To the Graduate Council:

I am submitting herewith a dissertation written by Sherwin Kouchekian-Sabour entitled "Unbounded Bergman Operators." I have examined the final copy of this dissertation for form and content and recommend that it be accepted in partial fulfillment of the requirements for the degree of Doctor of Philosophy, with a major in Mathematics.

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Unbounded Bergman Operators

A Thesis
Presented for the
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Sherwin Kouchekian-Sabour
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Dedication

To the memory of my father

M. Ali Kouchekian-Sabour

and

to the strength of my mother

Parwin Gharib

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Abstract

Let G be an open subset of the plane, and denote by $L_a^2(G)$ the Bergman space of all square integrable analytic functions with respect to the Lebesgue area measure. If $\operatorname{dom} S_G = \{f \in L_a^2(G)\colon\ zf \in L_a^2(G)\}$, define the Bergman operator $S_G\colon \operatorname{dom} S_G \to L_a^2(G)$, by $S_G\colon f(z) \mapsto zf(z)$. We show that the problem regarding the density of $\operatorname{dom} S_G$ in $L_a^2(G)$ is equivalent to the problem regarding the density of the range of the operator of multiplication by z on some open subset of the unit disc $\mathbb{D}$. If U is a open subset of $\mathbb{D}$ containing 0 in its topological boundary, using Wiener capacity we present two sufficient conditions such that $z\big[L_a^2(U)\big]$ is dense in $L_a^2(U)$. As a consequence, it follows that if G has finite area, or the component of the complement of G with respect to the extended plane containing ∞ does not equal the singleton $\{\infty\}$, then the Bergman operator S_G is densely defined. Furthermore, we prove that if G is a simply connected region of finite area, or a half plane, then the self-commutator of S_G is densely defined. It is also shown that for a Bergman operator S_G, the spectrum equals the closure of G, and its point spectrum is the empty set. Finally, we show that if the Bergman operator S_G is densely defined, then the set of all bounded operators in $L_a^2(G)$ that commute with S_G equals $\{M_\varphi\colon\ \varphi \in H^\infty(G)\}$, where M_φ denotes the operator of multiplication by φ on $L_a^2(G)$.

Preface

The theory of subnormal operators is well developed. During the past two decades this theory has provided a great deal of challenging problems in operator theory and function theory. Besides the unilateral shift, the Bergman operator has been a crucial source regarding the development of the theory of subnormal operators. It seems natural to ask about the extension of such a theory to the unbounded case. The theory of unbounded subnormal operators is very young and is growing. Janas, McDonald, Sundberg, Stochel, and Szafraniec are a few who have studied different aspects of the subject. As in the bounded case, the unbounded Bergman operator is an important concrete example in the class of unbounded subnormal operators. Perhaps this example alone motivates a further investigation with respect to the concept of subnormality in the unbounded case. It is within the context of the above remarks that this study has been undertaken.

Chapter One is mainly a collection of basic results from the theory of linear operators and some analytic function theory which will be used throughout this study. In this chapter, we also present the notion of subnormality in the unbounded case and a few concrete examples of unbounded subnormal operators are provided. Furthermore, we give

the definition of unbounded Bergman operators and some basic facts regarding these operators are proved in detail.

An unbounded subnormal operator is densely defined; that is, the domain of an unbounded subnormal operator is a dense subset of the underlying Hilbert space on which the operator is defined. In this regard, the first natural question for an unbounded Bergman operator is the "density problem". More precisely, the "density problem" refers to the classification of those open sets in the plane for which the corresponding Bergman operator is densely defined. Chapters Two and Three take up the density problem in detail.

Chapter Two describes an analytic approach to the density problem. First it is shown that a Bergman operator in general is not densely defined. Furthermore, by virtue of the notion of the essential and removable boundary points for Bergman spaces, we are able to describe a large class of open sets in the complex plane for which the Bergman operator has a dense domain. By providing three examples of different open subsets of the plane, we justify that our results in this chapter can not be further improved with the aid of the analytic approach. Nevertheless, this approach makes it possible to extend our result, regarding the density problem, to all positive integer powers of the Bergman operator.

Chapter Three addresses again the density problem. However, in this chapter we show that the problem of whether a Bergman operator is densely defined is equivalent to the problem of density of the range of a bounded Bergman operator. In the latter problem, the corresponding Bergman space consists of all square integrable analytic functions in a bounded open set having the origin as its boundary point. This problem has been studied by several authors including Aleman, Axler, Bourdon, and Hedberg. In light of the equivalent problem, using a technique similar to Hedberg, we provide a pair of sufficient conditions given in terms of Wiener capacity that describe open sets in the plane for which the corresponding Bergman operator is densely defined.

Chapter Four consists of some further results regarding the properties of the unbounded Bergman operators. In this chapter, we obtain the spectrum, the point spectrum, and the bounded commutant of an unbounded Bergman operator. Furthermore, we consider the problem of the density of the self-commutator of an unbounded Bergman operator. This problem is a consequence of the attempt to generalize the Berger-Shaw Theorem to the unbounded case where it is required that the Bergman operator and its self-commutator are both densely defined.

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CHAPTER 1

Introduction

Throughout this study all Hilbert spaces are assumed to be complex and separable. For a Hilbert space $\mathcal{H}$, its inner product and the resulting norm are denoted by $\langle .,. \rangle_{\mathcal{H}}$ and $\| \, . \, \|_{\mathcal{H}}$ respectively. Whenever there is no ambiguity about the underlying Hilbert space, the subscript $\mathcal{H}$ will be excluded from the notation. We start this chapter by recalling a few facts regarding unbounded linear operators.

1.1. Unbounded Operators

In what follows the word *operator* means a linear mapping between two Hilbert spaces. Let $T\colon \mathcal{H} \to \mathcal{K}$ be an operator, then its *domain* is a linear manifold in $\mathcal{H}$, and it will be denoted by $\operatorname{dom} T$. An operator $T\colon \mathcal{H} \to \mathcal{K}$ is said to be *densely defined* if its domain is a dense subset of $\mathcal{H}$. If T, S are operators from $\mathcal{H}$ into $\mathcal{K}$, then $T \pm S$ and TS are defined by $\operatorname{dom}(T \pm S) = \operatorname{dom} T \cap \operatorname{dom} S$ and $\operatorname{dom} TS = \{h \in \operatorname{dom} S\colon \ Sh \in \operatorname{dom} T\}$, respectively. T is said to be an *extension* of S, denoted by $S \subseteq T$, if $\operatorname{dom} S \subseteq \operatorname{dom} T$ and $Sh = Th$ for every h in $\operatorname{dom} S$.

DEFINITION 1.1.1. If $T\colon \mathcal{H} \to \mathcal{K}$ is densely defined, and

$$\operatorname{dom} T^* = \{k \in \mathcal{K}\colon h \mapsto \langle Th, k\rangle \text{ is a bounded functional on } \operatorname{dom} T\},$$

define the *adjoint* of T, $T^*\colon \operatorname{dom} T^* \to \mathcal{H}$, by

$$\langle Th, k\rangle = \langle h, T^*k\rangle$$

for every h in $\operatorname{dom} T$ and k in $\operatorname{dom} T^*$.

From the density of $\operatorname{dom} T$ in $\mathcal{H}$, it follows that the adjoint operator T^* is uniquely determined. Furthermore, an operator $T\colon \mathcal{H} \to \mathcal{K}$ is said to be *closed,* if its *graph* $\{h \oplus Th\colon \quad h \in \operatorname{dom} T\}$ is closed in $\mathcal{H} \oplus \mathcal{K}$. We call an operator *closable,* if it has a closed extension.

PROPOSITION 1.1.2. *If $T\colon \mathcal{H} \to \mathcal{K}$ is a densely defined operator, then the following statements hold.*

(a) *T^* is a closed operator.*

(b) *T^* is densely defined if and only if T is closable.*

If in addition T is closed then,

(c) *TT^* and T^*T are densely defined.*

PROOF. For a proof, see Conway [7], Proposition 1.6 on page 305, and Proposition 4.2 on page 319. □

DEFINITION 1.1.3. Let $T\colon \mathcal{H} \to \mathcal{K}$ be a densely defined operator.The *self-commutator* of T, $[T^*, T]$, is given by $[T^*, T] = T^*T - TT^*$, with $\operatorname{dom}[T^*, T] = \operatorname{dom} T^*T \cap \operatorname{dom} TT^*$.

1.2. Unbounded Subnormal Operators

In this section we introduce the notion of subnormality in the unbounded case. Furthermore, a few concrete examples of unbounded subnormal operators will be presented.

Denote by $B(\mathcal{H})$ the set of all bounded linear operators from $\mathcal{H}$ into $\mathcal{H}$. Recall that S in $B(\mathcal{H})$ is said to be a *(bounded) subnormal operator,* if there is a Hilbert space $\mathcal{K}$ containing $\mathcal{H}$ and a normal operator N in $B(\mathcal{K})$ such that $N\mathcal{H} \subseteq \mathcal{H}$ and $Sh = Nh$ for every h in $\mathcal{H}$.

DEFINITION 1.2.1. A closed, densely defined operator $N\colon \mathcal{K} \to \mathcal{K}$ is said to be *normal,* if $\operatorname{dom} N = \operatorname{dom} N^*$, and $\|Nk\| = \|N^*k\|$ for every k in $\operatorname{dom} N$. A closed, densely defined operator $S\colon \mathcal{H} \to \mathcal{H}$ is an *(unbounded) subnormal operator,* if there is a Hilbert space $\mathcal{K}$ containing $\mathcal{H}$ and a normal operator $N\colon \mathcal{K} \to \mathcal{K}$ such that

$$(*)\quad \operatorname{dom} S = \operatorname{dom} N \cap \mathcal{H},$$

and $Nh = Sh$ for every h in $\operatorname{dom} S$.

Note that if S is a bounded operator, the above definition agrees with the usual definition of bounded subnormal operators, see

Sundberg and McDonald [**16**]. There is, however, another less restrictive definition of subnormality, see Stochel and Szafraniec [**19**], where S is not required to be closed and the condition $(*)$ in Definition 1.2.1 is only the inclusion $\operatorname{dom} S \subseteq \operatorname{dom} N \cap \mathcal{H}$. This definition implies that S is closable; moreover, it follows that S and its closure have the same normal extension. We mention also that it is not known whether a (closed) subnormal operator, in accordance with Stochel and Szafraniec's definition, has a normal extension for which the equality $(*)$ in Definition 1.2.1 holds. For more detail on this topic, we refer the reader to Stochel and Szafraniec [**20**].

Since most of the known concrete examples of subnormal operators are naturally closed, we have adopted Definition 1.2.1 for the notion of subnormality in the unbounded case. Before presenting a few examples of unbounded subnormal operators, we prove a proposition which will be useful in Chapter 4.

PROPOSITION 1.2.2. *If S is a subnormal operator acting on a Hilbert space $\mathcal{H}$ with the normal extension N defined on a Hilbert space $\mathcal{K}$, and if $P_{\mathcal{H}}$ denotes the orthogonal projection of $\mathcal{K}$ onto $\mathcal{H}$, then*

(a) $\operatorname{dom} S \subseteq \operatorname{dom} S^*$;

(b) $S^*h = P_{\mathcal{H}} N^* h \quad$ *for all* h *in* $\operatorname{dom} S$.

PROOF. To prove (a), note that if h and f are in $\operatorname{dom} S$,

$$\langle Sf, h\rangle_{\mathcal{H}} = \langle Nf, h\rangle_{\mathcal{K}} = \langle f, N^*h\rangle_{\mathcal{K}}.$$

Now from Definition 1.1.1 it follows that h is in $\operatorname{dom} S^*$. Next for h in $\operatorname{dom} S$, using part(a), we have

$$\langle Sf, h\rangle_{\mathcal{H}} = \langle f, S^*h\rangle_{\mathcal{H}}.$$

On the other hand if $f \in \operatorname{dom} S$,

$$\begin{aligned}\langle Sf, h\rangle_{\mathcal{H}} &= \langle Nf, h\rangle_{\mathcal{K}} \\ &= \langle f, N^*h\rangle_{\mathcal{K}} \\ &= \langle f, P_{\mathcal{H}}N^*h\rangle_{\mathcal{H}},\end{aligned}$$

From the density of $\operatorname{dom} S$ in $\mathcal{H}$, we see that $P_{\mathcal{H}}N^*h = S^*h$. This proves (b). □

EXAMPLE 1.2.3 (Unbounded Toeplitz Operators). Denote by L^2 the space of square integrable functions with respect to normalized Lebesgue measure on the unit circle. Let H^2 denote the Hardy space of those functions in L^2 whose negative Fourier coefficients vanish. If φ is in L^2 and

$$\operatorname{dom} T_\varphi = \{f \in H^2 \colon\ \varphi f \in L^2\},$$

define the *unbounded Toeplitz operator*

$$T_\varphi \colon \operatorname{dom} T_\varphi \to H^2$$

with symbol φ by

$$T_\varphi \colon f \mapsto P\varphi f,$$

where P is the orthogonal projection of L^2 onto H^2. If φ is in H^2, then T_φ will be subnormal with the normal extension M_φ, the operator of multiplication by φ on L^2, defined with

$$\operatorname{dom} M_\varphi = \left\{ f \in L^2 \colon \ \varphi f \in L^2 \right\}.$$

EXAMPLE 1.2.4 (Bargmann-Segal Toeplitz Operators). Denote by $L^2(\mu)$ the space of square integrable functions with respect to the Gaussian measure defined on the Borel subsets of the complex plane $\mathbb{C}$ by $d\mu(z) \equiv \frac{1}{\pi} e^{-|z|^2} d\mathcal{A}(z)$, where $d\mathcal{A}(z)$ is the Lebesgue area measure. The *Bargmann space* $\mathcal{B}$ is the Hilbert space of entire functions which belong to $L^2(\mu)$. If φ is in $L^2(\mu)$ and

$$\operatorname{dom} T_\varphi = \left\{ f \in \mathcal{B} \colon \ \varphi f \in L^2(\mu) \right\},$$

define the *Bargmann-Segal Toeplitz* operator with symbol φ by

$$T_\varphi \colon f \mapsto P\varphi f,$$

where P is the orthogonal projection of $L^2(\mu)$ onto $\mathcal{B}$. If φ is an entire function and if N_φ denotes the operator of multiplication by φ on $L^2(\mu)$ with $\operatorname{dom} N_\varphi = \{f \in L^2(\mu): \ \varphi f \in L^2(\mu)\}$, then N_φ is a normal extension of T_φ. Hence T_φ is subnormal in the sense of Definition 1.2.1.

EXAMPLE 1.2.5 (The Creation Operator). If $\operatorname{dom} A_+$ denotes the Schwartz space on the real line, define the *creation operator*

$$A_+ : \operatorname{dom} A_+ \to L^2(\mathbb{R})$$

by

$$A_+ : f(x) \mapsto \frac{1}{\sqrt{2}}\left(x - \frac{d}{dx}\right) f(x).$$

From Bargmann [5] it follows that the normal extension of A_+ is unitarily equivalent to the normal operator of multiplication by the independent variable z on $L^2(\mu)$, where $L^2(\mu)$ is as in Example 1.2.4. Thus A_+ is an unbounded subnormal operator.

Our final example, unbounded Bergman operators, will be defined in the next section. Their properties will be the main concern of this study.

1.3. Unbounded Bergman Operators

Let G be an open subset of the complex plane $\mathbb{C}$, and denote by $L^2(G)$ the Hilbert space of square integrable functions with respect to

$d\mathcal{A}$, the Lebesgue area measure restricted to G. The inner product in $L^2(G)$ is defined by

$$\langle f, g\rangle_{L^2(G)} = \int_G f(z)\overline{g(z)}\, d\mathcal{A}(z),$$

for all f , g in $L^2(G)$. The *Bergman space* $L^2_a(G)$ consists of all those functions in $L^2(G)$ that are also analytic on G. It is well known that $L^2_a(G)$ is closed subspace of $L^2(G)$, thus a Hilbert space.

DEFINITION 1.3.1. If φ is an analytic function on G and

$$\operatorname{dom} M_{\varphi,G} = \{f \in L^2_a(G): \quad \varphi f \in L^2(G)\},$$

define

$$M_{\varphi,G}\colon \operatorname{dom} M_{\varphi,G} \to L^2_a(G)$$

by

$$M_{\varphi,G}\colon f \mapsto \varphi f.$$

In particular if $\varphi(z) = z$, then $M_{z,G}$ is called the *(unbounded) Bergman operator* and it will be denoted by S_G.

Whenever there is no ambiguity regarding the corresponding open set G, $M_{\varphi,G}$ and S_G will be denoted by M_φ and S respectively. Clearly if G is a bounded open set in $\mathbb{C}$, then S_G agrees with the usual definition of the Bergman operator.

Suppose $\mathcal{H}$ and $\mathcal{K}$ are Hilbert spaces. An *isometry* $V\colon \mathcal{H} \to \mathcal{K}$ is an operator such that $\|Vh\|_{\mathcal{K}} = \|h\|_{\mathcal{H}}$ for all h in $\mathcal{H}$. An *isomorphism* $U\colon \mathcal{H} \to \mathcal{K}$ of $\mathcal{H}$ onto $\mathcal{K}$ is a surjective isometry. $\mathcal{H}$ and $\mathcal{K}$ are said to be *isomorphic,* if there is an isomorphism U between $\mathcal{H}$ and $\mathcal{K}$. We will reserve the notation of a *unitary operator* for an isomorphism with the same domain and range. The next result is important in the sense that it will be used throughout this study.

THEOREM 1.3.2. *If G is an open subset of $\mathbb{C}$ and $\varphi\colon G \to \Omega$ is a one-to-one analytic function that maps G onto Ω, then the following statements hold.*

(a) *$L^2_a(\Omega)$ is isomorphic to $L^2_a(G)$ via the isomorphism*

$$U\colon L^2_a(\Omega) \to L^2_a(G),$$

defined by

$$\big(Uf\big)(z) = \varphi'(z) f(\varphi(z)).$$

(b) $U\big(\operatorname{dom} S_\Omega\big) = \operatorname{dom} M_{\varphi,G}$.

If in addition the Bergman operator S_Ω is densely defined ,then

(c) $U\big(\operatorname{dom} S^*_\Omega\big) = \operatorname{dom} M^*_{\varphi,G}$;

(d) $U\big(\operatorname{dom} [S^*_\Omega, S_\Omega]\big) = \operatorname{dom} [M^*_{\varphi,G}, M_{\varphi,G}]$.

PROOF. First note that by the Open Mapping Theorem, Ω is open. Furthermore since φ is one-to-one and analytic, φ' does not vanish on

G. If we introduce the Cartesian coordinates $w = u+iv$, then it follows easily that the Jacobian of the change of variables for the mapping $w = \varphi(z)$ is given by $|\varphi'(z)|^2$. For f in $L_a^2(\Omega)$,

$$\begin{aligned}\|Uf\|_{L_a^2(G)}^2 &= \int_G \big|(Uf)(z)\big|^2\,dx\,dy \\ &= \int_G \big|f(\varphi(z))\big|^2|\varphi'(z)|^2\,dx\,dy \\ &= \|f\|_{L_a^2(\Omega)}^2.\end{aligned}$$

Thus U is an isometry. Moreover if g is in $L_a^2(G)$, and if $f\colon \Omega \to \mathbb{C}$ is defined by $f(w) = \big[(\frac{1}{\varphi}\ g) \circ \varphi^{-1}\big](w)$, then it follows easily that f belongs to $L_a^2(\Omega)$ and $Uf = g$. Hence U is also surjective. This completes the proof of part(a). Part (b) follows from the equality

$$\begin{aligned}\int_\Omega |wf(w)|^2\,du\,dv &= \int_G |\varphi(z)f(\varphi(z))|^2|\varphi'(z)|^2\,dx\,dy \\ &= \int_G |\varphi(z)(Uf)(z)|^2\,dx\,dy.\end{aligned}$$

Before proving (c) and (d), we make a couple of observations.First note that, in accordance with Definition 1.1.1, the assumption regarding the density of $\operatorname{dom} S_\Omega$ in $L_a^2(\Omega)$ is necessary in order to have a unique adjoint operator S_Ω^*. Next if $\mathcal{F}$ is a dense subset of $\mathcal{H}$, then $U[\mathcal{F}]$ is dense in $\mathcal{K}$ for every isomorphism U from $\mathcal{H}$ onto $\mathcal{K}$. Thus it follows from (b) that $\operatorname{dom} M_{\varphi,G}$ is also dense in $L_a^2(G)$. Now if $f \in \operatorname{dom} S_\Omega^*$,

then for all g in $\operatorname{dom} S_\Omega$ we have

$$\begin{aligned}\langle Sg, f\rangle_{L^2_a(\Omega)} &= \int_\Omega wg(w)\overline{f(w)}\,du\,dv \\ &= \int_G \varphi(z)g(\varphi(z))\varphi'(z)\overline{f(\varphi(z))\varphi'(z)}\,dx\,dy \\ &= \langle M_\varphi Ug, Uf\rangle_{L^2_a(G)}.\end{aligned}$$

From the Cauchy-Schwartz inequality and (b), we see that Uf is in $\operatorname{dom} M_\varphi^*$. The other inclusion is proved in the same way. This proves (c). Actually the proofs of parts (b) and (c) show also that; $USf = M_\varphi Uf$ for all f in $\operatorname{dom} S_\Omega$, and $US^*g = M_\varphi^* Ug$ for all g in $\operatorname{dom} S_\Omega^*$. So if f is in $\operatorname{dom} S^*S$ or equivalently $f \in \operatorname{dom} S$ and $Sf \in \operatorname{dom} S^*$, then by (b) $Uf \in \operatorname{dom} M_\varphi$. Moreover by the above remark we have that

$$M_\varphi Uf = USU^{-1}Uf = USf.$$

Therefore $M_\varphi Uf$ is in $U\big(\operatorname{dom} S^*\big)$. Using part (c), we conclude that $Uf \in \operatorname{dom} M_\varphi^* M_\varphi$. To prove the other inclusion let $g \in \operatorname{dom} M_\varphi^* M_\varphi$, then we have that $U^{-1}g$ is in $\operatorname{dom} S$ and

$$SU^{-1}g = U^{-1}M_\varphi UU^{-1}g = U^{-1}M_\varphi g.$$

So $U^{-1}g$ is in $\operatorname{dom} S^*S$ and therefore

$$U\big(\operatorname{dom} S^*S\big) = \operatorname{dom} M_\varphi^* M_\varphi.$$

The proof of $U\big(\operatorname{dom} SS^*\big) = \operatorname{dom} M_\varphi M_\varphi^*$ follows by the same sort of argument. Thus

$$\operatorname{dom}[M_\varphi^*, M_\varphi] = \Big[U\big(\operatorname{dom} S^*S\big)\Big] \cap \Big[U\big(\operatorname{dom} SS^*\big)\Big]$$

$$= U\big(\operatorname{dom}[S^*, S]\big),$$

which proves (d). □

COROLLARY 1.3.3. *Assume G is an open subset of $\mathbb{C}$, and let $r \neq 0$, c denote complex numbers. If $T\colon \mathbb{C} \to \mathbb{C}$ is defined by $T\colon z \mapsto rz + c$, and if Ω denotes the image of G under the linear mapping T, then* $\operatorname{dom} S_G$ *is dense in* $L_a^2(G)$ *if and only if* $\operatorname{dom} S_\Omega$ *is dense in* $L_a^2(\Omega)$.

