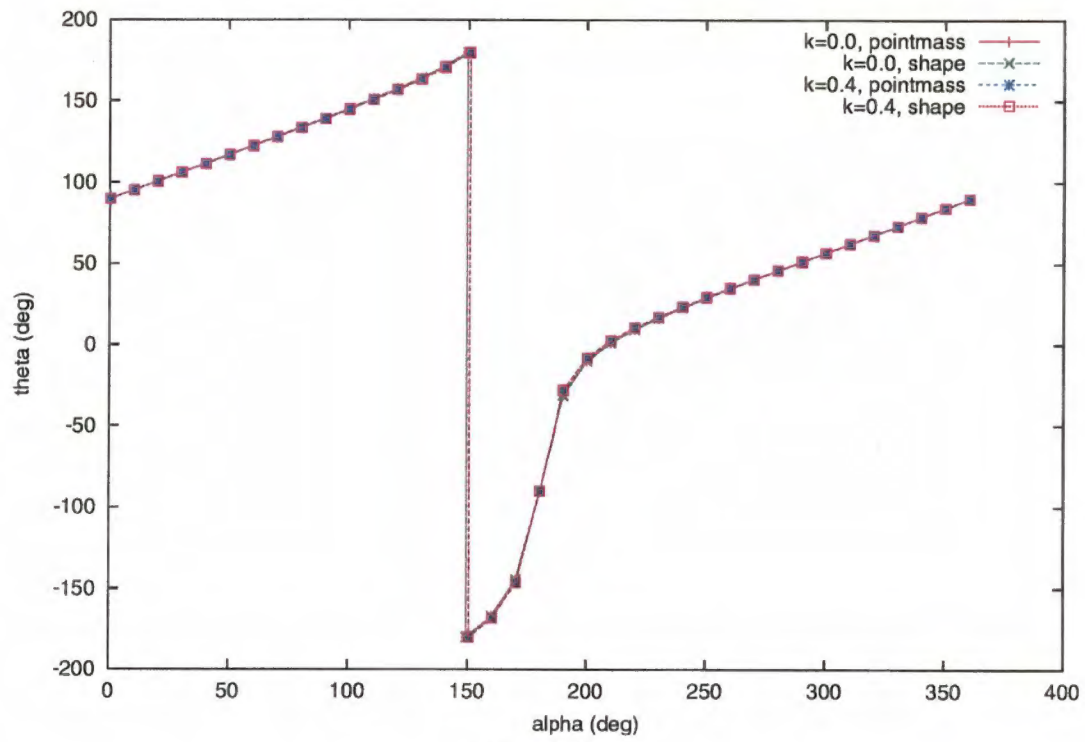


a) $l_3 = 5.0$ m.

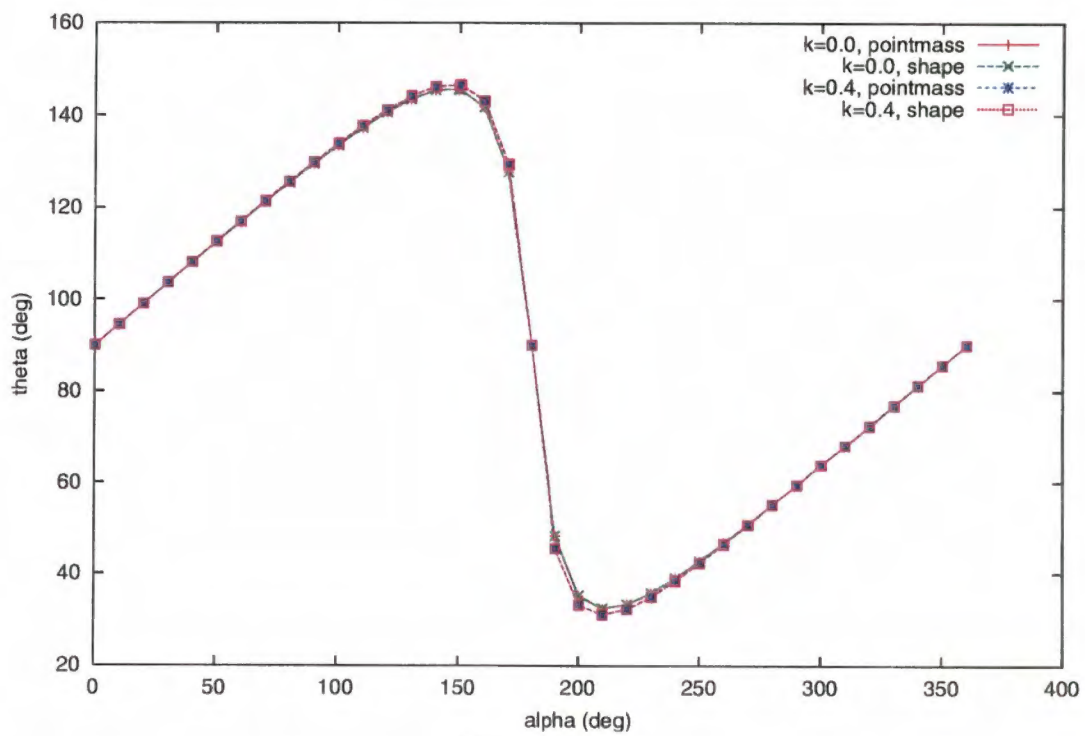


b) $l_3 = -5.0$ m.

Figure 4.20: Equilibrium values for $m_p = 50.0$ kg.



a) $l_3 = 5.0$ m.



b) $l_3 = -5.0$ m.

Figure 4.21: Equilibrium values for $m_p = 100.0$ kg.

For $180^\circ < \alpha < 360^\circ$ and $l_3 = 5.0$ m, the results are at initially below those for $m_p = 0.0$ kg and for higher α values above, and the difference increase with increasing payload mass. This is valid regardless of the value of the structural mass factor or the payload shape. For $l_3 = -5.0$ m, the opposite is the case. In most cases both sailplanes remain in the area in which both of them reflect the sunlight, except for $m_p = 5.0$ kg, $k = 0.4$ and both payload shapes. In this case, for $l_3 = 5.0$ m and α close to 360° , the equilibrium values enter the range in which plane 2 is absorbing the sunlight, while plane 1 is still reflecting it. For $l_3 = -5.0$ m, the equilibrium values enter the area in which plane 1 is absorbing with plane 2 reflecting the sunlight.

These results also show that for $l_3 = 0.0$ m, the results are identical to those for $m_p = 0.0$ kg, for all payload mass ranges, structural factors and payload shape (figures 4.3-c, 4.4-c, 4.10-c and 4.11-c).

For $m_p = 0.0$ kg, the results also do not change for the different structural mass factors and payload shapes (figures 4.5, 4.12, and 4.17). This is expected, since the location of the center of mass of the system does not depend on the structural mass factor (it divides out of equation 3.10), and the contribution to the total moment by the gravity gradient term can be neglected compared to the contribution by the solar radiation force.

When $m_p > 0.0$ kg, the influence of the structural mass factor is that of a slightly lower payload mass (figures 4.6 - 4.9, 4.13 - 4.16). This can be explained by examining the influence of the structural mass on the location of the center of mass of the system. When the

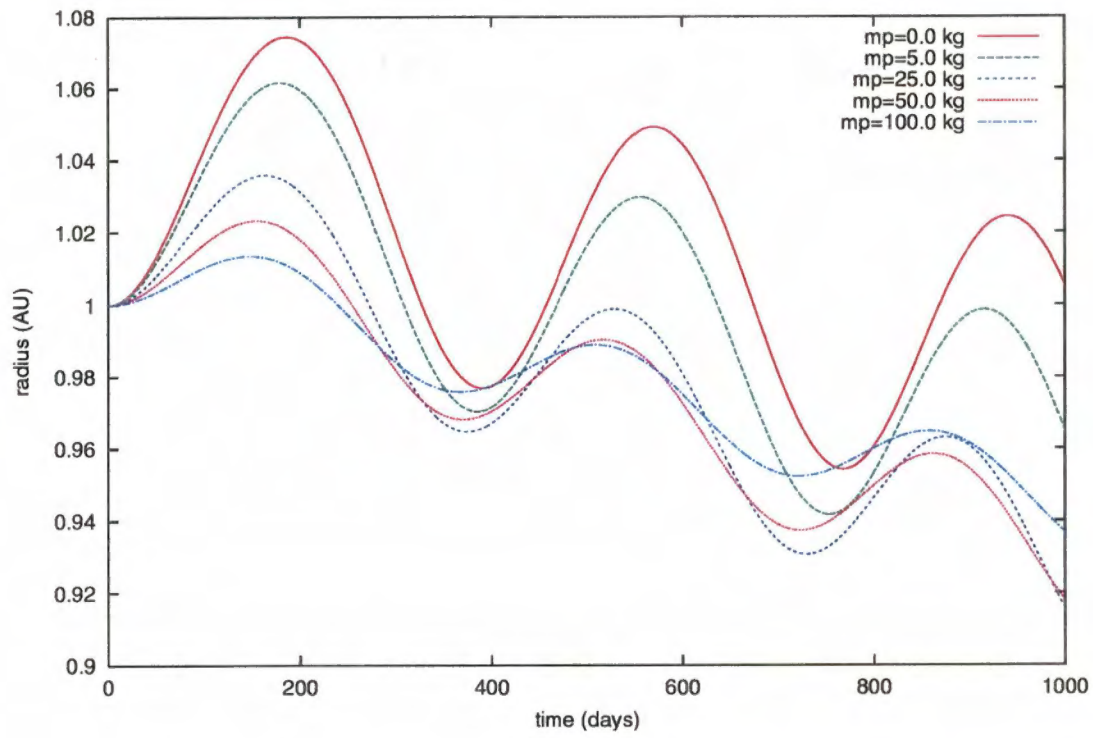
structural mass is included, the mass of the sail planes increases. This will move the total center of mass closer to the center of mass of the sail planes, which would also happen when the payload mass is reduced. In general, including the structural mass factor only has a small influence on the equilibrium position

Finally, when comparing the results for the different structural values and payload shapes for each payload mass, the results show that they are independent of the payload shape (figures 4.17 - 4.21). This is expected, since the gravity gradient terms can be neglected.

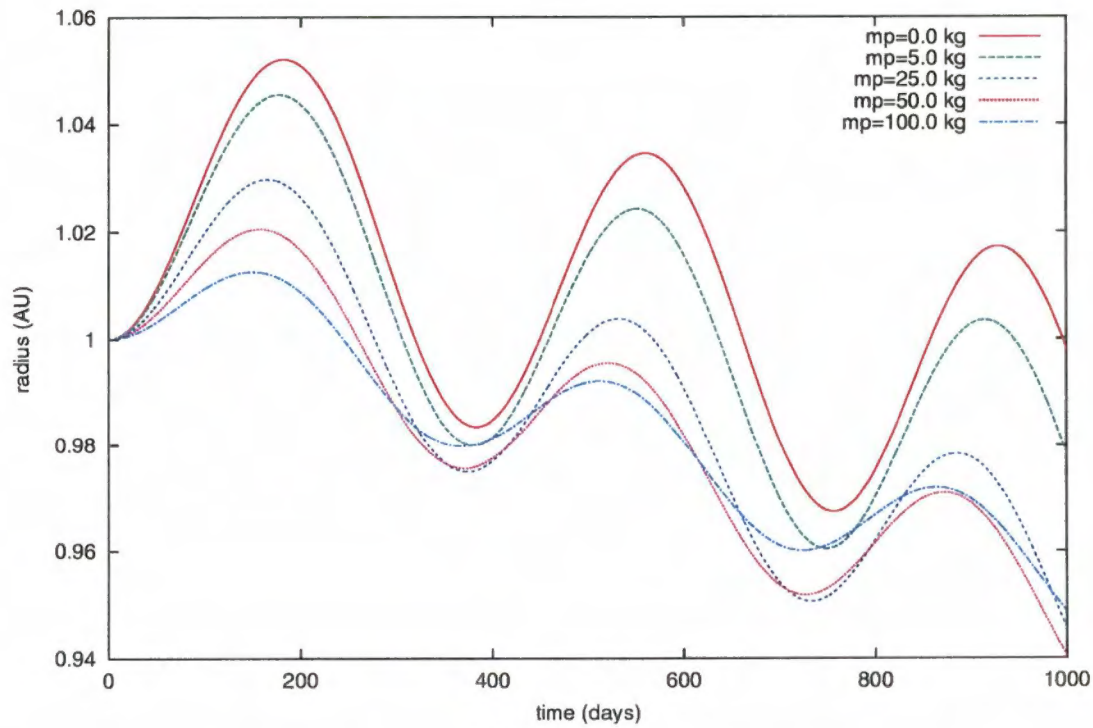
4.3 Trajectory types

Now that the equilibrium positions are known, it will be possible to calculate the position of the solar sail over time by incorporating the trajectory equations into the model. Since in the case of absorption the direction of the force vector is in the direction of the incident sunlight (see equation 3.26), which means that its direction cannot be controlled, the range of α values from 0° to 180° will not be considered here. Trajectories were calculated for α values of 220° , 250° and 270° . It was assumed that the initial orbit is a circular orbit with radius 1 AU and $V_\phi = 1.0 \sqrt{GM/AU}$.

Figure 4.22 shows the radial position of the solar sail as function of time for a structural mass factor k_s of 0.0 and 0.4, payload masses m_p of 0.0 kg, 5.0 kg, 25.0 kg, 50.0 kg and 100.0 kg, with sail dimensions $l_1 = 50.0$ m, $l_2 = 25.0$ m and $l_3 = 5.0$ m. The sail thickness is $24.23 \mu\text{m}$,



a) $k = 0.0$



b) $k = 0.4$

Figure 4.22: Radial position as function of time for $h = 24.23 \mu\text{m}$, $l_3 = 5.0 \text{ m}$

density is $1.3 \cdot 10^3 \text{ kg/m}^3$. This figure shows that over time the resulting trajectories are inward and that the flight time needed to reach a given radial position increases with increasing structural mass factor. These results are expected. An increase in structural mass increases the total mass of the system, thus reducing the acceleration. Also, when the second sail plane is the larger plane, the resulting orientation is such that the component of the solar sail force in the e_ϕ direction is negative, leading to inward trajectories. Initially, though, the trajectories show an increase in radial position. This can be explained by the solar sail force component in the e_r direction. This component is positive, and as can be seen from equation 4.5, this will lead to an increase in the radial velocity and radial position.

One remarkable results which is evident from this figure is that for payload masses of 5.0 kg and 25.0 kg the flight time needed to reach a certain radial position decreases. An explanation for this can be found in tables 4.1 and 4.2. These two tables list the resulting forces and accelerations at 1 AU for the different payload masses. The first row corresponds to the e_r direction, the second row to the e_ϕ direction for each payload. What these tables show, is that with increasing payload mass the resulting forces in the e_ϕ direction increase in magnitude and that the resulting accelerations in this direction increase in magnitude for payload masses of 5.0 kg and 25.0 kg.

Figure 4.23 show the results for the same case as in figure 4.22, but now the sail film thickness is $7.5 \mu\text{m}$. When comparing these two figures, it becomes evident that reducing the total sail mass by reducing the sail film thickness will reduce the flight time. The reduction

Table 4.1: Accelerations at 1 AU for $k = 0.0$, $h = 24.23 \mu\text{m}$, $l_3 = 5.0 \text{ m}$

m_p	theta	n_1	n_2	F_1	F_2	F_{tot}	a
0	0.332523	0.945222	0.933906	0.004750	0.005498	0.010248	2.36624E-04
		0.326429	-0.357518	0.001641	-0.002105	-0.000464	-1.07197E-05
5	0.305416	0.953722	0.923873	0.004880	0.005323	0.010202	2.11182E-04
		0.300690	-0.382699	0.001538	-0.002205	-0.000666	-1.37945E-05
25	0.240481	0.971223	0.897093	0.005153	0.004873	0.010026	1.46776E-04
		0.238170	-0.441842	0.001264	-0.002400	-0.001136	-1.66368E-05
50	0.201247	0.979818	0.879072	0.005291	0.004585	0.009877	1.05846E-04
		0.199891	-0.476689	0.001079	-0.002486	-0.001407	-1.50790E-05
100	0.165776	0.986291	0.861614	0.005397	0.004318	0.009714	6.77854E-05
		0.165017	-0.507565	0.000903	-0.002543	-0.001640	-1.14470E-05

Table 4.2: Accelerations at 1 AU for $k = 0.4$, $h = 24.23 \mu\text{m}$, $l_3 = 5.0 \text{ m}$

m_p	theta	n_1	n_2	F_1	F_2	F_{tot}	a
0	0.332523	0.945222	0.933906	0.004750	0.005498	0.010248	1.69017E-04
		0.326429	-0.357518	0.001641	-0.002105	-0.000464	-7.65690E-06
5	0.312487	0.951572	0.926556	0.004847	0.005369	0.010216	1.55648E-04
		0.307426	-0.376157	0.001566	-0.002180	-0.000614	-9.35412E-06
25	0.258257	0.966837	0.904805	0.005084	0.004500	0.010084	1.17751E-04
		0.255395	-0.425826	0.001343	-0.002353	-0.001010	-1.17969E-05
50	0.220361	0.975819	0.888022	0.005227	0.004727	0.009954	8.99676E-05
		0.218582	-0.459800	0.001171	-0.002447	-0.001277	-1.15398E-05
100	0.182031	0.983478	0.869750	0.005351	0.004441	0.009792	6.09568E-05
		0.181027	-0.493492	0.000985	-0.002520	-0.001535	-9.55538E-06

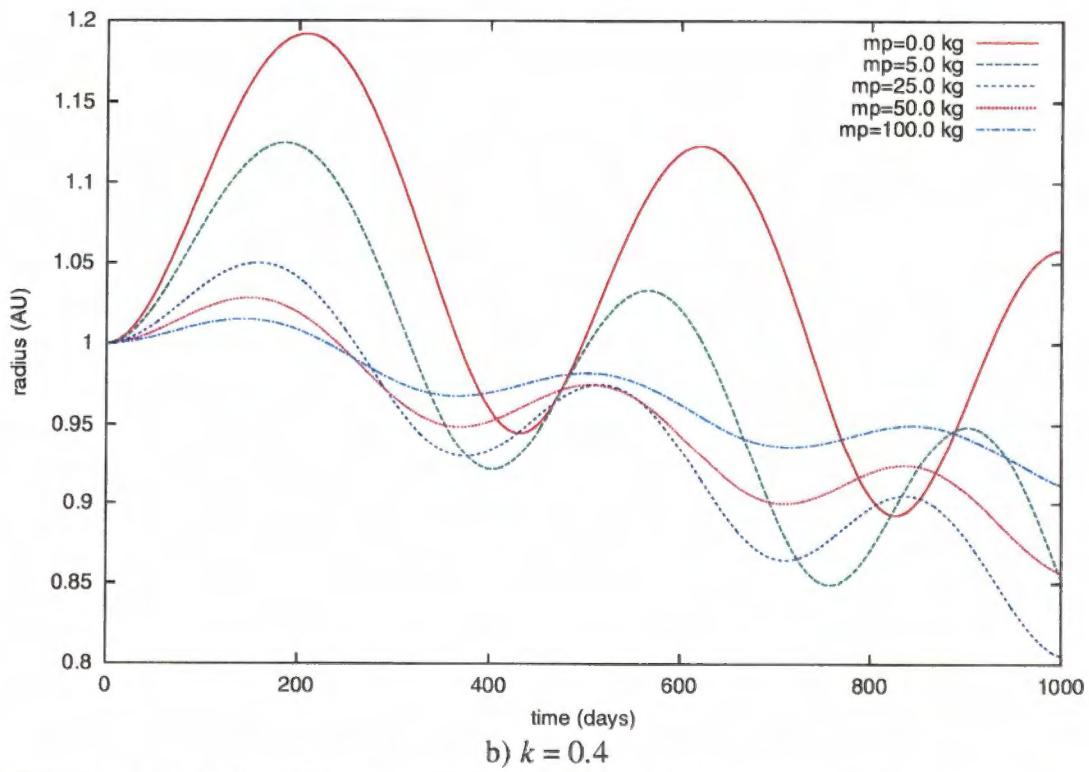
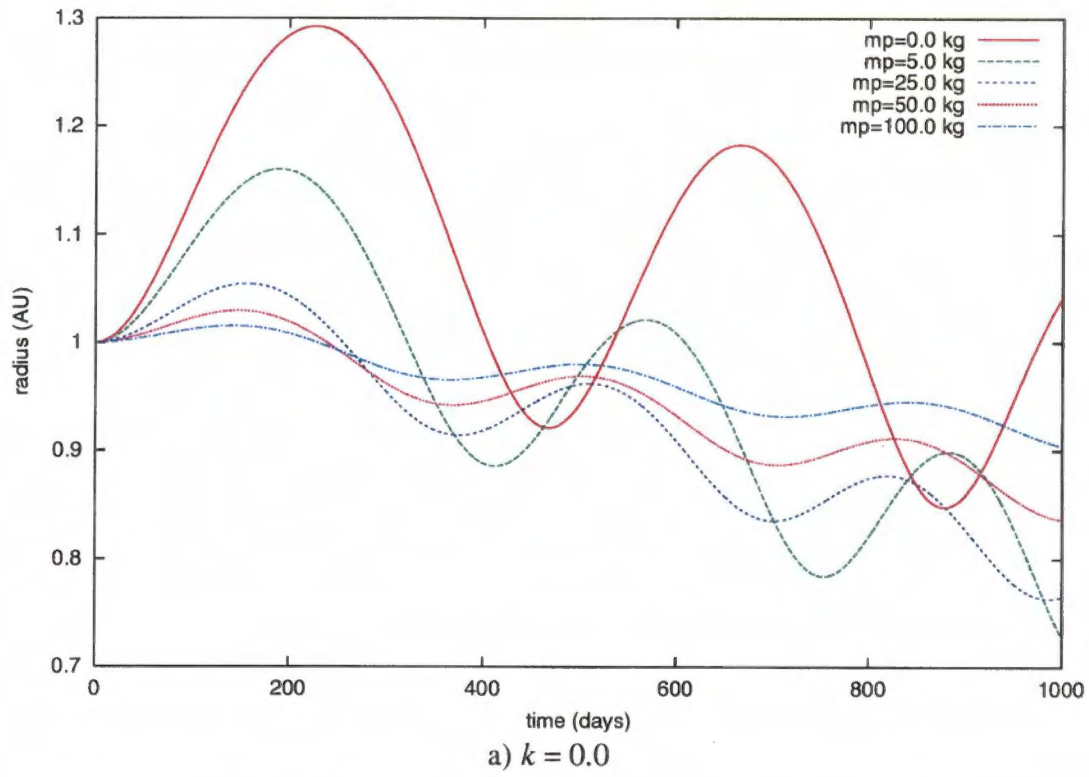
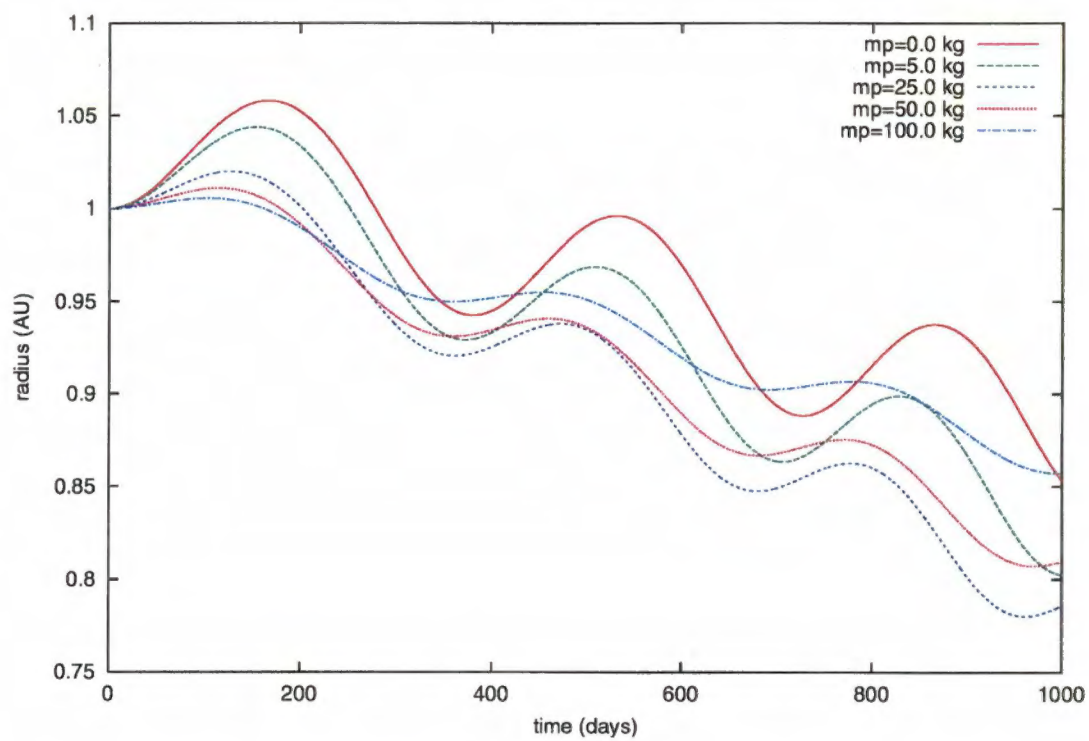


Figure 4.23: Radial position as function of time for $h = 7.5 \mu\text{m}$, $l_3 = 5.0 \text{ m}$

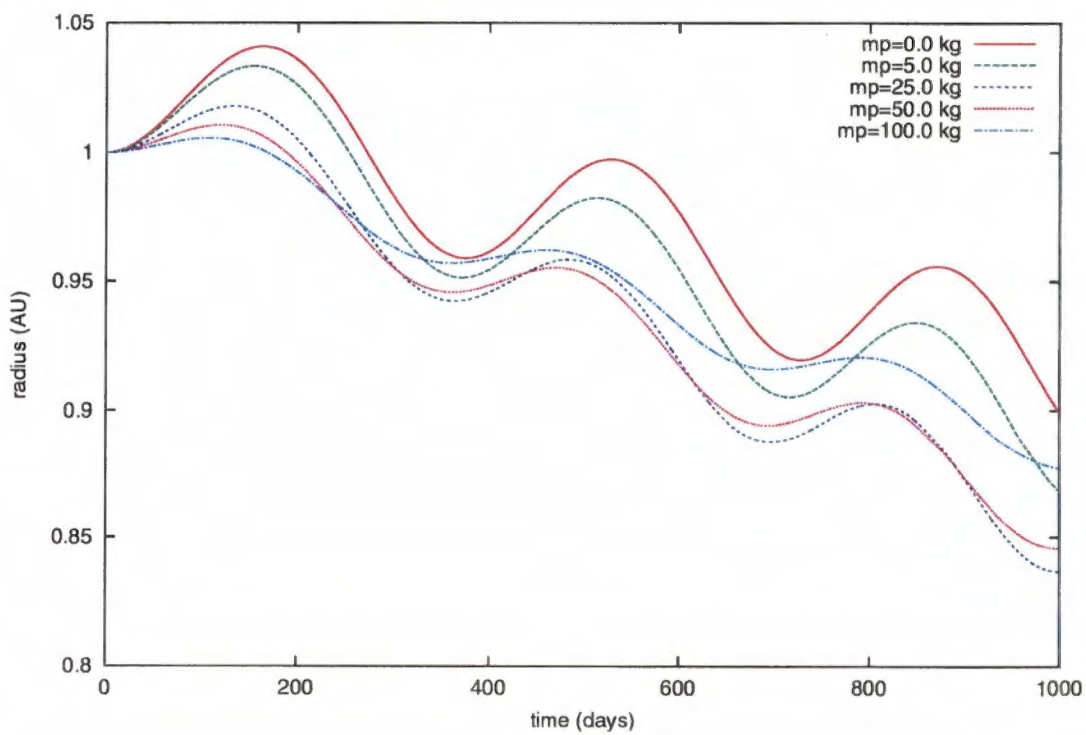
in mass leads to an increase in the magnitude of the acceleration. Again, including the structural mass factor will lead to increased flight times, and for a payload mass of 5.0 kg the flight time decreases compared to the flight time for no payload mass. For higher payload masses, the flight times increase again.