PROOF. It is easy to see that $\operatorname{dom} M_{T,G} = \operatorname{dom} S_G$. Now the result follows from part(b) of Theorem 1.3.2. □

Corollary 1.3.3 shows that problem of density of the Bergman operator is invariant under rotation, scaling, and translation. This fact will be used throughout.

1.4. Some Results from the Theory of H^p Spaces

DEFINITION 1.4.1. If $0 < p < \infty$, the *Hardy space* H^p is the set of all analytic functions in $\mathbb{D}$ such that

$$\sup_{r<1}\left\{\frac{1}{2\pi}\int_0^{2\pi} |f(re^{it})|^p\right\} < \infty.$$

In the case $p = \infty$, H^∞ is the set of all bounded analytic functions in $\mathbb{D}$.

THEOREM 1.4.2 (Factorization Theorem). *If $p > 0$ and $f \in H^p$, then f has a unique factorization of the form $f(z) = B(z)S(z)F(z)$, where $B(z)$ is a Blaschke product, $S(z)$ is a singular inner function, and $F(z)$ is an outer function in H^p defined by*

$$F(z) = \alpha \exp\left(\frac{1}{2\pi}\int_0^{2\pi} \frac{e^{it}+z}{e^{it}-z} \log|f(e^{it})|\, dt\right),$$

where $|\alpha| = 1$.

PROOF. For a proof see Duren [**10**], Theorem 2.8 on page 24. □

Next we state the well known Beurling's Theorem in the form that is suited for our purposes. It should be mentioned that the given version is only one of many consequences of the theorem. For a proof of theorem see Duren [**10**], Theorem 7.4 on page 114.

THEOREM 1.4.3 (Beurling). *If $1 \leq p < \infty$ and f is an outer function in H^p, then* $\operatorname{cl}\{pf\colon\ p \text{ is a polynomial}\} = H^p$.

DEFINITION 1.4.4. A function defined on an open set is said to be *univalent* if it is analytic and one-to-one. Denote by $\mathcal{S}$ the class of all univalent functions f in $\mathbb{D}$ normalized by the conditions $f(0) = 0$, and $f'(0) = 1$.

The next result for the functions in the $\mathcal{S}$ class is known as the Koebe Distortion Theorem.

THEOREM 1.4.5. *If $f \in \mathcal{S}$, then*

$$\frac{|z|}{(1+|z|)^2} \leq |f(z)| \leq \frac{|z|}{(1-|z|)^2}$$

for all z in $\mathbb{D}$.

PROOF. For a proof, see Conway [**9**], Theorem 7.9 on page 65. □

THEOREM 1.4.6. *If f is univalent in $\mathbb{D}$, then $f \in H^p$ for all $p < \frac{1}{2}$ and furthermore the singular factor of f is constantly* 1.

PROOF. A proof can be found in Duren [**10**], Theorems 3.16 and 3.17 on pages 50–51. □

COROLLARY 1.4.7. *If $f \in \mathcal{S}$, then f has the unique factorization $f(z) = zF(z)$, where $F(z)$ is an outer function in H^p for all $p < \frac{1}{2}$.*

PROOF. Note that since f is univalent and it only vanishes at 0, it follows that the Blaschke factor of f equals z. The rest of the proof follows from Theorem 1.4.6. □

1.5. Some Results from Analytic Function Theory

We start this section by recalling some basic definitions and facts that will be used throughout. For a set G in the complex plane $\mathbb{C}$, the

usual *boundary* of G is denoted by ∂G, and its *complement* is written as $\mathbb{C}\backslash G$. $\mathbb{C}_\infty$ denotes the extended complex numbers $\mathbb{C}\cup\{\infty\}$. If G is a set in $\mathbb{C}$, denote by $\partial_\infty G$ the *extended boundary* of G with respect to $\mathbb{C}_\infty$. In other words, $\partial_\infty G = \partial G$ when G is bounded and $\partial_\infty G = \partial G \cup \{\infty\}$, if G is unbounded. The *closure* of a set G in $\mathbb{C}$ is $\operatorname{cl} G$. If $r > 0$, denote by $B(z,r)$ the *open disc* in $\mathbb{C}$ with center at z and radius r. $\mathbb{D}$ stands for the *open unit disc*. Finally, for an open set G in $\mathbb{C}$, the class of all analytic functions in G will be denoted by $\operatorname{Hol}(G)$.

DEFINITION 1.5.1. A function $f(z)$ is said to be analytic in a neighborhood of ∞, if $f(\frac{1}{z})$ is analytic in a neighborhood of 0. If f is analytic in a neighborhood of infinity, then

$$f(z) = a_0 + a_1 z^{-1} + a_2 z^{-2} + \cdots$$

where the series converges uniformly for $|z| \geq R$ and R sufficiently large. Furthermore, we have that

$$f(\infty) = a_0,$$

and

$$f'(\infty) = \lim_{z\to\infty} z\big[f(z) - f(\infty)\big].$$

Recall that the Jordan Curve Theorem states that a simple closed curve in $\mathbb{C}$, say γ, divides the plane into two disjoint components where

one is bounded and the other is unbounded. Denote by $\operatorname{ins}\gamma$ and $\operatorname{out}\gamma$, the bounded and the unbounded components of $\mathbb{C}\setminus\{\gamma\}$, respectively.

The *winding number* of a point a in $\mathbb{C}\setminus\{\gamma\}$ with respect to γ is denoted by $n(\gamma;a)$. A curve γ is said to be *smooth* if γ is continuously differentiable function and $\gamma'(t)$ is non-zero for all t.

DEFINITION 1.5.2. By a *smooth positively oriented Jordan system*, we mean a collection $\gamma = \{\gamma_1,\ldots,\gamma_m\}$ of pairwise disjoint smooth Jordan curves such that for a in $\mathbb{C}\setminus\Gamma$ we have that

$$n(\Gamma;a)=\sum_{j=1}^{m} n(\gamma_j;a)=0 \quad \text{or} \quad 1.$$

Denote by $\operatorname{out}\Gamma$ the set of all points a in $\mathbb{C}\setminus\Gamma$ such that $n(\Gamma;z)=0$; and let $\operatorname{ins}\Gamma$ denote the set of points a in $\mathbb{C}\setminus\Gamma$ such that $n(\Gamma;z)=1$.

THEOREM 1.5.3. *If K is a compact subset of an open set G, then there is a positive Jordan system Γ in $G\setminus K$ such that $K\subseteq\operatorname{ins}\Gamma\subseteq G$.*

PROOF. A proof can be found in Conway [**9**], Proposition 1.8 on page 13. □

The next result is well known. We present, however, a proof to obtain (1.5.5), and (1.5.6) for later references.

THEOREM 1.5.4. *Let G be an open set in $\mathbb{C}$, and suppose that K is a compact subset of G. If f is in $\operatorname{Hol}(G\setminus K)$, then there are unique*

functions $f_1 \in \mathrm{Hol}(G)$ *and* $f_2 \in \mathrm{Hol}(\mathbb{C}_\infty \setminus K)$ *such that* $f_2(\infty) = 0$ *and* $f \equiv f_1 + f_2$ *on* $G \setminus K$.

PROOF. For z in G, let Γ_1 denote a positive Jordan system in $G \setminus K$ such that $K \cup \{z\}$ belongs to $\mathrm{ins}\,\Gamma_1$. Define $f_1 \colon G \to \mathbb{C}$ by

$$f_1(z) = \frac{1}{2\pi i} \int_{\Gamma_1} \frac{f(w)}{w - z}\, dw. \tag{1.5.5}$$

Next for z in $\mathbb{C} \setminus K$, let Γ_2 denote a positive Jordan system in $G \setminus K$ such that $z \in \mathrm{out}\,\Gamma_2$, and $K \subseteq \mathrm{ins}\,\Gamma_2$. Define $f_2 \colon \mathbb{C} \setminus K \to \mathbb{C}$ by

$$f_2(z) = \frac{-1}{2\pi i} \int_{\Gamma_2} \frac{f(w)}{w - z}\, dw. \tag{1.5.6}$$

From the Cauchy Integral Formula, we see that f_1 and f_2 are independent of the choice of contours and they satisfy the assertion of the theorem. □

COROLLARY 1.5.7. *If* $f \in \mathrm{Hol}(\mathbb{C}_\infty \setminus K)$ *and* $f(\infty) = 0$, *then*

$$f(z) = \frac{-1}{2\pi i} \int_{\Gamma} \frac{f(w)}{w - z}\, dw,$$

where Γ *is a positive Jordan system such that* $z \in \mathrm{out}\,\Gamma$ *and* $K \subseteq \mathrm{ins}\,\Gamma$.

PROOF. Letting $G = \mathbb{C}$ in the Theorem 1.5.4, we have that f_1 is an entire function. But since $f_1 \equiv f - f_2$ in $\mathbb{C} \setminus K$, it follows that $f_1(\infty) = 0$. By Liouville's Theorem $f_1 \equiv 0$. □

1.6. Logarithmic Capacity

This section contains the definition of logarithmic potential and its corresponding capacity. We will also present some basic properties of logarithmic capacity which are essential to this study. Almost all the results in this section are given without proofs. We will, however, give a reference to each stated result. For more extensive studying on this topic, we refer to any classical book on this subject. See, for example, Landkof [**15**], Carleson [**6**], and Conway [**9**].

First of all we fix the following notation which will be used throughout. For a compact set K in $\mathbb{C}$, by $\mathrm{M}(K)$ we mean the class of all complex-valued regular Borel measures that are supported on K. The cone of positive elements in $\mathrm{M}(K)$, that is the set of positive regular Borel measures on K, is denoted by $\mathrm{M}^+(K)$.

DEFINITION 1.6.1. If K is compact and $\mu \in M(K)$, then the *logarithmic potential* of μ is defined as the function

$$U_\mu(z) = \int \log \frac{1}{|z-w|}\, d\mu(w).$$

Note that since U_μ is the convolution of the locally integrable function $\log \frac{1}{|z|}$ with the compactly supported measure μ, then U_μ is defined everywhere on the complex plane.

DEFINITION 1.6.2. For a compactly supported real-valued measure μ on $\mathbb{C}$ such that U_μ belongs to $L_1(|\mu|)$, define the *energy integral* of μ by

$$\mathrm{I}(\mu) = \iint \log \frac{1}{|z-w|}\, d\mu(w)\, d\mu(z).$$

Now we can give the definition of logarithmic capacity.

DEFINITION 1.6.3. If K is a compact set, the *logarithmic capacity* of K is defined by

$$c(K) := \sup_{\mu} e^{-\mathrm{I}(\mu)},$$

where the supremum is taken over all Borel probability measures μ in $\mathrm{M}^+(K)$.

In above definition, it is understood that $e^{-\infty} = 0$; that is, $c(K) = 0$ exactly when $\mathrm{I}(\mu) = +\infty$ for all measures in $\mathrm{M}^+(K)$. For non-compact sets E, the definition of logarithmic capacity can be extended by

$$c(E) := \sup_{K}\ c(K),$$

where the supremum is taken over all compact subsets K of E.

The next proposition is a collection of some elementary properties of logarithmic capacity. A proof can be found in Conway [**9**], Proposition 7.4 on page 334.

PROPOSITION 1.6.4. *The following statements hold.*

(a) *If* $E_1 \subseteq E_2$, *then* $c(E_1) \leq c(E_2)$.

(b) *If* $T(z) = az + b$, $a \neq 0$, *then* $c\big(T(E)\big) = |a|\, c(E)$.

(c) *If* E *is a Borel set, then* E *has positive capacity if and only if there is a positive measure* μ *with* $\mathrm{I}(\mu) < \infty$ *and* $\mu(E) > 0$.

(d) *If* K *is a compact set having positive capacity, then there is a unique measure* ν *in* $\mathrm{M}^+(K)$ *such that* $c(K) = e^{-\mathrm{I}(\nu)}$.

The unique probability measure in Proposition 1.6.4, part(d), is called the *equilibrium measure* for K. Though logarithmic capacity is not a measure, nevertheless the next result can be considered as a statement regarding the finer structure of Borel sets of zero capacity compared to Borel sets with zero Lebesgue measure.

THEOREM 1.6.5. *Let* K *be a compact set with* $c(K) > 0$, *and suppose that* μ *is a compactly supported real-valued Borel measure on* K *such that* $\mathrm{I}(|\mu|) < \infty$. *If* E *is a Borel set such that* $c(E) = 0$, *then* $|\mu|(E) = 0$.

PROOF. See, for example, Conway [**9**], Corollary 7.6 on page 335.

□

For compact sets, there are methods which determine the logarithmic capacity either exactly or as an estimate. Some of these results are gathered in the next proposition.

PROPOSITION 1.6.6. *The following statements hold.*

(a) *If* $a \in \mathbb{C}$, *then* $c(\{a\}) = 0$.

(b) *If* $r \geq 0$, *then* $c\big(\operatorname{cl} B(0,r)\big) = r$.

(c) *If* K *is a connected compact set, then* $c(K) \geq \frac{1}{4} \operatorname{diam} K$.

(d) *If* Γ *is a rectifiable Jordan curve, then* $c(\Gamma) \leq \ell(\Gamma)/4$, *where* $\ell(\Gamma)$ *denotes the length of* Γ.

(e) *If* E *is a Borel set, then* $c(E) \geq \sqrt{\dfrac{\operatorname{Area}(E)}{\pi e}}$.

PROOF. See Conway [**9**], Proposition 7.8 on page 335, Corollary 10.4 on page 354, and Corollary 11.7 on page 359. □

We conclude this section by stating an important theorem. As we will see in Chapter 2, the next result gives an exact characterization of those open sets G for which the corresponding Bergman space $L^2_a(G)$ is trivial.

THEOREM 1.6.7. *If* K *is compact, then* $L^2_a(\mathbb{C} \setminus K)$ *is trivial if and only if* K *has zero logarithmic capacity.*

PROOF. For a proof, see Carleson [**6**], Theorem 1 on page 73. □

CHAPTER 2

An Analytic Approach to the Density Problem

2.1. Removable Boundary Points and the Density Problem

DEFINITION 2.1.1. Let G be an open set in $\mathbb{C}$. A point $\lambda \in \partial_\infty G$ is called a *removable boundary point* for $L_a^2(G)$ if there is a neighborhood U of λ such that each f in $L_a^2(G)$ has an analytic extension to $G \cup U$. Denote by $\partial_{2-r}(G)$ the set of all removable boundary points for $L_a^2(G)$. A point λ in $\partial_\infty G$ is an *essential boundary point* for $L_a^2(G)$ if it is not removable; and the set of all essential boundary points of $L_a^2(G)$ is denoted by $\partial_{2-e}(G)$.

The next theorem is a collection of results regarding the properties of the boundary points. For a proof, we refer the reader to Axler, Conway, and McDonald [**3**].

THEOREM 2.1.2. *Let G be an open set in $\mathbb{C}$.*

(a) *If λ is an an isolated point of $\partial_\infty G$, then $\lambda \in \partial_{2-r}(G)$.*

(b) *The set $G \cup \partial_{2-r}(G)$ is an open subset of $\mathbb{C}_\infty$.*

(c) *If $\lambda \in \partial_\infty G$, then $\lambda \in \partial_{2-r}(G)$ if and only if there is a neighborhood U of λ such that $\operatorname{cl} U \setminus G$ has zero logarithmic capacity.*

LEMMA 2.1.3. *If G is an unbounded open set in $\mathbb{C}$ and $\infty \in \partial_{2-r}(G)$, then there is a compact set K in $\mathbb{C} \setminus G$ such that $L^2_a(G) = L^2_a(\mathbb{C} \setminus K)$.*

PROOF. By Theorem 2.1.2, part(c), there is a neighborhood U_∞ of infinity such that $c(\operatorname{cl} U_\infty \setminus G) = 0$. Choose $R > 0$, large enough, that $U_R = \{z \in \mathbb{C}\colon |z| > R\}$ is a subset of U_∞.

If we set $K = G^c \cap \operatorname{cl} B(0, R)$, then it follows that K is compact, and $G \cup U_R = \mathbb{C} \setminus K$. Now let $f \in L^2_a(G)$, then by assumption f has an analytic extension to $G \cup U_R$. Denoting this extension also by f, we have that $f \in \operatorname{Hol}(\mathbb{C} \setminus K)$, and moreover

$$\begin{aligned}
\|f\|^2_{L^2_a(\mathbb{C}\setminus K)} &= \int_{G\cup U_R} |f(z)|^2 \, d\mathcal{A}(z) \\
&\leq \int_G |f(z)|^2 \, d\mathcal{A}(z) + \int_{U_R} |f(z)|^2 \, d\mathcal{A}(z) \\
&\leq \|f\|^2_{L^2_a(G)} + \int_G |f(z)|^2 \, d\mathcal{A}(z) + \int_{U_R \setminus G} |f(z)|^2 \, d\mathcal{A}(z) \\
&= 2\|f\|^2_{L^2_a(G)} + \int_{U_R \setminus G} |f(z)|^2 \, d\mathcal{A}(z).
\end{aligned}$$

But since $\operatorname{cl} U_R \setminus G \subseteq \operatorname{cl} U_\infty \setminus G$, it follows that $c(\operatorname{cl} U_R \setminus G) = 0$. By Proposition 1.6.6, part(e), Area $(\operatorname{cl} U_R \setminus G) = 0$, and consequently

$$\int_{U_R \setminus G} |f(z)|^2 \, d\mathcal{A}(z) = 0.$$

Thus $\|f\|^2_{L^2_a(\mathbb{C}\setminus K)} \leq 2\|f\|^2_{L^2_a(G)}$. Now since $f \in L^2_a(\mathbb{C} \setminus K)$ is arbitrary, we conclude that $L^2_a(G) \subseteq L^2_a(\mathbb{C} \setminus K)$. The other inclusion that

$L_a^2(\mathbb{C} \setminus K) \subseteq L_a^2(G)$, follows from the fact that $G \subseteq \mathbb{C} \setminus K$. This completes the proof. □

The next theorem states exactly when the Bergman space is non-trivial. This result is an extension of Theorem 1.6.7 . For an alternative proof see also Adams and Hedberg [1] , Proposition 11.1.1 on page 306.

THEOREM 2.1.4. *If G is an open set in $\mathbb{C}$, then $L_a^2(G)$ is trivial if and only if $\mathbb{C} \setminus G$ has zero logarithmic capacity.*

PROOF. First we assume that $L_a^2(G)$ is trivial. Let K be an arbitrary compact subset of $\mathbb{C} \setminus G$. Since $L_a^2(\mathbb{C} \setminus K) \subseteq L_a^2(G)$, it follows that $L_a^2(\mathbb{C} \setminus K)$ is trivial. By Theorem 1.6.7 , this happens only if $c(K) = 0$. Since K is arbitrary, from Definition 1.6.3, we see that $c(\mathbb{C} \setminus G) = 0$.

Next suppose $c(\mathbb{C} \setminus G) = 0$. It is clear that G is necessarily unbounded. For any neighborhood U of infinity, it follows from the assumption that $c(\mathrm{cl}\, U \setminus G) = 0$. Now by Theorem 2.1.2, part(c), $\infty \in \partial_{2-r}(G)$. Lemma 2.1.3 implies that $L_a^2(G) = L_a^2(\mathbb{C} \setminus K)$, for some compact set K belonging to $\mathbb{C} \setminus G$. But since $c(K) = 0$, we must have that $L_a^2(\mathbb{C} \setminus K)$ is trivial by Theorem 1.6.7. □

The following proposition shows that an unbounded Bergman operator is not necessarily densely defined.

PROPOSITION 2.1.5. *If K is a compact set in $\mathbb{C}$ such that $L^2_a(\mathbb{C}\setminus K)$ is non-trivial, then the Bergman operator $S_{\mathbb{C}\setminus K}$ is not densely defined.*

PROOF. First note that ∞ is an isolated point of $\partial_\infty(\mathbb{C}\setminus K)$, so by Theorem 2.1.2, part(a), we have that $\infty \in \partial_{2-r}(\mathbb{C}\setminus K)$. Let Γ denote a smooth Jordan system in $\mathbb{C}\setminus K$ such that $K \subseteq \operatorname{ins}\Gamma$. By Corollary 1.5.7, if $f \in L^2_a(\mathbb{C}\setminus K)$ and $z \in \operatorname{out}\Gamma$, then f can be written as

$$f(z) = \frac{-1}{2\pi i}\int_\Gamma \frac{f(w)}{w-z}\,dw.$$

Next choose $R > 0$ sufficiently large that $\operatorname{dist}(z,\Gamma) > 1$, whenever z is in $\mathbb{C}$ and $|z| \geq R$. It follows that for f in $L^2_a(\mathbb{C}\setminus K)$,

$$f(z) = \frac{1}{2\pi i}\,z^{-1}\int_\Gamma \frac{f(w)}{1-\frac{w}{z}}\,dw = \sum_{n=0}^{\infty}\left(\frac{1}{2\pi i}\int_\Gamma w^n f(w)\,dw\right)\frac{1}{z^{n+1}}$$

for all z in $\mathbb{C}$ such that $|z| \geq R$.

CLAIM. *If $\operatorname{dom} S_{\mathbb{C}\setminus K}$ is dense in $L^2_a(\mathbb{C}\setminus K)$, then it follows that $\int_\Gamma w^n f(w)\,dw = 0$ for all f in $L^2_a(\mathbb{C}\setminus K)$, and $n = 0, 1, 2, \ldots$.*

Assume for a moment that the claim holds, then each f in $L^2_a(\mathbb{C}\setminus K)$ must vanish in some neighborhood of infinity. If $\mathbb{C}\setminus K$ is connected, by analycity f must vanish identically on $\mathbb{C}\setminus K$; that is $L^2_a(\mathbb{C}\setminus K)$ is trivial, a contradiction. If $\mathbb{C}\setminus K$ is not connected and G_0 denotes the unbounded component of $\mathbb{C}\setminus K$, the above argument still implies that

every f in $L^2_a(\mathbb{C} \setminus K)$ must vanish on G_0. But the function f defined by $f = (z-a)^{-2}$ on G_0 and $f = 0$ on $(\mathbb{C} \setminus K) \setminus G_0$, where a is a point in a bounded component of $\mathbb{C} \setminus K$, clearly belongs to $L^2_a(\mathbb{C} \setminus K)$ and non-zero on G_0 which is absurd. This proves the proposition.

To prove the claim, note that for f in $L^2_a(\mathbb{C}\setminus K)$, f and its derivative vanish at infinity. But

$$\begin{aligned} f'(\infty) &= \lim_{z\to\infty} zf(z) \\ &= \lim_{z\to\infty} \frac{1}{2\pi i} \int_\Gamma \frac{f(w)}{1-\frac{w}{z}}\, dw \\ &= \frac{1}{2\pi i} \int_\Gamma f(w)\, dw, \end{aligned}$$

where in the last equality we have used the fact that $\dfrac{1}{1-\frac{w}{z}} \to 1$ uniformly on Γ as $z \to \infty$. Thus $\int_\Gamma f(w)\, dw = 0$.

Next assume that $\int_\Gamma w^n f(w)\, dw = 0$ for each f in $L^2_a(\mathbb{C} \setminus K)$ and all $n \geq 0$. Since $zg \in L^2_a(\mathbb{C} \setminus K)$ for all g in $\operatorname{dom} S_{\mathbb{C}\setminus K}$, it follows that $\int_\Gamma w^{n+1} g(w)\, dw = 0$ whenever $g \in \operatorname{dom} S_{\mathbb{C}\setminus K}$.

Now we use the density assumption. Given f in $L^2_a(\mathbb{C} \setminus K)$, denote by $\{g_m\}$ a sequence in $\operatorname{dom} S_{\mathbb{C}\setminus K}$ such that $g_m \to f$ in $L^2_a(\mathbb{C}\setminus K)$. Since it also follows that $g_m \to f$ uniformly on compact subsets of $\mathbb{C}\setminus K$, the inequality

$$\left| \int_\Gamma w^{n+1} f(w)\, dw \right| \leq \int_\Gamma |w|^{n+1} |f(w) - g_m(w)|\, dw + \left| \int_\Gamma w^{n+1} g_m(w)\, dw \right|$$

implies that $\int_\Gamma w^{n+1} f(w)\, dw = 0$. Thus the claim follows by induction on $n \geq 0$. □

Now we can state the main result of this section.

THEOREM 2.1.6. *If G is an unbounded open set in $\mathbb{C}$ for which $L^2_a(G)$ is non-trivial and if the Bergman operator S_G is densely defined, then ∞ belongs to $\partial_{2-e}(G)$.*

PROOF. The proof follows easily by contradiction. To see this, assume that ∞ is a removable boundary point for $L^2_a(G)$. Then by Lemma 2.1.3, $L^2_a(G) = L^2_a(\mathbb{C}\setminus K)$ for some compact subset K of $\mathbb{C}\setminus G$. Now if $L^2_a(\mathbb{C}\setminus K)$ is non-trivial, from Proposition 2.1.5, we see that $\operatorname{dom} S_G = \operatorname{dom} S_{\mathbb{C}\setminus K}$ is not dense in $L^2_a(G)$, a contradiction. □

2.2. A Geometric Condition for the Density Problem

In this section we prove one our main main results of this chapter, namely Theorem 2.2.1. For reasons that will become clear in the next section,Theorem 2.2.1 is the best sufficient condition one would expect from the geometrical point of view.

THEOREM 2.2.1. *If G is an open set in $\mathbb{C}$ and the component of its complement with respect to $\mathbb{C}_\infty$ that contains ∞ does not equal the singleton $\{\infty\}$, then the Bergman operator S_G is densely defined.*

Before proving the theorem we need a pair of lemmas.

LEMMA 2.2.2. *If Ω is an open subset of $\mathbb{D}$, and if F is a bounded outer function in $H^\infty(\mathbb{D})$, then $F[L^p_a(\Omega)]$ is a dense subset of $L^p_a(\Omega)$ for all $p \geq 1$.*

PROOF. In order to prove the lemma we need the following result due to Rubel and Shields [**17**], see Theorem 5.1 on page 265. *If F is a bounded outer function in $H^\infty(\mathbb{D})$, then $F[H^\infty(\mathbb{D})]$ is weak* sequentially dense in $H^\infty(\mathbb{D})$.*

Now in view of the Rubel and Shields's Theorem, there exists a sequence $\{h_n\}$ in $H^\infty(\mathbb{D})$ such that $Fh_n \to 1$ weak* in $L^\infty(\mathbb{D})$. It is a well known fact that a sequence $\{f_n\}$ in $H^\infty(\mathbb{D})$ converges weak* to f if and only if $\sup_{n\geq 1} \|f_n\|_\infty < \infty$ and $f_n(z) \to f(z)$ for all z in $\mathbb{D}$ (See Conway [**9**], Proposition 3.15 on page 283).

So in our case

$$\sup_{n\geq 1} \|Fh_n\|_\infty < \infty \quad \text{and} \quad F(z)h_n(z) \to 1 \quad \text{for all } z \text{ in } \mathbb{D}.$$

Now if $f \in L^p_a(\Omega)$ and if $g_n \colon \Omega \to \mathbb{C}$ is defined by $g_n \equiv Fh_nf$, then $g_n \in L^p_a(\Omega)$ for all $n \geq 1$. Thus

$$\|g_n - f\|^p_{L^p_a(\Omega)} = \int_\Omega |F(z)h_n(z) - 1|^p \, |f(z)|^p \, d\mathcal{A}(z).$$

For convenience put $\Theta_n(z) = |F(z)h_n(z) - 1|^p \, |f(z)|^p$. From the weak* convergence, we have

$$\Theta_n(z) \to 0 \quad \text{for all } z \text{ in } \mathbb{D}.$$

Furthermore

$$\begin{aligned} \Theta_n(z) &\leq \big(|F(z)h_n(z)| + 1\big)^p \, |f(z)|^p \\ &\leq \left(\sup_{n\geq 1} \|Fh_n\|_\infty + 1\right)^p |f(z)|^p . \end{aligned}$$

By assumption $|f|^p \in L^1(\Omega)$, thus from the Dominated Convergence Theorem it follows that $\Theta_n \to 0$ in $L^1(\Omega)$; that is, $g_n \to f$ in $L^p_a(\Omega)$. This proves the lemma. □

Recall that by $\mathcal{S}$ we denote the class of all one-to-one analytic functions f on $\mathbb{D}$ subject to conditions: $f(0) = 0$ and $f'(0) = 1$.

LEMMA 2.2.3. *If Ω denotes an open subset of $\mathbb{D}$ and φ belongs to $\mathcal{S}$, then* dom M_φ *is dense in $L^2_a(\Omega)$.*

PROOF. In view of Corollary 1.4.7, φ can be factorized as $zF(z)$, where F is an outer function in H^p for all $p \leq \frac{1}{2}$. Since F does not vanish in $\mathbb{D}$, $\dfrac{1}{F}$ is holomorphic in $\mathbb{D}$. Furthermore as a consequence of the Distortion Theorem 1.4.5

$$\left|\frac{1}{F(z)}\right| = \left|\frac{z}{\varphi(z)}\right| \leq \big(1 + |z|\big)^2,$$

for all z in $\mathbb{D}$. Hence $\frac{1}{F} \in H^\infty$.

Next if $f \in L^2_a(\Omega)$, then

$$\|\varphi \frac{1}{F} f\|^2_{L^2_a(\Omega)} = \int_\Omega \left| zF(z) \frac{1}{F(z)} f(z) \right|^2 d\mathcal{A}(z) \leq \|f\|^2_{L^2_a(\Omega)}.$$

So $\frac{1}{F} f$ is in $\operatorname{dom} M_\varphi$, and consequently $\frac{1}{F}[L^2_a(\Omega)] \subseteq \operatorname{dom} M_\varphi$. Now from the definition of an outer function, along with the fact that $\frac{1}{F} \in H^\infty$, we see that $\frac{1}{F}$ is a bounded outer function. Hence the result follows from Lemma 2.2.2 for the case $p = 2$. □

PROOF OF THEOREM 2.2.1. Let E denote the component of $\mathbb{C}_\infty \setminus G$ that contains ∞. If we set $\Lambda = \mathbb{C}_\infty \setminus E$, then it follows that Λ is a region in $\mathbb{C}_\infty$. Since E consists of more than one point we have that Λ is a simply connected region in $\mathbb{C}$ which is not the whole plane. Note also that by definition G is a subset of Λ.

Fix a point a in G. In light of the Riemann Mapping Theorem, there is a unique conformal mapping $\varphi\colon \mathbb{D} \to \Lambda$ from $\mathbb{D}$ onto Λ such that $\varphi(0) = a$ and $\varphi'(0) > 0$. Now if we put $\Omega = \varphi^{-1}(G)$, then it follows that Ω is an open subset of the unit disc. By Theorem 1.3.2, part(b), $\operatorname{dom} S_G$ is dense in $L^2_a(G)$ if and only if $\operatorname{dom} M_{\varphi,\Omega}$ is dense in $L^2_a(\Omega)$. Therefore in order to prove the theorem, it is enough to show that $\operatorname{dom} M_{\varphi,\Omega}$ is dense in $L^2_a(\Omega)$.

To see this let

$$\psi(z) = \frac{\varphi(z) - a}{\varphi'(0)}.$$

Thus $\psi \in \mathrm{Hol}(\mathbb{D})$, $\psi(0) = 0$, and $\psi'(0) = 1$. Since φ is one-to-one, it follows that ψ belongs to the class $\mathcal{S}$. So by Lemma 2.2.3, we have that $\mathrm{dom}\, M_{\psi,\Omega}$ is dense in $L^2_a(\Omega)$. But on the other hand $\mathrm{dom}\, M_{\varphi,\Omega}$ is same as $\mathrm{dom}\, M_{\psi,\Omega}$ (see Definition 1.3.1). Hence the assertion of the theorem. □

2.3. Essential Boundary Points and the Density Problem

The main purpose of this section is to provide three examples. We start by stating a result regarding the essential boundary points due to Axler, Conway, and McDonald [3].

PROPOSITION 2.3.1. *If G is an open set in $\mathbb{C}$, $\lambda \in \partial_\infty G$, and if the connected component of $\partial_\infty G$ containing λ consists of more than one point, then $\lambda \in \partial_{2-e}(G)$.*

PROOF. See Axler, Conway, and McDonald [3] , Proposition 14 on page 479. □

In view of Theorem 2.1.6, it seems reasonable to conjecture that the condition $\infty \in \partial_{2-e}(G)$ is also sufficient in order for a Bergman operator to be densely defined. For this reason, Theorem 2.1.6, along

with the preceding proposition, could be thought of as a step toward proving the conjecture. That is, we would only need to investigate the density problem for an unbounded open set where infinity is an essential boundary point and ,moreover, the component of the extended boundary containing ∞ equals the singleton $\{\infty\}$. Our first example uses essentially a result of Axler and Bourdon [4], and provides a counterexample to such a conjecture.

PROPOSITION 2.3.2. *Let* $x_n = e^{-n}$ *and* $r_n = e^{-n^4 e^{2n}}$, *where* $n \geq 1$. *If we define the set* Ω *by*

$$\Omega = \mathbb{D}^* \setminus \bigcup_{n=1}^{\infty} \operatorname{cl} B(x_n, r_n)$$

where $\mathbb{D}^*$ *denotes* $\mathbb{D} \setminus \{0\}$, *then* $\lim_{x \to 0-} f(x)$ *exists as a finite complex number for all* $f \in L_a^2(\Omega)$, *and, moreover, there is a constant* $C > 0$ *such that* $|\lim_{x \to 0-} f(x)| \leq C\|f\|_{L_a^2(\Omega)}$ *for all* $f \in L_a^2(\Omega)$.

PROOF. See Axler and Bourdon [4], Theorem 8 on page 14. □

EXAMPLE 2.3.3. There exists an unbounded region G in the plane for which the component of $\partial_\infty(G)$ containing ∞ equals the singleton $\{\infty\}$ and the Bergman operator S_G is not densely defined.

PROOF. If $\varphi(z) = \dfrac{1}{z}$ and $n \geq 1$, let $K_n = \varphi[\operatorname{cl} B(x_n, r_n)]$, where x_n and r_n are defined as in Proposition 2.3.2. In addition let K_0 denote

cl $\mathbb{D}$. We will show that

$$G = \mathbb{C} \setminus \bigcup_{n=0}^{\infty} K_n$$

has the desired properties.

First of all note that G is an unbounded region in the plane. It is easy to see that K_n's are pairwise disjoint closed discs in $\mathbb{C}$. From Proposition 2.3.1, it follows that $\partial K_n \subseteq \partial_{2-e}(G)$ for all $n \geq 0$. Now since the set of essential boundary points is closed in $\partial_\infty G$, (see Theorem 2.1.2(b)), we have that $\partial_{2-e}(G) = \partial_\infty G$. Hence $\infty \in \partial_{2-e}(G)$.

Next, as a consequence of the construction of the sets K_n, we find that

$$\varphi^{-1} \colon G \to \Omega$$

maps G conformally onto Ω, where Ω is as in Proposition 2.3.2. In view of Theorem 1.3.2, part(b), dom S_G is dense in $L^2_a(G)$ if and only if dom $M_{\frac{1}{z},\Omega}$ is dense in $L^2_a(\Omega)$. Since Ω is bounded, dom $M_{\frac{1}{z},\Omega} = zL^2_a(\Omega)$.

We will complete the proof by showing that $zL^2_a(\Omega)$ is not dense in $L^2_a(\Omega)$. For $f \in L^2_a(\Omega)$, define $T \colon L^2_a(\Omega) \to \mathbb{C}$ by

$$T \colon f \mapsto \lim_{x \to 0^-} f(x).$$

By Proposition 2.3.2, T is a bounded linear functional on $L^2_a(\Omega)$. Moreover $T(zf) = 0$ for all $f \in L^2_a(\Omega)$; that is, T annihilates the subspace

$zL_a^2(\Omega)$. So if $zL_a^2(\Omega)$ is dense in $L_a^2(\Omega)$, then T must be the zero functional. But $T(1) = 1$, a contradiction. □

The preceding example shows that condition $\infty \in \partial_{2-e}(G)$ is not sufficient for the density of the domain of the Bergman operator. So now it seems natural to ask the question whether Example 2.3.3 is the worst that can happen; that is, if $\infty \in \partial_{2-e}(G)$ and if $\{\infty\}$ is the component of $\partial_\infty G$ that contains ∞, can we conclude that $\operatorname{dom} S_G$ is not dense in $L_a^2(G)$? The next result answers the above question in the negative.

PROPOSITION 2.3.4. *If $G = \bigcup_n G_n$ is a countable union of pairwise disjoint and bounded open subsets of $\mathbb{C}$, then the Bergman operator S_G is densely defined.*

PROOF. The proof is trivial if there are only finitely many G_n. So we may assume that $G = \bigcup_{n=1}^{\infty} G_n$. Since G_n are disjoint, we can decompose $L_a^2(G)$ as a direct sum

$$L_a^2(G) = L_a^2(G_1) \oplus L_a^2(G_2) \oplus \cdots .$$

Consequently for f in $L_a^2(G)$

$$\|f\|_{L_a^2(G)}^2 = \sum_{n=1}^{\infty} \int_{G_n} |f(z)|^2 \, d\mathcal{A}(z) = \sum_{n=1}^{\infty} \|f|_{G_n}\|_{L_a^2(G_n)}^2 ,$$

where $f|_{G_n}$ denotes the restriction of f to the subset G_n. Next put

$$g_n(z) = \sum_{k=1}^{n} \chi_{G_k}(z) f(z).$$

In other words g_n is the restriction of f to $G_1 \cup \cdots \cup G_n$. Clearly g_n belongs to $L^2_a(G)$ and

$$\begin{aligned} \|zg_n\|^2_{L^2_a(G)} &= \sum_{i=1}^{n} \int_{G_i} |z|^2 \, |f(z)|^2 \, d\mathcal{A}(z) \\ &\leq C \sum_{i=1}^{n} \|f|_{G_i}\|^2_{L^2_a(G_i)}, \end{aligned}$$

where C is a positive constant given by $C = \sup\{|z|^2 \colon z \in G_1 \cup \cdots \cup G_n\}$.

It follows now that for each $n \geq 1$, g_n belongs to $\operatorname{dom} S_G$. Furthermore

$$\begin{aligned} \|f - g_n\|^2_{L^2_a(G)} &= \int_G |f(z) - g(z)|^2 \, d\mathcal{A}(z) \\ &= \sum_{i=n+1}^{\infty} \int_{G_i} |f(z)|^2 \, d\mathcal{A}(z) \\ &= \sum_{i=n+1}^{\infty} \|f|_{G_i}\|^2_{L^2_a(G_i)}. \end{aligned}$$

Finally since the sum $\sum_{i=1}^{\infty} \|f|_{G_i}\|^2_{L^2_a(G_i)}$ converges, we conclude that $\|f - g_n\|_{L^2_a(G)} \to 0$. □

EXAMPLE 2.3.5. For $n = 0, 1, 2, \ldots$, denote by G_n the annuli $\{z \colon 2^n < |z| < 2^{n+1}\}$. If $G = \bigcup_{n=0}^{\infty} G_n$, then a similar argument as in Example 2.3.3 yields that $\infty \in \partial_{2-e}(G)$. It is easy to see that $L^2_a(G)$

is non-trivial. For example consider the function

$$f(z) = \begin{cases} 1, & \text{if } z \in G_1 \ ; \\ 0, & \text{otherwise.} \end{cases}$$

Clearly $\{\infty\}$ is the component of $\partial_\infty G$ containing ∞. Finally from Proposition 2.3.4, it follows that the Bergman operator S_G is densely defined.

In Proposition 2.3.4 G does not contain an unbounded component. This observation raises the question whether Example 2.3.3 can be generalized to arbitrary unbounded regions in the plane where ∞ is an essential boundary point and the component of the extended boundary that contains ∞ equals the singleton $\{\infty\}$. Our final example addresses the above question and answers it in the negative. Before giving this final example, however, we shall need to recall the notion of continuous analytic capacity and a result due to Melnikov.

DEFINITION 2.3.6. If E is a subset of $\mathbb{C}$, the *continuous analytic capacity* of E is defined by

$$\alpha(E) = \sup |f'(\infty)| \ ,$$

where the supremum is taken over the class of all continuous functions f defined on $\mathbb{C}_\infty$ such that $f(\infty) = 0$, $\|f\|_\infty \leq 1$, and f is analytic off some compact subset of E.

The continuous analytic capacity α enjoys a lot of the properties which belong to a more general class of capacities. For more details on this topic, we refer the reader to Gamelin [**11**]. There is, however, one property of α that we state explicitly in the next theorem.

THEOREM 2.3.7. *If E in a bounded measurable subset of $\mathbb{C}$, then*

$$[\alpha(E)]^2 \geq \frac{1}{4\pi}\text{Area}\,(E).$$

PROOF. See Gamelin [**11**], Theorem 3.2 on page 204. □

For a bounded open set Ω in $\mathbb{C}$, denote by $\mathrm{A}(\mathrm{cl}\,\Omega)$ the algebra of all continuous functions on $\mathrm{cl}\,\Omega$ that are analytic in Ω. Recall also that a point $\lambda \in \partial\Omega$ is said to be a *peak point* for $\mathrm{A}(\mathrm{cl}\,\Omega)$, if there exists f in $\mathrm{A}(\mathrm{cl}\,\Omega)$ such that $|f(z)| < 1$ for all z in $\mathrm{cl}\,\Omega \setminus \{\lambda\}$ and $f(\lambda) = 1$. Furthermore for a point z_0 in $\mathbb{C}$, let $E_n(z_0)$ denote the annulus $\{z\colon\ 2^{-(n+1)} < |z - z_0| < 2^{-n}\}$.

THEOREM 2.3.8 (Melnikov). *If K is a compact set in $\mathbb{C}$ and $z_0 \in K$, then z_0 is a peak point for $A\,(K)$ if and only if*

$$\sum_{n=1}^{\infty} 2^n \alpha\big(E_n(z_0) \setminus \mathrm{int}\,K\big) = +\infty.$$

PROOF. See Gamelin [**11**], Theorem 4.5 on page 205. □

Now we can state our final example.

EXAMPLE 2.3.9. There exists an unbounded region G in $\mathbb{C}$ such that the component of $\partial_\infty G$ that contains ∞ equals the singleton $\{\infty\}$ and the Bergman operator S_G is densely defined.

PROOF. Let $F_n = \left\{z\colon\ 2^{-(n+1)} \leq |z| \leq 2^{-n}\ ;\ |\arg z| \leq \frac{\pi}{2}\right\}$, for $n \geq 1$. Define

$$\Omega = \mathbb{D}^* \setminus \bigcup_{n=0}^{\infty} F_{2n+1},$$

where $\mathbb{D}^*$ denotes $\mathbb{D} \setminus \{0\}$.

Clearly $E_{2n+1}(0) \setminus \operatorname{int}(\operatorname{cl}\Omega) = \operatorname{int} F_{2n+1}$, thus from Theorem 2.3.7 it follows that

$$\alpha\left(E_{2n+1}(0) \setminus \operatorname{int}(\operatorname{cl}\Omega)\right) \geq \frac{1}{2\sqrt{\pi}}\sqrt{\operatorname{Area}\left(F_{2n+1}\right)} \geq C2^{-n},$$

where $C > 0$ is a positive constant independent of n. Consequently

$$\sum_{n=1}^{\infty} 2^{n+1}\alpha\left(E_{2n+1}(z_0) \setminus \operatorname{int}(\operatorname{cl}\Omega)\right) = +\infty.$$

By Melnikov's Theorem, 0 is a peak point for the algebra $\mathrm{A}(\operatorname{cl}\Omega)$.

Next let $\varphi(z) = \dfrac{1}{z}$ and define $K_n = \varphi[E_{2n+1}]$, for $n \geq 1$. If we let

$$G = \mathbb{C} \setminus \bigcup_{n=0}^{\infty} K_n,$$

where K_0 is the closed unit disc, then the same sort of argument as in Example 2.3.3 shows that $\operatorname{dom} S_G$ is dense in $L_a^2(G)$ if and only if $zL_a^2(\Omega)$ is dense in $L_a^2(\Omega)$.

Aleman [**2**], see Lemma 2.1 on page 4, has shown that if 0 is a peak point for the algebra $\mathrm{A}(\operatorname{cl}\Omega)$, then $zL_a^2(\Omega)$ is dense in $L_a^2(\Omega)$. This observation concludes the proof of our example. □

2.4. Further Generalizations

Using the methods described in Section 2.2 we give a generalization of Theorem 2.2.1 to the operators S_G^n, where $n \geq 1$. We start by extending the Definition 1.3.1 to all positive integer powers of the multiplication operator.