Results for $l_3 = 15.0$ m and for the two different sail film thicknesses and structural mass factors are given in figures 4.24 and 4.25. These results again show that the trajectories are inward, including structural mass increase flight times and reducing mass by reducing the sail film thickness reduces the flight times. When compared to figures 4.22 and 4.23, the flight times decrease for this case. A larger value for l_3 increases the total area of the sail, thus increasing the solar sail force. Although the total mass also increases, the increase in force is the dominant factor here.

Figures 4.26 and 4.27 show the results for $l_3 = -5.0$ m, again for the two sail film thicknesses and structural mass factors. Now the trajectories are outward, which is expected. The second sail plane is now the smaller plane, and the total force in the e_ϕ direction will be positive, leading to an increase of the radius over time. Including the structural mass again increases the flight times and using a thinner sail film material will result in faster times. The flight times for no payload and a payload mass of 5.0 kg and 25.0 kg for $h = 24.23$ μm are almost identical, for the higher payload masses the flight times increase with increased payload masses. The same is the case for no payload and a payload mass of 5.0 kg for $h = 7.5$ μm .



a) $k = 0.0$



b) $k = 0.4$

Figure 4.24: Radial position as function of time for $h = 24.23 \mu\text{m}$, $l_3 = 15.0 \text{ m}$

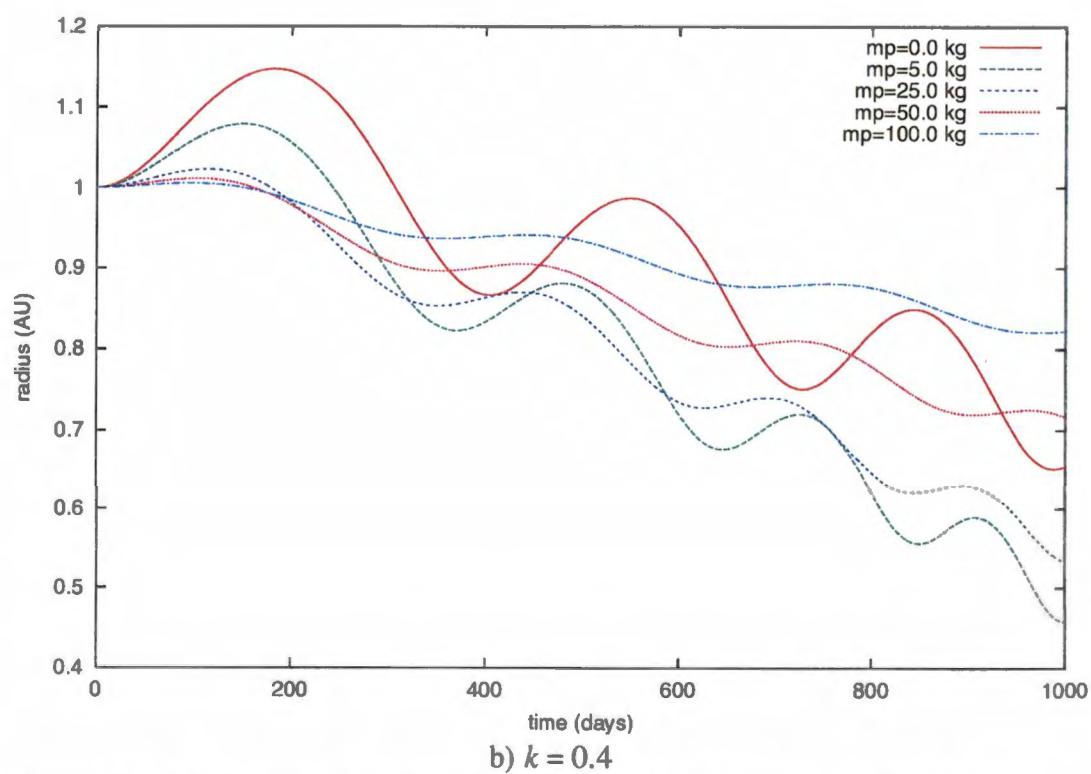
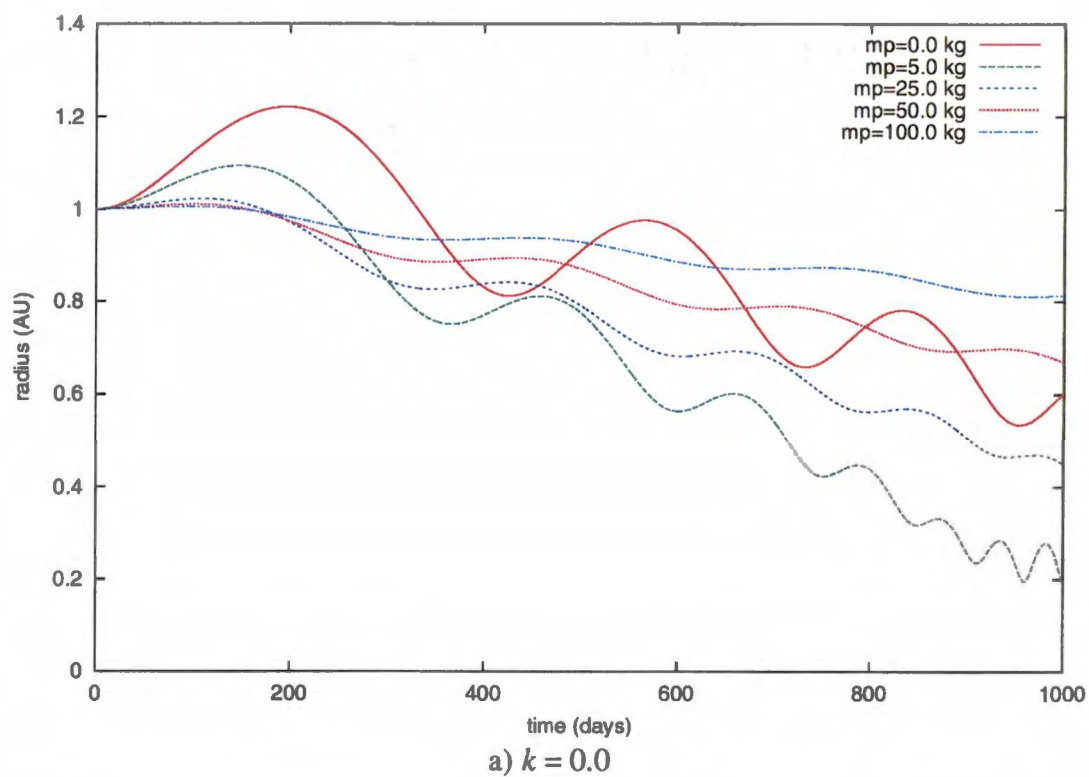


Figure 4.25: Radial position as function of time for $h = 7.5 \mu\text{m}$, $l_3 = 15.0 \text{ m}$

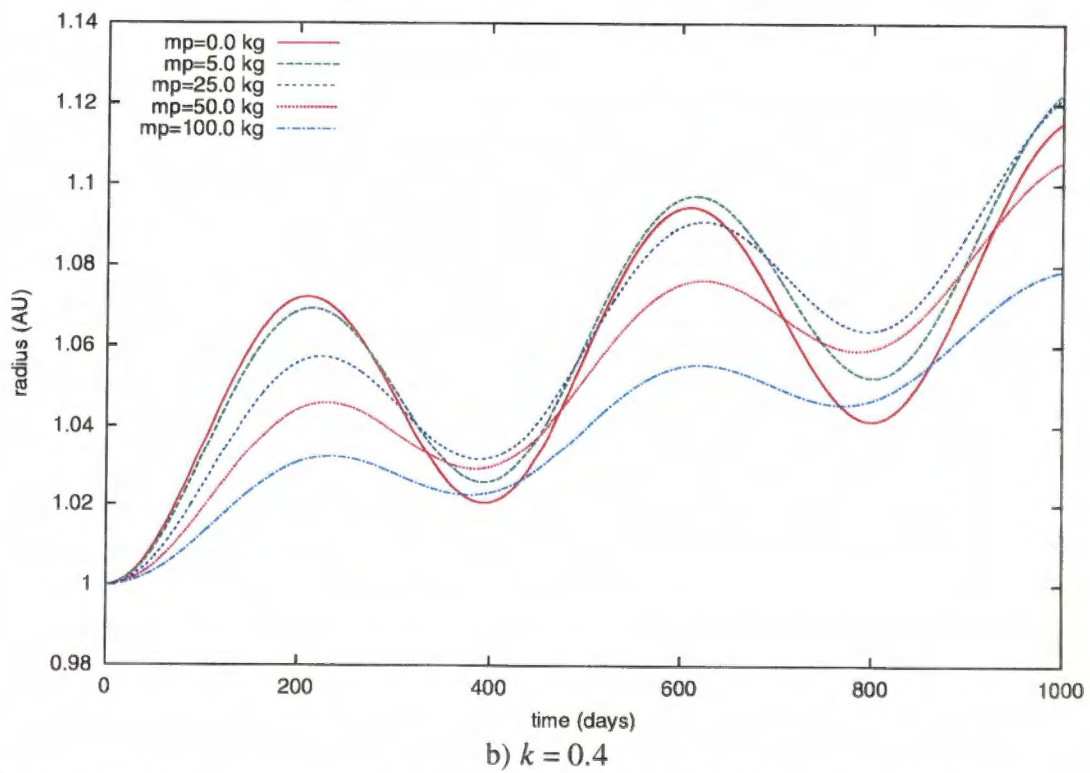
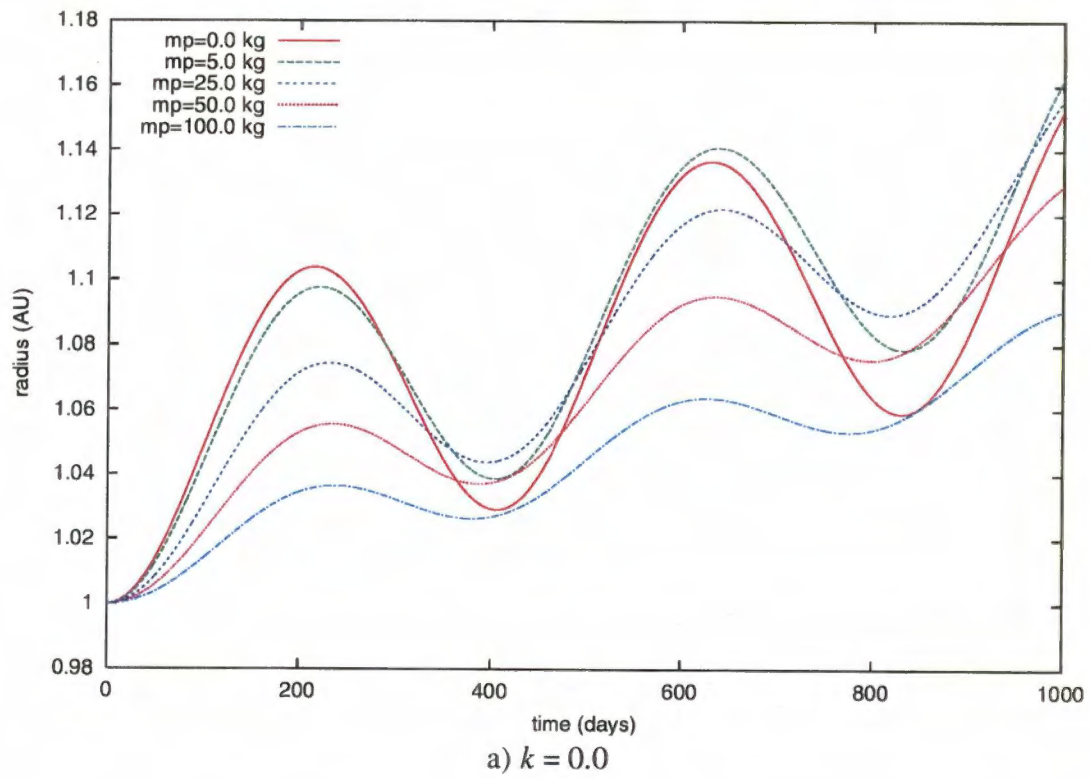


Figure 4.26: Radial position as function of time for $h = 24.23 \mu\text{m}$, $l_3 = -5.0 \text{ m}$

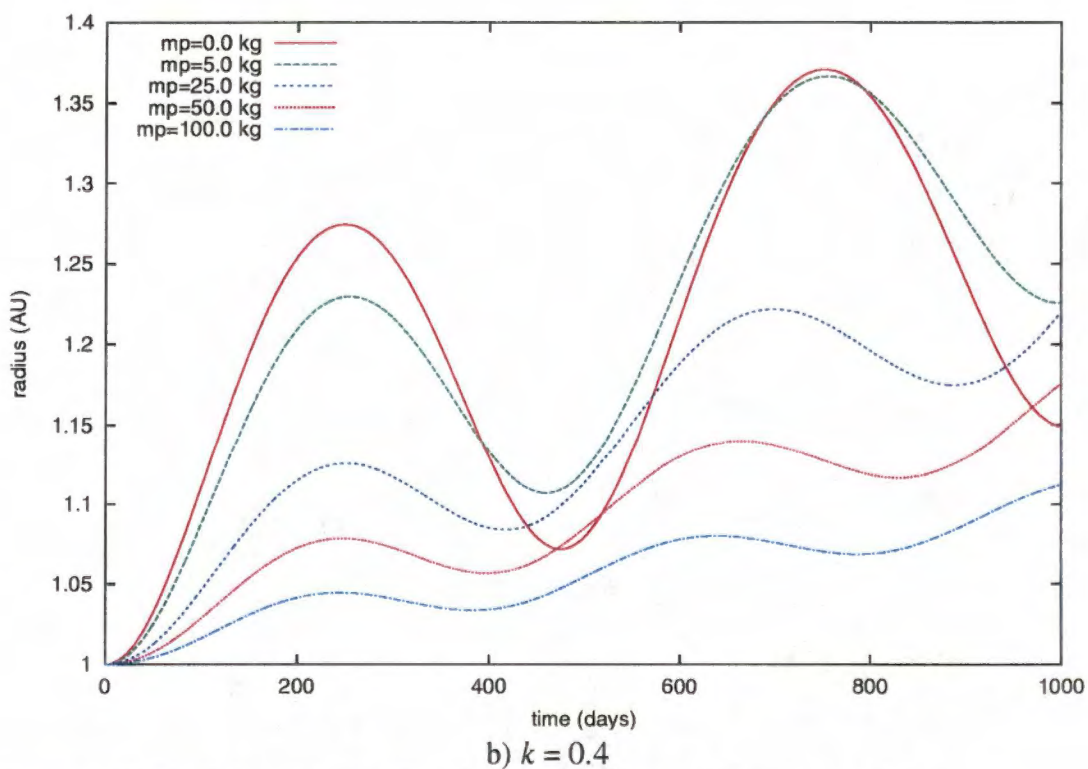
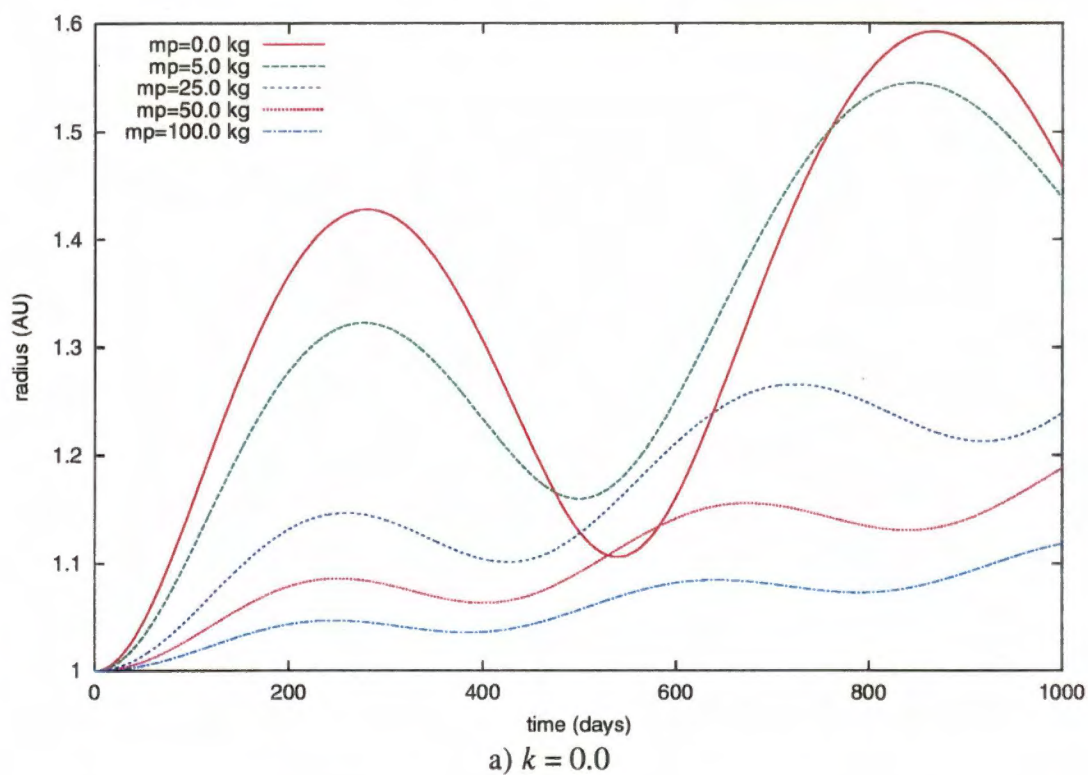


Figure 4.27: Radial position as function of time for $h = 7.5 \mu\text{m}$, $l_3 = -5.0 \text{ m}$

Again, a slight increase in acceleration in the e_ϕ direction can be found in these cases, see tables 4.3 and 4.4.

The results for $l_3 = -15.0$ m are given in figures 4.28 and 4.29. Again it is evident that a decrease in sail film thickness reduces the flight times whereas an increase in structural mass factor increases the flight time. Compared to figures 4.26 and 4.27, it is not very obvious which value of l_3 leads to the faster flight times. Tables 4.5 and 4.6 show that for $l_3 = -15.0$ m and $h = 7.5 \mu\text{m}$, although the total sail area is smaller compared to $l_3 = -5.0$ m, the magnitude of the force in the e_ϕ direction actually increases for some payload masses due to the change in orientation. This means that the magnitude of the acceleration in this direction will increase for those cases and the flight times will decrease, even though the total sail area is smaller.

In order to compare flight times for inward and outward orbits, trajectories were calculated with $l_1 = 50.0$ m, $l_2 = 40.0$ m, $l_3 = -15.0$ m, the two different sail thicknesses and structural mass factors. This means that the first sail plane has a height of 40.0 m, the second sail plane a height of 25.0 m. These results are given in figures 4.30 and 4.31. These results can be compared to the case of $l_1 = 50.0$ m, $l_2 = 25.0$ m and $l_3 = 15.0$ m, since in this case the first sail plane has a height of 25.0 m and the second sail plane a height of 40.0 m. These results were given in figures 4.24 and 4.25.

Table 4.3: Accelerations at 1 AU for $k = 0.0$, $h = 24.23 \mu\text{m}$, $l_3 = -5.0 \text{ m}$

m_p	theta	n_1	n_2	F_1	F_2	F_{tot}	a
0	0.369285	0.932586	0.946415	0.004562	0.003815	0.008377	2.363964E-04
		0.360949	-0.322952	0.001766	-0.001302	0.000464	1.309689E-05
5	0.408837	0.917584	0.958445	0.004346	0.003962	0.008308	2.054512E-04
		0.397542	-0.285277	0.001883	-0.001179	0.000703	1.739769E-05
25	0.495319	0.879817	0.979504	0.003831	0.004229	0.008060	1.333605E-04
		0.475313	-0.201425	0.002070	-0.000870	0.001200	1.985498E-05
50	0.542507	0.856417	0.987915	0.003533	0.004339	0.007872	9.213987E-05
		0.516285	-0.154997	0.002130	-0.000681	0.001449	1.696332E-05
100	0.582261	0.835221	0.993295	0.003277	0.004410	0.007687	5.676072E-05
		0.549914	-0.115611	0.002158	-0.000513	0.001645	1.214259E-05

Table 4.4: Accelerations at 1 AU for $k = 0.4$, $h = 24.23 \mu\text{m}$, $l_3 = -5.0 \text{ m}$

m_p	theta	n_1	n_2	F_1	F_2	F_{tot}	a
0	0.369285	0.932586	0.946415	0.004562	0.003815	0.008377	1.688545E-04
		0.360949	-0.322952	0.001766	-0.001302	0.000464	9.354924E-06
5	0.398727	0.921556	0.955512	0.004402	0.003926	0.008328	1.524991E-04
		0.388246	-0.294951	0.001855	-0.001212	0.000643	1.177208E-05
25	0.472735	0.890326	0.974706	0.003970	0.004167	0.008137	1.090578E-04
		0.455323	-0.223493	0.002030	-0.000955	0.001075	1.440440E-05
50	0.519961	0.867839	0.984169	0.003677	0.004290	0.007966	7.997312E-05
		0.496846	-0.177230	0.002105	-0.000772	0.001332	1.337576E-05
100	0.564369	0.844926	0.991067	0.003393	0.004380	0.007773	5.195772E-05
		0.534883	-0.133364	0.002148	-0.000589	0.001558	1.041674E-05

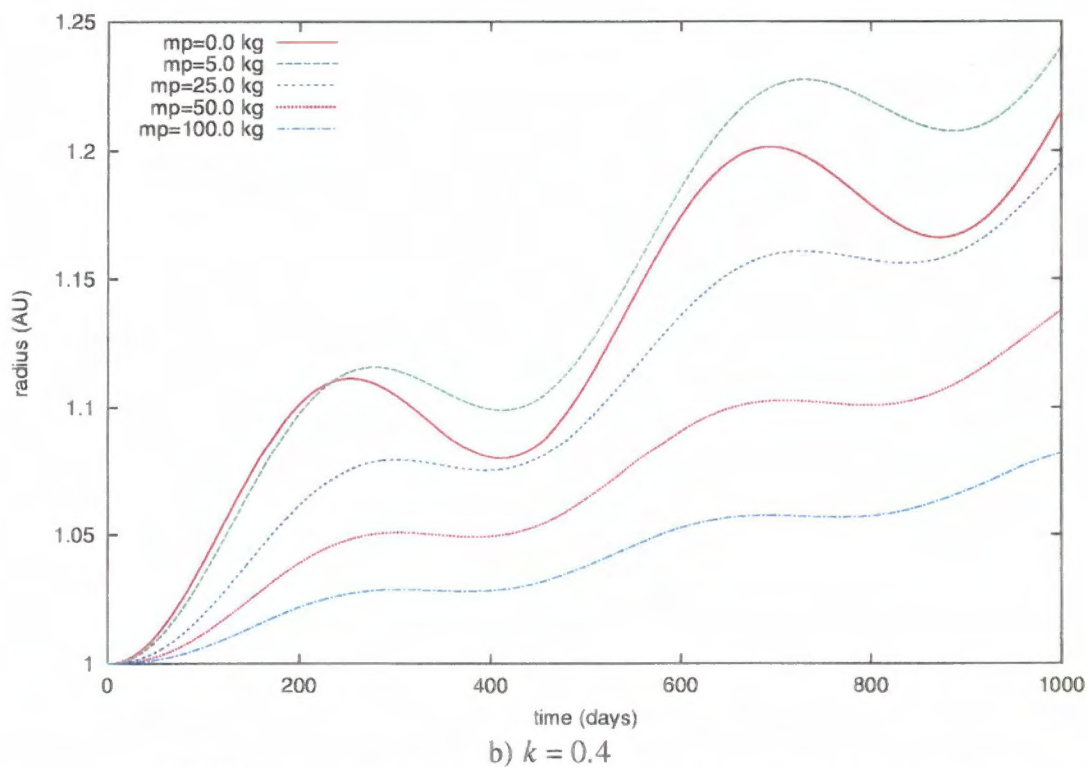
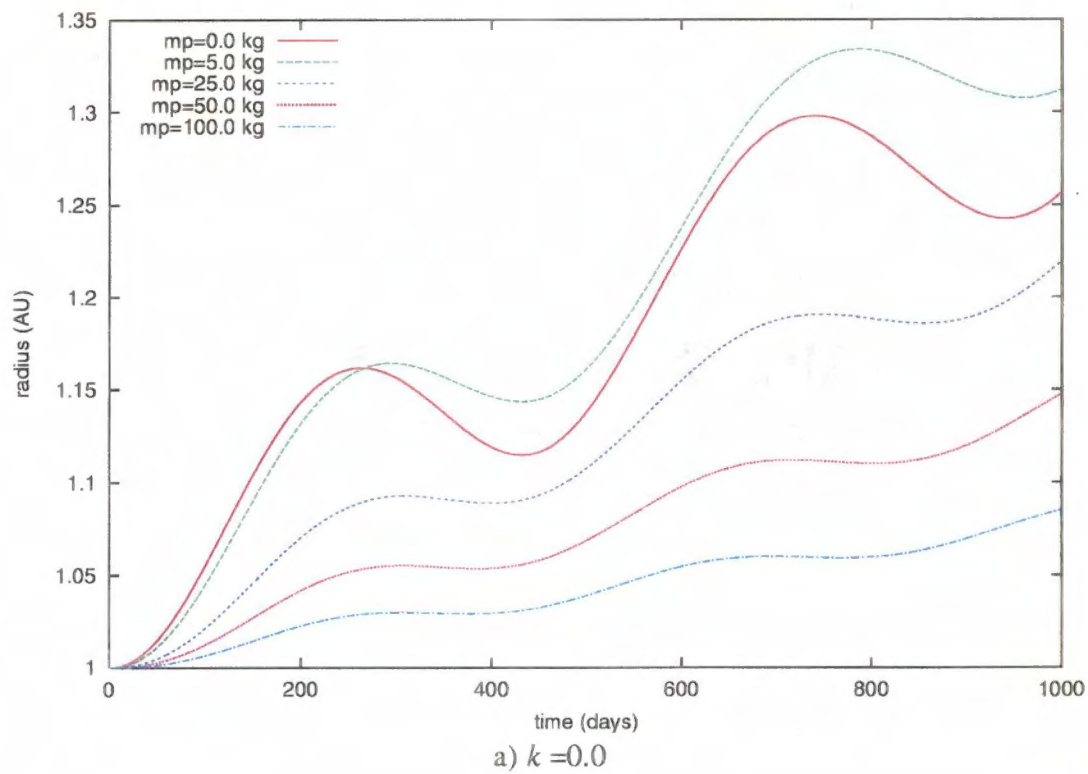


Figure 4.28: Radial position as function of time for $h = 24.23 \mu\text{m}$, $l_3 = -15.0 \text{ m}$