DEFINITION 2.4.1. If φ is an analytic function on an open set of the plane G, define the operator $M_{\varphi,G}^n\colon \operatorname{dom} M_{\varphi,G}^n \to L_a^2(G)$, $n \geq 1$, by

$$M_{\varphi,G}^n\colon f(z) \mapsto \varphi^n(z) f(z),$$

where

$$\operatorname{dom} M_{\varphi,G}^n = \left\{ f \in \operatorname{dom} M_{\varphi,G}^{n-1}\colon\ \varphi f \in L_a^2(G) \right\}.$$

If $n = 0$, denote by $M_{\varphi,G}^0$ the identity operator on the Bergman space $L_a^2(G)$.

In particular, if $\varphi(z) = z$, the operator $M^n_{z,G}$ is denoted by S^n_G (see Definition 1.3.1).

LEMMA 2.4.2. *If $n \geq 1$, then $M^n_{\varphi,G} = M_{\varphi^n,G}$.*

PROOF. It follows easily from Definition 2.4.1 that

$$\operatorname{dom} M^n_{\varphi,G} = \left\{f \in L^2_a(G) \colon \varphi f \in L^2_a(G), \ldots, \varphi^n f \in L^2_a(G)\right\}.$$

Obviously $M^n_{\varphi,G}$ is an extension of $M_{\varphi^n,G}$; furthermore, if $f \in \operatorname{dom} M_{\varphi^n,G}$, then

$$\begin{aligned}\int_G |\varphi^m(z)f(z)|^2\, d\mathcal{A}(z) &= \int_{\{z\in G\colon\, |\varphi(z)|\leq 1\}} |\varphi^m(z)f(z)|^2\, d\mathcal{A}(z) + \\ &\qquad \int_{\{z\in G\colon\, |\varphi(z)|>1\}} |\varphi^m(z)f(z)|^2\, d\mathcal{A}(z) \\ &\leq \|f\|^2_{L^2_a(G)} + \|\varphi^n f\|^2_{L^2_a(G)},\end{aligned}$$

for all $1 \leq m \leq n$. Thus $f \in \operatorname{dom} M^n_{\varphi,G}$ and, as a consequence, we have that $M_{\varphi^n,G} \subseteq M^n_{\varphi,G}$. This proves the lemma. □

THEOREM 2.4.3. *If G is an open subset of the plane and the component of its complement with respect to $\mathbb{C}_\infty$ that contains ∞ does not equal the singleton $\{\infty\}$, then the operators S^n_G, where $n \geq 1$, are densely defined.*

PROOF. Before giving the proof, note that, as a consequence of Lemma 2.4.2, we have $S^n_G = M_{z^n,G}$. By the same sort of argument as in

the proof of Theorem 2.2.1, there exists an open set $\Omega \subseteq \mathbb{D}$ containing 0 and a conformal mapping $\varphi\colon \Omega \to G$ from Ω onto G such that $\varphi'(0) > 0$ and $\varphi(0) = a$, where $a \in G$. From Theorem 1.3.2 it follows that $L^2_a(G)$ is isomorphic to $L^2_a(\Omega)$ via the the isomorphism $U\colon L^2_a(\Omega) \to L^2_a(G)$, defined by $\big(Uf\big)(z) = \varphi'(z) f(\varphi(z))$ for all $f \in L^2_a(G)$. Now, from the equality

$$\begin{aligned} \|w^n f\|^2_{L^2_a(G)} &= \int_G |w^n f(w)|^2 \, du\, dv \\ &= \int_\Omega |\varphi^n(z) f(\varphi(z))|^2 |\varphi'(z)|^2 \, dx\, dy \\ &= \|\varphi^n U f\|^2_{L^2_a(\Omega)}, \end{aligned}$$

along with the Lemma 2.4.2, one easily sees that

$$U\big[\operatorname{dom} M_{\varphi^n,\Omega}\big] = \operatorname{dom} S^n_G.$$

Hence the result follows if it can be shown that $\operatorname{dom} M_{\varphi^n,\Omega}$ is dense in $L^2_a(\Omega)$. Now if

$$\psi(z) = \frac{\varphi(z) - a}{\varphi'(0)},$$

Lemma 2.4.2 implies that $\operatorname{dom} M_{\psi^n,\Omega} = \operatorname{dom} M_{\varphi^n,\Omega}$; so without loss of generality, we may assume that φ belongs to to the class $\mathcal{S}$. In view of Corollary 1.4.7, φ can be factorized as $zF(z)$, where $F(z)$ is an outer function in $H^p(\mathbb{D})$ for all $p < \dfrac{1}{2}$. A similar argument as in the proof of

Lemma 2.2.3 shows that $\frac{1}{F^n}$ is a bounded outer function in $H^\infty(\mathbb{D})$.

Hence the result follows from Lemma 2.2.2. □

CHAPTER 3

The Density Problem

3.1. Preliminaries

In this section we will prove two useful results.

PROPOSITION 3.1.1. *Let G be an open set in $\mathbb{C}$ with finite area and assume that K is a compact subset of G. If $f \in L^2_a(G \setminus K)$, and if f_1, f_2 denote the corresponding functions in accordance with Theorem 1.5.4, then f_1 and f_2 belong to $L^2_a(G \setminus K)$.*

PROOF. Recall from Theorem 1.5.4 that f_1 , f_2 are the unique functions satisfying the conditions: $f \equiv f_1 + f_2$ on $G \setminus K$, $f_1 \in \mathrm{Hol}(G)$, $f_2 \in \mathrm{Hol}(\mathbb{C}_\infty \setminus K)$, and $f_2(\infty) = 0$.

For $\varepsilon > 0$, let $F_\varepsilon = \{z\colon\ \mathrm{dist}(z, K) < 2\varepsilon\}$. Fix ε such that $\mathrm{cl}\, F_\varepsilon \subseteq G$. Next define Γ to be a smooth positive Jordan system in $F_{\frac{\varepsilon}{2}} \setminus K$ such that $K \subseteq \mathrm{ins}\,\Gamma$. To show that f_1 belongs to $L^2_a(G \setminus K)$, we observe that

$$\|f_1\|^2_{L^2_a(G\setminus K)} = \int_{F_\varepsilon \setminus K} |f_1(z)|^2 \, d\mathcal{A}(z) + \int_{G \setminus F_\varepsilon} |f_1(z)|^2 \, d\mathcal{A}(z).$$

Now if $M_1 = \sup\{|f_1(z)| : z \in \operatorname{cl} F_\varepsilon\}$, then since $f_1 \in \operatorname{Hol}(G)$ and $\operatorname{cl} F_\varepsilon$ is compact, we have that

$$\int_{F_\varepsilon \setminus K} |f_1(z)|^2 \, d\mathcal{A}(z) \leq M_1 \cdot \operatorname{Area}(\operatorname{cl} F_\varepsilon) < \infty.$$

Moreover

$$\begin{aligned}\int_{G\setminus F_\varepsilon} |f_1(z)|^2 \, d\mathcal{A}(z) &= \int_{G\setminus F_\varepsilon} |f(z) - f_2(z)|^2 \, d\mathcal{A}(z) \\ &\leq 2\|f\|^2_{L^2_a(G\setminus K)} + 2\int_{G\setminus F_\varepsilon} |f_2(z)|^2 \, d\mathcal{A}(z).\end{aligned}$$

Thus $f_1 \in L^2_a(G \setminus K)$, if it can be shown that $\int_{G\setminus F_\varepsilon} |f_2|^2 \, d\mathcal{A}(z)$ is finite. But since Γ is in $F_{\frac{\varepsilon}{2}} \setminus K$, we have that $z \in \operatorname{out}\Gamma$ for all z in $G \setminus F_\varepsilon$. Using (1.5.6), we find that

$$\begin{aligned}\int_{G\setminus F_\varepsilon} |f_2(z)|^2 \, d\mathcal{A}(z) &= \int_{G\setminus F_\varepsilon} \left|\frac{-1}{2\pi i}\int_\Gamma \frac{f(w)}{w-z}\, dw\right|^2 d\mathcal{A}(z) \\ &\leq \frac{1}{4\pi^2}\int_{G\setminus F_\varepsilon}\left(\int_\Gamma \frac{|f(w)|}{|w-z|}\,|dw|\right)^2 d\mathcal{A}(z).\end{aligned}$$

Now since $\operatorname{dist}(z,\Gamma) \geq \varepsilon$ for all z in $G \setminus F_\varepsilon$, if M_2 denotes $\max\{|f(w)| : w \in \Gamma\}$, then it follows that

$$\int_{G\setminus F_\varepsilon} |f_2(z)|^2 \, d\mathcal{A}(z) \leq \frac{1}{4\pi^2}\left(\frac{M_2}{\varepsilon}\,\ell(\Gamma)\right) \cdot \operatorname{Area}(G \setminus F_\varepsilon).$$

By assumption $\operatorname{Area}(G) < \infty$, hence the above integral is finite. Finally since $f_2 \equiv f - f_1$ on $G \setminus K$, it follows that f_2 belongs also to $L^2_a(G \setminus K)$. □

For $R > 0$, denote by U_R the open set $\{z\colon\ |z| > R\}$. Recall also that f is said to be analytic in a neighborhood of infinity, if $f \in \mathrm{Hol}(U_R)$ for some large value of R, and $f(z) = \sum_{n=0}^{\infty} a_n z^{-n}$ in U_R. See Definition 1.5.1.

LEMMA 3.1.2. *If for some large value of $R > 0$, f is in $L^2_a(U_R)$, then f and its derivative vanish at infinity. Conversely, if f is analytic in a neighborhood of infinity such that $f(\infty) = 0$, $f'(\infty) = 0$, then $f \in L^2_a(U_R)$ for some large value of R.*

PROOF. Recall that if $f \in \mathrm{Hol}(U_R)$, then f has the power series expansion $f(z) = \sum_{n=0}^{\infty} a_n z^{-n}$ in U_R, where $a_0 = f(\infty)$, and $a_1 = f'(\infty)$; see Definition 1.5.1. Now if $f \in L^2_a(U_R)$, then from the identity

$$\|f\|^2_{L^2_a(U_R)} = \int_{|z|>R} |f(z)|^2 \, d\mathcal{A}(z) = 2\pi \sum_{n=0}^{\infty} |a_n|^2 \int_R^{\infty} \frac{1}{r^{2n-1}} \, dr,$$

it follows easily that $a_0 = a_1 = 0$.

For the converse, choose $\rho > 0$ large enough such that f is analytic in a neighborhood of $\mathrm{cl}\, U_\rho$. By assumption $f(z) = \sum_{n=2}^{\infty} a_n z^{-n}$, where the series converges absolutely and uniformly for $|z| > \rho$. So if $R > \rho$, we see that

$$\begin{aligned} \|f\|^2_{L^2_a(U_R)} &= \pi R^2 \sum_{n=2}^{\infty} \frac{1}{n-1} |a_n|^2 R^{-2n} \\ &\leq \pi R^2 \sum_{n=2}^{\infty} \left(|a_n| \rho^{-n}\right)^2 \left(\frac{\rho}{R}\right)^{2n}. \end{aligned}$$

For n sufficiently large, from the absolute convergence, we have $|a_n|\rho^{-n} < 1$. Thus f belongs to $L^2_a(U_R)$. □

3.2. The Cauchy Transform

The Cauchy transform is an essential tool in order to prove the main result of this chapter. Therefore, we devote this section to introduce and to give the most important properties of this transform.

DEFINITION 3.2.1. For a compactly supported measure μ on the plane, the *Cauchy transform* of μ, denoted by $\widehat{\mu}$, is defined by

$$\widehat{\mu}(z) = \int \frac{1}{w-z} d\mu(z).$$

The next result not only states the elementary properties of the Cauchy transform, but it also shows that the above definition makes sense.

PROPOSITION 3.2.2. *If μ is a compactly supported measure on $\mathbb{C}$, then:*

(a) *$\widehat{\mu}$ is locally integrable;*

(b) *$\widehat{\mu}$ is analytic on $\mathbb{C}_\infty \setminus \operatorname{supp}(\mu)$;*

(c) *The power series expansion of $\widehat{\mu}$ in a neighborhood of infinity is given by*

$$\widehat{\mu}(z) = \sum_{n=0}^{\infty} \left(- \int w^n \, d\mu(w) \right) \frac{1}{z^{n+1}}.$$

PROOF. For a proof see Conway[8], Proposition 3.2 on page 174.

□

The next lemma is stated in Carleson [6], see Lemma 4 on page 56; for a proof see also Conway [9], pages 348–349. Here we will, however, present another proof.

LEMMA 3.2.3. *If $R > 0$, and $\zeta_1 \neq \zeta_2$ belong to the disc $B(0,R)$, then there exists a constant $C > 0$, depending only on R, such that*

$$\int_{B(0,R)} \frac{d\mathcal{A}(z)}{|z-\zeta_1|\,|z-\zeta_2|} \leq C \log \frac{3R}{|\zeta_1-\zeta_2|}.$$

PROOF. Make the change of variable given by $w\colon z \mapsto z-\zeta_1$. Obviously w maps $B(0,R)$ onto a disc with radius R and centered at $-\zeta_1$. Since $\zeta_1 \in B(0,R)$, it follows that $w\big(B(0,R)\big) \subseteq B(0,2R)$. As a result

$$\int_{B(0,R)} \frac{d\mathcal{A}(z)}{|z-\zeta_1|\,|z-\zeta_2|} \leq \int_{B(0,2R)} \frac{d\mathcal{A}(w)}{|w|\,|w-(\zeta_2-\zeta_1)|}.$$

For simplicity put $|\zeta_1-\zeta_2| = t$ and $\arg(\zeta_2-\zeta_1) = \theta_0$, so that $\zeta_2-\zeta_1 = te^{i\theta_0}$. Using the polar coordinates $w = re^{i\theta}$ and substitution $\theta-\theta_0 = \phi$, we have that

$$\int_{B(0,2R)} \frac{d\mathcal{A}(w)}{|w|\,|w-(\zeta_2-\zeta_1)|} \leq \int_{\theta_0}^{2\pi-\theta_0} \int_0^{2R} \frac{1}{|re^{i\phi}-t|}\,dr\,d\phi.$$

Noting that the function $\phi \mapsto |re^{i\phi} - t|$ attains its minimum value at $\phi = 0$, it follows that

$$\int_{\theta_0}^{2\pi-\theta_0} \int_0^{2R} \frac{1}{|re^{i\phi} - t|}\, dr\, d\phi \leq 2\pi \log \left| \frac{2R - t}{t} \right|$$
$$\leq 2\pi \log \frac{3R}{|\zeta_1 - \zeta_2|}.$$

The proof of lemma is complete. □

The preceding lemma has the following important consequence. Recall also Definition 1.6.2 for the definition of energy integral $\mathrm{I}(\mu)$ corresponding to a compactly supported real-valued measure μ on $\mathbb{C}$.

LEMMA 3.2.4. *If K is a compact set in $\mathbb{C}$ with positive logarithmic capacity and if $\mu \in \mathrm{M}(K)$ has finite energy $\mathrm{I}(|\mu|)$, then $\widehat{\mu} \in L^2_{loc}(\mathbb{C})$.*

PROOF. Without loss of generality we may assume that $\mu \in \mathrm{M}^+(K)$. Choose $R > 0$, sufficiently large, such that $K \subseteq B(0, R)$. From the Fubini's Theorem one can see that

$$\int_{B(0,R)} |\widehat{\mu}(z)|^2\, d\mathcal{A}(z) = \int_{B(0,R)} \left| \int_K \frac{d\mu(w)}{w - z} \right|^2 d\mathcal{A}(z)$$
$$= \int_{B(0,R)} \int_K \frac{d\mu(u)}{u - z} \int_K \frac{d\mu(v)}{\overline{v} - \overline{z}}\, d\mathcal{A}(z)$$
$$\leq \iint_{K\times K} \int_{B(0,R)} \frac{d\mathcal{A}(z)}{|u - z|\,|v - z|}\, d\mu(u)\, d\mu(v).$$

Note that in the last inequality we have implicitly used the fact that the set $\{(z, z)\colon\ z \in K\}$ has zero Lebesgue area measure with respect

to $|\mu \times \mu|$. Now from Lemma 3.2.3, along with the fact that $\mathrm{I}(\mu) < \infty$, it follows that

$$\int_{B(0,R)} |\widehat{\mu}(z)|^2 \, d\mathcal{A}(z) \leq C \iint_{K\times K} \log \frac{3R}{|u-v|} \, d\mu(u) \, d\mu(v)$$

$$\leq C \left[\left(\mu(K)\right)^2 \log 3R + \mathrm{I}(\mu) \right] < \infty.$$

□

LEMMA 3.2.5. *If K is a compact set in $\mathbb{C}$ with positive logarithmic capacity, and if μ_1 , μ_2 denote probability measures in $\mathrm{M}^+(K)$ such that their corresponding energy integrals are finite, then the Cauchy transform of the measure μ defined by $\mu \equiv \mu_1 - \mu_2$ belongs to $L^2_a(\mathbb{C}\setminus K)$.*

PROOF. Clearly $\widehat{\mu} \in \mathrm{Hol}(\mathbb{C} \setminus K)$. By assumption $\mathrm{I}(\mu_i) < \infty$ for $i = 1, 2$; thus it follows from Lemma 3.2.4 that $\widehat{\mu} \in L^2_{loc}(\mathbb{C})$. On the other hand we have that

$$\widehat{\mu}(\infty) = \widehat{\mu}_1(\infty) - \widehat{\mu}_2(\infty) = 0.$$

Moreover by Proposition 3.2.2 part(d),

$$\widehat{\mu}'(\infty) = \widehat{\mu}'_1(\infty) - \widehat{\mu}'_2(\infty)$$

$$= \mu_2(K) - \mu_1(K) = 0.$$

Now the result follows from Lemma 3.1.2. □

There are some useful identities, especially for our purpose, regarding the Cauchy transform which have been collected in the next lemma.

LEMMA 3.2.6. *If K is compact and Γ denotes a positive Jordan system in $\mathbb{C}\backslash K$ such that $K \subseteq \operatorname{ins}\Gamma$, then for $\mu \in \mathrm{M}^+(K)$ the following identities hold.*

(a) $\displaystyle\int_\Gamma \widehat{\mu}(w)\,dw = -2\pi i\mu(K).$

(b) $\displaystyle\int_\Gamma w\widehat{\mu}(w)\,dw = -2\pi i\int_K w\,d\mu(w).$

(c) *If* $z \in \operatorname{out}\Gamma$, *then* $\displaystyle\frac{-1}{2\pi i}\int_\Gamma \frac{w\widehat{\mu}(z)}{w-z}\,dw = \mu(K) + z\widehat{\mu}(z).$

PROOF. Noting that $\mathrm{n}(\Gamma, z) = 1$ for all $z \in K$, the proof of parts (a)–(c) follows from Fubini's Theorem.

For (a) we have that

$$\begin{aligned}\int_\Gamma \widehat{\mu}(w)\,dw &= \int_\Gamma\int_K \frac{d\mu(z)}{z-w}\,dw \\ &= \int_K\int_\Gamma \frac{dw}{z-w}\,d\mu(z) \\ &= -2\pi i\,\mathrm{n}(\Gamma, z)\,\mu(K),\end{aligned}$$

Thus the result.

Part(b) follows by the same sort of argument as in (a), along with the identity $\displaystyle\frac{w}{z-w} = \frac{z}{z-w} - 1.$

To prove part(c), we see that

$$\int_\Gamma \frac{w\widehat{\mu}(z)}{w-z}\,dw = -2\pi i\mu(K) + z\int_\Gamma \frac{\widehat{\mu}(z)}{w-z}\,dw.$$

On the other hand

$$\begin{aligned}\int_\Gamma \frac{\widehat{\mu}(z)}{w-z}\,dw &= \int_\Gamma \frac{1}{w-z}\int_K \frac{d\mu(u)}{u-w}\,dw \\ &= \int_K \frac{1}{u-z}\int_\Gamma \left(\frac{1}{w-z}+\frac{1}{u-w}\right) dw\,d\mu(u) \\ &= -2\pi i\big[\mathrm{n}(\Gamma,u)-\mathrm{n}(\Gamma,z)\big]\,\widehat{\mu}(z)\,.\end{aligned}$$

Now part(c) follows easily by observing that $z \in \operatorname{out}\Gamma$ and $u \in \operatorname{ins}\Gamma$. □

The last result of this section is due to Hedberg [**13**].

THEOREM 3.2.7 (Hedberg). *If G is a bounded open set in $\mathbb{C}$, then the class of all Cauchy transforms $\widehat{\mu}$ in $L^2_a(G)$, where μ is a compactly supported measure on $\mathbb{C}\setminus G$, is dense in $L^2_a(G)$.*

PROOF. See Hedberg[**13**], Theorem 10 on page 314. □

It should be mentioned that the above result has been generalized to arbitrary open subsets of the plane, see Adams and Hedberg [**1**], Theorem 11.1.8 on page 309.

3.3. An Equivalence for the Density Problem

This section contains one of the main results of this study. In Theorem 3.3.1, it will be shown that the density of $\operatorname{dom} S_G$ in $L^2_a(G)$ is equivalent to the density of range of the operator of multiplication

by the independent variable z on some bounded open subset of $\mathbb{C}$. The main tool in proving Theorem 3.3.1 is, however, Theorem 3.3.4.

THEOREM 3.3.1. *Suppose that G is an open set in $\mathbb{C}$ for which the corresponding Bergman space $L_a^2(G)$ is non-trivial. Let $a \in G$ and $r > 0$ be small enough that $\operatorname{cl} B(a,r) \subseteq G$. If K denotes the image of $\mathbb{C}_\infty \setminus G$ under the mapping $\varphi\colon\ \mathbb{C}_\infty \to \mathbb{C}_\infty$ defined by $\varphi\colon\ z \mapsto \dfrac{r}{z-a}$, then $\operatorname{dom} S_G$ is dense in $L_a^2(G)$ if and only if $z\big[L_a^2(\mathbb{D} \setminus K)\big]$ is dense in $L_a^2(\mathbb{D} \setminus K)$.*

The proof of Theorem 3.3.1 requires several preliminary results. Theorem 3.3.4 can be considered as one of our principal results from which Theorem 3.3.1 follows as a consequence.

LEMMA 3.3.2. *If K is a compact set with positive logarithmic capacity, then there are points z_1, z_2 in K and positive numbers r_1, r_2 such that the following statements hold.*

(a) $\operatorname{cl} B(z_1,r_1) \cap \operatorname{cl} B(z_2,r_2) = \emptyset$.