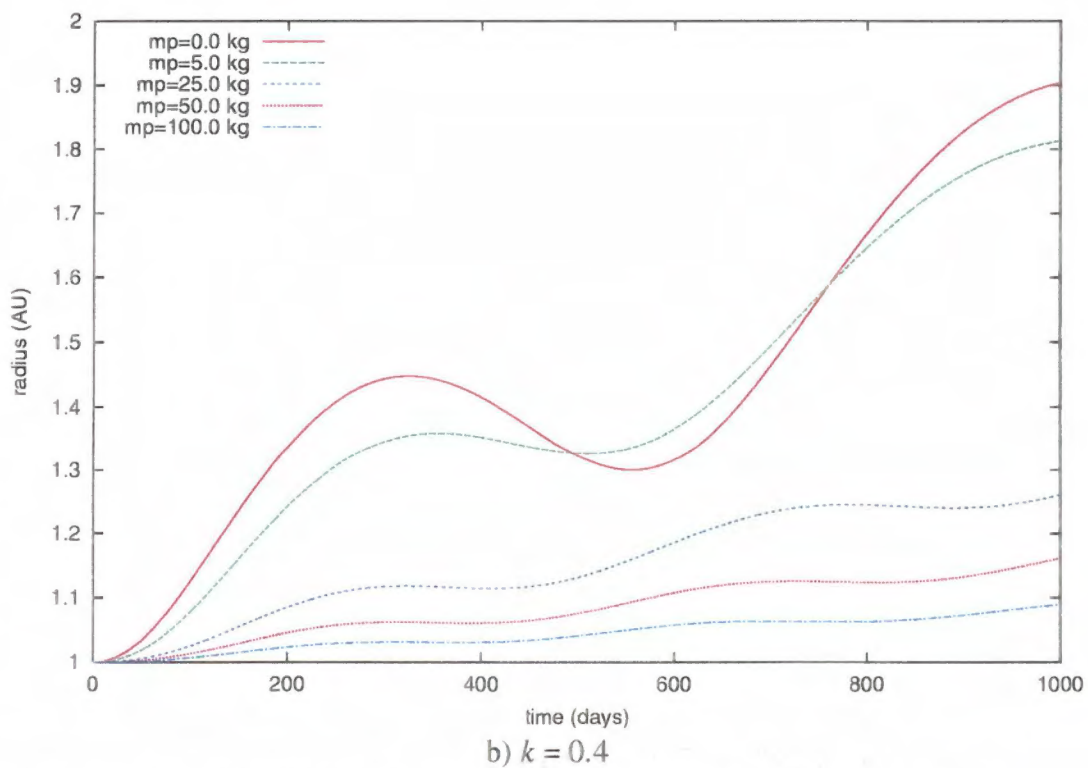
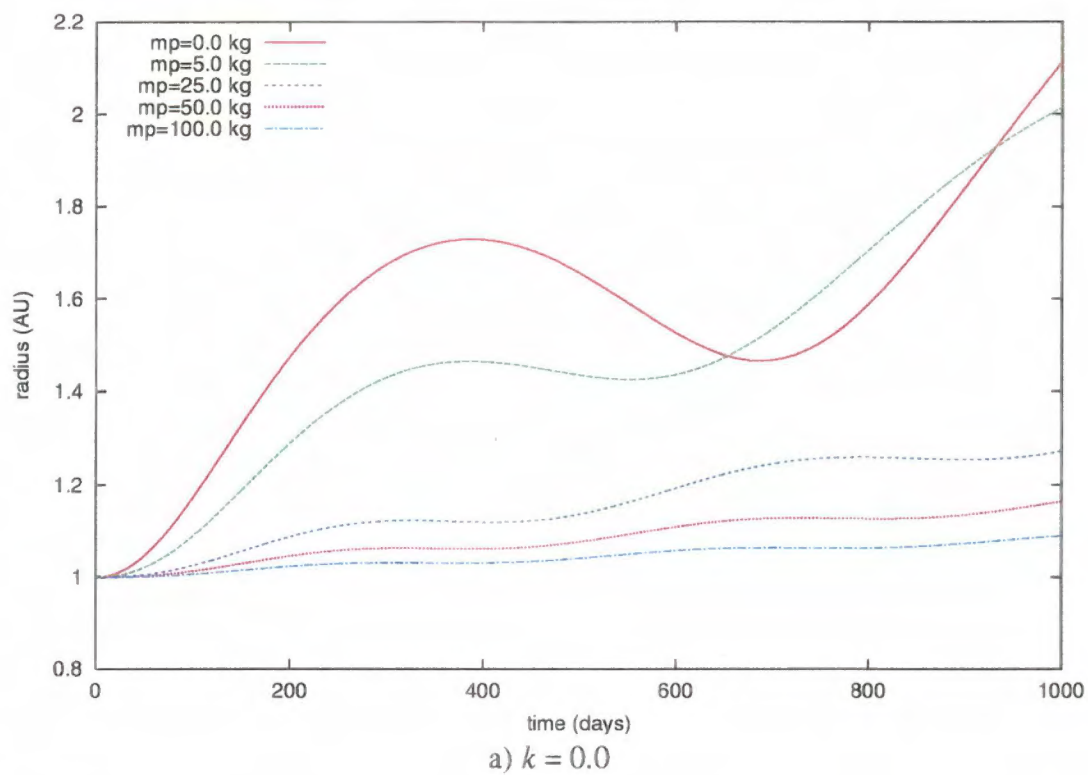


Figure 4.29: Radial position as function of time for $h = 7.5 \mu\text{m}$, $l_3 = -15.0 \text{ m}$

Table 4.5: Accelerations at 1 AU for $k = 0.0$, $h = 7.5 \mu\text{m}$, $l_3 = -5.0 \text{ m}$

m_p	theta	n_1	n_2	F_1	F_2	F_{tot}	a
0	0.369285	0.932586	0.946415	0.004562	0.003815	0.008377	7.637179E-04
		0.360949	-0.322952	0.001766	-0.001302	0.000464	4.231170E-05
5	0.466313	0.893232	0.973250	0.004009	0.004148	0.008157	5.108265E-04
		0.449596	-0.229748	0.002018	-0.000979	0.001038	6.503227E-05
25	0.571237	0.841233	0.991960	0.003349	0.004392	0.007741	2.152142E-04
		0.540673	-0.126554	0.002152	-0.000560	0.001592	4.425681E-05
50	0.602609	0.823860	0.995441	0.003145	0.004439	0.007584	1.243946E-04
		0.566794	-0.095377	0.002164	-0.000425	0.001739	2.851772E-05
100	0.622210	0.812592	0.997119	0.003018	0.004461	0.007479	6.740071E-05
		0.582832	-0.075849	0.002165	-0.000339	0.001825	1.644980E-05

Table 4.6: Accelerations at 1 AU for $k = 0.0$, $h = 7.5 \mu\text{m}$, $l_3 = -15.0 \text{ m}$

m_p	theta	n_1	n_2	F_1	F_2	F_{tot}	a
0	0.426964	0.910227	0.963459	0.004242	0.002012	0.006254	7.331001E-04
		0.414110	-0.267856	0.001930	-0.000559	0.001370	1.606417E-04
5	0.850864	0.659334	0.988359	0.001612	0.002172	0.003785	2.796939E-04
		0.751850	0.152139	0.001839	0.000334	0.002173	1.605834E-04
25	1.100195	0.453422	0.920256	0.000524	0.001754	0.002278	6.793276E-05
		0.891296	0.391318	0.001031	0.000746	0.001776	5.297692E-05
50	1.153206	0.405559	0.898228	0.000375	0.001631	0.002006	3.426890E-05
		0.914069	0.439529	0.000846	0.000798	0.001644	2.808035E-05
100	1.183108	0.378049	0.884686	0.000304	0.001558	0.001862	1.715509E-05
		0.925785	0.466187	0.000744	0.000821	0.001565	1.442191E-05

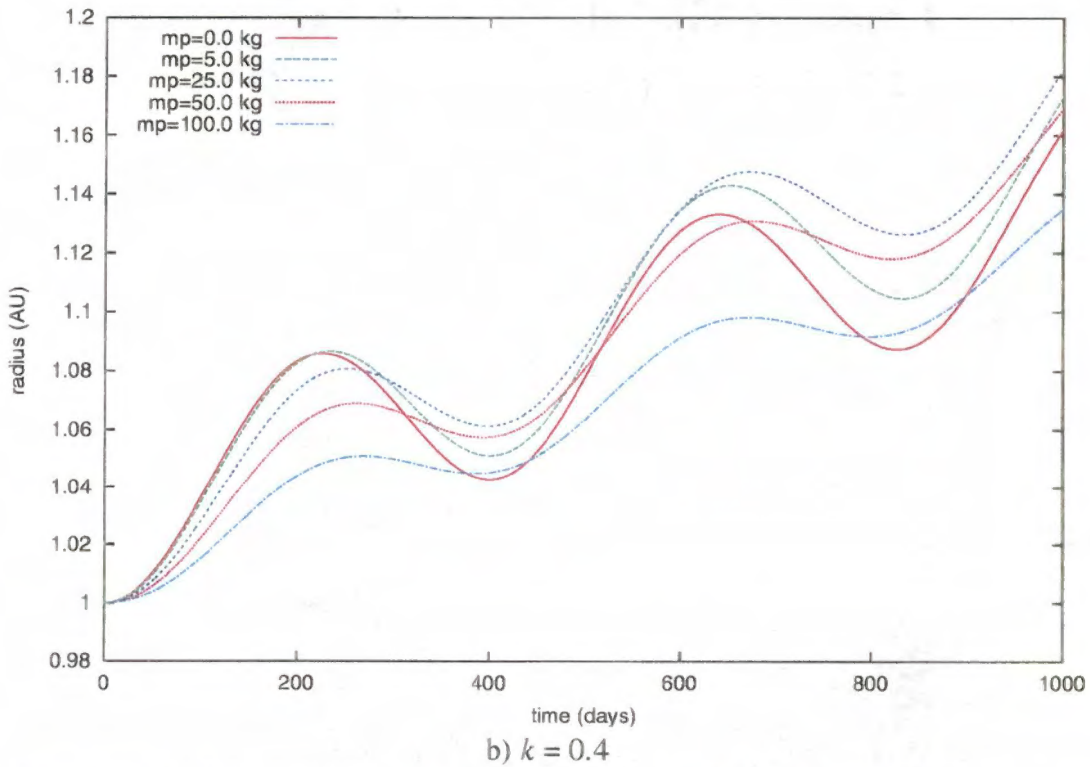
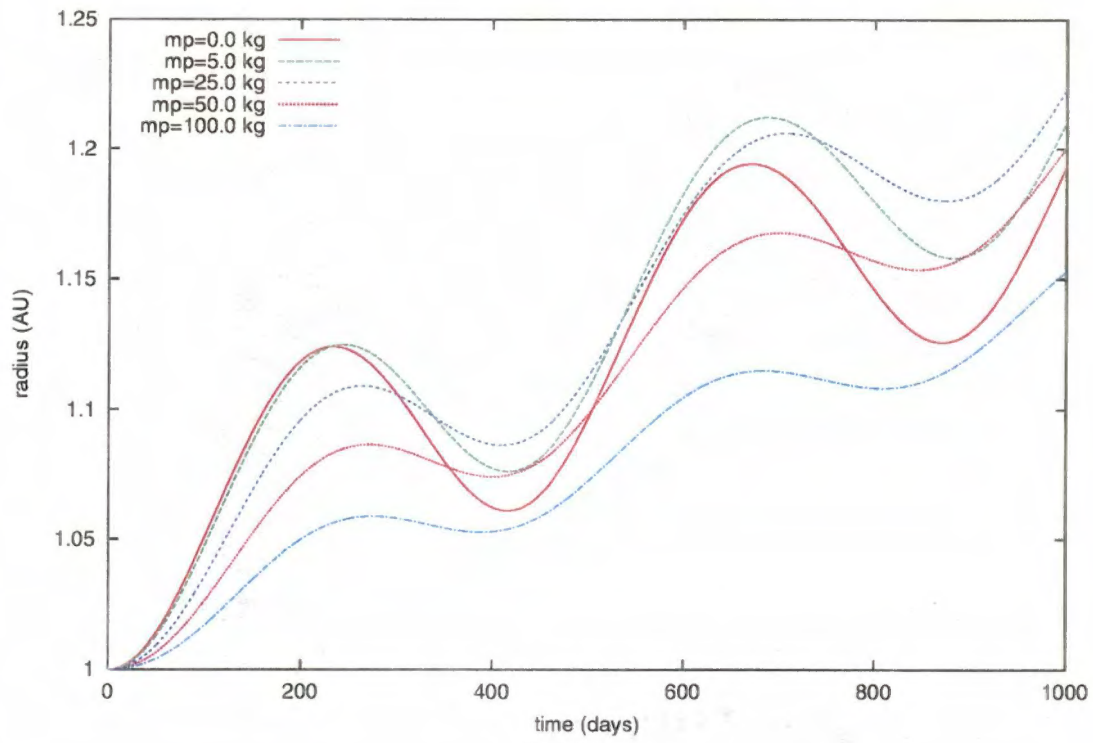


Figure 4.30: Radial position as function of time for $h = 24.23 \mu\text{m}$, $l_2 = 40.0 \text{ m}$, $l_3 = -15.0 \text{ m}$

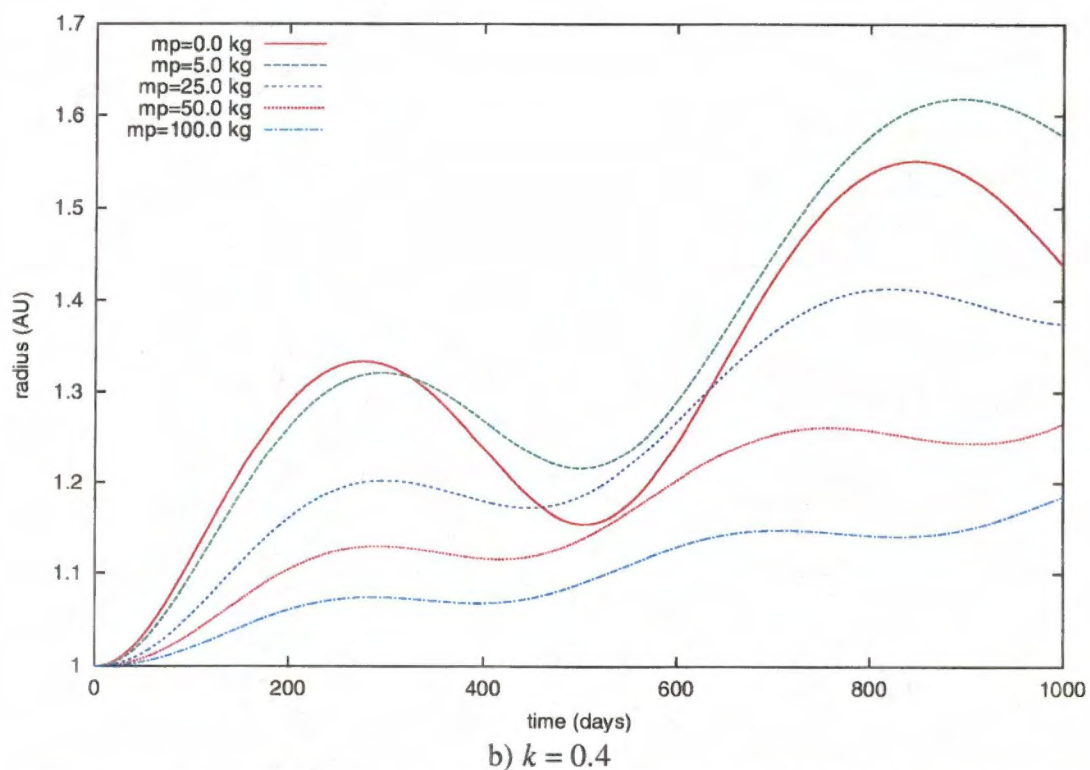
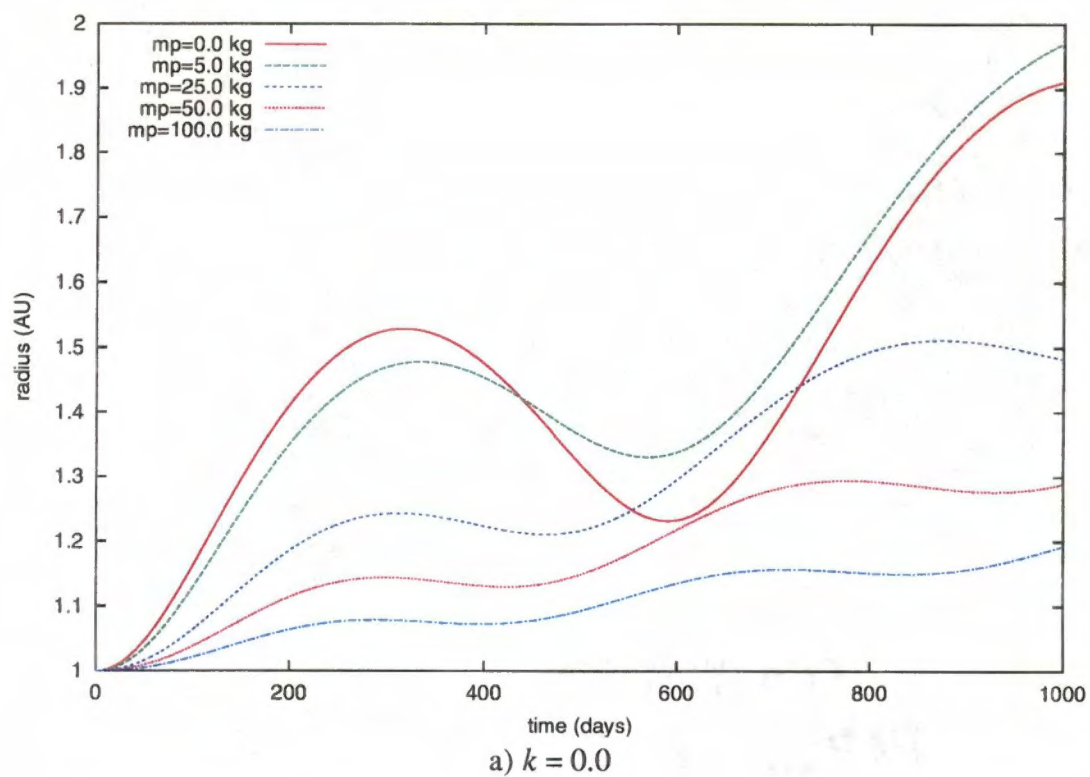


Figure 4.31: Radial position as function of time for $h = 7.5 \mu\text{m}$, $l_2 = 40.0 \text{ m}$, $l_3 = -15.0 \text{ m}$

Tables 4.7 and 4.8 give numerical values for the accelerations at 1 AU for these two cases for $h = 24.23 \mu\text{m}$ and $k = 0.0$. These tables show that the magnitude of the accelerations in the e_ϕ direction are of equal magnitude but opposite in sign. The accelerations in the e_r direction are of equal magnitude and of the same sign. The forces generated by the first sail plane in the first case and the second sail plane of the second case, which have identical dimensions, are of equal magnitude but opposite in sign in the e_ϕ direction. The same is valid for the forces generated by the second sail plane in the first case and the first sail plane in the second case.

It is evident from the figures that for the lower payload masses it takes less time to fly out a certain distance when comparing the flight times needed to fly a certain distance inward or outward. Although the solar sail forces generated become smaller with increasing radial position, the positive acceleration in the e_r direction can explain this. The outward trajectories are helped by this acceleration, the inward trajectories have to overcome this acceleration. For the higher payload masses there is hardly a detectable change in flight times needed. The tables show that the acceleration in the e_r direction become smaller with increasing payload mass, which means that its influence on the trajectories diminishes.

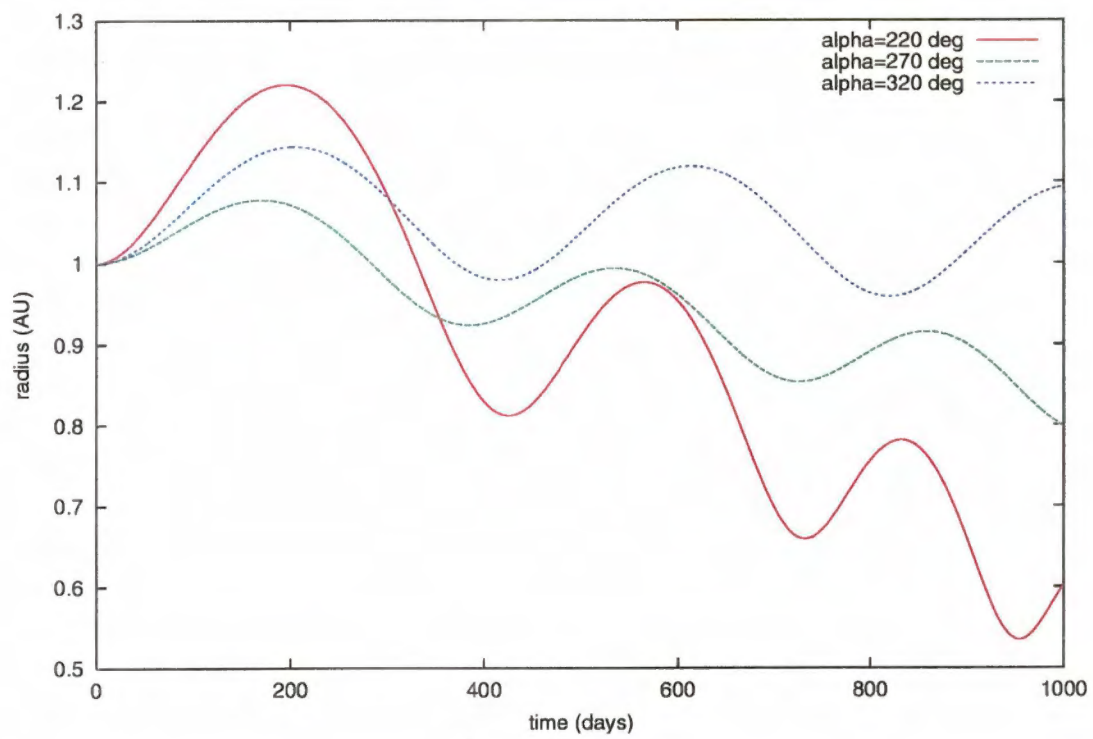
Finally, the influence of varying the angle α between the two sail planes on the trajectories was investigated. Figure 4.32 shows the trajectories as function of α for payload masses of 0.0 kg, 5.0 kg, 25.0 kg, 50.0 kg and 100.0 kg respectively. The sail film thickness is $7.5 \mu\text{m}$, the structural mass factor k is 0.0 and the sail dimensions are $l_1 = 50.0 \text{ m}$, $l_2 = 25.0 \text{ m}$ and

Table 4.7: Accelerations at 1 AU for $k = 0.0$, $h = 24.23 \mu\text{m}$, $l_2 = 25.0 \text{ m}$, $l_3 = 15.0 \text{ m}$

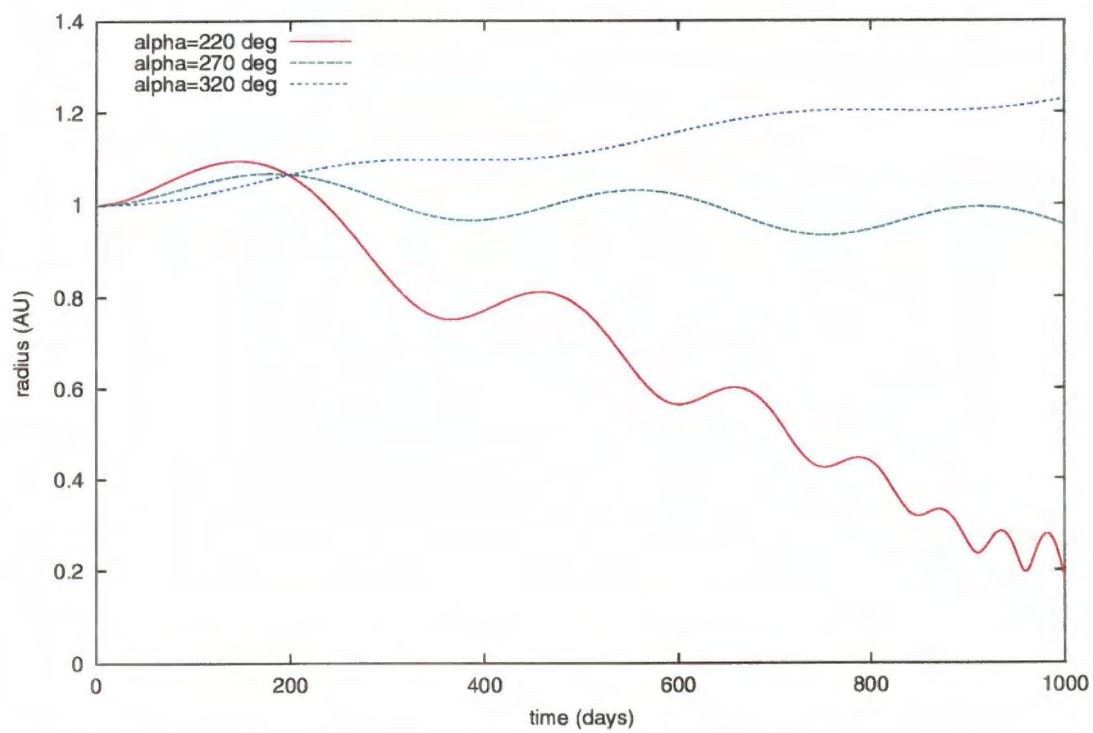
m_p	theta	n_1	n_2	F_1	F_2	F_{tot}	a
0	0.307084	0.953219	0.924510	0.004872	0.007112	0.011984	2.341212E-04
		0.302281	-0.381157	0.001545	-0.002932	-0.001387	-2.709883E-05
5	0.245926	0.969912	0.899485	0.005132	0.006550	0.011682	2.079196E-04
		0.243454	-0.436951	0.001288	-0.003182	-0.001893	-3.370000E-05
25	0.096988	0.995300	0.824689	0.005546	0.005048	0.010594	1.390546E-04
		0.096836	-0.565586	0.000540	-0.003462	-0.002922	-3.835837E-05
50	0.006291	0.999980	0.770073	0.005625	0.004110	0.009735	9.620545E-05
		0.006291	-0.637956	0.000035	-0.003405	-0.003369	-3.329967E-05
100	-0.075547	0.997148	0.715345	0.005577	0.003295	0.008872	5.867946E-05
		-0.075475	-0.698771	-0.000422	-0.003218	-0.003640	-2.407828E-05

Table 4.8: Accelerations at 1 AU for $k = 0.0$, $h = 24.23 \mu\text{m}$, $l_2 = 40.0 \text{ m}$, $l_3 = -15.0 \text{ m}$

m_p	theta	n_1	n_2	F_1	F_2	F_{tot}	a
0	0.391048	0.924510	0.953219	0.007112	0.004872	0.011984	2.341212E-04
		0.381157	-0.302280	0.002932	-0.001545	0.001387	2.709885E-05
5	0.452206	0.899485	0.969912	0.006550	0.005132	0.011682	2.079196E-04
		0.436951	-0.243454	0.003182	-0.001288	0.001893	3.370000E-05
25	0.601144	0.824689	0.995300	0.005048	0.005546	0.010594	1.390546E-04
		0.565586	-0.096836	0.003462	-0.000540	0.002922	3.835838E-05
50	0.691841	0.770073	0.999980	0.004110	0.005625	0.009735	9.620545E-05
		0.637956	-0.006291	0.003405	-0.000035	0.003369	3.329967E-05
100	0.773678	0.715345	0.997148	0.003295	0.005577	0.008872	5.867946E-05
		0.698771	0.075475	0.003218	0.000422	0.003640	2.407828E-05

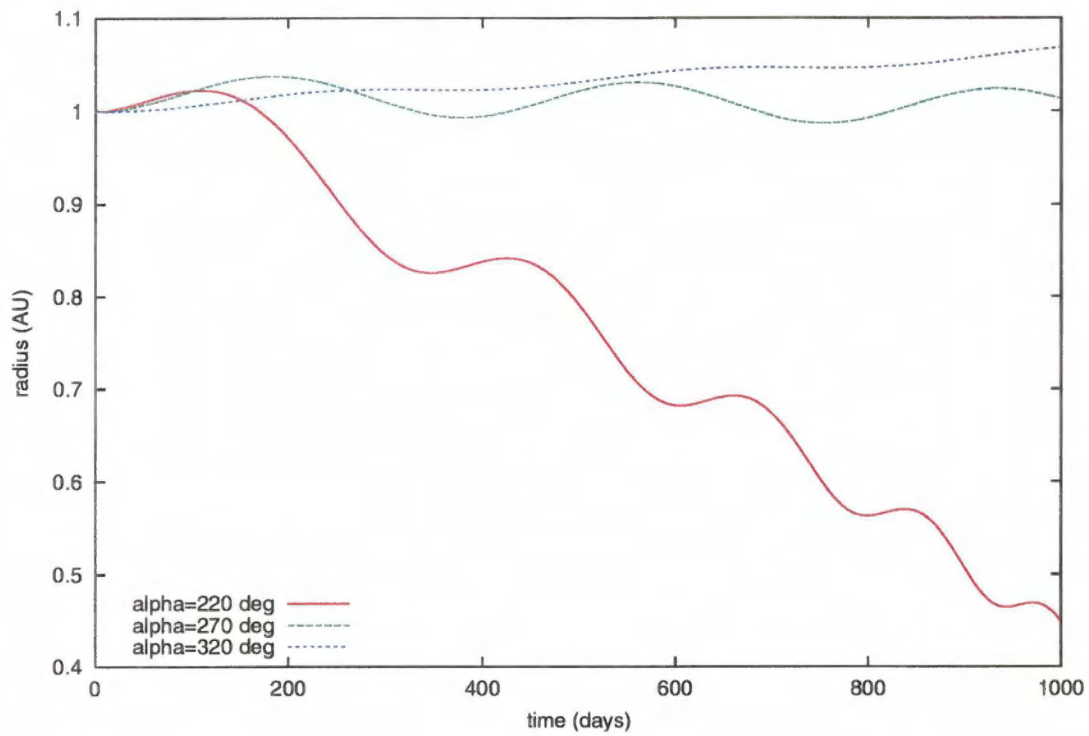


a) $m_p = 0.0$ kg.