(b) *If $K_i = \operatorname{cl} B(z_i,r_i) \cap K$ for $i = 1,2$, then $c(K_i) > 0$.*

PROOF. First we claim; *if K is compact and μ is a positive measure in $\mathrm{M}(K)$, then there exists a point $z \in K$ such that $\mu\big(B(z,r)\big) > 0$ for all $r > 0$.*

To see this, assume that the claim is not true. Then to each z in K there corresponds a $r_z > 0$ such that $\mu\big(B(z, r_z)\big) = 0$. Now since $\bigcup_{z\in K} B(z, r_z)$ is an open covering of K, by a compactness argument there must be a finite subcover of K, say $\bigcup_{i=1}^{n} B(z_i, r_{z_i})$. But then we have that

$$\begin{aligned}\mu(K) &\leq \mu\Big(\bigcup_{i=1}^{n} B(z_i, r_{z_i})\Big) \\ &\leq \sum_{i=1}^{n} \mu\big(B(z_i, r_{z_i})\big) = 0,\end{aligned}$$

a contradiction.

Next let μ be a positive measure in $\mathrm{M}(K)$ with finite energy. It is a well known fact from the measure theory that we can pick disjoint compact subsets of K each of which has positive measure with respect to μ. Therefore from the above claim, there are points $z_1 \neq z_2$ in K such that $\mu\big(B(z_i, r)\big) > 0$ for all $r > 0$ and $i = 1, 2$. Next choose r_1, and r_2 small enough so that $\mathrm{cl}\, B(z_1, r_1) \cap \mathrm{cl}\, B(z_2, r_2) = \emptyset$. Finally if we put $K_i = \mathrm{cl}\, B(z_i, r_i) \cap K$, then from Theorem 1.6.5 it follows that $c(K_i) > 0$ for $i = 1, 2$. □

Before presenting the next lemma, recall that if E is a subset of the plane, then $\overline{\mathrm{co}}(E)$ denotes the *closed convex hull* of E.

LEMMA 3.3.3. *If K is a compact set with positive logarithmic capacity, then there are measures μ_1, μ_2 in $\mathrm{M}(K)$ with finite energy integrals such that:*

(a) $\mu_1(K) = 0$ *and* $\displaystyle\int w\, d\mu_1(w) = 1$;

(b) $\mu_2(K) = 1$ *and* $\displaystyle\int w\, d\mu_2(w) = 0$.

PROOF. Let K_1, K_2 be as in Lemma 3.3.2. The index i will stand for 1 or 2. Since $c(K_i) > 0$, there are probability measures ν_1 and ν_2 in $\mathrm{M}^+(K)$ such that $\mathrm{I}(\nu_i) < \infty$ and $\mathrm{supp}\,(\nu_i) \subseteq K_i$, see Definition 1.6.3. Next let

$$C = \int_K w\, d\nu_1(w) - \int_K w\, d\nu_2(w).$$

We claim that C is a non-zero constant. In order to prove the claim first note that since $\varphi(w) = w$ is continuous in $\mathbb{C}$, it follows from a convexity argument that there are points $z_i \in \overline{\mathrm{co}}(K_i)$ such that

$$z_i = \int_K w\, d\nu_i,$$

see Conway [7]. From Lemma 3.3.2 part(a), on the other hand, we have that $\overline{\mathrm{co}}(K_1) \cap \overline{\mathrm{co}}(K_2) = \emptyset$. This proves the claim.

Now we can define the measure μ_1 in $\mathrm{M}(K)$ by

$$\mu_1 \equiv \frac{1}{C}(\nu_1 - \nu_2).$$

It is easy to check that $\mu_1(K) = 0$, and $\int_K w\, d\mu_1(w) = 1$. Furthermore since $\mathrm{I}(\nu_i) < \infty$, we have that $\mathrm{I}(|\mu_1|) < \infty$. The proof of part(a) is complete.

To prove part(b), let K_1 and K_2 be as above. Choose a probability measure ξ_1 in $\mathrm{M}^+(K_1)$ with finite energy integral $\mathrm{I}(\xi_1)$. Obviously, if $\int_K w\, d\xi_1(w) = 0$, then we are done. So we may assume that the value of the integral $\int_K w\, d\xi_1(w)$ is a non-zero constant, say C. In light of part(a), we can choose $\xi_2 \in \mathrm{M}(K_2)$ with finite energy integral $\mathrm{I}(|\xi_2|)$ such that

$$\xi_2(K_2) = 0 \text{ and } \int_K w\, d\xi_2(w) = 1.$$

Finally if we set $\mu_2 \equiv \xi_1 - C\xi_2$, then μ_2 is the desired measure. □

Now we are ready to state our main result.

THEOREM 3.3.4. *If K is a compact subset of $\mathbb{D}$ such that $L^2_a(\mathbb{C}\setminus K)$ is non-trivial and if $0 \in K$, then $z\big[\mathrm{dom}\, S_{\mathbb{C}\setminus K}\big]$ is dense in $L^2_a(\mathbb{C}\setminus K)$ if and only if $z\big[L^2_a(\mathbb{D}\setminus K)\big]$ is dense in $L^2_a(\mathbb{D}\setminus K)$.*

Note that in the easy case where K has a non-empty interior containing 0, the function $z \mapsto \dfrac{1}{z}$ belongs to $H^\infty(\mathbb{C}\setminus K)$. Consequently $M_{\frac{1}{z},\Omega\setminus K}$, where Ω denotes either the complex plane $\mathbb{C}$ or the open unit disc $\mathbb{D}$, is a bounded operator on $L^2_a(\Omega\setminus K)$; so the theorem holds trivially. Therefore *throughout the proof we may assume that* $0 \in \partial K$.

PROOF OF NECESSITY. Assume that $z\big[\operatorname{dom} S_{\mathbb{C}\setminus K}\big]$ is dense in $L^2_a(\mathbb{C}\setminus K)$.

Let U be a bounded open set in the complex plane and suppose that z_0 belongs to the boundary ∂U. Denote by $\mathrm{A}_{2,U}(z_0)$ the collection of all functions g in $L^2_a(U)$ for which there is a neighborhood U_g of z_0 such that g has an analytic extension to $U\cup U_g$. Hedberg [**13**] has shown that $\mathrm{A}_{2,U}(z_0)$ is dense in $L^2_a(U)$, see Lemma 3 on page 279.

For convenience put $\mathcal{F}=\mathrm{A}_{2,\mathbb{D}\setminus K}(0)$. If g belongs to $\mathcal{F}$, define h_g by $h_g\equiv\dfrac{g-g(0)}{z}$. Since g is analytic in a neighborhood U_g of 0, h_g belongs to $\operatorname{Hol}\big((\mathbb{D}\setminus K)\cup U_g\big)$. Now if we choose $r>0$ such that $\operatorname{cl}B(0,r)\subseteq U_g$, then

$$\int_{\mathbb{D}\setminus K}|h_g|^2\,d\mathcal{A}\le\int_{\operatorname{cl}B(0,r)}|h_g|^2\,d\mathcal{A}+\int_{\left(\mathbb{D}\setminus K\right)\setminus\operatorname{cl}B(0,r)}|h_g|^2\,d\mathcal{A}\,.$$

The first integral in the above inequality is bounded by $\max\{|h_g(z)|^2\colon\ z\in\operatorname{cl}B(0,r)\}\cdot\pi r^2$. For the second integral we have that

$$\int_{\left(\mathbb{D}\setminus K\right)\setminus\operatorname{cl}B(0,r)}|h_g|^2\,d\mathcal{A}\le\frac{2}{r^2}\left(\int_{\mathbb{D}\setminus K}|g|^2\,d\mathcal{A}+\int_{\mathbb{D}\setminus K}|g(0)|^2\,d\mathcal{A}\right)<\infty.$$

Hence $h_g\in L^2_a(\mathbb{D}\setminus K)$.

Next we make the assumption that there is a non-zero function f in $L^2_a(\mathbb{D}\setminus K)$ such that f is orthogonal to the linear manifold $z\big[L^2_a(\mathbb{D}\setminus K)\big]$;

that is $\langle f, g\rangle = 0$ for all g in $z\big[L^2_a(\mathbb{D}\backslash K)\big]$. Our goal is to reach a contradiction. Now by the preceding argument, for g in $\mathcal{F}$, the corresponding function h_g belongs to $L^2_a(\mathbb{D}\setminus K)$; so our assumption on f implies that

$$0 = \langle f, zh_g\rangle = \langle f, g\rangle - \overline{g(0)}\langle f, 1\rangle.$$

In other words the above inequality shows that

$$(\star) \qquad \langle f, g\rangle_{L^2_a(\mathbb{D}\backslash K)} = \overline{g(0)}\,\langle f, 1\rangle_{L^2_a(\mathbb{D}\backslash K)} \quad \text{for all} \quad g \in \mathcal{F}.$$

CLAIM. $\langle f, 1\rangle_{L^2_a(\mathbb{D}\backslash K)} = 0$.

If the claim holds, then f is orthogonal to $\mathcal{F}$. But since $\mathcal{F}$ is dense in $L^2_a(\mathbb{D}\setminus K)$, f must be zero; hence the proof of the necessity part.

In order to prove the claim we must use the assumption regarding the density of $z\big[\operatorname{dom} S_{\mathbb{C}\backslash K}\big]$ in $L^2_a(\mathbb{C}\setminus K)$. So if h is an arbitrary non-zero function in $L^2_a(\mathbb{C}\setminus K)$, we can find a sequence $\{h_n\}$ in $\operatorname{dom} S_{\mathbb{C}\backslash K}$ such that $zh_n \to h$ in $L^2_a(\mathbb{C}\setminus K)$. By the Cauchy-Schwartz inequality and our assumption on f, it follows that

$$\begin{aligned}
\big|\langle f, h\rangle_{L^2_a(\mathbb{D}\backslash K)}\big| &= \big|\langle f, h - zh_n\rangle_{L^2_a(\mathbb{D}\backslash K)} + \langle f, zh_n\rangle_{L^2_a(\mathbb{D}\backslash K)}\big| \\
&= \big|\langle f, h - zh_n\rangle_{L^2_a(\mathbb{D}\backslash K)}\big| \\
&\le \|f\|_{L^2_a(\mathbb{D}\backslash K)}\,\|h - zh_n\|_{L^2_a(\mathbb{D}\backslash K)}.
\end{aligned}$$

Consequently

$$(\star\star) \qquad \langle f, h\rangle_{L^2_a(\mathbb{D}\setminus K)} = 0 \quad \text{for all} \quad h \in L^2_a(\mathbb{C}\setminus K).$$

Since $L^2_a(\mathbb{C}\setminus K)$ is non-trivial, Theorem 1.6.7implies that K has positive logarithmic capacity. By Lemma 3.3.2, there are points z_1, z_2 in K and positive numbers r_1, r_2 such that $\operatorname{cl} B(z_1, r_1) \cap \operatorname{cl} B(z_2, r_2) = \emptyset$; and if $K_i = \operatorname{cl} B(z_i, r_i) \cap K$ then $c(K_i) > 0$ for $i = 1, 2$. By modifying the proof of Lemma 3.3.2, if necessary, we may assume that neither z_1 nor z_2 is 0. Again by adjusting r_1, r_2, if necessary, along with the fact that a single point has zero logarithmic capacity, we may further assume that $0 \notin \operatorname{cl} B(z_1, r_1) \cup \operatorname{cl} B(z_2, r_2)$. Since $c(K_i) > 0$ for $i = 1, 2$, we can choose probability measures μ_i in $\mathrm{M}^+(K_i)$ such that the energy integrals $\mathrm{I}(\mu_i)$ are finite.

Now if we let

$$\mu \equiv \mu_1 - \mu_2,$$

then from Lemma 3.2.5 we see that $\widehat{\mu} \in L^2_a(\mathbb{C}\setminus K)$. Furthermore, as a consequence of our construction, $0 \notin \operatorname{supp}(\mu)$; hence $\widehat{\mu}$ is analytic in a neighborhood of 0. By identities $(\star)$ and $(\star\star)$

$$\overline{\widehat{\mu}(0)}\, \langle f, 1\rangle_{L^2_a(\mathbb{D}\setminus K)} = 0.$$

So the proof of the claim is complete, if it can be shown that $\widehat{\mu}(0) \neq 0$. To see this, note that

$$\begin{aligned}\widehat{\mu}(0) &= \widehat{\mu}_1(0) - \widehat{\mu}_2(0) \\ &= \int_{K_1} \frac{1}{w}\, d\mu_1(w) - \int_{K_2} \frac{1}{w}\, d\mu_2(w).\end{aligned}$$

Since 0 does not belong to $\operatorname{cl} B(z_i, r_i)$, it follows that $\varphi\colon z \mapsto \dfrac{1}{z}$ is conformal on $\operatorname{cl} B(z_i, r_i)$. One easily sees that the image of $\operatorname{cl} B(z_i, r_i)$ under φ is a closed disc in $\mathbb{C}$. Thus

$$\overline{\operatorname{co}}\big[\varphi\big(\operatorname{cl} B(z_i, r_i)\big)\big] = \varphi\big(\operatorname{cl} B(z_i, r_i)\big).$$

Furthermore since φ is one-to-one, we get

$$\varphi\big(\operatorname{cl} B(z_1, r_1)\big) \cap \varphi\big(\operatorname{cl} B(z_2, r_2)\big) = \emptyset.$$

Since discs are obviously convex sets, there are points $c_i \in \overline{\operatorname{co}}\big(\varphi(K_i)\big)$ such that

$$c_i = \int_{K_i} \frac{1}{w}\, d\mu_i(w), \quad \text{for } i = 1, 2.$$

Finally noting that $\overline{\operatorname{co}}\big(\varphi(K_i)\big) \subseteq \overline{\operatorname{co}}\big[\varphi\big(\operatorname{cl} B(z_i, r_i)\big)\big]$, the above argument implies that $c_1 \neq c_2$. This concludes the proof of the claim and the necessity part. $\square$

PROOF OF THE SUFFICIENCY. In this part of the proof, we assume that $z\big[L_a^2(\mathbb{D}\backslash K)\big]$ is dense in $L_a^2(\mathbb{D}\backslash K)$. Let Δ denote $\mathbb{C}\backslash\mathbb{D}$, and define

$$\mathcal{N} = \left\{\widehat{\mu}\colon \mu \in \mathrm{M}(K\cup\Delta) \quad \text{and} \quad \widehat{\mu} \in L_a^2(\mathbb{D}\setminus K)\right\}.$$

By Theorem 3.2.7, $\mathcal{N}$ is a dense subset of $L_a^2(\mathbb{D}\setminus K)$. From the boundeness of $\mathbb{D}\setminus K$ combined with our assumption, it follows that $z[\mathcal{N}]$ is dense in $L_a^2(\mathbb{D}\setminus K)$.

Let f be an arbitrary non-zero function in $L_a^2(\mathbb{C}\setminus K)$. The last statement is justified due to the fact that $L_a^2(\mathbb{C}\setminus K)$ is assumed to be non-trivial. So we can pick a sequence of measures $\{\sigma_n\}$ in $\mathrm{M}(K\cup\Delta)$ such that

$$z\widehat{\sigma}_n \to f \quad \text{in} \quad L_a^2(\mathbb{D}\setminus K).$$

From the mean value property it follows also that

$$z\widehat{\sigma}_n(z) \to f(z) \quad \text{uniformly on compact subsets of } \mathbb{D}\setminus K.$$

Since $K\cap\Delta = \emptyset$, we can decompose each σ_n , $n \geq 1$, as

$$\sigma_n \equiv \mu_n + \gamma_n,$$

where $\mathrm{supp}\,(\mu_n) \subseteq K$ and $\mathrm{supp}\,(\gamma_n) \subseteq \Delta$. By linearity of the Cauchy transform it follows that

$$\widehat{\sigma}_n \equiv \widehat{\mu}_n + \widehat{\gamma}_n.$$

CLAIM. *For $n \geq 1$, the following statements hold.*

(a) $\widehat{\mu}_n(\infty) = 0$ *and* $\widehat{\mu}_n \in \mathrm{Hol}(\mathbb{C}_\infty \setminus K)$.

(b) $\widehat{\gamma}_n(\infty) = 0$ *and* $\widehat{\gamma}_n \in \mathrm{Hol}(\mathbb{C}_\infty \setminus \mathrm{supp}\,(\gamma_n))$.

(c) $\widehat{\mu}_n$ *and* $\widehat{\gamma}_n$ *belong to* $L^2_a(\mathbb{D} \setminus K)$.

(d) $\mu_n(K) \to 0$.

(e) $z\widehat{\mu}_n(z) \to f(z)$ *uniformly on compact subsets of* $\mathbb{C} \setminus K$.

(f) $\displaystyle\int w\, d\mu_n(w) \to 0$.

PROOF OF THE CLAIM. Properties (a) and (b) are obvious from the definitions of μ_n and γ_n. For each $n \geq 1$, in view of Theorem 1.5.4, along with (a) and (b), it follows that $\widehat{\mu}_n$, $\widehat{\gamma}_n$ are the unique corresponding functions in the decomposition of $\widehat{\sigma}_n$. Therefore by Proposition 3.1.1, $\widehat{\mu}_n$ and $\widehat{\gamma}_n$ belong to $L^2_a(\mathbb{D} \setminus K)$. This proves part(c).

To prove part(d), let Γ denote a positive Jordan system in $\mathbb{D} \setminus K$ such that K is contained in $\mathrm{ins}\,\Gamma$. By Lemma 3.2.5, part(a)

$$\begin{aligned}
\mu_n(K) &= \frac{-1}{2\pi i}\int_\Gamma \widehat{\mu}_n(w)\, dw \\
&= \frac{-1}{2\pi i}\int_\Gamma \big(\widehat{\sigma}_n(w) - \widehat{\gamma}_n(w)\big)\, dw \\
&= \frac{-1}{2\pi i}\int_\Gamma \widehat{\sigma}_n(w)\, dw.
\end{aligned}$$

In the last equality we have used the fact that $\widehat{\gamma}_n \in \mathrm{Hol}(\mathbb{D})$. Now since $0 \notin \Gamma$ and $\widehat{\sigma}_n(w) \to \dfrac{f(w)}{w}$ uniformly on Γ, it follows that

$$\mu_n(K) \to \frac{-1}{2\pi i} \int_\Gamma \frac{f(w)}{w}\, dw.$$

As a consequence of the Cauchy Integral Formula

$$\begin{aligned} \left| \int_\Gamma \frac{f(w)}{w}\, dw \right| &= \left| \int_{|w|=R} \frac{f(w)}{w}\, dw \right| \\ &\leq \frac{1}{R} \int_{|w|=R} |f(w)|\, d|w|, \end{aligned}$$

whenever $R > 1$. But since f vanishes at infinity, the above inequality implies that $\int_\Gamma \frac{f(w)}{w}\, dw = 0$. This proves part(d).

For part(e), let Γ be as above and for $n \geq 1$, define F_n by

$$F_n(z) = \frac{-1}{2\pi i} \int_\Gamma \frac{w\widehat{\mu}_n(w)}{w - z}\, dw,$$

where z belongs to $\operatorname{out}\Gamma$. It follows that the definition of F_n is independent of the choice of Γ. Moreover $F_n \in \mathrm{Hol}(\mathbb{C}_\infty \setminus K)$, for all $n \geq 1$. We will show that

$$F_n(z) \to f(z) \quad \text{uniformly on compact subsets of } \mathbb{C} \setminus K.$$

To see this, let E denote a compact set in $\mathbb{C} \setminus K$ and let Γ be defined as above with an additional property that $E \subseteq \operatorname{out}\Gamma$. For z in E, from Corollary 1.5.7, we see that

$$f(z) = \frac{-1}{2\pi i} \int_\Gamma \frac{f(w)}{w - z}\, dw.$$

Consequently

$$\begin{aligned}\left|f(z)-F_n(z)\right| &= \frac{1}{2\pi}\left|\int_\Gamma \frac{f(w)}{w-z}\,dw - \int_\Gamma \frac{w\big(\widehat{\sigma}_n(w)-\widehat{\gamma}_n(w)\big)}{w-z}\,dw\right| \\ &\leq \frac{1}{2\pi}\int_\Gamma \frac{\left|f(w)-w\widehat{\sigma}_n(w)\right|}{|w-z|}\,d|w| \\ &\leq C_E \int_\Gamma \left|f(w)-w\widehat{\sigma}_n(w)\right|\,d|w|,\end{aligned}$$

where $C_E > 0$ is a constant depending only on the compact set E. Now the result follows from a uniform convergence argument. In Lemma 3.2.6 part(c), it has been shown that

$$F_n(z) = \mu_n(K) + z\widehat{\mu}_n(z).$$

Hence the proof of part(e) follows by using the result of part(d).

It remains to show that (f) also holds. By Lemma 3.1.2, $f'(\infty) = 0$. Furthermore since ∞ is a removable singularity for f and F_n, it follows from the proof of part(e) that

$$F_n'(\infty) \to f'(\infty) = 0.$$

On the other hand since $F_n(\infty) = 0$, for $n \geq 1$, we have that

$$F_n'(\infty) = \lim_{z\to\infty} zF_n(z) = \frac{1}{2\pi i}\int_\Gamma w\widehat{\mu}_n(w)\,dw.$$

Now apply the identity established in Lemma 3.2.6, part(b). □

Having proved the claim, we will next construct a sequence of measures $\{\nu_n\}$ in $\mathrm{M}(K)$ such that $\{\widehat{\nu}_n\} \in \operatorname{dom} S_{\mathbb{C}\setminus K}$ and $z\widehat{\nu}_n \to f$ in $L^2_a(\mathbb{C} \setminus K)$, proving the sufficiency part.

By Lemma 3.3.3, there are measures ξ_1 , ξ_2 in $\mathrm{M}(K)$ with finite energy integrals $\mathrm{I}(|\xi_1|)$, $\mathrm{I}(|\xi_2|)$ such that :

$$\xi_1(K) = 1, \qquad \xi_2(K) = 0,$$

and

$$\int w\, d\xi_1(w) = 0; \qquad \int w\, d\xi_2(w) = 1.$$

Next for $n = 1, 2, \ldots$, define ν_n by

$$\nu_n \equiv \mu_n - A_n\xi_1 - B_n\xi_2,$$

where $A_n = \mu_n(K)$ and $B_n = \int w\, d\mu_n(w)$.

Fix $n \geq 1$; since

$$\widehat{\nu}_n = \widehat{\mu}_n - A_n\widehat{\xi_1} - B_n\widehat{\xi_2},$$

it follows that

$$\widehat{\nu}_n \in \mathrm{Hol}(\mathbb{C}_\infty \setminus K) \qquad \text{and} \qquad \widehat{\nu}_n(\infty) = 0.$$

In view of Lemma 3.2.4, $\widehat{\xi_1}$ and $\widehat{\xi_2}$ are in $L^2_{loc}(\mathbb{C})$. So from the Claim ,part(c), it follows that $\widehat{\nu}_n \in L^2_{loc}(\mathbb{C})$. Now the definition of ξ_1 and ξ_2,

along with Proposition 3.2.2 part(d), imply that

$$\widehat{\nu}_n'(\infty) = -\nu_n(K) = 0,$$

$$\widehat{\nu}_n''(\infty) = -\int w d\nu_n(w) = 0.$$

Finally from Lemma 3.1.2, we see that $\widehat{\nu}_n$ and $z\widehat{\nu}_n$ are in $L^2_a(U)$, where U is a neighborhood of infinity. Hence we conclude that $\widehat{\nu}_n \in \operatorname{dom} S_{\mathbb{C}\setminus K}$.

It is left to show that $z\widehat{\nu}_n \to f$ in $L^2_a(\mathbb{C}\setminus K)$. Choose $r < 1$ such that $K \subseteq B(0,r)$, and for convenience put $B_r = B(0,r)$. We see that

$$\begin{aligned}\left\|z\widehat{\nu}_n - f\right\|_{L^2_a(\mathbb{C}\setminus K)} &\leq \left\|z\widehat{\nu}_n - f\right\|_{L^2(B_r\setminus K)} + \\ &+ \left\|z\widehat{\nu}_n - f\right\|_{L^2(\{r\leq|z|\leq 1\})} + \left\|z\widehat{\nu}_n - f\right\|_{L^2(\Delta)} \\ &= I_n^{(1)} + I_n^{(2)} + I_n^{(3)}.\end{aligned}$$

From the Claim, we have that $z\widehat{\nu}_n \to f$ uniformly on compact subsets of $\mathbb{C}\setminus K$; thus $I_n^{(2)} \to 0$. Using the definition of $\widehat{\nu}_n$, we observe that

$$\begin{aligned}I_n^{(1)} &\leq \left\|z\widehat{\sigma}_n - f\right\|_{L^2(B_r\setminus K)} + \\ &+ \|z\widehat{\gamma}_n\|_{L^2(B_r\setminus K)} + |A_n|\,\|\widehat{\xi}_1\|_{L^2(B_r\setminus K)} + |B_n|\,\|\widehat{\xi}_2\|_{L^2(B_r\setminus K)}.\end{aligned}$$

In view of the Claim, part(e), it follows easily that $z\widehat{\gamma}_n \to 0$ uniformly on compact subsets of $\mathbb{D}\setminus K$. Now by analycity of $\widehat{\gamma}_n$ on $\operatorname{cl} B_r$, along with the Maximum Modulus Theorem, we conclude that $\|z\widehat{\gamma}_n\|_{L^2(B_r\setminus K)} \to 0$. Again by applying the Claim, parts (d) and (f), it follows that $I_n^{(1)} \to 0$.

For $I_n^{(3)}$ we will use the change of variable $w\colon z \to \dfrac{1}{z}$ to get

$$\left[I_n^{(3)}\right]^2 = \int_{\{0<|w|\leq 1\}} |\Phi_n(w)|^2 \, d\mathcal{A}(w),$$

where

$$\Phi_n(w) = \frac{1}{w^2}\left[\frac{1}{w}\widehat{\nu}_n(\frac{1}{w}) - f(\frac{1}{w})\right].$$

Clearly $\Phi_n \in \mathrm{Hol}(\mathrm{cl}\,\mathbb{D} \setminus \{0\})$, and moreover

$$\lim_{w\to 0} w\Phi_n(w) = \lim_{z\to\infty}\left[z^2\widehat{\nu}_n(z) - zf(z)\right] = \widehat{\nu}_n{}''(\infty) - f'(\infty) = 0.$$

Hence Φ_n is analytic on $\mathrm{cl}\,\mathbb{D}$. Consequently

$$I_n^{(3)} \leq \sqrt{\pi} \sup_{|w|\leq 1} |\Phi_n(w)|.$$

So the result follows from the fact that $z\widehat{\nu}_n(z) \to f(z)$ uniformly on $\partial\mathbb{D}$. The proof of the theorem is complete. □

Now we are in the position to present a proof for Theorem 3.3.1.

PROOF OF THEOREM 3.3.1. In Corollary 1.3.3, it has been shown that the density of $\mathrm{dom}\, S_G$ in $L_a^2(G)$ is equivalent to the density of $\mathrm{dom}\, S_\Omega$, where Ω is the image of G under the linear mapping , T, given by $T(z) = rz + c$, where $r > 0$. Therefore without loss of generality we may assume that $a = 0$, and $r = 1$. Set $G^* = G \setminus \{0\}$. Since 0 is an isolated point, it is a removable boundary point for $L_a^2(G^*)$; see

Theorem 2.1.2, part(a). Hence $L^2_a(G^*)$ is isomorphic to $L^2_a(G)$ via the natural restriction mapping.

Let K denote the image of $\mathbb{C}_\infty \setminus G^*$ under the mapping φ given by $\varphi\colon z \mapsto \frac{1}{z}$. Since G contains $\operatorname{cl}\mathbb{D}$, it is easy to see that K is a compact subset of $\mathbb{D}$ and obviously $0 \in \partial K$. Now from Theorem 1.3.2, part(b), we see that $\operatorname{dom} S_{G^*}$ is dense in $L^2_a(G^*)$ if and only if $\operatorname{dom} M_{\frac{1}{z},\mathbb{C}\setminus K}$ is dense in $L^2_a(\mathbb{C} \setminus K)$. On the other hand it is easy to show that $\operatorname{dom} M_{\frac{1}{z},\mathbb{C}\setminus K} = z\big[\operatorname{dom} S_{\mathbb{C}\setminus K}\big]$. To see this fact, first let $f \in \operatorname{dom} M_{\frac{1}{z},\mathbb{C}\setminus K}$, then $f/z = g$ belongs to $L^2_a(\mathbb{C} \setminus K)$; hence $f = zg$ where g clearly is in $\operatorname{dom} S_{\mathbb{C}\setminus K}$. For the proof of the other direction note that if $g \in z\big[\operatorname{dom} S_{\mathbb{C}\setminus K}\big]$, then $g = zf$ and f belongs to $\operatorname{dom} S_{\mathbb{C}\setminus K}$. Thus $g \in L^2_a(\mathbb{C}\setminus K)$ and, moreover, $g/z = f \in L^2_a(\mathbb{C}\setminus K)$. Now the result follows from Theorem 3.3.4. □

3.4. Wiener Capacity and the Density Problem

In light of Theorem 3.3.1, the density problem for the Bergman operator can now be considered as a local problem regarding the density of the range of operator of multiplication by z on the Bergman space of a bounded open set. Moreover, as it has been shown in Chapter 2, Theorem 2.2.1 is the best result in the sense of a topological condition. All these facts strongly indicate the usage of capacity and potential theory for additional refinements. In this section we introduce the

Wiener capacity and its relation to the density problem. Theorem 3.4.5, Theorem 3.4.6, and Theorem 3.4.9 are the main results of this section.

DEFINITION 3.4.1. For a compact subset K of the unit disc $\mathbb{D}$, we define the *Wiener capacity,* c_2, as

$$c_2(K) = \sup_{\mu} \mu(K),$$

where the supremum is taken over all measures μ in $\mathrm{M}^+(K)$ for which the potential $U_\mu(z) = \int \log \frac{1}{|z-w|}\, dw$ satisfies $U_\mu(z) \leq 1$ everywhere in the plane.

Next, for arbitrary sets E in $\mathbb{D}$, we define

$$c_2(E) = \sup\{c_2(K): \quad K \subseteq E\}.$$

It can be shown, see Landkof [**15**] pages 167–168, that if K is a compact subset of $\mathbb{D}$, then there exists a unique equilibrium measure μ in $\mathrm{M}^+(K)$ such that $c_2(K) = \mu(K)$ and $U_\mu(z) \leq 1$ everywhere in the plane. It turns out that c_2 is a true capacity in the sense of Choquet. More importantly, the Wiener capacity and the logarithmic capacity are closely related via the relation

$$c(K) = e^{-c_2^{-1}(K)} \tag{3.4.2}$$

when $K \subseteq \mathbb{D}$ is compact, see Landkof [**15**] page 167. Thus the two capacities are equivalent; that is, a subset E of $\mathbb{D}$ has logarithmic capacity zero if and only if it has Wiener capacity zero.

Before exhibiting how the Wiener capacity and the density problem are related to each other, we need a lemma.

LEMMA 3.4.3. *If K is a compact set in $\mathbb{D}$ which contains 0, then $z\big[L^2_a(\mathbb{D}\setminus K)\big]$, is dense in $L^2_a(\mathbb{D}\setminus K)$ if and only if the constant function 1 belongs to* $\mathrm{cl}\left(z\big[L^2_a(\mathbb{D}\setminus K)\big]\right)$, *where the closure is taken with respect to the L^2 norm.*

PROOF. The necessary part is obvious. To prove the sufficiency, assume that there is a non-zero function f in $L^2_a(\mathbb{D}\setminus K)$ such that f is orthogonal to $\mathrm{cl}\left(z\big[L^2_a(\mathbb{D}\setminus K)\big]\right)$. As in the proof of Theorem 3.3.4, it follows that

$$\langle f, g\rangle_{L^2_a(\mathbb{D}\setminus K)} = \overline{g(0)}\,\langle f, 1\rangle_{L^2_a(\mathbb{D}\setminus K)} \quad \text{for all} \quad g \in \mathrm{A}_{2,\mathbb{D}\setminus K}(0),$$

where $\mathrm{A}_{2,\mathbb{D}\setminus K}(0)$ denotes the set of all functions in $L^2_a(\mathbb{D}\setminus K)$ that can be analytically extended to some neighborhood of 0.

Now using the assumption that $1 \in \mathrm{cl}\left(z\big[L^2_a(\mathbb{D}\setminus K)\big]\right)$, we see that

$$\langle f, 1\rangle_{L^2_a(\mathbb{D}\setminus K)} = 0.$$

Hence f must be orthogonal to $\mathrm{A}_{2,\mathbb{D}\setminus K}(0)$. On the other hand, we have that $\mathrm{A}_{2,\mathbb{D}\setminus K}(0)$ is dense in $L^2_a(\mathbb{D}\setminus K)$, see the proof of Theorem 3.3.4. Thus f is the zero function, a contradiction. □

The next technical lemma can be considered as the main result from which Theorem 3.4.5, and Theorem 3.4.6 follow as consequences. Before presenting the lemma, we fix the following notation. If z_0 is a point in $\mathbb{C}$ and r_n , r_{n+1} are two positive numbers such that $r_{n+1} < r_n$, then denote by $A_{r_n}(z_0)$ the annulus $\{r_{n+1} < |z - z_0| \leq r_n\}$.

LEMMA 3.4.4. *Let K be a compact set in $\mathbb{D}$ which contains the point z_0, and suppose that $\{r_n\}_{n\geq 0}$ is a sequence of positive numbers such that $r_n \downarrow 0$, $r_0 = 1$, $r_n \geq 2\, r_{n+1}$, and $r_n \leq 2^{-n}$ for all $n \geq 1$. If*

$$\sum_{n=1}^{\infty} {r_n}^{-2} c_2\big(A_{r_n}(z_0) \cap K\big) = +\infty,$$

then $(z - z_0)\big[L^2_a(\mathbb{D}\setminus K)\big]$ is dense in $L^2_a(\mathbb{D}\setminus K)$.

PROOF. In view of Proposition 1.6.4 (b), along with (3.4.2), we may assume that $z_0 = 0$. For the sake of brevity, $A_{r_n}(0)$ will be denoted as A_n. By Definition 3.4.1, to each integer $n \geq 1$, there corresponds a compact set K_n in $A_n \cap K$ such that

$$c_2(K_n) \geq c_2\big(A_n \cap K\big) - \varepsilon_n,$$

where $\{\varepsilon_n\}$ is a sequence of positive number such that

$$\sum_{n=1}^{\infty} {r_n}^{-2}\varepsilon_n < \infty.$$

It follows that

$$\sum_{n=1}^{\infty} {r_n}^{-2}c_2(K_n) = +\infty.$$

For each K_n, where $n = 1, 2, \ldots$, let μ_n denote the corresponding equilibrium measure; that is,

$$\mu_n \in \mathrm{M}^+(K_n),\ c_2(K_n) = \mu_n(K_n),\ \text{and } U_{\mu_n}(z) \leq 1 \text{ for all } z \in \mathbb{C}.$$

Now given a positive integer m, we can find an integer $\rho = \rho(m)$ greater than m such that

$$\sum_{n=m}^{\rho} {r_n}^{-2}c_2(K_n) \geq m.$$

Set $C_m = \sum_{n=m}^{\rho} {r_n}^{-2}c_2(K_n)$, and for each $m = 1, 2, \ldots$ define the measure σ_m by

$$\sigma_m \equiv \frac{-1}{C_m}\sum_{n=m}^{\rho} {r_n}^{-2}\mu_n.$$

An easy computation shows that

$$-z\widehat{\mu}_n(z) = \mu_n(K) - f_n(z),$$

where

$$f_n(z) = \int_{K_n} \frac{w\, d\mu_n(w)}{w - z}.$$

Thus we have that

$$z\widehat{\sigma}_m(z) = 1 + \frac{1}{C_m}\sum_{n=m}^{\rho} {r_n}^{-2} f_n(z).$$

Clearly $\widehat{\sigma}_m$ belong to $L^2_a(\mathbb{D}\setminus K)$ for all $m \geq 1$. Now the result follows from Lemma 3.4.3 if it can be shown that $z\widehat{\sigma}_m \to 1$ in $L^2_a(\mathbb{D}\setminus K)$. First of all we have

$$\begin{aligned}
\left\|1 - z\widehat{\sigma}_m(z)\right\|^2_{L^2_a(\mathbb{D}\setminus K)} &= \int_{\mathbb{D}\setminus K} C_m^{-2}\left|\sum_{n=m}^{\rho} {r_n}^{-2} f_n(z)\right|^2 d\mathcal{A}(z) \\
&= C_m^{-2}\sum_{l=0}^{\infty}\int_{A_l\setminus K}\left|\sum_{n=m}^{\rho} {r_n}^{-2} f_n(z)\right|^2 d\mathcal{A}(z) \\
&\leq C_m^{-2}\sum_{l=0}^{\infty}\int_{A_l\setminus K}\left(\sum_{n=m}^{\rho} {r_n}^{-2}|f_n(z)|\right)^2 d\mathcal{A}(z).
\end{aligned}$$

We break the last sum into three parts, depending on ρ, and show that each part tends to 0 as $m \to \infty$. Throughout the proof, the index n always belongs to the set $\{m, m+1, \ldots, \rho-1, \rho\}$, and different constants will be denoted by F. Now using the above inequality, we get

$$\begin{aligned}
\left\|1 - z\widehat{\sigma}_m(z)\right\|^2_{L^2_a(\mathbb{D}\setminus K)} \leq\ & C_m^{-2}\sum_{l=0}^{m-2}\int_{A_l\setminus K}\left(\sum_{n=m}^{\rho} {r_n}^{-2}|f_n(z)|\right)^2 d\mathcal{A}(z) + \\
& C_m^{-2}\sum_{l=m-1}^{\rho+1}\int_{A_l\setminus K}\left(\sum_{n=m}^{\rho} {r_n}^{-2}|f_n(z)|\right)^2 d\mathcal{A}(z) + \\
& C_m^{-2}\sum_{l=\rho+2}^{\infty}\int_{A_l\setminus K}\left(\sum_{n=m}^{\rho} {r_n}^{-2}|f_n(z)|\right)^2 d\mathcal{A}(z) \\
=\ & S_m^{(1)} + S_m^{(2)} + S_m^{(3)}.
\end{aligned}$$

First we consider the sum $S_m^{(3)}$. If $l \geq \rho + 2$, $z \in A_l$, and $w \in A_n$, then it is easy to see that

$$|w - z| \geq r_{n+1} - r_{\rho+2} \geq F\, r_{n+1}.$$

As a result

$$|f_n(z)| \leq \int_{K_n} \frac{|w|\, d\mu_n(w)}{|w - z|} \leq F \frac{r_n}{r_{n+1}} c_2(K_n).$$

Noting that Area $(A_l \setminus K) \leq \pi r_l^2$, we get

$$S_m^{(3)} \leq F\, C_m^{-2} \sum_{l=\rho+2}^{\infty} \left(\sum_{n=m}^{\rho} \frac{r_n}{r_{n+1}} r_n^{-2} c_2(K_n) \right)^2 r_l^2.$$

Using the identity $\dfrac{r_l}{r_{n+1}} = \dfrac{r_l}{r_{l-1}} \cdot \dfrac{r_{l-1}}{r_{l-2}} \cdots \dfrac{r_{n+3}}{r_{n+2}} \cdot \dfrac{r_{n+2}}{r_{n+1}}$, it follows from the assumption

$$\frac{r_l}{r_{n+1}} r_n \leq 2^{-(l-n-1)} 2^{-n} \leq F\, 2^{-l}.$$

Thus

$$S_m^{(3)} \leq F\, C_m^{-2} \sum_{l=\rho+2}^{\infty} 2^{-2l} C_m^2 = F \sum_{l=\rho+2}^{\infty} 2^{-2l},$$

which clearly tends to 0 as $m \to \infty$.

For the first sum $S_m^{(1)}$, if $l \leq m - 2$, $z \in A_l$, and $w \in A_n$, then

$$|w - z| \geq r_{l+1} - r_n \geq F\, r_{l+1}.$$

By the same sort of reasoning as above it follows that

$$|f_n(z)| \leq F \frac{r_n}{r_{l+1}} c_2(K_n)$$

and

$$\frac{r_n}{r_{l+1}} \leq F\, 2^{-n+l}.$$

Consequently

$$\begin{aligned} S_m^{(1)} &\leq F\, C_m^{-2} \sum_{l=0}^{m-2} \left(\sum_{n=m}^{\rho} 2^{-n+l} {r_n}^{-2} c_2(K_n) \right)^2 r_l^2 \\ &\leq F\, C_m^{-2} \sum_{l=0}^{m-2} 2^{-2m} \left(\sum_{n=m}^{\rho} {r_n}^{-2} c_2(K_n) \right)^2 \\ &\leq F\, m\, 2^{-2m}. \end{aligned}$$

Hence $S_m^{(1)} \to 0 \quad \text{as} \quad m \to \infty$.

For convenience let $a_n(z) = r_n^{-2} |f_n(z)|$. We break $S_m^{(2)}$'s inner sum into three separate parts as

$$S_m^{(2)} = C_m^{-2} \sum_{l=m-1}^{\rho+1} \int_{A_l \setminus K} \left(\sum_{n=m}^{l-2} a_n(z) + \sum_{n=l-1}^{l+1} a_n(z) + \sum_{n=l+2}^{\rho} a_n(z) \right)^2 d\mathcal{A}(z).$$

Now noting that in the the middle sum $\sum_{n=l-1}^{l+1} a_n(z)$, we only sum over a fixed number of terms, it follows that

$$\begin{aligned} S_m^{(2)} \leq F \Bigg\{ & C_m^{-2} \sum_{l=m-1}^{\rho+1} \int_{A_l \setminus K} \left(\sum_{n=m}^{l-2} a_n(z) + \sum_{n=l+2}^{\rho} a_n(z) \right)^2 + \\ & C_m^{-2} \sum_{l=m-1}^{\rho+1} \int_{A_l \setminus K} \sum_{n=l-1}^{l+1} a_n^2(z)\, d\mathcal{A}(z) \Bigg\} \\ = F & \left\{ J_m^{(1)} + J_m^{(2)} \right\}. \end{aligned}$$

By the same sort of argument as it has been shown for the sums $S_m^{(1)}$, $S_m^{(3)}$, it is not hard to see that $J_m^{(1)} \to 0$. In order to complete

the proof, it is only left to show that $J_m^{(2)}$ also tends to 0 as $m \to \infty$.

To see this, note that

$$\begin{aligned} J_m^{(2)} &\leq F\, C_m^{-2} \sum_{l=m-1}^{\rho+1} \int_{A_l \setminus K} \sum_{n=m}^{\rho} a_n^2(z)\, d\mathcal{A}(z) \\ &\leq F\, C_m^{-2} \int_{\mathbb{D}\setminus K} \sum_{n=m}^{\rho} a_n^2(z)\, d\mathcal{A}(z) \\ &= F\, C_m^{-2} \sum_{n=m}^{\rho} r_n^{-4} \int_{\mathbb{D}\setminus K} |f_n(z)|^2\, d\mathcal{A}(z). \end{aligned}$$

From the definition of f_n, along with Fubini's Theorem, we see that

$$\begin{aligned} \int_{\mathbb{D}\setminus K} |f_n(z)|^2\, d\mathcal{A}(z) &= \int_{\mathbb{D}\setminus K} \left| \int_{K_n} \frac{w\, d\mu_n(w)}{w - z} \right|^2 d\mathcal{A}(z) \\ &\leq \iint |u||v| \int_{\mathbb{D}\setminus K} \frac{1}{|u - z||v - z|}\, d\mathcal{A}(z)\, d\mu_n(u)\, d\mu_n(v). \end{aligned}$$

In view of Lemma 3.2.3, and the fact that $U_{\mu_n}(z) \leq 1$ everywhere in the plane, it follows that

$$\begin{aligned} \int_{\mathbb{D}\setminus K} |f_n(z)|^2\, d\mathcal{A}(z) &\leq r_n^2 \iint \log \frac{3}{|u - z||v - z|}\, d\mu_n(u)\, d\mu_n(v) \\ &\leq F\, r_n^2 c_2(K_n). \end{aligned}$$

The above estimate yields

$$J_m^{(2)} \leq F\, C_m^{-2} \sum_{n=m}^{\rho} r_n^{-4} r_n^2 c_2(K_n) = F\, C_m^{-1} \leq \frac{F}{m}$$

This completes the proof of the lemma. □

THEOREM 3.4.5. *If K is a compact set in $\mathbb{D}$ that contains the point z_0 and if*

$$\sum_{n=1}^{\infty} 2^{2n} c_2\big(A_n(z_0) \cap K\big) = +\infty$$

where $A_n(z_0)$ is the annulus $\{z\colon 2^{-(n+1)} < |z - z_0| \leq 2^{-n}\}$, then $(z - z_0)\big[L_a^2(\mathbb{D} \setminus K)\big]$ is dense in $L_a^2(\mathbb{D} \setminus K)$.