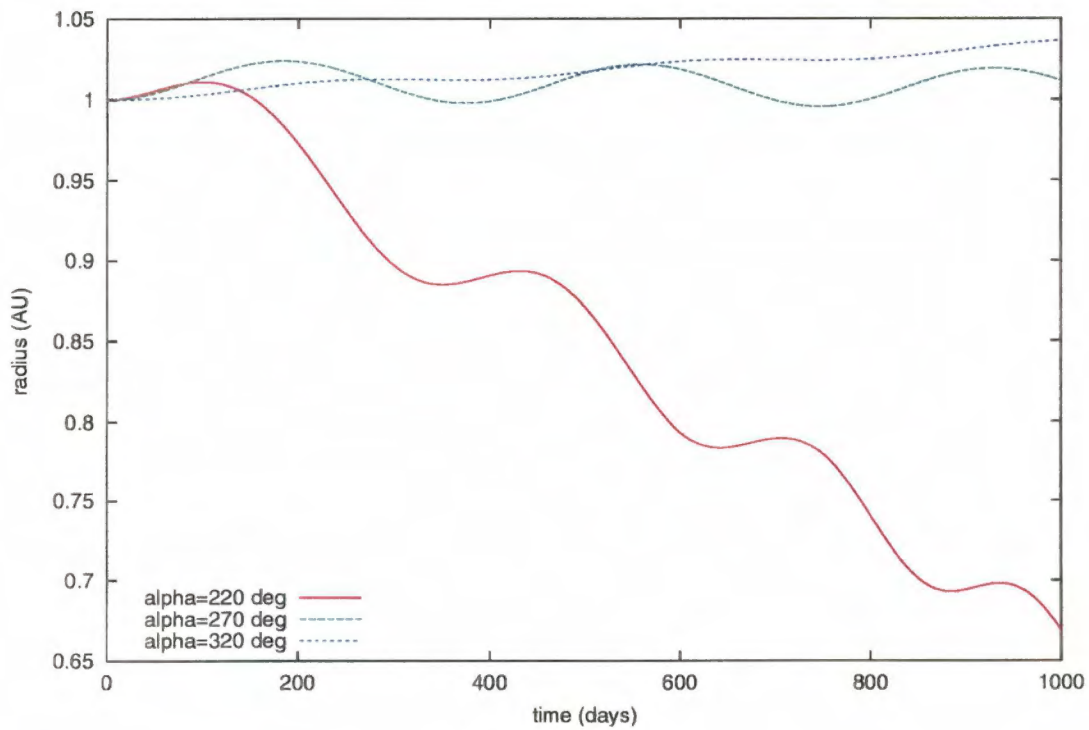


b) $m_p = 5.0$ kg

Figure 4.32: Radial position as function of time and α for $l_3 = 15.0$ m

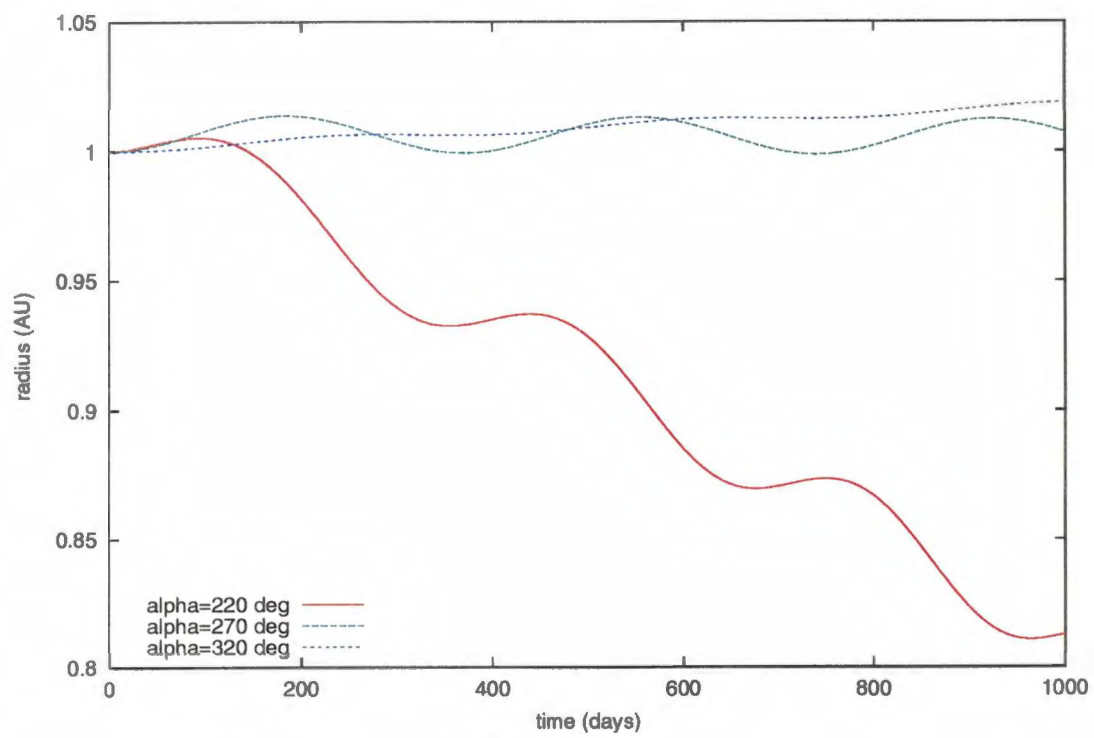


c) $m_p = 25.0$ kg.



d) $m_p = 50.0$ kg.

Figure 4.32 (continued)



e) $m_p = 100.0$ kg.

Figure 4.32 (continued)

$l_3 = 15.0$ m. Figure 4.33 shows the results for the same values of α , payload masses and sail film thickness, but here the sail dimensions are $l_1 = 50.0$ m, $l_2 = 40.0$ m and $l_3 = -15.0$ m. These figures show that for increasing α , the flight times increase significantly, and orbits that were inward for $\alpha=220^\circ$ become outward for $\alpha=270^\circ$ and 320° and vice versa. Tables 4.9 through 4.14 help to explain this.

Tables 4.9, 4.10 and 4.11 give the accelerations and forces at 1 AU for $l_3 = 15.0$ m and the three different values of α . In this case the second sail is the larger sail, and these tables show that for each payload mass, the component of the normal vector of the second plane in the radial direction reduces in size with increasing value of α . This leads to very small values for the dot product between the incident sunlight vector and the sail plane normal. As a result of this the force generated by the second sail reduces in size significantly, and the component in the $\mathbf{e}_\phi$ direction becomes much smaller. For $\alpha=320^\circ$ this actually leads to the force generated by the first sail in this direction being larger in magnitude than the force generated by the second sail plane, and the total force in the $\mathbf{e}_\phi$ direction becomes positive, leading to outward orbits. For $\alpha=270^\circ$ the total force in this direction is still negative, but much smaller than the total force in the radial direction, especially for the higher payload mass values. This leads to slightly outward trajectories for these higher payload masses.

For $l_3 = -15.0$ m, tables 4.12, 4.13 and 4.14 show the forces and accelerations at 1 AU. Now the first sail plane is the larger plane, and again it is observed from the tables that with increasing values of α the radial component of the normal vector of this plane reduces in size

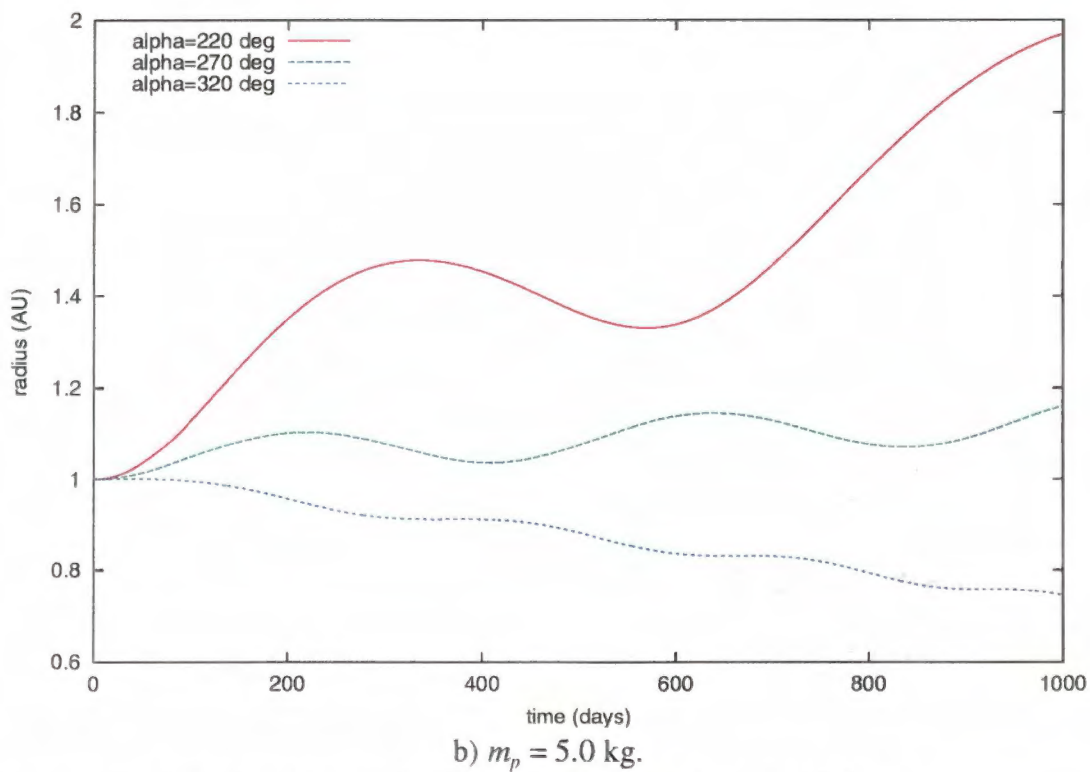
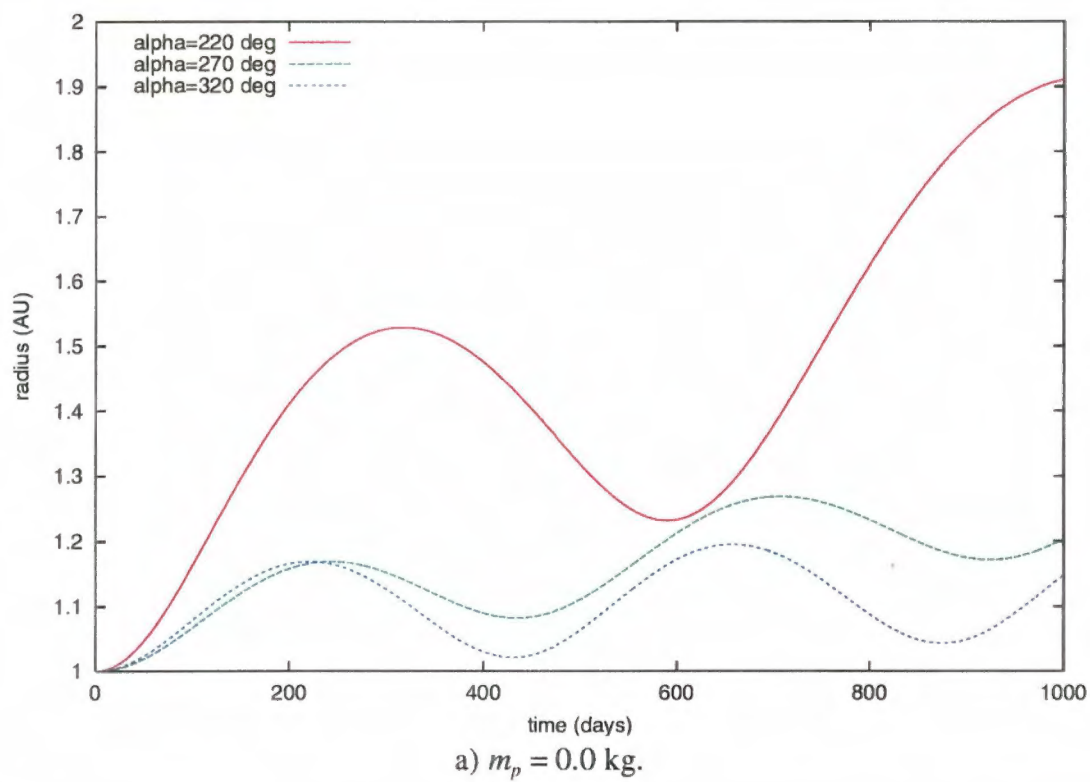


Figure 4.33: Radial position as function of time and α for $l_3 = -15.0$ m

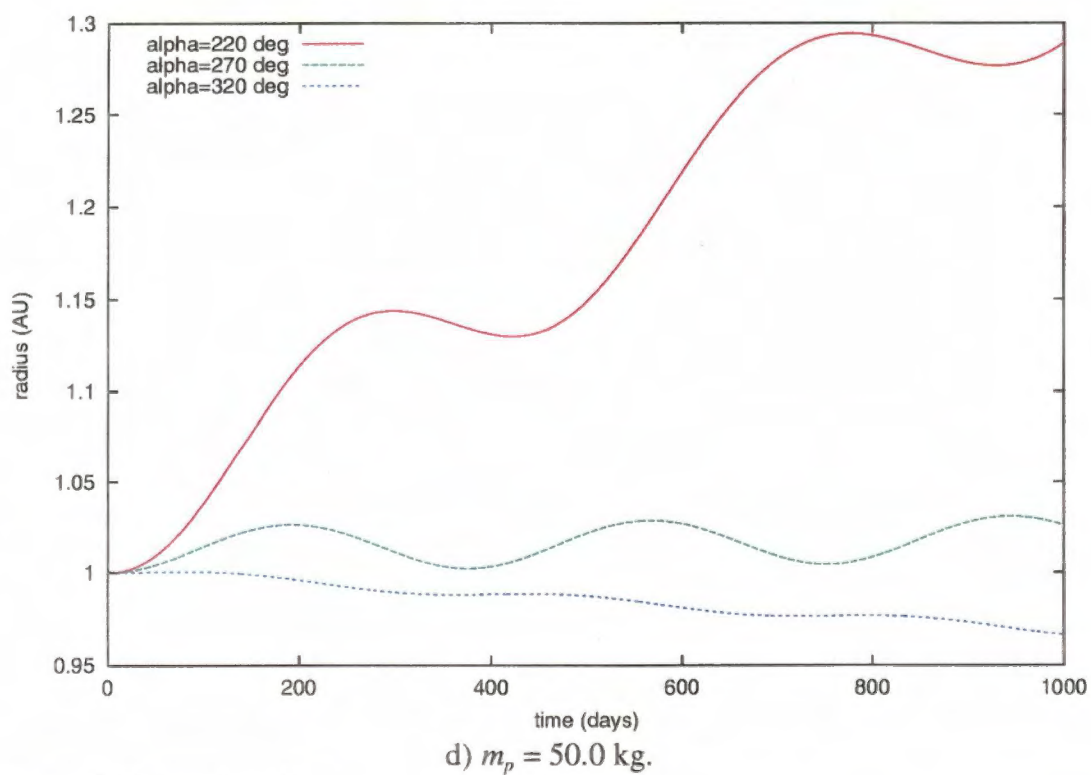
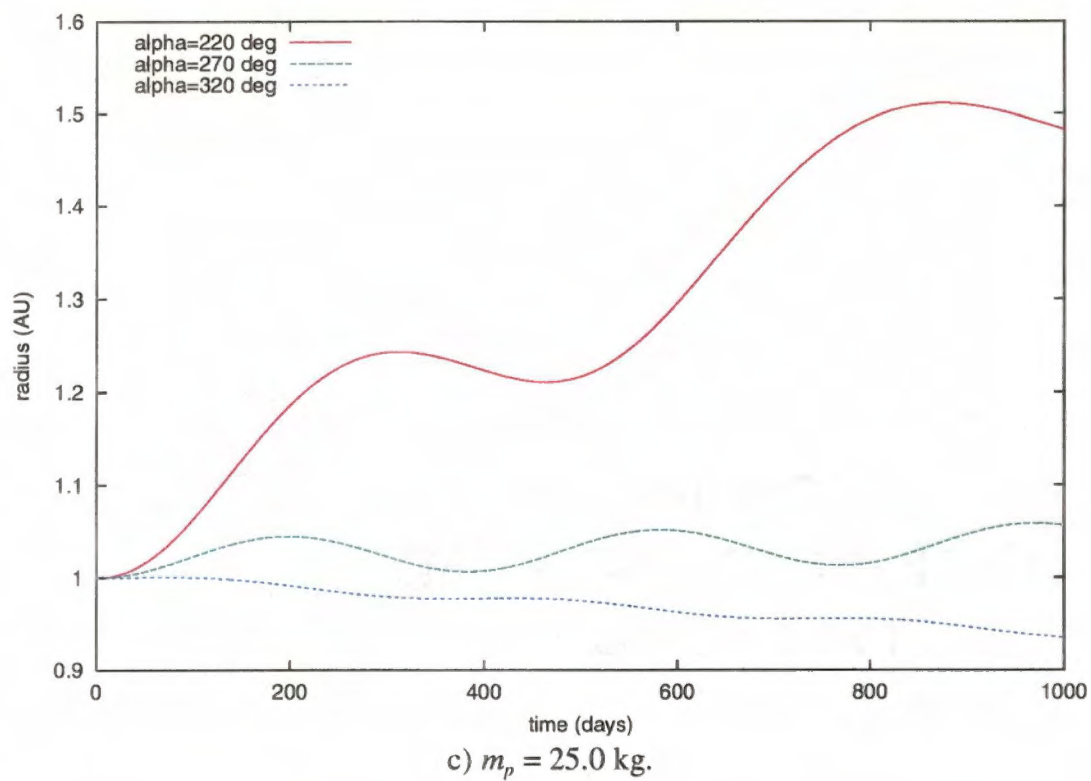


Figure 4.33 (continued)

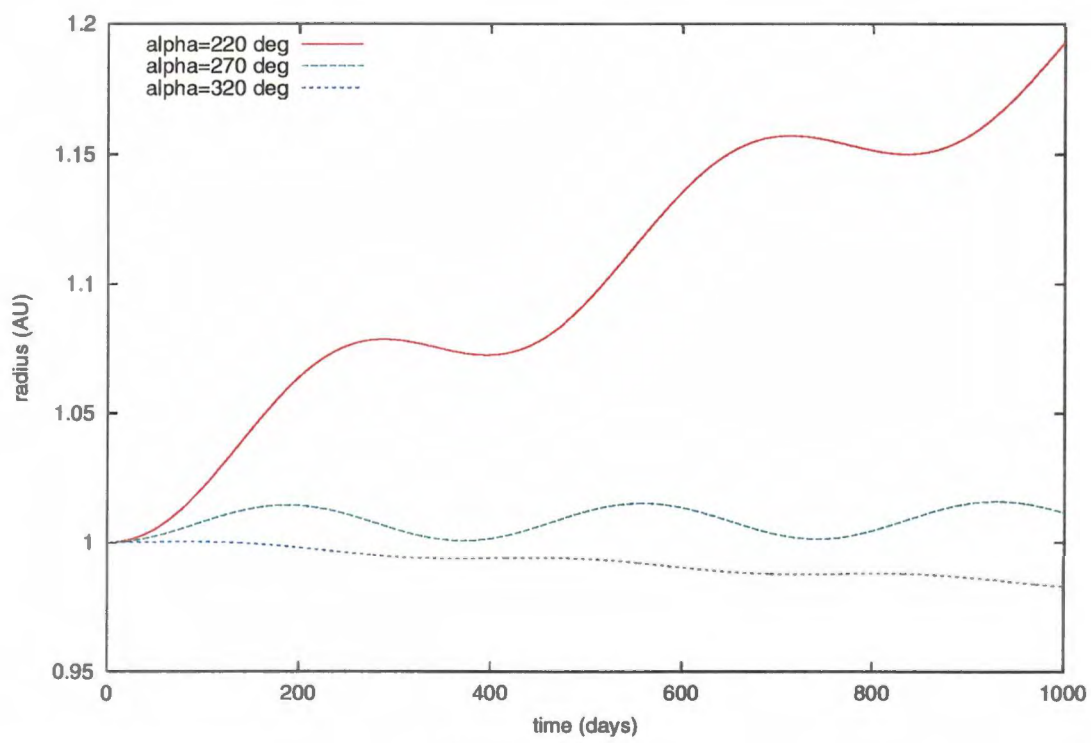


Figure 4.33 (continued)

e) $m_p = 100.0$ kg.

Table 4.9: Accelerations at 1 AU for $k = 0.0$, $h = 7.5 \mu\text{m}$, $l_2 = 25.0 \text{ m}$, $l_3 = 15.0 \text{ m}$, $\alpha = 220^\circ$

m_p	theta	n_1	n_2	F_1	F_2	F_{tot}	a
0	0.307084	0.953219	0.924510	0.004872	0.007112	0.011984	7.563674E-04
		0.302281	-0.381157	0.001545	-0.002932	-0.001387	-8.754730E-05
5	0.149369	0.988865	0.853171	0.005439	0.005589	0.011028	5.290982E-04
		0.148814	-0.521632	0.000819	-0.003417	-0.002599	-1.246763E-04
25	-0.052346	0.998630	0.731363	0.005602	0.003521	0.009123	2.233565E-04
		-0.052323	-0.681989	-0.000294	-0.003283	-0.003577	-8.756831E-05
50	-0.119404	0.992880	0.684021	0.005506	0.002880	0.008386	1.273635E-04
		-0.119120	-0.729462	-0.000661	-0.003072	-0.003732	-5.668392E-05
100	-0.162936	0.986755	0.651628	0.005404	0.002490	0.007895	6.814938E-05
		-0.162216	-0.758539	-0.000888	-0.002899	-0.003787	-3.269287E-05

Table 4.10: Accelerations at 1 AU for $k = 0.0$, $h = 7.5 \mu\text{m}$, $l_2 = 25.0 \text{ m}$, $l_3 = 15.0 \text{ m}$, $\alpha = 270^\circ$

m_p	theta	n_1	n_2	F_1	F_2	F_{tot}	a
0	0.668964	0.784465	0.620174	0.002715	0.002147	0.004862	3.068850E-04
		0.620174	-0.784465	0.002147	-0.002715	-0.000569	-3.589424E-05
5	0.624465	0.811276	0.584663	0.003004	0.001799	0.004802	2.303909E-04
		0.584663	-0.811276	0.002165	-0.002496	-0.000331	-1.589603E-05
25	0.583878	0.834331	0.551264	0.003267	0.001508	0.004775	1.169000E-04
		0.551264	-0.834331	0.002159	-0.002282	-0.000123	-3.020841E-06
50	0.572886	0.840340	0.542059	0.003338	0.001433	0.004771	7.246647E-05
		0.542059	-0.840340	0.002153	-0.002222	-0.000069	-1.048928E-06
100	0.566241	0.843923	0.536464	0.003381	0.001390	0.004770	4.117970E-05
		0.536464	-0.843923	0.002149	-0.002186	-0.000037	-3.169715E-07

Table 4.11: Accelerations at 1 AU for $k = 0.0$, $h = 7.5 \mu\text{m}$, $l_2 = 25.0 \text{ m}$, $l_3 = 15.0 \text{ m}$, $\alpha = 320^\circ$

m_p	theta	n_1	n_2	F_1	F_2	F_{tot}	a
0	-0.026672	0.999644	-0.782915	0.005619	???	???	> 0.0
		-0.026669	-0.622129	-0.000150	0.000000	-0.000150	-9.461592E-06
5	1.054279	0.493855	0.180618	0.000678	0.000053	0.000731	3.504875E-05
		0.869544	-0.983553	0.001193	-0.000289	0.000904	4.337744E-05
25	1.120086	0.435605	0.244905	0.000465	0.000132	0.000597	1.462024E-05
		0.900138	-0.969547	0.000961	-0.000523	0.000437	1.070909E-05
50	1.128881	0.427672	0.253422	0.000440	0.000146	0.000586	8.907155E-06
		0.903934	-0.967356	0.000930	-0.000559	0.000371	5.632378E-06
100	1.133372	0.423608	0.257764	0.000428	0.000154	0.000582	5.021546E-06
		0.905846	-0.966208	0.000914	-0.000578	0.000337	2.905287E-06

Table 4.12: Accelerations at 1 AU for $k = 0.0$, $h = 7.5 \mu\text{m}$, $l_2 = 40.0 \text{ m}$, $l_3 = -15.0 \text{ m}$, $\alpha = 220^\circ$

m_p	theta	n_1	n_2	F_1	F_2	F_{tot}	a
0	0.391048	0.924510	0.953219	0.007112	0.004872	0.011984	7.563674E-04
		0.381157	-0.302280	0.002932	-0.001545	0.001387	8.754736E-05
5	0.548763	0.853171	0.988865	0.005589	0.005439	0.011028	5.290982E-04
		0.521632	-0.148814	0.003417	-0.000819	0.002599	1.246763E-04
25	0.750478	0.731363	0.998630	0.003521	0.005602	0.009123	2.233565E-04
		0.681989	0.052322	0.003283	0.000294	0.003577	8.756831E-05
50	0.817535	0.684021	0.992880	0.002880	0.005506	0.008386	1.273635E-04
		0.729462	0.119120	0.003072	0.000661	0.003732	5.668392E-05
100	0.861068	0.651628	0.986755	0.002490	0.005404	0.007895	6.814938E-05
		0.758539	0.162216	0.002899	0.000888	0.003787	3.269287E-05

Table 4.13: Accelerations at 1 AU for $k = 0.0$, $h = 7.5 \mu\text{m}$, $l_2 = 40.0 \text{ m}$, $l_3 = -15.0 \text{ m}$, $\alpha = 270^\circ$

m_p	theta	n_1	n_2	F_1	F_2	F_{tot}	a
0	0.901832	0.620174	0.784465	0.002147	0.002715	0.004862	3.068850E-04
		0.784465	-0.620174	0.002715	-0.002147	0.000569	3.589421E-05
5	0.946332	0.584663	0.811276	0.001799	0.003004	0.004802	2.303909E-04
		0.811276	-0.584663	0.002496	-0.002165	0.000331	1.589603E-05
25	0.986918	0.551264	0.834331	0.001508	0.003267	0.004775	1.169000E-04
		0.834331	-0.551264	0.002282	-0.002159	0.000123	3.020845E-06
50	0.997911	0.542059	0.840340	0.001433	0.003338	0.004771	7.246647E-05
		0.840340	-0.542059	0.002222	-0.002153	0.000069	1.048930E-06
100	1.004555	0.536464	0.843923	0.001390	0.003381	0.004770	4.117970E-05
		0.843923	-0.536464	0.002186	-0.002149	0.000037	3.169727E-07

Table 4.14: Accelerations at 1 AU for $k = 0.0$, $h = 7.5 \mu\text{m}$, $l_2 = 40.0 \text{ m}$, $l_3 = -15.0 \text{ m}$, $\alpha = 320^\circ$

m_p	theta	n_1	n_2	F_1	F_2	F_{tot}	a
0	2.470131	-0.782913	0.999644	???	0.005619	???	> 0.0
		0.622131	0.026667	0.000000	0.000150	0.000150	9.460801E-06
5	1.389182	0.180618	0.493855	0.000053	0.000678	0.000731	3.504876E-05
		0.983553	-0.869544	0.000289	-0.001193	-0.000904	-4.337745E-05
25	1.323375	0.244905	0.435605	0.000132	0.000465	0.000597	1.462024E-05
		0.969547	-0.900138	0.000523	-0.000961	-0.000437	-1.070910E-05
50	1.314580	0.253422	0.427672	0.000146	0.000440	0.000586	8.907156E-06
		0.967356	-0.903934	0.000559	-0.000930	-0.000371	-5.632384E-06
100	1.310089	0.257764	0.423608	0.000154	0.000428	0.000582	5.021547E-06
		0.966208	-0.905846	0.000578	-0.000914	-0.000337	-2.905290E-06

significantly. This means that the same thing as discussed above will happen, but now leading to inward orbits with increasing values of α .