PROOF. Without loss of generality, we may assume $z_0 = 0$. Now the result follows from Lemma 3.4.4, by taking $r_n = 2^{-n}$ for $n \geq 0$. □

THEOREM 3.4.6. *If K is a compact set in $\mathbb{D}$ that contains the point z_0 and if*

$$\limsup_{r \to 0^+} \frac{c_2\big(\operatorname{cl} B(z_0, r) \cap K\big)}{r^2} > 0,$$

then $(z - z_0)\big[L_a^2(\mathbb{D} \setminus K)\big]$ is dense in $L_a^2(\mathbb{D} \setminus K)$.

PROOF. We may assume that $z_0 = 0$. By assumption, there is a constant $F > 0$ and a sequence $\{r_n\}$ of positive numbers such that: $r_n \downarrow 0$; and

$$c_2\big(\operatorname{cl} B(0, r_n) \cap K\big) \geq F r_n^2.$$

Taking a subsequence, if necessary, we may also assume that

$$r_n \leq 2^{-n}, \quad \text{for all} \quad n \geq 1.$$

Define $A_n = \{z\colon r_{n+1} < |z| \le r_n\}$, and let $n_1 = 1$. From the regularity of the Wiener capacity, we have

$$\lim_{m\to\infty} c_2\Big(\bigcup_{i=n_1}^{n_1+m} A_i \cap K\Big) = c_2\big(\mathrm{cl}\, B(0, r_{n_1}) \cap K\big).$$

Thus we can choose an integer $n_2 \ge n_1$ such that $r_{n_2} \le \frac{1}{2} r_{n_1}$ and

$$c_2\Big(\bigcup_{i=n_1}^{n_2} A_i \cap K\Big) \ge \frac{1}{2} c_2\big(\mathrm{cl}\, B(0, r_{n_1}) \cap K\big) \ge F r_{n_1}^2.$$

Repeating the above procedure, after the m-th step, we have a subsequence

$$r_1 = r_{n_1} > r_{n_2} > \cdots r_{n_m} > r_{n_{m+1}}$$

such that

$$r_{n_{i+1}} \le \frac{1}{2} r_{n_i}, \quad \text{for} \quad 1 \le i \le m$$

and

$$c_2\Big(\bigcup_{i=n_m}^{n_{m+1}} A_i \cap K\Big) \ge F\, r_{n_m}^2.$$

For convenience, let $n_i = i$ for $i \ge 1$, and set $r_0 = 1$. Now since

$$\bigcup_{i=m}^{m+1} A_i \cap K = A_m \cap K,$$

it follows from the above construction that

$$\sum_{n=1}^{\infty} r_n^{-2} c_2(A_n \cap K) = +\infty.$$

Hence the result follows from Lemma 3.4.4. □

COROLLARY 3.4.7. *If K is a compact set in $\mathbb{D}$ containing the point z_0 and if the component of K that contains z_0 consists of more than one point, then $(z - z_0)\big[L^2_a(\mathbb{D} \setminus K)\big]$ is dense in $L^2_a(\mathbb{D} \setminus K)$.*

PROOF. Let F denote the component of K containing z_0, and set $r_n = 2^{-n}$ for all $n \geq 0$.

Using the relation $c_2(K) = \big[\log \frac{1}{c(K)}\big]^{-1}$, together with the fact that F contains more than one point, it follows from Proposition 1.6.6 that

$$c_2\big(B(z_0, r_n) \cap K\big) \geq \big[\log \frac{4}{r_n}\big]^{-1},$$

for large values of n. Thus

$$\frac{c_2\big(\operatorname{cl} B(z_0, r_n) \cap K\big)}{r_n^2} \geq C\,\frac{2^{2n}}{n},$$

where C is some positive constant. Now the result follows from Theorem 3.4.6 or Theorem 3.4.5. □

Our next result, namely Theorem 2.2.1 proved in Chapter 2, follows as an easy consequence of Corollary 3.4.7 and Theorem 3.3.1.

COROLLARY 3.4.8. *If G is an open set in $\mathbb{C}$ for which the component of its complement with respect to $\mathbb{C}_\infty$ that contains ∞ does not equal the singleton $\{\infty\}$, then the Bergman operator S_G is densely defined.*

PROOF. Without loss of generality, we may assume that G contains the closed unit disc. In view of Theorem 3.3.1, dom S_G is dense in $L^2_a(G)$ if and only if $z\big[L^2_a(\mathbb{D}\setminus K)\big]$ is dense in $L^2_a(\mathbb{D}\setminus K)$, where K is the image of $\mathbb{C}_\infty \setminus G$ under the mapping $z \mapsto \dfrac{1}{z}$. Clearly $0 \in \partial K$ and, by our assumption, the component of K that contains 0 consists of more than one point. Now the results follows by applying Corollary 3.4.7. □

The next result shows one of the surprising consequences of Theorem 3.3.1. The author, however, has not been able to obtain this result by means of methods described in Chapter 2.

THEOREM 3.4.9. *If G is an open set of finite area, then the Bergman operator S_G is densely defined.*

PROOF. If G is bounded, then there is nothing to prove. So we may assume that G is an unbounded open set of finite area. By virtue of Corollary 1.3.3, we may also assume that G contains the closed unit disc.

Suppose that the Bergman operator S_G is not densely defined. Since $L^2_a(G)$ is obviously non-trivial, it follows from Theorem 3.3.1 that $z\big[L^2_a(\mathbb{D}\setminus K)\big]$ is not dense in $L^2_a(\mathbb{D}\setminus K)$, where $K \subseteq \mathbb{D}$ is the image of $\mathbb{C}_\infty \setminus G$ under the mapping $z \mapsto \dfrac{1}{z}$. By Theorem 3.4.5, we must have

$$\sum_{n=1}^{\infty} 2^{2n} c_2\big(A_n \cap K\big) < \infty,$$

where A_n is the annulus $\{z\colon 2^{-(n+1)} < |z| \leq 2^{-n}\}$. Consequently

$$2^{2n} c_2(A_n \cap K) \to 0.$$

Let K_n denote any compact subset of $A_n \cap K$. Use (3.4.2) along with Proposition 1.6.6 to conclude that

$$2^{2n} \ln \left(\frac{\pi e}{\text{Area}\,(K_n)} \right)^{-\frac{1}{2}} \to 0.$$

Hence

$$2^{2n} \sqrt{\text{Area}\,(K_n)} \to 0.$$

Since K_n is an arbitrary compact subset of $A_n \cap K$, we conclude that

$$2^{2n} \sqrt{\text{Area}\,(A_n \cap K)} \to 0.$$

Next by the change of variable $w = \dfrac{1}{z}$, we get

$$\begin{aligned} \text{Area}\,(A_n \cap K) &= \int_{A_n \cap K} d\mathcal{A}(w) \\ &= \int_{F_n \cap (\mathbb{C} \setminus G)} \frac{1}{|z|^4}\, d\mathcal{A}(z), \end{aligned}$$

where $F_n = \{z\colon 2^n \leq |z| < 2^{n+1}\}$.

Therefore

$$\text{Area}\,(A_n \cap K) \geq 2^{-4(n+1)} \text{Area}\,\big(F_n \cap (\mathbb{C} \setminus G)\big)$$

and, as a result,

$$\text{Area}\,\big(F_n \cap (\mathbb{C} \setminus G)\big) \to 0.$$

Now since G has finite area we must have that Area $(F_n \cap G) \to 0$, which together with the last expression imply that Area $(F_n) \to 0$. But this is absurd; therefore, the Bergman operator S_G must be densely defined. This completes the proof of the theorem. □

CHAPTER 4

Further Results

4.1. Bergman Projection

In this section, we collect some classical results regarding the Bergman projection and Bergman kernel functions. Since in this section we will only consider the Bergman space of the unit disc $\mathbb{D}$, spaces $L^2(\mathbb{D})$ and $L^2_a(\mathbb{D})$ will be referred to by L^2 and L^2_a respectively.

If $e_n(z) = \sqrt{n+1}\, z^n$, for $n \geq 0$, then it is well known that $\{e_n\}$ forms an *orthonormal basis* for L^2_a. It follows easily from the mean value property that for each point z in $\mathbb{D}$, the *point evaluation* at z is a bounded linear functional on L^2_a. Hence by Riesz representation theorem, there exists a unique function k_z in L^2_a such that $f(z) = \langle f, k_z \rangle$, for all f in L^2_a. Now for f in L^2_a, it follows that

$$\begin{aligned} f(z) &= \sum_{n=0}^{\infty} \langle f, e_n \rangle \, e_n(z) \\ &= \sum_{n=0}^{\infty} \int_{\mathbb{D}} f(w) \overline{e_n(w)} \, d\mathcal{A}(w) \, e_n(z) \\ &= \int_{\mathbb{D}} f(w) \sum_{n=0}^{\infty} \overline{e_n(w)} e_n(z) \, d\mathcal{A}(w) \\ &= \int_{\mathbb{D}} f(w) \overline{k_z(w)} \, d\mathcal{A}(w). \end{aligned}$$

The *Bergman kernel* $k(z,w)$ is a function on $\mathbb{D}\times\mathbb{D}$, defined by

$$k(z,w)=\overline{k_z(w)}.$$

Furthermore, it is easily seen that

$$k(z,w)=\sum_{n=0}^{\infty}(n+1)z^n\overline{w}^n=\frac{1}{(1-z\overline{w})^2}.$$

It should also be mentioned that the definition of $k(z,w)$ is independent of the choice of orthonormal basis, see for example Zhu [**21**] for more details. The orthogonal projection P from L^2 onto L^2_a is given by

$$\big(Pf\big)(z)=\int_{\mathbb{D}}f(w)k(z,w)\,d\mathcal{A}(w)=\langle f,k_z\rangle_{L^2}$$

for all f in L^2. The operator P is called the *Bergman projection.*

LEMMA 4.1.1. *If n and m are non-negative integers, then*

$$P\big(\overline{z}^m z^n\big)=\begin{cases}0, & \text{if } m>n;\\ \dfrac{n+1-m}{n+1}z^{n-m}, & \text{if } m\le n\ .\end{cases}$$

PROOF. From the definition of P, we find

$$P\big(\overline{z}^m z^n\big)=\sum_{J=0}^{\infty}\langle\overline{z}^m z^n,e_j\rangle\,e_j(z).$$

Moreover

$$\begin{aligned}\langle \overline{z}^m z^n, e_j \rangle &= \sqrt{j+1} \int_{\mathbb{D}} \overline{z}^m z^n \overline{z}^j \, d\mathcal{A}(z) \\ &= \frac{\sqrt{j+1}}{\pi} \int_0^1 r^{m+n+j+1} \, dr \int_0^{2\pi} e^{i(n-m-j)\theta} \, d\theta \\ &= \begin{cases} 0, & \text{if } n < m \text{ and } j = 0, 1, 2, \ldots; \\ 0, & \text{if } m \leq n \text{ and } j \neq n-m; \\ \dfrac{1}{n+1}, & \text{if } m \leq n \text{ and } j = n-m \, . \end{cases}\end{aligned}$$

Hence the assertion of the lemma. □

For our next lemma, we briefly discuss the notion of Bloch space.The *Bloch space* $\mathcal{B}$ of $\mathbb{D}$ is the class of all analytic functions on the unit disc for which

$$\|f\|_{\mathcal{B}} = \sup\{(1-|z|^2)|f'(z)| : \ z \in \mathbb{D}\} < +\infty.$$

It turns out that $\mathcal{B}$ can be made into a Banach space by introducing the norm

$$\|f\| = |f(0)| + \|f\|_{\mathcal{B}}.$$

THEOREM 4.1.2. *The Bergman projection P is a bounded operator from $L^\infty(\mathbb{D})$ onto $\mathcal{B}$.*

PROOF. For a proof see, for example, Zhu [**21**], Theorem 5.1.3 on page 77. □

We conclude this section by presenting a useful fact regarding the Bergman space L^2_a.

LEMMA 4.1.3. *If $f \in \mathrm{Hol}(\mathbb{D})$, then f belongs to L^2_a if and only if $(1-|z|^2)f'(z)$ is in L^2.*

PROOF. If the series expansion of f in $\mathbb{D}$ is given by $\sum_{n=0}^{\infty} a_n z^n$, then from the earlier remark, it follows that $\{\frac{a_n}{\sqrt{n+1}}\}$ are the Fourier coefficients of f. Thus f is in L^2_a if and only if

$$\sum_{n=0}^{\infty} \frac{|a_n|^2}{n+1} < +\infty.$$

A formal computation shows that

$$\begin{aligned}
\int_{\mathbb{D}} (1-|z|^2)^2 |f'(z)|^2 \, d\mathcal{A}(z) &= \int_{\mathbb{D}} (1-z\overline{z})^2 \sum_n n a_n z^{n-1} \overline{\sum_m m a_m z^{m-1}} \, d\mathcal{A}(z) \\
&= \sum_{n,m} nm a_n a_m \Big[\int_{\mathbb{D}} z^{n-1} \overline{z}^{m-1} \, d\mathcal{A}(z) - \\
&\quad - 2 \int_{\mathbb{D}} z^n \overline{z}^m \, d\mathcal{A}(z) + \int_{\mathbb{D}} z^{n+1} \overline{z}^{m+1} \, d\mathcal{A}(z) \Big] \\
&= \sum_n n^2 |a_n|^2 \big(\frac{1}{n} - \frac{2}{n+1} + \frac{1}{n+2} \big) \\
&= \sum_n \frac{2n}{(n+1)(n+2)} |a_n|^2.
\end{aligned}$$

Now the result follows easily from the inequalities

$$\sum_n \frac{|a_n|^2}{n+1} \le \sum_n \frac{2n}{(n+1)(n+2)} |a_n|^2 \le 2 \sum_n \frac{|a_n|^2}{n+1}.$$

□

4.2. The Density Problem for the Self-Commutator of the Bergman Operator

This section begins by presenting an extension of the Berger and Shaw Theorem to the unbounded Bergman operators. First we need a definition.

DEFINITION 4.2.1. A bounded operator A acting on a Hilbert space $\mathcal{H}$ is *trace class,* if there is a basis $\{e_n\}$ in $\mathcal{H}$ such that

$$\sum_n \langle |A| e_n, e_n \rangle < +\infty,$$

where $|A|$ is the unique square root of the operator A^*A. The set of trace class operators in $\mathcal{H}$ is denoted by $\mathcal{B}_1(\mathcal{H})$. If $\{e_n\}$ is an orthonormal basis for $\mathcal{H}$, then for $A \in \mathcal{B}_1(\mathcal{H})$, the *trace* of A is defined by

$$\operatorname{tr} A = \sum_n \langle Ae_n, e_n \rangle.$$

Moreover, the definition of trace class and the trace of an operator does not depend on the choice of a basis.

THEOREM 4.2.2. *If G is an open set in $\mathbb{C}$ of finite area such that the corresponding Bergman operator S and its self-commutator $[S^*, S]$ are densely defined, then $[S^*, S]$ is a trace class operator and furthermore*

$$\operatorname{tr} [S^*, S] \leq \frac{1}{\pi} \operatorname{Area} (G).$$

PROOF. For a proof, see Jin [**14**]. □

The above result is one of our main motivations to investigate the density problem regarding $[S^*, S]$; that is, we wish to determine the open subsets of the complex plane for which the corresponding Bergman operator and its self-commutator are both densely defined. Jin [**14**] has shown that if G is an open subset of $\mathbb{C}$ of finite area such that $\{\mathrm{Re}z\colon\ z \in G\}$ is bounded and $\{\mathrm{Im}z\colon\ z \in G\}$ is bounded below, then S_G and $[S^*, S]$ are densely defined.

In this section, we also present some sufficient conditions on an open subset G of the complex plane such that S_G and $[S^*, S]$ are densely defined. Throughout this section by M_φ and S, we mean the operators $M_{\varphi,\mathbb{D}}$ and S_G in accordance with Definition 1.3.1. Recall also that in Corollary 1.4.7 it has been shown that a function f in the class $\mathcal{S}$ has a unique factorization of the form $f = zF(z)$, where F is an outer function in the Hardy space H^p for all $p < \frac{1}{2}$.

LEMMA 4.2.3. *If $\varphi \in \mathcal{S}$, $zF(z)$ denotes φ's unique factorization, and if $P\big(\overline{\varphi}\,\frac{1}{F^2}\,z^n\big)$ belongs to $\mathrm{dom}\, M_\varphi$ for all $n \geq 0$, then the operator M_φ and its self-commutator $[M_\varphi^*, M_\varphi]$ are both densely defined.*

PROOF. First of all note that in view of Lemma 2.2.3, M_φ is densely defined. Therefore M_φ^* and, as a consequence, $[M_\varphi^*, M_\varphi]$ are well defined operators in accordance with Definition 1.1.1 and Definition 1.1.3. Again using Definition 1.1.3, we see that

$$\operatorname{dom}[M_\varphi^*, M_\varphi] = \operatorname{dom} M_\varphi^* M_\varphi \cap \operatorname{dom} M_\varphi M_\varphi^*,$$

where

$$\operatorname{dom} M_\varphi^* M_\varphi = \left\{ f \in L_a^2(\mathbb{D}) : \quad f \in \operatorname{dom} M_\varphi, \quad M_\varphi f \in \operatorname{dom} M_\varphi^* \right\}$$

and

$$\operatorname{dom} M_\varphi M_\varphi^* = \left\{ f \in L_a^2(\mathbb{D}) : \quad f \in \operatorname{dom} M_\varphi^*, \quad M_\varphi^* f \in \operatorname{dom} M_\varphi \right\}.$$

Thus if we set

$$\mathcal{D}_1 = \left\{ f \in L_a^2(\mathbb{D}) : \quad f \in \operatorname{dom} M_\varphi, \quad M_\varphi f \in \operatorname{dom} M_\varphi \right\}$$

and

$$\mathcal{D}_2 = \left\{ f \in L_a^2(\mathbb{D}) : \quad f \in \operatorname{dom} M_\varphi, \quad M_\varphi^* f \in \operatorname{dom} M_\varphi \right\},$$

then from Proposition 1.2.2 part(a), we have

$$\mathcal{D}_1 \cap \mathcal{D}_2 \subseteq \operatorname{dom}[M_\varphi^*, M_\varphi].$$

A similar argument as in the proof of Lemma 2.2.3 shows that $\frac{1}{F^2}$ is a bounded outer function in $H^\infty(\mathbb{D})$. So if we put

$$f_n(z) = \frac{1}{F^2(z)} z^n \quad \text{for} \quad n \geq 0,$$

then f_n belong to $L^2_a(\mathbb{D})$. Furthermore, it is easy to see that $f_n \in \operatorname{dom} M_\varphi$ and $M_\varphi f_n \in \operatorname{dom} M_\varphi$ for all $n \geq 0$. From Proposition 1.2.2 part(b), it follows that

$$M^*_\varphi h = P N^*_\varphi h \quad \text{for all} \quad h \in \operatorname{dom} M_\varphi,$$

where N_φ, the operator of multiplication by φ on $L^2(\mathbb{D})$, is a normal extension of M_φ. Since clearly $N^*_\varphi h = \overline{\varphi} h$ for all h in $\operatorname{dom} N^*_\varphi$, we have that

$$M^*_\varphi f_n = P\big(\overline{\varphi} f_n\big) \quad \text{for} \quad n \geq 0.$$

Finally, by using the assumption, we can conclude that

$$\frac{1}{F^2(z)} z^n \in \mathcal{D}_1 \cap \mathcal{D}_2 \quad \text{for} \quad n \geq 0.$$

Thus the result follows from the Beurling's Theorem (see Theorem 1.4.3).

□

Now we can state one our main results of this section.

THEOREM 4.2.4. *If G is a simply connected region of finite area, then the Bergman operator S_G and its self-commutator $[S^*, S]$ are densely defined.*

PROOF. The fact that S is densely defined is a consequence of Theorem 3.4.9. To show that $[S^*, S]$ is also densely defined, in view of Corollary 1.3.3, it can be assumed that G contains 0. Next let φ be the unique conformal mapping of $\mathbb{D}$ onto G such that $\varphi(0) = 0$ and $\varphi'(0) > 0$. By taking $\frac{\varphi}{\varphi'(0)}$, if necessary, we may also assume that $\varphi'(0) = 1$. From Theorem 1.3.2 part(d), it follows that $\operatorname{dom}[S^*, S]$ is dense in $L^2_a(G)$ if and only if $\operatorname{dom}[M^*_\varphi, M_\varphi]$ is dense in $L^2_a(\mathbb{D})$.

Now since φ belongs to $\mathcal{S}$, from Lemma 4.2.3 we see that it is enough to verify $P\big(\overline{\varphi}\frac{1}{F^2(z)}z^n\big)$ belongs to $\operatorname{dom} M_\varphi$ for all $n \geq 0$.

CLAIM. *If $\varphi \in \operatorname{Hol}(\mathbb{D})$ and $\varphi' \in L^2_a(\mathbb{D})$, then φPh belongs to $L^2_a(\mathbb{D})$ for all h in $L^\infty(\mathbb{D})$.*

PROOF OF THE CLAIM. By Lemma 4.1.3, it suffices to show that $(1-|z|^2)\big(\varphi Ph\big)'(z)$ belongs to $L^2(\mathbb{D})$ for all h in $L^\infty(\mathbb{D})$. Since

$$(1-|z|^2)\big(\varphi(z)\big(Ph\big)(z)\big)' = (1-|z|^2)\big(Ph\big)(z)\varphi'(z)+(1-|z|^2)\big(Ph\big)'(z)\varphi(z),$$

it is enough to show that both terms in the above equality belong to $L^2(\mathbb{D})$.

For the first term we have

$$\big|(1-|z|^2)\big(Ph\big)(z)\varphi'(z)\big| \leq (1-|z|^2)\int_{\mathbb{D}} \frac{|h(w)|}{|1-\overline{w}z|^2}\, d\mathcal{A}(w)\ |\varphi'(z)|.$$

Using Hölder's inequality, we get

$$\begin{aligned}\int_{\mathbb{D}} \frac{|h(w)|}{|1-\overline{w}z|^2}\, d\mathcal{A}(w) &\leq \|h\|_\infty^{\frac{1}{2}} \left(\int_{\mathbb{D}} \frac{1}{|1-\overline{w}z|^4}\, d\mathcal{A}(w)\right)^{\frac{1}{2}} \cdot \text{Area}\,(\mathbb{D})^{\frac{1}{2}} \\ &\leq C\,\|h\|_\infty^{\frac{1}{2}} \langle k_z, k_z\rangle^{\frac{1}{2}} \\ &= C\,\|h\|_\infty^{\frac{1}{2}} k(z,z)^{\frac{1}{2}},\end{aligned}$$

where C is a positive constant. Now since by assumption φ' is in $L^2_a(\mathbb{D})$, it follows that $(1-|z|^2)\big(Ph\big)(z)\varphi'(z) \leq C\,\|h\|_\infty^{\frac{1}{2}}|\varphi'(z)|$ belongs to $L^2(\mathbb{D})$.