Tables 4.11 and 4.14 also show that for $\alpha = 320^\circ$ and $m_p = 0.0$ kg, the value of θ is in the range where the larger sail plane is actually absorbing the light. The smaller plane is now the only plane producing a force in the $\mathbf{e}_\phi$ direction. Figure 4.34 shows that this leads to very different behavior of the trajectories, compared to the trajectories for the other payload masses.

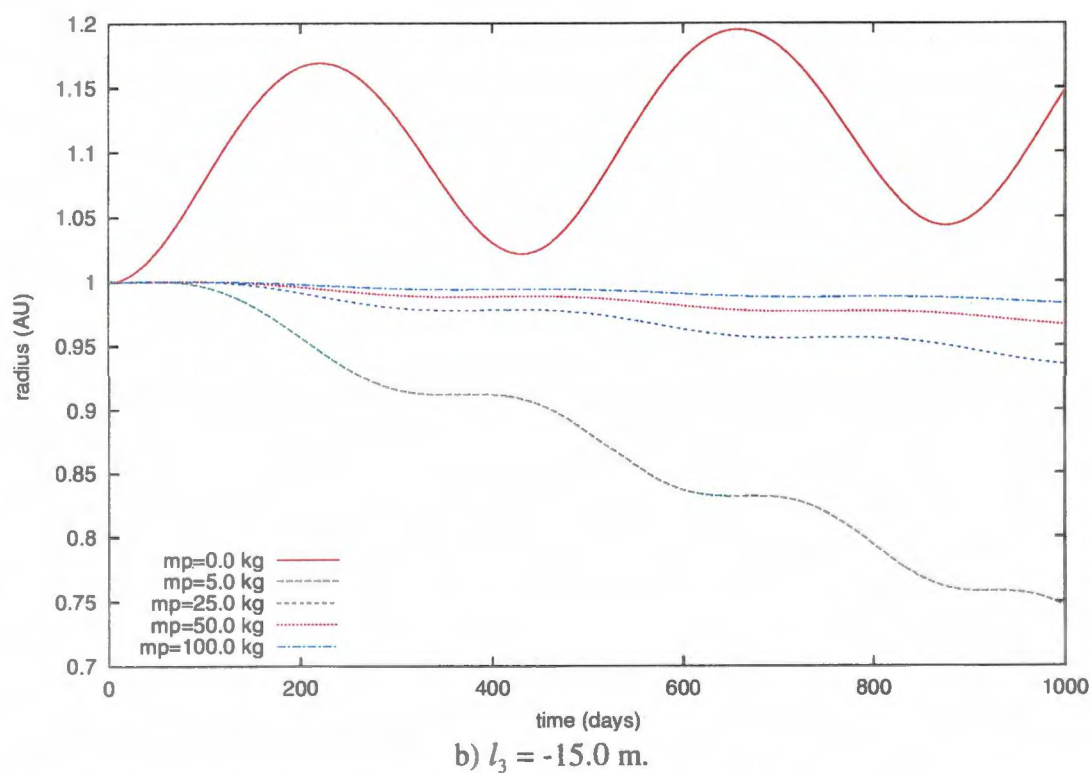
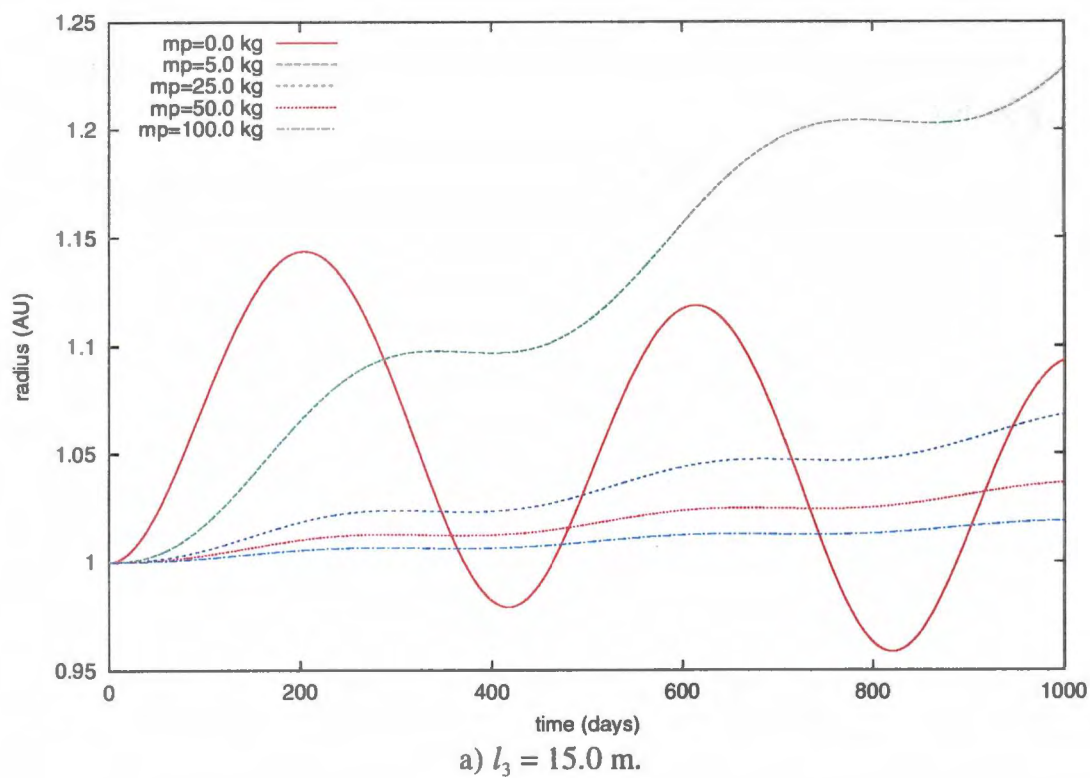


Figure 4.34: Radial position as function of time for $\alpha = 320^\circ$

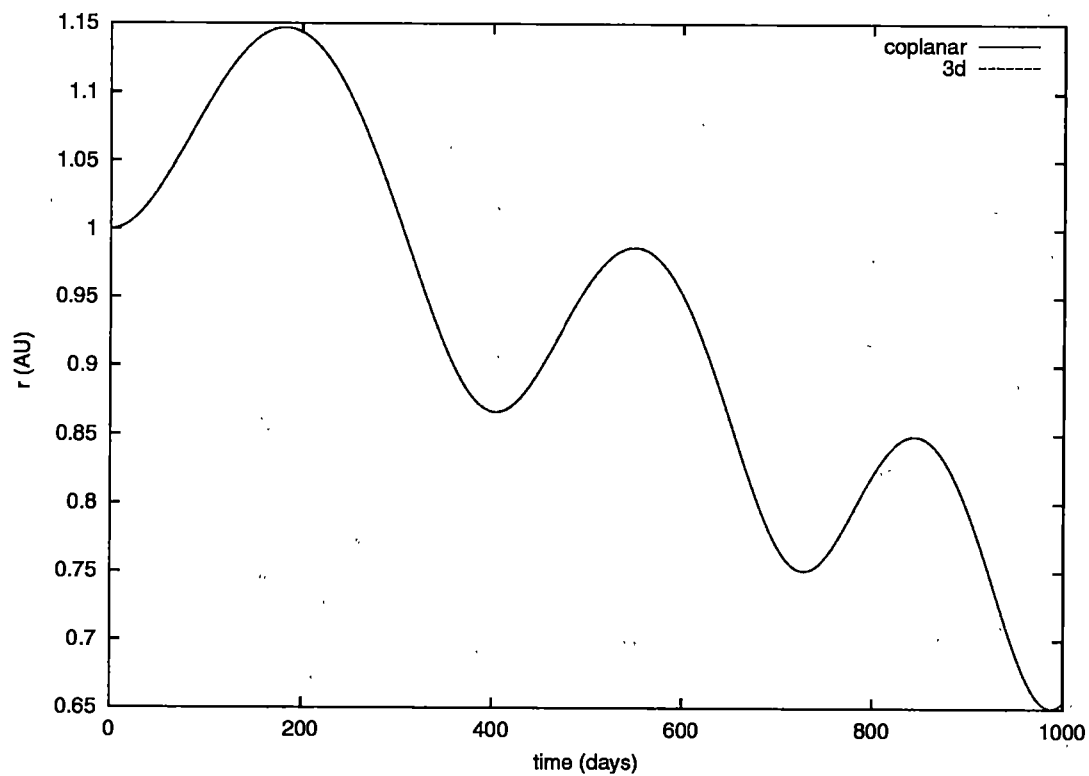
Chapter 5

The 3D Case

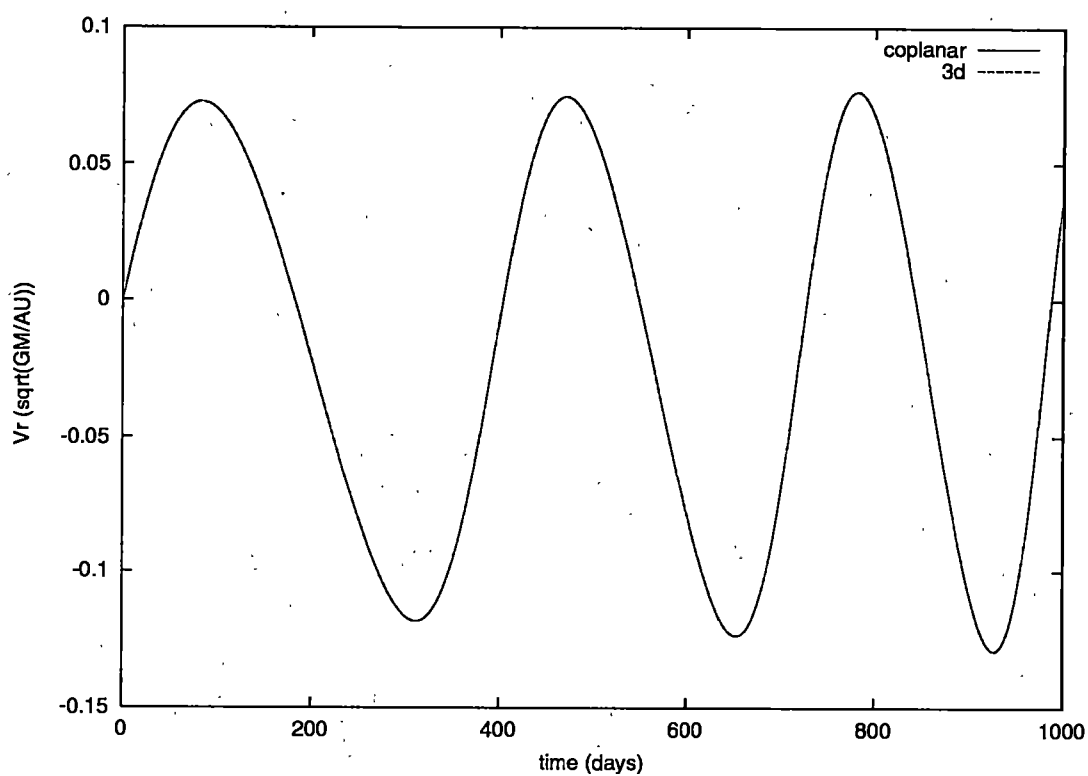
In this chapter the results of the full 3D-model will be compared with those of the simplified coplanar model in order to verify the 3D-model and trajectories for non coplanar cases will be investigated. The stability of the motion about all three axis of the solar sail will be investigated also.

5.1 Comparison with the coplanar model

First, the results of the full 3D-model were compared to those of the coplanar model. This was mainly done to verify the quaternion equations. For the case of $l_1 = 50.0$ m, $l_2 = 25.0$ m, $l_3 = 15.0$ m, $k = 0.4$, $h = 7.5$ μ m and $\alpha x = -40^\circ$, the results of the coplanar and the 3D-model are given in figure 5.1. This figure compares the radial distance and velocity r and V_r , the angular position ϕ and its corresponding velocity V_ϕ , the angular position ψ and its corresponding velocity V_ψ and angular velocity ω_z . As this figure shows, the results for the two models are identical.

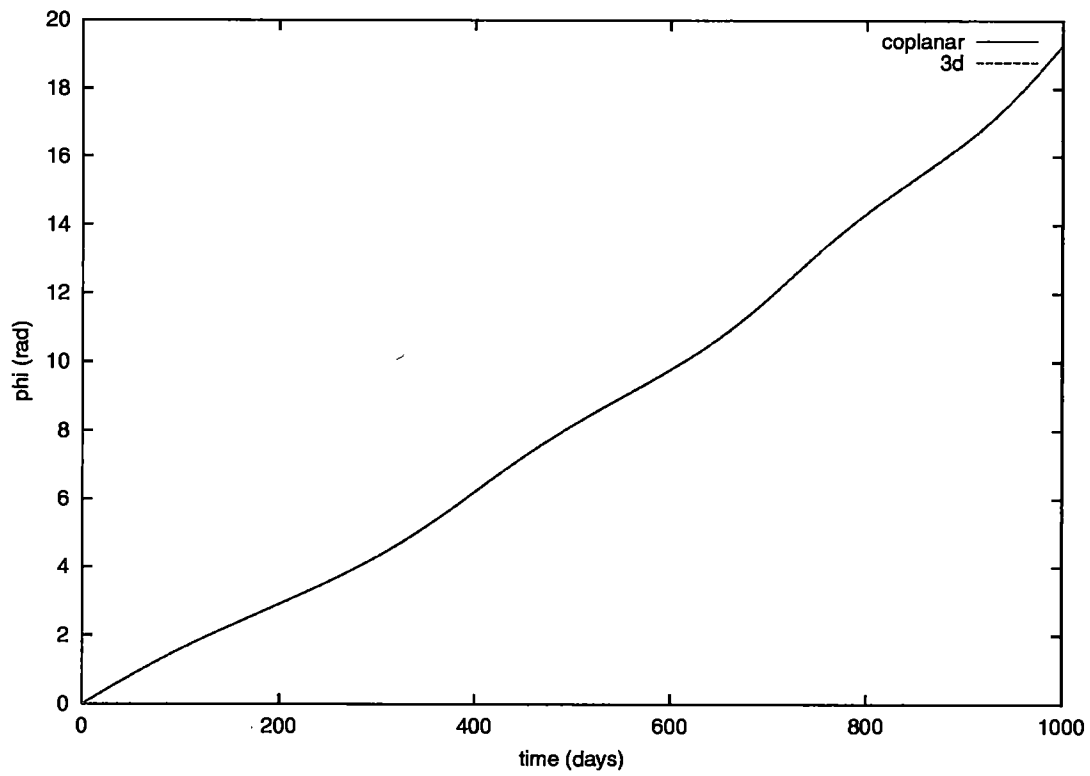


a) Radial position

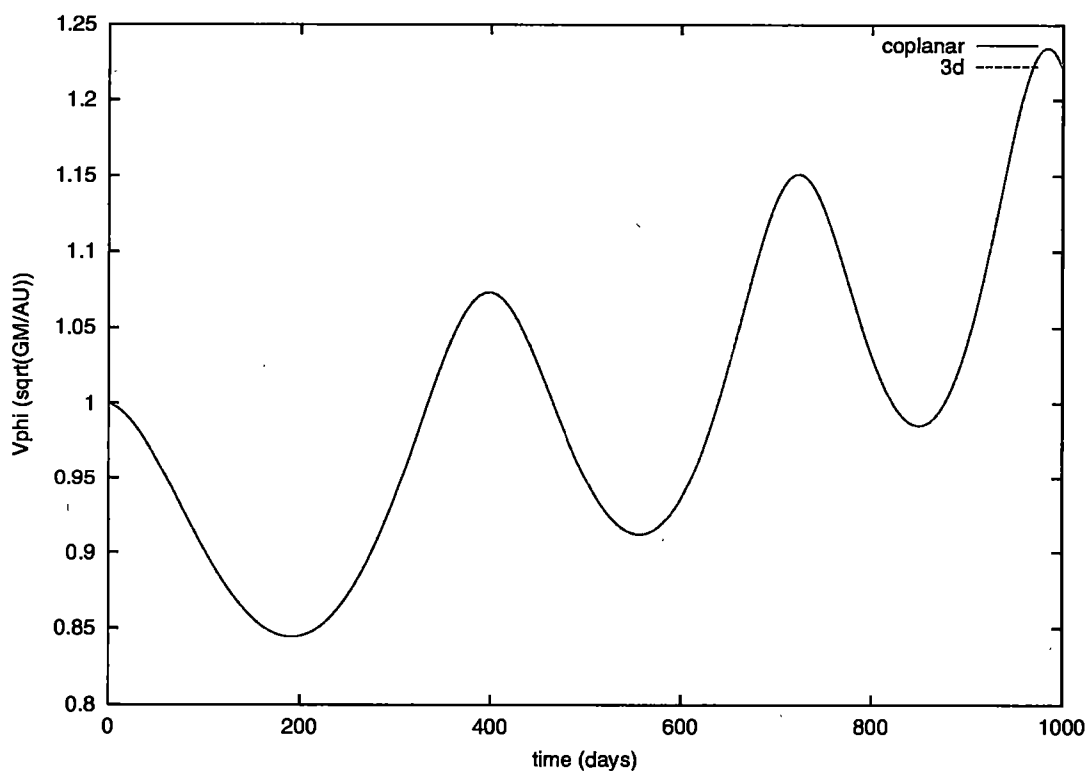


b) Radial velocity

Figure 5.1: Comparison between coplanar model and 3d model

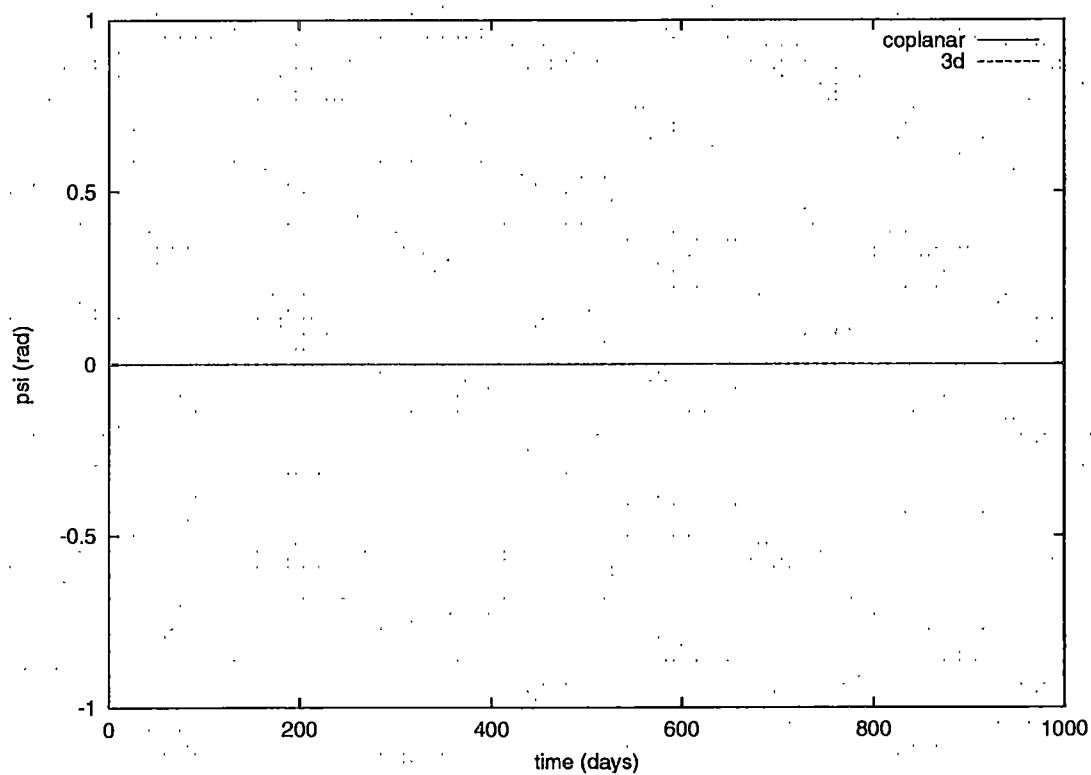


c) Angular position ϕ

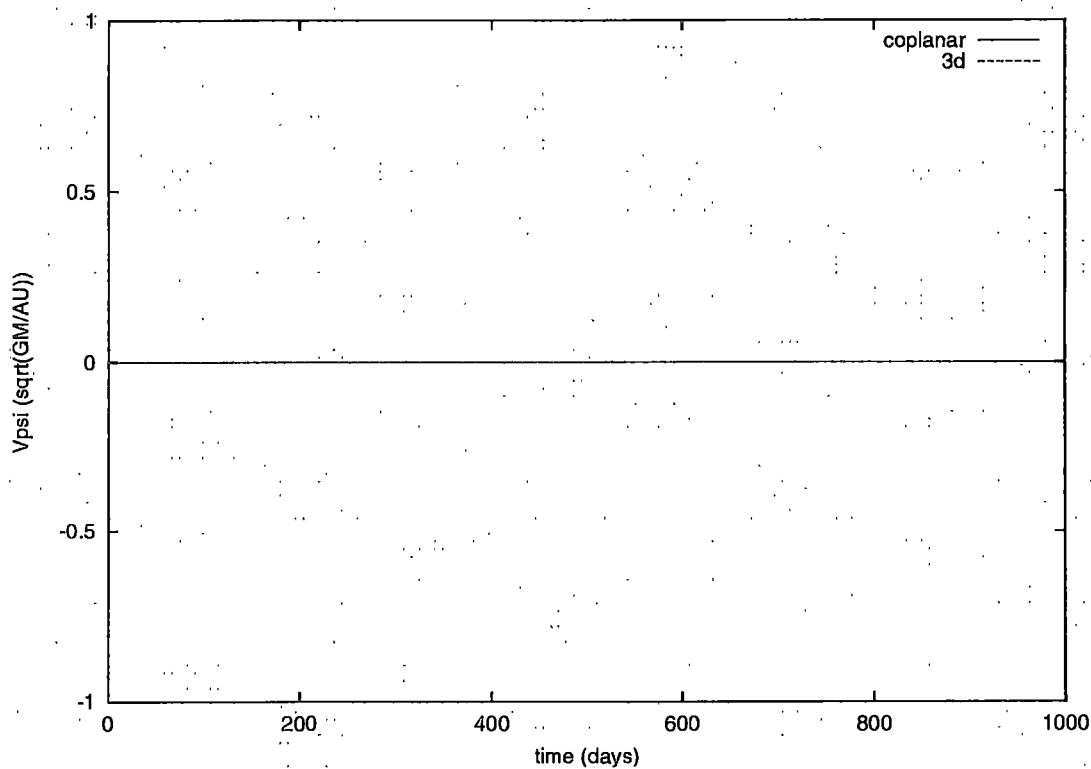


d) Angular velocity V_ϕ

Figure 5.1 (continued)



e) Angular position ψ



f) Angular velocity V_ψ

Figure 5.1 (continued)

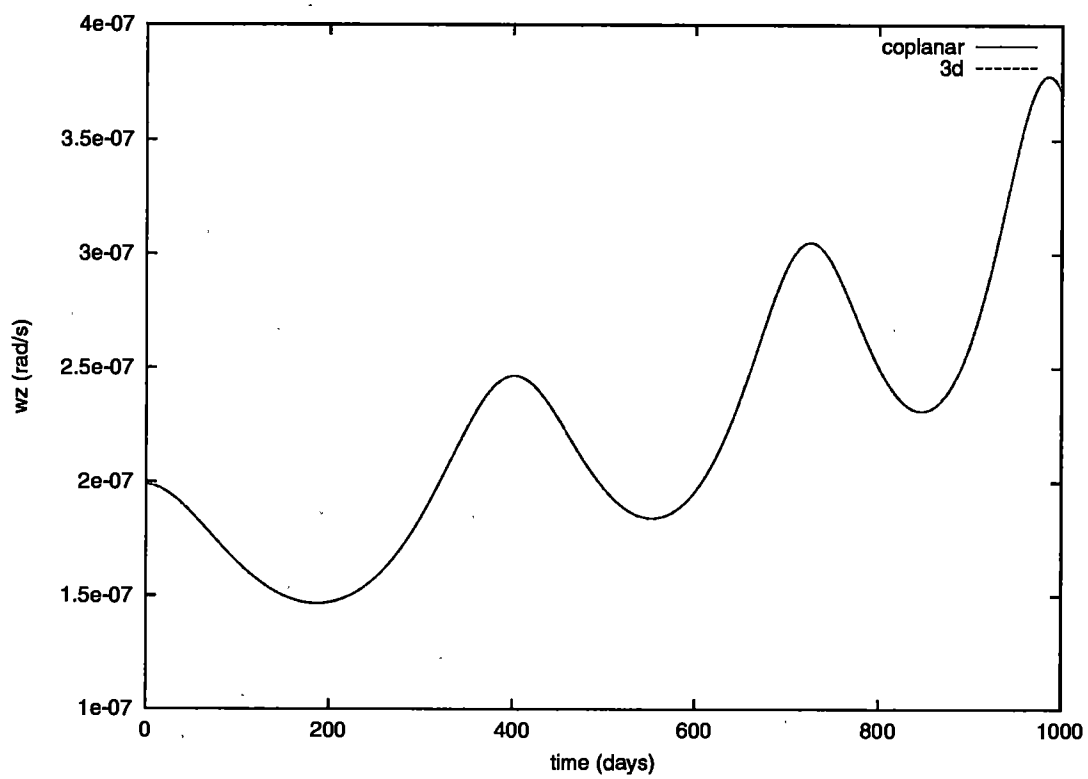


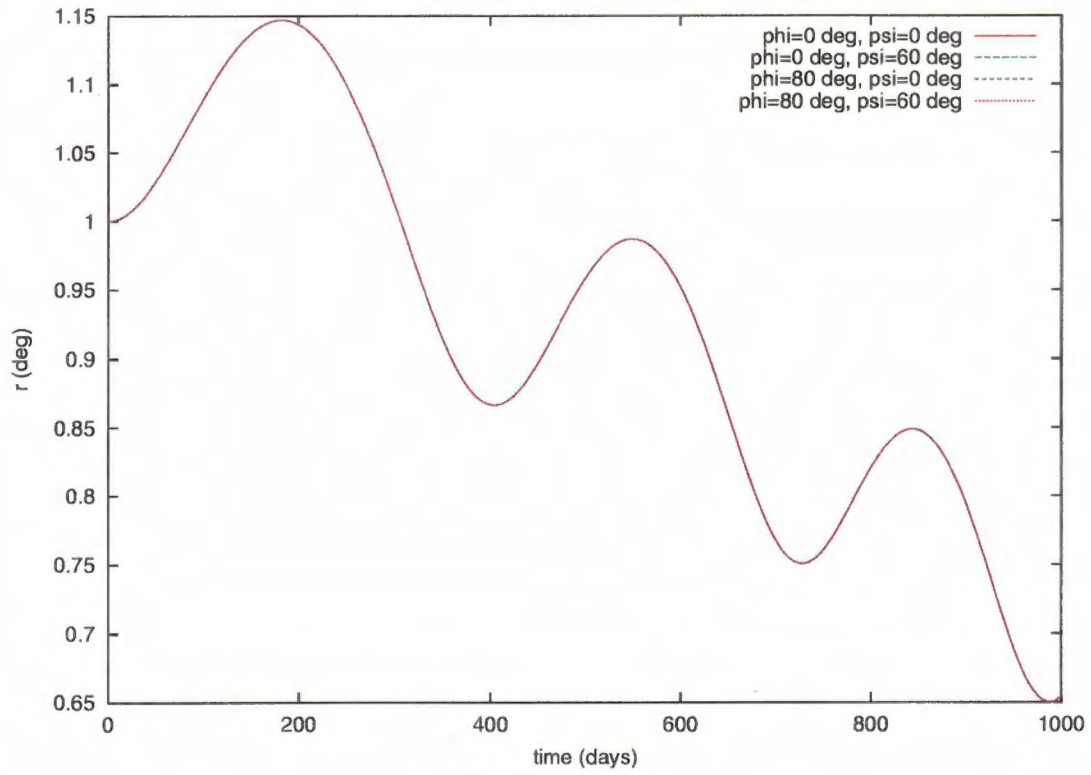
Figure 5.1 (continued)

g) Angular velocity ω_z

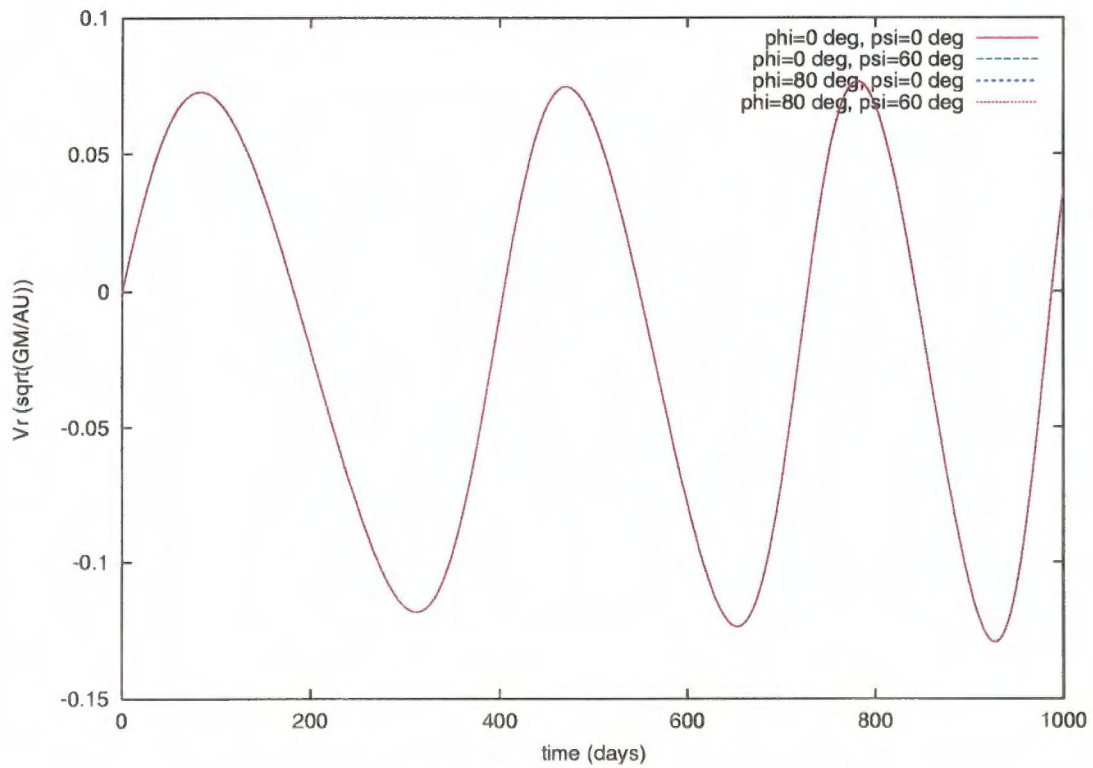
5.2 Non-coplanar trajectories

All forces and moments acting in the solar sail have a radial dependency only. Therefore it is expected that the results should be independent of the initial conditions of the angular positions ϕ and ψ , as long as the initial conditions for the radial position r , radial velocity V_r , tangential velocity component $\sqrt{V_\phi^2 + V_\psi^2}$, control angles θ, β, Ω and angular velocities $\omega_x, \omega_y, \omega_z$ remain identical.

As figure 5.2 shows, the radial position, radial velocity and tangential velocity do not change when the initial values of ψ and or ϕ are changed. A change in the initial position of ϕ to 80° alone leads to an offset of the angular position ϕ over time, but does not change the corresponding velocity V_ϕ . Changing the initial condition of ψ to 60° leads to a change in both angular positions and their corresponding velocities. This is expected, since a change in ψ leads to changes in ϕ and V_ϕ as can be seen from the differential equations, and the change in V_ϕ in its turn affects ψ and V_ψ . Changing both initial conditions leads to the same offset in angular position ϕ , but the other angular position and both velocities are exactly the same as for the change in ψ alone. The plots for ω_x, ω_y and $\sin\theta\sin\Omega$ show that they do not remain exactly zero. This is due to numerical error and dynamic instability and will be discussed in more detail in the next section. The control angle θ and Ω do not change, but the angle β does change with changes in the initial condition for ψ . The change in β can be explained by looking at the equation for ω_x , as given in 3.70. In order for ω_x to remain zero over time, $\dot{\Omega}, \dot{\phi}\sin\psi - \dot{\beta}$ and $\dot{\phi}\cos\psi\cos\beta - \dot{\psi}\sin\beta$ all have to remain zero over time also. For

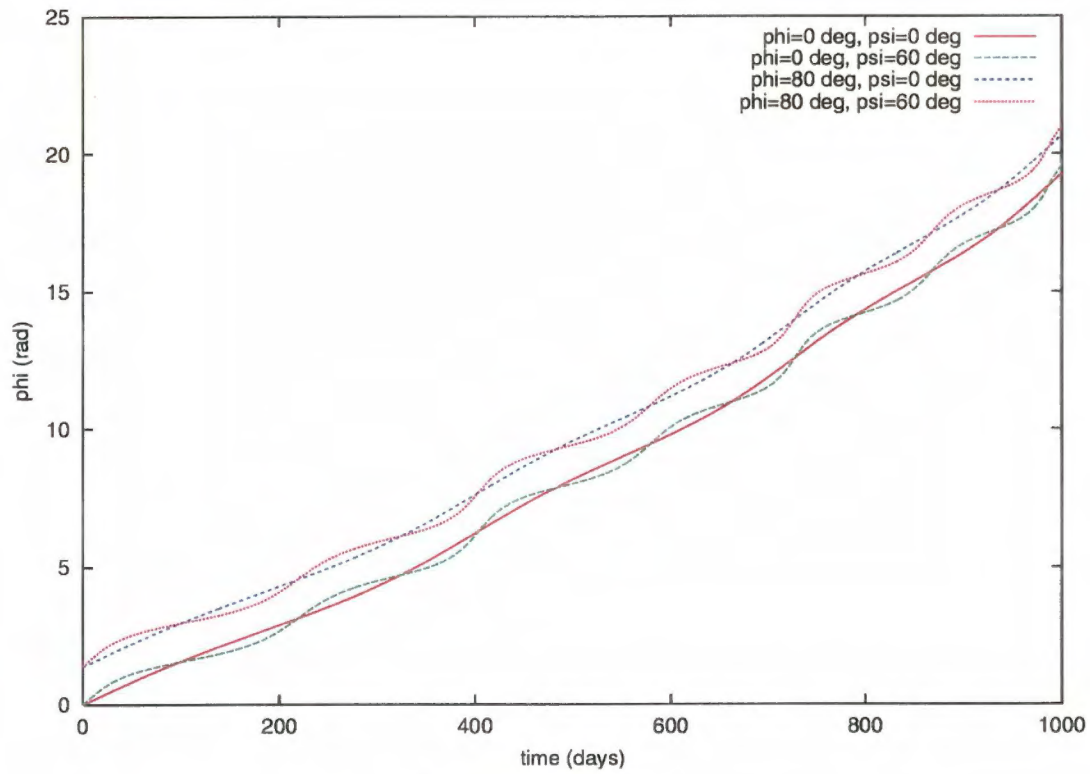


a) Radial position

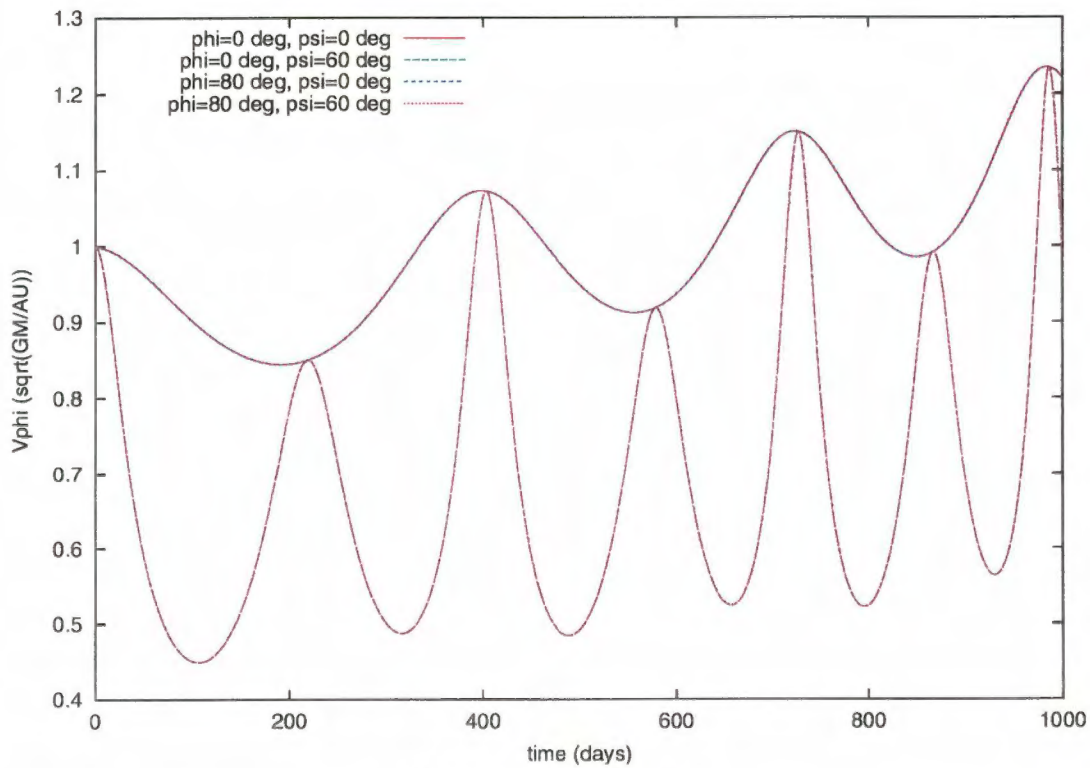


b) Radial velocity

Figure 5.2: Influence of changing the initial values of ϕ and ψ .

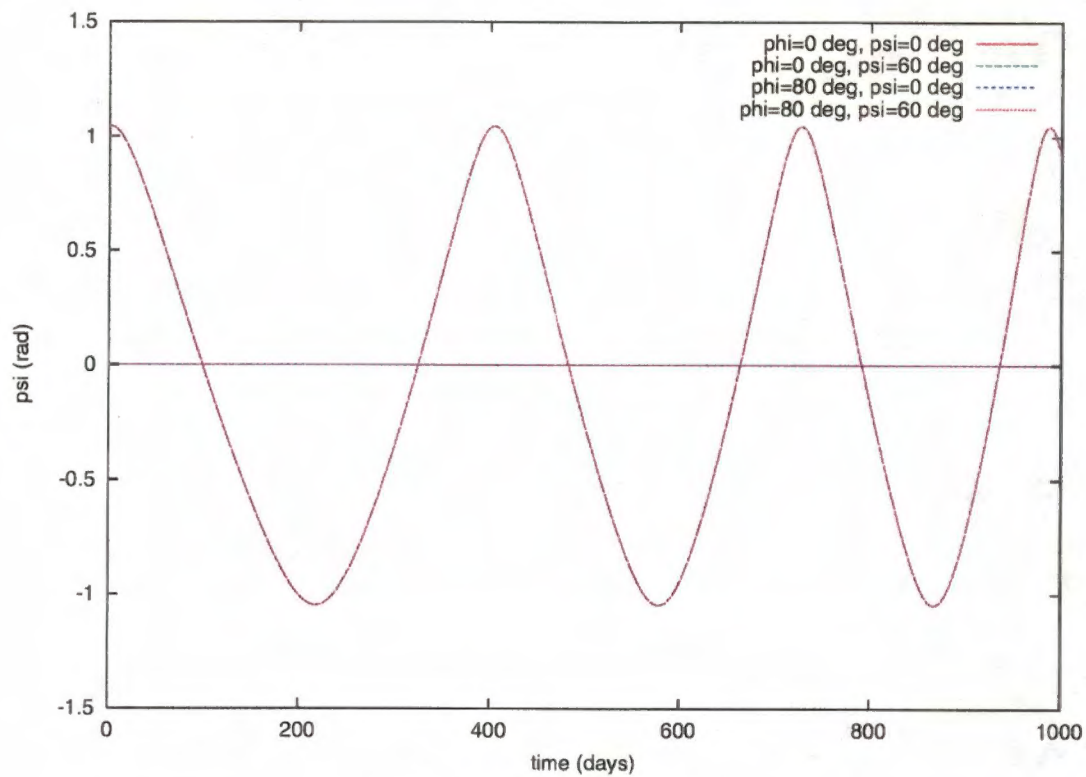


c) Angular position ϕ

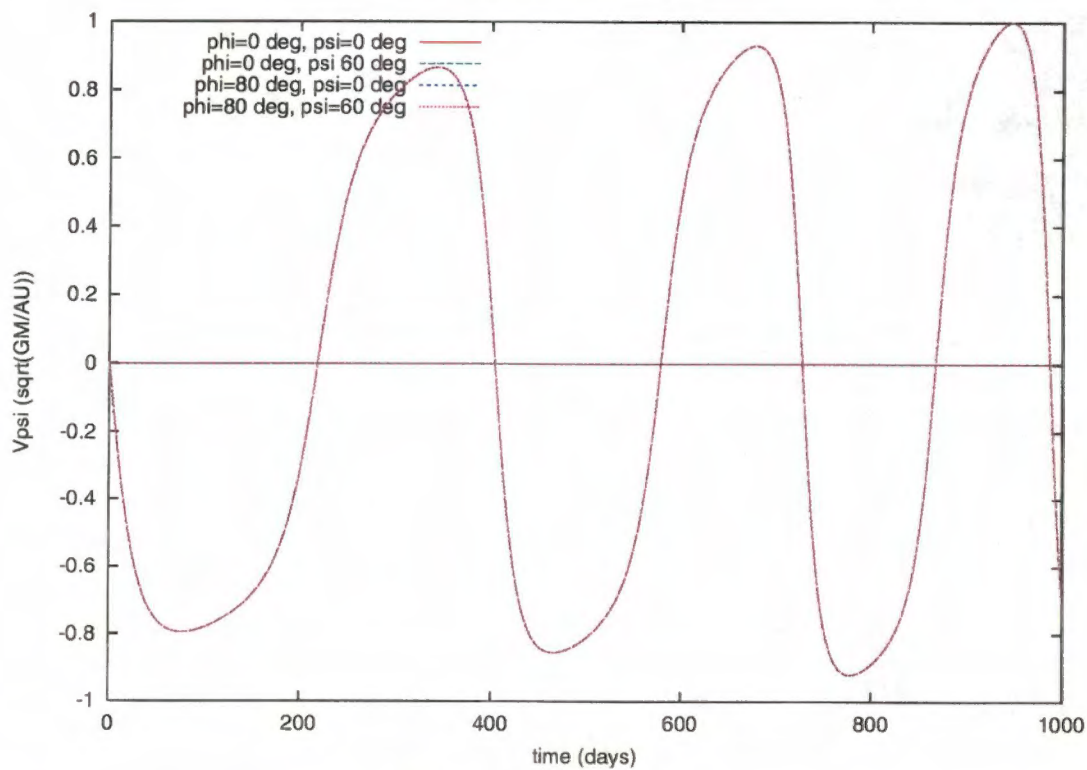


d) Angular velocity V_ϕ

Figure 5.2 (continued)

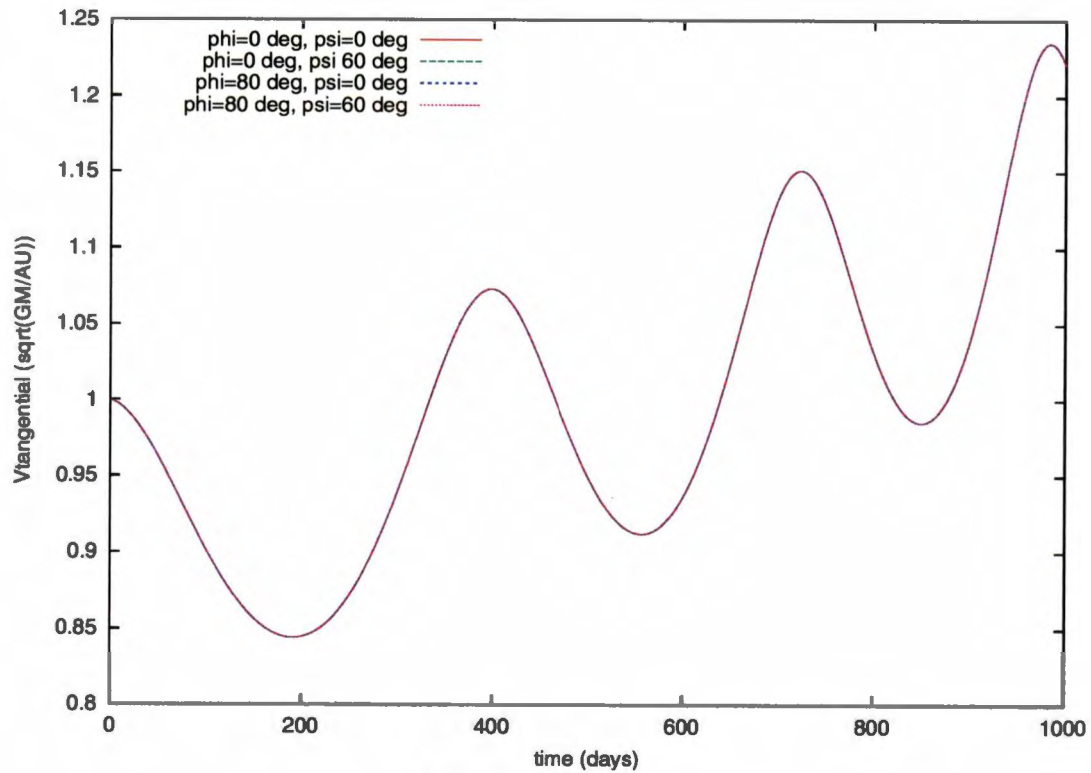


e) Angular position ψ

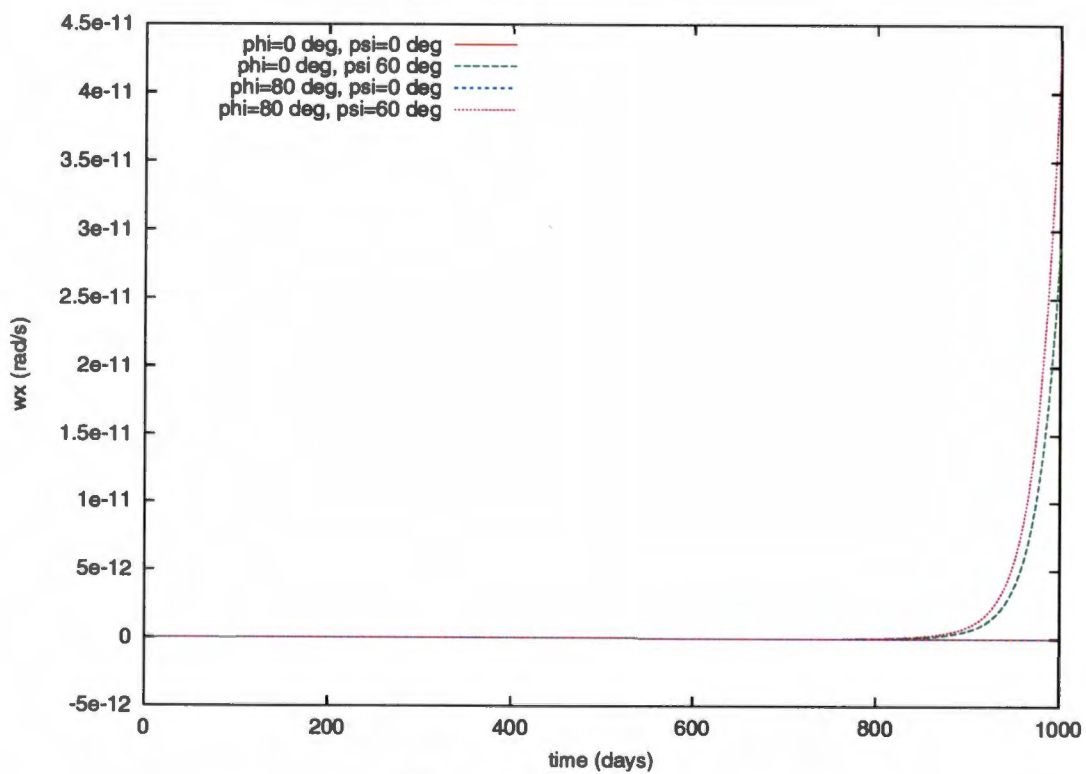


f) Angular velocity V_{ψ}

Figure 5.2 (continued)

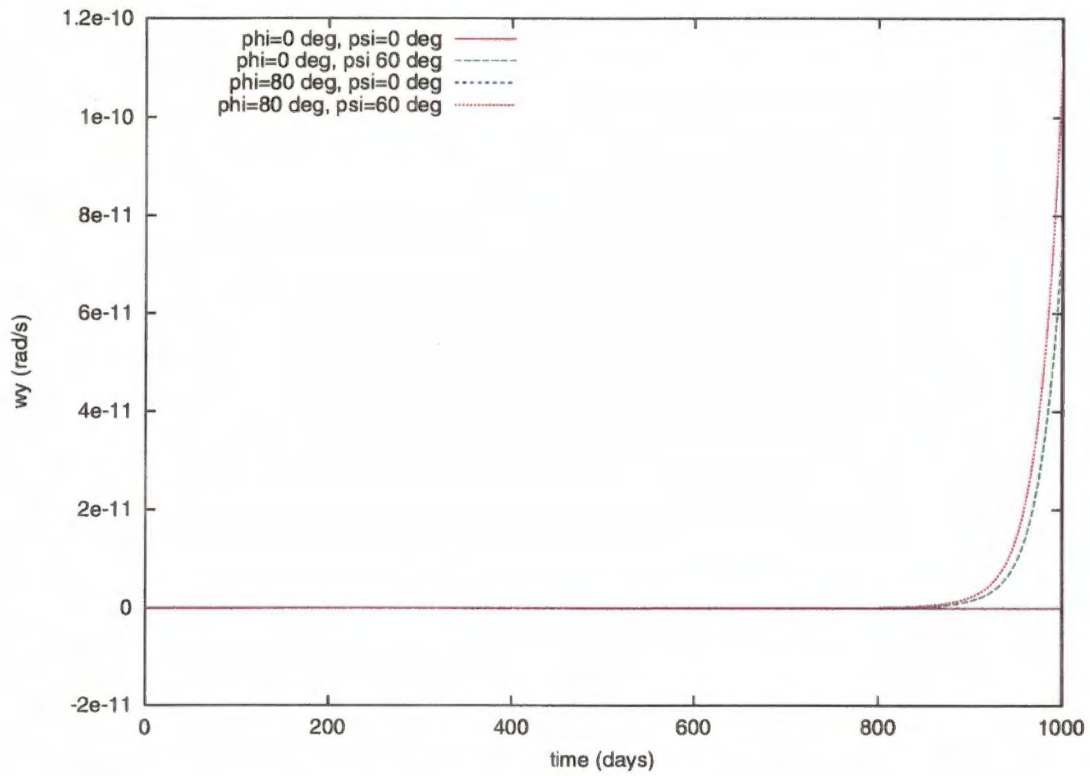


g) Tangential velocity

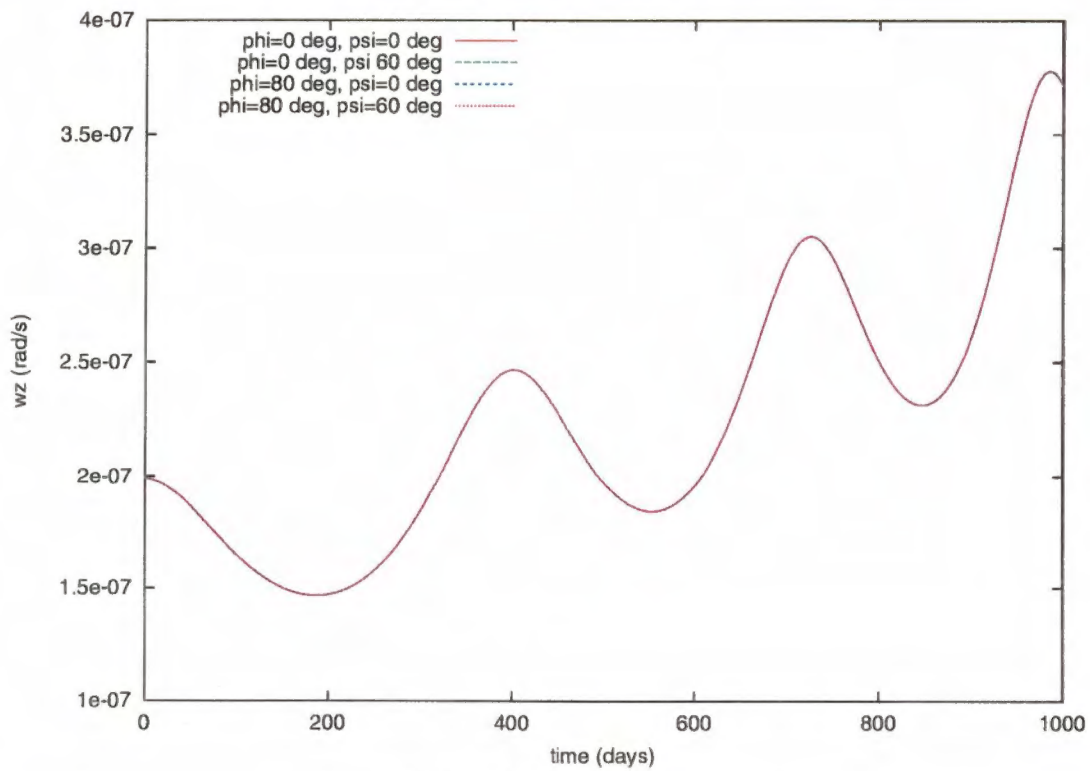


h) Angular velocity ω_x

Figure 5.2 (continued)



i) Angular velocity ω_y



j) Angular velocity ω_z

Figure 5.2 (continued)