Next by Theorem 4.1.2, Ph belongs to the Bloch space $\mathcal{B}$; that is

$$(1-|z|^2)\big|\big(Ph\big)'(z)\big| \leq \|Ph\|_{\mathcal{B}} < +\infty.$$

Thus

$$\big|(1-|z|^2)\big(Ph\big)'(z)\big||\varphi(z)| \leq \|Ph\|_{\mathcal{B}}|\varphi(z)|.$$

But since $\varphi' \in L^2_a(\mathbb{D})$, we also have that φ is in $L^2_a(\mathbb{D})$. This proves the claim. □

To complete the proof of the theorem note that

$$\Big|\overline{\varphi(z)}\,\frac{1}{F(z)^2}\,z^n\Big| = \Big|\frac{1}{F(z)}\,\overline{z}z^n\Big| \leq \Big|\frac{1}{F(z)}\Big|,$$

for all $n \geq 0$. As it has been shown earlier, $\frac{1}{F}$ belongs to $H^\infty(\mathbb{D})$ (see the proof of Lemma 2.2.3). Furthermore since G has finite area, from the identity

$$\int_G |\varphi'(z)|^2 \, d\mathcal{A}(z) = \text{Area}\,(G)$$

we have that φ' belongs to $L^2_a(\mathbb{D})$. Hence the result follows from the above claim. □

THEOREM 4.2.5. *If G is an open half plane in $\mathbb{C}$, then the corresponding Bergman operator and its self-commutator are densely defined.*

PROOF. By Theorem 2.2.1, the Bergman operator S is clearly densely defined. Next if T is a linear mapping defined on $\mathbb{C}$ by $T\colon z \mapsto rz + c$, where $r \neq 0$ and c are complex numbers, then one easily sees that

$$\text{dom}\, M^*_{T,T(G)} = \text{dom}\, S^*_{T(G)}.$$

Thus in view of Theorem 1.3.2, we may assume that G is the right half plane $\mathbb{H} = \{z\colon \ \text{Re}\, z > 0\}$.

If $\varphi = \dfrac{z}{1-z}$, then $\varphi \in \text{Hol}(\mathbb{D})$ and φ maps $\mathbb{D}$ conformally onto $\mathbb{H}$. Again from Theorem 1.3.2, it follows that $\text{dom}\,[S^*, S]$ is dense in $L^2_a(\mathbb{H})$ if and only if $\text{dom}\,[M^*_\varphi, M_\varphi]$ is dense in $L^2_a(\mathbb{D})$.

Using the fact that $\varphi(z) = \sum_{k=0}^{\infty} z^k$ in $\mathbb{D}$, if $z \in \mathbb{D}$ and $n \geq 0$, then

$$\begin{aligned}
\left| P\left(\overline{\varphi(z)}(1-z)^2\right) - \sum_{k=1}^{m} P\left(\overline{z}^k(1-z)^2 z^n\right) \right| &= \left| P\left(\left(\overline{\varphi(z)} - \sum_{k=1}^{m} \overline{z}^k\right)(1-z)^2 z^n\right) \right| \\
&= \left| P\left(\frac{\overline{z}^{m+1}}{1-\overline{z}}(1-z)^2 z^n\right) \right| \\
&\leq \left\| \frac{\overline{z}^{m+1}}{1-\overline{z}}(1-z)^2 z^n \right\|_{L^2(\mathbb{D})} \\
&\leq 2\|\overline{z}^{m+1}\|_{L^2(\mathbb{D})} \\
&= 2\sqrt{\frac{\pi}{m+2}}.
\end{aligned}$$

Since the last term in the above inequality tends to zero as $m \to \infty$, it follows that

$$P\left(\overline{\varphi(z)}(1-z)^2 z^n\right) = \sum_{k=1}^{\infty} P\left(\overline{z}^k(1-z)^2 z^n\right).$$

On the other hand, by Lemma 4.1.1

$$\begin{aligned}
\sum_{k=1}^{\infty} P\left(\overline{z}^k(1-z)^2 z^n\right) &= \sum_{k=1}^{\infty} P\left(\overline{z}^k z^n\right) - 2\sum_{k=1}^{\infty} P\left(\overline{z}^k z^{n+1}\right) + \sum_{k=1}^{\infty} P\left(\overline{z}^k z^{n+2}\right) \\
&= \sum_{k=1}^{n} \frac{n+1-k}{n+1} z^{n-k} - 2\sum_{k=1}^{n+1} \frac{n+2-k}{n+2} z^{n+1-k} + \\
&\quad \sum_{k=1}^{n+2} \frac{n+3-k}{n+3} z^{n+2-k} \\
&= p_{n+1}(z),
\end{aligned}$$

where $p_{n+1}(z)$ is a polynomial of degree $n+1$. An easy calculation shows that $p_{n+1}(1) = 0$. Hence

$$P\left(\overline{\varphi(z)}(1-z)^2 z^n\right) = (1-z)p_n(z),$$

where $p_n(z)$ is a polynomial in z of degree n. It follows easily now that $P\left(\overline{\varphi(z)}(1-z)^2 z^n\right)$ belongs to $\operatorname{dom} M_\varphi$ for all $n \geq 0$. Finally noting that φ belongs to the class $\mathcal{S}$, and $F(z) = \dfrac{1}{1-z}$ is the unique outer function in φ's factorization, the result follows from Lemma 4.2.3. □

4.3. Spectrum of the Bergman Operator

To avoid trivialities, throughout the rest of this study, *by an open set G it is always implicitly assumed that the resulting Bergman space $L^2_a(G)$ is non-trivial.* See Theorem 2.1.4 for an exact explanation of when $L^2_a(G)$ is trivial.

DEFINITION 4.3.1. An operator $A\colon \mathcal{H} \to \mathcal{H}$ is said to be *boundedly invertible*, if there exists a bounded operator B in $\mathcal{B}(\mathcal{H})$ such that $AB = 1$ and $BA \subseteq 1$, where $1\colon x \mapsto x$ denotes the identity operator. The operator B is called the *(bounded) inverse* of A.

The uniqueness assertion in the next result explains why we can speak of *the* (bounded) inverse of a boundedly invertible operator.

PROPOSITION 4.3.2. *An operator* $A\colon \mathcal{H} \to \mathcal{H}$ *is boundedly invertible if and only if* $\ker A = (0)$, $\operatorname{ran} A = \mathcal{H}$, *and the graph of* A *is closed. Furthermore, if* A *is boundedly invertible, then its (bounded) inverse is unique.*

PROOF. A proof can be found in Conway [7], Proposition 1.15 on page 307. □

DEFINITION 4.3.3. If $A\colon \mathcal{H} \to \mathcal{H}$ is an operator, denote by $\rho(A)$ the set of all points $\lambda \in \mathbb{C}$ such that the operator $\lambda - A$ is boundedly invertible. $\rho(A)$ is called the *resolvent* set for A. The *spectrum* of A, $\sigma(A)$, is defined by $\sigma(A) = \mathbb{C}\backslash\rho(A)$. The *point spectrum* of A, $\sigma_p(A)$, is defined as the set of all points $\lambda \in \mathbb{C}$ such that $Ax = \lambda x$ for some non-zero vector x in $\operatorname{dom} A$.

PROPOSITION 4.3.4. *If* $A\colon \mathcal{H} \to \mathcal{H}$ *is an operator, then* $\sigma(A)$ *is closed.*

PROOF. See, for example, Conway [7], Proposition 1.17 on page 307. □

Having introduced the basic concepts regarding the spectrum and the point spectrum of an operator, we can now state the main result of this section.

THEOREM 4.3.5. *If G is an open set in $\mathbb{C}$, and if S denotes the Bergman operator acting on $L_a^2(G)$, then $\sigma(S) = \mathrm{cl}\,(G)$ and $\sigma_p(S) = \emptyset$.*

The proof of Theorem 4.3.5 is essentially a generalization of the similar results for the case of (bounded) Bergman operators. One can see, for example, Conway [8], Theorem 8.5 on page 67. Before presenting the proof, however, we need a lemma.

LEMMA 4.3.6. *If G is an open set in $\mathbb{C}$ and $\lambda \in G$, then* $\mathrm{ran}\,(S-\lambda)$ *is a closed subset of $L_a^2(G)$, and*

$$\mathrm{ran}\,(S-\lambda) = \big\{f \in L_a^2(G)\colon\ f(\lambda) = 0\big\}.$$

PROOF. The fact that $\mathrm{ran}\,(S-\lambda)$ is closed, follows easily once the second assertion of the lemma has been proved.

Let $\mathcal{L}_\lambda = \big\{f \in L_a^2(G)\colon\ f(\lambda) = 0\big\}$. If $g \in \mathrm{ran}\,(S-\lambda)$, then $g(z) = (z-\lambda)f(z)$ for some $f \in \mathrm{dom}\,S$. Since $g(\lambda) = 0$, it follows that $\mathrm{ran}\,(S-\lambda) \subseteq \mathcal{L}_\lambda$. Conversely, if $f \in \mathcal{L}_\lambda$, then f has a zero at λ. It follows that $f(z) = (z-\lambda)g(z)$, where g belongs to $\mathrm{Hol}(G)$. Now the proof follows, if it can be shown that $g \in \mathrm{dom}\,S$. To see this, if $\delta > 0$

such that $\operatorname{cl} B = \operatorname{cl} B(\lambda, \delta) \subseteq G$, then

$$\begin{aligned}\int_G |g(z)|^2 \, d\mathcal{A}(z) &= \int_{\operatorname{cl} B} |g(z)|^2 \, d\mathcal{A}(z) + \int_{G \setminus \operatorname{cl} B} \frac{f(z)}{z-\lambda} \, d\mathcal{A}(z) \\ &\leq \max_{z \in \operatorname{cl} B} |g(z)|^2 \, \pi\delta^2 + \delta^{-2} \|f\|^2_{L^2_a(G)} \\ &< \infty.\end{aligned}$$

Thus g belongs to $L^2_a(G)$, and, moreover,

$$\begin{aligned}\|zg\|_{L^2_a(G)} &\leq \|(z-\lambda)g\|_{L^2_a(G)} + \|\lambda g\|_{L^2_a(G)} \\ &= \|f\|_{L^2_a(G)} + |\lambda| \|g\|_{L^2_a(G)} < +\infty.\end{aligned}$$

This completes the proof of the lemma. □

PROOF OFTHEOREM 4.3.5. If $\lambda \notin \operatorname{cl}(G)$, then $(z-\lambda)^{-1}$ is a bounded analytic function on G. It follows from Definition 4.3.1 that $M_{(z-\lambda)^{-1}}$, the operation of multiplication by $(z-\lambda)^{-1}$ on $L^2_a(G)$, is the (bounded) inverse of $S - \lambda$. Hence by Definition 4.3.3, we have that $\sigma(S) \subseteq \operatorname{cl}(G)$.

Next assume that $\lambda \in G$. We will show that $S - \lambda$ is not invertible; or equivalently, in view of Proposition 4.3.2, $\operatorname{ran}(S-\lambda) \neq L^2_a(G)$. To reach a contradiction, suppose that $\operatorname{ran}(S-\lambda) = L^2_a(G)$.

If G is connected, then there is a finite integer $n \geq 1$ such that $f(z) = (z-\lambda)^n g(z)$, where $g(\lambda) \neq 0$ and $g \in \operatorname{Hol}(G)$. A similar argument, as in the proof of Lemma 4.3.6, shows that $g \in \operatorname{dom} S$. Since $g(\lambda) \neq 0$, Lemma 4.3.6 implies that $g \notin \operatorname{ran}(S-\lambda)$; a contradiction.

If G is not connected, denote by G_λ the component of G which contains λ, and let a be a point in $G \setminus G_\lambda$. Define $f\colon G \to \mathbb{C}$ by

$$f(z) = \begin{cases} 0, & \text{if } z \in G \setminus G_\lambda; \\ \dfrac{1}{(z-a)^3}, & \text{if } z \in G_\lambda. \end{cases}$$

Clearly $f \in \mathrm{Hol}(G)$, and

$$\int_G |f(z)|^2 \, d\mathcal{A}(z) = \int_{G_\lambda} |f(z)|^2 \, d\mathcal{A}(z) \leq \int_{\mathbb{C}\setminus \mathrm{cl}\, B(a,\delta)} \frac{1}{|z-a|^6} \, d\mathcal{A}(z),$$

where $\delta > 0$ is chosen such that $\mathrm{cl}\, B(a,\delta) \subseteq G \setminus G_\lambda$. Since the last integral in the above inequality is finite, we have that $f \in L^2_a(G)$. But again, since $f(\lambda) \neq 0$, it follows from Lemma 4.3.6 that $f \notin \mathrm{ran}\,(S-\lambda)$; a contradiction.

Hence we have proved that $G \subseteq \sigma(S)$, and since $\sigma(S)$ is closed, one sees that $\mathrm{cl}\, G \subseteq \sigma(S)$. This proves the first claim.

Finally, if $\lambda \in \mathrm{cl}\, G$ and $\lambda \in \sigma_p(S)$, then $Sf = \lambda f$ for all f in $\mathrm{dom}\, S$; that is, $(z-\lambda)f(z) = 0$ for all $z \in G$. Hence f must vanish identically on G, and, as a consequence, $\ker(S-\lambda) = (0)$. □

4.4. The Bounded Commutant of the Bergman Operator

This section contains a generalization of a result regarding the commutant of the (bounded) Bergman operator to the unbounded case.

First we need to define the concept of commuting operators in a more general setting.

DEFINITION 4.4.1. If $A\colon \mathcal{H} \to \mathcal{H}$ is an operator, a bounded operator B is said to *commute* with A, if $BA \subseteq AB$; that is, $\operatorname{dom} A \subseteq \operatorname{dom} AB$ and $BAx = ABx$ for all $x \in \operatorname{dom} A$.

REMARK 4.4.2. From Definition 4.4.1 it follows that B in $\mathcal{B}(\mathcal{H})$ commutes with an operator $A\colon \mathcal{H} \to \mathcal{H}$ if and only if

$$B \operatorname{dom} A \subseteq \operatorname{dom} A \quad \text{and} \quad BAx = ABx \quad \forall x \in \operatorname{dom} A.$$

The next lemma will be used in the proof of the basic result of this section.

LEMMA 4.4.3. *Let $A\colon \mathcal{H} \to \mathcal{H}$ be a densely defined operator and suppose that $B \in \mathcal{B}(\mathcal{H})$. If $BA \subseteq AB$, then $B^*A^* \subseteq A^*B^*$.*

PROOF. If $x \in \operatorname{dom} A$ and $y \in \operatorname{dom} A^*$, then

$$\begin{aligned} |\langle Ax, B^*y\rangle| &= |\langle BAx, y\rangle| \\ &= |\langle ABx, y\rangle| \\ &= |\langle Bx, A^*y\rangle| \leq \|B\|\,\|A^*y\|\,\|x\|. \end{aligned}$$

Thus the operator $x \mapsto \langle Ax, B^*y\rangle$ is a bounded functional on $\operatorname{dom} A$. So, from Definition 1.1.1, it follows that $B^*y \in \operatorname{dom} A^*$; that is,

$$(\star) \qquad B^*\operatorname{dom} A^* \subseteq \operatorname{dom} A^*.$$

Furthermore, if $x \in \operatorname{dom} A$ and $y \in \operatorname{dom} A^*$, then

$$\begin{aligned}\langle B^*A^*y, x\rangle &= \langle A^*y, Bx\rangle \\ &= \langle y, ABx\rangle \\ &= \langle y, BAx\rangle \quad \text{(by assumption)} \\ &= \langle B^*y, Ax\rangle \\ &= \langle A^*B^*y, x\rangle \quad \text{(by } (\star)\text{)}.\end{aligned}$$

Since $\operatorname{dom} A$ is dense in $\mathcal{H}$, we have that

$$B^*A^*y = A^*B^*y \text{ for all } y \text{ in } \operatorname{dom} A^*.$$

Now the result follows from Remark 4.4.2. □

DEFINITION 4.4.4. If $A\colon \mathcal{H} \to \mathcal{H}$ is an operator, the *bounded commutant* of A, denoted by $\{A\}'_{bd}$, is the set of all bounded operators B such that B commutes with A; that is,

$$\{A\}'_{bd} = \{B \in \mathcal{B}(\mathcal{H})\colon \quad BA \subseteq AB\}.$$

Note that if the operator A is also bounded , then $\{A\}'_{bd}$ is the usual *commutant* of A, denoted by $\{A\}'$. Thus the notion of the bounded commutant can be regarded as a generalization of the classical definition of the commutant for a bounded operator. Now we can state the main result of this section.

THEOREM 4.4.5. *If G is an open set in $\mathbb{C}$ such that the corresponding Bergman operator S is densely defined, then*

$$\{A\}'_{bd} = \{M_\varphi : \quad \varphi \in H^\infty(G)\}.$$

The idea of the proof is due to Shields and Wallen [**18**]. First, we need a lemma.

LEMMA 4.4.6. *If G is an open set in $\mathbb{C}$ such that the Bergman operator S_G is densely defined and if $\lambda \in G$, then $k_\lambda \in \ker(S^* - \overline{\lambda})$, where k_λ denotes the Bergman kernel function.*

PROOF. Recall that if $\lambda \in G$, then k_λ is the unique element in $L^2_a(G)$ such that $f(\lambda) = \langle f, k_\lambda \rangle$ for all f in $L^2_a(G)$. First, it must be shown that k_λ belongs to $\operatorname{dom} S^*$. We observe that if $f \in \operatorname{dom} S$, then

$$\begin{aligned} |\langle Sf, k_\lambda \rangle| &= |(Sf)(\lambda)| \\ &= |\lambda f(\lambda)| \\ &\leq |\lambda| \, \|k_\lambda\| \, \|f\|. \end{aligned}$$

Thus $f \mapsto \langle Sf, k_\lambda \rangle$ is a bounded functional on $\operatorname{dom} S$, and consequently $k_\lambda \in \operatorname{dom} S^*$.

Next, if $f \in \operatorname{dom} S$, then we have that

$$\begin{aligned} \langle f, S^* k_\lambda \rangle &= \langle Sf, k_\lambda \rangle \\ &= \lambda \langle f, k_\lambda \rangle \\ &= \langle f, \overline{\lambda} k_\lambda \rangle. \end{aligned}$$

Now since $\operatorname{dom} S$ is dense in $L^2_a(G)$, it follows that $S^* k_\lambda = \overline{\lambda} k_\lambda$; that is $k_\lambda \in \ker (S^* - \overline{\lambda})$. □

PROOF OF THEOREM 4.4.5. First, we verify that if $\lambda \in L^2_a(G)$, then $L^2_a(G)$ can be written as a direct sum

$$(\star) \qquad L^2_a(G) = \operatorname{ran} (S - \lambda) \oplus \mathbb{C} k_\lambda,$$

where k_λ is the Bergman kernel function. Now if we define $h(z) = \dfrac{1}{\|k_\lambda\|^2} k_\lambda(z)$, then $h \in L^2_a(G)$, and, moreover,

$$h(\lambda) = \frac{1}{\|k_\lambda\|^2} k_\lambda(\lambda) = \frac{1}{\|k_\lambda\|^2} \langle k_\lambda, k_\lambda \rangle = 1.$$

If $f \in L^2_a(G)$, then f can be written as

$$f(z) = f(\lambda) h(z) + \big(f - f(\lambda) h\big)(z).$$

Noting that $\big(f - f(\lambda)h\big)(z)$ vanishes at λ, it follows from Lemma 4.3.6 that $f - f(\lambda)h$ belongs to $\operatorname{ran}(S - \lambda)$. Furthermore, one sees that $f(\lambda)h = \dfrac{f(\lambda)}{\|k_\lambda\|}\, k_\lambda \in \mathbb{C}k_\lambda$; hence $(\star)$ is established.

On the other hand, it is a known fact that

$$\ker(S^* - \overline{\lambda}) = \ker(S - \lambda)^* = \big[\operatorname{ran}(S - \lambda)\big]^{\perp}.$$

So by $(\star)$, along with Lemma 4.4.6, we find that

$$(\star\star) \qquad \ker(S^* - \overline{\lambda}) = \mathbb{C}k_\lambda.$$

Let B belong to $\big\{S\big\}'_{bd}$. In view of Lemma 4.4.3, we have that $B^*S^* \subseteq S^*B^*$; so from $(\star\star)$, it follows

$$\overline{\lambda}B^*k_\lambda = B^*S^*k_\lambda = S^*B^*k_\lambda.$$

Thus, it is shown that $B^*k_\lambda \in \ker(S^* - \overline{\lambda})$. Since λ is arbitrary, we conclude that there is a function $\varphi\colon G \to \mathbb{C}$ such that $B^*k_\lambda = \overline{\varphi(z)}k_\lambda$ Now if $f \in L^2_a(G)$, then

$$\big(Bf\big)(z) = \langle Bf, k_\lambda\rangle = \langle f, B^*k_\lambda\rangle = \varphi(z)f(z).$$

So we have $\varphi(z) \in \sigma_p(B)$. Next, using the fact that

$$\sigma(B) \subseteq \big\{z\colon |z| \leq \|B\|\big\},$$

we see that $|\varphi(z)| \leq \|B\|$ for all $z \in G$. Hence $\varphi \in L^\infty(G)$, and consequently $B \equiv M_\varphi$. Finally, we will show that φ belongs to $\mathrm{Hol}(G)$, proving

$$\{S\}'_{bd} \subseteq \{M_\varphi : \quad \varphi \in H^\infty(G)\}.$$

To see this, fix λ in G. From $(\star)$, along with Lemma 4.3.6, there exists a function f in $L^2_a(G)$ such that $f(\lambda) \neq 0$. Therefore, there is a neighborhood of λ, say U, in G where $f \not\equiv 0$ for all z in U. If h denotes $\frac{1}{f} \mid U$, then h is analytic on U. Moreover

$$\varphi(z) = h(z)\big(Bf\big)(z).$$

Thus φ is analytic in a neighborhood of λ; but since λ is arbitrary, we conclude that $\varphi \in \mathrm{Hol}(G)$.

The proof of the other inclusion follows easily by observing that if $\varphi \in H^\infty(G)$, then clearly $M_\varphi \mathrm{dom}\, S \subseteq \mathrm{dom}\, S$, and $M_\varphi S f = S M_\varphi f$ for all $f \in \mathrm{dom}\, S$. So, in view of Remark 4.4.2, $\mathrm{M}_\varphi \in \{S\}'_{bd}$. This completes the proof of the theorem. □

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Vita

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