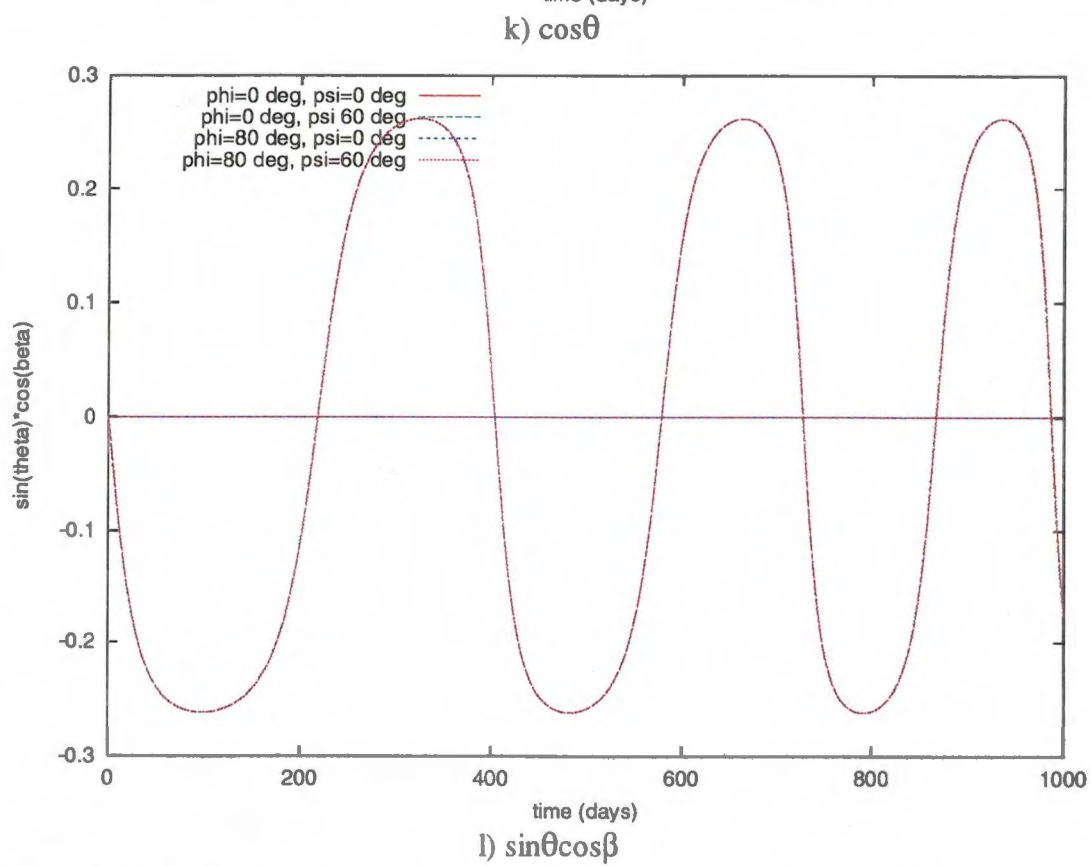
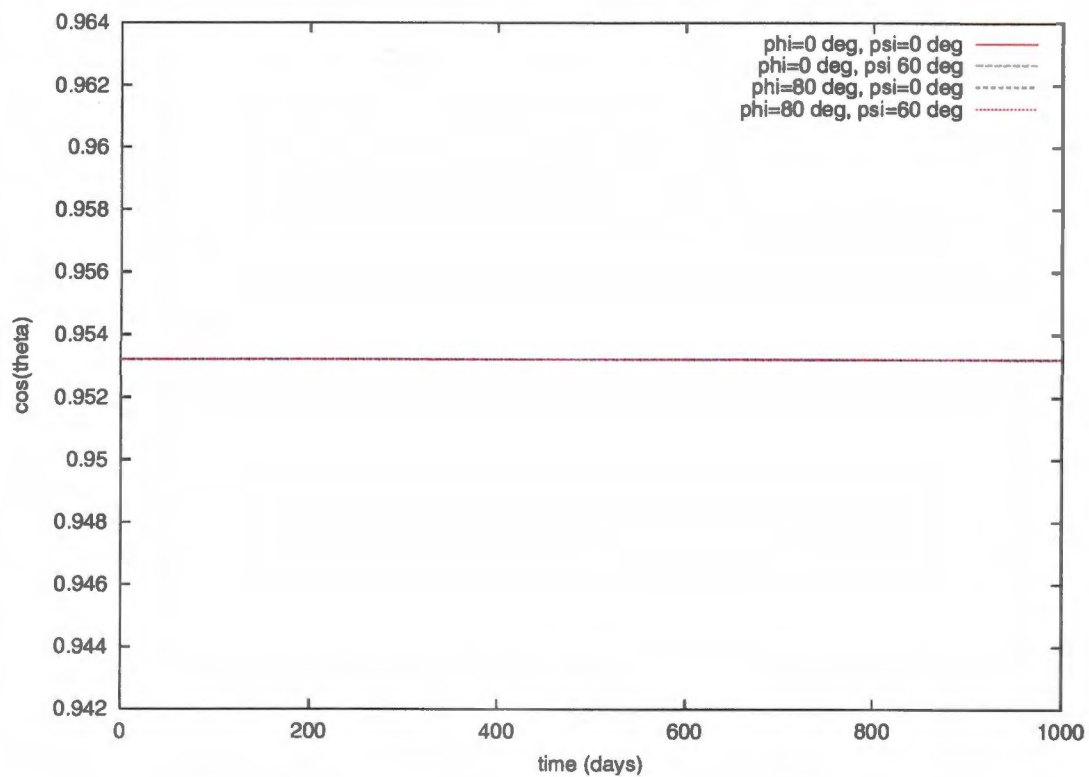


Figure 5.2 (continued)

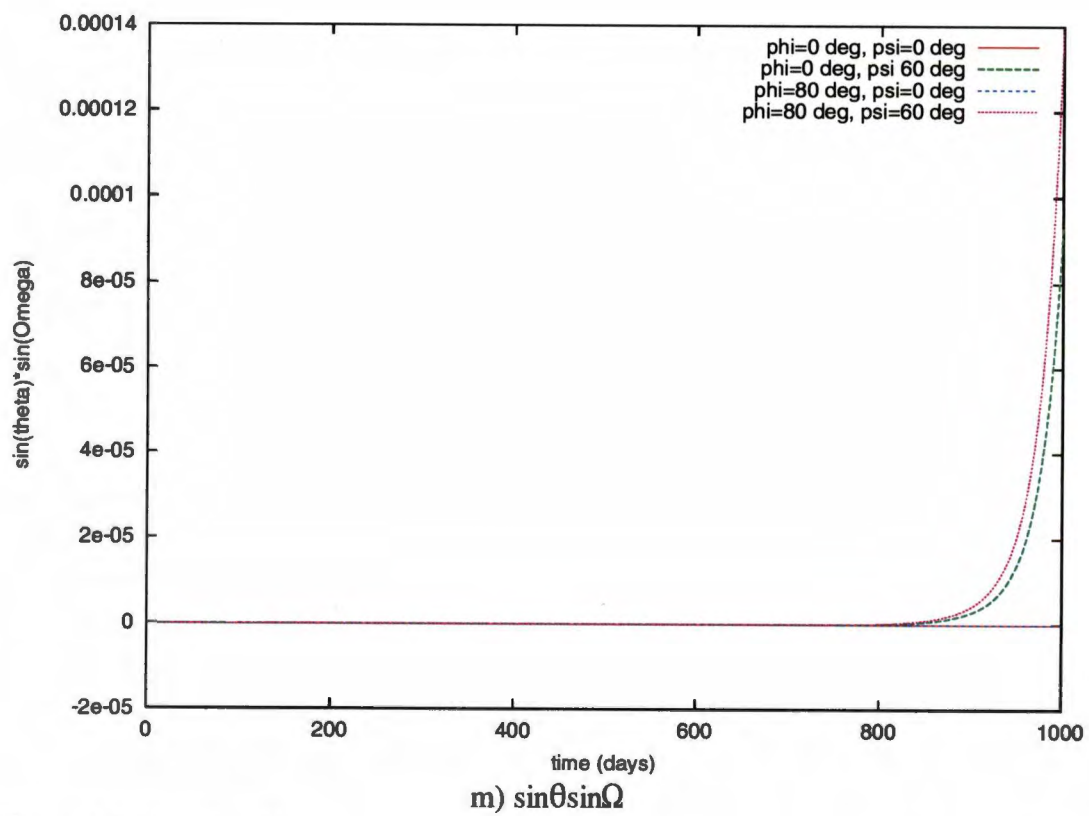


Figure 5.2 (continued)

$\psi \neq 0^\circ$, this means that β does not remain zero all the time, and also β can not remain 90° at all time. And since $\dot{\Omega}$ has to remain zero at all time, Ω will remain zero at all time.

5.3 Stability

As was mentioned in the previous section, the calculation of ω_x over time does not remain zero due to a combination of numerical error and dynamic instability present in the system. If the system were stable the accumulation of numerical error over time would not grow exponentially, but would remain stable.

In order to investigate the stability of the flight dynamics of the system more precisely the computer program was run with perturbations in the initial conditions for ω_x , ω_y and ω_z with the same sail dimension and initial conditions for the trajectory and control angles as in the previous section. The results of these simulations for the radial position over time are given in figure 5.3. This figure shows that when the initial conditions for the angular velocities are perturbed, the changes in the radial position are such that the sail remains in the same orbit over time.

The first attempt to improve the stability of the system was to add two small sail planes to the top and bottom of the two main sail planes. Figure 5.4 shows this new design. The angle between the normal vectors and the $x'y'$ -plane is γ for both normal vectors, and the angle between the projection onto the $x'y'$ -plane and the y' -axis is $\alpha/2$. This means the two normal

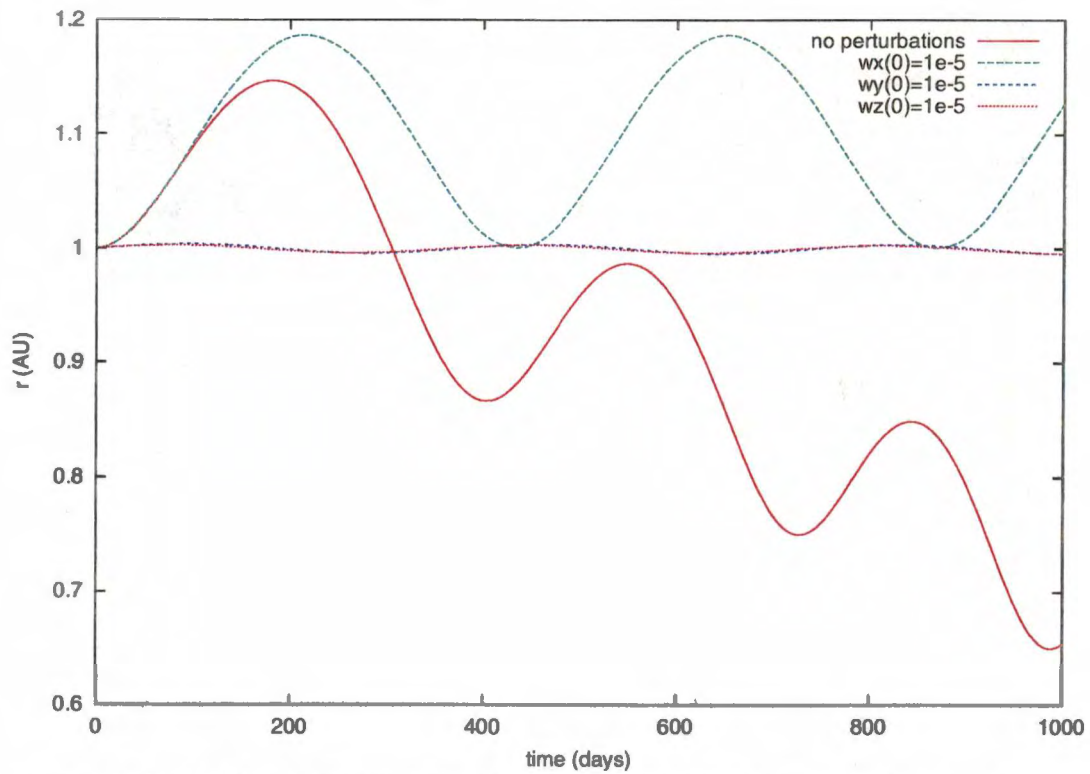


Figure 5.3: Radial position over time with perturbed initial angular velocities, no control.

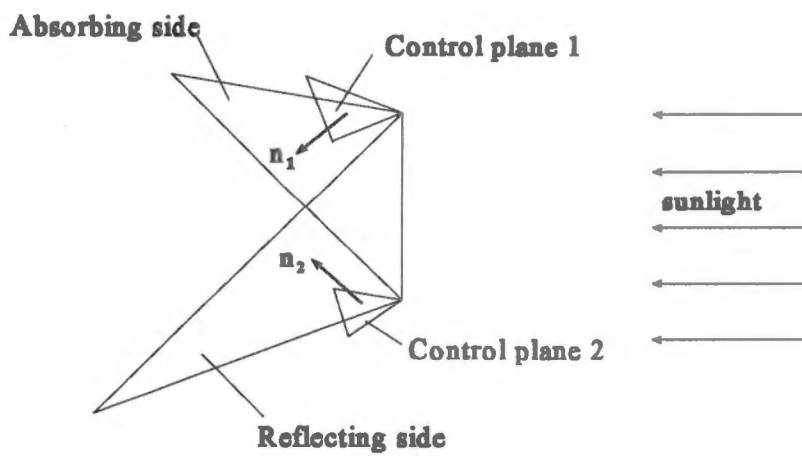


Figure 5.4: Two plane solar sail with two small control planes

vectors are given by:

$$n_1 = \begin{pmatrix} \cos \gamma \sin \frac{\alpha}{2} \\ \cos \gamma \cos \frac{\alpha}{2} \\ \sin \gamma \end{pmatrix}, \quad n_2 = \begin{pmatrix} \cos \gamma \sin \frac{\alpha}{2} \\ \cos \gamma \cos \frac{\alpha}{2} \\ -\sin \gamma \end{pmatrix} \quad (5.1)$$

If the base of the planes is given by l_4 and the height by l_3 (compare to l_1 and l_2), then in their local reference system their moments of inertia are given by (see figure 3.4 and equations 3.11 - 3.13):

$$I_{xx} = \frac{1}{24} m l_4^2, \quad I_{yy} = \frac{1}{18} m l_5^2, \quad I_{zz} = I_{xx} + I_{yy} \quad (5.2)$$

In order to calculate the moments and products of inertia of these two control planes in the local rotated frame, their own reference frames need to be rotated about their y-axis over $90^\circ - \gamma$. In this rotated frame, the moments and products of inertia are:

$$\begin{aligned} I_{x'x'} &= I_{xx} \cos^2(90^\circ - \gamma) + I_{zz} \sin^2(90^\circ - \gamma) \\ I_{y'y'} &= I_{yy} \\ I_{z'z'} &= I_{xx} \sin^2(90^\circ - \gamma) + I_{zz} \cos^2(90^\circ - \gamma) \\ I_{x'z'} &= -\sin(90^\circ - \gamma) \cos(90^\circ - \gamma) (I_{xx} - I_{zz}) \end{aligned} \quad (5.3)$$

For control plane 1, a rotation of $90^\circ + \alpha/2$ about the z' axis will bring the local frame of parallel to the local rotated frame from chapter 3. Thus the contribution from control plane 1 to the total moment and products of inertia are:

$$\begin{aligned}
I_{XX_1} &= I_{x'x'} \sin^2 \frac{\alpha}{2} + I_{y'y'} \cos^2 \frac{\alpha}{2} + m((y_1 - y_{cm})^2 + (z_1 - z_{cm})^2) \\
I_{YY_1} &= I_{x'x'} \cos^2 \frac{\alpha}{2} + I_{y'y'} \sin^2 \frac{\alpha}{2} + m((x_1 - x_{cm})^2 + (z_1 - z_{cm})^2) \\
I_{ZZ_1} &= I_{z'z'} + m((x_1 - x_{cm})^2 + (y_1 - y_{cm})^2) \\
I_{XY_1} &= \sin \frac{\alpha}{2} \cos \frac{\alpha}{2} (I_{y'y'} - I_{x'x'}) + m(x_1 - x_{cm})(y_1 - y_{cm}) \\
I_{XZ_1} &= -\sin \frac{\alpha}{2} I_{x'z'} + m(x_1 - x_{cm})(z_1 - z_{cm}) \\
I_{YZ_1} &= -\cos \frac{\alpha}{2} I_{x'z'} + m(y_1 - y_{cm})(z_1 - z_{cm})
\end{aligned} \tag{5.4}$$

For control plane 2, first a rotation of $\alpha/2$ about the z' -axis followed by a rotation about the y'' -axis of 180° will bring this plane's local frame parallel to the local rotated frame, and the contribution of this plane to the total moment and products of inertia is:

$$\begin{aligned}
I_{XX_2} &= I_{x'x'} \sin^2 \frac{\alpha}{2} + I_{y'y'} \cos^2 \frac{\alpha}{2} + m((y_2 - y_{cm})^2 + (z_2 - z_{cm})^2) \\
I_{YY_2} &= I_{x'x'} \cos^2 \frac{\alpha}{2} + I_{y'y'} \sin^2 \frac{\alpha}{2} + m((x_2 - x_{cm})^2 + (z_2 - z_{cm})^2) \\
I_{ZZ_2} &= I_{z'z'} + m((x_2 - x_{cm})^2 + (y_2 - y_{cm})^2) \\
I_{XY_2} &= \sin \frac{\alpha}{2} \cos \frac{\alpha}{2} (I_{y'y'} - I_{x'x'}) + m(x_2 - x_{cm})(y_2 - y_{cm}) \\
I_{XZ_2} &= \sin \frac{\alpha}{2} I_{x'z'} + m(x_2 - x_{cm})(z_2 - z_{cm}) \\
I_{YZ_2} &= \cos \frac{\alpha}{2} I_{x'z'} + m(y_2 - y_{cm})(z_2 - z_{cm})
\end{aligned} \tag{5.5}$$

The center of masses of the two control planes are:

$$\begin{aligned}
x_1 &= \frac{1}{3} l_5 \cos \gamma \sin \frac{\alpha}{2}, & x_2 &= \frac{1}{3} l_5 \cos \gamma \sin \frac{\alpha}{2} \\
y_1 &= \frac{1}{3} l_5 \cos \gamma \cos \frac{\alpha}{2}, & y_2 &= \frac{1}{3} l_5 \cos \gamma \cos \frac{\alpha}{2} \\
z_1 &= -\frac{1}{2} l_1 - \frac{1}{3} l_5 \sin \gamma, & z_2 &= \frac{1}{2} l_1 + \frac{1}{3} l_5 \sin \gamma
\end{aligned} \tag{5.6}$$

Since the masses of the two planes are identical, $z_1 = -z_2$ and $z_{cm} = 0.0$, the contributions of the two planes to the moments of inertia I_{xz} and I_{yz} will cancel out. This result is expected, since the $x'y'$ plane is still the symmetry plane of the system.

Equation 3.24 was used to calculate the contribution to the total solar radiation forces. The planes are assumed to be reflective. When $n \cdot i < 0$ for any of the planes, it is assumed that the two main planes will be shadowing that control plane totally.

Figure 5.5-a shows the effects on the radial position if these two control planes are added. It shows that now a perturbation in the initial condition for ω_z will lead to stable motion, but the system is still unstable with respects to perturbations in ω_x and ω_y . The results for $\cos\theta$, $\sin\theta\cos\beta$ and $\sin\theta\sin\Omega$ are given in figures 5.5-a, b and c respectively. Figure 5.5-b can be used to calculate $\sin\theta$, this is found to be approximately 0.2988. From figure 5.5-c it follows that $\cos\beta$ varies between -1 to 1, meaning β varies from -180° to 180° . This means that this motion is totally uncontrolled. Figure 5.5-d shows that Ω oscillates about 0, with a maximum amplitude of about 2° when $\omega_y(0) = 10^{-5}$ rad/s and an amplitude of about 0.8° when $\omega_x(0) = 10^{-5}$ rad/s.

The fact that the β -motion cannot be controlled by sail planes alone is not surprising. This is a roll motion about the e_r axis or the sail-sun axis. With respect to the local rotated frame the solar sail forces, and thus the corresponding moments, do not depend on this angle. It only plays a role when calculating the forces in the spherical reference system, thus

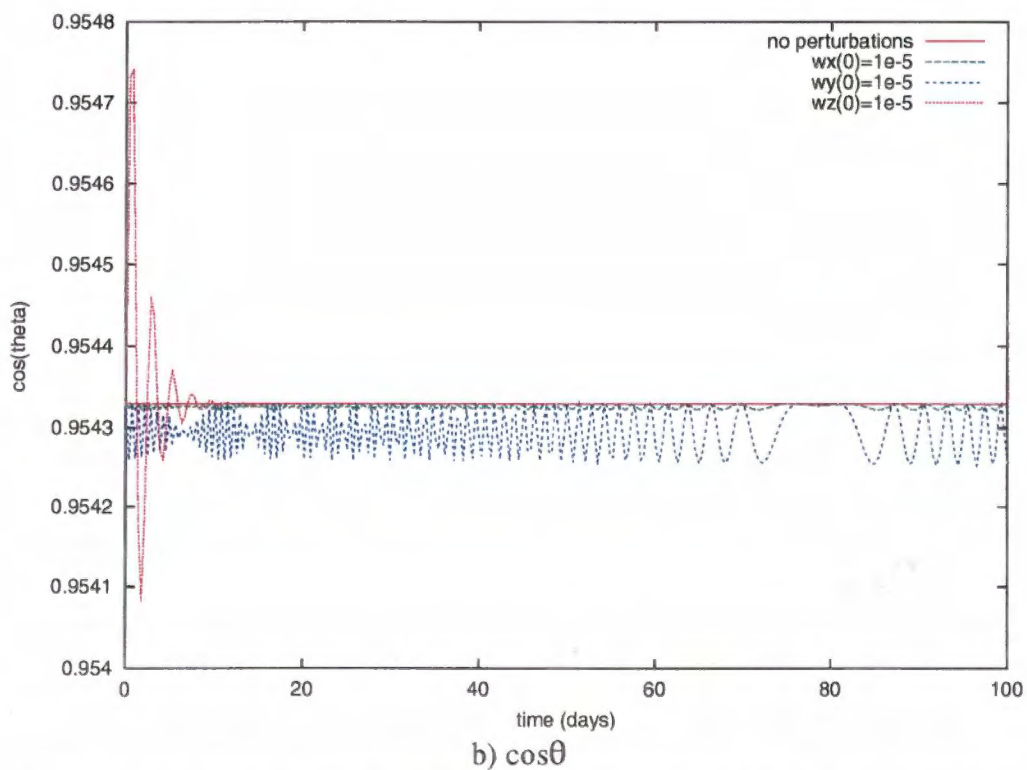
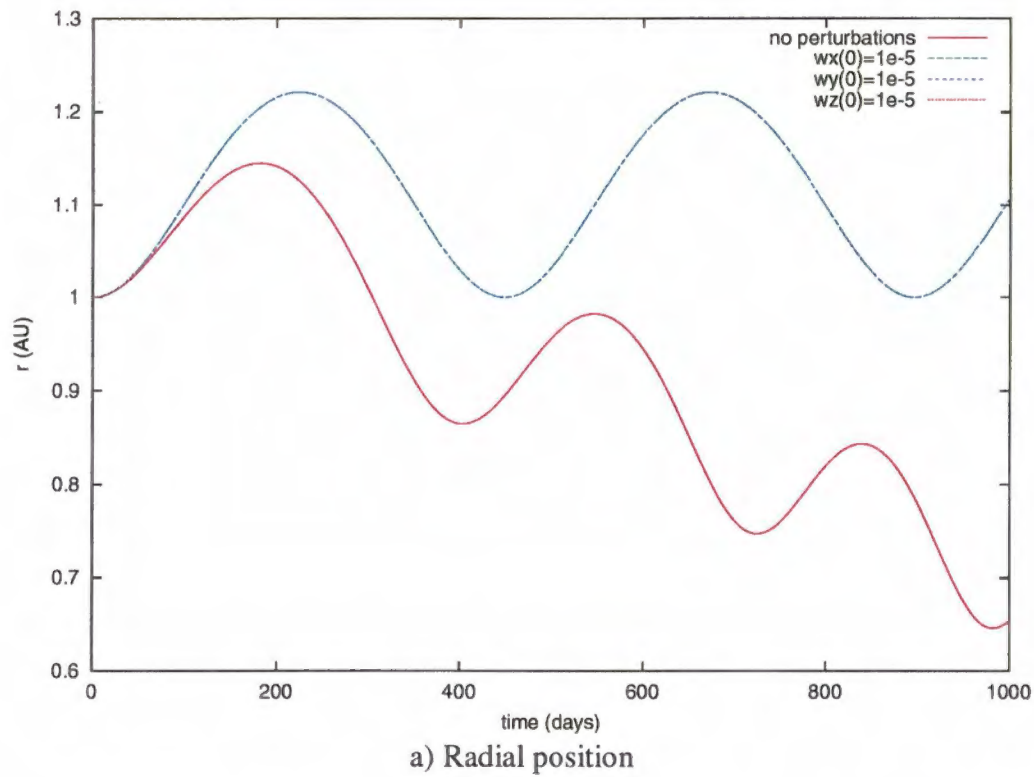


Figure 5.5: Influence of control planes on the perturbed initial angular velocities

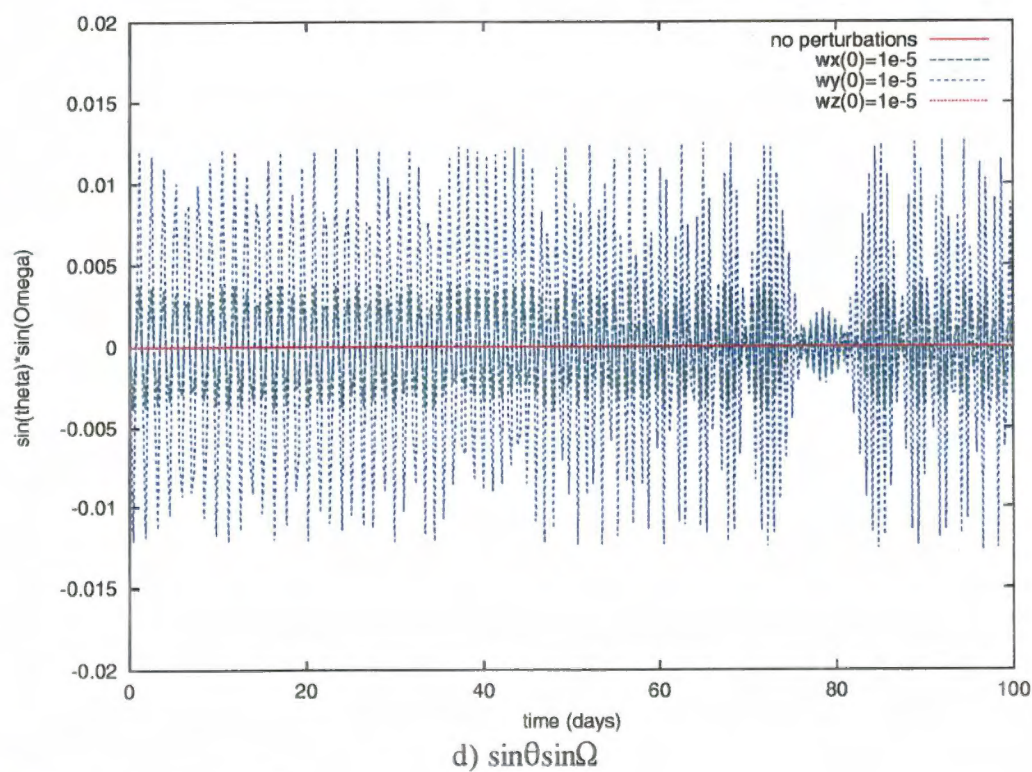
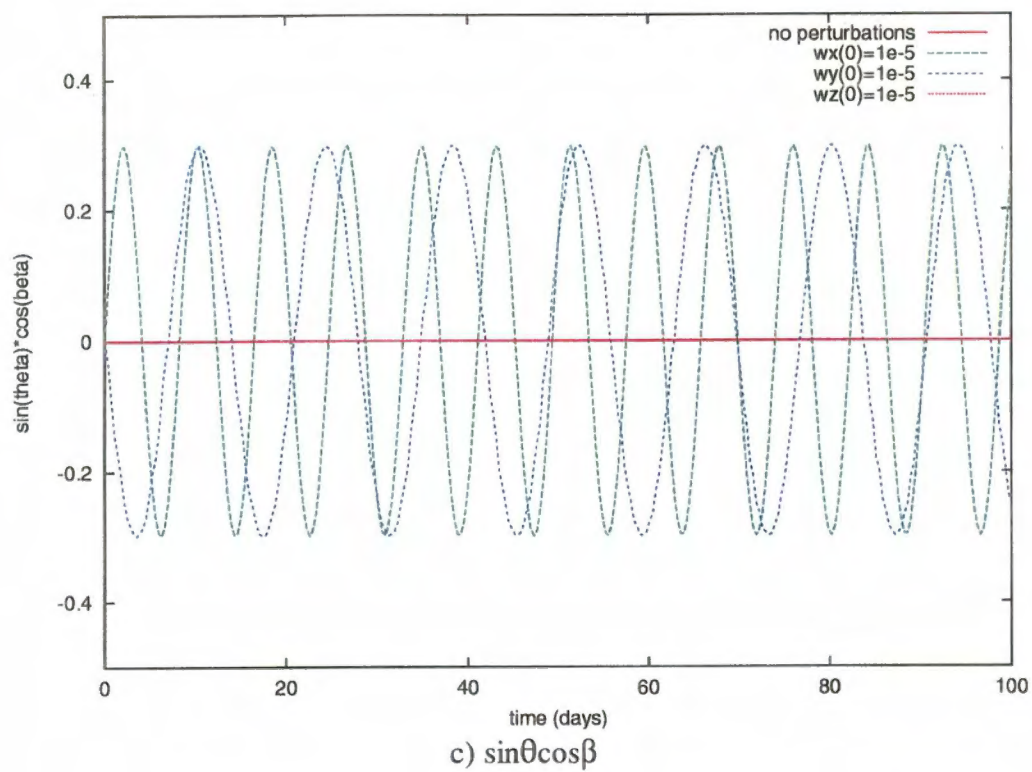


Figure 5.5 (continued)

influencing the trajectory only. An active form of control will be the best solution to stabilize this motion.

A small thruster was added which produces a control torque about the sail-sun axis. This torque will be $5 \cdot 10^{-8}$ Nm if $|\omega_x| > 10^{-15}$ rad/s and 0 Nm otherwise. The results of adding this thruster can be seen in figure 5.6. Although the radial position over time for $\omega_x(0) = 10^{-5}$ rad/s and $\omega_y(0) = 10^{-5}$ rad/s are not identical to that for the case with the correct initial conditions, the trajectories are inward again. $\cos\theta$ (figure 5.6-b) and $\sin\theta\sin\Omega$ (figure 5.6-d) reach their correct values within 100 days. The β -motion is also controlled now, and is approaching its correct value faster for the case with the error in ω_x compared to the case with the error in ω_y . This again explains why the radial position for the ω_x case is closer to the correct value compared to the ω_y case.

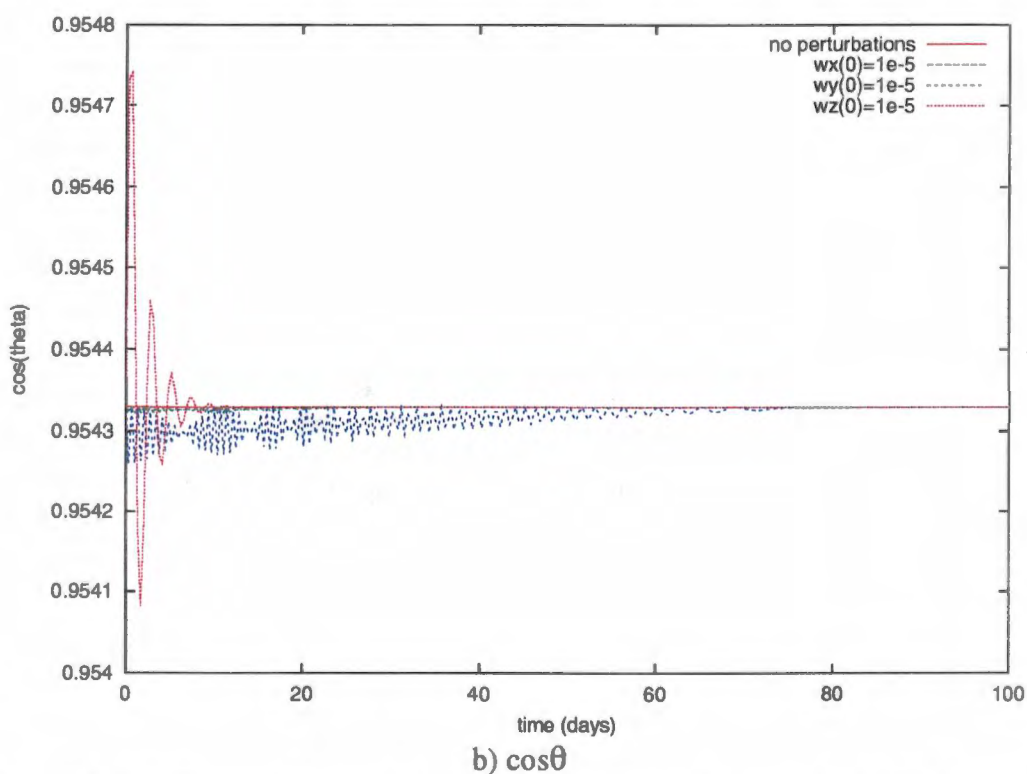
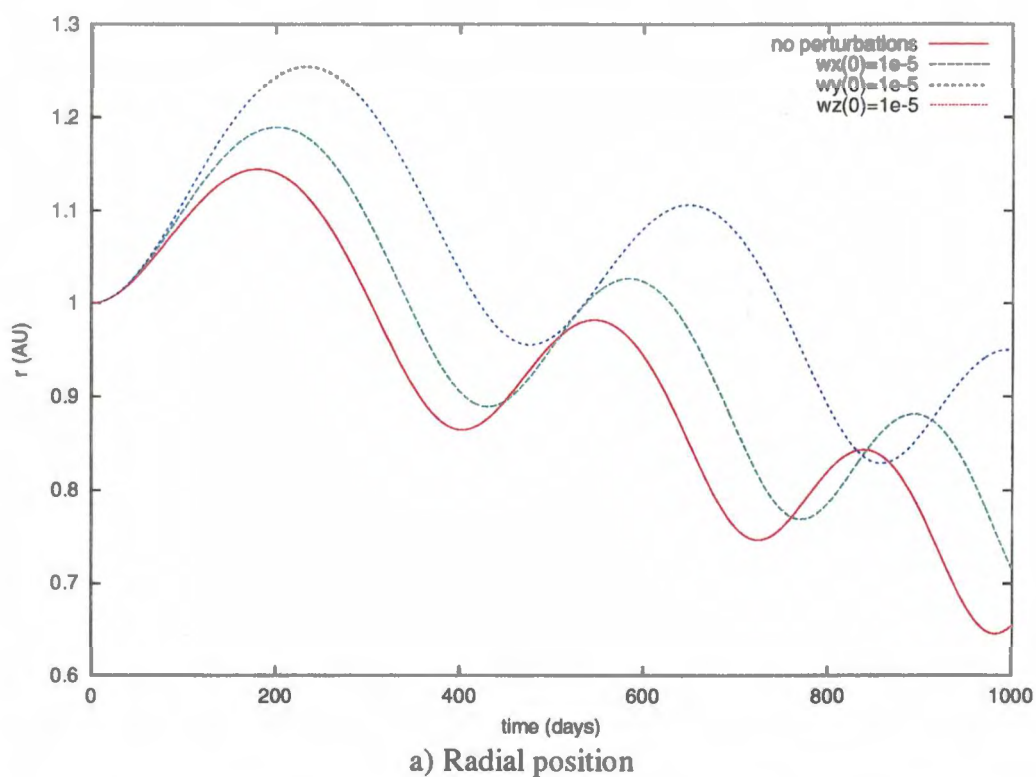


Figure 5.6: Influence of control planes and thruster on the perturbed initial angular velocities

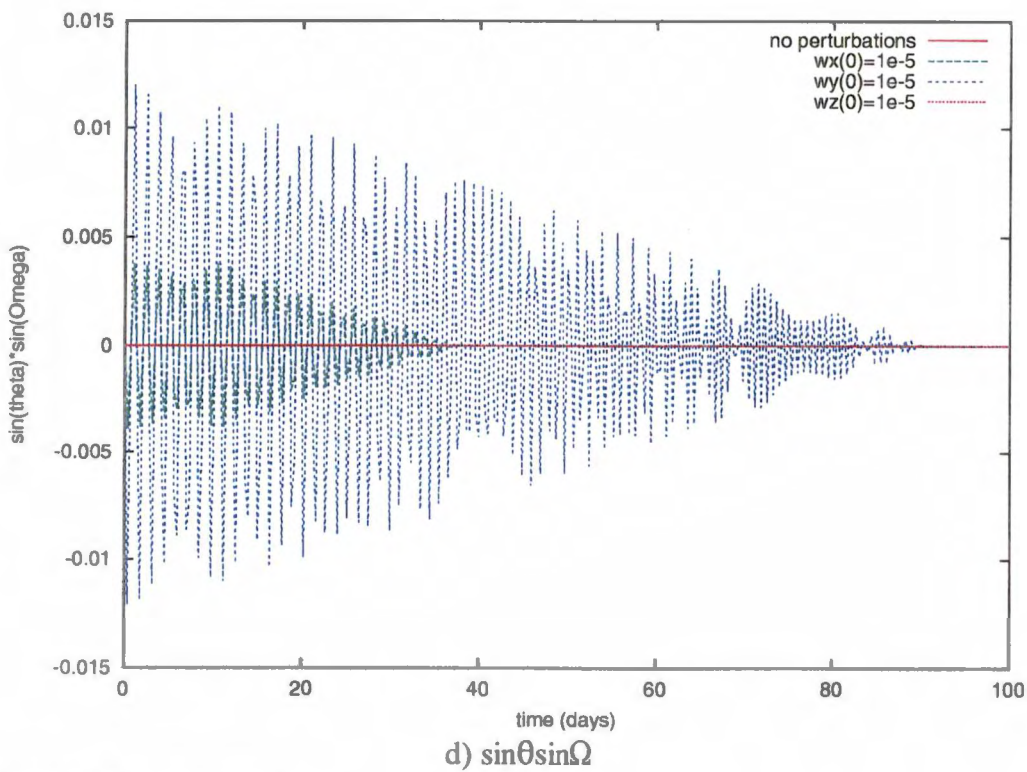
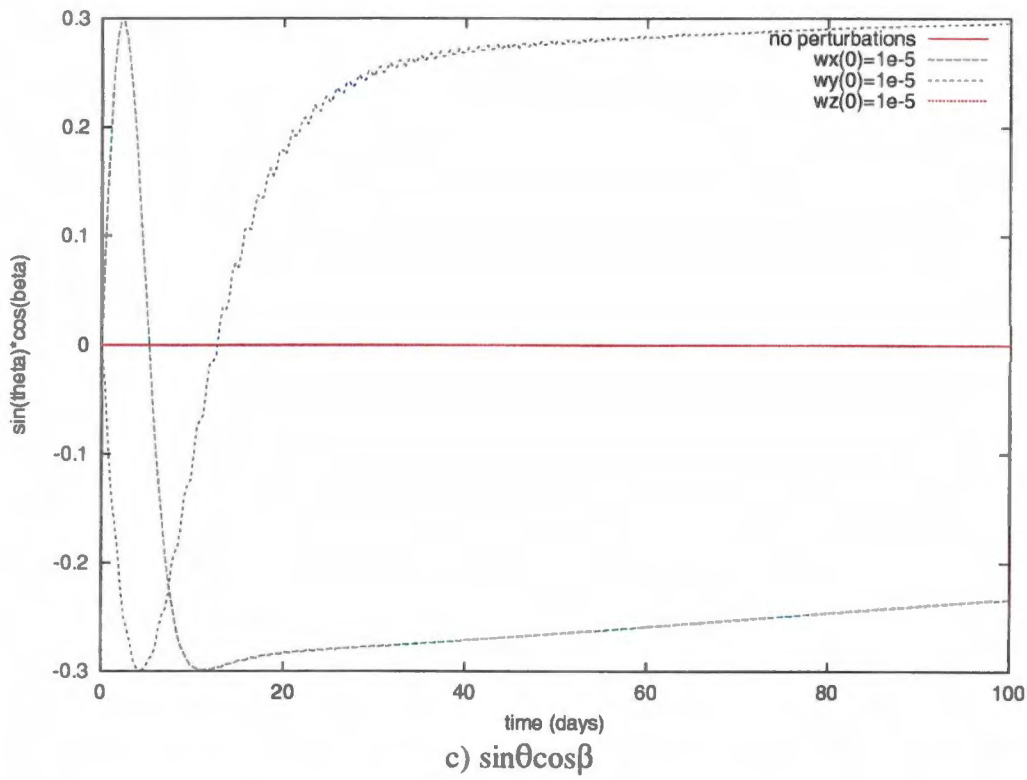


Figure 5.6 (continued)

Chapter 6

Conclusions and Recommendations

6.1 Conclusions

It was shown in chapter 4 that for two-plane solar sails with triangular sail planes, passive attitude stability and control can be achieved for the coplanar case. The type of trajectory and the flight times that can be achieved depends heavily upon the angle α between the two sail planes and the size of the individual sail planes, but both inward and outward trajectories are feasible.

Chapter 5 showed the results for the full 3D model. It was shown that the shape of the resulting trajectories do not depend on the orientation of the trajectory plane. When the stability of the flight dynamics was investigated, it became clear that the original design was not stable. Adding two small control planes, one at the top and one at the bottom, to the two-plane solar sail makes the motion about the z-axis stable. Adding a thruster in addition to the two small control planes makes the motion about the other two axes stable also.

6.2 Recommendations

Further investigation into the optimum size of the two sail planes and the angle between the two sail planes is still needed. Also the optimum size and orientation of the two control planes needs to be found. The type of the active control system needs a more detailed investigation also. The thruster was only used to show that control can be achieved, there might be a better solution.

When calculating the moments and products of inertia, it was assumed that the mass of the supporting structure was homogeneously distributed over the two sail planes. This will not be case in reality, and changing the model to accurately represent this will need to be done. It is very likely that there will be natural damping present in the solar sail structure. The flight dynamics model does not account for this yet. Incorporating damping into the model will have an influence on the stability of the solar sail.

Bibliography

Bibliography

References

DLR: <http://www.kp.dlr.de/solarsail>

Friedman, L. et al (1978), 'Solar Sailing-The Concept Made Realistic'. *Proceedings of the AIAA 16th Aerospace Sciences meeting*, Huntsville, Alabama. AIAA paper 78-82.

Garwin, R.L. (1958) 'Solar Sailing-A Practical Method of Propulsion Within the Solar System'. *Jet Propulsion*, pp 188-190.

Grodzovskii, G.L., Y.N. Ivanov and V.V. Tokarev (1969), *Mechanics of Spaceflight Low-Thrust*. Translated from Russian, Israel Program for Scientific Translations.

Kirpichnikov S. N., E. S. Kirpichnikova, E. N. Polyakhova and A. S. Shmyrov (1996), *Planar Heliocentric Roto-Translatory Motion of a Spacecraft with a Solar Sail of Complex Shape*, *Celest. Mech. & Dyn. Astron.*, vol 63, pp 255-269.

MacNeal, R.H., J.M. Hedgepeth and H.U. Schuerch (1969), *Heliogyro Solar Sailer Summary Report*. NASA CR-1329.

McInnes, C.R. (1999), *Solar Sailing: Technology, Dynamics and Mission Applications*. Chichester, UK: Springer-Praxis series in Space Science and Technology.

Wright, J.L. (1974), *Solar Sailing: Evaluation of Concept and Potential*. Batelle Memorial Institute Report No. BMI-NLVP-TM-74-3.

Wright, J.L.(1976), *Solar Sailing Study, Summary of results*.

Wright, J.L. (1992), *Space Sailing*. Amsterdam, the Netherlands: Gordon and Breach Science Publishers.

Bibliography

Kaplan, M.H. (1976), *Modern Spacecraft Dynamics & Control*. John Wiley & Sons, Inc.

Souza, D.M. (1994), *Space Sailing*, Minneapolis: Lerner Publications Company.

Wells, A.A. (1967), *Theory and Problems of Lagrangian Dynamics*. Schaum's outline series.

QUAT-S 2102 (1984) Advanced training series, Quaternions Supplemental Workbook.
NASA Lyndon B. Johnson Space Center, Houston Texas.

Appendices

Appendix A

Quaternion Algebra

A.1 Basic definitions

As was mentioned in Chapter 3, a quaternion is a four-parameter representation of a transformation matrix, consisting of a scalar plus a vector:

$$Q = q_1 + q_2 \hat{i} + q_3 \hat{j} + q_4 \hat{k} \\ = (q_1, q_2, q_3, q_4) \quad (\text{A.1})$$

The three elements describing the vector define an axis of rotation, the scalar defines the magnitude of rotation. Now, let $P = (p_1, p_2, p_3, p_4)$ and $Q = (q_1, q_2, q_3, q_4)$. The fundamental definitions are as follows:

Equality: $Q = P$ when and only when:

$$q_1 = p_1, q_2 = p_2, q_3 = p_3, q_4 = p_4 \quad (\text{A.2})$$

Addition:

$$Q + P = (q_1 + p_1, q_2 + p_2, q_3 + p_3, q_4 + p_4) \quad (\text{A.3})$$

Multiplication by a scalar a :

$$aQ = (aq_1, aq_2, aq_3, aq_4) \quad (\text{A.4})$$

A.2 Definition of quaternion product

The quaternion product

$$QP = (q_1 + q_2\hat{i} + q_3\hat{j} + q_4\hat{k})(p_1 + p_2\hat{i} + p_3\hat{j} + p_4\hat{k}) \quad (\text{A.5})$$

is obtained by distributing the terms on the right as in ordinary algebra, except that the order of the unit vectors must be preserved, and then replacing each product of unit vectors using the following rules:

$$\begin{aligned} \hat{i}\hat{i} &= \hat{j}\hat{j} = \hat{k}\hat{k} = -1 \\ \hat{i}\hat{j} &= -\hat{j}\hat{i} = \hat{k} \\ \hat{j}\hat{k} &= -\hat{k}\hat{j} = \hat{i} \\ \hat{k}\hat{i} &= -\hat{i}\hat{k} = \hat{j} \end{aligned} \quad (\text{A.6})$$

Therefore:

$$\begin{aligned} QP &= q_1p_1 - q_2p_2 - q_3p_3 - q_4p_4 + \\ &\quad (q_1p_2 + q_2p_1 + q_3p_4 - q_4p_3)\hat{i} \\ &\quad (q_1p_3 - q_2p_4 + q_3p_1 + q_4p_2)\hat{j} \\ &\quad (q_1p_4 + q_2p_3 - q_3p_2 + q_4p_1)\hat{k} \end{aligned} \quad (\text{A.7})$$

This can also be written as a multiplication of a matrix with a vector:

$$QP = \begin{pmatrix} q_1 & -q_2 & -q_3 & -q_4 \\ q_2 & q_1 & -q_4 & q_3 \\ q_3 & q_4 & q_1 & -q_2 \\ q_4 & -q_3 & q_2 & q_1 \end{pmatrix} \begin{pmatrix} p_1 \\ p_2 \\ p_3 \\ p_4 \end{pmatrix} = \begin{pmatrix} p_1 & -p_2 & -p_3 & -p_4 \\ p_2 & p_1 & p_4 & -p_3 \\ p_3 & -p_4 & p_1 & p_2 \\ p_4 & p_3 & -p_2 & p_1 \end{pmatrix} \begin{pmatrix} q_1 \\ q_2 \\ q_3 \\ q_4 \end{pmatrix} \quad (\text{A.8})$$

A.3 Transformation matrices

Let the quaternion Q_A^B define the rotation from a system A into a system B and the quaternion Q_B^C define the rotation from a system B into a system C . Then it is possible to

combine Q_A^B and Q_B^C by means of the quaternion product to generate the quaternion Q_A^C

describing the rotation from system A to C through a single rotation angle:

$$Q_A^C = Q_B^C Q_A^B \quad (\text{A.9})$$

Let T_A^B define the transformation matrix from a system A into a system B:

$$T_A^B = \begin{pmatrix} \cos \theta & \sin \theta & 0 \\ -\sin \theta & \cos \theta & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad (\text{A.10})$$

and T_B^C define the transformation matrix from a system B into a system C:

$$T_B^C = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos \beta & \sin \beta \\ 0 & -\sin \beta & \cos \beta \end{pmatrix} \quad (\text{A.11})$$

then the transformation matrix T_A^C from a system A into a system C is given by:

$$T_A^C = T_B^C T_A^B = \begin{pmatrix} \cos \theta & \sin \theta & 0 \\ -\sin \theta \cos \beta & \cos \theta \cos \beta & \sin \beta \\ \sin \theta \sin \beta & -\cos \theta \sin \beta & \cos \beta \end{pmatrix} \quad (\text{A.12})$$

The rotation matrix in terms of the quaternion elements is given by:

$$T_q = \begin{pmatrix} q_1^2 + q_2^2 - q_3^2 - q_4^2 & 2(q_2 q_3 - q_1 q_4) & 2(q_2 q_4 + q_1 q_3) \\ 2(q_2 q_3 + q_1 q_4) & q_1^2 - q_2^2 + q_3^2 - q_4^2 & 2(q_3 q_4 - q_1 q_2) \\ 2(q_2 q_4 - q_1 q_3) & 2(q_3 q_4 + q_1 q_2) & q_1^2 - q_2^2 - q_3^2 + q_4^2 \end{pmatrix} \quad (\text{A.13})$$

Solving equation A.10 for Q_A^B , equation A.11 for Q_B^C and equation A.12 for Q_A^C by using

equation A.13 gives:

$$\begin{aligned}
 Q_A^B &= (\cos \frac{\theta}{2}, 0, 0, -\sin \frac{\theta}{2}) \\
 Q_B^C &= (\cos \frac{\beta}{2}, -\sin \frac{\beta}{2}, 0, 0) \\
 Q_A^C &= (\cos \frac{\beta}{2} \cos \frac{\theta}{2}, -\sin \frac{\beta}{2} \cos \frac{\theta}{2}, -\sin \frac{\beta}{2} \sin \frac{\theta}{2}, -\cos \frac{\beta}{2} \sin \frac{\theta}{2})
 \end{aligned} \tag{A.14}$$

Substituting the results found here for Q_A^B and Q_B^C into equation A.9 gives the same result for Q_A^C .

A.4 Solving the quaternion problem of chapter 3

In chapter 3 a transformation matrix was derived that transformed an inertial system into the local rotated system. This was done by applying 5 successive rotations to this inertial systems. It is now possible to define the quaternion associated with each rotation, and the overall system quaternion can then be found by applying equation A.9 several times. The individual quaternions for each successive rotation are:

$$R_1 = \begin{pmatrix} \cos \phi & \sin \phi & 0 \\ -\sin \phi & \cos \phi & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad Q_1 = (\cos \frac{\phi}{2}, 0, 0, -\sin \frac{\phi}{2}) \tag{A.15}$$

$$R_2 = \begin{pmatrix} \cos \psi & 0 & -\sin \psi \\ 0 & 1 & 0 \\ \sin \psi & 0 & \cos \psi \end{pmatrix} \quad Q_2 = (\cos \frac{\psi}{2}, 0, -\sin \frac{\psi}{2}, 0) \tag{A.16}$$

$$R_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos \beta x & \sin \beta x \\ 0 & -\sin \beta x & \cos \beta x \end{pmatrix}, \quad Q_3 = (\cos \frac{\beta x}{2}, -\sin \frac{\beta x}{2}, 0, 0) \quad (\text{A.17})$$

$$R_4 = \begin{pmatrix} \cos \theta & \sin \theta & 0 \\ -\sin \theta & \cos \theta & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad Q_4 = (\cos \frac{\theta}{2}, 0, 0, -\sin \frac{\theta}{2}) \quad (\text{A.18})$$

$$R_5 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos \Omega & \sin \Omega \\ 0 & -\sin \Omega & \cos \Omega \end{pmatrix}, \quad Q_5 = (\cos \frac{\Omega}{2}, -\sin \frac{\Omega}{2}, 0, 0) \quad (\text{A.19})$$

Multiplying these results out leads to equation 3.70.

Vita

Christel Brigitte van de Kolk was born on June 25, 1972 in Hengelo, the Netherlands. She attended schools in Delden, Bunschoten and Amersfoort, all located in the Netherlands, graduating from Schothorst College Highschool in May 1990.

She attended the department of Aerospace Engineering at Delft University of Technology, the Netherlands, from September 1990 until June 1995. There she received a masters degree in Aerospace Engineering.

From August 1996 until May 2000 she attended the University of Tennessee Space Institute, where she received her PhD in Aerospace Engineering.